MATH323 - Tutorial 8 (March 8th | 2017)

1. (Exercise 4.165): Let Y have density function

$$f(y) = \begin{cases} cye^{-2y}, & 0 \leq y < \infty \\ 0, & \text{elsewhere} \end{cases}$$

a) Find the value of c that makes $f(y)$ a density function.
b) Give the mean and variance for Y.
c) Give the moment-generating function for Y.

Solution:

a) We require that $f(y) \geq 0$ for all y and that $\int_{-\infty}^{\infty} f(y)\,dy = 1$. So $c \geq 0$. Now, we solve the integral for c:

$$\int_0^\infty cye^{-2y}\,dy = c\int_0^\infty ye^{-2y}\,dy = c\left\{\frac{ye^{-2y}}{-2} + \frac{e^{-2y}}{-4}\right\}\Bigg|_0^\infty = 1$$

$$= c\left(0 + \frac{1}{4}\right) = 1$$

$$\Rightarrow c = 4 \quad \blacksquare$$

b) $$\mathbb{E}(Y) = \int_0^\infty y\, 4ye^{-2y}\,dy = 4\int_0^\infty y^2 e^{-2y}\,dy$$

$u = y^2 \quad dv = e^{-2y}$
$du = 2y \quad v = \frac{e^{-2y}}{-2}$

$$= 4\left\{\left(\frac{y^2e^{-2y}}{-2}\right)\Bigg|_0^\infty - \int_0^\infty (2y)\left(\frac{e^{-2y}}{-2}\right)dy\right\}$$

$$= 4\left\{0 + \int_0^\infty ye^{-2y}\,dy\right\} = 4\left(\frac{1}{4}\right) = 1$$

$$E(Y^2) = \int_0^\infty y^2\, 4ye^{-2y}\,dy = 4\int_0^\infty y^3 e^{-2y}\,dy \qquad \begin{matrix} u = y^3 & dv = e^{-2y} \\ du = 3y^2 & v = \frac{e^{-2y}}{-2} \end{matrix}$$

$$= 4\left\{ \left(\frac{y^3 e^{-2y}}{-2} \right)\Bigg|_0^\infty - \int_0^\infty \frac{3y^2 e^{-2y}}{-2}\,dy \right\}$$

$$= 4\left\{ 0 + \frac{3}{2}\int_0^\infty y^2 e^{-2y}\,dy \right\}$$

$$= 4\left\{ \left(\frac{3}{2}\right)\left(\frac{1}{4}\right) \right\} = \frac{3}{2}$$

$$V(Y) = E(Y^2) - (E(Y))^2 = \frac{3}{2} - 1^2 = \frac{3}{2} - \frac{2}{2} = \frac{1}{2} \quad \blacksquare$$

c) $$m(t) = \int_0^\infty e^{ty}\, 4ye^{-2y}\,dy = 4\int_0^\infty ye^{y(t-2)}\,dy$$

Observe that if $t \geq 2$, then the integral will not converge to a finite number. Thus, we require $t < 2$.

$$4\int_0^\infty ye^{y(t-2)}\,dy = 4\left\{ \frac{ye^{y(t-2)}}{t-2} - \frac{e^{y(t-2)}}{(t-2)^2} \right\}\Bigg|_0^\infty$$

$$= 4\left(0 + \frac{1}{(t-2)^2} \right)$$

$$= \frac{4}{(t-2)^2} = \frac{4}{4(1-0.5t)^2} = \frac{1}{(1-0.5t)^2} \quad \blacksquare$$

Note: the density function for Y is Gamma$(\alpha = 2, \beta = 0.5)$.

2. (Exercise 4.184): Let Y denote a random variable with probability density function given by $f(y) = \left(\frac{1}{2}\right)e^{-|y|}$, $-\infty < y < \infty$. Find the moment-generating function of Y and use it to find $E(Y)$.

Solution:

$$m(t) = \int_{-\infty}^\infty e^{ty} f(y)\,dy$$

$$= \frac{1}{2}\int_{-\infty}^\infty e^{ty} e^{-|y|}\,dy$$

$$= \frac{1}{2}\left\{\int_{-\infty}^{0} e^{ty}e^{y}\,dy + \int_{0}^{\infty} e^{ty}e^{-y}\,dy\right\}$$

$$= \frac{1}{2}\left\{\int_{-\infty}^{0} e^{y(t+1)}\,dy + \int_{0}^{\infty} e^{y(t-1)}\,dy\right\}$$

Observe the first integral is only finite for $t > -1$ and the second integral is only finite for $t < 1$. Thus the mgf exists for $|t| < 1$.

$$= \frac{1}{2}\left\{\frac{1}{t+1} e^{y(t+1)}\Big|_{-\infty}^{0} + \frac{1}{t-1} e^{y(t-1)}\Big|_{0}^{\infty}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{t+1} - \frac{1}{t-1}\right\} = \frac{1}{2}\left\{\frac{-2}{(t+1)(t-1)}\right\} = \frac{1}{1-t^2} = m(t)$$

$$E(Y) = m'(0) = \frac{d}{dt} m(t)\Big|_{t=0}$$

$$= \frac{2t}{(1-t^2)^2}\Big|_{t=0}$$

$$= 0$$

Check: $E(Y) = \int_{-\infty}^{\infty} \frac{1}{2} y e^{-|y|} = \frac{1}{2}\left\{\int_{-\infty}^{0} ye^{y}dy + \int_{0}^{\infty} ye^{-y}\,dy\right\}$

$$= \frac{1}{2}\left\{(ye^{y} - e^{y})\Big|_{-\infty}^{0} + (ye^{-y} - e^{-y})\Big|_{0}^{\infty}\right\}$$

$$= \frac{1}{2}\{(-1) + (1)\} = 0 \quad \blacksquare$$

3. (Exercise 4.144): Consider a random variable Y with density function given by $f(y) = ke^{-y^2/2}$, $-\infty < y < \infty$

a) Find k.

b) Find the moment-generating function of Y.

c) Find $E(Y)$ and $V(Y)$.

Solutions:

a) Clearly, since $f(y) \geq 0$, $k \geq 0$.

$$\int_{-\infty}^{\infty} ke^{-y^2/2}\, dy = k\int_{-\infty}^{\infty} e^{-y^2/2}\, dy = 1$$

If $X \sim \text{Normal}(0, 1)$, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$.

and $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

Thus, $k(\sqrt{2\pi}) = 1 \Rightarrow k = \frac{1}{\sqrt{2\pi}}$

b) $$m(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{ty} e^{-y^2/2}\, dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^2 - 2t)}\, dy$$

$$= e^{+\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^2 - 2t + t^2)}\, dy$$

$$= e^{\frac{1}{2}t^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-t)^2}\, dy}_{=1} = e^{\frac{1}{2}t^2}$$

c) $$E(Y) = m'(0) = te^{\frac{1}{2}t^2}\Big|_{t=0} = 0$$

$$E(Y^2) = m''(0) = \left(e^{\frac{1}{2}t^2} + t^2 e^{\frac{1}{2}t^2}\right)\Big|_{t=0} = 1$$

$$\Rightarrow V(Y) = E(Y^2) - (E(Y))^2 = 1 - 0 = 1$$

4. (Exercise 4.110): If Y has a probability density function given by $f(y) = \begin{cases} 4y^2 e^{-2y}, & y > 0 \\ 0, & \text{elsewhere} \end{cases}$

obtain $\mathbb{E}(Y)$ and $V(Y)$ by inspection.

<u>Solution</u>: Observe that if $X \sim \text{Gamma}(\alpha, \beta)$ then

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Take $\alpha = 3$ and $\beta = 1/2$ and we obtain the density of Y.

Thus, $\mathbb{E}(Y) = \alpha\beta = \dfrac{3}{2}$ and $V(Y) = \alpha\beta^2 = 3\left(\dfrac{1}{2}\right)^2 = \dfrac{3}{4}$. ∎