INF4820: Algorithms for Artificial Intelligence and Natural Language Processing

Hidden Markov Models

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Language Technology Group (LTG)

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Recap: Probabilistic Language Models

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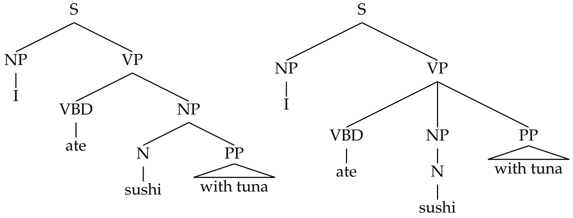
- An n-gram language model predicts the n-th word, conditioned on the $n-1$ previous words;
- Maximum Likelihood Estimation uses relative frequencies to approximate the conditional probabilities needed for an n-gram model;
- Smoothing techniques are used to avoid zero probabilities.

Today

Determining

- which string is most likely:
  - *She studies morphosyntax* vs. *She studies more faux syntax*
- which tag sequence is most likely for *flies like flowers*:
  - **NNS VB NNS** vs. **VBZ P NNS**
- which syntactic analysis is most likely:



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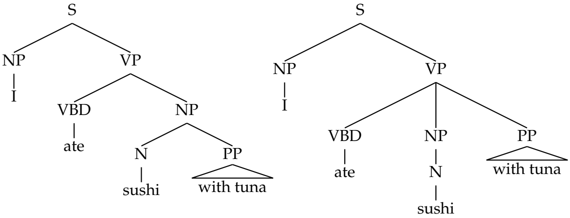

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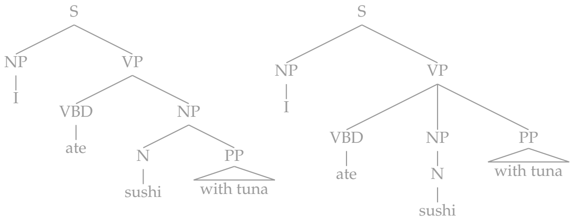

Parts of Speech

- Known by a variety of names: part-of-speech, POS, lexical categories, word classes, morphological classes, . . .
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- Open-classes
  - New words created/updated/deleted all the time
- Closed-classes
  - Smaller classes, relatively static membership
  - Usually function words

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- Nouns: dog, Oslo, scissors, snow, people, truth, cups
  - proper or common; countable or uncountable; plural or singular; masculine, feminine or neuter; ...

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- Adverbs: again, somewhat, slowly, yesterday, aloud
  - intersective; scopal; discourse; degree; temporal; directional; comparative or superlative; . . .

Closed Class Words

- Prepositions: *on*, *under*, *from*, *at*, *near*, *over*, . . .
- Determiners: *a*, *an*, *the*, *that*, . . .
- Pronouns: *she*, *who*, *I*, *others*, . . .
- Conjunctions: *and*, *but*, *or*, *when*, . . .
- Auxiliary verbs: *can*, *may*, *should*, *must*, . . .
- Interjections, particles, numerals, negatives, politeness markers, greetings, existential there . . .

(Examples from Jurafsky & Martin, 2008)

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  - Norwegian
    - Oslo-Bergen Tagset—multi-part: ⟨subst appell fem be ent⟩

  `http://tekstlab.uio.no/obt-ny/english/tags.html`

Labelled Sequences

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- Consider the POS tags to be underlying skeleton of the sentence, unseen but influencing the sentence shape.
- A structure like this, consisting of a hidden state sequence, and a related observation sequence can be modelled as a *Hidden Markov Model*.

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⟨S⟩

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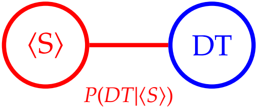

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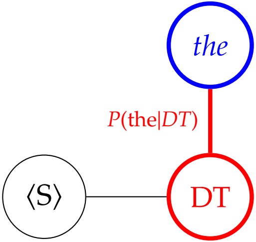

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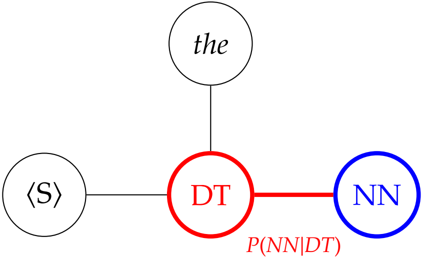

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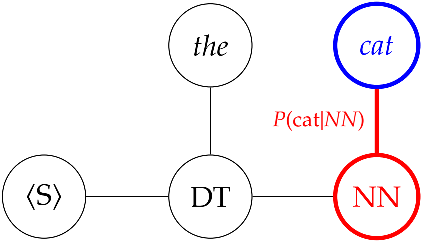

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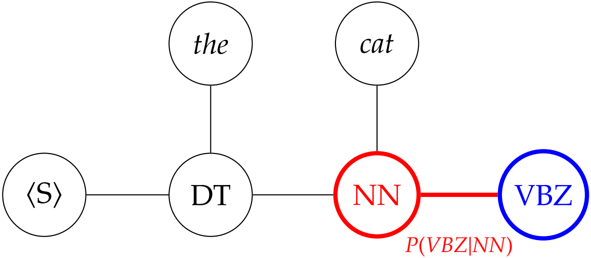

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Hidden Markov Models

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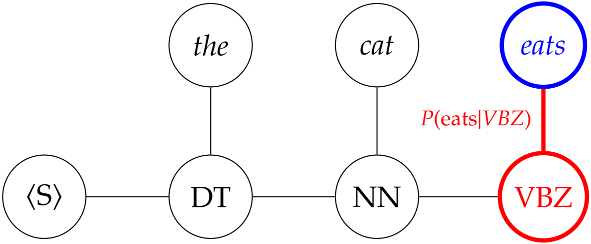

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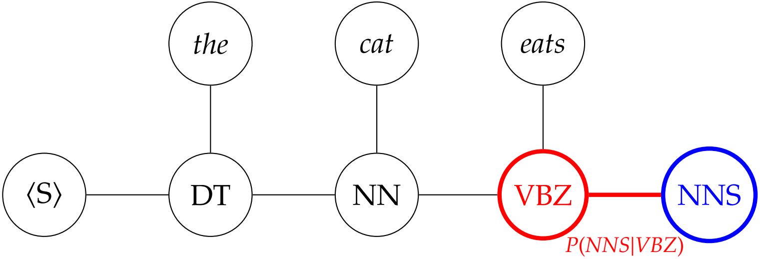

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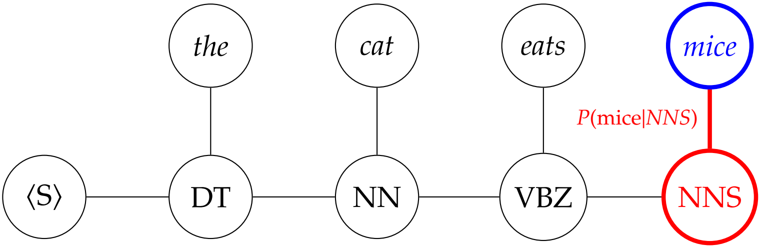

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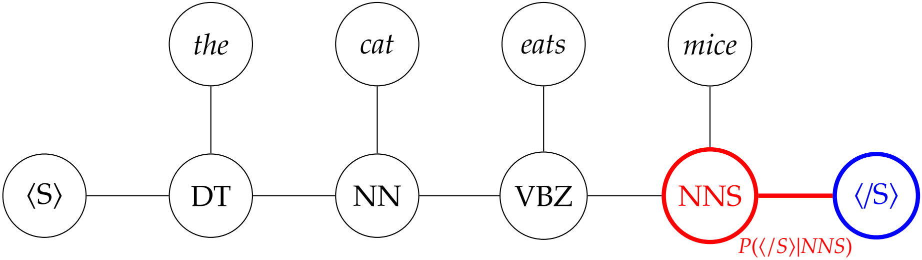

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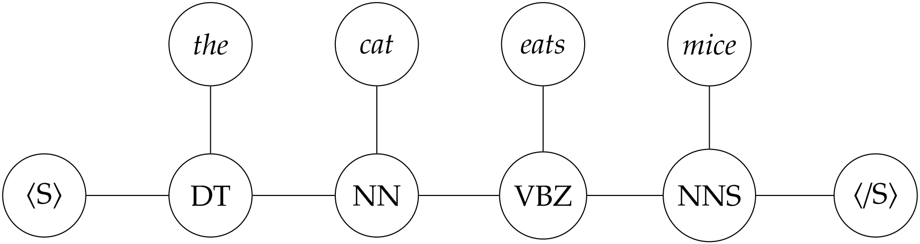

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For a bi-gram HMM, with O_1^N:

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- The transition probabilities model the probabilities of moving from state to state.
- The emission probabilities model the probability that a state *emits* a particular observation.

Using HMMs

The HMM models the process of generating the labelled sequence. We can use this model for a number of tasks:

- $P(S, O)$ given S and O
- $P(O)$ given O
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Our *observations* will be words (w_i), and our *states* PoS tags (t_i)

Estimation

As so often in NLP, we learn an HMM from labelled data:

Transition probabilities

Based on a training corpus of previously tagged text, with tags as our state, the MLE can be computed from the counts of observed tags:

$$P(t_i|t_{i-1}) = \frac{C(t_{i-1}, t_i)}{C(t_{i-1})}$$

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Emission probabilities

Computed from relative frequencies in the same way, with the words as observations:

$$P(w_i|t_i) = \frac{C(t_i, w_i)}{C(t_i)}$$

Implementation Issues

$$
\begin{aligned}
P(S, O) &= P(s_1|\langle S\rangle)P(o_1|s_1)P(s_2|s_1)P(o_2|s_2)P(s_3|s_2)P(o_3|s_3)\ldots \\
&= 0.0429 \times 0.0031 \times 0.0044 \times 0.0001 \times 0.0072 \times \ldots
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- Solution: work in log(arithmic) space:
  - $\log(AB) = \log(A) + \log(B)$
  - hence $P(A)P(B) = \exp(\log(A) + \log(B))$
  - $\log(P(S,O)) = -1.368 + -2.509 + -2.357 + -4 + -2.143 + \ldots$

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The issues related to MLE / smoothing that we discussed for n-gram models also applies here . . .

Ice Cream and Global Warming

Missing records of weather in Baltimore for Summer 2007

- Jason likes to eat ice cream.
- He records his daily ice cream consumption in his diary.
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A Hidden Markov Model!

with:

- Hidden states: $\{H, C\}$ (plus pseudo-states $\langle S \rangle$ and $\langle /S \rangle$)
- Observations: $\{1, 2, 3\}$

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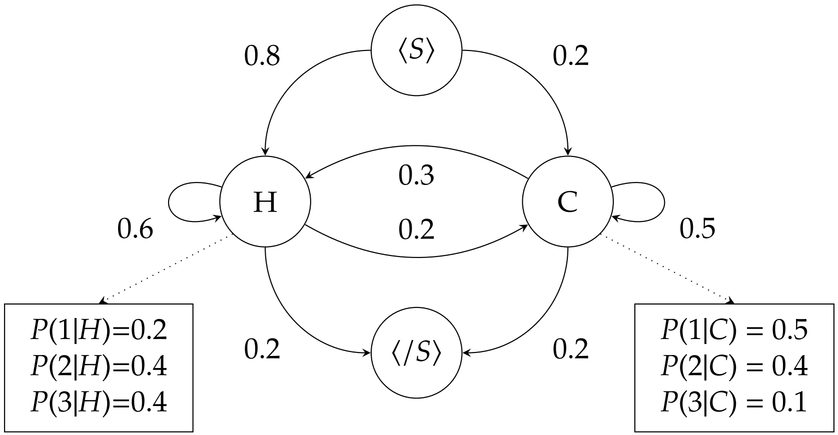

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Part-of-Speech Tagging

We want to find the tag sequence, given a word sequence. With tags as our states and words as our observations, we know:

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Since $P(O)$ always is the same, we can drop the denominator:

$$\hat{S} = \arg\max_S P(S, O)$$

Decoding

Task

What is the most likely state sequence S, given an observation sequence O and an HMM.

HMM

$$
\begin{array}{|rr|}
 & \\
P(H|\langle S\rangle) = 0.8 & P(C|\langle S\rangle) = 0.2 \\
P(H|H) = 0.6 & P(C|H) = 0.2 \\
P(H|C) = 0.3 & P(C|C) = 0.5 \\
P(\langle/S\rangle|H) = 0.2 & P(\langle/S\rangle|C) = 0.2 \\
 & \\
P(1|H) = 0.2 & P(1|C) = 0.5 \\
P(2|H) = 0.4 & P(2|C) = 0.4 \\
P(3|H) = 0.4 & P(3|C) = 0.1 \\
 & \\
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P(3|H) = 0.4 & P(3|C) = 0.1
\end{array}
$$

if $O = 3\ 1\ 3$

Decoding

Task

What is the most likely state sequence S, given an observation sequence O and an HMM.

HMM

$$P(H|\langle S\rangle) = 0.8 \qquad P(C|\langle S\rangle) = 0.2$$
$$P(H|H) = 0.6 \qquad P(C|H) = 0.2$$
$$P(H|C) = 0.3 \qquad P(C|C) = 0.5$$
$$P(\langle /S\rangle|H) = 0.2 \qquad P(\langle /S\rangle|C) = 0.2$$

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if $O = 3\ 1\ 3$

$\langle S\rangle$ H H H $\langle /S\rangle$

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Task

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HMM		if $O = 3\ 1\ 3$
$P(H\|\langle S\rangle) = 0.8$	$P(C\|\langle S\rangle) = 0.2$	$\langle S\rangle$ H H H $\langle /S\rangle$ 0.0018432
$P(H\|H) = 0.6$	$P(C\|H) = 0.2$	
$P(H\|C) = 0.3$	$P(C\|C) = 0.5$	
$P(\langle /S\rangle\|H) = 0.2$	$P(\langle /S\rangle\|C) = 0.2$	
$P(1\|H) = 0.2$	$P(1\|C) = 0.5$	
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if $O = 3\ 1\ 3$

$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$\langle S\rangle$	H	H	C	$\langle /S\rangle$	

Decoding

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 & \\
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P(3|H) = 0.4 & P(3|C) = 0.1
\end{array}
$$

if $O = 3\ 1\ 3$

$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$\langle S\rangle$	H	H	C	$\langle /S\rangle$	0.0001536

Decoding

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\end{array}
$$

if O = 3 1 3

$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$\langle S\rangle$	H	H	C	$\langle /S\rangle$	0.0001536
$\langle S\rangle$	H	C	H	$\langle /S\rangle$	0.0007680

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if $O = 3\ 1\ 3$

$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$\langle S\rangle$	H	H	C	$\langle /S\rangle$	0.0001536
$\langle S\rangle$	H	C	H	$\langle /S\rangle$	0.0007680
$\langle S\rangle$	H	C	C	$\langle /S\rangle$	0.0003200

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What is the most likely state sequence S, given an observation sequence O and an HMM.

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if $O = 3\ 1\ 3$

$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$\langle S\rangle$	H	H	C	$\langle /S\rangle$	0.0001536
$\langle S\rangle$	H	C	H	$\langle /S\rangle$	0.0007680
$\langle S\rangle$	H	C	C	$\langle /S\rangle$	0.0003200
$\langle S\rangle$	C	H	H	$\langle /S\rangle$	0.0000576

Decoding

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if $O = 3\ 1\ 3$

$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$\langle S\rangle$	H	H	C	$\langle /S\rangle$	0.0001536
$\langle S\rangle$	H	C	H	$\langle /S\rangle$	0.0007680
$\langle S\rangle$	H	C	C	$\langle /S\rangle$	0.0003200
$\langle S\rangle$	C	H	H	$\langle /S\rangle$	0.0000576
$\langle S\rangle$	C	H	C	$\langle /S\rangle$	0.0000048

Task

What is the most likely state sequence S, given an observation sequence O and an HMM.

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if $O = 3\ 1\ 3$

$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$\langle S\rangle$	H	H	C	$\langle /S\rangle$	0.0001536
$\langle S\rangle$	H	C	H	$\langle /S\rangle$	0.0007680
$\langle S\rangle$	H	C	C	$\langle /S\rangle$	0.0003200
$\langle S\rangle$	C	H	H	$\langle /S\rangle$	0.0000576
$\langle S\rangle$	C	H	C	$\langle /S\rangle$	0.0000048
$\langle S\rangle$	C	C	H	$\langle /S\rangle$	0.0001200

Decoding

Task

What is the most likely state sequence S, given an observation sequence O and an HMM.

HMM

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$P(H\vert H) = 0.6$	$P(C\vert H) = 0.2$
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if $O = 3\ 1\ 3$

$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$\langle S\rangle$	H	H	C	$\langle /S\rangle$	0.0001536
$\langle S\rangle$	H	C	H	$\langle /S\rangle$	0.0007680
$\langle S\rangle$	H	C	C	$\langle /S\rangle$	0.0003200
$\langle S\rangle$	C	H	H	$\langle /S\rangle$	0.0000576
$\langle S\rangle$	C	H	C	$\langle /S\rangle$	0.0000048
$\langle S\rangle$	C	C	H	$\langle /S\rangle$	0.0001200
$\langle S\rangle$	C	C	C	$\langle /S\rangle$	0.0000500

Decoding

Task

What is the most likely state sequence S, given an observation sequence O and an HMM.

HMM		if $O = 3\ 1\ 3$					
$P(H\|\langle S\rangle) = 0.8$	$P(C\|\langle S\rangle) = 0.2$	$\langle S\rangle$	H	H	H	$\langle /S\rangle$	0.0018432
$P(H\|H) = 0.6$	$P(C\|H) = 0.2$	$\langle S\rangle$	H	H	C	$\langle /S\rangle$	0.0001536
$P(H\|C) = 0.3$	$P(C\|C) = 0.5$	$\langle S\rangle$	H	C	H	$\langle /S\rangle$	0.0007680
$P(\langle /S\rangle\|H) = 0.2$	$P(\langle /S\rangle\|C) = 0.2$	$\langle S\rangle$	H	C	C	$\langle /S\rangle$	0.0003200
$P(1\|H) = 0.2$	$P(1\|C) = 0.5$	$\langle S\rangle$	C	H	H	$\langle /S\rangle$	0.0000576
$P(2\|H) = 0.4$	$P(2\|C) = 0.4$	$\langle S\rangle$	C	H	C	$\langle /S\rangle$	0.0000048
$P(3\|H) = 0.4$	$P(3\|C) = 0.1$	$\langle S\rangle$	C	C	H	$\langle /S\rangle$	0.0001200
		$\langle S\rangle$	C	C	C	$\langle /S\rangle$	0.0000500

Dynamic Programming

For (only) two states and a (short) observation sequence of length three, comparing all possible sequences is workable, but ...

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- for N observations and L states, there are L^N sequences
- we do the same partial calculations over and over again

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Dynamic Programming:

- records sub-problem solutions for further re-use
- useful when a complex problem can be described recursively
- examples: Dijkstra's shortest path, minimum edit distance, longest common subsequence, Viterbi algorithm

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Viterbi Algorithm

Recall our problem:

$$\text{maximise } P(s_1 \dots s_n | o_1 \dots o_n) = P(s_1|s_0)P(o_1|s_1)P(s_2|s_1)P(o_2|s_2)\dots$$

Viterbi Algorithm

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Our recursive sub-problem:

$$v_i(x) = \max_{k=1}^{L} [v_{i-1}(k) \cdot P(x|k) \cdot P(o_i|x)]$$

The variable $v_i(x)$ represents the maximum probability that the i-th state is x, given that we have seen O_1^i.

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The variable $v_i(x)$ represents the maximum probability that the i-th state is x, given that we have seen O_1^i.

Viterbi Algorithm

Recall our problem:

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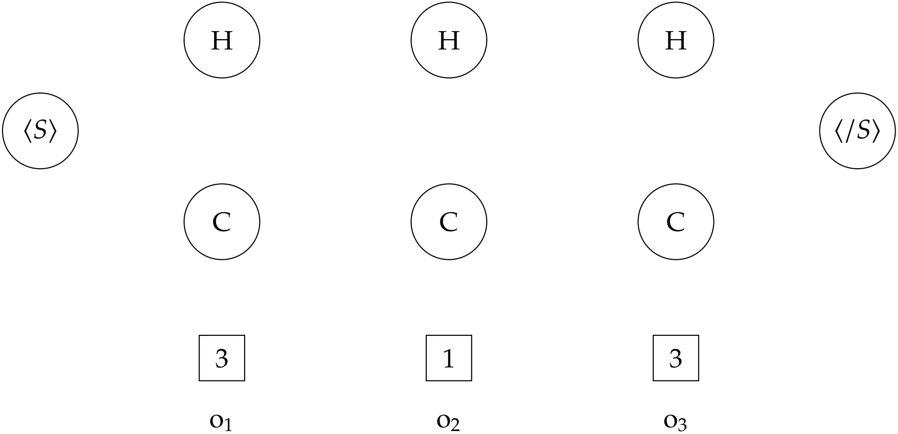
The variable $v_i(x)$ represents the maximum probability that the i-th state is x, given that we have seen O_1^i.

At each step, we record backpointers showing which previous state led to the maximum probability.

An Example of the Viterbi Algorithm



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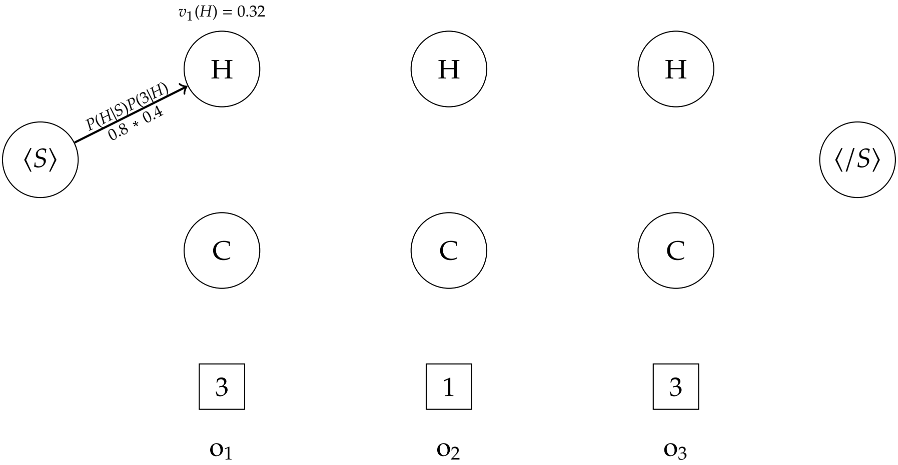

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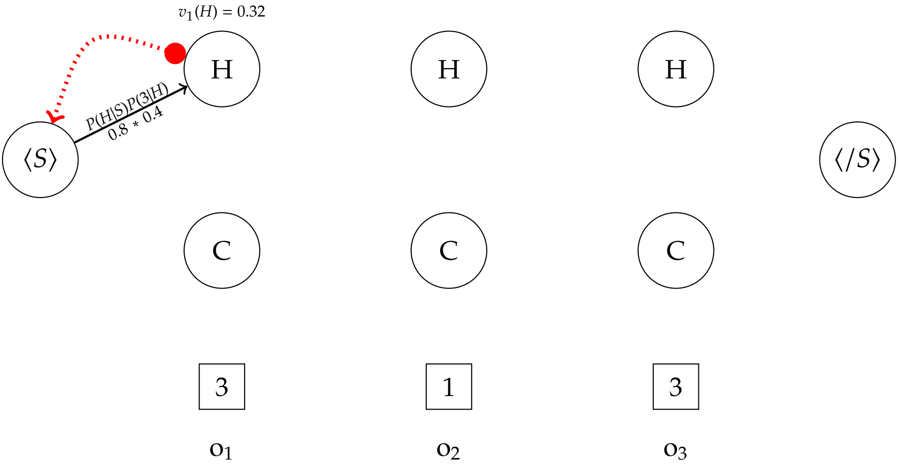

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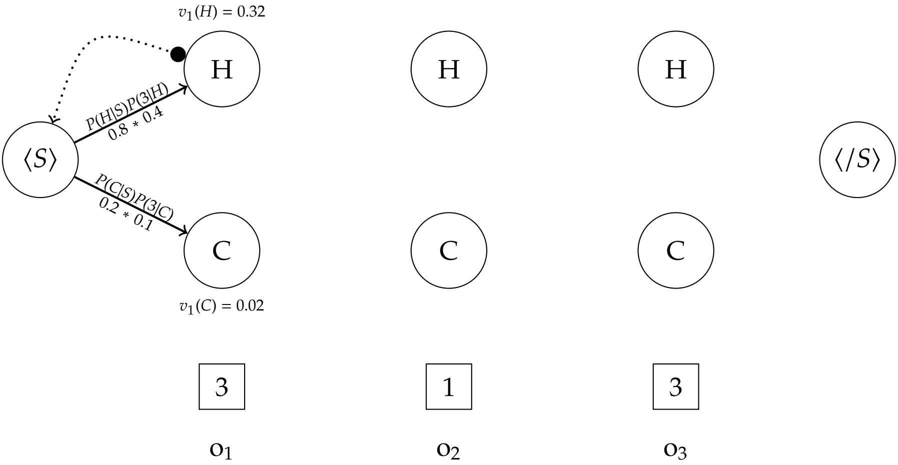

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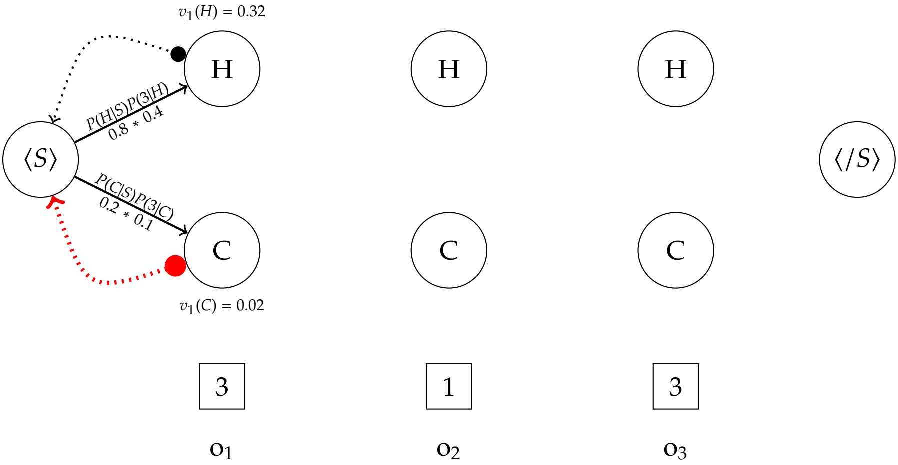

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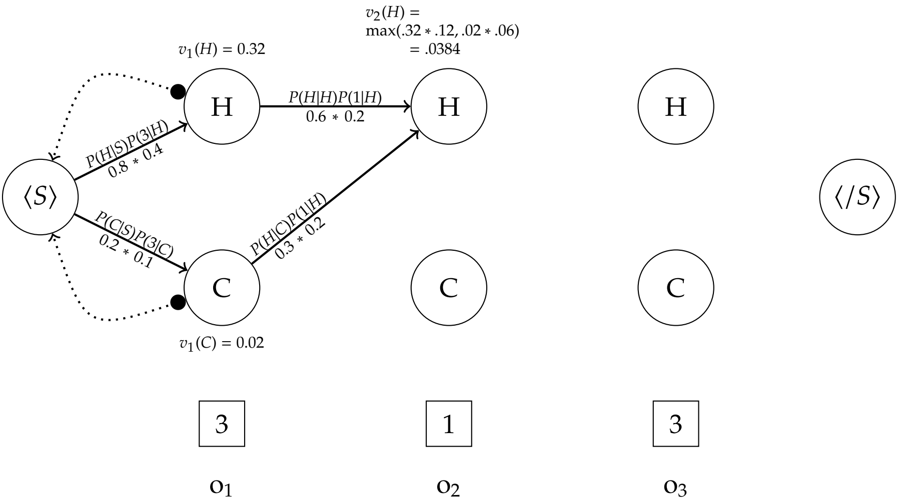

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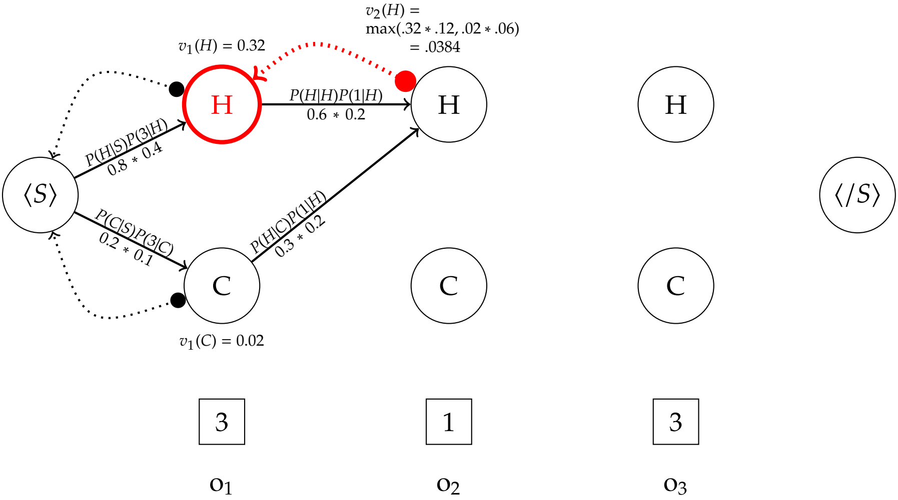

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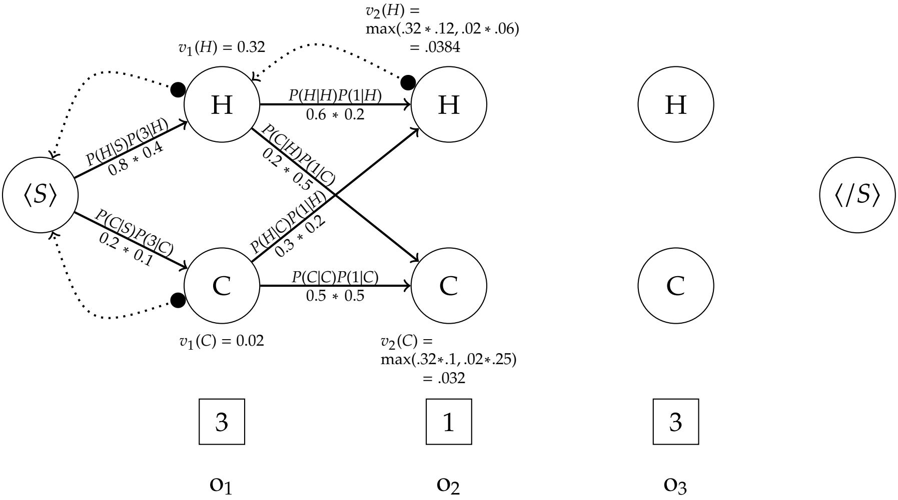

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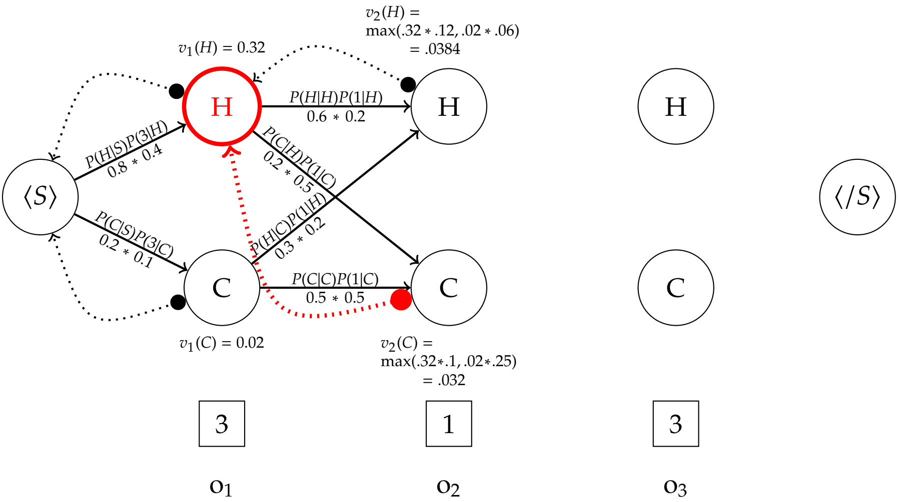

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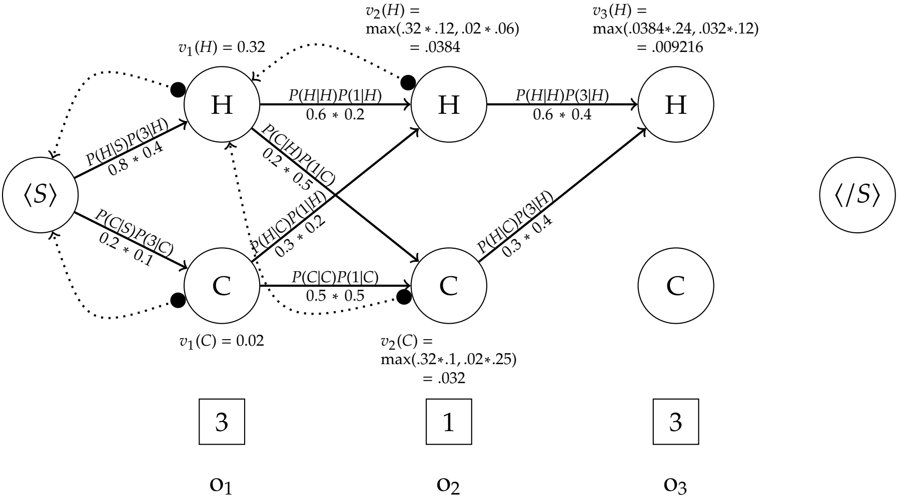

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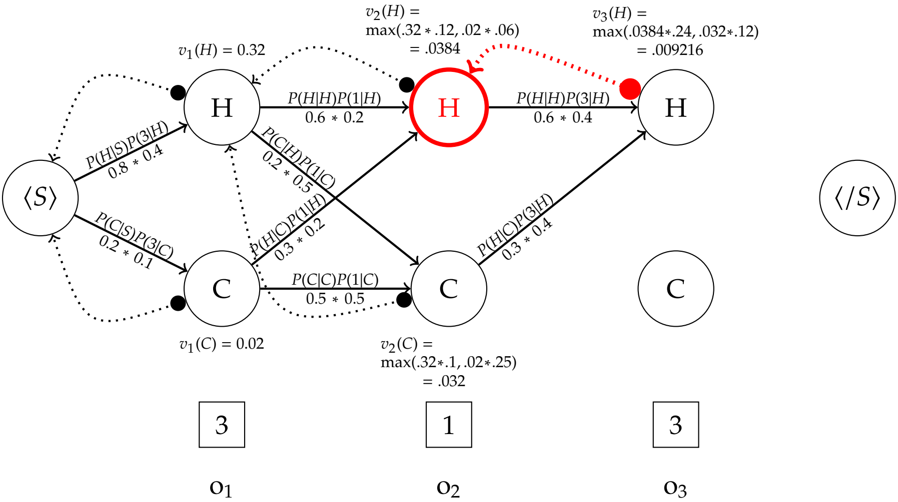

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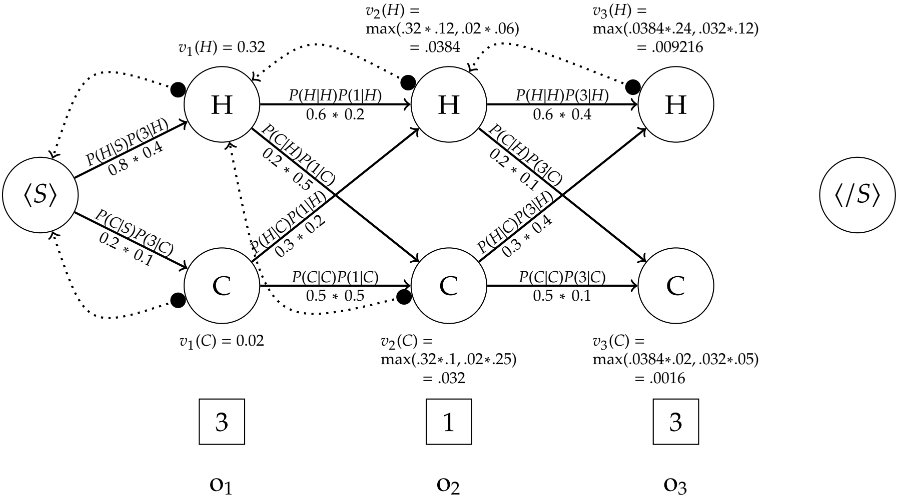

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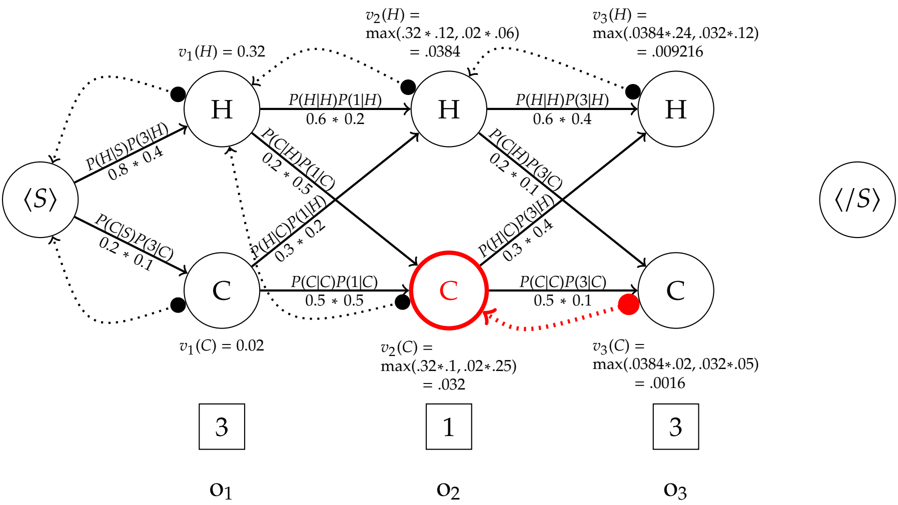

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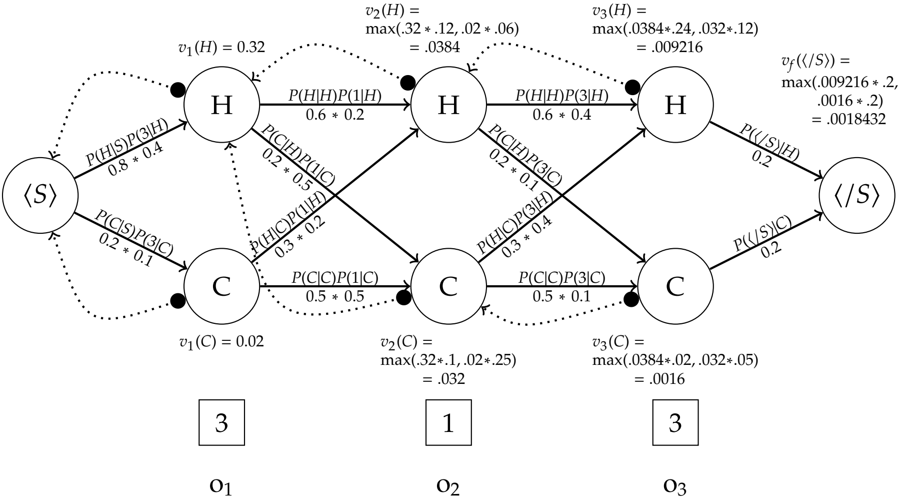

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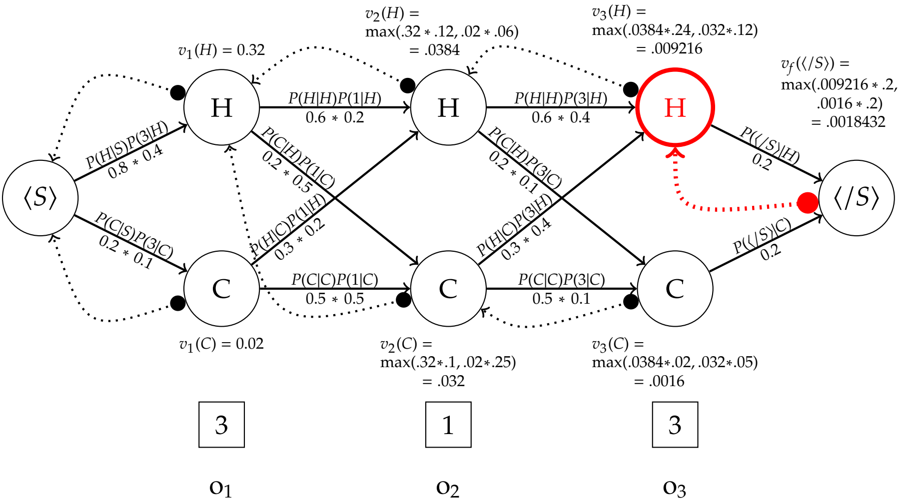

H

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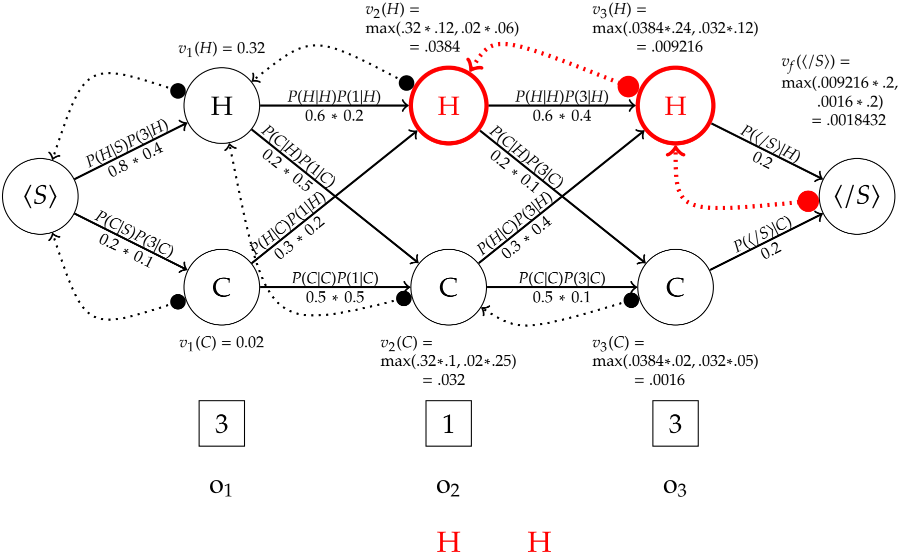

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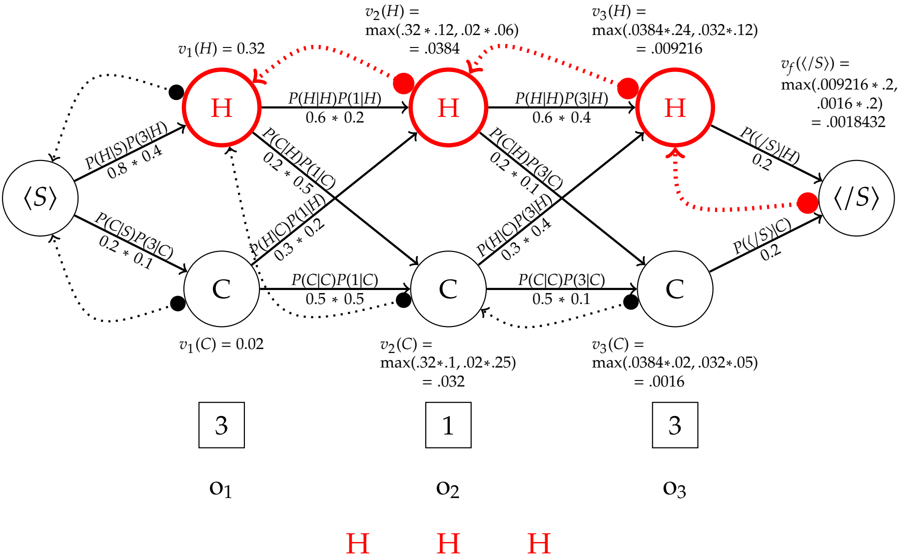

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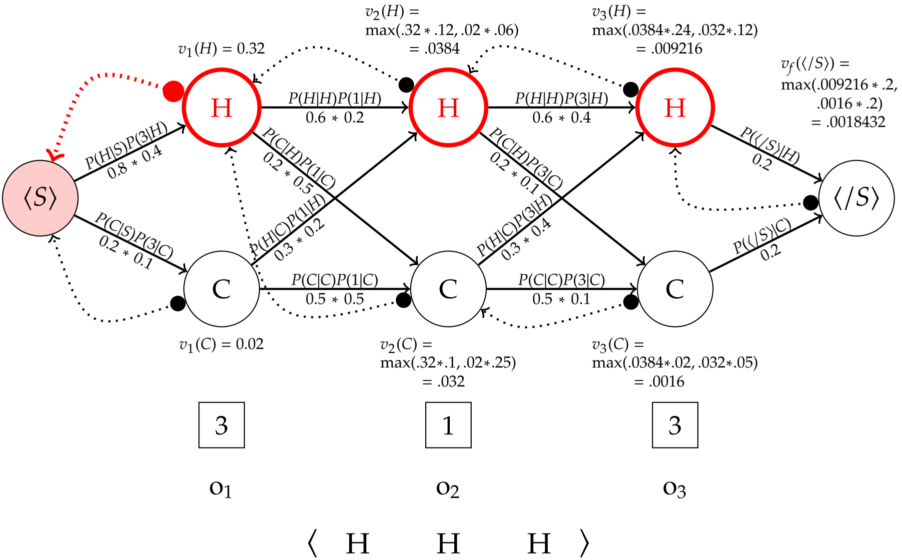

Pseudocode for the Viterbi Algorithm

Input: *observations* of length N, *state set* of size L
Output: *best-path*
create a path probability matrix $viterbi[N, L+2]$
create a path backpointer matrix $backpointer[N, L+2]$
for each *state s from 1 to L* **do**
    $viterbi[1, s] \leftarrow trans(\langle S \rangle, s) \times emit(o_1, s)$
    $backpointer[1, s] \leftarrow 0$
end
for each *time step i from 2 to N* **do**
    for each *state s from 1 to L* **do**
        $viterbi[i, s] \leftarrow \max_{s'=1}^{L} viterbi[i-1, s'] \times trans(s', s) \times emit(o_i, s)$
        $backpointer[i, s] \leftarrow \arg\max_{s'=1}^{L} viterbi[i-1, s'] \times trans(s', s)$
    end
end
$viterbi[N, L+1] \leftarrow \max_{s=1}^{L} viterbi[s, N] \times trans(s, \langle /S \rangle)$
$backpointer[N, L+1] \leftarrow \arg\max_{s=1}^{L} viterbi[N, s] \times trans(s, \langle /S \rangle)$
return the path by following backpointers from $backpointer[N, L+1]$

Diversion: Complexity and O(N)

Big-O notation describes the complexity of an algorithm.

- it describes the worst-case *order of growth* in terms of the size of the input
- only the largest order term is represented
- constant factors are ignored
- determined by looking at loops in the code

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    end
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Pseudocode for the Viterbi Algorithm

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$$O(L^2 N)$$

Using HMMs

The HMM models the process of generating the labelled sequence. We can use this model for a number of tasks:

- $P(S, O)$ given S and O
- $P(O)$ given O
- S that maximises $P(S|O)$ given O
- $P(s_x|O)$ given O
- We can also learn the model parameters, given a set of observations.

Computing Likelihoods

Task

Given an observation sequence O, determine the likelihood $P(O)$, according to the HMM.

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Compute the sum over all possible state sequences:

$$P(O) = \sum_S P(O, S)$$

For example, the ice cream sequence 3 1 3:

$$\begin{aligned} P(3\ 1\ 3) = \quad & P(3\ 1\ 3, \text{cold cold cold}) + \\ & P(3\ 1\ 3, \text{cold cold hot}) + \\ & P(3\ 1\ 3, \text{hot hot cold}) + \ldots \quad \Rightarrow O(L^N N) \end{aligned}$$

The Forward Algorithm

Again, we use dynamic programming—storing and reusing the results of partial computations in a trellis α.

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Each cell in the trellis stores the probability of being in state s_x after seeing the first i observations:

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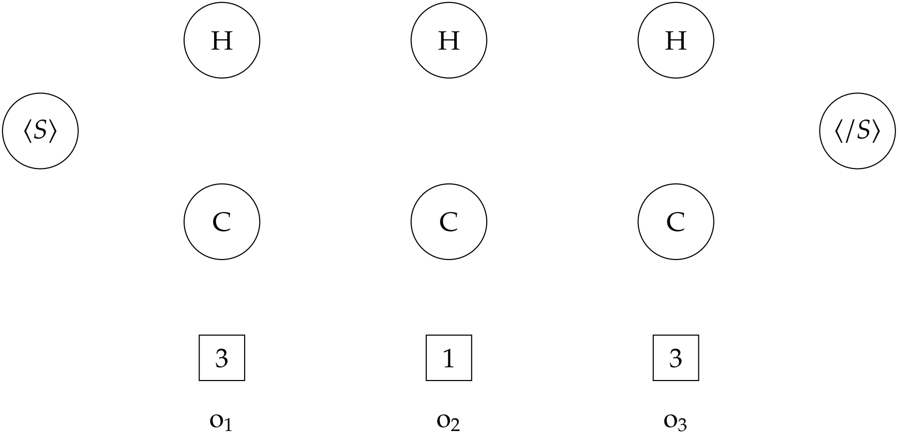
Note $\sum$, instead of the max in Viterbi.

An Example of the Forward Algorithm



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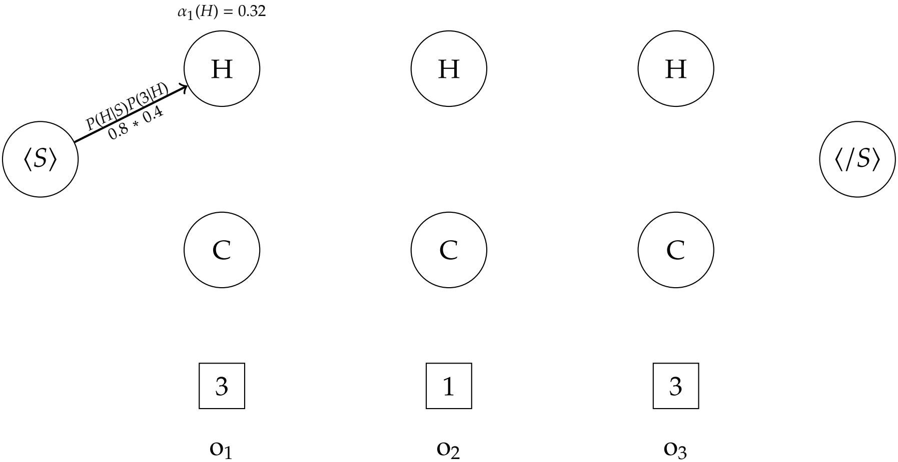

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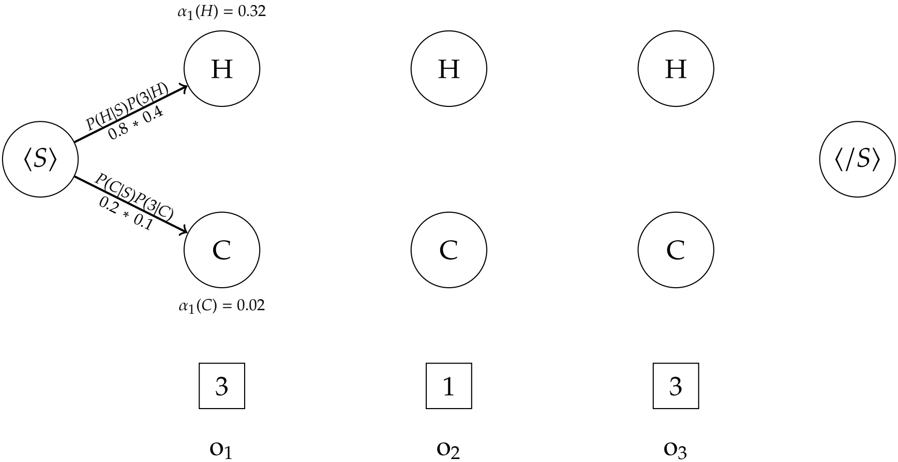

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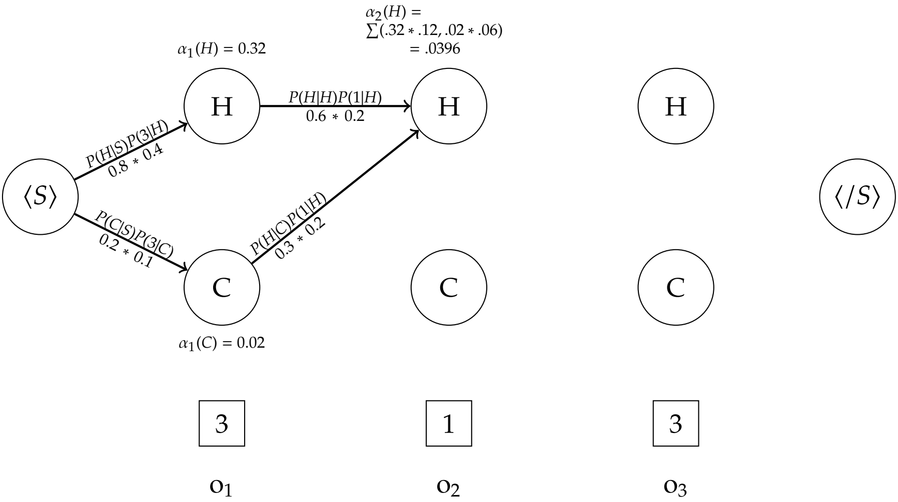

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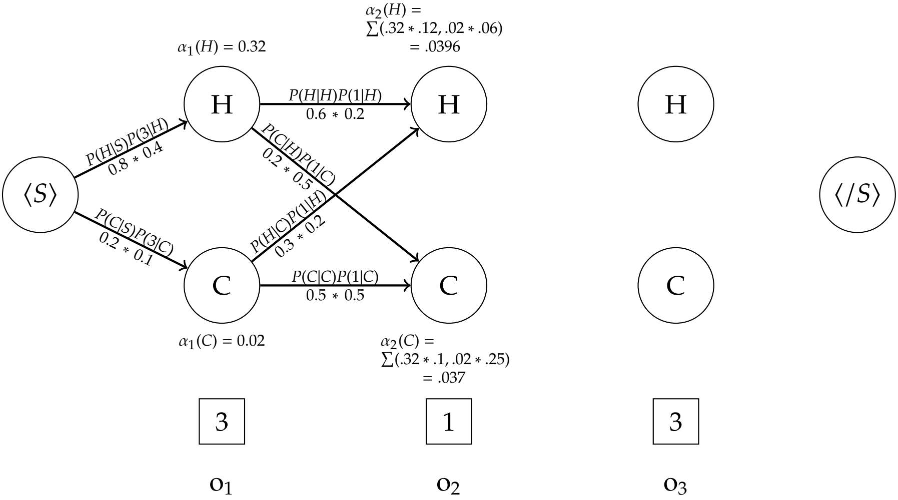

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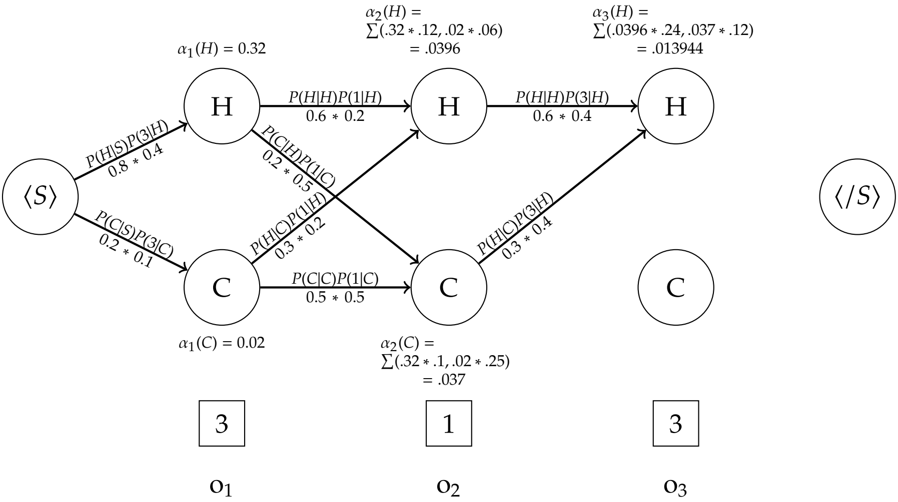

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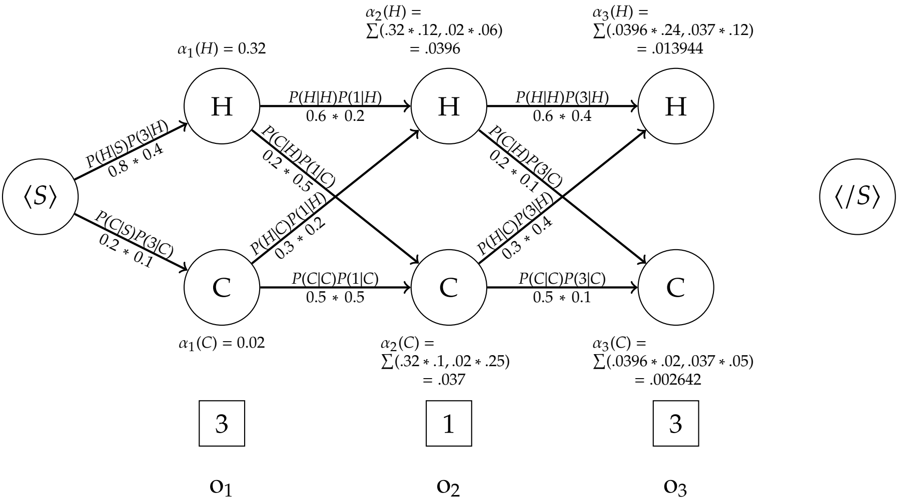

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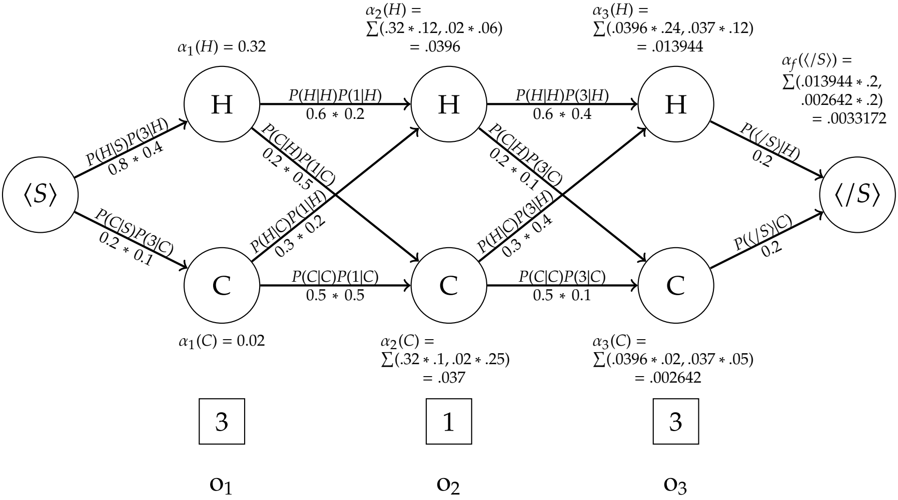

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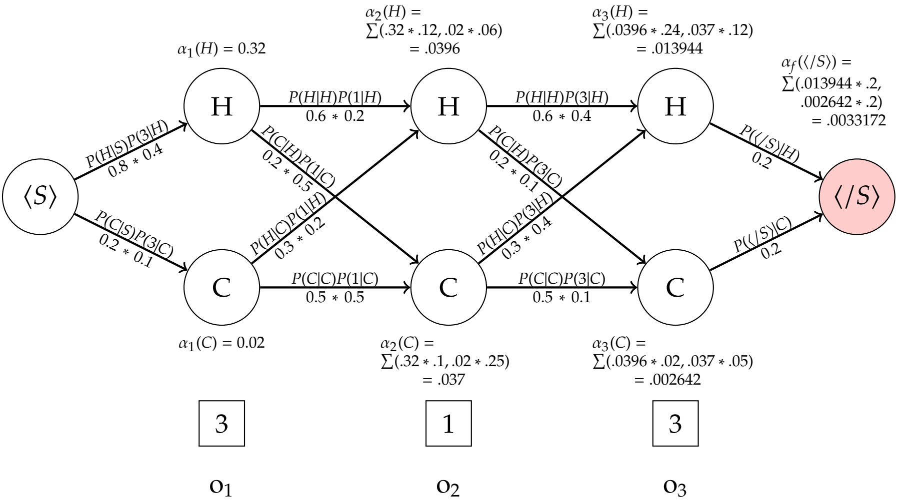

$$P(3\,1\,3) = 0.0033172$$

Pseudocode for the Forward Algorithm

Input: *observations* of length N, *state set* of length L
Output: *forward-probability*
create a probability matrix $forward[N, L+2]$
for each *state s from 1 to L* **do**
  $forward[1, s] \leftarrow trans(\langle S \rangle, s) \times emit(o_1, s)$
end
for each *time step i from 2 to N* **do**
  for each *state s from 1 to L* **do**
    $forward[i, s] \leftarrow$
    $\sum_{s'=1}^{L} forward[i-1, s] \times trans(s', s) \times emit(o_t, s)$
  end
end
$forward[N, L+1] \leftarrow \sum_{s=1}^{L} forward[N, s] \times trans(s, \langle /S \rangle)$
return $forward[N, L+1]$

Tagger Evaluation

To evaluate a part-of-speech tagger (or any classification system) we:

- train on a labelled training set
- test on a *separate* test set

For a POS tagger, the standard evaluation metric is tag accuracy:

$$Acc = \frac{\text{number of correct tags}}{\text{number of words}}$$

The other metric sometimes used is *error rate*:

$$\textit{error rate} = 1 - Acc$$

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- Learn the parameters of HMM (i.e. transition and emission probabilities) using MLE.
- Use Viterbi for decoding, i.e.: S that maximises $P(S|O)$ given O.
- Use Forward for computing likelihood, i.e.: $P(O)$ given O