MATH 254 Tutorial 9 (Cluster Points and Limit of Functions):

Problem 1: Let $A \subseteq \mathbb{R}$ and B be the set of all cluster points of A. Prove that any cluster point of B is in B. (Remark: A subset of real numbers is called "closed" if it contains all of its cluster points. You can easily show that $A \subseteq \mathbb{R}$ is closed if and only if $\mathbb{R} - A$ is open. Recall that $A \subseteq \mathbb{R}$ is open if for any $a \in A$ there exists $r > 0$ such that $(a - r, a + r) \subseteq A$.)

Problem 2: Let $A \subseteq \mathbb{R}$ be infinite and bounded. Prove that there exists a cluster point of A.

Problem 3: Consider the function $f : (0, 1) \to \mathbb{R}$ defined by:

$$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{n} & \exists \quad m, n \in \mathbb{N} \quad (m, n) = 1 \wedge x = \frac{m}{n} \end{cases}$$

Prove that the limit of f at all points of $[0, 1]$ exists and is zero.

Problem 4: Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x + y) = f(x) + f(y)$ for all real numbers x and y (this property is called being additive).

a) Prove that there is a constant $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{Q}$.

b) Prove that if f has limit at one point, then it has limit at all points in $\mathbb{R}$.

c) Prove that if f has limit at 0, then the limit should be 0.

d) Prove that if f has limit at $a \in \mathbb{R}$, then the limit should be $f(a)$.

e) Find all additive functions that has limit at least at one point.

Problem 5: Either find the limit directly using the $\epsilon\delta$-definition of limit or prove that the limit doesn't exist using the equivalent sequential definition of limit.

a) $lim_{x \to -2} \frac{x^3+2x^2+x+1}{2x+3}$

b) $lim_{x \to 0} sin(\frac{1}{x^n}), \qquad n \in \mathbb{N}$