**Problem 1.** Let $Q \subset \mathbb{R}^d$ be a cube, and suppose $f, g : Q \to \mathbb{R}$ are integrable functions. Prove that $|f|$ and $fg$ are integrable.

**Solution.** Fix $\epsilon > 0$. Since $f$ is integrable, there exists a partition of $Q$ and simple functions $h, k$ subordinate to this partition so that $h \le f \le k$ and $\int_Q k(x) - h(x)\, dx < \epsilon$.

Define two functions $M(x) := \max(|k(x)|, |h(x)|)$ and $m(x) := \min(|k(x)|, |h(x)|)$. It is clear that $M$ and $m$ are simple functions subordinate to our partition, and that

$$m(x) \le |f(x)| \le M(x).$$

Furthermore, we estimate $M(x) - m(x) \le k(x) - h(x)$ (this is intuitively obvious and can easily proved using the reverse triangle inequality $||b| - |a|| \le |b - a|$). Thus

$$\int_Q M(x) - m(x)\, dx \le \int_Q k(x) - h(x)\, dx < \epsilon,$$

and since $\epsilon$ was arbitrary, we conclude $|f(x)|$ is integrable.

To prove that $fg$ is integrable, we note that both $f, g$ are bounded (since they can be bounded by simple functions, which are obviously bounded), and so we let $B/2$ be a common bound so that

$$\max(|f|, |g|) \le B/2.$$

In particular, letting $F := f + B/2$ and $G := g + B/2$, we have $F, G$ nonnegative, integrable and bounded above by $B$.

Then we let $h, j, k, \ell$ be non-negative simple functions so that (i) $h \le F \le j$, $k \le G \le \ell$, (ii) $j, k \le B$ and

(iii) $$\int_Q j(x) - h(x)\, dx + \int_Q \ell(x) - k(x)\, dx \le \epsilon/B.$$

(Why can (i), (ii) and (iii) be satisfied?). Then we estimate

$$h(x)k(k) \le F(x)G(x) \le j(x)\ell(x),$$

where we use the fact that $h, k, j, \ell, F, G$ are all non-negative! We compute

$$\begin{aligned}\int_Q j(x)\ell(x) - h(x)k(x)\, dx &= \int_Q j(x)\ell(x) - j(x)k(x) + j(x)k(x) - h(x)k(x)\, dx \\ &\le B \int_Q \ell(x) - k(x) + j(x) - h(x)\, dx \le \epsilon,\end{aligned}$$

where we have used (ii) in the second inequality, and (iii) in the third inequality. Since $\epsilon$ was arbitrary, we conclude that $FG$ is integrable. Noting that

$$FG = (f + B/2)(g + B/2) = fg + B(f + g)/2 + B^2/4,$$

we conclude

$$fg = FG - B(f + g)/2 - B^2/4,$$

and consequently $fg$ is integrable if $f+g$ is integrable. Proving that $f+g$ is integrable is easy, and is left to the reader. □

**Problem 2.** Compute the volume of the four dimensional ball with radius $r$.

**Solution.** Let $B$ be the ball $\left\{x \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 < r^2\right\}$. Consider the cube $Q := (0,r)\times(0,\pi)\times(0,\pi)\times(0,2\pi)$, with generic element of $Q$ denoted $(\rho,\phi,\psi,\theta) \in Q$ and consider the map $P : Q \to B$ defined by

$$P(\rho,\phi,\psi,\theta) = (\rho\sin\phi\sin\psi\sin\theta, \rho\sin\phi\sin\psi\cos\theta, \rho\sin\phi\cos\psi, \rho\cos\phi).$$

First we compute that

$$DP = \begin{pmatrix} \sin\phi\sin\psi\sin\theta & \rho\cos\phi\sin\psi\sin\theta & \rho\sin\phi\cos\psi\sin\theta & \rho\sin\phi\sin\psi\cos\theta \\ \sin\phi\sin\psi\cos\theta & \rho\cos\phi\sin\psi\cos\theta & \rho\sin\phi\cos\psi\cos\theta & -\rho\sin\phi\sin\psi\sin\theta \\ \sin\phi\cos\psi & \rho\cos\phi\cos\psi & -\rho\sin\phi\sin\psi & 0 \\ \cos\phi & -\rho\sin\phi & 0 & 0 \end{pmatrix},$$

and a straightforward computation yields

$$|\det DP| = \rho^3\sin^2\phi\sin\psi.$$

Note that $\det DP(\rho,\phi,\psi,\theta) \neq 0$ whenever $(\rho,\phi,\psi,\theta) \in Q$.

Now we check that $P(\rho,\phi,\psi,\theta) \in B$ whenever $(\rho,\phi,\psi,\theta) \in Q$ (this is left to reader). We also check that $P$ is injective: clearly $P(\rho,\phi,\psi,\theta) = P(\rho',\phi',\psi',\theta')$ implies that $\rho = \rho'$. Then using $\rho = \rho'$ and $\rho\cos\phi = \rho'\cos\phi'$, and $\phi,\phi' \in (0,\pi)$ we deduce $\phi = \phi'$. Similarly we use $\rho = \rho'$ and $\phi = \phi'$ to deduce $\psi = \psi'$ and ultimately $\theta = \theta'$ (in this order; the details are left to the reader). In particular, $P : Q \to P(Q)$ is a $C^1$-bijection, and thus $P$ has an inverse $P^{-1} : P(Q) \to Q$ (a priori, we do not know that $P^{-1}$ is continuous or differentiable). However, since $\det DP \neq 0$ everywhere in $Q$, we conclude by the "inverse function theorem" that it *locally* has a $C^1$ inverse, but any local inverse must agree with the pre-existing inverse $P^{-1}$; by this argument we deduce that $P^{-1}$ is locally $C^1$. Since $C^1$-differentiability is a local property, we conclude that $P^{-1}$ is $C^1$. Therefore $P : Q \to P(Q)$ is a $C^1$ diffeomorphism, and we may apply the change of variables formula for integrals to conclude

$$\mathrm{Vol}(P(Q)) := \int_{P(Q)} dx_1\,dx_2\,dx_3\,dx_4 = \int_Q |DP(\rho,\phi,\psi,\theta)|\;d\rho\,d\phi\,d\psi\,d\theta.$$

Next we prove that $P(Q)$ is *almost* all of $B$. In particular we will show that if $x \in B$ satisfies $x_1 \neq 0$, then there is $(\rho,\phi,\psi,\theta)$ so that $P(\rho,\phi,\psi,\theta) = x$. To show this, pick $\rho > 0$ so that $x_1^2 + x_2^2 + x_3^2 + x_4^2 = \rho^2$, then pick $\phi \in (0,\pi)$ so that $\rho\cos\phi = x_4$, this is possible since $x_4 < \rho$ (since $x_1 \neq 0$). Then pick $\psi \in (0,\pi)$ so that $\rho\sin\phi\cos\psi = x_3$. This is possible since $x_3^2 + x_4^2 < \rho^2$ (since $x_1 \neq 0$), so $\rho\sin\phi > x_3$. Then

$$x_1^2 + x_2^2 = \rho^2(1 - \cos^2\phi - \sin^2\phi\cos^2\psi) = \rho^2\sin^2\phi\sin^2\psi,$$

and so $x_1 = \rho \sin\phi \sin\psi \sin\theta$ and $x_2 = \rho \sin\phi \sin\psi \cos\theta$ for some $\theta$. Since $x_1 \neq 0$, $\theta$ can be chosen in $(0, 2\pi)$. Therefore $B \setminus P(Q)$ is contained in the hyperplane $\{x_1 = 0\}$. Since $\mathrm{Vol}(B) = \mathrm{Vol}(B \setminus P(Q)) + \mathrm{Vol}(P(Q))$, and $\mathrm{Vol}(\{x_1 = 0\}) = 0$ (prove this last part explicitly, its not hard), we conclude

$$\mathrm{Vol}(B) := \int_0^r \int_0^\pi \int_0^\pi \int_0^{2\pi} \rho^3 \sin^2\phi \sin\psi \, d\rho \, d\phi \, d\psi \, d\theta = \frac{\pi^2 r^4}{2}.$$

□

**Problem 3.** Compute the mass of the solid ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1$ if its mass density is given by $\mu(x, y, z) = x^2 + y^2 + z^2$.

**Solution.** Let $E$ denote the solid ellipsoid, and let $B$ denote the unit ball in $\mathbb{R}^3$. Consider the map $L : B \to E$ given by $L(u, v, w) = (au, bv, cw)$. It is exceedingly easy to verify that $L$ is a $C^1$ diffeomorphism with constant Jacobian equal to $abc$, and hence we may apply the change of variables formula for integrals to write

$$\int_E x^2 + y^2 + z^2 \, dx \, dy \, dz = abc \int_B (au)^2 + (bv)^2 + (cw)^2 \, du \, dv \, dw.$$

Consider the cube $Q := (0, 1) \times (0, \pi) \times (0, 2\pi)$ and map $P : Q \to B$ by

$$P(\rho, \phi, \theta) = (\rho \sin\phi \sin\theta, \rho \sin\phi \cos\theta, \rho \cos\phi).$$

Similarly to the last problem, we deduce that $P : Q \to P(Q)$ is a $C^1$ diffeomorphism with $|DP(\rho, \phi, \theta)| = \rho^2 \sin\phi$, and that $B \setminus P(Q)$ is a neglible set, so that

$$\int_B (au)^2 + (bv)^2 + (cw)^2 \, du \, dv \, dw = \int_Q ((a \sin\phi \sin\theta)^2 + (b \sin\phi \cos\theta)^2 + (c \cos\phi)^2) \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Using the facts that

(i) $$\int_0^\pi \sin^3\phi \, d\phi = \frac{4}{3} \quad \text{and} \quad \int_0^\pi \cos^2\phi \sin\phi \, d\phi = \frac{2}{3}$$

(ii) $$\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \cos^2\theta \, d\theta = \pi,$$

and (iii) $\int_0^1 \rho^4 \, d\rho = 1/5$, we deduce that

$$\text{mass of ellipsoid} = \frac{4\pi}{15} abc(a^2 + b^2 + c^2).$$

□

**Problem 4.** Let $B$ be a ball of radius $a > 0$ centered at the origin, let $\mu : B \to [0, \infty)$ be a radially symmetric function, and compute the vector-valued function

(∗) $$F(y) := \int_B \frac{\mu(x)}{|x - y|^2} \frac{x - y}{|x - y|} \, d^3x$$

**Solution.** This is undeniably a hard problem. The integrals can be computed by brute force, but it is very ugly in my opinion.

To compute $F(y)$, we will show that $F$ is the gradient vector field of a simpler (scalar) function $\varphi$. The gradient of a scalar function is $\nabla\varphi := (D\varphi)^T$.

**Claim.** The vector-valued function $F(y)$ defined in $(*)$ satisfies $-\nabla\varphi(y) = F(y)$, where

$$\varphi(y) := \int_B \frac{\mu(x)}{|x-y|}\, d^3x. \tag{$**$}$$

Before we prove this claim, we wish to emphasize one thing: $F$ is a vector-valued function *defined* by an "integral expression," and it happens that is the gradient of a function $\varphi$ which is also defined by an "integral expression." To prove our claim, we require one lemma from advanced calculus (this lemma is crucial for working this functions defined by "integral expressions")

**Lemma** (differentiation under the integral sign)**.** Let $K$ be a compact Jordan region, let $U$ be an open region, and let $a : K \times U \to \mathbb{R}$ be a $C^1$-function. Define

$$\varphi(y) := \int_K a(x,y)\, d^3x.$$

Then $\varphi$ is $C^1$ and

$$\nabla\varphi(y) = \int_K \nabla a(x,y)\, d^3x.$$

**Proof.** To check this, it suffices to prove that the partial derivative $\partial\varphi/\partial y_1$ is continuous on $U$ and is given by

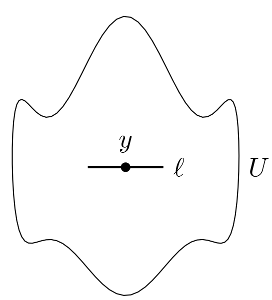
$$\frac{\partial\varphi(y)}{\partial y_1} = \int_K \frac{\partial a(x,y)}{\partial y_1}\, d^3x. \tag{1}$$

It can be shown that the right hand side of (1) is continuous (the proofs I know are beyond the scope of this course: but it is certainly within the scope of the course to consider the problem).

We will show that (1) holds. We fix $y \in U$, and consider a line segment

$$\ell := [y_1 - \delta_-, y_1 + \delta_+] \times \{y_2\} \times \cdots \times \{y_n\}$$

which we may assume lies entirely in $U$.



Consider the double integral

$$\int_\ell \int_K \frac{\partial a(x,y)}{\partial y^1}\, d^3x\, dy = \int_K \int_\ell \frac{\partial a(x,y)}{\partial y^1}\, dy\, d^3x,$$

where we have used Fubini's theorem. Using the fundamental theorem of calculus in the second integral we conclude

$$\int_K a(x, y_1+\delta_+, y_2, \cdots) - a(x, y_1-\delta_-, y_2, \cdots)\, d^3x = \int_K \int_\ell \frac{\partial a(x,y)}{\partial y^1}\, dy\, d^3x,$$

and then using

$$\varphi(y_1+\delta_+, y_2, \cdots) - \varphi(y_1-\delta_-, y_2, \cdots) = \int_K a(x, y_1+\delta_+, y_2, \cdots) - a(x, y_1-\delta_-, y_2, \cdots)\, d^3x,$$

we conclude

$$\varphi(y_1+\delta_+, y_2, \cdots) - \varphi(y_1-\delta_-, y_2, \cdots) = \int_\ell \int_K \frac{\partial a(x,y)}{\partial y_1}\, d^3x\, dy = \int_{y_1-\delta_-}^{y_1+\delta_+} \int_K \frac{\partial a(x,y)}{\partial y_1}\, d^3x\, dy.$$

The fundamental theorem of calculus proves that $\varphi$ is differentiable in the $y_1$ direction (we need to know that the right hand side of (1) is continuous to apply the fundamental theorem) and

$$\frac{\partial \varphi}{\partial y_1}(y) := \int_K \frac{\partial a(x,y)}{\partial y_1}\, d^3x.$$

And since the right hand side of the above is continuous, we conclude that $\varphi$ is continuously differentiable on $U$ and that (1) holds throughout. This completes the proof. $\square$

Now we apply this "differentiation under the integral sign" result to the problem at hand. Since $y$ lies outside the sphere, we know that $f \in B \times \mathbb{R}^3 \backslash B \to \mathbb{R}$ given by $f(x,y) := 1/|x-y|$ satisfies the assumptions of the lemma (i.e. we assume in this problem that $B$ is closed), and so we have

$$\varphi(y) := \int_B \frac{\mu(x)}{|x-y|}\, d^3x \implies \nabla\varphi(y) = -\int_B \frac{\mu(x)}{|x-y|^2} \frac{x-y}{|x-y|}\, d^3x =: F(y),$$

where we have used the fact that

$$\nabla \frac{1}{|x-y|} = \frac{y-x}{|x-y|^3} \qquad \text{(check this!)}$$

Now we compute $\varphi(y)$, and then take its gradient to obtain $F(y)$. First we note that $\varphi(y)$ is also radially symmetric and so its gradient must always point in the radial direction; hence along the $z$-axis we have

$$\nabla\varphi(0,0,\xi) = (0, 0, \frac{\partial \varphi}{\partial \xi}).$$

With this in mind, we let $y = (0,0,\xi)$ and parametrize $B$ by spherical coordinates $x = (\rho \sin\phi \sin\theta, \rho \sin\phi \cos\theta, \rho\cos\phi)$. Hence

$$\varphi(y) := \int_0^a \int_0^\pi \int_0^{2\pi} \frac{g(\rho)\rho^2 \sin\phi\, d\theta\, d\phi\, d\rho}{\sqrt{(\rho\sin\phi\sin\theta)^2 + (\rho\sin\phi\cos\theta)^2 + (\rho\cos\phi - \xi)^2}},$$

simplifying the denominator yields

$$\varphi(y)=\int_0^a\int_0^\pi\int_0^{2\pi}\frac{g(\rho)\rho\sin\phi\,d\theta\,d\phi\,d\rho}{\sqrt{\rho^2-2\rho\xi\cos\phi+\xi^2}},$$

integrating over $\theta$, and noting that

$$\frac{\partial}{\partial\theta}(\rho^2-2\rho\xi\cos\phi+\xi^2)^{1/2}=\frac{\rho\xi\sin\theta}{\sqrt{p^2-2\rho\xi\cos\phi+\xi^2}},$$

we obtain

$$\varphi(y)=\frac{2\pi}{\xi}\int_0^a g(\rho)\rho\int_0^\pi\frac{\partial}{\partial\theta}(\rho^2-2\rho\xi\cos\phi+\xi^2)^{1/2}\,d\phi\,d\rho.$$

Using the fundamental theorem of calculus, this becomes

$$\varphi(y)=\frac{2\pi}{\xi}\int_0^a g(\rho)\rho\left((\rho^2+2\rho\xi+\xi^2)^{1/2}-(\rho^2-2\rho\xi+\xi^2)^{1/2}\right),$$

and noting that $(\rho\pm\xi)^2=\rho^2\pm2\rho\xi+\xi^2$, we obtain

$$\varphi(y)=\frac{2\pi}{\xi}\int_0^a g(\rho)\rho(|\rho+\xi|-|\xi-\rho|)=\frac{2\pi}{\xi}\int_0^a g(\rho)\rho^2\,d\theta=\frac{1}{\xi}\int_B\mu(x)\,d^3x,$$

where we use the fact that $2=\int_0^\pi\sin\phi\,d\phi$. Then we obtain

$$F(0,0,\xi)=-\nabla\varphi(0,0,\xi)=\left(0,0,-\frac{\partial\varphi(0,0,\xi)}{\partial\xi}\right):=\frac{1}{\xi^2}\int_B\mu(x)d^3x. \tag{2}$$

Establishing (2) was the hardest part of the problem, and so we leave the rest to the reader. The second part of the problem can be solved in a similar fashion. □