# Tutorial 12 (April $5^{th}$/2017) - Final Tutorial

In earlier tutorials, we focused on continuous bivariate distributions. Here, we examine a bivariate discrete problem.

1. Consider the following probability function for $Y_1$ and $Y_2$:

| $y_2$ \ $y_1$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $1/9$ | $2/9$ | $1/9$ |
| 1 | $2/9$ | $2/9$ | 0 |
| 2 | $1/9$ | 0 | 0 |

a) Find the marginal pmfs of $Y_1$ and $Y_2$.

b) Find the conditional pmf $\mathbb{P}(Y_1 = y_1 \mid Y_2 = 0)$.

c) Find $\mathbb{P}(Y_1 + Y_2 = 2)$.

d) Find $Cov(Y_1, Y_2)$.

e) Find $\mathbb{E}(Y_1^3 \mid Y_2 = 0)$.

Solution:

a) By definition, $p_1(y_1) = \sum_{\text{all } y_2}' p(y_1, y_2)$, $p_2(y_2) = \sum_{\text{all } y_1}' p(y_1, y_2)$

$$p_1(0) = \sum_{\text{all } y_2}' p(0, y_2) = p(0,0) + p(0,1) + p(0,2) = 1/9 + 2/9 + 1/9 = 4/9$$

$$p_1(1) = \sum_{\text{all } y_2}' p(1, y_2) = 2/9 + 2/9 + 0 = 4/9$$

$$p_1(2) = \sum_{\text{all } y_2}' p(2, y_2) = 1/9 + 0 + 0 = 1/9$$

$$p_2(0) = \sum_{\text{all } y_1}' p(y_1, 0) = p(0,0) + p(1,0) + p(2,0) = 1/9 + 2/9 + 1/9 = 4/9$$

$$p_2(1) = \sum_{\text{all } y_1}' p(y_1, 1) = 2/9 + 2/9 + 0 = 4/9$$

$$p_2(2) = \sum_{\text{all } y_1}' p(y_1, 2) = 1/9 + 0 + 0 = 1/9$$

b) $\mathbb{P}(Y_1 = 0 \mid Y_2 = 0) = \dfrac{\mathbb{P}(Y_1 = 0, Y_2 = 0)}{\mathbb{P}(Y_2 = 0)}$

$= \dfrac{1/9}{4/9} = \dfrac{1}{4}$

$\mathbb{P}(Y_1 = 1 \mid Y_2 = 0) = \dfrac{\mathbb{P}(Y_1 = 1, Y_2 = 0)}{\mathbb{P}(Y_2 = 0)}$

$= \dfrac{2/9}{4/9} = \dfrac{2}{4}$

$\mathbb{P}(Y_1 = 2 \mid Y_2 = 0) = \dfrac{\mathbb{P}(Y_1 = 2, Y_2 = 0)}{\mathbb{P}(Y_2 = 0)}$

$= \dfrac{1/9}{4/9} = \dfrac{1}{4}$

c) $Y_1 + Y_2 = 2$ is satisfied for the following pairs of $(Y_1, Y_2)$:
$(0,2)$, $(1,1)$, $(2,0)$ which are all disjoint from each other:

$\mathbb{P}(Y_1 + Y_2 = 2) = \mathbb{P}\{(Y_1 = 0, Y_2 = 2) \cup (Y_1 = 1, Y_2 = 1) \cup (Y_1 = 2, Y_2 = 0)\}$
$= \mathbb{P}(Y_1 = 0, Y_2 = 2) + \mathbb{P}(Y_1 = 1, Y_2 = 1) + \mathbb{P}(Y_1 = 2, Y_2 = 0)$
$= 1/9 + 2/9 + 1/9 = 4/9$

d) $\mathrm{Cov}(Y_1, Y_2) = \mathbb{E}(Y_1 Y_2) - \mathbb{E}(Y_1)\mathbb{E}(Y_2)$

$= \underbrace{\sum_{\text{all } y_1, y_2} y_1 y_2\, p(y_1, y_2)}_{1} - \Big(\underbrace{\sum_{\text{all } y_1, y_2} y_1\, p(y_1, y_2)}_{2}\Big)\Big(\underbrace{\sum_{\text{all } y_1, y_2} y_2\, p(y_1, y_2)}_{3}\Big)$

1 $= (0)(0)p(0,0) + (0)(1)p(0,1) + (0)(2)p(0,2) +$
$(1)(0)p(1,0) + (1)(1)p(1,1) + (1)(2)p(1,2) +$
$(2)(0)p(2,0) + (2)(1)p(2,1) + (2)(2)p(2,2)$
$= p(1,1) + 2p(1,2) + 2p(2,1) + 4p(2,2)$
$= 2/9 + 0 + 0 + 0 = 2/9$

2 $= (0)p(0,0) + (0)p(0,1) + (0)p(0,2)$
$+ (1)p(1,0) + (1)p(1,1) + (1)p(1,2)$
$+ (2)p(2,0) + (2)p(2,1) + (2)p(2,2)$
$= 2/9 + 2/9 + 0 + 2(1/9) + 2(0) + 2(0) = 6/9$

$③ = (0)p(0,0) + (0)p(1,0) + (0)p(2,0) +$
$(1)p(0,1) + (1)p(1,1) + (1)p(2,1) +$
$(2)p(0,2) + (2)p(1,2) + (2)p(2,2)$
$= 2/9 + 2/9 + 0 + 2(1/9) + 0 + 0 = 6/9$

$$\mathrm{Cov}(Y_1, Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$$
$$= 2/9 - (6/9)(6/9)$$
$$= 18/81 - 36/81$$
$$= -18/81 = -2/9$$

e) $E(Y_1^3 \mid Y_2 = 0) = \sum_{\text{all } y_1} y_1^3 \, p(y_1 \mid Y_2 = 0)$

$$= 0^3\left(\tfrac{1}{4}\right) + 1^3\left(\tfrac{2}{4}\right) + 2^3\left(\tfrac{1}{4}\right)$$
$$= 0 + \tfrac{2}{4} + \tfrac{8}{4}$$
$$= \tfrac{10}{4}$$
$$= 2.5$$

Sums and Differences of Normal distributions are still Normal

2. Let $X_1 \sim N(5, 1)$ and let $X_2 \sim N(6, 2)$. Find $P(X_1 - \tfrac{1}{6}X_2 < 2.5)$, under the assumption $X_1 \perp X_2$.

Solution:

$\tfrac{1}{6}X_2 \sim N(1, \tfrac{1}{36})$ since

$E(\tfrac{1}{6}X_2) = \tfrac{1}{6}E(X_2) = \tfrac{1}{6}(6) = 1$

$V(\tfrac{1}{6}X_2) = \tfrac{1}{36}V(X_2) = \tfrac{1}{36}(2) = \tfrac{1}{18}$

$$E(X_1 - \tfrac{1}{6}X_2) = E(X_1) - E(\tfrac{1}{6}X_2)$$
$$= 5 - 1 = 4$$
$$V(X_1 - \tfrac{1}{6}X_2) = V(X_1) + V(\tfrac{1}{6}X_2) = 1 + \tfrac{1}{18} = \tfrac{19}{18}$$

$$\Rightarrow X_1 - \tfrac{1}{6}X_2 \sim N(4, \tfrac{19}{18})$$

**Note**: If $Z \sim N(0,1)$ then we have that for any $z \in \mathbb{R}$, $\mathbb{P}(Z \le z) = \Phi(z)$.

Let $Y = X_1 - \frac{1}{6}X_2 \sim N(4, \frac{19}{18})$

$$\mathbb{P}(X_1 - \tfrac{1}{6}X_2 < 2.5) = \mathbb{P}(Y < 2.5)$$
$$= \mathbb{P}(Y - 4 < 2.5 - 4)$$
$$= \mathbb{P}\left(\frac{Y-4}{\sqrt{\frac{19}{18}}} < \frac{2.5-4}{\sqrt{\frac{19}{18}}}\right)$$

Notice that $\frac{Y-4}{\sqrt{\frac{19}{18}}} \sim N(0,1)$.

Let $Z = \frac{Y-4}{\sqrt{19/18}}$ $\Rightarrow \mathbb{P}\left(\frac{Y-4}{\sqrt{\frac{19}{18}}} < \frac{2.5-4}{\sqrt{\frac{19}{18}}}\right)$

$$= \mathbb{P}\left(Z < 4.5\sqrt{\tfrac{2}{19}}\right)$$
$$= \Phi\left(4.5\sqrt{\tfrac{2}{19}}\right).$$

**Note**: Similar sum concept holds for Poisson, Binomial & Gamma.

3. (Exercise 5.136): Let $Y \sim \text{Poisson}(\lambda)$ and assume $\lambda$ is a random variable with a density function given by

$$f(\lambda) = \begin{cases} e^{-\lambda}, & \lambda \ge 0 \\ 0, & \text{elsewhere} \end{cases}$$

a) Find $\mathbb{E}(Y)$.

b) Find $V(Y)$.

Solution:

a) Using the conditional expectation formula, we have the following:

$$\mathbb{E}(Y) = \mathbb{E}(\mathbb{E}(Y|\lambda))$$

$\mathbb{E}(Y|\lambda)$ is "the expectation of $Y$ when $\lambda$ is considered fixed". In this scenario, $Y \sim \text{Poisson}(\lambda)$ and $\mathbb{E}(Y|\lambda) = \lambda$

$$\Rightarrow \mathbb{E}(\mathbb{E}(Y|\lambda)) = \mathbb{E}(\lambda)$$
$$= \int_0^\infty \lambda e^{-\lambda}\, d\lambda$$
$$= \left(-\lambda e^{-\lambda} - e^{-\lambda}\right)\Big|_0^\infty$$
$$= 1$$

Therefore, $\mathbb{E}(Y) = 1$.

b) By definition, $V(Y) = \mathbb{E}(V(Y|\lambda)) + V(\mathbb{E}(Y|\lambda))$
$\qquad\qquad\qquad\qquad\qquad\qquad 1 \qquad\qquad\qquad 2$

1 $V(Y|\lambda)$ is "the variance of $Y$ when $\lambda$ is considered fixed" so
$$V(Y|\lambda) = \lambda$$
$$\Rightarrow \mathbb{E}(V(Y|\lambda)) = \mathbb{E}(\lambda) = 1 \quad \text{from part (a)}$$

2 $\mathbb{E}(Y|\lambda) = \lambda \quad \Rightarrow V(\mathbb{E}(Y|\lambda)) = V(\lambda)$

$$V(\lambda) = \mathbb{E}(\lambda^2) - (\mathbb{E}(\lambda))^2$$
$$= \int_0^\infty \lambda^2 e^{-\lambda}\, d\lambda - (1)^2$$
$$= \left(-\lambda^2 e^{-\lambda} - 2\lambda e^{-\lambda} - 2e^{-\lambda}\right)\Big|_0^\infty - 1$$
$$= 2 - 1 = 1$$

$$\Rightarrow V(Y) = \mathbb{E}(V(Y|\lambda)) + V(\mathbb{E}(Y|\lambda))$$
$$= 1 + 1$$
$$= 2$$