THE STOKES THEOREM

TSOGTGEREL GANTUMUR

ABSTRACT. Integration on manifolds, differential forms, and the Stokes theorem.

Contents

1. Line integrals and 1-forms

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a continuously differentiable function, representing a curve L in $\mathbb{R}^n$. Suppose that $u : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function. Then $f(t) = u(\gamma(t))$ is a function in $[a, b]$, and by the fundamental theorem of calculus, we have

$$u(\gamma(b)) - u(\gamma(a)) = f(b) - f(a) = \int_a^b f'(t)\mathrm{d}t = \int_a^b Du(\gamma(t))\gamma'(t)\mathrm{d}t. \tag{1}$$

This quantity does not depend on the curve L, let alone the parameterization γ, as long as the endpoints $\gamma(a)$ and $\gamma(b)$ stay fixed. We are going to interpret the integral in the right hand side as the integral of Du over the curve L, and attempt to generalize it to a class of objects broader than the derivatives of scalar functions. We make the following observations.

- The integral must depend not only on the curve L as a subset of $\mathbb{R}^n$, but also on a directionality property of the curve, since switching the endpoints $\gamma(a)$ and $\gamma(b)$ would flip the sign of (1).
- The derivative Du is a row vector at each point $x \in \mathbb{R}^n$, that is, $Du(x) \in \mathbb{R}^{n*}$ for $x \in \mathbb{R}^n$, or $Du : \mathbb{R}^n \to \mathbb{R}^{n*}$. Thus it might be possible to generalize (1) from integration of Du to that of $\alpha : \mathbb{R}^n \to \mathbb{R}^{n*}$.

Let $\alpha : \mathbb{R}^n \to \mathbb{R}^{n*}$ be smooth, and define

$$\int_\gamma \alpha = \int_a^b \alpha(\gamma(t))\gamma'(t)\mathrm{d}t. \tag{2}$$

The right hand side makes sense, because for each $t \in [a, b]$, we have $\alpha(\gamma(t)) \in \mathbb{R}^{n*}$ and $\gamma'(t) \in \mathbb{R}^n$, and so $\alpha(\gamma(t))\gamma'(t) \in \mathbb{R}$. Objects such as α are called differential 1-forms.

Now let us check if the integral (2) depends on the parameterization γ. Suppose that $\phi : [c, d] \to [a, b]$ is a continuously differentiable function, with $\phi([c, d]) = [a, b]$, $\phi(c) = a$ and $\phi(d) = b$. Then by applying the change of variables formula

$$\int_{\phi(c)}^{\phi(d)} f = \int_c^d (f \circ \phi)\phi', \tag{3}$$

Date: 2016/12/14 02:05 (EST). Incomplete draft.

to $f(t) = \alpha(\gamma(t))\gamma'(t)$, we have

$$\int_a^b \alpha(\gamma(t))\gamma'(t)\mathrm{d}t = \int_c^d \alpha(\gamma(\phi(s)))\gamma'(\phi(s))\phi'(s)\mathrm{d}s = \int_c^d \alpha(\eta(s))\eta'(s)\mathrm{d}s, \tag{4}$$

where $\eta(s) = \gamma(\phi(s))$ is the new parameterization. Hence the integral does not depend on the parameterization, as long as the endpoints of the curve are kept fixed. On the other hand, if $\phi(c) = b$ and $\phi(d) = a$, then we have

$$\int_a^b \alpha(\gamma(t))\gamma'(t)\mathrm{d}t = -\int_b^a \alpha(\gamma(t))\gamma'(t)\mathrm{d}t = -\int_c^d \alpha(\eta(s))\eta'(s)\mathrm{d}s, \tag{5}$$

which tells us that if the endpoints of the curve get switched under reparameterization, then the sign of the integral flips. Therefore, the integral (2) depends only on those aspects of the parameterization γ that specify a certain "directionality" property of the underlying curve. This "directionality" property is called orientation.

Intuitively, and in practice, an *oriented curve* is a curve given by some concrete parameterization γ, with the understanding that one can freely replace it by any other parameterization $\eta = \gamma \circ \phi$, as long as $\phi' > 0$. To define it precisely, we need to spend a bit more effort. Let $L \subset \mathbb{R}^n$ be a curve, admitting a parameterization $\gamma : [a, b] \to \mathbb{R}^n$ which is a continuously differentiable function with $\gamma' \neq 0$ in (a, b). Suppose that P is the set of all such parameterizations of L. Then for any two parameterizations $\gamma_1 : [a_1, b_1] \to \mathbb{R}^n$ and $\gamma_2 : [a_2, b_2] \to \mathbb{R}^n$ from P, there exists a continuously differentiable function $\phi : [a_2, b_2] \to [a_1, b_1]$ with $\phi' \neq 0$ in (a_2, b_2), such that $\gamma_2 = \gamma_1 \circ \phi$. This gives a way to decompose P into two mutually disjoint classes P_1 and P_2: If $\phi' > 0$, then γ_1 and γ_2 are in the same class, and if $\phi' < 0$, then γ_1 and γ_2 are in different classes. The curve L, together with a choice of P_1 or P_2, is called an *oriented curve*. So the classes P_1 and P_2 are the possible *orientations* of the curve L. As mentioned before, in practice, we specify an orientated curve simply by giving a concrete parameterization.

Example 1.1. Let $\gamma : [0, \pi] \to \mathbb{R}^2$ be given by $\gamma(t) = (\cos t, \sin t)$, and let $\eta : [0, \sqrt{\pi}] \to \mathbb{R}^2$ be given by $\eta(s) = (\cos s^2, \sin s^2)$. Then we have $\eta = \gamma \circ \phi$ with $\phi(s) = s^2$ for $s \in [0, \sqrt{\pi}]$, and since $\phi' > 0$ in $(0, \sqrt{\pi})$, these two parameterizations define the same oriented curve. On the other hand, $\xi(\tau) = (-\cos \tau, \sin \tau)$, $\tau \in [0, \pi]$, gives the same curve as γ and η, but ξ is in the orientation opposite to that of γ and η, and thus as an *oriented* curve, ξ is different than γ.

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be an open set. Then a *(differential) 1-form* on Ω is a continuously differentiable function $\alpha : \Omega \to \mathbb{R}^{n*}$.

For convenience, we restate the definition of integration of 1-forms over oriented curves.

Definition 1.3. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let α be a 1-form on Ω. Let $\gamma : [a, b] \to \Omega$ be an oriented curve. Then we define

$$\int_\gamma \alpha = \int_a^b \alpha(\gamma(t))\gamma'(t)\mathrm{d}t. \tag{6}$$

We have already established that the integral in the right hand side does not depend on the parameterization γ, as long as the orientation is kept fixed.

Example 1.4. Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^{2*}$ be given by $\alpha(x, y) = (-y, x)$, and let $\gamma(t) = (\cos t, \sin t)$, $t \in [0, \pi]$. Then we have

$$\int_\gamma \alpha = \int_0^\pi \begin{pmatrix} -\sin t & \cos t \end{pmatrix} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \mathrm{d}t = \int_0^\pi \mathrm{d}t = \pi. \tag{7}$$

A convenient way to generate 1-forms is to differentiate scalar functions. That is, if $\Omega \subset \mathbb{R}^n$ is an open set, and $u : \Omega \to \mathbb{R}$ is a smooth function, then Du is a 1-form on Ω. In the following, we will use the notation

$$\mathrm{d}u \equiv Du. \tag{8}$$

For example, if $u(x,y) = x^2 + y$, then $\mathrm{d}u(x,y) = (2x,1) \in \mathbb{R}^{2*}$ for $(x,y) \in \mathbb{R}^2$.

Remark 1.5. If $u(x,y) = x$, then $\mathrm{d}u = (1,0)$. In other words, we have $\mathrm{d}x = (1,0)$. If $u(x,y) = y$, then $\mathrm{d}u = (0,1)$, or $\mathrm{d}y = (0,1)$. Hence any 1-form $\alpha(x,y) = (\alpha_1(x,y), \alpha_2(x,y))$ can be written as

$$\alpha(x,y) = \alpha_1(x,y)\mathrm{d}x + \alpha_2(x,y)\mathrm{d}y. \tag{9}$$

For example, if $u(x,y) = x^2+y$, then $\mathrm{d}u(x,y) = 2x\mathrm{d}x+\mathrm{d}y$. In general, any 1-form $\alpha : \Omega \to \mathbb{R}^{n*}$ with components $\alpha(x) = (\alpha_1(x), \ldots, \alpha_n(x)) \in \mathbb{R}^{n*}$ can be written as

$$\alpha(x) = \alpha_1(x)\mathrm{d}x_1 + \ldots + \alpha_n(x)\mathrm{d}x_n. \tag{10}$$

Example 1.6. Let $\alpha = x\mathrm{d}x + (x+y)\mathrm{d}y$, and let $\gamma(t) = (t, 2t)$, $t \in [0,1]$. Then we have

$$\int_\gamma x\mathrm{d}x + (x+y)\mathrm{d}y = \int_0^1 (t + (t+2t)\cdot 2)\mathrm{d}t = \int_0^1 7t\,\mathrm{d}t = \frac{7}{2}. \tag{11}$$

Definition 1.7. Let $\Omega \subset \mathbb{R}^n$ be an open set. Then a *vector field* on Ω is a continuously differentiable function $V : \Omega \to \mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ be open sets, and let $\Psi : \Omega \to U$ be a differentiable map. Then any oriented curve $\gamma : [a,b] \to \Omega$ gets sent to an oriented curve $\tilde{\gamma} = \Psi \circ \gamma$ in U. By the chain rule, the velocity vectors transform as

$$\tilde{\gamma}'(t) = D\Psi(\gamma(t))\gamma'(t). \tag{12}$$

Since any vector is a velocity vector of some curve, we are led to the transformation rule

$$\tilde{V}(\tilde{x}) = D\Psi(x)V(x), \qquad \text{or in components,} \qquad \tilde{V}_i = \sum_{k=1}^n \frac{\partial \tilde{x}_i}{\partial x_k} V_k, \quad (i = 1, \ldots, m), \tag{13}$$

for vector fields, where $V : \Omega \to \mathbb{R}^n$ and $\tilde{V} : U \to \mathbb{R}^m$ are vector fields, $x \in \Omega$ is arbitrary, and $\tilde{x} = \Psi(x) \in U$. The vector field $\tilde{V}$ is called the *push-forward* of V under the mapping Ψ, and denoted by $\tilde{V} = \Psi_* V$. In particular, taking $V = e_j \in \mathbb{R}^n$, the j-th standard basis vector in $\mathbb{R}^n$, we get $V_k = \delta_{jk}$, and hence[1]

$$\Psi_* e_j = \frac{\partial \tilde{x}_1}{\partial x_j}\,\tilde{e}_1 + \ldots + \frac{\partial \tilde{x}_m}{\partial x_j}\,\tilde{e}_m, \tag{14}$$

where $\tilde{e}_k \in \mathbb{R}^m$ denotes the m-th standard basis vector in $\mathbb{R}^m$.

Now suppose that $\tilde{\alpha} : U \to \mathbb{R}^{m*}$ is a 1-form on U. Then it is natural to define a 1-form α on Ω by requiring

$$\alpha(x)V(x) = \tilde{\alpha}(\tilde{x})\tilde{V}(\tilde{x}) = \tilde{\alpha}((\tilde{x}))D\Psi(x)V(x), \qquad (x \in \Omega), \tag{15}$$

[1]If we identify a vector V with the directional derivative operator D_V, then the standard basis vectors are simply the partial derivative operators $\frac{\partial}{\partial x_j}$, etc. With this convention, (14) takes the convenient form

$$\Psi_* \frac{\partial}{\partial x_j} = \frac{\partial \tilde{x}_1}{\partial x_j}\,\frac{\partial}{\partial \tilde{x}_1} + \ldots + \frac{\partial \tilde{x}_m}{\partial x_j}\,\frac{\partial}{\partial \tilde{x}_m}.$$

for any vector field $V : \Omega \to \mathbb{R}^n$. By choosing V to be the standard basis vectors of $\mathbb{R}^n$, we get the transformation law

$$\alpha(x) = \tilde{\alpha}(\tilde{x}) D\Psi(x), \qquad \text{or in components,} \qquad \alpha_k = \sum_{i=1}^{m} \frac{\partial \tilde{x}_i}{\partial x_k} \tilde{\alpha}_i, \quad (k = 1, \ldots, n), \tag{16}$$

The 1-form α is called the *pull-back* of $\tilde{\alpha}$ under the mapping Ψ, and denoted by $\alpha = \Psi^* \tilde{\alpha}$. In particular, if $\tilde{\alpha} = \mathrm{d}\tilde{x}_j$, then $\tilde{\alpha}_i = \delta_{ij}$, and hence

$$\Psi^* \mathrm{d}\tilde{x}_j = \frac{\partial \tilde{x}_j}{\partial x_1} \mathrm{d}x_1 + \ldots + \frac{\partial \tilde{x}_j}{\partial x_n} \mathrm{d}x_n. \tag{17}$$

Remark 1.8. The aforementioned transformation laws include coordinate transformation formulas as special cases. Suppose that we have the relations $\tilde{x} = \tilde{x}(x)$ and $x = x(\tilde{x})$ between two coordinate systems, and we want to express vector fields and 1-forms in the $\tilde{x}$-coordinate system, assuming that they are available in the x-coordinate system. Then vector fields transform as

$$\tilde{V}_i = \frac{\partial \tilde{x}_i}{\partial x_1} V_1 + \ldots + \frac{\partial \tilde{x}_i}{\partial x_n} V_n, \qquad (i = 1, \ldots, n). \tag{18}$$

In this setting, (14) should be thought of as the expression of the vector field e_j in the $\tilde{x}$-coordinate system

$$e_j = \frac{\partial \tilde{x}_1}{\partial x_j} \tilde{e}_1 + \ldots + \frac{\partial \tilde{x}_n}{\partial x_j} \tilde{e}_n, \qquad (j = 1, \ldots, n). \tag{19}$$

On the other hand, for 1-forms, we need to switch the roles of x and $\tilde{x}$ in (16), to infer the transformation law

$$\tilde{\alpha}_k = \frac{\partial x_1}{\partial \tilde{x}_k} \alpha_1 + \ldots + \frac{\partial x_n}{\partial \tilde{x}_k} \alpha_n, \qquad (k = 1, \ldots, n). \tag{20}$$

Using (17), we can also derive the expression of $\mathrm{d}x_j$ in the $\tilde{x}$-coordinate system

$$\mathrm{d}x_j = \frac{\partial x_j}{\partial \tilde{x}_1} \mathrm{d}\tilde{x}_1 + \ldots + \frac{\partial x_j}{\partial \tilde{x}_n} \mathrm{d}\tilde{x}_n, \qquad (j = 1, \ldots, n). \tag{21}$$

We note that while the Jacobian matrix of the transformation $x = x(\tilde{x})$ enters in the transformation law for 1-forms (20), the inverse of the same matrix is needed in the transformation law for vectors (18).

Example 1.9. Consider $(x, y) = (r \cos\phi, r \sin\phi)$. Invoking (21), we have

$$\begin{aligned} \mathrm{d}x &= \cos\phi \, \mathrm{d}r - r \sin\phi \, \mathrm{d}\phi, \\ \mathrm{d}y &= \sin\phi \, \mathrm{d}r + r \cos\phi \, \mathrm{d}\phi. \end{aligned} \tag{22}$$

So, for example, the 1-form $\alpha = x\mathrm{d}x + xy\mathrm{d}y$ given in the xy-coordinate system can immediately be written in the $r\phi$-coordinate system as

$$\begin{aligned} \alpha &= r\cos\phi(\cos\phi \, \mathrm{d}r - r \sin\phi \, \mathrm{d}\phi) + r^2 \sin\phi \cos\phi(\sin\phi \, \mathrm{d}r + r\cos\phi \, \mathrm{d}\phi) \\ &= (r \cos^2\phi + r^2 \sin^2\phi \cos\phi)\mathrm{d}r + r^2 \sin\phi\cos\phi(r\cos\phi - 1)\mathrm{d}\phi. \end{aligned} \tag{23}$$

Definition 1.10. For $V, W \in \mathbb{R}^n$, their *Euclidean inner product* (or *dot product*) is

$$V \cdot W = V_1 W_1 + \ldots + V_n W_n. \tag{24}$$

Furthermore, the *Euclidean norm* or the *2-norm* of a vector $V \in \mathbb{R}^n$ is

$$|V| = \sqrt{V \cdot V}. \tag{25}$$

If we fix $V \in \mathbb{R}^n$, and consider $f(W) = V \cdot W$ as a function of $W \in \mathbb{R}^n$, then it is a linear function: $f(\lambda W_1 + W_2) = \lambda f(W_1) + f(W_2)$ for $W_1, W_1 \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, and can be represented by the row vector V^T, in the sense that $f(W) = V^\mathsf{T} W$. Hence the Euclidean inner product establishes the correspondence between $\mathbb{R}^n$ and $\mathbb{R}^{n*}$ given by $\kappa(V) = V^\mathsf{T}$. Using this correspondence, we can go between 1-forms and vector fields. Thus given a 1-form α, the vector field α^T is called the vector field *associated to* α. Similarly, given a vector field F, the 1-form F^T is called the 1-form *associated to* F. One of the applications of this is that we can make sense of the integral of a vector field F over an oriented curve γ as

$$\int_\gamma F \cdot \mathrm{d}\ell := \int_\gamma F^\mathsf{T} = \int_a^b F(\gamma(t))^\mathsf{T} \gamma'(t) \mathrm{d}t = \int_a^b F(\gamma(t)) \cdot \gamma'(t) \mathrm{d}t. \tag{26}$$

Remark 1.11. The expression (24) for the Euclidean inner product is valid only in a special class of coordinate systems, and *cannot* be valid in general coordinate systems. Even under the simple scaling $x = \lambda \tilde{x}$ for a constant $\lambda \neq 0$, we get

$$V \cdot W = V_1 W_1 + \ldots + V_n W_n = \lambda^2 (\tilde{V}_1 \tilde{W}_1 + \ldots + \tilde{V}_n \tilde{W}_n) = \lambda^2 V^\mathsf{T} W, \tag{27}$$

because $V = \lambda \tilde{V}$ and $W = \lambda \tilde{W}$ by (18). Note that we have to apply (18) with the roles of the quantities with tilde and the ones without reversed. More generally, if $x = \Phi(\tilde{x})$, and V and W are vector fields, then

$$V(x) \cdot W(x) = \tilde{V}(\tilde{x})^\mathsf{T} (D\Phi(\tilde{x}))^\mathsf{T} D\Phi(\tilde{x}) \tilde{V}(\tilde{x}). \tag{28}$$

This suggests that in a general coordinate system, the expression $V(x)^\mathsf{T} M(x) W(x)$, where $M(x)$ is some symmetric matrix, is more natural than the Euclidean inner product. Riemannian geometry studies spaces with such inner product structures.

2. Higher degree forms and oriented manifolds

Our next goal is to define integration on surfaces and higher dimensional manifolds. In the case of line integrals, what guided us were properties of the derivative of a scalar function and the change of variables formula. It is not clear what the former would be in higher dimensions, so we start our investigation by looking once again at the change of variables formula. Let $U \subset \mathbb{R}^n$ be a bounded open set, and let $\Phi : Q \to \mathbb{R}^n$ be a map for which the change of variables formula holds, where $Q \subset \mathbb{R}^n$ is an oriented rectangle. We assume that $U \subset \Phi(Q)$ and that U has the same orientation as that of $\Phi(Q)$. Then for any integrable function $u : \mathbb{R}^n \to \mathbb{R}$ with $u = 0$ outside U, we have

$$\int_U u = \int_{\Phi(Q)} u = \int_Q (u \circ \Phi) \det(D\Phi). \tag{29}$$

The one dimensional case

$$\int_{\phi(a)}^{\phi(b)} f = \int_a^b (f \circ \phi) \phi' = \int_a^b f(\phi(t)) \phi'(t) \mathrm{d}t, \tag{30}$$

can be interpreted as the line integral of f over the 1-dimensional oriented curve $\phi : [a, b] \to \mathbb{R}$. In this context, the function f must be interpreted as a 1-form, and the integrand $f(\phi(t))\phi'(t)$ should be understood as the "row vector" $f(\phi(t))$ applied to the velocity vector $\phi'(t)$. Hence if (29) is to be a special case of integration over manifolds in the same say (30) is a special case of line integrals, then it might be fruitful to think of $(u \circ \Phi) \det(D\Phi)$ as "something" applied to the Jacobian matrix $D\Phi$. Of course, we know what that "something" is. It is the scalar function $u(x)$, multiplied by the determinant as a function $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$.

Definition 2.1. An *alternating k-linear form in $\mathbb{R}^n$* is a multilinear function

$$\omega : \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{k \text{ times}} \to \mathbb{R}, \tag{31}$$

that is linear in each of its arguments, and satisfies

$$\omega(\ldots, V, \ldots, V, \ldots) = 0 \qquad \text{for any} \quad V \in \mathbb{R}^n. \tag{32}$$

The set of all alternating k-linear form in $\mathbb{R}^n$ is denoted by $\mathrm{Alt}^k\,\mathbb{R}^n$.

An alternating k-linear form in $\mathbb{R}^n$ can also be though of as a function $\omega : \mathbb{R}^{n\times k} \to \mathbb{R}$, by thinking of a matrix $A \in \mathbb{R}^{n\times k}$ as the collection of its columns.

Remark 2.2. The requirement (32) is vacuous when $k = 1$, and therefore any linear function $\omega : \mathbb{R}^n \to \mathbb{R}$ is an alternating 1-linear form (or simply linear form). These are exactly the elements of $\mathbb{R}^{n*}$, that is, $\mathrm{Alt}^1\mathbb{R}^n = \mathbb{R}^{n*}$. Furthermore, it is a standard result in linear algebra that any $\omega \in \mathrm{Alt}^n\mathbb{R}^n$ satisfies

$$\omega(A) = \omega(I)\det(A), \qquad A \in \mathbb{R}^{n\times n}. \tag{33}$$

In particular, $\mathrm{Alt}^n\mathbb{R}^n$ is 1-dimensional.

With this preparation, we recognize that at each $\xi \in Q$, $(u \circ \Phi)\det(D\Phi)$ in (29) is an expression of the type (33). Thus we define $\omega(x) \in \mathrm{Alt}^n\mathbb{R}^n$ for each $x \in \mathbb{R}^n$ by

$$\omega(x)(A) = u(x)\det(A), \qquad x \in \mathbb{R}^n, \tag{34}$$

and (29) becomes

$$\int_U \omega(x)(I)\mathrm{d}^n x = \int_Q \omega(\Phi(\xi))(D\Phi(\xi))\mathrm{d}^n\xi. \tag{35}$$

Note that both sides are in the same form, since we can think of I as the Jacobian matrix of the the trivial parameterization $\Psi(x) = x$. Note also that ω is indeed a function $\omega : \mathbb{R}^n \to \mathrm{Alt}^n\mathbb{R}^n$. Such functions are called simply *n-forms*. For any n-form ω, if we let $u(x) = \omega(x)(I)$, then according to (33), we have (34). We *define* the integral of ω over the oriented domain U by

$$\int_U \omega = \int_Q \omega(\Phi(\xi))(D\Phi(\xi))\mathrm{d}^n\xi. \tag{36}$$

The right hans side does not depend on the map Φ, as long as the orientation of $\Phi(Q)$ agrees with that of U.

We are now ready to start defining integrals over manifolds.

Definition 2.3. Let $G \subset \mathbb{R}^n$ be an open set. Then a *(differential) k-form* on G is a function $\omega : G \to \mathrm{Alt}^k\mathbb{R}^n$, such that $\omega(x)(A)$ is continuously differentiable as a function of $x \in G$, for any matrix $A \in \mathbb{R}^{n\times k}$.

Let M be a k-dimensional manifold in $\mathbb{R}^n$, and let $\Psi : \Omega \to M$ be a smooth parameterization, with $\Omega \subset \mathbb{R}^k$ open. Suppose that $G \subset \mathbb{R}^n$ is an open set, such that $M \subset G$, and let ω be a k-form on G. Let $K \subset \Omega$ be a closed and bounded set, and assume that $\omega(\Psi(\xi)) = 0$ whenever $\xi \in \Omega \setminus K$. Finally, we define

$$\int_U \omega = \int_\Omega \omega(\Psi(\xi))(D\Psi(\xi))\mathrm{d}^k\xi, \tag{37}$$

provided the function $\omega(\Psi(\xi))(D\Psi(\xi))$ is integrable.

Lemma 2.4. *The right hand side in* (37) *is preserved under reparameterization* $\Xi = \Psi \circ \Phi$*, as long as* $\det(D\Phi) > 0$*.*

Proof. Let $\Phi : \Omega' \to \Omega$. The change of variables formula gives

$$\int_{\Omega} \omega(\Psi(\xi))(D\Psi(\xi))\mathrm{d}^k\xi = \int_{\Omega'} \omega(\Xi(\eta))(D\Psi(\Phi(\eta)))\det(D\Phi(\eta))\mathrm{d}^k\eta. \tag{38}$$

We compute $D\Psi = D(\Xi \circ \Phi^{-1})$ by the chain rule as

$$D\Psi = D(\Xi \circ \Phi^{-1}) = (D\Xi \circ \Phi^{-1})D(\Phi^{-1}) = (D\Xi \circ \Phi^{-1})(D\Phi \circ \Phi^{-1})^{-1}. \tag{39}$$

Finally, the property (33) yields

$$\omega(\Xi(\eta))(D\Psi(\Phi(\eta))) = \omega(\Xi(\eta))(D\Xi(\eta)(D\Phi)^{-1}) = \omega(\Xi(\eta))(D\Xi(\eta))\det((D\Phi)^{-1}), \tag{40}$$

completing the proof. $\square$