COMP 360 - Fall 2015 - Sample Final Exam

There are in total 105 points, but your grade will be considered out of 100.

1. (10 points) Prove that the following problem belongs to P: Given a graph G, we want to know whether G has an independent set of size 100.

2. (10 Points) Prove that the following problem belongs to PSPACE: Given a graph G and an integer k, we want to know whether the number of independent sets in G is equal to k.

3. (10 Points) Show that the following problem is NP-complete:

   - Input: An undirected graph G and an edge e.
   - Question: Does G have a Hamiltonian cycle that passes through the edge e.

4. (10 Points) Consider the following optimization problem:

   - Input: A 4CNF ϕ where the variables appearing in each clause are distinct.
   - Question: Find a truth assigment that maximizes the number of clauses that receive at least one true and one false term.

   Prove that a random truth assignment (i.e. every variable is independently set to either T or F with equal probability) provides a $\frac{7}{8}$-factor approximation algorithm for this problem. In other words, the expected value of the output is at least $\frac{7}{8}$ of the optimal solution.

5. (10 points) Prove that the following algorithm is a 2-factor approximation algorithm for the minimum vertex cover problem:

   - While there is still an edge e left in G:
   - Delete all the two endpoints of e from G
   - EndWhile
   - Output the set of the deleted vertices

6. (10 points) Prove that the following algorithm is a 2-factor approximation algorithm for the MAX-SAT problem: Given a CNF ϕ on n variables $x_1, \ldots, x_n$:

   - For $i = 1, \ldots, n$ do
   - IF x_i appears in more clauses than $\overline{x_i}$ THEN
   - Set $x_i = T$
   - Else
   - Set $x_i = F$
   - Remove all True clauses from ϕ and remove x_i and $\overline{x_i}$ from all the other clauses

- EndFor

7. (15 Points) Let G be a 4-regular graph on n vertices (4-regular means that every vertex is adjacent to 4 edges). We want to color the *edges* of G with two colors Red and Blue such that the number of vertices that are adjacent to exactly two Red and two Blue edges is maximized. If we color the edges at random, then what is the expected number of vertices that satisfy the above condition?

8. Consider a graph $G = (V, E)$. The chromatic number of G is the minimum number of colors required to color the vertices of G properly. Let $\mathcal{I}$ be the set of all independent sets in G (Note that every element in $\mathcal{I}$ is a set).

   (a) (10 Points) Prove that the solution to the following linear program provides a lower-bound for the chromatic number of G.

$$\begin{array}{lll} \max & \sum_{I \in \mathcal{I}} x_I & \\ \text{s.t.} & \sum_{I : v \in I} x_I \leq 1 & \forall v \in V \\ & x_I \geq 0 & \forall I \in \mathcal{I} \end{array}$$

   (b) (10 Points) Write the dual of the above linear program.

   (c) (10 Points) Prove that every clique in G provides a solution to the dual linear program.