COMP 531: Assignment 2

Winter 2016

Due February 10th

Each question is worth 10 points. You can submit the assignment in class on the due date, or, if you TeX it, by e-mail to harrison.humphrey@mail.mcgill.ca before midnight.

Question 1

Show that if both f and g are computable in logarithmic space, $f \circ g$ is also computable in logarithmic space. (Note this is not trivial).

Question 2

Consider the following algorithm to detect cycles in a undirected graph $G = (V, E)$. Imagine that a father and son are travelling along the edges of G. The father sits at a vertex v (for $v = 1, 2, \ldots, |V|$), while the son traverses the graph according to the so-called "cycle searching principle":

- For every vertex $u \in G$, give all edges out from u an order accoring to their end vertex. Thus we can say that u has a first edge, second edge, and so on. Note that an edge (u, v) may be the ith edge for u but the jth edge for v, where $i \neq j$.
- Whenever entering a vertex u of fanout k along the ith edge of u, the son leaves through edge $i + 1$ (here $k + 1$ is taken to be 1)

The father remembers the edge along which the son departed and sees if he comes back along the same edge. If, for every edge adjacent to v, the son does so, the the father takes his son to vertex $v + 1$ (if $v < |V|$), or declares that G has no cycle (if $v = |V|$). Otherwise, he declares that there must be a cycle.
Prove that this algorithm terminates in finite time, and that it is correct. What is the space complexity of the algorithm?

Question 3

A boolean function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is called symetric if its value does not change when permuting its input bits. We denote by $f \circ g$ the class of depth 2 circuits where the output gate is f and the gates at the first level are g. For example, $AND \circ MAJ$ denotes the class of depth 2 circuits where the output gate is AND and the first level has only majority gates.

Show that any symmetric function can be computed by linear size $MAJ \circ MAJ$ circuits.

Question 4

Let $C = (C_n)$ be a family of circuits constructed with binary AND and OR gates. Assume that C has polynomial size and the graph of each C_n is a tree. Show that the induced boolean function is actually in NC^1. In other words, show that the circuit can be modified to have $O(logn)$ depth. Hint: Divide and Conquer.

Question 5

Let $\omega = \{1, 2, 3, \ldots\}$. We say that a function $f : \{0,1\}^\omega \to \{0,1\}$ is an ω-parity function if it has the property that, for any input assignment, flipping a bit will flip the answer. Show that an ω-parity function cannot be computed by a Boolean circuit of of constant depth and countable size.

To do so, consider the following hint. First, show directly that such a function cannot be computed in depth 1 or 2 with countable size. Then, show that for $k \geq 3$, if there is an ω-parity function that is computed in depth k and countable size, then there is another such function which is computed by a similar circuit with the additional restriction that the level 1 gates have finite fan-in. Next show that this resriction can be made to hold of the first two levels. Deduce the result using the distributivity of AND over OR (or vice versa) and applying the induction hypothesis.