MATH 254 Tutorial 7 (Monotone Sequences and Subsequences):

Problem 1: Do the following tasks for these four sequences:
(i) $x_1 := 2, \qquad x_{n+1} := 2 - 1/x_n$ for all $n \in \mathbb{N}$
(ii) $x_1 := 2, \qquad x_{n+1} := 2x_n - 1$ for all $n \in \mathbb{N}$
(iii) $x_1 := 3, \qquad x_{n+1} := 1 + \sqrt{x_n - 1}$ for all $n \in \mathbb{N}$
(iv) $x_1 := 1, \qquad x_{n+1} := \sqrt{2 + x_n}$ for all $n \in \mathbb{N}$

a) Assuming that the sequence (x_n) is convergent to x, use limit theorems to find an equation for x and solve that equation to find the limit of (x_n) under the assumption of convergency. Explain why this doesn't prove that the sequence is convergent to x.

b) Either prove that the sequence (x_n) is divergent so the computation in part a is useless and meaningless since its assumption of convergency is not satisfied, or using mathematical induction and monotone convergence theorem prove that the sequence (x_n) is convergent so the computation in part a is valid (since its assumption of convergency is satisfies) and therefore the limit of the sequence is what we have found in part a.

Problem 2: a) Let A be a subset of real numbers such that $supA \in \mathbb{R}$ $[infA \in \mathbb{R}]$ exists. Prove that there exists an increasing [decreasing] sequence (x_n) with values in A being convergent to $supA$ $[infA]$. (If $supA$ $[infA]$ is not in A, then the sequence can be constructed to be strictly increasing [decreasing].)

b) Let (x_n) be a sequence and $A := \{x_n : n \in \mathbb{N}\}$. Given a sequence (y_n) of distinct elements of A i.e. $y_n \in A$ and $y_i \neq y_j$ (note that in general, (y_n) is not a subsequence of (x_n)), prove that there is a subsequence of (y_n) which is also a subsequence of (x_n) i.e. there are subsequences (x_{n_k}) and $(y_{n'_k})$ of (x_n) and (y_n) respectively such that $x_{n_k} = y_{n'_k}$ for all $k \in \mathbb{N}$.

c) Let (x_n) be a bounded sequence and let $s := sup\{x_n : n \in \mathbb{N}\}$. Using part a and b, show that if s is not in the set $\{x_n : n \in \mathbb{N}\}$, then there is a strictly increasing subsequence that converges to s.

Solution: b) The assumptions $y_n \in A$ and $y_i \neq y_j$ imply that A is an infinite set. We will construct the subsequences inductively.

Base Step: Take $n'_1 = 1$. Since $y_1 \in A$ there is an $n_1 \in \mathbb{N}$ such that $x_{n_1} = y_{n'_1}$.

Inductive Step: Having n_k and n'_k, I will find n_{k+1} and n'_{k+1}. The set $B := \{x_n : n \leq n_k\}$ is finite so there is $n'_{k+1} > n'_k$ such that $y_{n'_{k+1}} \in A - B$ (we are using the assumption $y_i \neq y_j$, here). Since $y_{n'_{k+1}} \in A$ there is an $n_{k+1} \in \mathbb{N}$ such that $x_{n_{k+1}} = y_{n'_{k+1}}$. Then we must have $n_{k+1} > n_k$ because of $x_{n_{k+1}} \in A - B$ and definition of B.

c) This is just combining the results in part a (inside the parantheses) and part b.

Problem 3: Explain why there is a correspondence between subsequences of a given sequence and infinite subsets of $\mathbb{N}$. Let (x_n) be a sequence and $N, N' \subseteq \mathbb{N}$ be two infinite sets such that the corresponding subsequences converge to x and x', respectively. Also, assume that $\mathbb{N} - (N \cup N')$ is finite. Prove that:

a) If we have $x = x'$, then the sequence (x_n) is convergent to x. (One special corollary of this part is that if (x_{2n}) and (x_{2n-1}) converge to the same limit x, then (x_n) is also convergent to x.)

b) If $N \cap N'$ is infinite, then the sequence (x_n) is convergent.

c) Extend the statement in this problem (part a and b) for finitely many subsequences and the proof is completely similar.

d) Give a counter example to show that statements in part a and b are not true for countably many subsequences.

Solution: In this and next problems, I will abuse the terminology and say the subsequence N of (x_n), where $N \subseteq \mathbb{N}$ is infinite. If the k-th element of N is n_k, then by this I mean the subsequence (x_{n_k}).

a) Given $\epsilon > 0$. Since the subsequence N of (x_n) converges to x, I have $x_n \in (x - \epsilon, x + \epsilon)$ for all $n \in N$ except finitely many of them. Similarly, since the subsequence N' of (x_n) converges to x, I have $x_n \in (x - \epsilon, x + \epsilon)$ for all $n \in N'$ except finitely many of them. These two together implies that I have $x_n \in (x - \epsilon, x + \epsilon)$ for all $n \in N \cup N'$ except finitely many of them. Since $\mathbb{N} - (N \cup N')$ is finite, I have $x_n \in (x - \epsilon, x + \epsilon)$ for all $n \in \mathbb{N}$ except finitely many of them. So, the proof is complete.

b) Take $M := N \cap N'$. By assumption, it is infinite. Note that the subsequence M of (x_n) is also a subsequence of the subsequence N and N' of (x_n) since $M \subseteq N$ and $M \subseteq N'$. Therefore the subsequence M of (x_n) should converge to both x and x'. This proves $x = x'$ and we can use part a.

d) Define $x_n := 1$ if n is a prime number and otherwise 0. Since there are infinitely many prime numbers the sequence (x_n) is divergent. If p_i is the i-th prime number, note that the sequence $(x_{k.p_i}) = (1, 0, 0, 0, ...)$ is convergent to 0 for all $i \in \mathbb{N}$. Also, we have $\mathbb{N} = \cup_{i=1}^{\infty} \{k.p_i : k \in \mathbb{N}\}$ and $\{k.p_i : k \in \mathbb{N}\} \cap \{k.p_j : k \in \mathbb{N}\}$ is infinite for all $i, j \in \mathbb{N}$. So, this is a counter example for both part a and b for countably many subsequences.

Problem 4: In this problem, we want to prove a generalization of the Bolzano-Weierstrass theorem for countably many sequences. Let $(x_n^i : n \in \mathbb{N})$ be a bounded sequence of real numbers (the i-th sequence) for each $i \in \mathbb{N}$. We want to prove that there is an infinite set $N \subseteq \mathbb{N}$ such that the corresponding subsequence of the i-th sequence is convergent for all $i \in \mathbb{N}$.

a) First, assume that there are finitely many sequences i.e. $i = 1, ..., k$ for some $k \in \mathbb{N}$. Use the Bolzano-Weierstrass theorem repeatedly (mathematical induction) to find infinite set $N_j \subseteq \mathbb{N}$ such that the corresponding subsequence of the i-th sequence is convergent for $i = 1, ..., j$ and $N_k \subseteq ... \subseteq N_1$. The set $N := N_k = \cap_{j=1}^{k} N_j$ has the desired property.

b) Explain why the same proof as part a fails for countably many sequences ($i \in \mathbb{N}$).

c) Modify the proof in part a for countably many sequences ($i \in \mathbb{N}$). Use the Bolzano-Weierstrass theorem and mathematical induction to find infinite set $N_j \subseteq \mathbb{N}$ such that the corresponding subsequence of the i-th sequence is convergent for $i = 1, ..., j$ and $... \subseteq N_j \subseteq ... \subseteq N_1$. Satisfying this additional property that the first j elements of N_j and N_{j+1} are the same for all $j \in \mathbb{N}$ (or equivalently the first j elements of N_j belong to $N := \cap_{j=1}^{\infty} N_j$). Then, the set $N \subseteq \mathbb{N}$ is infinite and proves what we want.

Solution: c) Base Step (Finding N_1): Apply the Bolzano-Weierstrass theorem to the first sequence to find N_1.

Inductive Step (Finding N_{j+1} using N_j): Apply the Bolzano-Weierstrass theorem to the **subsequence** N_j of the $(j+1)$-th sequence to find an infinite set $N'_{j+1} \subseteq N_j$. Define N_{j+1} to be the union of N'_{j+1} and the set of first j elements of N_j.

Verify that the construction above have the required properties and $N := \cap_{j=1}^{\infty} N_j$ is an infinite set with the desired property.

Problem 5: a) Prove that the sequence (x_n) is unbounded if and only if there is a nonzero subsequence (x_{n_k}) such that the sequence $(1/x_{n_k})$ converges to zero.

b) Give an example of a divergent sequence such that every convergent subsequence converges to zero.

Solution: b) Consider the sequence $(0, 1, 0, 2, 0, 3, 0, 4, 0, 5, ...)$.