Tutorial 10 (March 22nd / 2017)

1. Let Y be normally distributed with mean μ and variance $\sigma^2 < 1/6$. Find $\mathbb{E}(e^{3Y^2})$.

Solution:

By definition, $\mathbb{E}(e^{3Y^2}) = \int_{-\infty}^{\infty} e^{3y^2} f(y)\,dy$

$$= \int_{-\infty}^{\infty} e^{3y^2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy$$

, we will simplify notation by using $\exp(y) = e^y$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(3y^2)\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right) dy$$

Expanding the second term and multiplying by the first term:

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{y^2}{2\sigma^2} + 3y^2 + \frac{2\mu y}{2\sigma^2} - \frac{\mu^2}{2\sigma^2}\right) dy$$

Simplifying and factoring the terms that do not contain y:

$$= \exp\left(\frac{-\mu^2}{2\sigma^2}\right) \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-y^2\left(\frac{1}{2\sigma^2} - 3\right) + 2y\left(\frac{\mu}{2\sigma^2}\right)\right) dy$$

$$= \exp\left(\frac{-\mu^2}{2\sigma^2}\right) \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} \exp\left(-y^2\left(\frac{1-6\sigma^2}{2\sigma^2}\right) + 2y\left(\frac{\mu}{2\sigma^2}\right)\right) dy$$

How to solve the integral? Examine $\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)$

We will complete the square to try and obtain another normal pdf. $= \exp\left(\left(\frac{1}{2\sigma^2}\right)\left(-y^2 + 2y\mu - \mu^2\right)\right)$

$$\Rightarrow \exp\left(-y^2\left(\frac{1-6\sigma^2}{2\sigma^2}\right) + 2y\left(\frac{\mu}{2\sigma^2}\right)\right)$$

$$= \exp\left(\left(\frac{1-6\sigma^2}{2\sigma^2}\right)\left\{-y^2 + 2y\left(\frac{\mu}{2\sigma^2} \cdot \frac{2\sigma^2}{1-6\sigma^2}\right)\right\}\right)$$

$$= \exp\left(\left(\frac{1-6\sigma^2}{2\sigma^2}\right)\left\{-y^2 + 2y\left(\frac{\mu}{1-6\sigma^2}\right)\right\}\right)$$

completing the square

$$= \exp\left(+\left(\frac{1-6\sigma^2}{2\sigma^2}\right)\left(\frac{\mu}{1-6\sigma^2}\right)^2\right) \exp\left(\left(\frac{1-6\sigma^2}{2\sigma^2}\right)\left\{-y^2 + 2y\left(\frac{\mu}{1-6\sigma^2}\right) - \left(\frac{\mu}{1-6\sigma^2}\right)^2\right\}\right)$$

$$= \exp\left(\left(\frac{1-6\sigma^2}{2\sigma^2}\right)\left(\frac{\mu}{1-6\sigma^2}\right)^2\right)\exp\left(1\Big/\left(\frac{2\sigma^2}{1-6\sigma^2}\right)\left\{-\left(y-\left(\frac{\mu}{1-6\sigma^2}\right)\right)^2\right\}\right)$$

So back to the integration:

$$= \exp\left(-\frac{\mu^2}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{\mu^2}{2\sigma^2(1-6\sigma^2)}\right)\underbrace{\int_{-\infty}^{\infty} e^{-\frac{\left(y-\left(\frac{\mu}{1-\sigma^2}\right)\right)^2}{\left(\frac{2\sigma^2}{1-6\sigma^2}\right)}}\,dy}_{\text{Look familiar?}}$$

its a pdf of a $N\left(\frac{\mu}{1-\sigma^2}, \frac{\sigma^2}{1-6\sigma^2}\right)$

missing $\frac{1}{\sqrt{2\pi\left(\frac{\sigma^2}{1-6\sigma^2}\right)}}$

$$= \exp\left(-\frac{\mu^2}{2\sigma^2}\right)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{\mu^2}{2\sigma^2(1-6\sigma^2)}\right)\times$$

$$\sqrt{2\pi\left(\frac{\sigma^2}{1-6\sigma^2}\right)}\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\left(\frac{\sigma^2}{1-6\sigma^2}\right)}} e^{-\frac{\left(y-\left(\frac{\mu}{1-\sigma^2}\right)\right)^2}{\left(\frac{2\sigma^2}{1-6\sigma^2}\right)}}\,dy}_{=1}$$

$=1$ since it is an integration of a pdf

$$= \exp\left(\frac{-\mu^2(1-6\sigma^2)+\mu^2}{2\sigma^2(1-6\sigma^2)}\right)\sqrt{\frac{1}{1-6\sigma^2}}$$

$$= \sqrt{\frac{1}{1-6\sigma^2}}\exp\left(\frac{3\mu^2}{1-6\sigma^2}\right)$$

2. Let X be continuous random variable with probability density function:

$$f(x;\mu,s) = \begin{cases} \frac{1}{2s}\left[1 + \cos\left(\left(\frac{x-\mu}{s}\right)\pi\right)\right], & \mu - s \le x \le \mu + s \\ 0, & \text{elsewhere.} \end{cases}$$

$\mu \in \mathbb{R}$
$s > 0$

Show that $f(x;\mu,s)$ is symmetric about μ.

Solution:

If $f(x;\mu,s)$ is symmetric about μ then for all $\delta > 0$

$$f(\mu+\delta\ ;\mu,s) = f(\mu-\delta\ ;\mu,s).$$

Clearly, if $\delta > s$ then $\mu+\delta > \mu+s \Rightarrow f(\mu+\delta) = 0$

$\mu-\delta < \mu-s \Rightarrow f(\mu-\delta) = 0$.

So, we only need to consider $0 < \delta < s$.

Observe in the density, that the choice of x does not affect the scaling of $\frac{1}{2s}$ or the shift of $+1$ so we can consider $\cos\left(\frac{x-\mu}{s}\pi\right)$ by itself.

Case of $x = \mu - \delta$

$$\cos\left(\frac{(\mu-\delta)-\mu}{s}\pi\right) = \cos\left(-\frac{\delta}{s}\pi\right)$$

Case of $x = \mu + \delta$

$$\cos\left(\frac{(\mu+\delta)-\mu}{s}\pi\right) = \cos\left(\frac{\delta}{s}\pi\right)$$

But $\cos(-x) = \cos(x)$ for all $x \in \mathbb{R}$ so

$\cos\left(\frac{\delta}{s}\pi\right) = \cos\left(-\frac{\delta}{s}\pi\right)$. Therefore, we have proven that f is symmetric about μ.

3. (Exercise 5.32) : Suppose that the random variables Y_1 and Y_2 have joint probability density function, $f(y_1, y_2)$, given by

$$f(y_1,y_2) = \begin{cases} 6y_1^2 y_2 , & 0 \le y_1 \le y_2 , \ y_1 + y_2 \le 2 \\ 0 , & \text{elsewhere} \end{cases}$$

a) Show that the marginal density of Y_1 is a beta density with $\alpha = 3$ and $\beta = 2$.

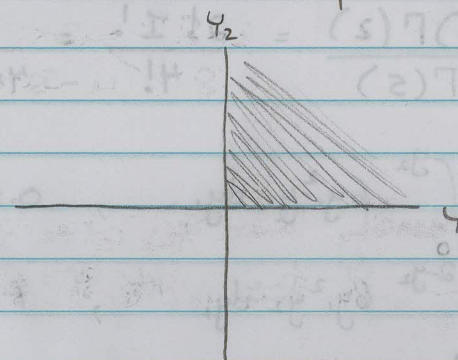
b) Derive the marginal density of Y_2.

c) Derive the conditional density of Y_2 given $Y_1 = y_1$.

d) Find $\mathbb{P}(Y_2 < 1.1 \mid Y_1 = 0.60)$

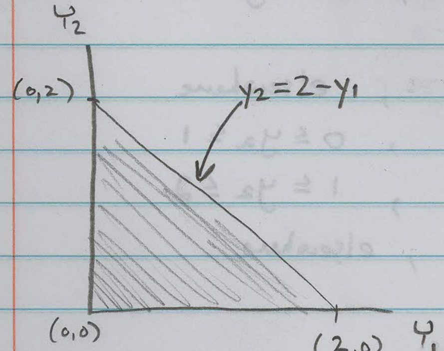
Solution:

First, we will draw the space of (Y_1, Y_2) where $f(y_1, y_2)$ is nonzero.



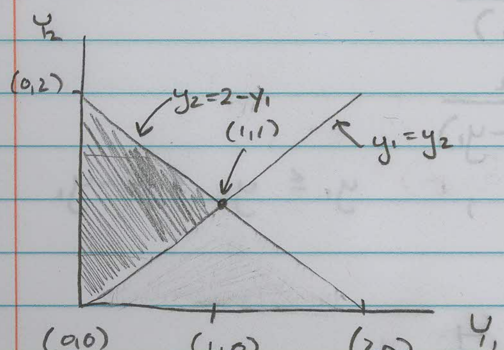
$0 \le y_1 \le y_2$

$\Rightarrow$ we are in the shaded quadrant as $0 \le y_1$ and $0 \le y_2$



$y_1 + y_2 \le 2$

$\Rightarrow$ we are in the shaded region



Thus, the shaded region is the region defined by $0 \le y_1 \le y_2$, $y_1 + y_2 \le 2$ where $f(y_1, y_2) \neq 0$.

a) $f_1(y_1) = \int_{y_1}^{2-y_1} 6y_1^2 y_2 \, dy_2$

$= 6y_1^2 \left\{ \int_{y_1}^{2-y_1} y_2 \, dy_2 \right\} = 6y_1^2 \left\{ \frac{1}{2} y_2^2 \Big|_{y_1}^{2-y_1} \right\}$

$= 3y_1^2 \left\{ (2-y_1)^2 - y_1^2 \right\} = 3y_1^2 \left\{ 4 - 4y_1 + y_1^2 - y_1^2 \right\}$

$= 3y_1^2 \left(4(1-y_1) \right)$

$= 12y_1^2(1-y_1) \quad , \quad 0 \le y_1 \le 1$

A Beta(3,2) pdf has the form $f(y) = \begin{cases} \dfrac{y^2(1-y)}{B(3,2)} , & 0 \le y_1 \le 1 \\ 0 , & \text{elsewhere} \end{cases}$

$B(3,2) = \dfrac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \dfrac{2!\,1!}{4!} = \dfrac{2}{4 \times 3 \times 2} = \dfrac{1}{12}$. ■

b) $f_2(y_2) = \begin{cases} \int_0^{y_2} 6y_1^2 y_2 \, dy_1 , & 0 \le y_2 \le 1 \\ \int_0^{2-y_2} 6y_1^2 y_2 \, dy_1 , & 1 \le y_2 \le 2 \\ 0 , & \text{elsewhere} \end{cases}$

$= \begin{cases} 2y_2^4 , & 0 \le y_2 \le 1 \\ 2y_2(2-y_2)^3 , & 1 \le y_2 \le 2 \\ 0 , & \text{elsewhere.} \end{cases}$

c) $f(y_2 | y_1) = \dfrac{f(y_1, y_2)}{f_1(y_1)}$

$= \dfrac{6y_1^2 y_2}{12y_1^2(1-y_1)}$

$= \dfrac{y_2}{2(1-y_1)} \quad , \quad y_1 \le y_2 \le 2-y_1$

Check: $\int_{y_1}^{2-y_1} f(y_2|y_1) \, dy_2 = 1$

d) $f(y_2 | y_1 = 0.60) = \dfrac{y_2}{2(0.4)} = \dfrac{y_2}{0.8}, \quad 0.60 \le y_2 \le 1.40$

$$\Rightarrow \mathbb{P}(Y_2 < 1.1 \mid Y_1 = 0.60) = \int_{0.60}^{1.1} \frac{y_2}{0.8}\, dy_2$$

$$= \frac{1}{0.8}\left(\frac{1}{2} y_2^2 \Big|_{0.60}^{1.1} \right) = \frac{1}{0.8}\left(\frac{1}{2}\left((1.1)^2 - (0.6)^2\right)\right)$$

$$= 0.53125$$