# Assignment 5 – COMP 527: Computation and Logic

Winter 2016
Due 12 April, 2016

**Exercise 1 (15 points)** Prove the following inversion lemma using cut.

**Theorem** : If $\Gamma \Longrightarrow A \supset B$ then $\Gamma, A \Longrightarrow B$.

**Exercise 2 (35 points)** Previously, we have defined $A \overline{\wedge} B$ in the natural deduction calculus as follows:

$$\dfrac{\begin{array}{c}\overline{A}^{\,u} \quad \overline{B}^{\,v} \\ \vdots \\ p\end{array}}{A \overline{\wedge} B}\ \overline{\wedge} I^{p,u,v} \qquad \dfrac{A \overline{\wedge} B \qquad A \qquad B}{C}\ \overline{\wedge} E$$

10 points Turn the natural deduction rules into sequent calculus rules deriving $\overline{\wedge} R$ and $\overline{\wedge} L$ rules.

20 points Are any of the rules for $\overline{\wedge}$ invertible? - If yes, prove that this is the case. If not, give a counter example.

5 points Given your previous analysis regarding invertibility of the left and right rule for $\overline{\wedge}$, explain in which phase you would add these rules to the focusing calculus. Should they be in the asynchronous or synchronous phase.

**Exercise (50 points)** In class we proved weak normalization. Here we ask you to extend the proof to cover products (i.e. conjunctions).

$$\begin{array}{lll} \text{Terms } M, N & ::= & \ldots \mid \langle M,\ N\rangle \mid \mathsf{fst}\ M \mid \mathsf{snd}\ M \\ \text{Types } A & ::= & \ldots \mid A \times B \end{array}$$

The proof of weak normalization was done in two steps:

- Define a notion of reducibility $\mathcal{R}_A$.

$$\begin{array}{lll} \mathcal{R}_{\mathsf{unit}} & = & \{M \mid M \text{ halts}\} \\ \mathcal{R}_{A \to B} & = & \{M \mid \forall N.\text{if } N \in \mathcal{R}_A\ \mathit{then} M\ N \in \mathcal{R}_B\} \end{array}$$

  where $M$ halts, iff $\exists V.\ M \longrightarrow^* V$ where $V$ is a value

- Prove that evaluation of well-typed halts.

1 If $M \in \mathcal{R}_A$ then $M$ halts.

2 If $M : A$ then $M \in \mathcal{R}_A$.

Recall that in order to prove the second statement, we proved a generalization, called the fundamental lemma:

**Lemma 1** [Fundamental Lemma]
If $\Gamma \vdash M : A$ and $\sigma \in \mathcal{R}_\Gamma$ then $[\sigma]M \in \mathcal{R}_A$.

where

$$\begin{array}{lcl} \mathcal{R}_{\cdot} & = & \{\cdot\} \\ \mathcal{R}_{\Gamma, x:A} & = & \{\sigma, M \mid M \in \mathcal{R}_A \text{ and } \sigma \in \mathcal{R}_\Gamma\} \end{array}$$

and we relied on the property that reductions were backwards closed.

**Lemma 2** [Backwards Closed]

1. If $M \longrightarrow M'$ and $M' \in \mathcal{R}_A$ then $M \in \mathcal{R}_A$.
2. If $M \longrightarrow^* M'$ and $M' \in \mathcal{R}_A$ then $M \in \mathcal{R}_A$.

10 points Extend the definition of reducibility to $A \times B$.

20 points Show the additional cases for the Backwards closed lemma (Part 1).

20 points Show the additional cases in the fundamental lemma.

**Exercise 5 (0 points)** Fill out your course evaluations!