MATH 254 – Hon. Analysis I
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Solutions for Assignment II.

1. Let $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$ be two sets bounded from above. The sum of A and B is the set

$$A + B = \{a + b \,:\, a \in A, b \in B\},$$

Prove that $A + B$ is bounded from above and that

$$\sup(A + B) = \sup A + \sup B.$$

Solution:

Since A and B are bounded from above, $\sup A$ and $\sup B$ exist and

$$a \leq \sup A \qquad \forall a \in A \tag{1}$$
$$b \leq \sup B \qquad \forall b \in B. \tag{2}$$

Let $c \in A + B$ be arbitrary, then $\exists a \in A, b \in B$ such that $c = a + b$. By (1) and (2),

$$c = a + b \leq \sup A + \sup B.$$

Since $c \in A + B$ was arbitrary, we conclude

$$c \leq \sup A + \sup B \qquad \forall c \in A + B$$

and therefore that $\sup A + \sup B$ is an upper bound of $A + B$. This also shows $A + B$ is bounded from above.

Note that $\sup(A + B) = \sup A + \sup B$ if and only if the folloing hold:

(i) $\sup A + \sup B$ is an upper bound for $A + B$, and
(ii) for any $\epsilon > 0$, $\sup A + \sup B - \epsilon$ is not an upper bound for $A + B$.

We already showed (i), so we need only show (ii). Let $\epsilon > 0$ be arbitrary. Since $\sup A$ is the supremum of A, $\sup A - \epsilon/2$ is not an upper bound of A. Similarily, $\sup B - \epsilon/2$ is not an upper bound of B. Therefore, $\exists a_\epsilon \in A, b_\epsilon \in B$ such that

$$a_\epsilon > \sup A - \epsilon/2 \tag{3}$$
$$b_\epsilon > \sup B - \epsilon/2. \tag{4}$$

Let $c_\epsilon = a_\epsilon + b_\epsilon$ (note that $c_\epsilon \in A + B$). Then, by (3) and (4),

$$c_\epsilon = a_\epsilon + b_\epsilon > \sup A + \sup B - \epsilon.$$

Hence, $\sup A + \sup B - \epsilon$ is not an upper bound of $A + B$. Since ϵ was arbitrary, we have (ii) and hence,

$$\sup A + \sup B = \sup(A + B).$$

Q.E.D.

Remarks:

(i) The quantifiers (such as "$\exists$" (there exists) and "$\forall$" (for all)) and the order in which they appear is very important in these proofs. To illustrate this, consider the two statements

$$\exists M \in \mathbb{R} \text{ such that } a \leq M \quad \forall a \in A \tag{5}$$

$$\forall a \in A, \exists M \in \mathbb{R} \text{ such that } a \leq M. \tag{6}$$

The first one (*i.e.* (5)) is the statement that M is an upper bound of the set A (it is therefore true whenever A is bounded from above). Note that since "$\exists M \in \mathbb{R}$" appears before "$a \leq M \quad \forall a \in A$", M does not (cannot) depend on a. The second one (*i.e.* (5)) is always true (tautology). Indeed, since "$\exists M \in \mathbb{R}$" appears after "$\forall a \in A$", M is allowed to depend on a and therefore $M = a \in A \subset \mathbb{R}$ trivially satisfies the inequality.

(ii) The supremum of a set S is not necessarily an element of the set S. It is not true that the supremum of S is the "maximum of the set S". To see this, consider $S := \{1 - \frac{1}{n} : n \in \mathbb{N}\}$. One can show $\sup S = 1$, but $1 \notin S$. Also note that S has no maximum.

2. Using only field axioms of $\mathbb{R}$ (Definition 2.1.1 in the book), prove that

$$(-1) \cdot (-1) = 1.$$

Write every step of the proof carefully indicating which field property you are using.

Solution:

There were many ways to do this, but here is one

$$\begin{aligned}
1 &= 1 \cdot 1 && \text{(M3)} \\
&= (1+0) \cdot (1+0) && \text{(A3)} \\
&= (1+(1+(-1))) \cdot (1+(1+(-1))) && \text{(A4)} \\
&= 1 \cdot (1+(1+(-1))) + (1+(-1)) \cdot (1+(1+(-1))) && \text{(D)} \\
&= 1+(1+(-1)) + (1+(-1)) \cdot (1+(1+(-1))) && \text{(M3)} \\
&= 1+(1+(-1)) + (1+(-1)) \cdot 1 + (1+(-1)) \cdot (1+(-1)) && \text{(D)} \\
&= 1+(1+(-1)) + (1+(-1)) + (1+(-1)) \cdot (1+(-1)) && \text{(M3)} \\
&= 1+0+0+(1+(-1)) \cdot (1+(-1)) && \text{(A4)} \\
&= 1+(1+(-1)) \cdot (1+(-1)) && \text{(A3,A3)} \\
&= 1+1 \cdot (1+(-1)) + (-1) \cdot (1+(-1)) && \text{(D)} \\
&= 1+1+(-1)+(-1) \cdot (1+(-1)) && \text{(M3, A2)} \\
&= 1+1+(-1)+(-1) \cdot 1 + (-1) \cdot (-1)) && \text{(D)} \\
&= 1+1+(-1)+(-1)+(-1) \cdot (-1)) && \text{(M3)} \\
&= 1+(1+(-1))+(-1)+(-1) \cdot (-1)) && \text{(A2)} \\
&= 1+((-1)+1)+(-1)+(-1) \cdot (-1)) && \text{(A1)} \\
&= (1+(-1))+(1+(-1))+(-1) \cdot (-1)) && \text{(A2)} \\
&= 0+0+(-1) \cdot (-1) && \text{(A4)} \\
&= (-1) \cdot (-1). && \text{(A3,A3)}
\end{aligned}$$

Q.E.D.

Remarks:

The axiom of existence of the 0 element (A3) is states that there $\exists 0 \in \mathbb{R}$ such that $a + 0 = 0 + a = a$ for any $a \in \mathbb{R}$. It does not state that $a \cdot 0 = 0 \cdot a = 0$ for any $a \in \mathbb{R}$ (if you wanted to use it, you had to show it). Also, the existence of negative elements (A4) does not state uniqueness of the negative element to a real number a. Finally, it is not given as an axiom that $-(-a) = a$.

3. Show that there exists no rational number r such that $r^2 = 3$.

Solution:

We proceed by contradiction. Assume $r^2 = 3$ and $r \in \mathbb{Q}$. Then we can write r as an irreducible fraction $r = n/m$ where $m \in \mathbb{N}$ and $n \in \mathbb{Z}$ are coprime ($\gcd(m,n) = 1$). Then, $3 = r^2 = (n/m)^2 = n^2/m^2 \Leftrightarrow 3m^2 = n^2$.

If m is even, then m^2 is even and $3m^2$ is even so that n^2 is even and n is even, contradicting the fact that m and n are coprime. We conlude m is odd, which implies m^2 and $3m^2$ are odd so that n^2 and n are odd.

We can therefore write $m = 2k+1$ for some $k \in \mathbb{Z}$, $n = 2\ell + 1$ for some $\ell \in \mathbb{Z}$. Then, we must have

$$\begin{aligned}
& & 3(2k+1)^2 = (2\ell+1)^2 \\
&\Leftrightarrow & 12k^2 + 12n^2 + 3 = 4\ell^2 + 4\ell + 1 \\
&\Leftrightarrow & 12k^2 + 12n^2 + 3 + (-1) = 4\ell^2 + 4\ell + 1 + (-1) \\
&\Leftrightarrow & 12k^2 + 12n^2 + 2 = 4\ell^2 + 4\ell \\
&\Leftrightarrow & 6k^2 + 6k + 1 = 2\ell^2 + 2\ell \\
&\Leftrightarrow & 2(2k^2 + 3k) + 1 = 2(\ell^2 + \ell).
\end{aligned}$$

Note that the LHS is an odd integer equal to an even integer on the RHS. This is a contradiction. We conclude that there is no $r \in \mathbb{Q}$ such that $r^2 = 3$.

Q.E.D.

4. Let $x, y, z \in \mathbb{R}$. Show that $|x-y|+|y-z|=|x-z|$ if and only if $x \leq y \leq z$ or $x \geq y \geq z$.

Solution:

It is useful to first analyze under what condition equality holds in the triangle inequality $|a+b| \leq |a|+|b|$:

$$\begin{aligned}
& |a+b| = |a|+|b| \\
\Leftrightarrow\quad & |a+b|^2 = (|a|+|b|)^2 \qquad \text{(since both sides are non-negative)} \\
\Leftrightarrow\quad & (a+b)^2 = |a|^2+2|a||b|+|b|^2 \\
\Leftrightarrow\quad & a^2+2ab+b^2 = a^2+2|a||b|+b^2 \\
\Leftrightarrow\quad & 2ab = 2|ab| \\
\Leftrightarrow\quad & ab = |ab| \\
\Leftrightarrow\quad & ab \geq 0
\end{aligned}$$

Thus $|a+b|=|a|+|b| \Leftrightarrow ab \geq 0$.

Now we prove the problem: Let $a := x-y$ and $b := y-z$. Then $a+b=(x-y)+(y-z)=x-z$. Thus, as shown above:

$$\begin{aligned}
|\underbrace{x-y}_{a}|+|\underbrace{y-z}_{b}|=|\underbrace{x-z}_{a+b}| & \Leftrightarrow (x-y)(y-z) \geq 0 \\
& \Leftrightarrow (x-y \geq 0 \text{ and } y-z \geq 0) \text{ or } (x-y \leq 0 \text{ and } y-z \leq 0) \\
& \Leftrightarrow (x \geq y \text{ and } y \geq z) \text{ or } (x \leq y \text{ and } y \leq z) \\
& \Leftrightarrow (x \geq y \geq z) \text{ or } (x \leq y \leq z).
\end{aligned}$$

Q.E.D.

5. If $a \in \mathbb{R}$, $a > -1$, prove by induction that

$$(1+a)^n \geq 1 + na$$

for all $n \in \mathbb{N}$.

Solution:

Base case: For $n = 1$, we indeed have $(1+a)^1 = (1+a) \geq 1 + 1 \cdot a$.

Induction step: Assume $(1+a)^k \geq 1 + ka$, $k \in \mathbb{N}$. We want to show $(1+a)^{k+1} \geq 1 + (k+1)a$. Indeed,

$$\begin{aligned}
(1+a)^{k+1} &= (1+a)^k \cdot (1+a) && \text{by associativity} \\
&\geq (1+ka) \cdot (1+a) && \text{by induction hypothesis and since } a > -1 \\
&= 1 \cdot (1+a) + (ka) \cdot (1+a) && \text{by distributivity} \\
&= 1 + a + ka + ka^2 && \text{by distributivity and multiplicative identity} \\
&= 1 + a(1+k) + ka^2 && \text{by associativity and distributivity} \\
&\geq 1 + a(1+k). && \text{since } ka^2 > 0
\end{aligned}$$

Note that in the second step $a > -1 \Rightarrow a + 1 > 0$ and therefore $(1+a)^k \geq 1 + ka \Rightarrow (1+a)^k \cdot (1+a) \geq (1+ka) \cdot (1+a)$.

By induction, we have shown that $(1+a)^n \geq 1 + an$ for all $n \in \mathbb{N}$.

Q.E.D.

6. For any $A \subseteq \mathbb{R}$ we define

$$-A = \{-a : a \in A\}$$

Suppose that A is bounded from above. Prove that $-A$ is bounded from below and that

$$\inf(-A) = -\sup A$$

Solution:

We start by showing that $-\sup A$ is a lower bound for $-A$. Let $a \in A$ be arbitrary. Then $a \leq \sup A$ and thus $-\sup A \leq -a$. Since a was arbitrarily chosen this means that $-\sup A \leq -a$ for all $a \in A$ i.e. $-\sup A$ is a lower bound for $-A$. This especially proves that $-A$ is bounded below.

In order to prove that $\inf(-A) = -\sup A$ we need to show two things:

(i) $-\sup A$ is a lower bound for $-A$, and

(ii) For any $\epsilon > 0$, $-\sup A + \epsilon$ is not a lower bound for $-A$.

We just proved (i) above, so all that remains to do is to show (ii). Let $\epsilon > 0$ be arbitrary. Since $\sup A$ is the least upper bound for A, $\sup A - \epsilon < \sup A$ is not an upper bound for A i.e. there exists an $a \in A$ with $\sup A - \epsilon < a$. Then $-\sup A + \epsilon > -a$ which means that $-\sup A + \epsilon$ is not a lower bound for $-A$. This proves (ii) and therefore that $\inf(-A) = -\sup A$.

Q.E.D.

Remark:

By substituting $-A$ for A we obtain the result that if A is bounded below then $\inf(A) = -\sup(-A)$. This especially shows that the infimum exists. So the completeness property of $\mathbb{R}$ also implies that any subset of $\mathbb{R}$ that is bounded below has an infimum in $\mathbb{R}$.

7. Let $x \in \mathbb{R}$ be irrational and $r \in \mathbb{Q}$, $r \neq 0$, be rational. Prove that $x + r$ and $x \cdot r$ are irrational.

   Solution:

   We prove both statements via proof by contradiction, using the fact that the set $\mathbb{Q}$ of all rational numbers is a field. We will prove all statements directly from the field axioms.

   $\underline{x + r}$: Assume that $x + r$ is rational. Since r is rational, its additive inverse $-r$ is rational. Since $\mathbb{Q}$ is closed under addition, $(x + r) + (-r) \overset{\text{assoc.}}{=} x + (r + (-r)) = x + 0 = x$ is rational, which is a contradiction. Thus the assumption is wrong and $x + r$ is irrational. (Note that the statement is trivially valid in the case $r = 0$ i.e. the condition $r \neq 0$ is not needed for this part of the problem.)

   Q.E.D.

   $\underline{x \cdot r}$: Assume that $x \cdot r$ is rational. Since $r \neq 0$ is rational, its multiplicative inverse $1/r$ is rational. Since $\mathbb{Q}$ is closed under multiplication, $(x \cdot r) \cdot (1/r) \overset{\text{assoc.}}{=} x \cdot (r \cdot (1/r)) = x \cdot 1 = x$ is rational, which is a contradiction. Thus the assumption is wrong and $x \cdot r$ is irrational. (Note that the condition $r \neq 0$ is essential for this part of the problem: it can be shown (using field axioms only, see Theorem 2.1.2 of Bartle and Sherbert) that $x \cdot 0 = 0$ and thus rational for all $x \in \mathbb{R}$.)

   Q.E.D.