Math 201 (Winter)
Differential Equations

Solution #4

1. (Section 4.6, Problem 7) Find a general solution to

$$y'' + 4y' + 4y = e^{-2t} \log t$$

using variation of parameters.

Solution: The auxiliary equation of the associated homogeneous equation is

$$0 = r^2 + 4r + 4 = (r+2)^2,$$

so that two linearly independent solutions of $y'' + 4y' + 4y = 0$ are

$$y_1(t) = e^{-2t} \qquad \text{and} \qquad y_2(t) = te^{-2t}$$

and thus

$$y_1'(t) = -2e^{-2t} \qquad \text{and} \qquad y_2'(t) = e^{-2t} - 2te^{-2t} = (1-2t)e^{-2t}.$$

Solve

$$e^{-2t}v_1'(t) + te^{-2t}v_2'(t) = 0 \qquad \text{and} \qquad -2e^{-2t}v_1'(t) + (1-2t)e^{-2t}v_2'(t) = e^{-2t}\log t$$

for $v_1'(t)$ and $v_2'(t)$. Multiply the first equation with $2e^{2t}$ and the second one with e^{2t}, and obtain

$$2v_1'(t) + 2tv_2'(t) = 0 \qquad \text{and} \qquad -2v_1'(t) + (1-2t)v_2'(t) = \log t.$$

Adding both equations yields $v_2'(t) = \log t$ and consequently $v_1'(t) = -t\log t$. Integrating, we obtain

$$v_2(t) = t\log t - t \qquad \text{and}$$
$$v_1(t) = -\int t\log t\,dt = -\frac{t^2}{2}\log t + \frac{1}{2}\int t\,dt = -\frac{t^2}{2}\log t + \frac{1}{4}t^2.$$

Thus, a particular solution $y_p(t)$ is given by

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) =$$
$$\left(-\frac{t^2}{2}\log t + \frac{1}{4}t^2\right)e^{-2t} + (t^2\log t - t^2)e^{-2t} = \frac{2\log t - 3}{4}t^2e^{-2t}.$$

A general solution is therefore of the form

$$y(t) = \frac{2\log t - 3}{4}t^2e^{-2t} + c_1\,e^{-2t} + c_2\,te^{-2t}.$$

2. (Section 4.6, Problem 12) Find a general solution to

$$y'' + y = \tan t + e^{3t} - 1.$$

Solution: Two linearly independent solutions to $y'' + y = 0$ are $y_1(t) = \cos t$ and $y_2(t) = \sin t$. In class, it was show that

$$y_{p,1}(t) = -(\cos t)\log(\sec t + \tan t)$$

is a particular solution to $y''+y = \tan t$. The method of undertermined coeffiencents yields the particular solutions $y_{p,2}(t) = \frac{1}{10}e^{3t}$ and $y_{p,3}(t) = 1$ to $y'' + y = e^{3t}$ and $y'' + y = 1$. By the superposition principle, a particular solution of $y'' + y = \tan t + e^{3t} - 1$ is thus given by

$$y_p(t) = y_{p,1}(t) + y_{p,2}(t) - y_{p,3}(t) = -(\cos t)\log(\sec t + \tan t) + \frac{1}{10}e^{3t} - 1$$

and a general solution is

$$y(t) = -(\cos t)\log(\sec t + \tan t) + \frac{1}{10}e^{3t} - 1 + c_1\,\cos t + c_2\,\sin t.$$

3. (Section 4.6, Problem 14) Find a general solution to

$$y''(\theta) + y(\theta) = \sec^3\theta.$$

Solution: Again, two linearly independent solutions to $y''(\theta) + y(\theta) = 0$ are $y_1(\theta) = \cos\theta$ and $y_2(\theta) = \sin\theta$. To find a particular solution, use variation of parameters, i.e., solve

$$(\cos\theta)v_1'(\theta) + (\sin\theta)v_2'(\theta) = 0 \qquad \text{and} \qquad (-\sin\theta)v_1'(\theta) + (\cos\theta)v_2'(\theta) = \sec^3\theta$$

for $v_1'(\theta)$ and $v_2'(\theta)$. Multiplying the first equation with $\sin\theta$ and the second one with $\cos\theta$ and then adding them, we obtain $v_2'(\theta) = \sec^2\theta$. Plugging into the first equation and then solving for $v_1'(\theta)$ yields $v_1'(\theta) = -(\tan\theta)\sec^2\theta$. Integrating, we obtain that

$$v_1(\theta) = -\frac{\tan^2\theta}{2} \qquad \text{and} \qquad v_2(\theta) = \tan\theta.$$

A particular solution is thus given by

$$y_p(\theta) = v_1(\theta)y_1(\theta) + v_2(\theta)y_2(\theta) = -\frac{(\tan\theta)\sin\theta}{2} + (\tan\theta)\sin\theta = \frac{(\tan\theta)\sin\theta}{2},$$

and a general solution is of the form

$$y(\theta) = \frac{(\tan\theta)\sin\theta}{2} + c_1\,\cos\theta + c_2\,\sin\theta.$$

4. (Section 4.6, Problem 18) Find a general solution to

$$y'' - 6y' + 9y = t^{-3}e^{3t}.$$

Solution: The auxiliar equation of the associated homogeneous equation is

$$0 = r^2 - 6r + 9 = (r-3)^3,$$

so that

$$y_1(t) = e^{3t} \qquad \text{and} \qquad y_2(t) = te^{3t}$$

are two linearly independent solutions of $y'' - 6y' + 9 = 0$. Apply variation of parameters, i.e., solve

$$e^{3t}v_1'(t) + te^{3t}v_2'(t) = 0 \qquad \text{and} \qquad 3e^{3t}v_1'(t) + (3t+1)e^{3t}v_2'(t) = t^{-3}e^{3t}.$$

Multiply the first equation with $-3e^{-3t}$, the second one with e^{-3t}, and add them: we obtain $v_2'(t) = t^{-3}$ and, consequently, $v_1'(t) = -t^{-2}$, so that

$$v_1(t) = t^{-1} \qquad \text{and} \qquad v_2(t) = -\frac{1}{2}t^{-2}.$$

It follows that

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = t^{-1}e^{3t} - \frac{1}{2}t^{-2}te^{3t} = \frac{1}{2t}e^{3t}$$

is a particular solution, and a general solution is

$$y(t) = \frac{1}{2t}e^{3t} + c_1\,e^{3t} + c_2\,te^{3t}.$$

5. (Section 4.7, Problem 16) Find a general solution for $t < 0$ to

$$t^2y''(t) - 3ty'(t) + 6y(t) = 0.$$

Solution: First, find a general solution for $t > 0$ to

$$t^2z''(t) - 3tz'(t) + 6z(t) = 0.$$

Its characteristic equation is

$$0 = r^2 - 4r + 6 = (r-2)^2 + 2$$

which has the complex conjugate roots $r_{1,2} = 2 \pm \sqrt{2}\,i$. Two linearly independent solutions for $t > 0$ are thus

$$\tilde{y}_1(t) = t^2\cos(\sqrt{2}\log t) \qquad \text{and} \qquad \tilde{y}_2(t) = t^2\sin(\sqrt{2}\log t).$$

Replacing t by $-t$, we obtain two linearly independent solutions for $t < 0$, namely

$$\tilde{y}_1(t) = t^2 \cos(\sqrt{2}\log(-t)) \qquad \text{and} \qquad \tilde{y}_2(t) = t^2 \sin(\sqrt{2}\log(-t)),$$

and a general solution is of the form

$$y(t) = c_1\, t^2 \cos(\sqrt{2}\log(-t)) + c_2\, t^2 \sin(\sqrt{2}\log(-t)).$$

6. (Section 4.7, Problem 20) Solve the initial value problem

$$t^2 y''(t) + 7ty'(t) + 5y(t) = 0; \qquad y(1) = -1, \quad y'(1) = 13.$$

Solution: The characteristic equation is

$$0 = r^2 + 6r + 5 = (r+3)^2 - 4$$

which has the roots $r_1 = -1$ and $r_2 = -5$. Two linearly independent solutions to $t^2 y''(t) + 7ty'(t) + 5y(t) = 0$ for $t > 0$ are therefore

$$y_1(t) = t^{-1} \qquad \text{and} \qquad y_2(t) = t^{-5}$$

and a general solution is

$$y(t) = c_1\, t^{-1} + c_2\, t^{-5}$$

so that

$$y'(t) = -c_1\, t^{-2} - 5c_2\, t^{-6}.$$

The initial conditions yield

$$-1 = y(1) = c_1 + c_2 \qquad \text{and} \qquad 13 = y'(1) = -c_1 - 5c_2.$$

Adding both equations yields $12 = -4c_2$, i.e., $c_2 = -3$ and consequently $c_1 = 2$. The solution is thus

$$y(t) = 2t^{-1} - 3t^{-5}.$$

7. Use variation of parameters to find a general solution for $t > 0$ to

$$t^2 y'' - 4ty' + 6y = t^3 + 1$$

given that $y_1 = t^2$ and $y_2 = t^3$ are linearly independent solutions.

Solution: Bring the equation in standard form

$$y'' - \frac{4}{t}y' + \frac{6}{t^2}y = t + \frac{1}{t^2}.$$

Solve

$$t^2 v_1'(t) + t^3 v_2'(t) = 0 \qquad \text{and} \qquad 2tv_1'(t) + 3t^2 v_2'(t) = t + \frac{1}{t^2}$$

for $v_1'(t)$ and $v_2'(t)$. Multiply the first equation with -2, the second one with t, and add: we obtain $v_2'(t) = \frac{1}{t} + \frac{1}{t^4}$ and, consequently, $v_1'(t) = -1 - \frac{1}{t^3}$. Integrating, we obtain

$$v_1(t) = -t + \frac{2t^2}{}\text{and} \qquad v_2'(t) = \log t - \frac{1}{3t^3},$$

so that

$$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) = -t^3 + \frac{1}{2} + t^3 \log t - \frac{1}{3} = t^3(\log t - 1) + \frac{1}{6}$$

is a particular solution. A general solution for $t > 0$ is thus

$$y(t) = t^3(\log t - 1) + \frac{1}{6} + c_1\, t^2 + c_2\, t^3.$$

8. (Section 4.7, Problem 48) For $t > 0$, the function $f(t) = e^t$ is a non-trivial solution to

$$ty'' + (1 - 2t)y' + (t - 1)y = 0.$$

Find a second, linearly independent solution using reduction of order.

Solution: Bring the equation in standard form

$$y'' + \left(\frac{1}{t} - 2\right) y' + \left(1 - \frac{1}{t}\right) y = 0,$$

so that $p(t) = \frac{1}{t} - 2$ and therefore

$$e^{-\int p(t)\, dt} = e^{-\log t + 2t} = \frac{1}{t} e^{2t}.$$

From reduction of order, it then follows that

$$g(t) = e^t \int \frac{1}{t}\, dt = e^t \log t$$

is a second linearly independent solution for $t > 0$.