INTRODUCTION TO MANIFOLDS

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Abstract. We discuss manifolds embedded in $\mathbb{R}^n$ and the implicit function theorem.

Contents

1. Differentiable curves

Intuitively, a differentiable curves is a curve with the property that the tangent line at each of its points can be defined. To motivate our definition, let us look at some examples.

Example 1.1. (a) Consider the function $\gamma : (0, \pi) \to \mathbb{R}^2$ defined by

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}. \tag{1}$$

As t varies in the interval $(0, \pi)$, the point $\gamma(t)$ traces out the curve

$$C = [\gamma] \equiv \{\gamma(t) : t \in (0, \pi)\} \subset \mathbb{R}^2, \tag{2}$$

which is a semicircle (without its endpoints). We call γ a *parameterization* of C. If $\gamma(t)$ represents the coordinates of a particle in the plane at time t, then the "instantaneous velocity vector" of the particle at the time moment t is given by

$$\gamma'(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}. \tag{3}$$

Obviously, $\gamma'(t) \neq 0$ for all $t \in (0, \pi)$, and γ is smooth (i.e., infinitely often differentiable). The direction of the velocity vector $\gamma'(t)$ defines the direction of the line tangent to C at the point $p = \gamma(t)$.

(b) Under the substitution $t = s^2$, we obtain a different parameterization of C, given by

$$\eta(s) \equiv \gamma(s^2) = \begin{pmatrix} \cos s^2 \\ \sin s^2 \end{pmatrix}. \tag{4}$$

Note that the parameter s must take values in $(0, \sqrt{\pi})$.

(c) Now consider the function $\xi : [0, \pi] \to \mathbb{R}^2$ defined by

$$\xi(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}. \tag{5}$$

The only difference between the curve $\bar{C} = [\xi]$ and the curve $C = [\gamma]$ from (a) is that $\bar{C}$ contains its endpoints, while C does not. In order to make sense of the velocity vector of

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ξ at the endpoints of the interval $[0, \pi]$, we may think of ξ as the *restriction* of another function $\tilde{\xi} : (-\varepsilon, \pi + \varepsilon) \to \mathbb{R}^2$ to the interval $[0, \pi]$, where $\varepsilon > 0$ is a small number, and

$$\tilde{\xi}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \qquad t \in (-\varepsilon, \pi + \varepsilon). \tag{6}$$

We call $\tilde{\xi}$ an *extension* of ξ. With such an extension at hand, the velocity vector of ξ at the endpoints of the interval $[0, \pi]$ can be defined as $\xi'(0) = \tilde{\xi}'(0)$ and $\xi'(\pi) = \tilde{\xi}'(\pi)$.

Exercise 1.1. Let $\delta : \mathbb{R} \to \mathbb{R}^2$ be given by

$$\delta(t) = \begin{pmatrix} (1 - \theta(t))t^3 \\ \theta(t)t^3 \end{pmatrix}, \qquad \text{where} \quad \theta(t) = \begin{cases} 1 & \text{for } t > 0, \\ 0 & \text{for } t \leq 0. \end{cases} \tag{7}$$

Show that δ is continuously differentiable in $\mathbb{R}$. Sketch the curve defined by δ. Why there is a "corner" at the origin?

The preceding discussions motivate us to state the following points.

- A curve is a set that admits a parameterization γ.
- In order to have a tangent line at every point of the curve, we require that γ is differentiable and $\gamma' \neq 0$ everywhere.
- By introducing an extension if necessary, we can always assume that γ is defined on some open interval (a, b).

In addition, we require that the tangent lines vary continuously as we traverse along the curve, i.e., we want velocity vector $\gamma'(t)$ to depend continuously on t. This gets rid of the pathological curves such as the graph of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

Definition 1.2. A set $L \subset \mathbb{R}^n$ is called an *open curve* if there exists a continuously differentiable function $\gamma : (a, b) \to \mathbb{R}^n$ with $-\infty \leq a < b \leq \infty$, such that $L = \{\gamma(t) : t \in (a, b)\}$ and $\gamma'(t) \neq 0$ for all $t \in (a, b)$. In this setting, γ is called a *parameterization* of L.

Remark 1.3. Strictly speaking, the preceding definition is that of *differentiable* open curves. However, all curves in these notes will be assumed to be differentiable, and we will simply omit the adjective "differentiable."

Example 1.4. (a) Consider the function $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t) = \begin{pmatrix} t^2 - 1 \\ t(t^2 - 1) \end{pmatrix}. \tag{8}$$

We have

$$\gamma'(t) = \begin{pmatrix} 2t \\ 3t^2 - 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{for} \quad t \in \mathbb{R}, \tag{9}$$

and thus γ defines an open curve in $\mathbb{R}^2$. However, we have $\gamma(-1) = \gamma(1)$, indicating that the curve intersects with itself. At the self-intersection point, we have two possible tangent directions $\gamma'(1) = (2, 2)$ and $\gamma'(-1) = (-2, 2)$. This is not a particularly serious problem, but it is useful to introduce a concept that rules out self-intersecting curves. An idea would be to require injectivity of the parameterization, that is, to require that $\gamma(s) = \gamma(t)$ implies $s = t$.

(b) Consider the unit circle. We may try to parameterize it by

$$\xi(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \qquad t \in (-\varepsilon, 2\pi), \tag{10}$$

but this is not injective, as $\xi(t) = \xi(t + 2\pi)$ for $t \in (-\varepsilon, 0)$. This cannot be avoided if we want a parameterization with an open interval as its domain. A way out would be to "cover" the circle by using multiple parameterizations, meaning that we consider the circle as multiple arcs glued together "nicely."

Definition 1.5. A set $L \subset \mathbb{R}^n$ is called an *(embedded) curve* if for each $p \in L$, there exists $\delta > 0$ such that $L \cap Q_\delta(p)$ is an open curve admitting an injective parameterization. Recall that a function γ is called injective if $\gamma(s) = \gamma(t)$ implies $s = t$.

Example 1.6. Let us show that the unit circle $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a curve in the sense of the preceding definition. Pick an arbitrary $p = (x_*, y_*) \in C$. We consider a few cases. First, assume $y_* > 0$. In this case, we choose $\delta > 0$ so small that $Q_\delta(p) \subset \{(x, y) : -1 < x < 1,\, y > 0\}$, and use the parameterization $\gamma : (x_* - \delta, x_* + \delta) \to \mathbb{R}^2$ defined by $\gamma(t) = (t, \sqrt{1 - t^2})$. We can check that this is an injective parameterization of $Q_\delta(p) \cap C$. The second case, where we assume $y_* < 0$, can be treated similarly, by using the parameterization $\gamma(t) = (t, -\sqrt{1 - t^2})$. The remaining case is $y = 0$, which can be separated into two subcases: $x_* = 1$ and $x_* = -1$. For $x_* = 1$, we use $\gamma(t) = (\sqrt{1 - t^2}, t)$ for $t \in (-1, 1)$, which parameterizes $Q_1(p) \cap C$ injectively. Similarly, for $x_* = -1$, we can use the paremeterization $\gamma(t) = (-\sqrt{1 - t^2}, t)$.

Definition 1.7. Given a parameterization $\gamma : (a, b) \to \mathbb{R}^n$ of a curve, the *velocity vector* of γ at the point $p = \gamma(t)$ is $\gamma'(t) \in \mathbb{R}^n$.

Remark 1.8. Let $\gamma : (a, b) \to \mathbb{R}^n$ be a parameterization of a curve L, and let $\bar{\gamma}(s) = \gamma(\phi(s))$ be another parameterization of L, where $\phi : (\bar{a}, \bar{b}) \to (a, b)$ is continuously differentiable. One can think of ϕ as a *reparameterization* or a *coordinate change* on the curve. Under this reparameterization, the velocity vector at $p = \bar{\gamma}(s)$ transforms as

$$\bar{\gamma}'(s) = \gamma'(\phi(s))\phi'(s). \tag{11}$$

Since $\phi'(s) \in \mathbb{R}$, we see that even though the velocity vector may change during reparameterization, its *direction* stays the same. This direction defines the *tangent line* of L at p, which is an intrinsic property of the curve L independent of parameterization.

2. Manifolds

In this section, we will generalize the concept of curves to higher dimensional objects called manifolds. Curves, surfaces, and hypersurfaces will be special cases of manifolds. We will only be concerned with smooth (or differentiable) manifolds, but in practice, non-smooth objects such as the surface of a cube do not cause much trouble because they can be treated as consisting of a number of smooth pieces.

Example 2.1. The defining characteristic of a curve is that near any of its points, it can be parameterized "nicely" by a single parameter. Intuitively, to parameterize a surface, we need to use two parameters. Let $\Omega = (-1, 1)^2 \subset \mathbb{R}^2$, and let $\Psi : \Omega \to \mathbb{R}^3$ be continuously differentiable in Ω. We imagine that the set $S = \{\Psi(x) : x \in \Omega\}$ is a piece of a surface in $\mathbb{R}^3$, so that Ψ is its parameterization. Consider the 2-dimensional curve $\gamma_\alpha(t) = \alpha t$, $t \in (-1, 1)$, where $\alpha \in \mathbb{R}^2$ is a fixed vector. Under the parameterization Ψ, this curve becomes the 3-dimensional curve $\eta_\alpha(t) = \Psi(\alpha t)$, which is contained in the surface S. The velocity vector of η_α at $\eta_\alpha(0) \in S$ is

$$\eta'_\alpha(0) = D\Psi(0)\gamma'_\alpha(0) = D\Psi(0)\alpha = \frac{\partial \Psi}{\partial x_1}(0)\alpha_1 + \frac{\partial \Psi}{\partial x_2}(0)\alpha_2, \tag{12}$$

where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$. If S is a smooth surface, then we expect that the velocity vectors $\eta'_\alpha(0)$ with different $\alpha \in \mathbb{R}^2$ are *not* all aligned to each other. In light of the preceding formula, this means that the columns of $D\Psi(0)$ are expected to be *linearly independent*.

The linear independence condition discussed in the preceding example will appear in the definition of manifolds later. Before that, we need to introduce the concept of open sets.

Definition 2.2. A set $\Omega \subset \mathbb{R}^n$ is called *open* if for any $p \in \Omega$, there is $\delta > 0$ such that $Q_\delta(p) \subset \Omega$.

Example 2.3. (a) The square $\Omega = (0,1)^2$ is open, because given any $(x,y) \in \Omega$, we have $(x-\delta, x+\delta) \times (y-\delta, y+\delta) \subset \Omega$ for $\delta = \min\{x, 1-x, y, 1-y\}$.
(b) $\Omega = [0,1)^2$ is *not* open, because taking $p = (0,0) \in \Omega$, there is no $\delta > 0$ with $Q_\delta(p) \subset \Omega$.
(c) The disk $\Omega = \{(x,y) : x^2 + y^2 < 1\}$ is open, because given any $(x,y) \in \Omega$, we have $(x-\delta, x+\delta) \times (y-\delta, y+\delta) \subset \Omega$ for $\delta = \sqrt{1-x^2-y^2}/\sqrt{2}$.

Definition 2.4. A set $M \subset \mathbb{R}^N$ is called an *n-dimensional manifold* (embedded in $\mathbb{R}^N$) if for each $p \in M$, there exist open sets $U \subset \mathbb{R}^N$, $\Omega \subset \mathbb{R}^n$, and a map $\Psi : \Omega \to \mathbb{R}^N$ such that

(i) $U \cap M = \Psi(\Omega)$ and $p \in U \cap M$.
(ii) Ψ is injective, and continuously differentiable.
(iii) For each $x \in \Omega$, the columns of $D\Psi(x)$ are linearly independent.

In this setting, Ψ is called a *local parameterization*, and the triple $(\Psi, \Omega, U \cap M)$ is called a *coordinate chart*. Since Ψ is injective, the inverse $\Psi^{-1} : U \cap M \to \Omega$ exists, and it is called a *local coordinate system* on M.

Remark 2.5. A manifold of dimension 1 is a curve, and a 2-dimensional manifold is called a *surface*. If $n = N - 1$, the manifold is called a *hypersurface in* $\mathbb{R}^N$.

Example 2.6. Let $M \subset \mathbb{R}^n$ be an open set. Given any $p \in M$, we use the coordinate chart $\Omega = U = M$, and $\Psi(x) = x$ for $x \in \Omega$. This makes M an n-dimensional manifold.

Example 2.7. Let us show that the 2-sphere $S^2 = \{y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ is a manifold (i.e., a surface). It will be done similarly to Example 1.6. Pick an arbitrary point $y^* \in S^2$. We will consider 6 different cases, corresponding to 6 coordinate charts covering S^2. The first case is $y_3^* > 0$. In this case, we set $U = \{y \in \mathbb{R}^3 : y_3 > 0\}$, $\Omega = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$, and $\Psi(x) = (x_1, x_2, \sqrt{1 - x_1^2 - x_2^2})$. It is easy to see that $U \cap S^2 = \Psi(\Omega)$, and Ψ is injective. Moreover, we have

$$D\Psi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x_1/y_3 & -x_2/y_3 \end{pmatrix}, \tag{13}$$

where $y_3 = \sqrt{1 - x_1^2 - x_2^2}$, which shows that Ψ is continuously differentiable in Ω, and that the columns of $\Psi(x)$ are linearly independent for each $x \in \Omega$. The remaining 5 cases are (ii) $y_3^* < 0$, (iii) $y_3^* = 0$ and $y_2^* > 0$, (iv) $y_3^* = 0$ and $y_2^* < 0$, (v) $y_3^* = y_2^* = 0$ and $y_1^* > 0$, and finally, (vi) $y_3^* = y_2^* = 0$ and $y_1^* < 0$. All these cases can be handled similarly to the first case, with each case corresponding to the positive or the negative half of a coordinate axis, and its associated hemisphere.

Definition 2.8. Given a manifold $M \subset \mathbb{R}^N$ and its point $p \in M$, the *tangent space* of M at p is defined as

$$T_pM = \{\gamma'(0) : \gamma \in \mathscr{C}^1((-\varepsilon, \varepsilon), M) \text{ for some } \varepsilon > 0,\ \gamma(0) = p\}, \tag{14}$$

where $\gamma \in \mathscr{C}^1((-\varepsilon, \varepsilon), M)$ means that $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^N$ is continuously differentiable in $(-\varepsilon, \varepsilon)$ and $\gamma(t) \in M$ for all $t \in (-\varepsilon, \varepsilon)$.

Example 2.9. Let us identify the tangent space T_pS^2, for $p = (x, y, z)$, $z > 0$. Consider an arbitrary function $\gamma \in \mathscr{C}^1((-\varepsilon, \varepsilon), S^2)$, with $\gamma(0) = p$. Taking the derivative of the relation $\gamma_1(t)^2 + \gamma_2(t)^2 + \gamma_3(t)^2 = 1$ with respect to t, we get

$$\gamma_1(t)\gamma_1'(t) + \gamma_2(t)\gamma_2'(t) + \gamma_3(t)\gamma_3'(t) = 0, \tag{15}$$

and therefore

$$x\gamma_1'(0) + y\gamma_2'(0) + z\gamma_3'(0) = 0. \tag{16}$$

This shows that $T_pS^2 \subset X$, where $X = \{V \in \mathbb{R}^3 : V^Tp = 0\}$. Geometrically, X is the space perpendicular to the vector p. Now let $V = (a, b, c)$ be an arbitrary element of X, meaning that $ax + by + cz = 0$, and let

$$\gamma(t) = \begin{pmatrix} x + at \\ y + bt \\ \sqrt{1 - (x + at)^2 - (y + bt)^2} \end{pmatrix}. \tag{17}$$

By construction, we have $\gamma(0) = p$. We also have

$$\frac{\mathrm{d}\sqrt{1 - (x + at)^2 - (y + bt)^2}}{\mathrm{d}t}\bigg|_{t=0} = \frac{-a(x + at) - b(y + bt)}{\sqrt{1 - (x + at)^2 - (y + bt)^2}}\bigg|_{t=0} = \frac{-ax - by}{\sqrt{1 - x^2 - y^2}}, \tag{18}$$

and hence

$$\gamma'(0) = \begin{pmatrix} a \\ b \\ -(ax + by)/z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = V, \tag{19}$$

where we have used the fact that $ax + by + cz = 0$. This implies $V \in T_pS^2$, and as V is an arbitrary element of X, we have $X \subset T_pS^2$. Since we already established $T_pS^2 \subset X$, we conclude that $T_pS^2 = X \equiv \{V \in \mathbb{R}^3 : V^Tp = 0\}$.

Remark 2.10 (Coordinate transformation in $\mathbb{R}^n$)**.** If Ω is an open set of $\mathbb{R}^n$, then $T_x\Omega = \mathbb{R}^n$ for each $x \in \Omega$, since for any $V \in \mathbb{R}^n$, we can take $\gamma(t) = x + Vt$ for $t \in (-\varepsilon, \varepsilon)$, with $\varepsilon > 0$ small enough, and we will have $\gamma'(0) = V$. This situation may seem trivial, but we can appreciate the "manifold aspect" of Ω by introducing a "curvilinear" coordinate system in Ω. Let $U \subset \mathbb{R}^n$ be an open set, and let $\Psi : U \to \Omega$ be a continuously differentiable and invertible map, whose inverse $\Psi^{-1} : \Omega \to U$ is also continuously differentiable. By default, the points in Ω will be denoted by $x = (x_1, \ldots, x_n)$, and the points in U will be denoted by $y = (y_1, \ldots, y_n)$. We can and should think of y as a new coordinate system in Ω, with $y = \Psi^{-1}(x)$ being the y-coordinates of the point $x \in \Omega$. It will sometimes be convenient to write $y = y(x)$ and $x = x(y)$ instead of $y = \Psi^{-1}(x)$ and $x = \Psi(y)$, respectively. Thus a curve $y = y(t)$ in U corresponds to the curve $x = x(y(t))$ in Ω, and

$$x'(t) = D\Psi(y(t))y'(t), \tag{20}$$

which tells us how the components of a vector should transform under change of coordinates:

$$\alpha = D\Psi(y)\beta, \qquad \text{i.e.,} \qquad \alpha_i = \sum_{k=1}^{n} \frac{\partial x_i}{\partial y_k}(y)\beta_k, \quad (i = 1, \ldots, n), \tag{21}$$

where $y \in U$, $x = x(y)$, $\alpha \in T_x\Omega$, and $\beta \in T_yU$. In fact we see that the columns of $D\Psi(y)$ plays the role of a *new basis* in $T_x\Omega$, and $\beta_1, \ldots, \beta_n$ are the coordinates of α with respect to this basis. On the other hand, from the point of view of the domain U, we would have

$$\beta = D\Psi^{-1}(x)\alpha, \qquad \text{i.e.,} \qquad \beta_k = \sum_{i=1}^{n} \frac{\partial y_k}{\partial x_i}(y)\alpha_i, \quad (k = 1, \ldots, n), \tag{22}$$

which means that $\alpha_1 \ldots, \alpha_n$ are the coordinates of $\beta \in T_yU$ when the columns of $D\Psi^{-1}(x)$ are used as a basis of T_yU.

Example 2.11 (Polar coordinates)**.** Consider the map $\Psi : U \to \Omega$, defined by

$$\Psi(r, \phi) = \begin{pmatrix} x(r, \phi) \\ y(r, \phi) \end{pmatrix} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}, \tag{23}$$

where $U = (0, \infty) \times (-\pi, \pi)$ and $\Omega = \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\}$. If $\Psi(r, \phi) = (x, y) \in \Omega$, then $x^2 + y^2 = r^2$, or $r = \sqrt{x^2 + y^2} > 0$. This yields $\cos\phi = \frac{x}{r} \in [-1, 1]$, and hence with the function $\arccos t \in [0, \pi]$, we have $\arccos \frac{x}{r} = \phi$ or $\arccos \frac{x}{r} = -\phi$, depending on the sign of ϕ. In other words, knowing x and r determines ϕ up to a sign. The sign of ϕ can be determined with the help of the conditions $\sin\phi = \frac{y}{r}$ and $-\pi < \phi < \pi$, because these imply that the sign of ϕ is the same as the sign of y. To conclude, $(x, y) \in \Omega$ determines $(r, \phi) \in U$ uniquely, i.e., the map Ψ is invertible. We can compute

$$D\Psi(r, \phi) = \begin{pmatrix} \partial_r x & \partial_\phi x \\ \partial_r y & \partial_\phi y \end{pmatrix} = \begin{pmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{pmatrix}, \tag{24}$$

which is nonsingular everywhere in U. Then by the inverse function theorem, $\Psi^{-1} : \Omega \to U$ is continuously differentiable in Ω, with

$$D\Psi^{-1}(x, y) = \begin{pmatrix} \partial_x r & \partial_y r \\ \partial_x \phi & \partial_y \phi \end{pmatrix} = \frac{1}{r} \begin{pmatrix} r\cos\phi & r\sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \tag{25}$$

where $r = r(x, y)$ and $\phi = \phi(x, y)$ are now understood to be the components of Ψ^{-1}. Now, in view of the preceding remark, any vector $\alpha \in T_p\Omega \equiv \mathbb{R}^2$ with $p \in \Omega$ can be written as

$$\alpha = D\Psi(q)\beta, \tag{26}$$

with $\beta \in T_qU \equiv \mathbb{R}^2$, where $q = \Psi^{-1}(p)$. Conversely, any $\beta \in T_qU \equiv \mathbb{R}^2$ can be written as

$$\beta = D\Psi^{-1}(p)\alpha, \tag{27}$$

for some $\alpha \in T_p\Omega \equiv \mathbb{R}^2$. If we give names to the components of α and β according to $\alpha = (\alpha_x, \alpha_y)$ and $\beta = (\beta_r, \beta_\phi)$, then we have

$$\alpha = \beta_r \hat{e}_r + \beta_\phi \hat{e}_\phi, \qquad \beta = \alpha_x \hat{e}_x + \alpha_y \hat{e}_y, \tag{28}$$

where $\hat{e}_r$ and $\hat{e}_\phi$ are the columns of $D\Psi(q)$, and $\hat{e}_x$ and $\hat{e}_y$ are the columns of $D\Psi^{-1}(p)$. For example, the vector $\beta = (1, 0)$ in the (r, ϕ)-coordinate system becomes $\alpha = \hat{e}_r$ in the (x, y)-coordinate system, and the vector $\beta = (0, 1)$ in the (r, ϕ)-coordinate system becomes $\alpha = \hat{e}_\phi$ in the (x, y)-coordinate system. Similarly, the vector $\alpha = (1, 0)$ in the (x, y)-coordinate system becomes $\beta = \hat{e}_x$ in the (r, ϕ)-coordinate system, and the vector $\alpha = (0, 1)$ in the (x, y)-coordinate system becomes $\beta = \hat{e}_y$ in the (r, ϕ)-coordinate system.

3. The implicit function theorem

In this section, we want to investigate if the equation $g(x, y) = 0$ can be solved as $y = y(x)$. The results will be applied in the next section to derive a convenient criterion to recognize if a set of the form $\{z : \phi(z) = 0\}$ is a manifold. Our approach will be based on differentiability, meaning that we fix some point (x_*, y_*), and approximate g as

$$g(x, y) \approx g(x_*, y_*) + \partial_x g(x_*, y_*)(x - x_*) + \partial_y g(x_*, y_*)(y - y_*), \tag{29}$$

for $y \approx y_*$ and $x \approx x_*$. If $\partial_y g(x_*, y_*) \neq 0$, this approximate equation can be solved for y:

$$y - y_* \approx \frac{g(x, y) - g(x_*, y_*) - \partial_x g(x_*, y_*)(x - x_*)}{\partial_y g(x_*, y_*)}. \tag{30}$$

Solving this equation for $g(x, y) \neq g(x_*, y_*)$ would not yield a good approximation, because then $x \approx x_*$ would *not* imply $y \approx y_*$. Thus we put $g(x, y) = g(x_*, y_*) = 0$, and get

$$y - y_* \approx -\frac{\partial_x g(x_*, y_*)}{\partial_y g(x_*, y_*)}(x - x_*). \tag{31}$$

This suggests that the conditions $g(x_*, y_*) = 0$ and $\partial_y g(x_*, y_*) \neq 0$ might be sufficient to solve $g(x, y) = 0$ for a function $y = y(x)$, at least when x is in a small interval containing x_*. In the following remark, we will justify this expectation in full detail.

Remark 3.1. Let $Q_a = (-a, a)^2 \subset \mathbb{R}^2$ be an open square, with $a > 0$, and let $g : Q_a \to \mathbb{R}$ be a continuously differentiable function, satisfying $g(0, 0) = 0$ and $\partial_y g(0, 0) \neq 0$. We want to find a function $y = h(x)$, defined for $x \in (-\delta, \delta)$ with some $\delta > 0$, such that $g(x, h(x)) = 0$ for all $x \in (-\delta, \delta)$. Note that the point (x_*, y_*) from the previous discussion is now the origin. This is no loss of generality, since we may think of $g(x, y)$ as $\tilde{g}(x_* + x, y_* + y)$ for some function $\tilde{g}$. To proceed further, we introduce the auxiliary map $f : Q_a \to \mathbb{R}^2$, given by $f(x, y) = (x, g(x, y))$ for $(x, y) \in Q_a$. The motivation for considering such a map is that if we can solve $f(x, y) = (\alpha, 0)$ for (x, y) depending on α, then we would have $x = \alpha$ and $g(\alpha, y(\alpha)) = 0$. In order to invert f near the origin, we shall invoke the inverse function theorem. The Jacobian of f is

$$J(x, y) = \begin{pmatrix} 1 & 0 \\ \partial_x g(x, y) & \partial_y g(x, y) \end{pmatrix}, \tag{32}$$

and since g is continuously differentiable, J is continuous in Q_a, and hence we conclude that f is continuously differentiable in Q_a with $Df = J$. At the origin, Df is invertible, and

$$(Df)^{-1} = \begin{pmatrix} 1 & 0 \\ -\partial_x g/\partial_y g & 1/\partial_y g \end{pmatrix}, \tag{33}$$

where all functions are evaluated at the origin $0 \in \mathbb{R}^2$. Now the inverse function theorem guarantees that there exist of $r > 0$ and $f^{-1} : f(Q_r) \to \mathbb{R}^2$, satisfying $f^{-1}(f(x, y)) = (x, y)$ for all $(x, y) \in Q_r$. Note that $f^{-1}(0, 0) = (0, 0)$. Moreover, $Df(x, y)$ is nonsingular for each $(x, y) \in Q_r$, and f^{-1} is continuously differentiable with $Df^{-1} \circ f = (Df)^{-1}$ in Q_r. If we let $f^{-1}(\alpha, \beta) = (x(\alpha, \beta), y(\alpha, \beta))$, then from $f(f^{-1}(\alpha, \beta)) = (\alpha, \beta)$, we infer that $x(\alpha, \beta) = \alpha$ and $g(\alpha, y(\alpha, \beta)) = \beta$ for $(\alpha, \beta) \in f(Q_r)$. In addition to what we have already mentioned, the inverse function theorem tells us that there is $\delta > 0$ such that $Q_\delta \in f(Q_r)$, implying that we have $g(\alpha, y(\alpha, \beta)) = \beta$ for all $(\alpha, \beta) \in Q_\delta$. In particular, setting $h(\alpha) = y(\alpha, 0)$, we get $g(\alpha, h(\alpha)) = 0$ for all $\alpha \in (-\delta, \delta)$. From $f^{-1}(0, 0) = (0, 0)$, we get $h(0) = 0$.

The function h we found in the preceding paragraph in fact solves our problem, but our assumptions are strong enough to yield additional results. As a component of f^{-1}, the function $y = y(\alpha, \beta)$ is *continuously differentiable* in Q_δ, and we have

$$Df^{-1} = \begin{pmatrix} 1 & 0 \\ \partial_\alpha y & \partial_\beta y \end{pmatrix}. \tag{34}$$

Comparing this with (33), we get $\partial_\alpha y \circ f = -\partial_x g/\partial_y g$ and $\partial_\beta y \circ f = 1/\partial_y g$. In particular, taking into account that $h'(\alpha) = \partial_\alpha y(\alpha, 0)$, we conclude that

$$h'(x) = -\frac{\partial_x g(x, h(x))}{\partial_y g(x, h(x))}, \qquad \text{for} \quad x \in (-\delta, \delta). \tag{35}$$

Before closing this remark, we make one crucial observation. Fix $x \in (-\delta, \delta)$, and consider $I = \{(x, y) : y \in (-r, r)\}$. The map f sends I to $f(I) = \{(x, g(x, y)) : y \in (-r, r)\} \subset f(Q_r)$. Since f is invertible in Q_r, the only point $(x, y) \in I$ with $g(x, y) = 0$ is $(x, h(x))$. In other words, apart from the curve $\{(x, h(x)) : x \in (-\delta, \delta)\}$, there are *no other* points (x, y) exist in the rectangle $(-\delta, \delta) \times (-r, r)$ satisfying $g(x, y) = 0$.

The preceding remark is the *implicit function theorem* in two dimensions.

Example 3.2. (a) Let us apply the implicit function theorem to the equation $x^2 + y^2 = 1$. Thus we set $g(x, y) = x^2 + y^2 - 1$, and compute $\partial_y g(x, y) = 2y$. This means that as long as (x_*, y_*) satisfies $g(x_*, y_*) = 0$ and $y_* \neq 0$, we can apply the result at the point (x_*, y_*), and infer the existence of $\delta > 0$ and $h : (x_* - \delta, x_* + \delta) \to \mathbb{R}$ such that $g(x, h(x)) = 0$ for all $x \in (x_* - \delta, x_* + \delta)$. We can also compute the derivative of h as

$$h'(x) = -\frac{\partial_x g(x,y)}{\partial_y g(x,y)} = -\frac{2x}{2y} = -\frac{x}{h(x)}, \qquad \text{for} \quad x \in (x_* - \delta, x_* + \delta). \tag{36}$$

The intuitive reason why the case $y_* = 0$ must be excluded is the fact that then the derivative $h'(x_*)$ would have to become infinity.

(b) Let $g(x,y) = y^3 - x$, and let us try to solve $g(x,y) = 0$ for $y = y(x)$ near $(x,y) = (0,0)$. We have $g(0,0) = 0$, but $\partial_y g(0,0) = (3y^2)|_{y=0} = 0$. Therefore the implicit function theorem *cannot* be applied, even though we can explicitly solve the equation as $y(x) = \sqrt[3]{x}$. This has of course to do with the fact that $\sqrt[3]{x}$ is *not* differentiable at $x = 0$.

(c) Let $g(x,y) = x^2 - y^2$, and let us try to solve $g(x,y) = 0$ for $y = y(x)$ near $(x,y) = (0,0)$. We have $g(0,0) = 0$, but $\partial_y g(0,0) = (-2y)|_{y=0} = 0$, and hence the implicit function theorem *cannot* be applied. A close inspection reveals that the solution of $g(x,y) = 0$ is $y = \pm x$, which *cannot* be written as a function $y = y(x)$ near $(x,y) = (0,0)$.

Let $\Omega \subset \mathbb{R}^n$ and $\Sigma \subset \mathbb{R}^m$ be open sets. Then their *product* $\Omega \times \Sigma \subset \mathbb{R}^{n+m}$ is given by

$$\Omega \times \Sigma = \{(x,y) : x \in \Omega,\, y \in \Sigma\}, \tag{37}$$

where $(x,y) = (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{n+m}$. Let $g : \Omega \times \Sigma \to \mathbb{R}^m$ be a differentiable function. The value of g at $(x,y) \in \Omega \times \Sigma$ is denoted by $g(x,y) \in \mathbb{R}^m$. For any fixed $x \in \Omega$, the correspondence $y \mapsto g(x,y)$ is a function of $y \in \Sigma$, and its derivative will be denoted by $D_y g$ Similarly, we can introduce $D_x g$. In the following, sometimes it will be convenient to specify the dimension of a cube in the notation, as in $Q^n_r(a) = (a - r, a + r)^n \subset \mathbb{R}^n$.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ and $\Sigma \subset \mathbb{R}^m$ be open sets, and let $g : \Omega \times \Sigma \to \mathbb{R}^m$ be continuously differentiable. Suppose that $(a,b) \in \Omega \times \Sigma$ satisfies $g(a,b) = 0$, and that $D_y g(a,b)$ is nonsingular. Then there exist $\delta > 0$ and $h : Q^n_\delta(a) \to \mathbb{R}^m$ with $h(a) = b$, such that $g(x,h(x)) = 0$ for all $x \in Q^n_\delta(a)$. Moreover, h is continuously differentiable in $Q^n_\delta(a)$, with*

$$Dh(x) = -(D_y g(x,h(x)))^{-1} D_x g(x,h(x)), \qquad x \in Q^n_\delta(a), \tag{38}$$

and we have $\{(x,h(x)) : x \in Q^n_\delta(a)\} = \{(x,y) \in Q^n_\delta(a) \times Q^m_r(b) : g(x,y) = 0\}$ for some $r > 0$.

Proof. Let $f : \Omega \times \Sigma \to \mathbb{R}^{n+m}$ be defined by $f(x,y) = (x, g(x,y))$. This function is continuously differentiable, and

$$Df(x,y) = \begin{pmatrix} I & 0 \\ D_x g & D_y g \end{pmatrix}, \tag{39}$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix. Since $\det Df(a,b) = \det D_y g(a,b) \neq 0$, the matrix $Df(a,b)$ is invertible. Consequently, the inverse function theorem guarantees that there exist of $r > 0$ and $f^{-1} : f(Q_r) \to \mathbb{R}^{n+m}$, satisfying $f^{-1}(f(x,y)) = (x,y)$ for all $(x,y) \in Q_r$, where $Q_r = Q^{n+m}_r(a,b)$. Note that $f^{-1}(a,0) = (a,b)$. Moreover, $Df(x,y)$ is nonsingular for each $(x,y) \in Q_r$, and f^{-1} is continuously differentiable with $Df^{-1} \circ f = (Df)^{-1}$ in Q_r. If we let $f^{-1}(\alpha,\beta) = (x(\alpha,\beta), y(\alpha,\beta))$, then from $f(f^{-1}(\alpha,\beta)) = (\alpha,\beta)$, we infer that $x(\alpha,\beta) = \alpha$ and $g(\alpha, y(\alpha,\beta)) = \beta$ for $(\alpha,\beta) \in f(Q_r)$. In addition to what we have already mentioned, the inverse function theorem tells us that there is $\delta > 0$ such that $Q^{n+m}_\delta(a,0) \in f(Q_r)$, implying that we have $g(\alpha, y(\alpha,\beta)) = \beta$ for all $(\alpha,\beta) \in Q^{n+m}_\delta(a,0)$. In particular, setting $h(\alpha) = y(\alpha,0)$, we get $g(\alpha,h(\alpha)) = 0$ for all $\alpha \in Q^n_\delta(a)$. From $f^{-1}(a,0) = (a,b)$, we get $h(a) = b$. Furthermore, since $(\alpha,h(\alpha)) = f^{-1}(\alpha,0)$ for all $\alpha \in Q^n_\delta(a)$, we infer that the points $(x,y) \in Q^n_\delta(a) \times Q^m_r(b)$ with $g(x,y) = 0$ are only of the form $(x,h(x))$. In other words, we have $\{(x,h(x)) : x \in Q^n_\delta(a)\} = \{(x,y) \in Q^n_\delta(a) \times Q^m_r(b) : g(x,y) = 0\}$.

As a collection of components of f^{-1}, the function $y = y(\alpha, \beta)$ is continuously differentiable in $Q_\delta^{n+m}(a, 0)$, and we have

$$Df^{-1} = \begin{pmatrix} I & 0 \\ D_\alpha y & D_\beta y \end{pmatrix}. \tag{40}$$

Comparing this with

$$(Df)^{-1} = \begin{pmatrix} I & 0 \\ -(D_y g)^{-1} D_x g & (D_y g)^{-1} \end{pmatrix}, \tag{41}$$

we infer

$$D_\alpha y \circ f = -(D_y g)^{-1} D_x g, \qquad D_\beta y \circ f = (D_y g)^{-1}. \tag{42}$$

In particular, taking into account that $Dh(\alpha) = D_\alpha y(\alpha, 0)$, we conclude that

$$Dh(x) = -(D_y g(x, h(x)))^{-1} D_x g(x, h(x)), \tag{43}$$

for all $x \in Q_\delta^n(a)$. □

Example 3.4. (a) Consider the equation

$$g(x, y, z) \equiv \sin(xy + z) + \log(yz^2) = 0. \tag{44}$$

The triple $p = (x, y, z) = (1, 1, -1)$ is a solution: $g(1, 1, -1) = 0$, and g is continuously differentiable in the open set $\{(x, y, z) : x \in \mathbb{R},\ y > 0,\ z < 0\}$. Can we express z as a function of x and y near p? This is exactly the kind of question that could be answered by the implicit function theorem. We have

$$\partial_z g(x, y, z) = \cos(xy + z) + \frac{2z}{yz^2} = \cos(xy + z) + \frac{2}{yz}, \tag{45}$$

and hence

$$\partial_z g(1, 1, -1) = \cos 0 - 2 = -1 \neq 0. \tag{46}$$

Thus there exist $\delta > 0$ and a continuously differentiable function $h : Q_\delta^2(1, 1) \to \mathbb{R}$ such that $g(x, y, h(x, y)) = 0$ for all $(x, y) \in Q_\delta^2(1, 1)$.

(b) Can we solve

$$\begin{aligned} xu^2 + yzv + x^2 z &= 3, \\ yv^5 + zu^2 - xv &= 1, \end{aligned} \tag{47}$$

for (u, v) near $(1, 1)$ as a function of (x, y, z) near $(1, 1, 1)$? We can formulate the problem as solving $g(\alpha, \beta) = 0$ for $\beta = \beta(\alpha)$, where $\alpha = (x, y, z)$, $\beta = (u, v)$, and

$$g(\alpha, \beta) = g(x, y, z, u, v) = \begin{pmatrix} xu^2 + yxv + x^2 z - 3 \\ yv^5 + 2zu - v^2 - 2 \end{pmatrix}. \tag{48}$$

Obviously, g is continuously differentiable in $\mathbb{R}^5$, and $g(1, 1, 1, 1, 1) = 0$. We can compute the relevant derivative as

$$D_\beta g(\alpha, \beta) = \begin{pmatrix} 2xu & yz \\ 2zu & 5yv^4 - x \end{pmatrix}. \tag{49}$$

so that the matrix

$$D_\beta g(1, 1, 1, 1, 1) = \begin{pmatrix} 2 & 1 \\ 2 & 4 \end{pmatrix}, \tag{50}$$

is invertible. Thus there exist $\delta > 0$ and $h : Q_\delta^3(1, 1, 1) \to \mathbb{R}^2$ continuously differentiable, such that $g(\alpha, h(\alpha)) = 0$ for all $\alpha \in Q_\delta^3(1, 1, 1)$.

4. The preimage theorem

With the implicit function theorem at hand, we are now ready to answer the question when the equation $\phi(x) = 0$ defines a manifold. We discuss the two dimensional case first, as it involves most of the main ideas. Thus let $g : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function, and let $L = \{(x, y) \in \mathbb{R}^2 : g(x, y) = 0\}$. We assume that $Dg(x, y) \neq 0$ for all $(x, y) \in L$, that is, at least one component of $Dg(x, y) \in \mathbb{R}^{1\times 2}$ is nonzero whenever (x, y) satisfies $g(x, y) = 0$. Under these assumptions, we want to show that L is a manifold. Recall from the definition that L would be a 1-dimensional manifold (or a curve) if for each $(\bar{x}, \bar{y}) \in L$, there exist an open set $U \subset \mathbb{R}^2$, an interval $I = (a, b)$, and a map $\Psi : I \to \mathbb{R}^2$ such that

(i) $U \cap L = \Psi(I)$ and $(\bar{x}, \bar{y}) \in U \cap L$.
(ii) Ψ is injective, and continuously differentiable.
(iii) For each $t \in I$, the derivative $D\Psi(t) \in \mathbb{R}^2$ has at least one nonzero component.

Now pick an arbitrary $(\bar{x}, \bar{y}) \in L$, and we shall build the triple (Ψ, I, U) satisfying the aforementioned conditions. We will consider the cases $\partial_y g(\bar{x}, \bar{y}) \neq 0$ and $\partial_y g(\bar{x}, \bar{y}) = 0$ separately.

Case 1. We assume that $\partial_y g(\bar{x}, \bar{y}) \neq 0$. Then the implicit function theorem guarantees that we can write y in terms of x at least when x is near $\bar{x}$. Namely, there exist $\delta > 0$ and a continuously differentiable function $h : I \to \mathbb{R}$ such that $g(x, h(x)) = 0$ for all $x \in I$, with $I = (\bar{x} - \delta, \bar{x} + \delta)$. Moreover, apart from the curve $\{(x, h(x)) : x \in I\}$, there are *no other* points (x, y) exist in the rectangle $U = I \times (\bar{y} - r, \bar{y} + r)$ satisfying $g(x, y) = 0$, where $r > 0$ is some constant. Therefore, with $\Psi : I \to \mathbb{R}^2$ defined by $\Psi(t) = (t, h(t))$, conditions (i) and (ii) are satisfied. Condition (iii) is also satisfied, since we have $\Psi'(t) = (1, h'(t)) \neq 0$.

Case 2. We assume that $\partial_y g(\bar{x}, \bar{y}) = 0$. In this case, we must have $\partial_x g(\bar{x}, \bar{y}) \neq 0$, because $Dg(\bar{x}, \bar{y}) \neq 0$, and hence the preceding arguments apply with the roles of x and y switched.

We have proved the following result.

Lemma 4.1 (Level curve theorem). *Let $A \subset \mathbb{R}^2$ be an open set, and let $g : A \to \mathbb{R}$ be a continuously differentiable function. Suppose that $Dg(x, y) \neq 0$ whenever $(x, y) \in A$ satisfies $g(x, y) = 0$. Then the set $L = \{(x, y) \in A : g(x, y) = 0\}$ is a differentiable curve.*

Example 4.2. Consider $g(x, y) = x^2 + y^2 - \rho$, where $\rho \in \mathbb{R}$ is some constant. This function is continuously differentiable in $\mathbb{R}^2$, with $Dg(x, y) = (2x, 2y) \in \mathbb{R}^{1\times 2}$. We see that $Dg(x, y) = 0$ if and only if $x = y = 0$. Now let $L = \{(x, y) : g(x, y) = 0\}$.

- If $\rho = 0$, then L is a single point $\{(0, 0)\}$, and $Dg = 0$ there. Hence the level curve theorem cannot be applied.
- If $\rho < 0$, then $L = \varnothing$. Since $(0, 0) \notin L$, we have $Dg(x, y) \neq 0$ for all $(x, y) \in L$. Thus the level curve theorem *can* be applied, to conclude that L is a curve. This example suggests that it is always a good idea to explicitly check if a manifold is nonempty, in order not to waste efforts working with an empty set.
- If $\rho > 0$, then L is nonempty, because, for example, we have $(0, \sqrt{\rho}) \in L$. Moreover, we have $(0, 0) \notin L$, implying that $Dg(x, y) \neq 0$ for all $(x, y) \in L$. This means that L is a differentiable curve.

Theorem 4.3 (Preimage theorem). *Let $A \subset \mathbb{R}^N$ be an open set, and let $\phi : A \to \mathbb{R}^k$ be a continuously differentiable function. Suppose that for each $x \in A$ satisfying $\phi(x) = 0$, there is a $k \times k$ submatrix of $D\phi(x)$ that is nonsingular. Then the set $M = \{x \in A : \phi(x) = 0\}$ is an $(N - k)$-dimensional manifold.*

Proof. In view of the definition of a manifold, what we need to do is to show that for each $y \in M$, there exist open sets $U \subset \mathbb{R}^N$, $\Omega \subset \mathbb{R}^{N-k}$, and a map $\Psi : \Omega \to \mathbb{R}^N$ such that

(i) $U \cap M = \Psi(\Omega)$ and $y \in U \cap M$.
(ii) Ψ is injective, and continuously differentiable.

(iii) For each $x \in \Omega$, the columns of $D\Psi(x)$ are linearly independent.

Thus we pick $y \in M$ arbitrary, and write $D\phi(y) = [b_1\ b_2\ \ldots\ b_N]$, where $b_i \in \mathbb{R}^k$ are the columns of $D\phi(y)$. Without loss of generality, we assume that the matrix $B = [b_{N-k+1}\ \ldots\ b_N]$ is invertible. Given $x \in \mathbb{R}^N$, let us introduce the notation $x' = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $x'' = (x_{n+1}, \ldots, x_N) \in \mathbb{R}^k$, where $n = N - k$. Now by the implicit function theorem, there exist $\delta > 0$ and a continuously differentiable function $h : Q^n_\delta(y') \to \mathbb{R}^k$ with $h(y') = y''$, such that $\phi(\alpha, h(\alpha)) = 0$ for all $\alpha \in Q^n_\delta(y')$. Moreover, there exists $r > 0$ such that

$$Q^n_\delta(y') \times Q^k_r(y'') \cap M = \{(\alpha, h(\alpha)) : \alpha \in Q^n_\delta(y')\}. \tag{51}$$

Thus if we set $U = Q^n_\delta(y') \times Q^k_r(y'')$, $\Omega = Q^n_\delta(y')$, and $\Psi(\alpha) = (\alpha, h(\alpha))$, then conditions (i) and (ii) are satisfied. As for (iii), we have

$$D\Psi(\alpha) = \begin{pmatrix} I \\ Dh(\alpha) \end{pmatrix}, \tag{52}$$

where $I \in \mathbb{R}^{n\times n}$ is the identity matrix, which makes it clear that the columns of $D\Psi(\alpha)$ are linearly independent. $\square$

If $k = 1$ in the preceding theorem, then $D\phi(x)$ is a $1 \times N$ matrix, and the existence of a nonsingular $k \times k$ submatrix of $D\phi(x)$, simply means that $D\phi(x)$ has a nonzero entry. This special case is important enough to deserve a separate display.

Corollary 4.4 (Level surface theorem)**.** *Let $A \subset \mathbb{R}^n$ be an open set, and let $\phi : A \to \mathbb{R}$ be a continuously differentiable function. Suppose that $D\phi(x) \neq 0$ whenever $x \in A$ satisfies $\phi(x) = 0$. Then the set $M = \{x \in A : \phi(x) = 0\}$ is a hypersurface in $\mathbb{R}^n$.*

Example 4.5. Let $a \in \mathbb{R}^n$ be a nonzero vector, and let

$$M = \{x \in \mathbb{R}^n : a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 = 1\}. \tag{53}$$

We would like to show that M is a hypersurface. Thus we let

$$\phi(x) = a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2 - 1, \tag{54}$$

so that $M = \{\phi = 0\}$, and compute

$$D\phi(x) = (2a_1x_1, 2a_2x_2, \ldots, 2a_nx_n). \tag{55}$$

Since a is a nonzero vector, $D\phi(x) = 0$ if and only if $x = 0$. We know that $0 \notin M$, because $\phi(0) = -1$, and hence $D\phi(x) \neq 0$ for all $x \in M$. Then the level surface theorem implies that M is a hypersurface in $\mathbb{R}^n$.

Remark 4.6. A matrix $B \in \mathbb{R}^{k\times N}$ has an invertible $k \times k$ submatrix if and only if it has k linearly independent columns. The latter is equivalent to the condition that B is surjective, i.e., that for any $s \in \mathbb{R}^k$ there exists $y \in \mathbb{R}^N$ such that $By = s$. Therefore in the preimage theorem (Theorem 4.3), the condition "there is a $k\times k$ submatrix of $D\phi(x)$ that is nonsingular" can be replaced by "$D\phi(x)$ is surjective." Now, the surjectivity of $D\phi(x)$ is equivalent to saying that for any $s \in \mathbb{R}^k$ there exists $V \in \mathbb{R}^N$ such that $D\phi(y)V = s$, i.e., such that $D_V\phi(y) = s$.

Example 4.7. Consider the set

$$M = \{X \in \mathbb{R}^{n\times n} : X^TX = I\}, \tag{56}$$

which is called the group of orthogonal matrices. This can be written as the zero set of $\phi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, which is given by

$$\phi(X) = X^TX - I. \tag{57}$$

Although ϕ sends $n \times n$ matrices to $n \times n$ matrices, the output $\phi(X)$ has fewer than n^2 independent components, because $\phi(X)$ is always a symmetric matrix. Thus we think of ϕ as a mapping $\phi : \mathbb{R}^N \to \mathbb{R}^k$, with $N = n^2$ and $k = \frac{1}{2}n(n+1)$. In view of Remark 4.6, our first task is to compute the directional derivative of ϕ along a matrix $B \in \mathbb{R}^{n\times n}$. Let us denote the components of ϕ, X, and B by ϕ_{ij}, x_{lm}, and b_{lm}, respectively. Then we have

$$\frac{\partial\phi_{ij}}{\partial x_{lm}}(X) = \frac{\partial}{\partial x_{lm}}\sum_{q=1}^{n} x_{qi}x_{qj} = \sum_{q=1}^{n}(\delta_{ql}\delta_{im}x_{qj} + x_{qi}\delta_{ql}\delta_{jm}) = \delta_{im}x_{lj} + x_{li}\delta_{jm}, \tag{58}$$

for the partial derivatives, and

$$D_B\phi_{ij}(X) = \sum_{l,m=1}^{n}(\delta_{im}x_{lj} + x_{li}\delta_{jm})b_{lm} = \sum_{l=1}^{n}(x_{lj}b_{li} + x_{li}b_{lj}) = (B^TX + X^TB)_{ij}, \tag{59}$$

for the directional derivative, yielding

$$D_B\phi(X) = X^TB + B^TX. \tag{60}$$

Our next task is to show that for each $X \in M$ and for any symmetric matrix $S \in \mathbb{R}^{n\times n}$, there exists $B \in \mathbb{R}^{n\times n}$ such that $D_B\phi(X) = S$. This would guarantee that $D\phi(X)$, as a linear map sending $\mathbb{R}^{n\times n}$ into the space of symmetric $n \times n$ matrices, is surjective. Let $S \in \mathbb{R}^{n\times n}$ be a symmetric matrix. We observe that $(X^TB)^T = B^TX$, and so the equation $X^TB + B^TX = S$ is of the form $C + C^T = S$. It is not difficult to construct a matrix C satisfying $C + C^T = S$. For example, one can check that the following works.

$$C_{ij} = \begin{cases} s_{ij} & \text{for } i < j, \\ \frac{1}{2}s_{ii} & \text{for } i = j, \\ 0 & \text{for } i > j. \end{cases} \tag{61}$$

Now that we have C, we need to solve $X^TB = C$. At this point, we recall that $X \in M$, that is, $X^TX = I$. This means that $(X^T)^{-1} = X$, and hence $B = XX^TB = XC$. We can also independently check that

$$X^TB + B^TX = X^TXC + (XC)^TX = C + C^TX^TX = C + C^T = S. \tag{62}$$

We conclude that the orthogonal group $M = \{X \in \mathbb{R}^{n\times n} : X^TX = I\}$ is a manifold of dimension $N - k = \frac{1}{2}n(n-1)$. The standard notation for this manifold is $O(n) = M$ (not to be confused with the big-O notation).