MATH 254 Tutorial 3 (Countable and Uncountable Sets):

Problem 1: Assuming you know that $\mathbb{N} \times \mathbb{N}$ is countable, prove that:

a) A countable union of countable sets is countable, i.e if all sets A_α and I are countable, then the set $\cup_{\alpha \in I} A_\alpha$ is countable.

b) A finite product of countable sets is countable, i.e if all sets $A_1, ..., A_n$ are countable, then the set $A_1 \times ... \times A_n$ is countable

Problem 2: If the set A is uncountable and the set B is countable, prove that $A - B$ is uncountable.

Problem 3: Let the set A be all sequences $a_1, a_2, ...$ with the values in the set $\{0, 1\}$.Prove that A is uncountable.

Problem 4: Let A and B be two sets. Consider the set C of all functions $f : A \to B$.

a) If A and B are finite with n and m elements, respectively, compute the number of elements of C in terms of number of n and m.

b) If A is finite and B is countable, guess and prove your conjecture about countablity or uncountablity of C.

c) If A is countably infinite and B has at least two elements, guess and prove your conjecture about countablity or uncountablity of C.

Solution: b) C is countable. Proof: If A is empty then C is also empty and countable. Otherwise, if A has $n \in \mathbb{N}$ elements, then there is a bijection $h : \{1, ..., n\} \to A$. Let D be the set of all functions $g : \{1, ..., n\} \to B$, which is the same as $B \times ... \times B$, n times. By part b of problem 1, D is countable. We will give a bijection $\phi : C \to D$, which proves countablity of C. For a given $f \in C$ define $\phi(f) := f \circ h$. ϕ is injective:

$$\phi(f) = \phi(f') \Rightarrow f \circ h = f' \circ h \Rightarrow f = f \circ h \circ h^{-1} = f' \circ h \circ h^{-1} = f'.$$

ϕ is surjective: Given $g \in D$, we have:

$$\phi(g \circ h^{-1}) = g \circ h^{-1} \circ h = g.$$

c) C is uncountable. Proof: Since A is countably infinite, there is a bijection $h : \mathbb{N} \to A$. Let D be the set of all functions $g : \mathbb{N} \to B$, which is the same as all sequences with values in B. Define $\phi : C \to D$ like part b and with the same proof it is bijective. So, it remains to prove that D is uncountable. B has at least two elements b_0 and b_1. Take the subset $E \subseteq D$ containing all sequences with values in the set $\{b_0, b_1\}$ instead of B. In problem 3, we proved that E is uncountable, so D and C are also uncountable.

Problem 5: Consider the set Σ as our alphabet, so elements of Σ are our characters or letters. A word of length n over this alphabet is a finite sequence or string $a_1 a_2 ... a_n$ of letters, where $a_i \in \Sigma$. Let W be the set of all words of any finite length.

a) If Σ is finite, guess and prove your conjecture about countablity or uncountablity of W.

b) If Σ is countable, guess and prove your conjecture about countablity or uncountablity of W.

c) Justify the argument that there are countably many computer programs can be written in a computer programming language.

Problem 6: Consider the set C of all convergent sequences of natural numbers. Guess and prove your conjecture about countablity or uncountablity of C.

Solution: C is countable. Proof: Let W be the set of all finite sequences of natural numbers. In part b of problem 5, we proved that W is countable (here we have $\Sigma = \mathbb{N}$). So, $W \times \mathbb{N}$ is also countable. Now, to prove countablity of C, it is enough to give a surjection $f : W \times \mathbb{N} \to C$. For given $a = (a_1, ..., a_n) \in W$ and $b \in \mathbb{N}$, define $f(a, b) := (a_1, ..., a_n, b, b, b, ...) \in C$ (note that the sequence is convergent so indeed it is an element of C). f is surjective: Given a sequence $(a_1, a_2, a_3, ...) \in C$ which converges to b. An easy proof shows that there is an $N \in \mathbb{N}$ such that we must have $a_n = b$ for all $n > N$ (you can prove this by your previous knowledge from calculus about convergent sequences or you can wait a few weeks until we cover the convergent sequences in this course). Now, if you take $a = (a_1, ..., a_N) \in W$, then $f(a, b)$ is the given sequence, which proves surjectivity of f.