RICE'S THEOREM

Let Prog be the set of all programs in some Turing-complete programming language. The set of such programs can be enumerated effectively. For simplicity, assume all programs take a natural number as input and produce a natural number as output. Every $P \in$ Prog defines a CE set (or a Turing-recognizable set) of pairs

$$[\![P]\!] = \{ (x,y) \mid P(x) = y \}$$

$P(x) = y$ means: P executed with input x terminates with output y. Two different programs may compute exactly the same set: $P_1 \neq P_2$ but $[\![P_1]\!] = [\![P_2]\!]$ may occur.

We say $P_1 \sim P_2$ if $[\![P_1]\!] = [\![P_2]\!]$: clearly an equivalence relation.

In terms of Turing machines viewed as acceptors we can say $M_1 \sim M_2$ if $L(M_1) = L(M_2)$.

We say Q : Prog $\to \{T, F\}$ is a property of programs.

We say Q is an extensional property of programs if

$$P_1 \sim P_2 \Rightarrow Q(P_1) \Leftrightarrow Q(P_2)$$

for TM's $\quad M_1 \sim M_2 \Rightarrow Q(M_1) \Leftrightarrow Q(M_2)$

Examples (i) This program runs in $O(n^2)$: NOT extensional

(ii) This program sorts its input : extensional

(iii) This program is 100K lines of code : NOT extensional.

An extensional property is only sensitive to IO behaviour. It ignores the actual text or running time or any performance characteristics. In terms of CE sets we say it is a property of the CE set & not of the TM. In software engineering we call it a <u>functional</u> <u>specification</u>.

Two TRIVIAL PROPERTIES: $Q_F(P) = F$ for every P and $Q_T(P) = T$ for every P.

THM (RICE) Every non-trivial extensional property (i.e. a property of CE sets) is undecidable.

<u>PROOF</u> Let Q be a non-trivial property of CE sets i.e. $\exists P$ s.t. $Q(P) = T$ & $\exists P'$ s.t. $Q(P') = F$.

Assume $EMPTY = \{\langle M\rangle \mid L(M) = \phi\}$ does not satisfy Q i.e. $\forall M \quad L(M) = \phi \Rightarrow \neg Q(M)$. [HARMLESS ASSUMPTION]

Let M_0 be such that $Q(M_0) = T$. Of course $L(M_0) \neq \phi$.

I will show $A_{TM} \leq_m L_Q = \{\langle M\rangle \mid Q(M) = T\}$.

$\longrightarrow$ INPUT: $\langle M, w\rangle$, HAVE GADGET to SOLVE (DECIDE) $x \in L_Q$?

CONSTRUCT M' from $\langle M, w\rangle$ as follows

M' on input x:
1. Simulate M on w
2. If M accepts w then simulate M_0 on x.

FEED $\langle M'\rangle$ to L_Q? GADGET.

If M accepts w $\quad L(M') = L(M_0)$
otherwise $\quad L(M') = \phi$

Since Q is an extensional property

$L(M') = L(M_0) \Longrightarrow Q(M') = T$; $\quad L(M') = \phi \Rightarrow Q(M') = F$

So my L_Q gadget <u>decides</u> whether M accepts w.

$A_{TM} \leq L_Q$?

Thus L_Q must be undecidable. END OF PROOF