MATH323 - Calculus Exercise Sheet (Univariate Integration)

1. $\int_2^3 \frac{1}{4}\,dx$ 2. $\int_1^3 \frac{x}{2}\,dx$ 3. $\int_1^3 \frac{x^2}{2}\,dx$ 4. $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x e^{-\frac{x^2}{2}}\,dx$ 5. $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x^2 e^{-\frac{x^2}{2}}\,dx$

6. $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x e^{-\frac{(x-3)^2}{2}}\,dx$ 7. $\int_{-\infty}^{\infty} \frac{1}{3\sqrt{2\pi}}\, x e^{-\frac{x^2}{18}}\,dx$ 8. $\int_0^{\infty} \frac{1}{3} e^{-y/3}\,dy$ 9. $\int_0^{\infty} \frac{1}{3} y e^{-y/3}\,dy$

10. $\int_0^{\infty} \frac{1}{3} e^{ty} e^{-y/3}\,dy$; $t < \frac{1}{3}$ 11. $\int_0^{\infty} \frac{1}{5} x^2 e^{-x/5}\,dx$ 12. Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Show $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, x e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \mu$

Solutions :

1. $$\int_2^3 \frac{1}{4}\,dx = \frac{1}{4}x\Big|_{x=2}^{x=3} = \frac{1}{4}(3) - \frac{1}{4}(2) = \frac{1}{4}$$

2. $$\int_1^3 \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_1^3 = \frac{3^2}{4} - \frac{1^2}{4} = \frac{8}{4} = 2$$

3. $$\int_1^3 \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_1^3 = \frac{3^3}{6} - \frac{1^3}{6} = \frac{26}{6} = \frac{13}{3}$$

4. $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x e^{-\frac{x^2}{2}}\,dx = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\Big|_{-\infty}^{\infty} = \lim_{x\to\infty}\left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) - \lim_{x\to-\infty}\left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\right) = 0 - 0 = 0$$

5. $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x^2 e^{-\frac{x^2}{2}}\,dx \underset{\uparrow}{=} -\frac{1}{\sqrt{2\pi}}\, x e^{-\frac{x^2}{2}}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \lim_{x\to\infty}\left(-\frac{1}{\sqrt{2\pi}}\, x e^{-\frac{x^2}{2}}\right) - \lim_{x\to-\infty}\left(-\frac{1}{\sqrt{2\pi}}\, x e^{-\frac{x^2}{2}}\right) + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$$

integration by parts

$du = xe^{-\frac{x^2}{2}}$ $v = x$

$u = -e^{-\frac{x^2}{2}}$ $dv = 1$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = 1 \text{ (a result learned later)}$$

6. $$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, x e^{-\frac{(x-3)^2}{2}}\,dx$$ Let $v = x-3 \Rightarrow dv = dx$, $x = v+3$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} (v+3) e^{-\frac{v^2}{3}}\,dv$$
$$= \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, v e^{-\frac{v^2}{3}}\,dv}_{=0 \text{ by Q.4}} + 3\underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{3}}\,dv}_{=1 \text{ by Q.5}} = 0 + 3(1) = 3$$

7. $$\int_{-\infty}^{\infty} \frac{1}{3\sqrt{2\pi}}\, x e^{-x^2/18}\,dx = \left(\frac{-1}{3\sqrt{2\pi}}\right) 9 e^{-x^2/18}\Big|_{-\infty}^{\infty} = \lim_{x\to\infty}\left(\frac{-9}{3\sqrt{2\pi}} e^{-x^2/18}\right) - \lim_{x\to-\infty}\left(-\frac{9}{3\sqrt{2\pi}} e^{-x^2/18}\right) = 0 + 0 = 0$$

8. $$\int_0^{\infty} \frac{1}{3} e^{-y/3}\,dy = -e^{-y/3}\Big|_0^{\infty} = \lim_{y\to\infty}\left(-e^{-y/3}\right) - \left(-e^{-0/3}\right) = 1$$

9. $$\int_0^{\infty} \frac{1}{3} y e^{-y/3}\,dy \underset{\uparrow}{=} \frac{1}{3}\left(-3ye^{-y/3}\right)\Big|_0^{\infty} - \int_0^{\infty} \frac{1}{3}\left(-3e^{-y/3}\right)dy = \int_0^{\infty} e^{-y/3}\,dy = -3e^{-y/3}\Big|_0^{\infty} = 3$$

integration by parts

$du = e^{-y/3}$ $v = y$

$u = -3e^{-y/3}$ $dv = 1$

10. $$\int_0^{\infty} \frac{1}{3} e^{ty - y/3}\,dy = \frac{1}{3(t - 1/3)} e^{ty - y/3}\Big|_0^{\infty} = \frac{1}{1/3 - t}$$

11. $$\int_0^{\infty} \frac{1}{5} x^2 e^{-x/5}\,dx \underset{\uparrow}{=} \frac{1}{5}\left(-5e^{-x/2}x^2\right)\Big|_0^{\infty} - \int_0^{\infty} \frac{1}{5}\left(-5e^{-x/5}\right)(2x)\,dx = \int_0^{\infty} 2xe^{-x/5}\,dx = 2(5)\underbrace{\int_0^{\infty} \frac{1}{5} x e^{-x/5}\,dx}_{=5}$$

$$= 50$$

integration by parts

$du = e^{-x/5}$ $v = x^2$

$u = -5e^{-x/5}$ $dv = 2x$

12. Left as Exercise

1. $\displaystyle\int_0^\infty\int_0^\infty e^{-y_1-3y_2}\,dy_1\,dy_2$ 2. $\displaystyle\int_0^\infty\int_0^\infty y_1 e^{-y_1-y_2}\,dy_1\,dy_2$ 3. $\displaystyle\int_2^3\int_{-1}^5 \frac{1}{6}\,dx_1\,dx_2$

4. Let $f(y_1,y_2)=\frac{1}{3}e^{-y_1-y_2}$. Calculate the volume projected by $f(y_1,y_2)$ onto the region enclosed by the intersection of the regions: $y_1>0$, $y_1<y_2$, $y_1<5$, $y_2>0$.

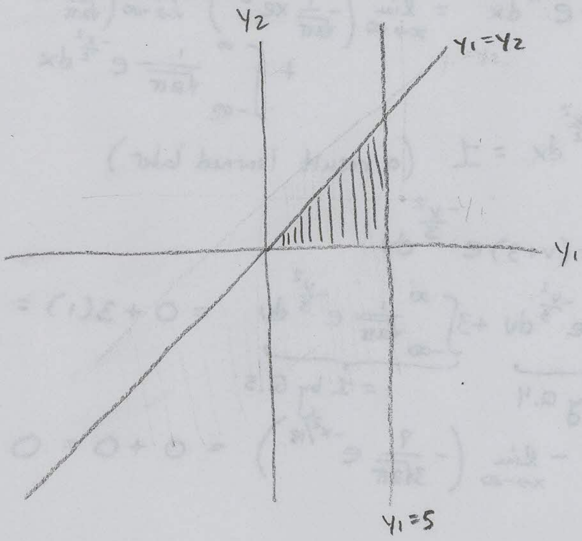
Solutions:

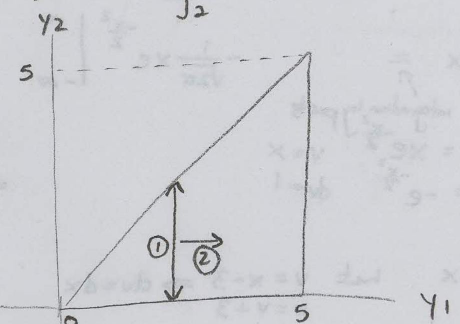
1. $$\int_0^\infty \left(-e^{-y_1-3y_2}\right)\Big|_{y_1=0}^{y_1=\infty} dy_2 = \int_0^\infty \left\{\lim_{y_1\to\infty}\left(-e^{-y_1}e^{-3y_2}\right)\right\} - \left(-e^{0}e^{-3y_2}\right) dy_2 = \int_0^\infty e^{-3y_2}\,dy_2$$
$$= -\frac{1}{3}e^{-3y_2}\Big|_0^\infty = \frac{1}{3}$$

2. $$\int_0^\infty\int_0^\infty y_1e^{-y_1}e^{-y_2}\,dy_1\,dy_2 = \int_0^\infty e^{-y_2}\Big(\underbrace{\int_0^\infty y_1e^{-y_1}\,dy_1}_{\text{by Q.9 of prev sheet}\;=1}\Big)dy_2 = \int_0^\infty e^{-y_2}\,dy_2 = -e^{-y_2}\Big|_0^\infty = 1$$

3. $$\int_2^3\int_{-1}^5 \frac{1}{6}\,dx_1\,dx_2 = \int_2^3\left(\frac{x_1}{6}\Big|_{-1}^5\right)dx_2 = \int_2^3 \frac{5}{6}-\left(-\frac{1}{6}\right)dx_2 = \int_2^3 1\,dx_2 = 3-2 = 1$$

4.





Integrate from $y_2=0$ to $y_2=y_1$ then integrate that line segment from $y_1=0$ to $y_1=5$.

$$\int_0^5\int_0^{y_1} f(y_1,y_2)\,dy_2\,dy_1 = \int_0^5\int_0^{y_1}\frac{1}{3}e^{-y_1-y_2}\,dy_2\,dy_1 = \int_0^5\left(-\frac{1}{3}e^{-y_1-y_2}\right)\Big|_{y_2=0}^{y_2=y_1} dy_1$$

$$= \int_0^5 -\frac{1}{3}e^{-2y_1}+\frac{1}{3}e^{-y_1}\,dy_1 = \frac{1}{6}e^{-2y_1}-\frac{1}{3}e^{-y_1}\Big|_0^5$$
$$= \left(\frac{1}{6}e^{-10}-\frac{1}{3}e^{-5}\right)-\left(\frac{1}{6}-\frac{1}{3}\right)$$
$$= \frac{1}{6}-\frac{1}{3}e^{-5}+\frac{1}{6}e^{-10}.$$