MATH323 - Calculus Exercise Sheet (Derivatives)

1. Let $0 \leq p \leq 1$, and $m(t) = 1-p+pe^t$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
2. Let $0 \leq p \leq 1$, and $m(t) = \dfrac{pe^t}{1-(1-p)e^t}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
3. Let $n>0$ and $0 \leq p \leq 1$, and $m(t) = (1-p+pe^t)^n$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
4. Let $\lambda > 0$, and $m(t) = e^{\lambda(e^t-1)}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
5. Let $b>a$, and $m(t) = \dfrac{e^{tb}-e^{ta}}{t(b-a)}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ for $t \neq 0$.
6. Let $b>a$, and $m(t) = \dfrac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ for $t \neq 0$.
7. Let $\mu, \sigma \in \mathbb{R}$, and $m(t) = e^{t\mu + \frac{1}{2}\sigma^2 t^2}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
8. Let $k>0$, and $m(t) = (1-2t)^{-k/2}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
9. Let $k, \theta > 0$, and $m(t) = (1-t\theta)^{-k}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
10. Let $\lambda > 0$, and $m(t) = \dfrac{1}{1-t\lambda}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
11. Let $a \in \mathbb{R}$, and $m(t) = e^{ta}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.
12. Let $\mu \in \mathbb{R}$ and $b>0$, and $m(t) = \dfrac{e^{t\mu}}{1-b^2t^2}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$.
13. Let $r>0$ and $0 \leq p \leq 1$, and $m(t) = \dfrac{(1-p)^r}{(1-pe^t)^r}$. Evaluate $\frac{d}{dt}m(t)$ and $\frac{d^2}{dt^2}m(t)$ at $t=0$.

Solutions:

1. $$\frac{d}{dt}(1-p+pe^t) = 0 - 0 + pe^t = pe^t \qquad \frac{d}{dt}m(t)\Big|_{t=0} = p$$
$$\frac{d^2}{dt^2}(1-p+pe^t) = \frac{d}{dt}(pe^t) = pe^t \qquad \frac{d^2}{dt^2}m(t)\Big|_{t=0} = p$$

2. $$\frac{d}{dt}\left(\frac{pe^t}{1-(1-p)e^t}\right) = \frac{(pe^t)(1-(1-p)e^t) + (pe^t)((1-p)e^t)}{(1-(1-p)e^t)^2} = \frac{pe^t}{(1-(1-p)e^t)^2}$$
$$\frac{d^2}{dt^2}\left(\frac{pe^t}{1-(1-p)e^t}\right) = \frac{d}{dt}\left(\frac{pe^t}{(1-(1-p)e^t)^2}\right) = \frac{(pe^t)(1-(1-p)e^t)^2 + (pe^t)(2(1-(1-p)e^t))((1-p)e^t)}{(1-(1-p)e^t)^4}$$
$$\frac{d}{dt}m(t)\Big|_{t=0} = \frac{1}{p}, \qquad \frac{d^2}{dt^2}m(t)\Big|_{t=0} = \frac{2-p}{p^2}$$

3. $$\frac{d}{dt}(1-p+pe^t)^n = npe^t(1-p+pe^t)^{n-1}$$
$$\frac{d^2}{dt^2}(1-p+pe^t)^n = \frac{d}{dt}npe^t(1-p+pe^t)^{n-1} = np\{e^t(1-p+pe^t)^{n-1} + e^t(n-1)pe^t(1-p+pe^t)^{n-2}\}$$
$$\frac{d}{dt}m(t)\Big|_{t=0} = np, \qquad \frac{d^2}{dt^2}m(t)\Big|_{t=0} = np(1+p(n-1))$$

4. $\frac{d}{dt}\left(e^{\lambda(e^t-1)}\right) = e^{\lambda(e^t-1)}\,\lambda e^t = \lambda e^{t+\lambda(e^t-1)}$ $\quad \frac{d}{dt}m(t)\Big|_{t=0} = \lambda$

$\frac{d^2}{dt^2}\left(e^{\lambda(e^t-1)}\right) = \frac{d}{dt}\left(\lambda e^{t+\lambda(e^t-1)}\right) = \lambda e^{t+\lambda(e^t-1)}\,(1+\lambda e^t)$ $\quad \frac{d^2}{dt^2}m(t)\Big|_{t=0} = \lambda(1+\lambda)$

5. $\frac{d}{dt}\left(\frac{e^{tb}-e^{ta}}{t(b-a)}\right) = \frac{1}{b-a}\left\{\frac{(be^{tb}-ae^{ta})(t)-(e^{tb}-e^{ta})}{t^2}\right\}$

$\frac{d^2}{dt^2}\left(\frac{e^{tb}-e^{ta}}{t(b-a)}\right) = \frac{d}{dt}\left(\frac{1}{b-a}\left\{\frac{(be^{tb}-ae^{ta})t-e^{tb}+e^{ta}}{t^2}\right\}\right)$

$= \frac{1}{b-a}\left(\frac{(t^2)\left(\left((be^{tb}-ae^{ta})+t(b^2e^{tb}-a^2e^{ta})\right)-be^{tb}+ae^{ta}\right) - (2t)\left((be^{tb}-ae^{ta})t-e^{tb}+e^{ta}\right)}{t^4}\right)$

6. $\frac{d}{dt}\left(\frac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}\right) = \left(\frac{1}{b-a+1}\right)\left\{\frac{(ae^{at}-(b+1)e^t)(1-e^t)+e^t(e^{at}-e^{(b+1)t})}{(1-e^t)^2}\right\}$

$\frac{d^2}{dt^2}\left(\frac{e^{at}-e^{(b+1)t}}{(b-a+1)(1-e^t)}\right) = \left(\frac{1}{b-a+1}\right)\left\{\frac{\left(a^2e^{at}-(b+1)e^t-a(a+1)e^{at+t}+2(b+1)e^{2t}+(a+1)e^{at+t}-(b+2)e^{(b+2)t}\right)\times(1-e^t)^2 + 2e^t(1-e^t)\left((ae^{at}-(b+1)e^t)(1-e^t)+e^t(e^{at}-e^{(b+1)t})\right)}{(1-e^t)^4}\right\}$

7. $\frac{d}{dt}\left(e^{t\mu+\frac{1}{2}\sigma^2t^2}\right) = (\mu+\sigma^2 t)\,e^{t\mu+\frac{1}{2}\sigma^2t^2}$ $\quad \frac{d}{dt}m(t)\Big|_{t=0} = \mu$

$\frac{d^2}{dt^2}\left(e^{t\mu+\frac{1}{2}\sigma^2t^2}\right) = \sigma^2 e^{t\mu+\frac{1}{2}\sigma^2t^2} + (\mu+\sigma^2t)^2 e^{t\mu+\frac{1}{2}\sigma^2t^2}$ $\quad \frac{d^2}{dt^2}m(t)\Big|_{t=0} = \sigma^2-\mu^2$

8. $\frac{d}{dt}(1-2t)^{-k/2} = \left(-\frac{k}{2}\right)(1-2t)^{-\frac{k}{2}-1}(-2) = k(1-2t)^{-\frac{k}{2}-1}$ $\quad \frac{d}{dt}m(t)\Big|_{t=0} = k$

$\frac{d^2}{dt^2}(1-2t)^{-k/2} = 2k\left(\frac{k}{2}+1\right)(1-2t)^{-\frac{k}{2}-2}$ $\quad \frac{d^2}{dt^2}m(t)\Big|_{t=0} = 2k\left(\frac{k}{2}+1\right)$

9. $\frac{d}{dt}(1-t\theta)^{-k} = -k(1-t\theta)^{-k-1}(-\theta) = k\theta(1-t\theta)^{-k-1}$ $\quad \frac{d}{dt}m(t)\Big|_{t=0} = k\theta$

$\frac{d^2}{dt^2}(1-t\theta)^{-k} = k\theta(k+1)\theta(1-t\theta)^{-k-2}$ $\quad \frac{d^2}{dt^2}m(t)\Big|_{t=0} = k(k+1)\theta^2$

10. $\frac{d}{dt}\left(\frac{1}{1-t\lambda}\right) = \frac{\lambda}{(1-t\lambda)^2}$, $\frac{d^2}{dt^2}\left(\frac{1}{1-\lambda t}\right) = \frac{2\lambda^2}{(1-t\lambda)^3}$, $\frac{d}{dt}m(t)\Big|_{t=0} = \lambda$, $\frac{d^2}{dt^2}m(t)\Big|_{t=0} = 2\lambda^2$

11. $\frac{d}{dt}(e^{ta}) = ae^{ta}$, $\frac{d^2}{dt^2}(e^{ta}) = a^2e^{ta}$, $\frac{d}{dt}m(t)\Big|_{t=0} = a$, $\frac{d^2}{dt^2}m(t)\Big|_{t=0} = a^2$

12. $\frac{d}{dt}\left(\frac{e^{t\mu}}{1-b^2t^2}\right) = \frac{(\mu e^{t\mu})(1-b^2t^2)+(e^{t\mu})(2b^2t)}{(1-b^2t^2)^2}$

$\frac{d^2}{dt^2}\left(\frac{e^{t\mu}}{1-b^2t^2}\right) = \frac{\left\{(\mu^2e^{t\mu})(1-b^2t^2)+(\mu e^{t\mu})(-2b^2t)+(\mu e^{t\mu})(2b^2t)+(e^{t\mu})(2b^2)\right\}(1-b^2t^2)^2 - \left\{(\mu e^{t\mu})(1-b^2t^2)-(e^{t\mu})(2b^2t)\right\}\left(2(1-b^2t^2)(-2b^2t)\right)}{(1-b^2t^2)^4}$

13. Left as exercise.