SÉRIES E EQUAÇÕES DIFERENCIAIS

LISTA 6 - SÉRIES DE FOURIER

July 2, 2017

Matrícula: 378592

Rayon Lindraz Nunes

Engenharia de Computação

UFC - Sobral

Ache as séries de Fourier das seguintes funções:

1

$f(x) = x, \quad -\pi \le x \le \pi, \quad T = 2\pi$

S. $t = 2\pi = 2l \quad \Rightarrow \quad l = \pi$

$$f(x) \quad = \quad \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_{\text{n}}\cos\left(\frac{n\pi x}{l}\right) + b_{\text{n}}\sin\left(\frac{n\pi x}{l}\right)\right] \tag{1}$$

$$a_0 \quad = \quad \frac{1}{l}\int_{-l}^{l} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\,dx \quad = \quad \frac{1}{\pi}\left.\frac{x^2}{2}\right|_{-\pi}^{\pi} \;=\; 0 \tag{2}$$

$$\begin{aligned} a_{\text{n}} &= \frac{1}{l}\int_{-l}^{l} f(x)\,\cos\left(\frac{n\pi x}{l}\right)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx \\ &= \frac{1}{\pi}\left[\frac{\cos nx}{n^2} + \frac{x\,\sin nx}{n}\right]\Bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos(n\pi)}{n^2} + \frac{\pi\,\sin n\pi}{n} - \frac{\pi\,\sin n\pi}{n}\right] \;=\; 0 \end{aligned} \tag{3}$$

$$\begin{aligned} b_{\text{n}} &= \frac{1}{l}\int_{-l}^{l} f(x)\,\sin\left(\frac{n\pi x}{l}\right)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx \\ &= \frac{1}{\pi}\left[\frac{\sin nx}{n^2} + \frac{x\,\sin nx}{n}\right]\Bigg|_{-\pi}^{\pi} \\ &= \frac{1}{\pi}\left[\frac{\sin n\pi}{n^2} - \frac{\pi\,\cos(n\pi)}{n} - \frac{\sin(-n\pi)}{n^2} - \frac{\pi\,\cos(-n\pi)}{n}\right] \end{aligned} \tag{4}$$

$$\begin{aligned} &\left[\frac{\sin n\pi}{n^2} + \frac{\sin n\pi}{n^2} - \frac{\pi\,\cos(n\pi)}{n} + \frac{\pi\,\cos(n\pi)}{n}\right] \\ &= \left[2\frac{\sin n\pi}{n^2} - 2\frac{\cos n\pi}{n}\right] = 0 - \frac{2\pi(-1^n)}{n}(*-1) = \frac{2\pi(-1)^{n+1}}{n} \end{aligned} \tag{5}$$

Substituindo (5) em (4) obtemos:

$$b_{\text{n}} \;=\; \frac{1}{\pi}\,\frac{2\pi(-1)^{n+1}}{n} \;=\; (-1)^{n+1}\frac{2}{n} \tag{6}$$

Portanto, substituindo (2),(3) e (6) em (1) temos a Série:

$$f(x) \quad = \quad \sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{2}{n}\,\sin nx \quad = \quad 2\sum_{n=1}^{\infty}(-1)^{n+1}\,\frac{\sin nx}{n} \tag{7}$$

2

$$f(x) = \begin{cases} -x, & -\pi \leq x \leq 0, \quad T = 2\pi \\ x, & 0 \leq x \leq \pi \end{cases}$$

S. $t = 2\pi = 2l \quad \Rightarrow \quad l = \pi$

$$a_0 \quad = \quad \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\,dx + \int_{0}^{\pi} x\,dx\right] \quad = \quad \pi \tag{1}$$

$$a_{\mathrm{n}} = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right] \tag{2}$$

$$\begin{aligned} \int_{-\pi}^{0}(-x)\cos nx\,dx &= \left[-\frac{\cos nx}{n^2} - \frac{x\,\sin nx}{n}\right]\Bigg|_{-\pi}^{0} \\ = \frac{\cos(0\,n)}{n^2} - \frac{0\,\sin n\,0}{n} &+ \frac{\cos(-n\pi)}{n^2} - \frac{\pi\sin(-n\pi)}{n} \\ &= \frac{\cos(-n\pi)}{n^2} \end{aligned} \tag{3}$$

$$\begin{aligned} \int_{0}^{\pi} x\cos nx\,dx &= \left[\frac{\cos nx}{n^2} + \frac{x\,\sin nx}{n}\right]\Bigg|_{0}^{\pi} \\ = \frac{\cos(n\pi)}{n^2} + \frac{\pi\,\sin n\pi}{n} &- \frac{\cos(n\,0)}{n^2} - \frac{0\,\sin(0\,n)}{n} \\ &= \frac{\cos n\pi}{n^2} \end{aligned} \tag{4}$$

Substituindo (4) e (3) em (2) temos:

$$a_n = \frac{1}{\pi}\left[-\frac{2}{n^2} + 2\frac{\cos n\pi}{n^2}\right] \tag{5}$$

$$\cos(n\pi) = \begin{cases} 1, & n \quad \text{é } par \\ -1, & n \quad \text{é } impar \end{cases}$$

Então temos que a equação (5) corresponde à:

$$a_n = \frac{1}{\pi}\left[-\frac{2}{n^2} + 2\frac{\cos n\pi}{n^2}\right] = \begin{cases} 0, & n \quad \text{é } par \\ -\frac{4}{\pi n^2}, & n \quad \text{é } impar \end{cases} \tag{6}$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right] \quad = \quad 0 \tag{7}$$

Temos portanto, a Série de Fourier a partir dos coeficientes em (1),(6) e (7) como:

$$\begin{aligned} f(x) &= \frac{\pi}{2} + \sum_{n=0}^{\infty}\left[-\frac{4}{\pi n^2}\cos\left(\frac{n\pi x}{l}\right)(*-1)\right] \\ &= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos[(2n+1)x]}{(2n+1)^2} \end{aligned} \tag{8}$$

3

$$f(x) = \begin{cases} -1, & -\pi < x < 0, \quad T = 2\pi \\ 1, & 0 \le x \le \pi \end{cases}$$

S. $t = 2\pi = 2l \quad \Rightarrow \quad l = \pi$

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\,dx + \int_{0}^{\pi} dx\right] = 0 \tag{1}$$

$$a_{\mathrm{n}} = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx\right] \tag{2}$$

$$\begin{aligned} \int_{-\pi}^{0}(-1)\cos nx\,dx &= -1\left[\frac{\sin nx}{n\pi}\right]\Bigg|_{-\pi}^{0} \\ = \frac{\sin(0\,n)}{n\pi} - \frac{\sin n\pi}{n\pi} = -\frac{\sin n\pi}{n\pi} &= 0 \end{aligned} \tag{3}$$

$$\begin{aligned} \int_{0}^{\pi}\cos nx\,dx &= \left[\frac{\sin nx}{n\pi}\right]\Bigg|_{0}^{\pi} \\ = \frac{\sin(0\,n)}{n\pi} - \frac{\sin n\pi}{n\pi} = -\frac{\sin n\pi}{n\pi} &= 0 \end{aligned} \tag{4}$$

Substituindo (4) e (3) em (2) temos:

$$a_n = 0 \tag{5}$$

$$b_{\mathrm{n}} = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nx\,dx + \int_{0}^{\pi}\sin nx\,dx\right] \tag{6}$$

$$\begin{aligned} \int_{-\pi}^{0}(-1)\sin nx\,dx &= \left(\frac{\cos nx}{n}\right)\Bigg|_{-\pi}^{0} \\ &= \frac{\cos n\,0}{n} - \frac{\cos(-n\pi)}{n} \end{aligned} \tag{7}$$

$$\begin{aligned} \int_{0}^{\pi}\sin nx\,dx &= \left(\frac{\cos nx}{n}\right)\Bigg|_{0}^{\pi} \\ &= \frac{\cos n\,0}{n} - \frac{\cos n\pi}{n} \end{aligned} \tag{8}$$

Substituindo (8) e (7) em (6) temos:

$$b_n = \frac{2}{n\pi}\ 1 - \cos(n\pi)\ = \begin{cases} 0, & n \quad \text{é}\, par \\ \frac{4}{\pi n}, & n \quad \text{é}\, impar \end{cases} \tag{9}$$

Temos portanto, a Série de Fourier a partir dos coeficientes em (1),(5) e (9) como:

$$\begin{aligned} f(x) \quad &= \quad \sum_{n=1}^{\infty} \frac{4}{\pi} \frac{\sin[(2n+1)x]}{2n+1} (* - 1) \\ &= \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin[(2n+1)x]}{(2n+1)} \end{aligned} \tag{10}$$

4

$f(x) = x^2, \quad -\pi \leq x \leq \pi, \quad T = 2\pi$

S. A função quadrática é tida como uma função par, ou seja $g(-x) = g(x)$ e de acordo com o teorema: $b_n = 0$
então:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx \ = \ \frac{2}{\pi}\left[\frac{x^3}{3}\right]\Bigg|_0^{\pi} = 2\frac{\pi^2}{3} \tag{1}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx = \frac{2}{\pi}\left[\ \frac{x^2 \sin nx}{n}\Bigg|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x \sin nx dx\right] \tag{2}$$

$$\int_0^{\pi} x \sin nx dx = \ \frac{\sin nx}{n^2} - \frac{x \cos nx}{n}\Bigg|_0^{\pi} \tag{3}$$

Substituindo (3) em (4) obtemos:

$$\begin{aligned} a_n &= \frac{2}{\pi}\left[\ \frac{x^2 \sin nx}{n}\Bigg|_0^{\pi} - \ \frac{\sin nx}{n^2} - \frac{x \cos nx}{n}\Bigg|_0^{\pi}\right] \\ &= \frac{2}{\pi}\left[\frac{\pi^2 \sin\ n\pi}{n} - \frac{0^2 \sin\ n\,0}{n} - \frac{2}{n}\left(\frac{\sin n\pi}{n^2} - \frac{\pi \cos n\pi}{n} - \frac{\sin nx}{n^2} - \frac{0 \cos 0\, x}{n}\right)\right] \\ Como: &\quad \frac{\pi^2 \sin\ n\pi}{n} = \frac{0^2 \sin\ n\,0}{n} = \frac{\sin n\pi}{n^2} = \frac{0 \cos 0\, x}{n} \ = 0 \\ Então: &\quad a_n = \frac{4}{n^2\pi}(\pi \cos n\pi) = \frac{4\ \cos n\pi}{n^2} = \begin{cases} \frac{4}{n^2}, & n \quad \text{é}\, par \\ -\frac{4}{\pi n}, & n \quad \text{é}\, impar \end{cases} \end{aligned} \tag{4}$$

Logo, podemos montar a Série de Fourier a partir dos coeficientes em (1) e (4)

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} \tag{5}$$

5

$$f(x) = \begin{cases} 0, & -\pi \leq x \leq \pi\,, \quad T = 2\pi \\ x, & 0 < x \leq \pi \end{cases}$$

S.

$$a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} 0dx + \int_{-\pi}^{\pi} xdx\right] \tag{1}$$

$$\begin{gathered} \int_{-\pi}^{\pi} 0dx = 0; \\ \int_{-\pi}^{\pi} xdx = \left.\frac{x^2}{2}\right|_0^{\pi} = \frac{\pi^2}{2} \end{gathered} \tag{2}$$

Substituindo os valores de (2) em (1) obtem-se:

$$a_0 = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2} \tag{3}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} x\cos nx\, dx = \frac{1}{\pi}\left.\left[\frac{\cos nx}{n^2} + \frac{x\,\sin nx}{n}\right]\right|_0^{\pi} \\ &= \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} + \frac{\pi\,\sin n\pi}{n} - \frac{\cos n\,0}{n^2} + \frac{0\,\sin n\,0}{n}\right] \\ &= \frac{1}{\pi\, n^2}(\cos n\pi - 1) = \begin{cases} 0, & n \quad \text{é}\; par \\ -\frac{2}{\pi n^2}, & n \quad \text{é}\; impar \end{cases} \end{aligned} \tag{4}$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{\pi} x\,\sin nx\, dx = \frac{1}{\pi}\left.\left[\frac{\sin nx}{n^2} - \frac{x\,\cos nx}{n}\right]\right|_0^{\pi} \\ &= \frac{1}{\pi}\left(\frac{\sin \pi n}{n^2} - \pi\,\frac{\cos n\pi}{n} - \frac{\sin 0\,\pi}{n^2} + 0\,\frac{\cos 0\,\pi}{n}\right) \\ &= -\frac{1}{n}\cos n\,\pi = \begin{cases} -\frac{1}{n}, & n \quad \text{é}\; par \\ \frac{1}{n}, & n \quad \text{é}\; impar \end{cases} \end{aligned} \tag{5}$$

Utilizando os coeficientes (3),(4) e (5), obtemos a Série de Fourier

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2} + \sum_{m=1}^{\infty}(-1)^{m+1}\frac{\sin mx}{m} \tag{6}$$

6

$$f(t)\begin{cases} 0, & -2<t<-1 \\ k, & -1<t<1, \quad T=4 \\ 0, & 1<t<2 \end{cases} \tag{1}$$

S. $T=4=2l \quad \Rightarrow \quad l=2$

$$\begin{gathered} a_0=\frac{2}{l}\int_0^l f(t)\,dt \\ a_0=\frac{2}{2}\left(\int_0^1 k\,dt+\int_1^2 kdt\right)=k+0=k \end{gathered} \tag{2}$$

$$\begin{gathered} a_n=\frac{2}{l}\int_0^l f(t)\cos\left(\frac{n\pi t}{l}\right)dt \\ a_n=\int_0^1 k\cos\left(\frac{n\pi t}{2}\right)\,dt=\frac{2k}{n\pi}\sin\frac{n\pi}{2}=\begin{cases} 0, & n \quad \acute{e}\, par \\ \frac{2k}{n\pi}, & n \quad = 1,5,9,\ldots \\ -\frac{2k}{n\pi}, & n \quad = 3,7,11,\ldots \end{cases} \end{gathered} \tag{3}$$

$$b_n=0 \tag{4}$$

Utilizando os coeficientes em (2),(3) e (4), obtemos a Série de Fourier:

$$f(x)=\frac{k}{2}+\frac{2k}{\pi}\left(\cos\frac{\pi}{2}t-\frac{1}{3}\cos\frac{3\pi}{2}t+\frac{1}{5}\cos\frac{5\pi}{2}t-\ldots\right) \tag{5}$$

7

A função periódica $2l$ definida no segmento $[-l,l]$ pela igualdade $f(x)=|x|$ e de $T=2l$

S.

$$\begin{gathered} T=2l \\ f(x)=|x| \Rightarrow f(x)=f(-x): função par \end{gathered} \tag{1}$$

$$a_0=\frac{2}{l}\int_0^l x\,dx=\frac{2}{l}\left(\frac{x^2}{2}\right)\Bigg|_0^l=l \tag{2}$$

$$a_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right) = \frac{2}{l}\left[\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} + \frac{x\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right]\Bigg|_0^l$$
$$= \frac{2}{l}\frac{l^2}{n^2\pi^2}(\cos n\pi - 1) = \frac{2l}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} 0, & n \quad \acute{e}\, par \\ -\frac{4l}{n^2\pi^2}, & n \quad \acute{e}\, impar \end{cases} \tag{3}$$

$$b_n = 0 \tag{4}$$

Utilizando os coeficientes em (2),(3) e (4), obtemos a Série de Fourier:

$$f(x) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos\left[\frac{(2k+1)\pi\, x}{l}\right]}{(2k+1)^2} \tag{5}$$

8

Para $f(t) = 1 + 2t$ no intervalo $-1 < t < 1$, com $T = 2L = 2$

S.

$$T = 2 = 2l \Rightarrow l = 1 \tag{1}$$

$$a_0 = \frac{l}{1}\int_{-1}^{1}(1+2t)\; dt = \; t + t^2\big|_{-1}^{1} = (1+1) - (-1+1) = 2 \tag{2}$$

$$a_n = \int_{-1}^{1}(1+2t)\cos(n\pi t)dt = \int_{-1}^{1}\cos(n\pi t)dt + 2\int_{-1}^{1} t\cos(n\pi t)dt \tag{3}$$

$$\int_{-1}^{1}\cos(n\pi t)dt = \left.\frac{\sin n\pi t}{n\pi}\right|_{-1}^{1} = 0 \tag{4}$$

$$\int_{-1}^{1} t\cos(n\pi t)dt = -\frac{\cos(-n\pi)}{(n\pi)^2} + \left.\frac{t\sin(-n\pi)}{(n\pi)}\right|_{1}^{-1}$$
$$= \frac{\cos(n\pi)}{(n\pi)^2} + \frac{\sin(n\pi)}{(n\pi)} - \frac{\cos(-n\pi)}{(n\pi)^2} + \frac{\sin(-n\pi)}{(n\pi)} = 0 \tag{5}$$

$$b_n = \int_{-1}^{1}(1+2t)\sin(n\pi t)\; dt = \int_{-1}^{1}\sin n\pi t dt + 2\int_{-1}^{1} t\sin n\pi t\; dt \tag{6}$$

$$\int_{-1}^{1}\sin n\pi t dt = -\left.\frac{\cos n\pi t}{n\pi}\right|_{-1}^{1} = -\frac{\cos n\pi}{n\pi} + \frac{\cos - n\pi}{n\pi} = 0 \tag{7}$$

$$\int_{-1}^{1} t\sin n\pi t\; dt = -\frac{\sin n\pi t}{(n\pi)^2} - \left.\frac{t\cos n\pi t}{(n\pi)^2}\right|_{-1}^{1}$$
$$= \frac{\sin n\pi}{(n\pi)^2} - \frac{\cos n\pi}{(n\pi)} - frac\sin - n\pi(n\pi)^2 - \frac{\cos - n\pi}{(n\pi)}$$
$$= -\frac{2cosn\pi}{n\pi} \tag{8}$$

Substituindo (7) e (8) em (6) temos:

$$b_n = 2(-2)\frac{\cos n\pi}{n\pi} = \begin{cases} \frac{4}{n\pi}, & n \quad \text{é}\, par \\ -\frac{4}{n\pi}, & n \quad \text{é}\, impar \end{cases} \tag{9}$$

E finalmente, utilizando os coeficientes em (2),(5) e (9), obtemos a Série de Fourier:

$$f(t) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\pi t}{n}, \quad -1 < t < 1 \tag{10}$$