McGill University
Department of Mathematics and Statistics
MATH 254, Fall 2015

Assignment 3: Solutions

1. Let x be a real number. Show that, for every $\varepsilon > 0$, there exist two rational numbers q and q' such that $q < x < q'$ and $|q - q'| < \varepsilon$.

   Solution:

   We have seen the following in class:

   $(\star)$ for any $x, y \in \mathbb{R}$ with $x < y$ there exists a rational number $r \in \mathbb{Q}$ with $x < r < y$.

   Let $\varepsilon > 0$. Applying $(\star)$ to the real numbers $x - \frac{\varepsilon}{2}$ and x yields that there exists a rational number $q \in \mathbb{Q}$ with $x - \frac{\varepsilon}{2} < q < x$. Similarly, if we apply $(\star)$ to the real numbers x and $x + \frac{\varepsilon}{2}$ this yields that there exists a rational number $q' \in \mathbb{Q}$ with $x < q' < x + \frac{\varepsilon}{2}$.

   Combining both inequalities, we obtain that there exist $q, q' \in \mathbb{Q}$ with $x - \frac{\varepsilon}{2} < q < x < q' < x + \frac{\varepsilon}{2}$. Thus we have especially that $q < x < q'$. Furthermore, we have

   $$\begin{aligned} q' &< x + \frac{\varepsilon}{2} \\ -q &< -x + \frac{\varepsilon}{2} \end{aligned}$$

   Adding both inequalities yields $q' - q < 2\,\frac{\varepsilon}{2} = \varepsilon$ and thus $|q - q'| < \varepsilon$. This is what we had to show.

2. Let A and B be two nonempty subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded above if and only if both A and B are bounded above. In this case, prove that $\sup(A \cup B) = \sup\{\sup A, \sup B\}$.

   Solution:

   Note that the supremum of $\{\sup A, \sup B\}$ is just the maximum of $\sup A$ and $\sup B$. We will use $\max(\sup A, \sup B)$ in the solution below.

   <u>Boundedness</u>: Let A and B be bounded above. Then there exist $u, v \in \mathbb{R}$ with $a \leq u$ for all $a \in A$ and $b \leq v$ for all $b \in B$. Then $a \leq \max(u, v)$ and $b \leq \max(u, v)$ for all $a \in A$ and all $b \in B$ i.e. $c < \max(u, v)$ for all $c \in A \cup B$. Thus $\max(u, v)$ is an upper bound of $A \cup B$; especially, $A \cup B$ is bounded above.

   Now let $A \cup B$ be bounded above. Then there exists $w \in \mathbb{R}$ with $c \leq w$ for all $c \in A \cup B$. Especially, we have $a \leq w$ for all $a \in A$ and $b \leq w$ for all $b \in B$. This means that both A and B are bounded above.

   <u>Proof of the supremum formula</u>: Let A and B be bounded; as seen above $A \cup B$ is then also bounded and $\sup A$, $\sup B$ and $\sup(A \cup B)$ exist.

   $\sup(A \cup B) \leq \max(\sup A, \sup B)$: We have shown above that if u is any upper bound for A and v is any upper bound for B then $\max(u, v)$ is an upper bound for $A \cup B$. Since $\sup A$ resp. $\sup B$

are upper bounds for A resp. B we have that $\max(\sup A, \sup B)$ is an upper bound for $A \cup B$ which implies $\sup(A \cup B) \leq \max(\sup A, \sup B)$.

$\sup(A \cup B) \geq \max(\sup A, \sup B)$: We have also seen above that any upper bound for $A \cup B$ is also an upper bound for both A and B. Thus $\sup(A \cup B)$ is an upper bound for both A and B. This implies that both $\sup A \leq \sup(A \cup B)$ and $\sup B \leq \sup(A \cup B)$ which implies that $\max(\sup A, \sup B) \leq \sup(A \cup B)$ which is what we had to show.

Finally, this proves $\sup(A \cup B) = \max(\sup A, \sup B)$.

3. Let S be a nonempty and bounded subset of $\mathbb{R}$.

   (a) Prove that $S \subseteq [\inf S, \sup S]$.

   (b) Prove that if J is a closed interval containing S, then $[\inf S, \sup S] \subseteq J$.

   Solution:

   (a) Let $x \in S$ be arbitrary. Then $\inf S \leq x \leq \sup S$ by the definitions of inf and sup. Thus $x \in [\inf S, \sup S]$ and (since $x \in S$ was arbitrary) $S \subseteq [\inf S, \sup S]$.

   (b) We first assume additionally that J is bounded i.e. that $J = [a, b]$ for some $a, b \in \mathbb{R}$. Let $x \in S$ be arbitrary. Since $S \subseteq J$ we have $a \leq x \leq b$. Thus a is a lower bound for S and b is an upper bound for S. Consequently, $a \leq \inf S$ and $\sup S \leq b$ which we can combine into $a \leq \inf S \leq \sup S \leq b$. But this just means that $[\inf S, \sup S] \subseteq [a, b] = J$. q.e.d.

   It remains to prove the statement in case that J is unbounded. Since S is bounded, there exists a closed and bounded interval I with $S \subseteq I$. Let $\tilde{J} := J \cap I$. Then $\tilde{J}$ is closed; it is also bounded since I is bounded. Furthermore, since $S \subseteq I$ and $S \subseteq J$ we have $S \subseteq \tilde{J}$. We can thus apply our result above to $\tilde{J}$ instead of J and obtain $[\inf S, \sup S] \subseteq \tilde{J}$ which implies $[\inf S, \sup S] \subseteq J$. This completes the proof.

4. For any $n \in \mathbb{N}$ let $I_n = (0, 1/n)$ and $J_n = [0, 1/n]$. Show that $\bigcap_{n \in \mathbb{N}} I_n = \emptyset$ and $\bigcup_{n \in \mathbb{N}} J_n = \{0\}$.

   Solution:

   For the first part, assume $\exists m \in \mathbb{R}, m \in \bigcap_{n \in \mathbb{N}}$. Then, $0 < m < \frac{1}{n}, \forall n \in \mathbb{N}$. By the Archimedean property, $\forall \epsilon > 0, \exists n : \epsilon > \frac{1}{n}$. Therefore, $|0 - \frac{1}{n}| < \epsilon, \forall \epsilon > 0$. Since $|0 - m| < |0 - \frac{1}{n}| < \epsilon$, we have $|0 - m| < \epsilon, \forall \epsilon > 0$ which implies $m = 0$. Contradiction.

   For the second part, it is clear by definition of a closed interval that $0 \in [0, 1/n], \forall n \in \mathbb{N}$. Therefore, $0 \in \bigcap_{n \in \mathbb{N}} I_n$. Assume that $\exists m > 0 : m \in \bigcap_{n \in \mathbb{N}} I_n$. Then by the same reasoning as above (replace $|0 - m| < |0 - \frac{1}{n}| < \epsilon$ with $|0 - m| \leq |0 - \frac{1}{n}| < \epsilon$), $m = 0$. Contradiction.

5. If f is a function $f : D \to \mathbb{R}$ one says that f is bounded above (resp. bounded below, bounded) if the image of D under f i.e. $f(D) = \{f(x) : x \in D\}$ is bounded above (resp. bounded below, bounded). If f is bounded above (resp. bounded below), then one denotes by $\sup f$ the supremum of $f(D)$ (resp. by $\inf f$ the infimum of $f(D)$).

   Assume that two functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded above.

   (a) Prove that $f(x) \leq g(x)$ for all $x \in D$ implies $\sup f \leq \sup g$.

   (b) Show that the converse it not true by providing a concrete counterexample.

(c) Prove that $f(x) \le g(y)$ for all $x, y \in D$ if and only if $\sup f \le \inf g$.

Solution:

Let f and g be bounded.

(a) Let $x \in D$ be arbitrary. $g(x) \in g(D) \Rightarrow g(x) \le \sup g(D) = \sup g$. Furthermore, since $f(x) \le g(x)$ we have $f(x) \le \sup g$. Since $x \in D$ was arbitrary this means that $f(x) \le \sup g$ for all $x \in D$ i.e. $\sup g$ is an upper bound for $f(D)$. Thus $\sup f = \sup f(D) \le \sup g$.

(b) Let $D = [-1, 1]$ and let $f(x) := 0$, $g(x) := x$ for all $x \in D$. Then $\sup f$=0 and $\sup g = g(1) = 1$ (since g is increasing on D). We thus have $0 = \sup f \le \sup g = 1$.

But $f(x) \le g(x) \Leftrightarrow x \ge 0$ i.e. $f(x) \le g(x)$ does not hold on all of D (e.g. it does not hold at $x = -1$).

(c) Let $f(x) \le g(y)$ for all $x, y \in D$. Fixing an arbitrary y we get $f(x) \le g(y)$ for all $x \in D$ which means that $g(y)$ is an upper bound for $f(D)$. Thus $\sup f \le g(y)$ for all $y \in D$. This, in return, means that $\sup f$ is a lower bound for $g(D)$ (especially, $g(D)$ is bounded below) and, consequently, $\sup f \le \inf g$.

Now assume conversely that $\sup f \le \inf g$. Let $x, y \in D$ be arbitrary. Then $f(x) \le \sup f$ and $\inf g \le g(y)$. Combining both inequalities yields $f(x) \le \sup f \le \inf g \le g(y)$ and thus $f(x) \le g(y)$ for all $x, y \in D$.

6. Define a sequence $(x_n)_{n\in\mathbb{N}}$ by $x_1 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for any $n \in \mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ is decreasing and bounded below by $\sqrt{2}$. Prove that $(x_n)_{n\in\mathbb{N}}$ is a sequence of rational numbers converging to $\sqrt{2}$.

Solution:

To show that the sequence is decreasing, we must show that $x_{n+1} \le x_n, \forall n \ge 1$. We first note that $x_2 = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$. So indeed, $x_2 \le x_1$. In general for $n \ge 1$,

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} = \frac{x_n^2 + 2}{2x_n}.$$

Thus, we have a quadratic

$$x_n^2 - 2x_{n+1}x_n + 2 = 0$$

for which x_n is a (real) root. This means that our discriminant (our $b^2 - 4ac$ thing under the square root) must be ≥ 0. Which means

$$4x_{n+1}^2 - 4 \times 2 \ge 0 \implies x_{n+1}^2 \ge 2.$$

Now,

$$x_n - x_{n+1} = \frac{2x_n^2 - x_n^2 - 2}{2x_n} = \frac{x_n^2 - 2}{2x_n}.$$

Since $x_{n+1}^2 \ge 2$ for $n \ge 1$, we have for $n \ge 2$

$$x_n - x_{n+1} = \frac{x_n^2 - 2}{2x_n} \ge 0.$$

But we have already shown that $x_n \geq x_{n+1}$ for $n = 1$. Therefore, for $n \geq 1$,

$$x_{n+1} \leq x_n.$$

As we wanted.

Now, it is clear that the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded below by 0, as it is decreasing but every term in the sequence is positive. We have shown in class that monotonically decreasing sequences that are bounded below converge. Hence, the limit of the sequence exists. Let x be the limit of our sequence $(x_n)_{n\in\mathbb{N}}$. Then $(x_{n+1})_{n\in\mathbb{N}}$ is a tail of our sequence and must converge to the same limit as $(x_n)_{n\in\mathbb{N}}$, giving us the relation:

$$x = \frac{x^2 + 2}{2x}$$

Then, solving for x, we get

$$x = \sqrt{2}.$$

So our sequence $(x_n)_{n\in\mathbb{N}}$ converges to $\sqrt{2}$. Since our sequence is monotonically decreasing and converges to $\sqrt{2}$, $x_n \geq \sqrt{2}, \forall n \geq 1$. Thus, $\sqrt{2}$ is an lower bound for $(x_n)_{n\in\mathbb{N}}$. Finally, we can see inductively that $(x_n)_{n\in\mathbb{N}}$ is a sequence of rational numbers. $x_1 = 2$, is clearly rational. Assuming that x_n is rational, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ is also rational, as $x_n/2$, $1/x_n$ are rational by hypothesis and the sum of two rational numbers is rational.