MATH 254 Tutorial 5 (The Completeness Property):

Remark: For almost all of the following problems, you will see two similar questions at the same time. One is constructed by omitting all []s and the other is constructed by replacing the phrases inside []s with the phrases just before []s. The proof of these are completely similar to each other, so most of the time we will only prove one of them.

Problem 1: For $A \subseteq \mathbb{R}$, prove that:

a) The converse of the completeness property of real numbers is also true. So we have: "$supA \in \mathbb{R}$ [$infA \in \mathbb{R}$] exists if and only if A is nonempty and bounded above [below]."

b) $maxA$ [$minA$] exists if and only if $supA$ [$infA$] exists and is in A. Moreover in this case, we have $maxA = supA$ [$minA = infA$]. In the sense of this question, we can say that the supremum [infimum] is a generalization of the maximum [minimum].

c) If $supA$ and $infA$ both exist (according to part a, we should have that A is nonempty and bounded), then $infA \leq supA$. Moreover, prove that the equality happens if and only if $A = \{a\}$ for some $a \in \mathbb{R}$.

Problem 2: Assuming that $A, B \subseteq \mathbb{R}$ are nonempty and bounded above [below], prove the followings:

a) If $A \subseteq B$, then $supA \leq supB$ [$infB \leq infA$].

b) $sup(A \cup B)$ [$inf(A \cup B)$] exists and equals to
$max\{supA, supB\}$ [$min\{infA, infB\}$].

c) If $A \cap B$ is nonempty, prove that $sup(A \cap B) \leq min\{supA, supB\}$
[$inf(A \cap B) \geq max\{infA, infB\}$]. Give an example for strict inequality.

d) If we have $a \leq b$ for all $a \in A$ and $b \in B$, then without any boundedness assumption prove that $supA$ and $infB$ both exist and $supA \leq infB$.

e) $sup(A + B)$ [$inf(A + B)$] exists and equals to
$supA + supB$ [$infA + infB$]. Note that $A + B := \{a + b : a \in A, b \in B\}$.

f) If all members of A and B are non-negative, then show that $sup(A.B)$ [$inf(A.B)$] exists and equals to $supA.supB$ [$infA.infB$]. Note that $A.B := \{a.b : a \in A, b \in B\}$.

Problem 3: Assume that $X \subseteq \mathbb{R}$ is nonempty and $f, g : X \to \mathbb{R}$ are bounded above [bellow] (i.e. their range $f(X)$ and $g(X)$ are bounded above [below]). Defining $supf = sup_x f(x) = sup\{f(x) : x \in X\} = supf(X)$ [similar definition for infimum], prove that:

a) Any restriction of f to a nonempty set $X' \subseteq X$ will decrease [increase] its supremum [infimum]. Compare with part a of problem 2.

b) If we have $f \leq g$, then we have $supf \leq supg$ [$inff \leq infg$]. Compare with part d of problem 2.

c) $sup(f + g)$ [$inf(f + g)$] exists and
$sup(f + g) \leq supf + supg$ [$inff + infg \leq inf(f + g)$]. Compare with part e of problem 2 and give an example for strict inequality.

d) If we have $f, g \geq 0$, then we have $sup(f.g)$ [$inf(f.g)$] exists and

$sup(f.g) \leq supf.supg$ [$inff.infg \leq inf(f.g)$]. Compare with part f of problem 2 and give an example for strict inequality.

Problem 4: Let $A, B \subseteq \mathbb{R}$ be nonempty, $g : B \to \mathbb{R}$ be bounded above [bellow] and $f : A \to B$ be arbitrary.

a) Prove that $sup(g \circ f) \leq supg$ [$inf(g \circ f) \geq infg$]. Give an example for strict inequality.

b) Prove that if f is surjective, then we have the equality. Is the converse of this statement true? Why?

Problem 5: Find supremum and infimum of the set $\{1/n - 1/m : m, n \in \mathbb{N}\}$.

Problem 6: Assume that $X, Y \subseteq \mathbb{R}$ are nonempty and $f : X \times Y \to \mathbb{R}$ is bounded (i.e. its range $f(X \times Y)$ is bounded). Define

$supf = sup_{x,y}f(x, y) = sup\{f(x, y) : x \in X, y \in Y\} = supf(X \times Y)$,

$sup_x f(x, y) = sup\{f(x, y) : x \in X\} = supf(X \times \{y\})$,

and other similar definitions. Prove that:

a) $supf = sup_{x,y}f(x, y) = sup_x sup_y f(x, y) = sup_y sup_x f(x, y)$ [same equation for infimum].

b) $sup_x inf_y f(x, y) \leq inf_y sup_x f(x, y)$ [$sup_y inf_x f(x, y) \leq inf_x sup_y f(x, y)$]. Give an example for strict inequality.

Problem 7: Prove that we can define $supA$ [$infA$] for $A \subseteq \mathbb{R}$ equivalently by conditions:

- $supA$ [$infA$] is an upper [lower] bound for A.
- For any real number $x < supA$ [$x > infA$], there is an element $a \in A$ such that $a > x$ [$a < x$].