Exercise 1. Induction Principles for binary trees [35 pts]

1. Induction rule [6 pts]

$$\frac{\Gamma \vdash t : \texttt{tree} \qquad \Gamma \vdash A(\texttt{Empty})\ \texttt{true} \qquad \Gamma, N : \texttt{nat}, T_1 : \texttt{tree}, T_2 : \texttt{tree}, \texttt{ih1} : A(T_1)\ \texttt{true}, \texttt{ih2} : A(T_2)\ \texttt{true} \vdash A(\texttt{Node}(N, T_1, T_2))\ \texttt{true}}{\Gamma \vdash A(t)\ \texttt{true}}\ \texttt{binE}^{\texttt{N, ih1, ih2}}$$

2. Proof term [6 pts]

$$\frac{\Gamma \vdash t : \texttt{tree} \qquad \Gamma \vdash M_{\texttt{Empty}} : A(\texttt{Empty})\ \texttt{true} \qquad \Gamma, N : \texttt{nat}, T_1 : \texttt{tree}, T_2 : \texttt{tree}, \texttt{f}\ \ T_1 : A(T_1)\ \texttt{true}, \texttt{f}\ \ T_2 : A(T_2)\ \texttt{true} \vdash M_{\texttt{Node}} : A(\texttt{Node}(N, T_1, T_2))\ \texttt{true}}{\Gamma \vdash \texttt{rec}^{\forall x:\texttt{tree}}\ t\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to M_{\texttt{Node}} : A(t)\ \texttt{true}}\ \texttt{binE}^{\texttt{N, ih1, ih2}}$$

3. Reduction rules [8 pts]

$\texttt{rec}^{\forall x:\texttt{tree}}\ \texttt{Empty}\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to M_{\texttt{Node}} \Rightarrow M_z$

$\texttt{rec}^{\forall x:\texttt{tree}}\ \texttt{Node}(t, t_1, t_2)\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to M_{\texttt{Node}} \Rightarrow$
$[t/n][t_1/T_1][t_2/T_2][r_1/\texttt{f}\ \ T_1][r_2/\texttt{f}\ \ T_2]M_{\texttt{Node}}$
where $r_1 = \texttt{rec}^{\forall x:\texttt{tree}}\ t_1\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to M_{\texttt{Node}}$,
$r_2 = \texttt{rec}^{\forall x:\texttt{tree}}\ t_2\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to M_{\texttt{Node}}$

4. Subject Reduction [10 pts]

Theorem 1. *If $\mathcal{D} : M \Rightarrow M'$ and $\mathcal{E} : \Gamma \vdash M : C$, then $\Gamma \vdash M' : C$.*

Proof. By structural induction on $\mathcal{D}$

Case. $\mathcal{D} =$
$\texttt{rec}^{\forall x:\texttt{tree}}\ \texttt{Empty}\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to M_{\texttt{Node}} \Rightarrow M_z$
$\texttt{rec}^{\forall x:\texttt{tree}}\ \texttt{Empty}\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to M_{\texttt{Node}} :$
$A(\texttt{Empty})\texttt{true}$ by exhaustion and $\texttt{binE}^{N,ih1,ih2}$
$M_{\texttt{Empty}} : A(\texttt{Empty})\texttt{true}$ by inversion on $\texttt{binE}^{N,ih1,ih2}$

Case. $\mathcal{D} =$
$\texttt{rec}^{\forall x:\texttt{tree}}\ \texttt{Node}(t, t_1, t_2)\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to M_{\texttt{Node}} \Rightarrow$
$[t/n][t_1/T_1][t_2/T_2][r_1/\texttt{f}\ \ T_1][r_2/\texttt{f}\ \ T_2]M_{\texttt{Node}}$
$\texttt{rec}^{\forall x:\texttt{tree}}\ \texttt{Node}(t, t_1, t_2)\ \texttt{with}\ (\texttt{f Empty}) \to M_{\texttt{Empty}} \mid (\texttt{f Node}(N, T_1, T_2)) \to$

$M_{\texttt{Node}} : A(\texttt{Node}(n, t_1, t_2))\texttt{true}$
$\Gamma, N : \texttt{nat}, T_1 : \texttt{tree}, T_2 : \texttt{tree}, \texttt{f}\ T_1 : A(T_1)\ \texttt{true}, \texttt{f}\ T_2 : A(T_2)\ \texttt{true} \vdash$
$M_{\texttt{Node}} : A(\texttt{Node}(N, T_1, T_2))\ \texttt{true}$
$\Gamma \vdash n : \texttt{nat}$, $\Gamma \vdash t_1 : \texttt{tree}$, $\Gamma \vdash t_2 : \texttt{tree}$ by exhaustion, inv on node-I
$\Gamma, t_1 : \texttt{tree} \vdash r_1 : A(t_1)\texttt{true}$, $\Gamma, t_2 : \texttt{tree} \vdash r_2 : A(t_2)\texttt{true}$ by $\texttt{binE}$
$\Gamma \vdash [t/n][t_1/T_1][t_2/T_2][r_1/\texttt{f}\ T_1][r_2/\texttt{f}\ T_2]M_{\texttt{Node}} : A(\texttt{Node}(n, t_1, t_2))\texttt{true}$ by subs. lemma

□

5. Primitive Recursive Program [5 pts]

$\lambda a.\texttt{rec}^{\forall x:\texttt{tree}}\ \texttt{with}\ (\texttt{f}\ \texttt{Empty}) \to z \mid (\texttt{f}\ \texttt{Node}(N, T_1, T_2)) \to s(\texttt{plus(}\ \texttt{f}\ T_1\texttt{,}\ \texttt{f}\ T_2))$

1 2. Sequent Calculus [65 pts]

1.1 1. Practice [10 pts]

1.1.1 a

Let $\Gamma = (A \wedge (B \vee C))$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, A, B \vee C, B \Longrightarrow A}\,\mathit{init} \quad \dfrac{}{\Gamma, A, B \vee C, B \Longrightarrow B}\,\mathit{init}}{\Gamma, A, B \vee C, B \Longrightarrow A \wedge B}\,\wedge R}{\Gamma, A, B \vee C, B \Longrightarrow (A \wedge B) \vee (A \wedge C)}\,\vee R_1 \quad \dfrac{\dfrac{\dfrac{}{\Gamma, A, B \vee C, B \Longrightarrow A}\,\mathit{init} \quad \dfrac{}{\Gamma, A, B \vee C, B \Longrightarrow C}\,\mathit{init}}{\Gamma, A, B \vee C, C \Longrightarrow (A \wedge C)}\,\wedge R}{\Gamma, A, B \vee C, C \Longrightarrow (A \wedge B) \vee (A \wedge C)}\,\vee R_2}{\dfrac{\Gamma, A, B \vee C \Longrightarrow (A \wedge B) \vee (A \wedge C)}{\Gamma, A \Longrightarrow (A \wedge B) \vee (A \wedge C)}\,\wedge L_2}\,\vee L}{(\Gamma \Longrightarrow (A \wedge B) \vee (A \wedge C)}\,\wedge L_1
$$

1.1.2 b

Let $\Gamma = (D \supset B) \supset C \supset D, B \supset A, (C \supset D) \supset B$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, D, C \Longrightarrow D}\,\mathit{init}}{\Gamma, D \Longrightarrow C \supset D}\,\supset R \quad \dfrac{}{\Gamma, D, B \Longrightarrow B}\,\mathit{init}}{\Gamma, D \Longrightarrow B}\,\supset L}{\Gamma \Longrightarrow D \supset B}\,\supset R \quad \dfrac{}{\Gamma, C \supset D \Longrightarrow C \supset D}\,\mathit{init}}{\Gamma \Longrightarrow C \supset D}\,\supset L \quad \dfrac{}{\Gamma, B \Longrightarrow B}\,\mathit{init}}{\Gamma \Longrightarrow B}\,\supset L \quad \dfrac{}{\Gamma, A \Longrightarrow A}\,\mathit{init}}{\Gamma \Longrightarrow A}\,\supset L
$$

1.2 2. Optimization [25 pts]

Lemma 1. *For all propositions A, we have $\Gamma, A \Rightarrow A$*

Proof. By structural induction on A

Case. $A = P$

$\Gamma, P \Longrightarrow P$ by init'

Case. $A = \top$

$\Gamma, \top \Longrightarrow \top$ by $\top$R

Case. $A = \bot$

$\Gamma, \bot \Longrightarrow \bot$ by $\bot$L

Case. $A = B \wedge C$ $\mathcal{D}_1 : \Gamma, B \Longrightarrow B$ by i.h. on B

$\mathcal{E}_1 : \Gamma, B \wedge C, B \Longrightarrow B$ by weakening

$\mathcal{D}_2 : \Gamma, C \Longrightarrow C$ by i.h. on C

$\mathcal{E}_2 : \Gamma, B \wedge C, C \Longrightarrow C$ by weakening

$\mathcal{F}_1 : \Gamma, B \wedge C \Longrightarrow B$ by $\wedge L_1$ with $\mathcal{E}_1$

$\mathcal{F}_2 : \Gamma, B \wedge C \Longrightarrow C$ by $\wedge L_2$ with $\mathcal{E}_2$

$\mathcal{F} : \Gamma, B \wedge C \Longrightarrow B \wedge C$ by $\wedge R$ with $\mathcal{F}_1$ and $\mathcal{F}_2$

Case. $A = B \vee C$

$\mathcal{D}_1 : \Gamma, B \Longrightarrow B$ by i.h. on B

$\mathcal{E}_1 : \Gamma, B \vee C, B \Longrightarrow B$ by weakening

$\mathcal{D}_2 : \Gamma, C \Longrightarrow B$ by i.h. on C

$\mathcal{E}_2 : \Gamma, B \vee C, C \Longrightarrow C$ by weakening

$\mathcal{F}_1 : \Gamma, B \vee C, B \Longrightarrow B \vee C$ by $\vee R_1$

$\mathcal{E}_2 : \Gamma, B \vee C, C \Longrightarrow B \vee C$ by $\vee R_2$

$\mathcal{F} : \Gamma, B \vee C \Longrightarrow B \vee C$ by $\vee L$ with $\mathcal{F}_1$ and $\mathcal{F}_2$

Case. $A = B \supset C$

$\mathcal{D}_1 : \Gamma, B \Longrightarrow B$ by i.h. on B

$\mathcal{E}_1 : \Gamma, B \supset C, B \Longrightarrow B$ by weakening

$\mathcal{D}_2 : \Gamma, C \Longrightarrow C$ by i.h. on C

$\mathcal{E}_2 : \Gamma, B \supset C, B, C \Longrightarrow C$ by weakening

$\mathcal{F}' : \Gamma, B \supset C, B \Longrightarrow C$ by $\supset L$ with $\mathcal{E}_1$ and $\mathcal{E}_2$

$\mathcal{F} : \Gamma, B \supset C \Longrightarrow B \supset C$ by $\supset R$ with $\mathcal{F}'$

$\square$

1.3 3. Proving Cuts [30 pts]

Theorem 2. *If* $\mathcal{D} : \Gamma \Longrightarrow A$ *and* $\mathcal{E} : \Gamma, A \Longrightarrow C$*, then* $\Gamma \Longrightarrow C$

Proof. By nested induction on the structure of A, the derivation of $\mathcal{D}$ and $\mathcal{E}$

Case. $B_1 \vee B_2$ is the principal formula of the final inference of both $\mathcal{D}$ and $\mathcal{E}$

Subcase. $\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \Longrightarrow B_1\end{matrix}}{\Gamma \Longrightarrow B_1 \vee B_2}$

$$\mathcal{E} : \dfrac{\begin{matrix}\mathcal{E}_1 \\ \Gamma, B_1 \vee B_2, B_1 \Longrightarrow C\end{matrix} \qquad \begin{matrix}\mathcal{E}_2 \\ \Gamma, B_1 \vee B_2, B_2 \Longrightarrow C\end{matrix}}{\Gamma, B_1 \vee B_2 \Longrightarrow C}$$

$\mathcal{D}' : \Gamma, B_1 \Longrightarrow B_1 \vee B_2$ by weakening (size of derivation preserved)
$\mathcal{F}_1 : \Gamma, B_1 \Longrightarrow C$ by i.h. on $B_1 \vee B_2, \mathcal{D}', \mathcal{E}_1$
$\Gamma \Longrightarrow C$ by i.h. on $B_1, \mathcal{D}_1, \mathcal{F}_1$

Subcase. $\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma \Longrightarrow B_2\end{matrix}}{\Gamma \Longrightarrow B_1 \vee B_2}$
Similar to previous subcase.

Case. $B_1 \vee B_2$ is not the principal formula of the final inference of $\mathcal{D}$

$$\mathcal{D} = \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Gamma, B_1 \vee B_2, B_1 \Longrightarrow A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ \Gamma, B_1 \vee B_2, B_2 \Longrightarrow A\end{matrix}}{\Gamma', B_1 \vee B_2 \Longrightarrow A}$$

$\mathcal{G} : \Gamma = \Gamma', B_1 \vee B_2$ by assumption
$\mathcal{F}_1 : \Gamma, B_1 \Longrightarrow C$ by i.h. on $A, \mathcal{D}_1, \mathcal{E}$
$\mathcal{F}_2 : \Gamma, B_1 \Longrightarrow C$ by i.h. on $A, \mathcal{D}_2, \mathcal{E}$
$\mathcal{F} : \Gamma \Longrightarrow C$ by $\vee$L with $\mathcal{F}_1, \mathcal{F}_2$ and contraction with $\mathcal{G}$

Case. $B_1 \vee B_2$ is not the principal formula of the final inference of $\mathcal{D}$

$$\textbf{SubCase.}\quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A \Longrightarrow B_1\end{array}}{\Gamma, A \Longrightarrow B_1 \vee B_2}$$

$\mathcal{F}_1 : \Gamma \Longrightarrow B_1$ by i.h. on $A, \mathcal{D}, \mathcal{E}_1$
$\mathcal{F} : \Gamma \Longrightarrow B_1 \vee B_2$ by $\vee R_1$ with $\mathcal{F}_1$

$$\textbf{SubCase.}\quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A \Longrightarrow B_2\end{array}}{\Gamma, A \Longrightarrow B_1 \vee B_2}$$

Similar to previous subcase

$$\textbf{SubCase.}\quad \mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, B_1 \vee B_2, B_1, A \Longrightarrow C\end{array} \qquad \begin{array}{c}\mathcal{E}_2\\ \Gamma, B_1 \vee B_2, B_2, A \Longrightarrow C\end{array}}{\Gamma', B_1 \vee B_2, A \Longrightarrow C}$$

$\mathcal{G} : \Gamma = \Gamma', B_1 \vee B_2$ by assumption
$\mathcal{F}_1 : \Gamma, B_1 \Longrightarrow C$ by i.h. on $A, \mathcal{D}, \mathcal{E}_1$
$\mathcal{F}_2 : \Gamma, B_2 \Longrightarrow C$ by i.h. on $A, \mathcal{D}, \mathcal{E}_2$
$\mathcal{F} : \Gamma \Longrightarrow C$ by $\vee L$ with $\mathcal{F}_1, \mathcal{F}_2$ and contraction with $\mathcal{G}$

$\square$