MATH 254 Assignment 5 McGill University
Fall 2016

You should work carefully on all problems. However you only have to hand in solutions to problems 6 and 7. This assignment is due on Thursday, November 17 in class and will be the last graded assignment for this course.

Exercise 1. Using the "ε-δ definitions" of limits of functions, show that

(1) $\lim\limits_{x\to a} \dfrac{x}{1+x} = \dfrac{a}{1+a}$

(2) $\lim\limits_{x\to +\infty} \dfrac{x}{1+x} = 1$

(3) $\lim\limits_{x\to -1^+} \dfrac{x}{1+x} = -\infty$

(4) $\lim\limits_{x\to -1^-} \dfrac{x}{1+x} = +\infty$

Exercise 2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Show that

(1) $\lim\limits_{x\to 0} f(x) = 0$

(2) $\lim\limits_{x\to a} f(x)$ does not exist for all $a \neq 0$.

Exercise 3. Let $A \subseteq \mathbb{R}$, $l \in \mathbb{R}$, c be a cluster point of A, and $f : A \to \mathbb{R}$ be a function. Show that

$$\lim_{x\to c} f(x) = l \quad \Longleftrightarrow \quad \lim_{x\to c} |f(x) - l| = 0 \quad \Longleftrightarrow \quad \lim_{x\to 0} f(x+c) = l.$$

Exercise 4. Let $A \subseteq \mathbb{R}$, c be a cluster point of A, and $f, g : A \to \mathbb{R}$ be two functions such that $\lim\limits_{x\to c} g(x) = 0$ and f is bounded on a neighborhood of c, namely such that there exists $M, \delta > 0$ such that $|f(x)| < M$ for all $x \in (c-\delta, c+\delta)$. Show that $\lim\limits_{x\to c} (f(x)\, g(x)) = 0$.

Exercise 5. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim\limits_{x\to +\infty} f(x) = l \in \mathbb{R}$ and f is periodic, namely such that there exists $T > 0$ such that $f(x+T) = f(x)$ for all $x \in \mathbb{R}$. Show that f is constant.

Exercise 6. [10 points] Determine at which numbers $x \in \mathbb{R}$ the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \lfloor x \rfloor + \sqrt{x - \lfloor x \rfloor}$ is continuous.

Exercise 7. [10 points] Show that the function $f : [0,1] \to [0,1]$ defined as

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap (0,1] \setminus \{1/2\} \\ 1-x & \text{if } x \in (0,1) \setminus \mathbb{Q} \\ 0 & \text{if } x = 1/2 \\ 1/2 & \text{if } x = 0. \end{cases}$$

is bijective but not continuous at any point of the interval $[0,1]$.