MATH323 - Tutorial 4 (Feb. 1st)

1. (Exercise 3.11) Persons entering a blood bank are selected such that 1 in 3 have type O^+ blood and 1 in 15 have type O^- blood. Consider three randomly selected donors for the blood bank. Let X denote the number of donors with type O^+ blood and let Y denote the number with type O^- blood. Find the probability distributions for X, Y and $X+Y$. The expectation of X, Y and $X+Y$ and the variance of X, Y and $X+Y$.

Solution:

X takes values $0, 1, 2, 3$. We make a table:

x	$p(x)$	
0	$\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^0 = \frac{8}{27}$	
1	$3\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{12}{27}$	(O^+ NotO^+ NotO^+, NotO^+ O^+ NotO^+, NotO^+ NotO^+ O^+ = 3)
2	$3\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^2 = \frac{6}{27}$	(" ")
3	$\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^3 = \frac{1}{27}$	

Observe $\sum_x p(x) = 1$, so we have found the probability distribution of X. Similarly for Y.

y	$p(y)$
0	$\left(\frac{14}{15}\right)^3\left(\frac{1}{15}\right)^0 = \frac{2744}{3375}$
1	$3\left(\frac{14}{15}\right)^2\left(\frac{1}{15}\right) = \frac{588}{3375}$
2	$3\left(\frac{14}{15}\right)\left(\frac{1}{15}\right)^2 = \frac{42}{3375}$
3	$\left(\frac{14}{15}\right)^0\left(\frac{1}{15}\right)^3 = \frac{1}{3375}$

Likewise, $\sum_y p(y) = 1$ so we have found the probability distribution of Y

Let $Z = X + Y$. Then Z takes the values $0, 1, 2, 3$, since we must respect the constraint of $0 \le Z \le 3$ in the context of this problem as there are only 3 donors selected for the blood bank. We make a table:

Note: Z counts individuals with type O blood $\Rightarrow$ the probability of having type O blood is $\frac{1}{15} + \frac{1}{3} = \frac{2}{5}$

z	$p(z)$
0	$\left(\frac{3}{5}\right)^3 = 27/125$
1	$3\left(\frac{3}{5}\right)^2\left(\frac{2}{5}\right) = 54/125$
2	$3\left(\frac{3}{5}\right)\left(\frac{2}{5}\right)^2 = 36/125$
3	$\left(\frac{2}{5}\right)^3 = 8/125$

Again, one can check that $\sum_z' p(z) = 1$ so we have a probability distribution for Z.

Expectations:

$$\mathbb{E}(X) = \sum_x' x\,p(x) = (0)\left(\tfrac{8}{27}\right) + (1)\left(\tfrac{12}{27}\right) + (2)\left(\tfrac{6}{27}\right) + (3)\left(\tfrac{1}{27}\right) = 1$$

$$\mathbb{E}(Y) = \sum_y' y\,p(y) = (0)\left(\tfrac{2744}{3375}\right) + (1)\left(\tfrac{588}{3375}\right) + (2)\left(\tfrac{42}{3375}\right) + (3)\left(\tfrac{1}{3375}\right) = 0.2$$

$$\mathbb{E}(Z) = \mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 1 + 0.2 = 1.2$$

the expectation acts linearly

(exercise: using the distribution of Z, check that $\mathbb{E}(Z) = 1.2$)

Variances:

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \left(\sum_x' x^2 p(x)\right) - (1)^2$$
$$= (0)^2\left(\tfrac{8}{27}\right) + (1)^2\left(\tfrac{12}{27}\right) + (2)^2\left(\tfrac{6}{27}\right) + (3)^2\left(\tfrac{1}{27}\right) - 1$$
$$= \tfrac{45}{27} - \tfrac{27}{27} = \tfrac{18}{27}$$

$$Var(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2 = \left(\sum_y y^2 p(y)\right) - (0.2)^2$$
$$= (0)^2\left(\tfrac{2744}{3375}\right) + (1)^2\left(\tfrac{588}{3375}\right) + (2)^2\left(\tfrac{42}{3375}\right) + (3)^2\left(\tfrac{1}{3375}\right) - 0.04$$
$$= \tfrac{765}{3375} - \tfrac{135}{3375} = \tfrac{630}{3375}$$

$$Var(Z) = \mathbb{E}(Z^2) - (\mathbb{E}(Z))^2 = 270/125 - (150/125)^2$$
$$= 0.72$$
$$= 0.72 \neq \tfrac{18}{27} + \tfrac{630}{3375}$$

2. (Exercise 2.166) : Eight tires of different brands are ranked from 1 to 8 (best to worst) according to mileage performance. If four of these tires are chosen at random by a customer, find the probability that the best tire among those selected by the customer is actually ranked third among the original eight.

<u>Solution</u>:

$$|S| = \binom{8}{4} = \frac{8 \times 7 \times \cancel{6}^{2} \times 5}{4 \times 3 \times 2 \times 1} = 70$$

Let E be the event that the best tire among those selected by the customer is actually ranked third among the original eight.

$$\underset{\text{Best}}{T_1} \quad T_2 \quad \underset{\text{selected}}{\underset{\uparrow}{T_3}} \quad \underbrace{T_4 \quad T_5 \quad T_6 \quad T_7 \quad \overset{\text{Worst}}{T_8}}_{\text{other 3 are among these}}$$

Thus, $|E| = \binom{5}{3} = \frac{5 \times 4}{2 \times 1} = 10$

Under the assumption of equal probability for every sample point, thus

$$\mathbb{P}(E) = \frac{|E|}{|S|} = \frac{10}{70} = \frac{1}{7}$$

3. (An old 203 question) : Consider the following experiment : A ball is drawn from an urn containing an equal number of red, blue and white balls. If the ball drawn is white, a fair coin is flipped and the outcome recorded. If the ball drawn is blue, a card is drawn from a deck of cards and if the ball is red, a die is rolled and the top number is recorded.

a) List the sample points in this experiment. What is the probability of each sample point?

b) What is the probability that the die rolls an odd number?

c) What is the probability that a blue ball is drawn and a heart is drawn?

d) Are the events {Blue ball drawn} and {Heart is drawn} independent?

Solution

a) Let R, B, W represent the color of ball drawn.

C_0^H, C_0^T " " side of coin

$C^C_{1-13}, C^H_{1-13}, C^D_{1-13}, C^S_{1-13}$ " " the type of card drawn

$D_1, \ldots, D_6$ " " the top number of the die.

$$S = \{WC_0^H, WC_0^T, BC^C_{1-13}, BC^H_{1-13}, BC^D_{1-13}, BC^S_{1-13}, RD_1, \ldots, RD_6\}$$

with respectively probabilities:

$$\frac{1}{6}, \frac{1}{6}, \underbrace{\left(\tfrac{1}{3}\right)\left(\tfrac{1}{52}\right), \ldots, \left(\tfrac{1}{3}\right)\left(\tfrac{1}{52}\right)}_{52 \text{ sample points}}, \underbrace{\left(\tfrac{1}{3}\right)\left(\tfrac{1}{6}\right), \ldots, \left(\tfrac{1}{3}\right)\left(\tfrac{1}{6}\right)}_{6 \text{ sample points}}$$

b) Let E be the event that the die rolls an odd number.

Thus, $E = \{RD_1, RD_3, RD_5\}$ each with probability $\frac{1}{18}$.

$\Rightarrow \mathbb{P}(E) = 3\left(\frac{1}{18}\right) = \frac{1}{6}$

c) Let A be the event that a blue ball is drawn and let B be " " a heart is drawn.

$A \cap B = \{BC_1^H, BC_2^H, \ldots, BC_{13}^H\}$ each with probability $\left(\frac{1}{3}\right)\left(\frac{1}{52}\right)$

$\Rightarrow \mathbb{P}(A \cap B) = 13\left(\frac{1}{3}\right)\left(\frac{1}{52}\right) = \left(\frac{1}{3}\right)\left(\frac{1}{4}\right) = \frac{1}{12}$

d) We must check if $\mathbb{P}(A \mid B) \stackrel{?}{=} \mathbb{P}(A)$.

LHS: $\mathbb{P}(A \mid B)$ is the probability that a blue ball is drawn if a heart is drawn. If we condition on the fact that a heart was drawn, this necessarily implies a blue ball is drawn.

Thus, $\mathbb{P}(A \mid B) = 1$.

RHS: $\mathbb{P}(A) = 1/3$

So $\mathbb{P}(A \mid B) \neq \mathbb{P}(A) \Rightarrow$ the events are not independent!