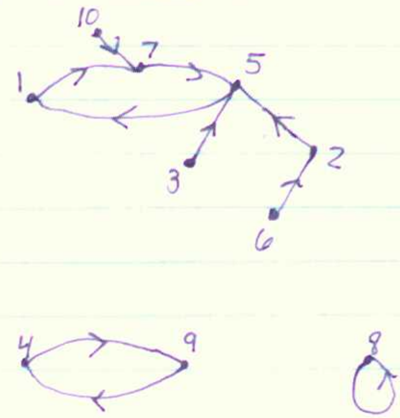
Math 340 - Mar. 22

Bijection between $\{f:[n]\to[n]\}$ and $\{$trees on n vertices with specified red and blue vertices$\}$

x	f(x)
1	7
2	5
3	5
4	9
5	1
6	2
7	5
8	8
9	4
10	7

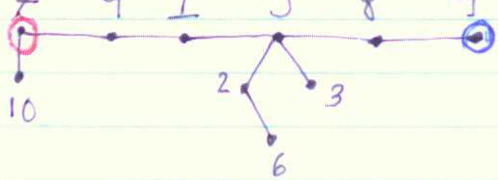
$\Rightarrow$ Graph (directed)



Tree

set of vertices in cycles (in order)

x	1	4	5	7	8	9
F(x)	7	9	1	5	8	4



Let us show that the resulting graph is a tree.
By induction on $[n]$.
Base case: $n=1$ → single vertex is automatically a tree
Inductive step: (if the statement holds $\forall k<n$ we need to show it for n)
Case 1: f is a bijection
- all vertices are in cycles
- the result is a path from red to blue vertices $\Rightarrow$ a tree (ordered paths on n vertices)

Case 2: f is not a bijection

- then f is not a surjection

$\Rightarrow \exists\ x \in [n] = \{1, 2, \ldots, n\}$ s.t. $f(y) \neq x \ \forall y \in [n]$

Let us assume $x = n$ has this property.

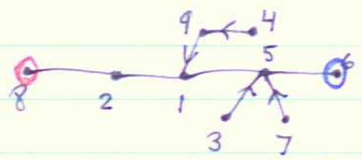
used fact not a surjection → Let $f|_{[n-1]} : [n-1] \to [n-1]$ it produces a tree on $n-1$ vertices using our procedure by the induction hypothesis

And n is joined by an edge to $f(n)$.

So the constructed graph is obtained from a tree by adding a leaf.

This produces a tree. □

To show that the constructed map is a bijection, let us demonstrate the inverse function



x	1	2	5	6	8	← vertices in red/blue path; increasing order
$f(x)$	8	2	1	5	6	← vertices in red to blue path in order along the path

- direct remaining edges towards the path

For every vertix, y, not in the red/blue path, the edge directed from it points to $f(y)$

x	1	2	3	4	5	6	7	8	9
$f(x)$	8	2	5	9	1	5	5	6	1

This is the inverse function, thus it must have a bijection. □

Catalan Numbers

How many sequence of n +'s and n −'s are there such that every initial subsequence of this sequence has at least as many +'s as it has −'s?
(Can also think of +'s as "+1"'s and −'s as "−1"s then want the subsequence to be positive).

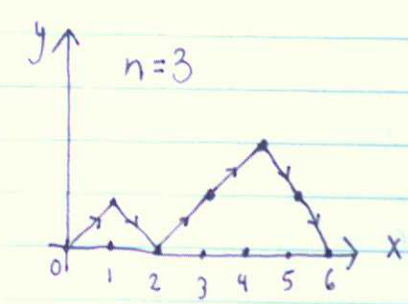
$n=1$	# ways = 1	$+1-1$ = + −
$n=2$	# ways = 2	+ + − − or + − + −
$n=3$	# ways = 5	+ − + + − − , + + − − + − , + − + − + − , + + − + − − , + + + − − −

Let C_n be the # ways or the number of sequences that can occur if we have n +'s and n −'s.

$\binom{2n}{n}$ sequences without restrictions.

Dyck paths (walks)

- walks from $(0,0)$ to $(2n, 0)$ in $\mathbb{R}^2$ using steps $(+1, +1)$ $(x,y) \to (x+1, y+1)$ and $(+1, -1)$ $(x,y) \to (x+1, y-1)$ such that the y-coordinate is always non-negative.



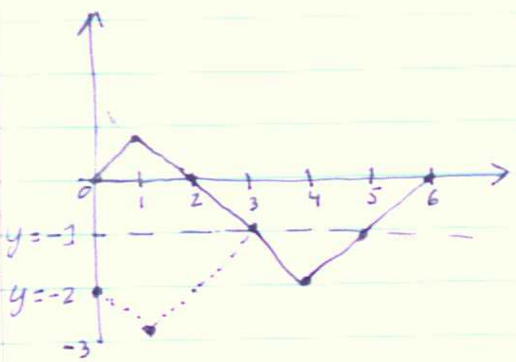
+ − + + − −

Dyck paths are in bijection with sequences defined earlier.
Step $(+1, +1) \Rightarrow +$
Step $(+1, -1) \Rightarrow -$

So there are C_n such walks.

__Theorem:__ $C_n = \frac{1}{n+1}\binom{2n}{n}$ Catalan Numbers

Proof:
To count Dyck paths, let us instead first count all the paths from $(0,0)$ to $(2n,0)$ using the same steps which "dip" below $y=0$ axis.



Choose the first point $(x,-1)$ on the path and reflect the segment of the path from $(0,0)$ to $(x,-1)$ with respect to $y=-1$ line.

The resulting reflected walk uses the same steps and goes from $(0,-2)$ to $(2n,0)$.

In fact this reflection is a bijection between paths I am counting and paths from $(0,-2)$ to $(2n,0)$ using steps $(+1,+1)$ and $(+1,+1)$ and $(+1,-1)$ (Encoding this sequence as before, we have $n+1$ +'s and $n-1$ -'s, so $\binom{2n}{n+1}$ possible sequences).

Total number of paths is $\binom{2n}{n}$.

So there are $\binom{2n}{n} - \binom{2n}{n+1}$ Dyck paths.

Now we just have to show that
$$\binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$$

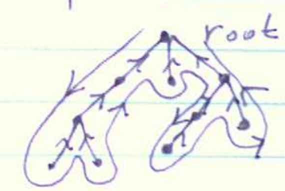
Want to show:
$$\frac{n}{n+1}\binom{2n}{n} = \binom{2n}{n+1}$$

$$\frac{n}{n+1}\,\frac{(2n)!}{n!\,n!} = \frac{(2n)!}{(n+1)!\,(n-1)!} \checkmark$$

□

This theorem shows that if we take a random walk from $(0,0)$ to $(2n,0)$ then the probability that the walk does not go below $y=0$ axis is $\frac{1}{n+1}$.

Other series of objects enumerated by Catalan numbers
1) Sequences with non-negative partial sums
2) Dyck paths
3) Rooted plane trees on $(n+1)$ vertices



tree with 10 vertices

There are C_n such trees with $(n+1)$ vertices

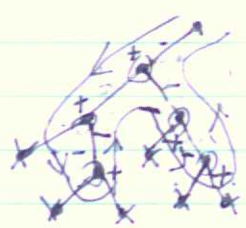
Bijection 3) → 1):
Walk around the tree starting from the root and going to the left of the leftmost edge emanating from the root ending to the right of the rightmost. Direct all the edges of the tree towards the root. Every time our walk traverses the edge away from the root record a "+" otherwise a "-"

+ + + - + - - - + - + + + - - + - -   n=9

This is a function.
The partial sum denotes distance from the root which is always ≥ 0.

It is fairly straightforward to show that it is in fact a bijection.

4) Planted trivalent trees on $(2n+2)$ vertices
trivalent – every vertex is a leaf (no children) or has degree 3 (2 children
planted – rooted at a leaf and plane

T: 

12 vertices $\Rightarrow n=5$
$n=5$

Bijection 4) $\to$ 1)
Suppose there are k vertices of degree 3 in T

$$\sum_{v\in V(T)} \deg(v) = 3k + (2n+2-k) = 2|E(T)| = 2((2n+2)-1)$$

$$2k+2n+2 = 4n+2$$
$$\Rightarrow 2k = 2n$$
$$k=n$$

So there are always n vertices of degree 3

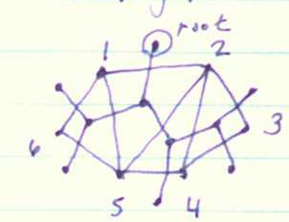
Erase non-root leaves, remember if edges go to the left or right.

Walk around the tree as before.
(assume the edge from root goes left).

Record + whenever we see a new vertex and minus if we go along an edge to the right

+ + − + − − + − + −   Bijection 4) $\to$ 1)

5) Triangulations of polygons on $(n+2)$ vertices



Bijection 5) $\to$ 4)

Put a vertex in every triangle of triangulation and a vertex outside each edge. Join two vertices if they are accross an edge or diagonal.

The vertex corresponding to the edge 17 is the root.
This is a planted trivalent tree on $2n+2$ vertices.