Vision 3D artificielle Session 1: Projective geometry, camera matrix, panorama

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Pinhole camera model

Projective geometry

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Panorama

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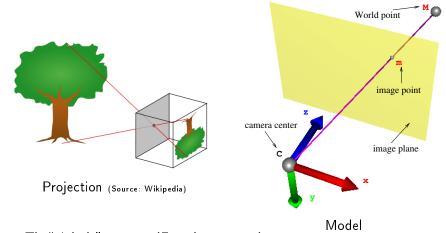
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The "pinhole" camera model



The "pinhole" camera (French: sténopé):

- ▶ Ideal model with an aperture reduced to a single point.
- ► No account for blur of out of focus objects, nor for the lens geometric distortion.

Central projection in camera coordinate frame

- **Proof** Rays from *C* are the same: $\vec{Cm} = \lambda \vec{CM}$
- ▶ In the camera coordinate frame CXYZ:

$$\begin{pmatrix} x \\ y \\ f \end{pmatrix} = \lambda \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

▶ Thus $\lambda = f/Z$ and

$$\begin{pmatrix} x \\ y \end{pmatrix} = f \begin{pmatrix} X/Z \\ Y/Z \end{pmatrix}$$

► In pixel coordinates:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \alpha x + c_x \\ \alpha y + c_y \end{pmatrix} = \begin{pmatrix} (\alpha f)X/Z + c_x \\ (\alpha f)Y/Z + c_y \end{pmatrix}$$

ightharpoonup lpha f: focal length *in pixels*, (c_x, c_y) : position of principal point P in pixels.

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Projective plane

We identify two points of \mathbb{R}^3 on the same ray from the origin through the equivalence relation:

$$\mathcal{R}: \mathbf{x}\mathcal{R}\mathbf{y} \Leftrightarrow \exists \lambda \neq \mathbf{0}: \mathbf{x} = \lambda \mathbf{y}$$

- Projective plane: $\mathbb{P}^2 = (\mathbb{R}^3 \setminus O)/\mathcal{R}$
- Point $(x \ y \ z) = (x/z \ y/z \ 1)$ if $z \neq 0$.
- ▶ The point $(x/\epsilon \ y/\epsilon \ 1) = (x \ y \ \epsilon)$ is a point "far away" in the direction of the line of slope y/x. The limit value $(x \ y \ 0)$ is the infinite point in this direction.
- ▶ Given a plane of \mathbb{R}^3 through O, its equation is aX + bY + cZ = 0. It corresponds to a line in \mathbb{P}^2 represented in homogeneous coordinates by $\begin{pmatrix} a & b & c \end{pmatrix}$. Its equation is:

$$(a b c)(X Y Z)^{\top} = 0.$$

Projective plane

► Line through points x₁ and x₂:

$$\ell = \mathbf{x_1} \times \mathbf{x_2} \text{ since } (\mathbf{x_1} \times \mathbf{x_2})^{\top} \mathbf{x_i} = |\mathbf{x_1} \times \mathbf{x_2} \times \mathbf{x_i}| = 0$$

▶ Intersection of two lines ℓ_1 and ℓ_2 :

$$\mathbf{x} = \ell_1 imes \ell_2$$
 since $\ell_i^{ op}(\ell_1 imes \ell_2) = |\ell_i op \ell_1 op \ell_2| = 0$

Line at infinity:

$$\ell_{\infty} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ since } \ell_{\infty}^{\top} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = 0$$

► Intersection of two "parallel" lines:

$$egin{pmatrix} a \ b \ c_1 \end{pmatrix} imes egin{pmatrix} a \ b \ c_2 \end{pmatrix} = (c_2 - c_1) egin{pmatrix} b \ -a \ 0 \end{pmatrix} \in \ell_{\infty}$$

Calibration matrix

Let us get back to the projection equation:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} fX/Z + c_x \\ fY/Z + c_y \end{pmatrix} = \frac{1}{Z} \begin{pmatrix} fX + c_x Z \\ fY + c_y Z \end{pmatrix}$$

(replacing αf by f)

► We rewrite:

$$Z\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} := \mathbf{x} = \begin{pmatrix} f & c_{x} \\ f & c_{y} \\ & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

► The 3D point being expressed in another orthonormal coordinate frame:

$$\mathbf{x} = \begin{pmatrix} f & c_{x} \\ f & c_{y} \\ & 1 \end{pmatrix} \begin{pmatrix} R & T \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ 1 \end{pmatrix}$$

Calibration matrix

▶ The (internal) calibration matrix (3×3) is:

$$K = \begin{pmatrix} f & c_x \\ f & c_y \\ & 1 \end{pmatrix}$$

▶ The projection matrix (3×4) is:

$$P = K (R T)$$

ightharpoonup If pixels are trapezoids, we can generalize K:

fy
$$K = \begin{pmatrix} f_x & s & c_x \\ & f_y & c_y \\ & & 1 \end{pmatrix}$$
 (with $s = -f_x \cot \theta$)

Theorem

Let P be a 3×4 matrix whose left 3×3 sub-matrix is invertible.

There is a unique decomposition $P = K(R \mid T)$.

Proof: Gram-Schmidt on rows of left sub-matrix of P starting from last row (RQ decomposition), then $T = K^{-1}P_4$.

Projective plane (perspective effect)

► Lines parallel in space project to a line bundle (set of lines parallel or concurrent). Let **d** be a fixed direction vector:

$$\begin{split} \mathcal{K}(\mathbf{X} + \lambda \mathbf{d}) &= \mathcal{K}\mathbf{X} + \lambda \mathcal{K}\mathbf{d} \\ \ell_{\mathbf{X}} &= (\mathcal{K}\mathbf{X}) \times (\mathcal{K}\mathbf{d}) \\ \forall \mathbf{X}, \ell_{\mathbf{X}}^{\top}\mathbf{v} &= 0 \quad \text{for } \mathbf{v} := \mathcal{K}\mathbf{d} \end{split}$$

- **v** is the vanishing point of lines of direction **d**.
- If $\mathbf{v_1} = K\mathbf{d_1}$ and $\mathbf{v_2} = K\mathbf{d_2}$ are vp of "horizontal" lines, another set of horizontal lines has direction $\alpha \mathbf{d_1} + \beta \mathbf{d_2}$, hence its vp $\alpha \mathbf{v_1} + \beta \mathbf{v_2}$, which belongs to line $\mathbf{v_1} \times \mathbf{v_2}$, the "horizon".

Projective plane (perspective effect)



Projective plane (perspective effect)



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Homographies

Let us see what happens when we take two pictures in the following particular cases:

1. Rotation around the optical center (and maybe change of internal parameters).

$$\mathbf{x}' = K'RK^{-1}\mathbf{x} := H\mathbf{x}$$

2. The world is flat. We observe the plane Z=0:

$$\mathbf{x}' = K \begin{pmatrix} R_1 & R_2 & R_3 & T \end{pmatrix} \begin{pmatrix} X \\ Y \\ 0 \\ 1 \end{pmatrix} = K \begin{pmatrix} R_1 & R_2 & T \end{pmatrix} \mathbf{x} := H\mathbf{x}$$

In both cases, we deal with a 3×3 invertible matrix H, a homography.

Property: a homography preserves alignment. If x_1, x_2, x_3 are aligned, then

$$|Hx_1 Hx_2 Hx_3| = |H||x_1 x_2 x_3| = 0$$

Homographies

Type Matrix

Rigid (Rot.+Trans.)
$$H = \begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix}$$
 $(c^2 + s^2 = 1)$

Similarity $H = \begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix}$

Affine $H = \begin{pmatrix} a & b & t_x \\ c & d & t_y \end{pmatrix}$

Homography

$$\begin{pmatrix} t_x \\ t_y \end{pmatrix}$$

$$\begin{pmatrix} y \\ 1 \end{pmatrix}$$

$$\frac{1}{s}$$

$$H = \begin{pmatrix} c & -s & t_x \\ s & c & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$$0 t_{\downarrow}$$

$$H = \begin{pmatrix} a & b & t_x \\ c & d & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

H invertible

Invariants

distances parallelism

cross-

ratio 4 aligned points

Given 4 aligned points A, B, C, D, their cross-ratio is:

$$(A,B;C,D) = \frac{AC}{BC} : \frac{AD}{BD}$$

Homographies: estimation from point correspondences

Theorem Let $e_1, \dots, e_{d+1}, f_1, \dots, f_{d+1} \in \mathbb{R}^d$ such that any d vectors e_i (resp. f_i) are linearly independent. Then there are (up to scale) a unique isomorphism H and a unique set of scalars $\lambda_i \neq 0$ so that $\forall i, He_i = \lambda_i f_i$.

1. Analysis: writing $e_{d+1} = \sum_{i=1}^d \mu_i e_i$ and $f_{d+1} = \sum_{i=1}^d \nu_i f_i$,

$$\sum_{i=1}^{d} \nu_i \lambda_{d+1} f_i = \lambda_{d+1} f_{d+1} = He_{d+1} = \sum_{i=1}^{d} \mu_i \lambda_i f_i$$

so that $\forall i = 1, \dots, d : \mu_i \lambda_i = \nu_i \lambda_{d+1}$.

- 2. $\forall i, \mu_i \neq 0$ and $\nu_i \neq 0$, since $\mu_i = 0 \Rightarrow \{e_j\}_{j \neq i}$ are linearly dependent.
- 3. Therefore we get $\lambda_i = \frac{\nu_i}{\mu_i} \lambda_{d+1}$.
- 4. Synthesis: fix $\lambda_{d+1} = 1$ and $\forall i : \lambda_i = \frac{\nu_i}{\mu_i} \neq 0$.
- 5. There is a unique H mapping basis $\{e_i\}_{i=1,...,d}^{\mu_i}$ to basis $\{\lambda_i f_i\}_{i=1,...,d}$

Application: given n+2 pairs $(\mathbf{x_i}, \mathbf{x_i'}) \in \mathbb{P}^n$, there is a unique homography mapping $\mathbf{x_i}$ to $\mathbf{x_i'}$.

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Panorama construction

- We stitch together images by correcting homographies. This assumes that the scene is flat or that we are rotating the camera.
- ► Homography estimation:

$$\lambda \mathbf{x}' = H\mathbf{x} \Rightarrow \mathbf{x}' \times (H\mathbf{x}) = 0,$$

which amounts to 2 independent linear equations per correspondence (x, x').

▶ 4 correspondences are enough to estimate *H* (but more can be used to estimate through mean squares minimization).







Panorama from 14 photos

Algebraic error minimization

- $\mathbf{x}_{\mathbf{i}}' \times (H\mathbf{x}_{\mathbf{i}}) = 0$ is a system of three linear equations in H.
- \blacktriangleright We gather the unknown coefficients of H in a vector of 9 rows

$$h = \begin{pmatrix} H_{11} & H_{12} & \dots & H_{33} \end{pmatrix}^{\top}$$

ightharpoonup We write the equations as $A_i h = 0$ with

$$A_{i} = \begin{pmatrix} x_{i} & y_{i} & 1 & 0 & 0 & 0 & -x'_{i}x_{i} & -x'_{i}y_{i} & -x'_{i} \\ 0 & 0 & 0 & x_{i} & y_{i} & 1 & -y'_{i}x_{i} & -y'_{i}y_{i} & -y'_{i} \\ -x_{i}y'_{i} & -y_{i}y'_{i} & -y'_{i} & x'_{i}x_{i} & x'_{i}y_{i} & x'_{i} & 0 & 0 & 0 \end{pmatrix}$$

- \triangleright We can discard the third line and stack the different A_i in A.
- ightharpoonup h is a vector of the kernel of A (8 imes 9 matrix)
- We can also suppose $H_{3,3} = h_9 = 1$ and solve

$$A_{:,1:8}h_{1:8} = -A_{:,9}$$

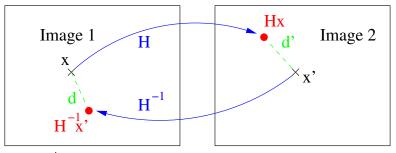
Geometric error

► When we have more than 4 correspondences, we minimize the algebraic error

$$\epsilon = \sum_{i} \|\mathbf{x}_{i}' \times (H\mathbf{x}_{i})\|^{2},$$

but it has no geometric meaning.

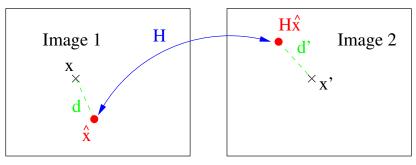
► A more significant error is geometric:



► Either $d'^2 = d(\mathbf{x}', \mathbf{H}\mathbf{x})^2$ (transfer error) or $d^2 + d'^2 = d(\mathbf{x}, H^{-1}\mathbf{x}')^2 + d(\mathbf{x}', H\mathbf{x})^2$ (Symmetric transfer error)

Gold standard error/Maximum likelihood estimator

Actually, we can consider x and x' as noisy observations of ground truth positions x and x' = Hx.



$$\epsilon(H, \mathbf{\hat{x}}) = d(\mathbf{x}, \mathbf{\hat{x}})^2 + d(\mathbf{x}', H\mathbf{\hat{x}})^2$$

Problem: this has a lot of parameters: $H, \{\hat{\mathbf{x}}_i\}_{i=1...n}$

Sampson error

► A method that linearizes the dependency on **x** in the gold standard error so as to eliminate these unknowns.

$$0 = \epsilon(H, \mathbf{\hat{x}}) = \epsilon(H, \mathbf{x}) + J(\mathbf{\hat{x}} - \mathbf{x}) \text{ with } J = \frac{\partial \epsilon}{\partial \mathbf{x}}(H, \mathbf{x})$$

- Find $\hat{\mathbf{x}}$ minimizing $\|\mathbf{x} \hat{\mathbf{x}}\|^2$ subject to $J(\mathbf{x} \hat{\mathbf{x}}) = \epsilon$
- ▶ Solution: $\mathbf{x} \mathbf{\hat{x}} = J^{\top} (JJ^{\top})^{-1} \epsilon$ and thus:

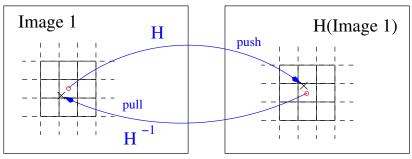
$$\|\mathbf{x} - \hat{\mathbf{x}}\|^2 = \epsilon^{\top} (JJ^{\top})^{-1} \epsilon \tag{1}$$

- ▶ Here, $\epsilon_i = A_i h = \mathbf{x}_i' \times (H\mathbf{x}_i)$ is a 3-vector.
- ▶ For each i, there are 4 variables (x_i, x_i') , so J is 3 × 4.
- ▶ This is almost the algebraic error $\epsilon^{\top}\epsilon$ but with adapted scalar product.
- ► The resolution, through iterative method, must be initialized with the algebraic minimization.

Applying homography to image

Two methods:

- 1. push pixels to transformed image and round to the nearest pixel center.
- 2. pull pixels from original image by interpolation.



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Camera calibration by resection

[R.Y. Tsai, An efficient and accurate camera calibration technique for 3D machine vision, CVPR'86] We estimate the camera internal parameters from a known rig, composed of 3D points whose coordinates are known.

- We have points X_i and their projection x_i in an image.
- In homogeneous coordinates: $x_i = PX_i$ or the 3 equations (but only 2 of them are independent)

$$x_i \times (PX_i) = 0$$

- ► Linear system in unknown P. There are 12 parameters in P, we need 6 points in general (actually only 5.5).
- \triangleright Decomposition of P allows finding K.



Restriction: The 6 points cannot be on a plane, otherwise we have a degenerate situation; in that case, 4 points define the homography and the two extra points yield no additional constraint.

Calibration with planar rig

- [Z. Zhang A flexible new technique for camera calibration 2000]
 - Problem: One picture is not enough to find K.Solution: Several snapshots are used.
- ► For each one, we determine the homography *H* between the rig and the image.
- ► The homography being computed with an arbitrary multiplicative factor, we write

$$\lambda H = K \begin{pmatrix} R_1 & R_2 & T \end{pmatrix}$$

► We rewrite:

$$\lambda K^{-1}H = \lambda (K^{-1}H_1 \quad K^{-1}H_2 \quad K^{-1}H_3) = (R_1 \quad R_2 \quad T)$$

 \triangleright 2 equations expressing orthonormality of R_1 and R_2 :

$$H_1^{\top}(K^{-\top}K^{-1})H_1 = H_2^{\top}(K^{-\top}K^{-1})H_2$$

 $H_1^{\top}(K^{-\top}K^{-1})H_2 = 0$

▶ With 3 views, we have 6 equations for the 5 parameters of $K^{-\top}K^{-1}$; then Cholesky decomposition.

The problem of geometric distortion

- At small or moderate focal length, we cannot ignore the geometric distortion due to lens curvature, especially away from image center.
- This is observable in the non-straightness of certain lines:



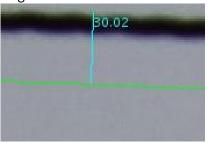


Photo: 5600×3700 pixels

Deviation of 30 pixels

The classical model of distortion is radial polynomial:

$$\begin{pmatrix} x_d \\ y_d \end{pmatrix} - \begin{pmatrix} d_x \\ d_y \end{pmatrix} = \left(1 + a_1 r^2 + a_2 r^4 + \dots\right) \begin{pmatrix} x - d_x \\ y - d_y \end{pmatrix}$$

Estimation of geometric distortion

- ► If we integrate distortion coefficients as unknowns, there is no more closed formula estimating *K*.
- We have a non-linear minimization problem, which can be solved by an iterative method.
- To initialize the minimization, we assume no distortion $(a_1 = a_2 = 0)$ and estimate K with the previous linear procedure.

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Linear least squares problem

► For example, when we have more than 4 point correspondences in homography estimation:

$$A_{m\times 8}h=B_m \quad m\geq 8$$

▶ In the case of an overdetermined linear system, we minimize

$$\epsilon(\mathbf{X}) = \|A\mathbf{X} - B\|^2 = \|f(\mathbf{X})\|^2$$

ightharpoonup The gradient of ϵ can be easily computed:

$$\nabla \epsilon(\mathbf{X}) = 2(A^{\top}A\mathbf{X} - A^{\top}B)$$

▶ The solution is obtained by equating the gradient to 0:

$$\mathbf{X} = (A^{\top}A)^{-1}A^{\top}B$$

- **Remark 1**: this is correct only if $A^{\top}A$ is invertible, that is A has full rank.
- **Remark 2**: if A is square, it is the standard solution $\mathbf{X} = A^{-1}B$
- Remark 3: $A^{(-1)} = (A^{\top}A)^{-1}A^{\top}$ is called the pseudo-inverse of A, because $A^{(-1)}A = I_n$.

Non-linear least squares problem

We would like to solve as best we can f(X) = 0 with f non-linear. We thus minimize

$$\epsilon(\mathbf{X}) = \|f(\mathbf{X})\|^2$$

 \blacktriangleright Let us compute the gradient of ϵ :

$$\nabla \epsilon(\mathbf{X}) = 2J^{\top} f(\mathbf{X}) \text{ with } J_{ij} = \frac{\partial f_i}{\partial x_i}$$

► Gradient descent: we iterate until convergence

$$\triangle \mathbf{X} = -\alpha J^{\mathsf{T}} f(\mathbf{X}), \ \alpha > 0$$

▶ When we are close to the minimum, a faster convergence is obtained by Newton's method:

$$\epsilon(\mathbf{X_0}) \sim \epsilon(\mathbf{X}) + \nabla \epsilon(\mathbf{X})^\top (\triangle \mathbf{X}) + (\triangle \mathbf{X})^\top (\nabla^2 \epsilon) (\triangle \mathbf{X})$$
 and minimum is for $\triangle \mathbf{X} = -(\nabla^2 \epsilon)^{-1} \nabla \epsilon$

Levenberg-Marquardt algorithm

- This is a mix of gradient descent and quasi-Newton method (quasi since we do not compute explicitly the Hessian matrix, but approximate it).
- ightharpoonup The gradient of ϵ is

$$\nabla \epsilon(\mathbf{X}) = 2J^{\mathsf{T}} f(\mathbf{X})$$

so the Hessian matrix of ϵ is composed of sums of two terms:

- 1. Product of first derivatives of f.
- 2. Product of f and second derivatives of f.
- The idea is to ignore the second terms, as they should be small when we are close to the minimum ($f \sim 0$). The Hessian is thus approximated by

$$H = 2J^{T}J$$

Levenberg-Marquardt iteration:

$$\triangle \mathbf{X} = -(J^{\top}J + \lambda I)^{-1}J^{\top}f(\mathbf{X}), \lambda > 0$$

Levenberg-Marquardt algorithm

- Principle: gradient descent when we are far from the solution (λ large) and Newton's step when we are close (λ small).
- 1. Start from initial **X** and $\lambda = 10^{-3}$.
- 2. Compute

$$\triangle \mathbf{X} = -(J^{\top}J + \lambda I)^{-1}J^{\top}f(\mathbf{X}), \lambda > 0$$

3. Compare $\epsilon(\mathbf{X} + \triangle \mathbf{X})$ and $\epsilon(\mathbf{X})$:

3a If $\epsilon(\mathbf{X} + \triangle \mathbf{X}) \sim \epsilon(\mathbf{X})$, finish.

3b If $\epsilon(\mathbf{X} + \triangle \mathbf{X}) < \epsilon(\mathbf{X})$,

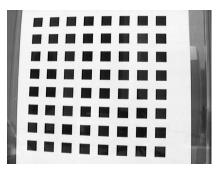
$$\mathbf{X} \leftarrow \mathbf{X} + \triangle \mathbf{X} \quad \lambda \leftarrow \lambda/10$$

3c If
$$\epsilon(\mathbf{X} + \triangle \mathbf{X}) > \epsilon(\mathbf{X})$$
, $\lambda \leftarrow 10\lambda$

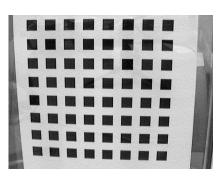
4. Go to step 2.

Example of distortion correction

Results of Zhang:



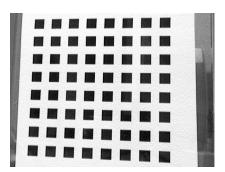
Snapshot 1



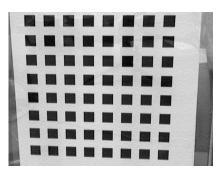
Snapshot 2

Example of distortion correction

Results of Zhang:



Corrected image 1



Corrected image 2

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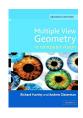
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Conclusion

- ▶ Camera matrix K (3 × 3) depends only on internal parameters of the camera.
- Projection matrix $P(3 \times 4)$ depends on K and position/orientation.
- Homogeneous coordinates are convenient as they linearize the equations.
- ► A homography between two images arises when the observed scene is flat or the principal point is fixed.
- 4 or more correspondences are enough to estimate a homography (in general)

References





Hartley & Zisserman (2004)

- Chapter 2: Projective Geometry and Transformations of 2D
- Chapter 4: Estimation–2D Projective Transformations
- Chapter 6: Camera Models
- ► Chapter 7: Computation of the Camera Matrix P

Semple & Kneebone (1962)

- Chapter IV: Projective Geometry of Two Dimensions
- Appendix: Two Basic Algebraic Theorems

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Practical session: panorama construction

Objective: the user clicks 4 or more corresponding points in left and right images. After a right button click, the program computes the homography and shows the resulting panorama in a new window.

- Install Imagine++
 (http://imagine.enpc.fr/~monasse/Imagine++/) on your
 machine.
- Let the user click the matching points.
- ▶ Build the linear system to solve Ax = b and find x.
- Compute the bounding box of the panorama.
- ➤ Stitch the images: on overlapping area, take the average of colors at corresponding pixels in both images.

Useful Imagine++ types/functions: Matrix, Vector, Image, anyGetMouse, linSolve