Magnetic Fields and Forces

Magnetism and Magnetic Fields $\vec{B}$

Magnetic Forces on Moving Charges $\vec{F}_m = q\vec{v} \times \vec{B}$ $\vec{F}_{Lorentz} = q(\vec{E} + \vec{v} \times \vec{B})$

Magnetic Field Lines and Magnetic Flux $\Phi_B \equiv \iint \vec{B} \cdot d\vec{A}$ $B = \dfrac{d\Phi_B}{dA_\perp}$

- Gauss's Law for Magnetism $\oint\!\!\oint \vec{B} \cdot d\vec{A} = 0$ (no magnetic monopoles)

Motion of Charged Particles in a Magnetic Field (and applications) $|q| vB = m\dfrac{v^2}{R}$

Magnetic Force on a Current-Carrying Conductor $d\vec{F} = Id\vec{l} \times \vec{B}$

Force and Torque on a Current Loop $\vec{\tau} = \vec{\mu} \times \vec{B}$, $U = -\vec{\mu} \cdot \vec{B}$

(Time-permitting)

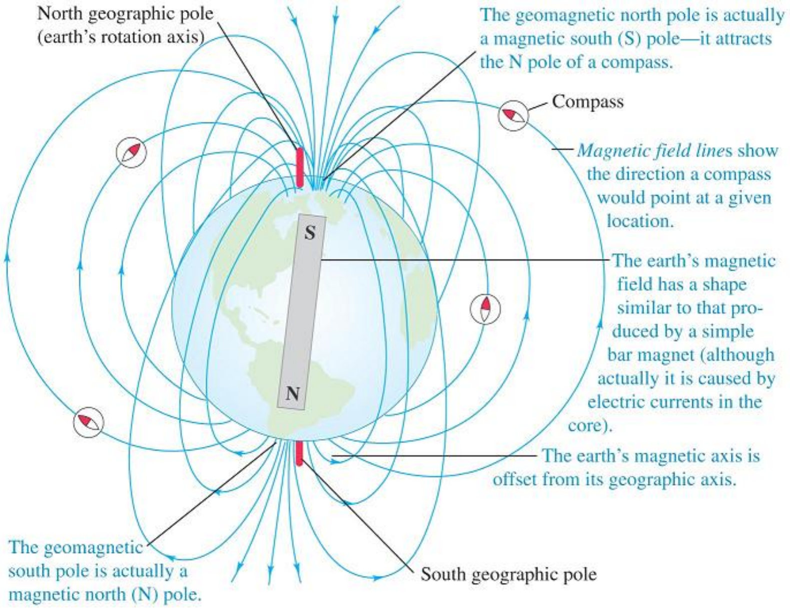
The Electric DC Motor

<u>A Brief History of Magnetism</u>

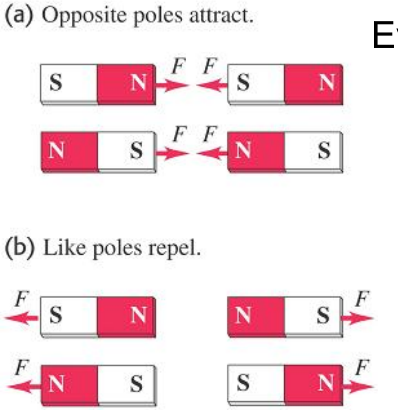
Some historians believe the compass which uses a *magnetic needle* was used as early as the 13^{th} Century BC in China – the needle aligns so that it points in an approximately constant north-south direction.

Ancient Greeks (800 BC) observed the stone magnetite (Fe_3O_4) – a permanent magnet – attracts pieces of iron, and can attract or repel other stones of magnetite.

In 1269, de Maricourt found that the directions of a needle near a spherical natural magnet formed lines that encircled the sphere and passed through two points diametrically opposite each other, called the poles.



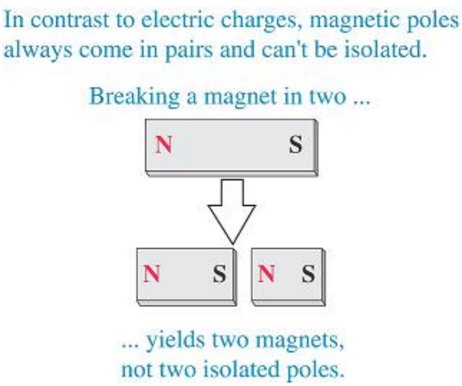
A Brief History of Magnetism



Every magnet (independent of shape) has two poles, called north (N) and south (S).

Opposite poles attract, like poles repel.

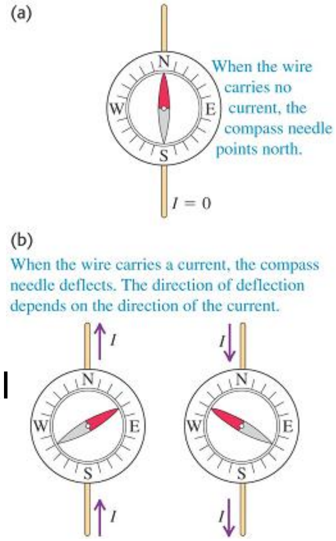
This is analogous to the two kinds of electric charge electric charge, with a key difference: while positive and negative 'charges' can exist in isolation (e.g. protons and electrons), a single magnetic pole (a monopole) has never been isolated. Magnetic poles, thus far, are always found in pairs.[2]



In 1600, Gilbert extended de Maricourt's experiments, and suggested that the Earth itself is a large permanent magnet.

[2]Nevertheless, grand unified theories (GUTs) *generically predict* (during their spontaneous symmetry breaking) the existence of magnetic **mono**poles. (String theories generically predict them as well.) In fact, their conspicuous experimental absence poses theoretical problems with standard big bang cosmology that inflationary models aim to solve.

A Brief History of Magnetism



The first connection between magnetism and electricity was discovered in 1819 by the Danish physicist Oersted – who observed (during a lecture!) that a magnetic compass could be deflected by an electrical current in a nearby wire.

Further connections were discovered by Ampère in France, Faraday in England and Henry in the United Sates. We will discuss these in turn.

The classical story ends with Maxwell's (1864) vaunted equations – they represent a theoretical synthesis of all classical electrical and magnetic phenomena.

For our immediate purposes, the experimental observation of interest is this:

Moving electrical charges can be deflected (i.e. experience forces) by magnets (i.e. magnetic fields).

Magnetic Fields and Forces

Recall: In our discussion of electrical forces, we introduced the notion of an E-field – a vector field (i.e. a vector defined at each point in space and time) as follows:

1) A static charge distribution creates an E-field in the surrounding space .
2) The E-field influences other charges present through $\vec{F}_{es} = q\vec{E}$

Magnetic interactions can be handled analogously:

1) Moving charges (and currents) create magnetic fields denoted $\vec{B}$ in the surrounding space.
2) The B-field influences other *moving charges (or currents)* present in the field through magnetic forces.

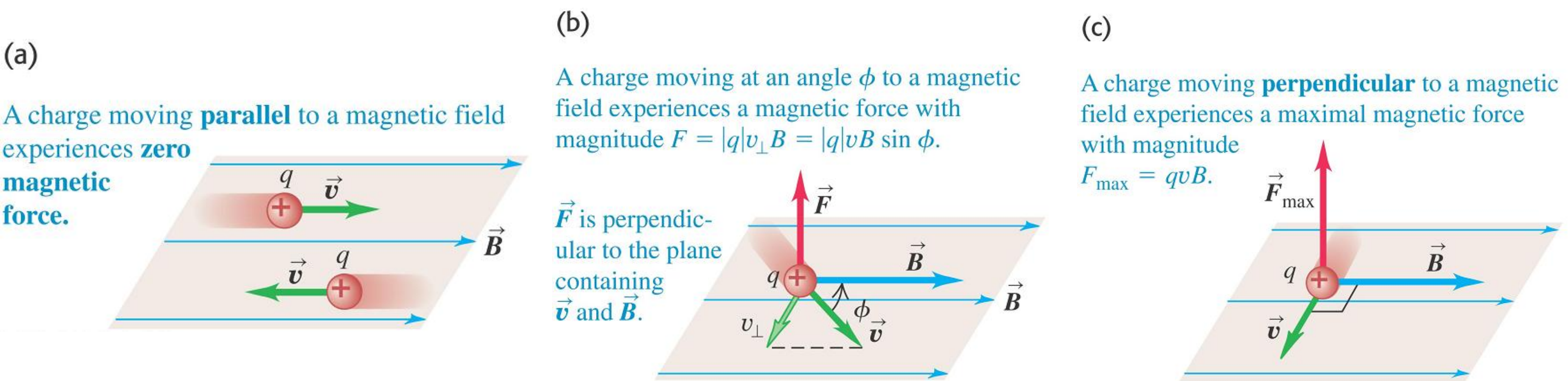
In this chapter we focus on the 2nd part of this interaction – how moving charges and currents respond to B-fields.
In the next chapter we turn to the issue of how to compute the B-field from sources of moving charges/currents.

Magnetic Fields and Forces *An Equation is Worth a Thousand Words*

Experimentally:

1) Magnetic forces act only on charged particles, and are proportional to their charge.
2) Magnetic forces act only on moving charged particles, and are proportional to their speed.
3) Magnetic forces on negative charges are opposite in direction to positive charges.
4) Magnetic forces are proportional to the strength (i.e. magnitude) of the magnetic field
5) If the velocity of the charge particle is parallel or antiparallel to the B-field, $F_m = 0$.
6) If $\mathbf{F}_m \neq 0$, then the magnetic force is perpendicular to BOTH the velocity and the B-field.
7) $|\mathbf{F}_m|$ is proportional to $\sin(\theta)$, where θ is the angle between the velocity and B-field.

Mathematically summarized by: $\boxed{\vec{F}_m = q\vec{v}\times\vec{B}}$ or $\boxed{\left|\vec{F}_m\right| = |q|vB\sin\theta}$ *plus the 'right hand rule' for positive charges.*



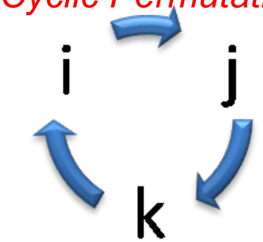
$$\vec{v}\times\vec{B} \equiv (v_y B_z - v_z B_y)\hat{i} + (v_z B_x - v_x B_z)\hat{j} + (v_x B_y - v_y B_x)\hat{k}$$

but there's easier ways to 'remember'/work with this.

Magnetic Fields and Forces – Two Equivalent Right Hand Rules

$$\hat{i} \times \hat{j} = \hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j}$$

Cyclic Permutation



$$\boxed{\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}}$$

Cross-products are anti-commutative

The right hand rules are memory mnemonics for cross-products.

The variant motivated from mathematics goes like this:

Point your fingers of your right hand in the direction of the velocity, curl them towards the direction of the magnetic field. Your thumb then points in the direction of the magnetic force.

The one you probably learned in high school goes like this:

Point your thumb in the direction of the velocity, point your fingers in the direction of the magnetic field. Your palm points in the direction of the magnetic force.

They are equivalent to each other, but remember they only apply for positive charges. Direction is reversed for negative charges. For ‘general’ orientations of **v** and **B**, it’s easiest to use the basic cross-product relations.

<u>Magnetic Fields and Forces</u> – Other Notes $\vec{F}_m = q\vec{v} \times \vec{B}$

This equation can be used to operationally define a magnetic field at a point in space: by measuring the force a charged particle experiences, we can determine B.

Units of B: 1 tesla = 1 T = 1 N·s/(C·m) = 1 N/A·m 1 gauss = 10^{-4} T

Earth's B-field ~ 10^{-4} T
Large Lab B-field ~ 10 T
Surface of Neutron star ~ 10^{8} T

In the presence of BOTH electric and magnetic fields the instantaneous force on a moving, charged particle is:

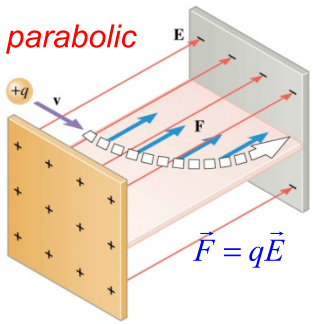
$$\boxed{\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})}$$ *Lorentz Force*

There is no way to avoid the three-dimensional character of the physics of magnetism. By convention, to represent directions perpendicular to the page/whiteboard we denote:

$\odot$ out of the page (arrowhead points towards you)

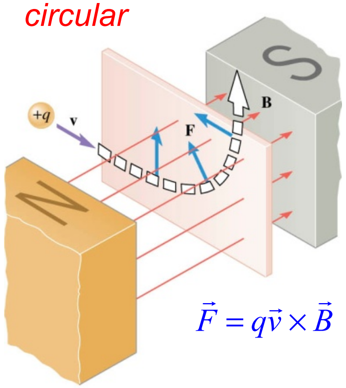
$\otimes$ into the page (fletchings as arrow points away from you)

parabolic

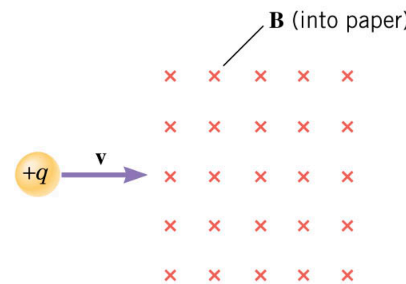


(a)

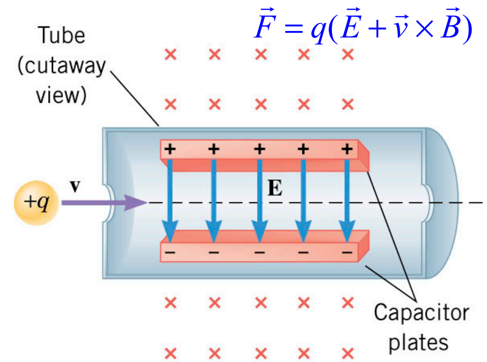
circular



(b)



(a)

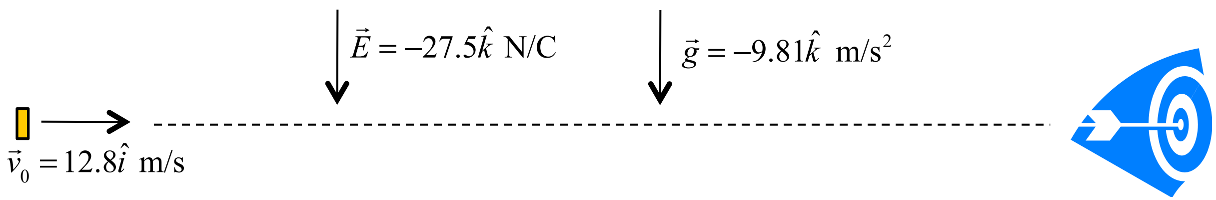


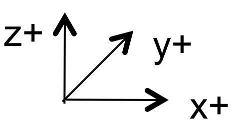
(b)

SJ Ex 1 A proton moves with a velocity of $\vec{v} = 2\hat{i} - 4\hat{j} + \hat{k}$ m/s in a region in which the magnetic field is $\vec{B} = \hat{i} + 2\hat{j} - 3\hat{k}$ T. What is the magnitude of the magnetic force this particle experiences?

(Cross-product review)

YF Ex 1 You wish to hit a target from several meters away with a charged coin having a mass of 5.0 g and a charge of +2500 μC. The coin is given an initial velocity of 12.8 m/s, and a downward uniform electric field with field strength 27.5 N/C exists throughout the region. If you aim directly at the target and fire the coin horizontally, what magnitude and direction of uniform magnetic field are needed in the region for the coin to hit the target.





a proper right-handed coordinate system

point fingers towards +x, curl towards +y, thumb is +z

Magnetic Field Lines and Flux

Somewhat analogous to electric fields and field lines, we can draw magnetic field lines in such a way that the B-field vector is tangent to the field line at a given point, and the magnitude of the B-field is encoded in the density of the lines drawn.

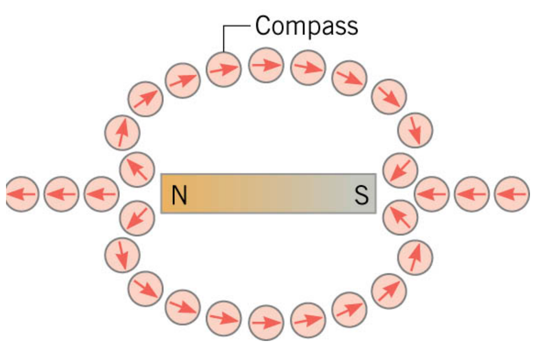
Operationally, a set of compasses placed at various points in the magnetic field will align themselves so that they are parallel to the B-field at that point. The field lines point away from north poles and towards south poles.

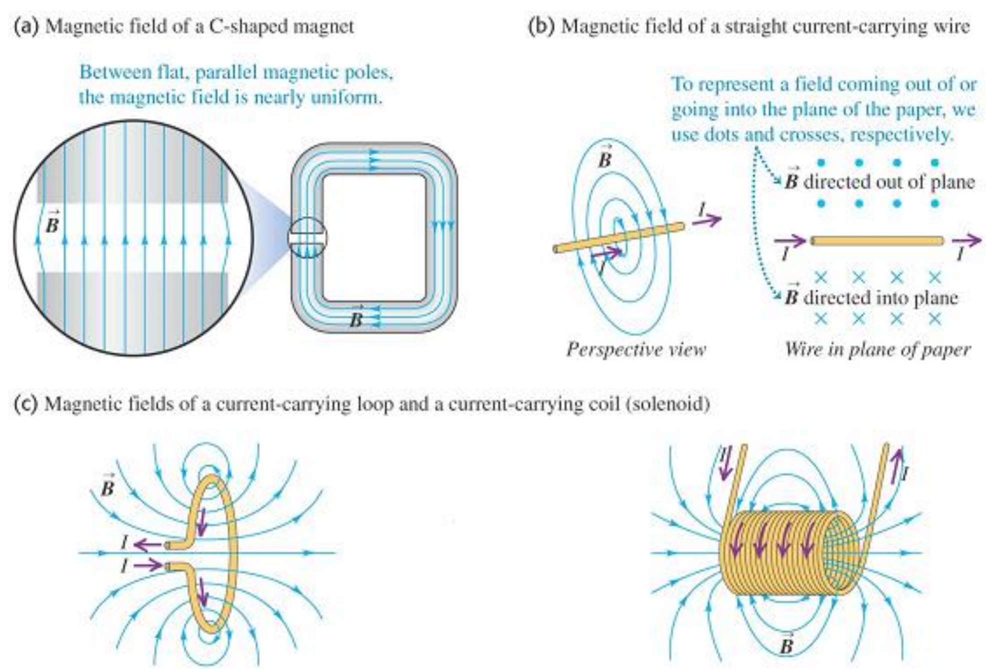
HOWEVER, MAGNETIC FIELD LINES ARE **NOT** LINES OF FORCE. F_{mag} is perp. to B.

Also because there are no magnetic monopoles (isolated north and south poles),

MAGNETIC FIELD LINES HAVE NO ENDS – THEY ALWAYS FORM CLOSED LOOPS

While we draw them *as if* they emanate from a north pole and go to a south pole, they continue through the interior of the magnet. That 'emanation/absorption' is used only to establish the direction of the B-field.



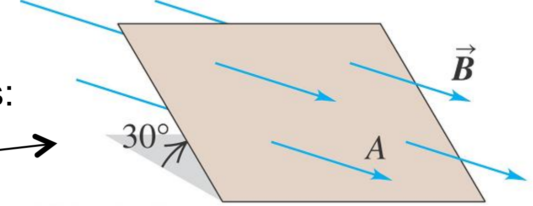


Magnetic Field Lines and Flux

Recall: $\vec{A} \equiv A\hat{n}$

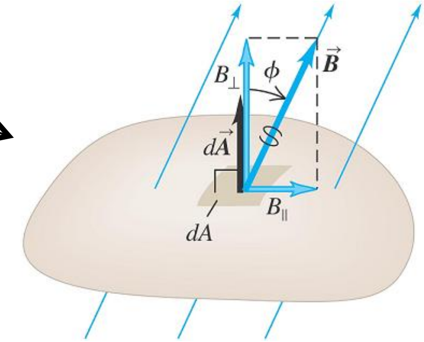
Analogous to electric flux, we can define magnetic flux through a surface as follows:

If the B-field is uniform over an open surface of area A, then $\Phi_B = \vec{B} \cdot \vec{A}$



If the B-field is not uniform over the surface and/or the surface is curved, then as usual, break the surface into infinitesimal patches of area dA and integrate:

$$\Phi_B \equiv \iint_S \vec{B} \cdot d\vec{A} = \iint_S B\cos\phi \, dA = \iint_S B_\perp \, dA$$



Because there are no magnetic monopoles (magnetic 'charges'), there is never any net magnetic flux through a ***closed*** surface:

$$\boxed{\oint\!\!\oint \vec{B} \cdot d\vec{A} = 0}$$

Gauss's Law for Magnetism

(One of the four Maxwell equations)

$[\Phi_B]$ = T·m^2 = Wb (Weber)

Alternate way of thinking of B: $B = \dfrac{d\Phi_B}{dA_\perp}$ "magnetic flux *density*"

Gauss's law deals with closed surfaces, and d**A** is defined to point out of the surface. However, we will usually be using magnetic flux in the context of Ampere's law (Ch 28), which deals with open surfaces – then one must choose a positive direction for d**A** (by choosing a positive side to the open surface), and work with it consistently.

YF Ex 2 (Gauss's Law for Magnetism) In a certain region of space the magnetic field is not uniform. The magnetic field has both a z-component and a component that points radially away from or toward the z-axis. The z-component is given by $B_z(z) = \beta z$, where β is a positive constant. The radial component B_r depends only on *r*, the radial distance from the z-axis. (a) Use Gauss's law for magnetism to find the radial component B_r as a function of *r*. (Hint: Try a cylindrical Gaussian surface of radius r, concentric with the z-axis, with one end at z=0 and the other and z=L.)

Motion of Charged Particles in a Magnetic Field

Case Study 1: Uniform B-field, no E-field, initial velocity is perpendicular to the B-field

The magnetic force on the particle is perpendicular to both the velocity and the B-field, and the particle will execute uniform circular motion. Note that the magnetic field cannot change the speed of the particle, only its direction of motion, i.e. its velocity. In this case Newton's second law gives us:

$$|q|vB = \frac{mv^2}{R} \quad \Leftrightarrow \quad R = \frac{mv}{|q|B}$$

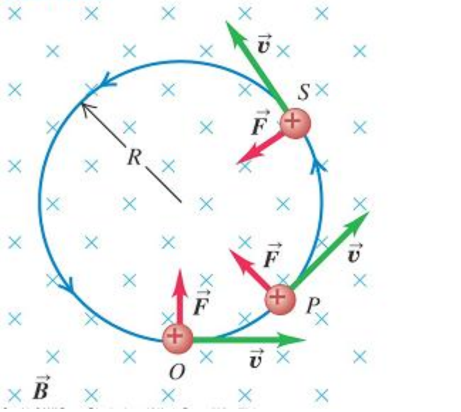
The sign of the charge determines which direction the particle circulates (RHR).

The frequency of revolution (cyclotron frequency) is given by: $f = \dfrac{\omega}{2\pi} = \dfrac{v}{2\pi R} = \dfrac{|q|B}{2\pi m}$

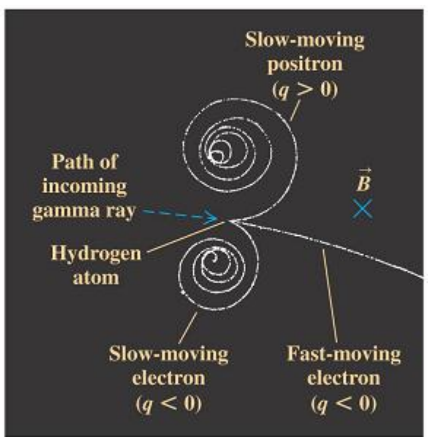
and is independent of the radius of the orbit.

(a) The orbit of a charged particle in a uniform magnetic field

A charge moving at right angles to a uniform $\vec{B}$ field moves in a circle at constant speed because $\vec{F}$ and $\vec{v}$ are always perpendicular to each other.



Particle tracks in an accelerator



In much greater mathematical detail:

Suppose $\vec{B} = B\hat{k}$

Then $q\vec{v} \times \vec{B} = m\vec{a}$ becomes:

$$qB[v_x \hat{i} \times \hat{k} + v_y \hat{j} \times \hat{k}] = m\vec{a}$$
$$qB[-v_x \hat{j} + v_y \hat{i}] = m\vec{a}$$

so

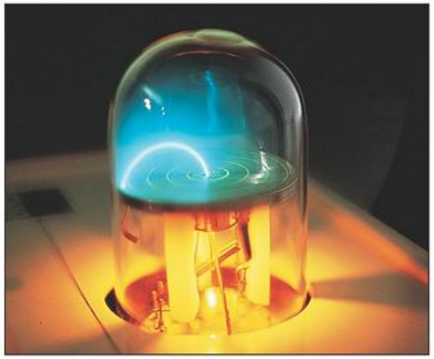
$$m\frac{dv_x}{dt} = qBv_y \qquad m\frac{dv_y}{dt} = -qBv_x$$

diff. and subst.

fancy trick

$$m\frac{d^2v_x}{dt^2} = -\frac{q^2B^2}{m}v_x \qquad m\frac{d^2v_y}{dt^2} = -\frac{q^2B^2}{m}v_y$$

(b) An electron beam (seen as a blue arc) curving in a magnetic field



These can be solved by:

$$v_x(t) = v_0 \sin(qBt/m)\ ,\ v_y(t) = v_0 \cos(qBt/m)$$

integrate

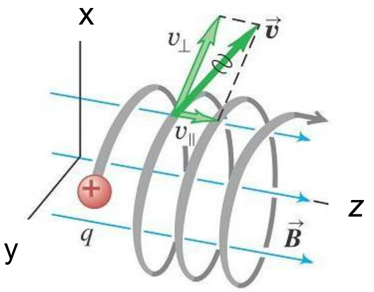
$$x(t) = -(mv_0/qB)\cos(qBt/m)\ ,\ y(t) = (mv_0/qB)\sin(qBt/m)$$

which are the parametric equations for a circle of radius mv/qB ! $x^2 + y^2 = (mv_0/qB)^2$

Motion of Charged Particles in a Magnetic Field

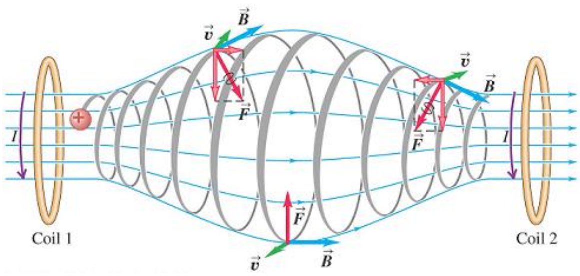
Case Study 2: Uniform B-field, no E-field, initial velocity is NOT perp to the B-field

The component of velocity parallel to the B-field is unaffected by the B-field. So we have the 'superposition' of the circular trajectory of the previous case[1], with uniform straight line motion in the direction parallel to the B-field: the trajectory is a helix.



Case Study 3: NON-Uniform B-field, no E-field

Very difficult to study in general. An example is the magnetic bottle which 'trap' moving charged particles so that they spiral towards one coil, and then towards the other. These magnetic bottles are used to confine plasmas at temperatures of millions of degrees in the study of controlled nuclear fusion.

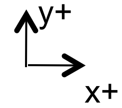


[1]Continuing the mathematical discussion on the previous slide, we didn't write the z-component of Newton II (since our choice of a magnetic field directed along the z-axis implied this was trivial). But here it is:

$$ma_z = 0 \Longrightarrow z(t) = z_0 + v_{0z} t$$

so if the initial velocity had a z-component, then the particle's motion along the z-axis is uniform. We still have circular motion in the x-y plane.

<u>Motion of Charged Particles in a Magnetic Field/Applications</u>



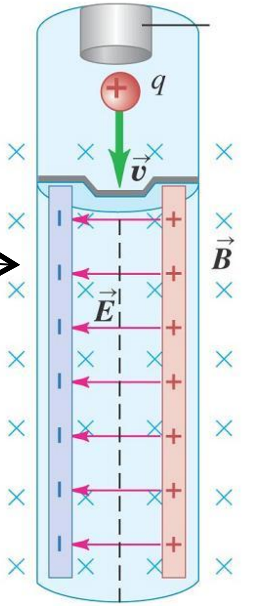
<u>Case Study 4 – Crossed Uniform B and E fields</u>
<u>a) Velocity Selectors</u>

Now we use the full Lorentz force: $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

$\vec{E} = -E\hat{i}$

$\vec{B} = -B\hat{k}$

$\vec{v} = -v\hat{j}$



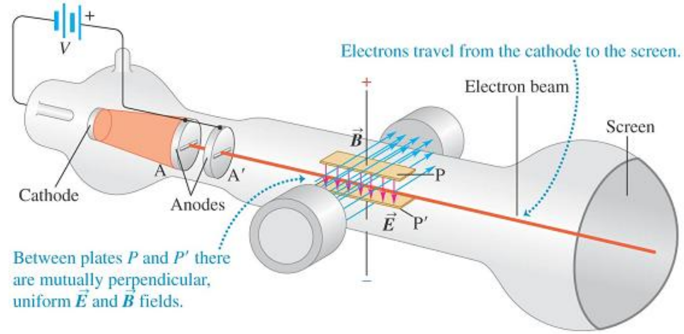
For the fields and initial velocity shown: $\vec{F} = q(-E\hat{i} + (-v)(-B)\hat{j} \times \hat{k}) = q(vB - E)\hat{i}$

so if $vB - E = 0 \iff \boxed{v = \frac{E}{B}}$ then the net force on a charged particle will equal zero,

and the particle will pass through the crossed fields undeflected in the x-direction.
In this way, by tuning the electric field (for example) we can selectively pass particles through a crossed E & B field region that only have a given desired velocity. (Most sources of charged particles like hot cathodes, radioactive sources produce particles with a random distribution of velocities.)

Motion of Charged Particles in a Magnetic Field/Applications

b) Cathode Ray Tubes (and Thomson's e/m expt.)



Heated 'cathodes' will boil off electrons. If these are passed through an accelerating E-field (i.e. potential difference), they acquire speeds determined through E-cons:

$$\Delta K = -\Delta U \Rightarrow \frac{1}{2}mv^2 = eV_{AA'} \Rightarrow v = \sqrt{\frac{2eV}{m}}$$

They then pass through a region of crossed B & E fields. In CRT televisions/monitors, by adjusting the voltage (and hence E-field) in this crossed region, we can deflect electrons so they can strike a scintillation screen and "illuminate a pixel". In Thomson's original application, the E-field was tuned so that the then unknown particles were undeflected:

$$\frac{E}{B} = \sqrt{\frac{2eV}{m}} \Rightarrow \boxed{\frac{e}{m} = \frac{E^2}{2VB^2}}$$

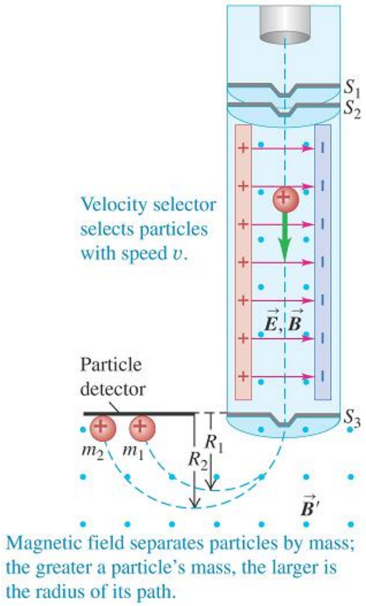
This allowed Thomson to observe[1] the first ***sub*** atomic particle (the electron$^-$) and determine its charge-mass ratio.

By measuring two voltages, and a magnetic field strength, you can determine a fundamnetal parameter of nature!

[1] The most important aspect of this historical expt was that this single ratio that kept arising did not depend on the cathode material, the residual gas in the tube, or anything else in the experiment: whatever these particles were, they were a common element to all of matter. Milikan, 15 years later was able to determine the charge itself precisely (in another historical experiment: the oil drop expt.), which then fixes the mass of the electron: $9.1093826 \times 10^{-31}$ kg.

Motion of Charged Particles in a Magnetic Field/Applications

c) Mass Spectrometer (Bainbridge)



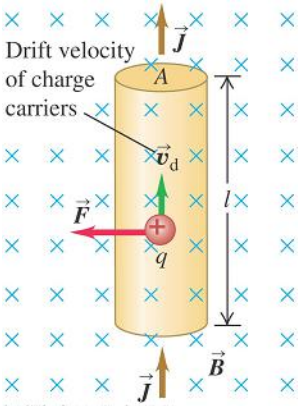
Another early application of uniform B and E fields is in the determination of atomic masses (and the discovery of isotopes of a given element). Here ions in a narrow beam are first passed through a velocity selector before emerging into a uniform perpendicular B-field where they undergo circular motion. Since the radius of orbit is proportional to the ion mass via

$$R = \frac{mv}{|q|B}$$

a measurement of where they strike a detector after traveling through 180° gives a measurement of their mass.

Magnetic Force on a Current-Carrying Conductor

At the basis of countless of applications of magnetism, from electrical motors to loudspeakers, is the effect magnetic fields have on current carrying wires.



Consider current carriers moving at their drift velocities inside a conductor immersed in a perpendicular B-field. Each carrier experiences a magnetic force of magnitude qv_dB. We want the total force on all the moving carriers in a length l of wire:

$$F = (nAl)qv_dB$$

n = volume density of carriers
A = cross-section of wire
l = length of wire

Recall however the current density J (current per unit area) is given by: $J = I / A = nqv_d$

Thus the magnitude of the force on the segment of wire is given by $\boxed{F = JAlB = IlB}$

[1]This is the force is transmitted to the wire itself when the carriers collide with the metal lattice.

A more elegant derivation of this result goes like this:

The force element on the wire dF due to a charge element dq is:

$$dF = dq\,v_dB = dq\frac{dy}{dt}B = \frac{dq}{dt}dyB = IBdy$$

Integrating both sides gives us immediately: $F = IB\int dy = IlB$

Magnetic Force on a Current-Carrying Conductor

If the conductor makes an angle ϕ instead with the B-field then only the perp component of B generates a force, so we can generalize this to:

$$F = IlB\sin\phi$$

We can write this in vector form as: $\boxed{\vec{F} = I\vec{l} \times \vec{B}}$ (This is valid for both signs of charge carrier.)

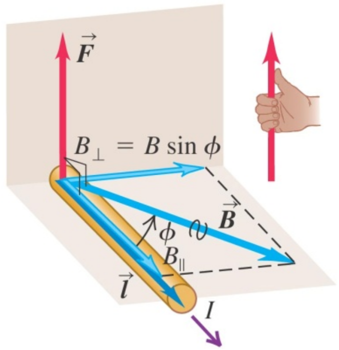
where $\vec{l}$ is a vector whose magnitude is the length l of the conductor, and whose direction is tangent to the conductor in the direction of current.[1]

If the wire is curved, or the B-field non-uniform then we break it into infinitesimal straight line segments and use the differential form:

$$\boxed{d\vec{F} = I\,d\vec{l} \times \vec{B}} \qquad \text{i.e.} \qquad \vec{F} = I\int d\vec{l} \times \vec{B}$$

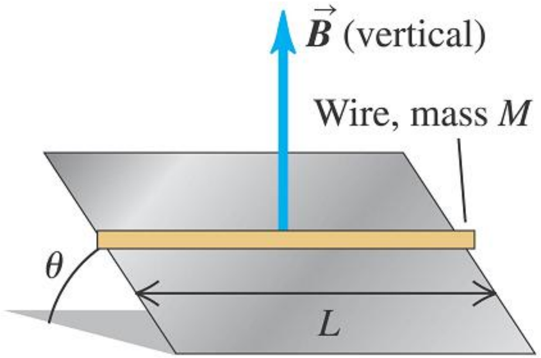
Force $\vec{F}$ on a straight wire carrying a positive current and oriented at an angle ϕ to a magnetic field $\vec{B}$:

- Magnitude is $F = IlB_\perp = IlB\sin\phi$.
- Direction of $\vec{F}$ is given by the right-hand rule.



[1]Current is not a vector. In a curved conductor (aka a non-straight wire), the current is the same at each point. However the differential path element $d\vec{l}$ is always tangent to the wire at a given point, and so varies if the wire is not straight.

<u>YF Ex 3</u> A straight piece of conducting wire with mass M and length L is placed on a frictionless incline tilted at an angle θ from the horizontal. There is a uniform vertical magnetic field at all points (produced by an arrangement of magnets not shown). To keep the wire from sliding down the incline, a voltage source is attached to the ends of the wire. When just the right amount of current flows through the wire, the wire remains at rest. Determine the magnitude and direction of the current in the wire that will cause the wire to remain at rest. Show your free-body diagram all the forces that act on the bar.



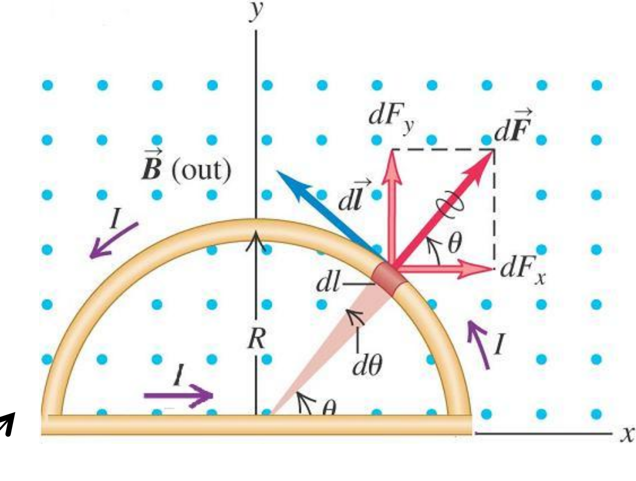
SJ Ex 2

Consider a closed semicircular wire of radius R that carries a current I. The wire lies in the xy-plane, and there is a uniform B-field pointing in the +z direction.

(a) What is the net force on the curved portion?

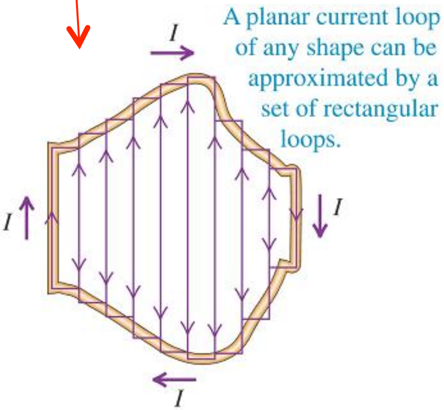
(b) What is the net force on the straight portion?

(c) What is the net force on the whole loop?



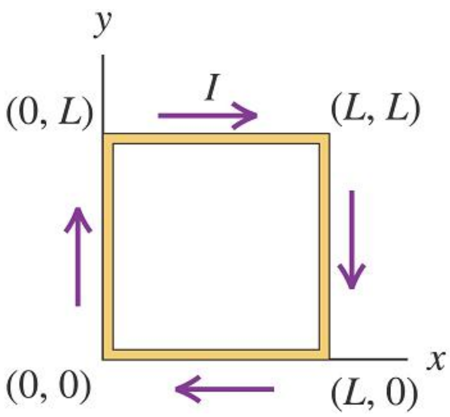
$$dl = Rd\theta$$
$$d\vec{l} = Rd\theta(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})$$

Claim: This result is general: the net magnetic force on a closed current loop in a **UNIFORM** B-field is zero.



YF Ex 4 (Nonuniform B-field) The square loop carries a constant current in the clockwise direction. The magnetic field has no x-component, but has both y and z components:

$$\vec{B} = \left(\frac{B_0 z}{L}\right)\hat{j} + \left(\frac{B_0 y}{L}\right)\hat{k}$$



where B_0 is a positive constant.

(a) Sketch the magnetic field lines in the *y-z plane*.

(b) Find the magnitude and direction of the magnetic force exerted on each of the sides of the loop by integration.

(c) Find the magnitude and direction of the net magnetic force on the loop.

*SJ Ex 3 (**NONUNIFORM** B-fields exert net forces on Current Loops)

(aka "how a magnet attracts iron objects")

A strong magnet is placed under a horizontal conducting ring of radius *r* that carries a current I as shown. If the magnetic field makes an angle θ with the vertical *at each point at the ring's location*, what are the magnitude and direction of the resultant magnetic force on the ring?

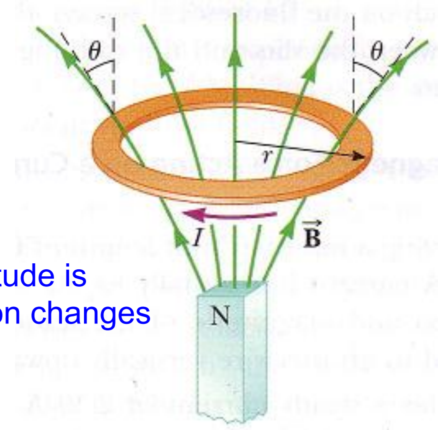


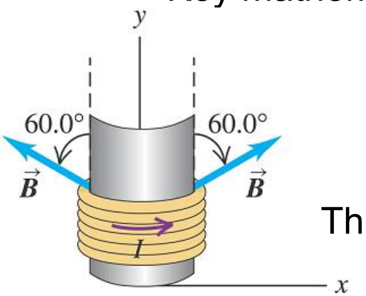
Figure P29.31

Key mathematics:

$$d\vec{l} = R\,d\alpha(+\sin\alpha, -\cos\alpha, 0)$$

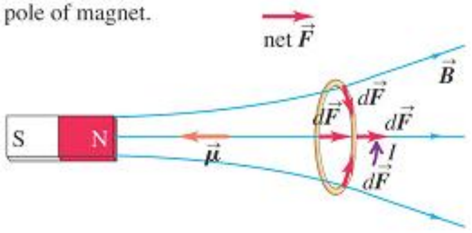
$$\vec{B} = B(\sin\theta\cos\alpha, \sin\theta\sin\alpha, \cos\theta)$$

Although the magnitude is constant the direction changes as α changes.

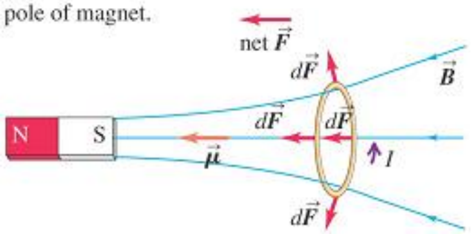


This is also how a voice coil (present in every microphone) works (YF27.73)

(a) Net force on this coil is away from north pole of magnet.



(b) Net force on same coil is toward south pole of magnet.



Torque on a Current Loop

We've found that in uniform fields, the net force on a current loop is zero. (We've also seen in nonuniform fields the net force can be nonzero) What about the torque on a current loop? This is the basis of electrical motors.

Consider the setup and notation in the figure. With the magnetic field along the +z axis, taking torques about y:

$$\vec{F}_1 = Ib(-\cos\phi\,\hat{i} - \sin\phi\,\hat{k}) \times B\hat{k} = +IbB\cos\phi\,\hat{j} \Rightarrow \vec{\tau}_{about\ y} = \vec{r}_1 \times \vec{F}_1 \propto \hat{j} \times \hat{j} = 0$$

$$\vec{F}_2 = I(a\hat{j}) \times (B\hat{k}) = IaB\hat{i} \Rightarrow \vec{\tau}_{about\ y} = \vec{r}_2 \times \vec{F}_2 = \frac{b}{2}(\cos\phi\,\hat{i} + \sin\phi\,\hat{k}) \times (IaB\hat{i}) = Ia\frac{b}{2}B\sin\phi\,\hat{j}$$

$$\vec{F}_3 = Ib(\cos\phi\,\hat{i} + \sin\phi\,\hat{k}) \times B\hat{k} = -IbB\cos\phi\,\hat{j} \Rightarrow \vec{\tau}_{about\ y} = \vec{r}_3 \times \vec{F}_3 \propto \hat{j} \times \hat{j} = 0$$

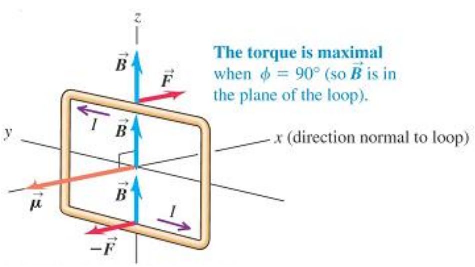
$$\vec{F}_4 = I(-a\hat{j}) \times (B\hat{k}) = -IaB\hat{i} \Rightarrow \vec{\tau}_{about\ y} = \vec{r}_4 \times \vec{F}_4 = \frac{b}{2}(-\cos\phi\,\hat{i} - \sin\phi\,\hat{k}) \times (-IaB\hat{i}) = Ia\frac{b}{2}B\sin\phi\,\hat{j}$$

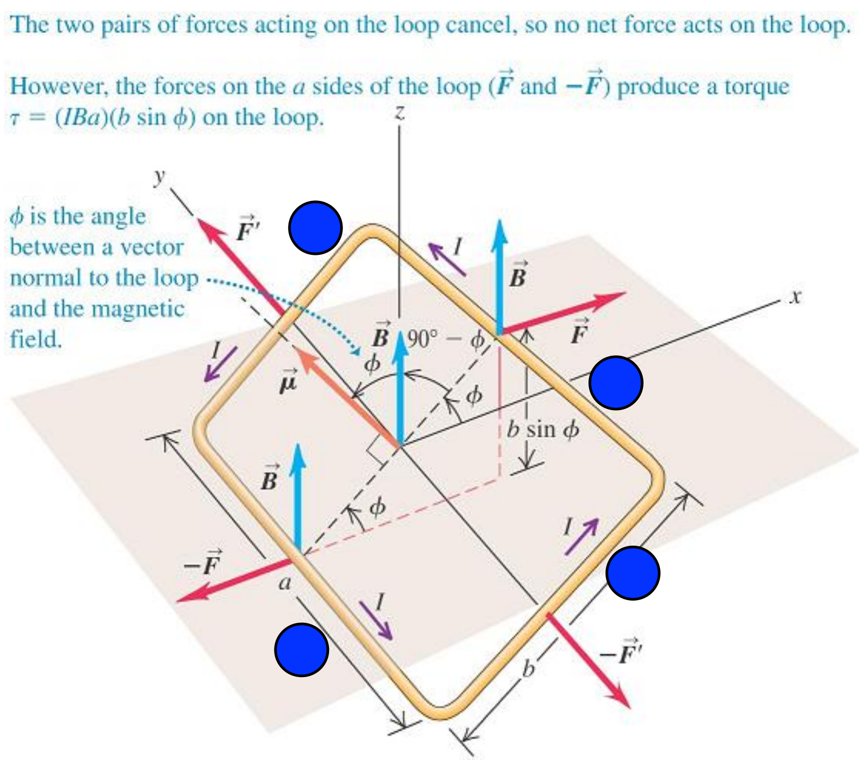
$$\vec{F}_{net} = \vec{0}$$

$$\boxed{\vec{\tau}_{net} = IAB\sin\phi\ \hat{j}}$$

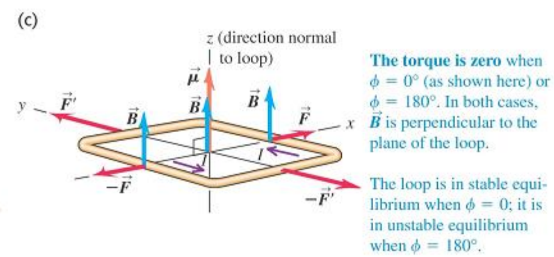
where A = ab is the area of the loop

Maximal Torque

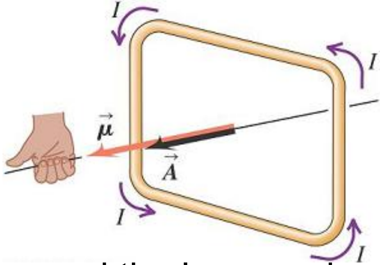




Zero Torque



Torque on a Current Loop: Vector Form and Magnetic Moment



Define the magnetic (dipole) moment of the loop as the product: $\mu \equiv IA$

We can go further and introduce the vector magnetic moment as: $\boxed{\vec{\mu} = I\vec{A}}$

where the positive direction is defined by the right hand rule: curl fingers in the direction of I around the loop, and the magnetic moment points in the direction of the thumb.

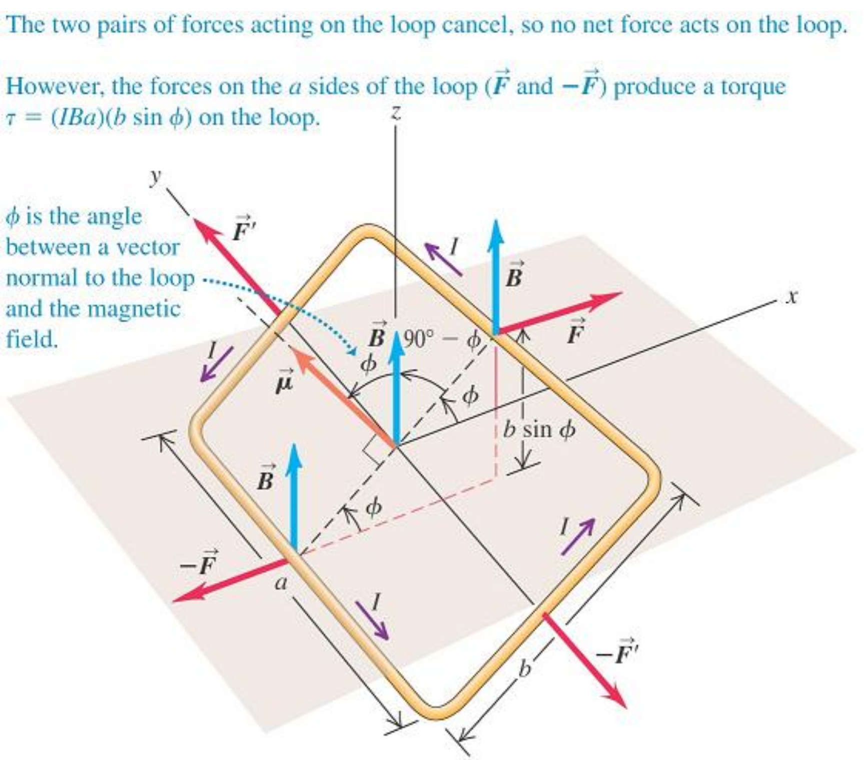
This allows us the write the vector torque on a current loop as: $\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$

Note, as with any cross-product: $\tau = |\vec{\tau}| = IAB\sin\phi$

Compare this with the equivalent expression for an electric dipole (Ch 21): $\vec{\tau} = \vec{p} \times \vec{E}$

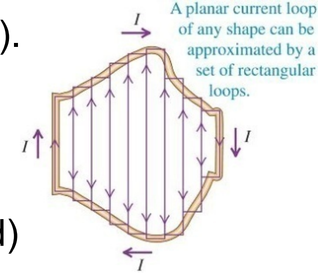
For example, considering again our setup from the previous slide: $\vec{\mu} = Iab(-\sin\phi\,\hat{i} + \cos\phi\,\hat{k})$ $\quad \vec{B} = B\hat{k}$

so $\vec{\tau} = \vec{\mu} \times \vec{B} = -IabB\sin\phi\,\hat{i} \times \hat{k} = IabB\sin\phi\,\hat{j}$



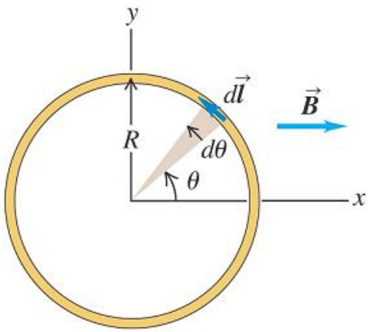
The claim is this relationship $\vec{\tau} = \vec{\mu} \times \vec{B}$ holds for any shape of loop (for **UNIFORM fields only**).

We'll content ourselves to consider verifying it holds also for a circular loop...



YF Ex 5

A wire ring lies in the xy-plane with its center at the origin. The ring carries a counterclockwise current I. An uniform magnetic field is in the positive +x direction (the result is easily generalized)



a) Show the line element is $d\vec{l} = -Rd\theta(-\sin\theta\hat{i} + \cos\theta\hat{j})$ and find $d\vec{F} = Id\vec{l} \times \vec{B}$.

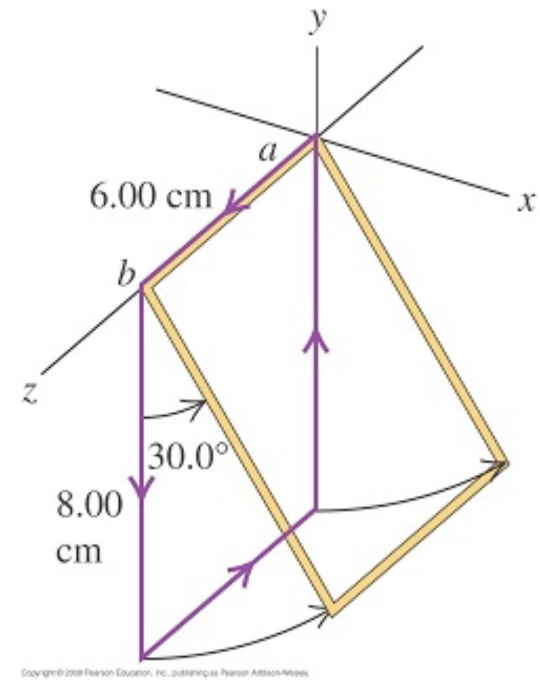
b) Integrate $d\vec{F}$ around the loop to show the net force is zero.

c) From part (a), find $d\vec{\tau} = \vec{r} \times d\vec{F}$. (Note that $\vec{r} = R(\cos\theta\hat{i} + \sin\theta\hat{j})$.)

d) Integrate $d\vec{\tau}$ over the loop to find the total torque on the loop. Show that it can be written in the standard form: $\vec{\tau} = \vec{\mu} \times \vec{B}$

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x\ ,\ \int \cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x\ ,\ \int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x$$

YF Ex 6 The rectangular loop of wire shown has a mass of 0.15 g/cm of length and is pivoted about side *ab* on a frictionless axis. The current in the wire is 8.2 A in the direction shown. Find the magnitude and direction of the magnetic field parallel to the *y*-axis that will cause the loop to swing up until its plane makes an angle of 30.0° with the *yz*-plane (i.e. when the torque due to the magnetic field cancels the torque due to gravity).



Be careful to note the positive direction of the axes: in problems where you're given only two axes always use the right hand rule, or cyclic permutation to determine the positive direction of the suppressed third axis.

Magnetic Torque: Other Notes

The torque is maximal when **μ** and **B** are perpendicular, zero when they are (anti)parallel. The stable equilibrium position occurs when **μ** and **B** are parallel.

Work and Potential Energy

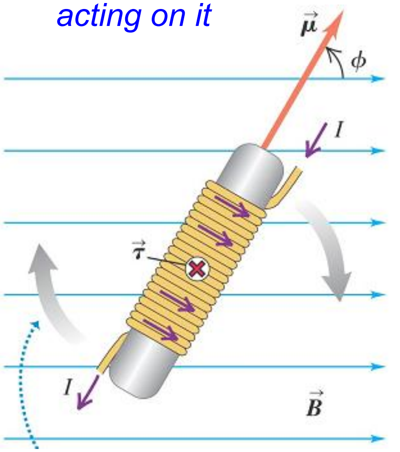
Recall: for electric dipoles, the expression $\vec{\tau} = \vec{p} \times \vec{E}$ implied: $U = -\vec{p} \cdot \vec{E}$

The symmetry here between electric and magnetic dipoles therefore immediately leads to: the potential energy for a magnetic dipole in a uniform field[1]: $\boxed{U = -\vec{\mu} \cdot \vec{B}}$

Coils and Solenoids

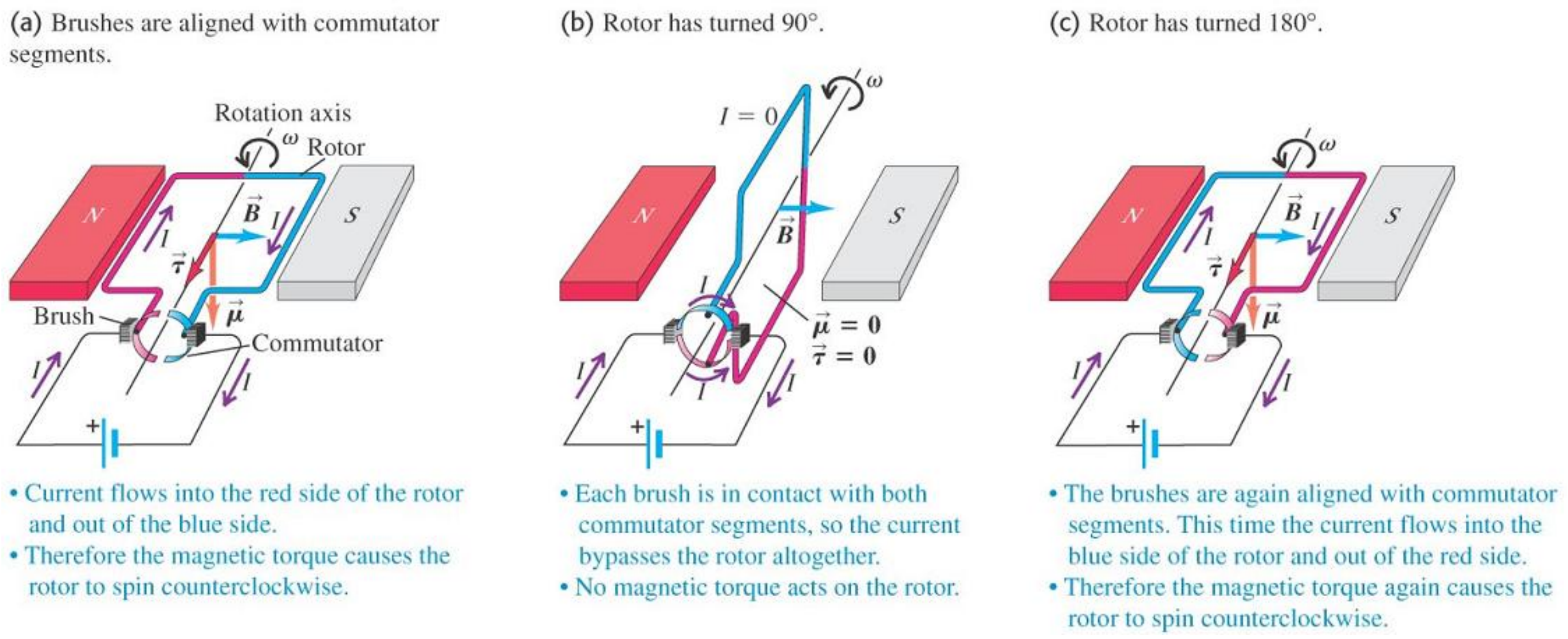
If we have N loops of wire, the previous equations are multiplied by N. Since the wires are of finite thickness, we can form a solenoid – a helical winding of wire) which we can approximate by N circular loops. So for example: $\tau = NIAB \sin\phi$

A solenoid, its magnetic moment, and the torque acting on it



[1] With this definition, the reference potential is zero when **μ** is perp. to **B**.

The DC Electrical Motor (aka the reason Mechanical Engineers take E&M)



Torque is maximal. Current flow is from red to blue. Loop starts rotating ccw.

Torque is momentarily zero. No current flow through loop due to brush-commutator. Inertia of the loop ($\omega \neq 0$) ensures it continues rotating past this point.

Torque is again maximal, but current flow is from blue to red. Loop continues rotating ccw.

This system will continue to undergo an angular acceleration (remember $\tau = I\alpha$) until the magnetic torque is cancelled by frictional torques in the system. At this point the angular speed will be constant.