(1) [a.] Total number of ways of distributing the balls is n^{2n}.

P(at least one box contains three or more balls)

$= 1 -$ P(every box contains at most two balls)

$= 1 -$ P(every box contains exactly two balls)

$$= 1 - \binom{2n}{2\ 2\ \ldots 2} \Big/ n^{2n}$$

$$= 1 - \frac{(2n)!}{(2n^2)^n}. \quad \blacksquare$$

[b] Let $A_i = \{ i\text{-th box is empty} \}$.

Then $P(A_1) = (n-1)^{2n} / n^{2n}$, and

$P(A_1 \cap A_2) = (n-2)^{2n} / n^{2n}$. Hence

$$P(A_2 | A_1) = \left(\frac{n-2}{n-1}\right)^{2n}. \quad \blacksquare$$

(2) [a] The event $\{X_1 = n\}$ occurs if
the first n tosses result in a run of heads of length n
which is then terminated by a tail at the
$(n+1)$-st toss, or if

the first n tosses " " " " " tails of length n
" " " " " ~~~~~~ a head at the
$(n+1)$-st toss. Hence,

$$P(X_1 = n) = p^n (1-p) + (1-p)^n p, \qquad n = 1, 2, 3, \ldots .$$

[b] Let $Y_{n,i} = 1_{\{\text{the } i\text{-th letter and the } n\text{-th letter are part of the same run}\}}$,

for $i, n = 1, 2, 3, \ldots$.

Then $X_n = \sum_{i=1}^{\infty} Y_{n,i}$. Since $Y_{n,i}$ are nonnegative,

$$E(X_n) = \sum_{i=1}^{\infty} E(Y_{n,i}) = \sum_{i=1}^{\infty} P(Y_{n,i} = 1).$$

Now $P(Y_{n,i} = 1) = p^{|n-i|+1} + (1-p)^{|n-i|+1}$.

Hence, $E(X_n) = p^n + p^{n-1} + \ldots + p + p^2 + p^3 + \ldots$

$+ q^n + q^{n-1} + \ldots + q + q^2 + q^3 + \ldots$

(where $q = 1-p$)

$$= \frac{p - p^{n+1} + p^2}{q} + \frac{q - q^{n+1} + q^2}{p}$$

$$= \frac{p^2 - p^{n+2} + p^3 + q^2 - q^{n+2} + q^3}{pq}$$

$\blacksquare$

(3) [a]

$\{X=2\} = \{10 \text{ tosses result in } 6 \text{ heads and } 4 \text{ tails}\}$.

Since the number of heads in 10 tosses $\sim \text{Bin}(10, 1/2)$,

$$P(X=2) = \binom{10}{6}\left(\frac{1}{2}\right)^{10} = \frac{210}{2^{10}} = \frac{105}{512}.$$

[b] Let p = prob. of winning a single game.

Then $p = P(X \geq 8) = P(X=8) + P(X=10)$

$$= \left[\binom{10}{9} + \binom{10}{10}\right]\frac{1}{2^{10}}$$

$$= 11/1024.$$

Now,

$$Y+4 \sim \text{NBin}(4, p)$$

$$\Rightarrow E(Y) = \frac{4}{p} - 4 = 4\left(\frac{1024}{11} - 1\right)$$

$$= \frac{4 \times 1013}{11} = \frac{4052}{11}. \blacksquare$$

(4) [a] Assume that $X \sim Poi(\lambda)$.

To find the value of λ, note that the probability that a manuscript is free of errors is $P(X=0) = e^{-\lambda}$.

Hence, among 100 manuscripts, expected number of manuscripts that are free of errors is

$100 e^{-\lambda}$.

$\Rightarrow \quad 100 e^{-\lambda} = 50 \quad \Rightarrow \lambda = \ln 2$.

$\Rightarrow V(X) = \ln 2$.

[b] Expected fine is

$$E(2^X) = \sum_{k=0}^{\infty} 2^k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{2\lambda}$$

$$= e^{\lambda} = 2.$$ ■

(5) Let $p = P(X=6)$.

Then $P(X=i) = \frac{1-p}{5}$, for $i=1,\dots,5$.

$\Rightarrow E(X) = \left(\frac{1-p}{5}\right)\sum_{i=1}^{5} i + 6p = 4$

$\Rightarrow 3(1-p) + 6p = 4$

$\Rightarrow 3 - 3p + 6p = 4$

$\Rightarrow p = 1/3$. ∎