COMP 360 - Fall 2015 - Assignment 5

Due: 6pm Dec 8th.

Guildenstern: The law of probability, as it has been oddly asserted, is something to do with the proposition that if six monkeys (he has surprised himself)... if six monkeys were...
Guildenstern: [...] The law of averages, if I have got this right, means that if six monkeys were thrown up in the air for long enough they would land on their tails about as often as they would land on their -
Rosencrantz: Heads. (He picks up the coin.)
Guildenstern: Which at first glance does not strike one as a particularly rewarding speculation, in either sense, even without the monkeys. I mean you wouldn't bet on it. I mean I would, but you wouldn't...

Tom Stoppard. Rosencrantz and Guildenstern Are Dead

General rules: In solving these questions you may collaborate with other students but each student has to write his/her own solution. There are in total 110 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

1. (20 Points) Given a set P of n points on the plane, consider the problem of finding the smallest circle containing all the points in P. Design a PTAS algorithm for this problem. In other words, given any fixed $\epsilon > 0$, design an algorithm whose running time is polynomial in n, and its output is at most $1+\epsilon$ times the optimal output.

2. (10 Points) Consider the matrix multiplication verification problem (whether $AB = C$ or not). Consider a modification of the Freivalds' algorithm where now at each round we pick a random vector $b \in \{-1, 0, 1\}^n$ and compare $A(Bb)$ and Cb. What is the probability of error for this algorithm in a single round?

3. (20 Points) In the MAX-k-COL problem we are given a graph G and we want to color the vertices of the graph with k-colors so as to maximize the number of edges whose endpoints are in different colors.

   Give an efficient randomized $(1 - \frac{1}{k})$-factor approximation algorithm for the MAX-k-COL problem. (That is the expected value of the output of the algorithm is at least $\frac{k-1}{k}$ times the optimal solution.)

4. (20 Points) Consider a k-CNF (each clause contains k terms) with $m < 2^{k-1}$ clauses. Suppose further that the variables involved in any given clause are distinct. Show that the probability that the following randomized algorithm does <u>not</u> succeed in satisfying ϕ is at most 2^{-1000}.

   - For $i = 1, \ldots, 1000$:
   - Pick a random truth assignment σ.
   - If σ satisfies the CNF, then output σ and terminate.

5. (20 Points) Consider a 4-regular graph (i.e. each vertex is incident to 4 edges) on n vertices. The following algorithm produces a set S such that every vertex either belongs to S or has at least one neighbour in S:

   - Include every vertex in S independently with probability p.
   - Add to S all the vertices that have no neighbours in S.
   - Output S.

   What is the expected size of the outputted S? What value of p maximizes this expected value?

6. (20 Points) Let $v_1, \ldots, v_n$ be *unit* vectors in $\mathbb{R}^n$. Prove that there exists $\epsilon_1, \ldots, \epsilon_n = \pm 1$ such that

$$|\epsilon_1 v_1 + \ldots + \epsilon_n v_n| \leq \sqrt{n}.$$

   Hint:

$$|\epsilon_1 v_1 + \ldots + \epsilon_n v_n|^2 = \langle \epsilon_1 v_1 + \ldots + \epsilon_n v_n, \epsilon_1 v_1 + \ldots + \epsilon_n v_n \rangle.$$