Tutorial 1 Problems (Mathematical Induction):

Problem 1: For all $n \in \mathbb{N}$, prove that:

$$1^2 - 2^2 + ... + (-1)^{n+1}n^2 = (-1)^{n+1}\frac{n(n+1)}{2}$$

Problem 2: Guess a formula for the sum $1 + 3 + ... + (2n - 1)$ and prove your conjecture using mathematical induction.

Problem 3: Guess a formula for the below sum and prove your conjecture using mathematical induction:

$$\frac{1}{1.3} + \frac{1}{3.5} + ... + \frac{1}{(2n-1).(2n+1)}$$

Problem 4: Prove that $5^n - 4n - 1$ is divisible by 16 for all $n \in \mathbb{N}$.

Problem 5: Prove that $n^2 + (n+1)^2 + (n+2)^2$ is divisible by 9 for all $n \in \mathbb{N}$.

Problems 6: Find all natural numbers n such that $n^2 < 2^n$.

Problem 7: Find the largest natural number m such that $n^3 - n$ is divisible by m for all $n \in \mathbb{N}$.

Problem 8: For all natural numbers $n > 1$, prove that:

$$\frac{1}{\sqrt{1}} + ... + \frac{1}{\sqrt{n}} > \sqrt{n}$$

Problem 9: Only using the definition of prime numbers, prove that any natural number $n > 1$ can be written as a product of prime numbers.

Problem 10: Consider Fibonacci numbers ($F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2} n > 2$), prove that F_{3k} is even for all $k \in \mathbb{N}$.

Problem 11: Guess the number of elements of the power set of a set with n elements and prove your conjecture using mathematical induction.

Problem 12: Prove that you can change n cents using only 6-cent and 11-cent coins for all natural numbers $n \geq 60$.

Problem 13: Prove that for all $n \in \mathbb{N}$, if any one square is removed from a $2^n \times 2^n$ checkerboard, then the remaining squares can be completely and exactly covered by L-shaped tiles of 3 squares.

Problem 14: Prove that for all $n \in \mathbb{N}$, if one corner triangle is removed from an equilateral traingle devided to 4^n congruent equilateral triangles, then the remaining $4^n - 1$ triangles can be completely and exactly covered by trapezoidal shaped tiles of 3 triangles.

Problem 15: For all $n \in \mathbb{N}$, prove that the arithmetic mean of n non-negative real numbers cannot be less than their geometric mean:

$$\frac{a_1 + ... + a_n}{n} \geq \sqrt[n]{a_1 \times ... \times a_n}$$

Problem 16: Consider a game with 3 rods and $n \in \mathbb{N}$ number of disks of different sizes. The game starts with the disks in a neat stack in ascending order of size (the smallest at the top) on the first rod. The goal is to move the entire stack to the second rod, obeying the following rules:

- Only one disk that is the uppermost disk in its stack can be moved to the top of another stack at a time.

- No disk can be placed on top of a smaller disk.

Guess the least number of moves is needed to finish the game and prove your conjecture.

Problem 17: There are n couples living in a town (no man is married to more than one woman and vice versa). One day the king announced that:

"There exist some women who are cheating on their husband and everybody knows them as a cheater except their own husband. No other woman should cheat on her husband. All men should deeply think about their own wife without talking to other people and the midnight right after understanding that his wife is definitely a cheater the couple should leave the town without noticing others. Each morning, we will announce to the public the couples that left the town in the previous midnight."

Assuming all people in the town are wise enough, prove that eventually the town will have no cheater.