

(1) [a] Total number of ways of distributing the balls is n^{2n} .

$P(\text{at least one box contains three or more balls})$

$$= 1 - P(\text{every box contains at most two balls})$$

$$= 1 - P(\text{every box contains exactly two balls})$$

$$= 1 - \frac{\binom{2n}{2 \ 2 \ \dots \ 2}}{n^{2n}}$$

$$= 1 - \frac{(2n)!}{(2n^2)^n} \quad \square$$

[b] Let $A_i = \{i\text{-th box is empty}\}$.


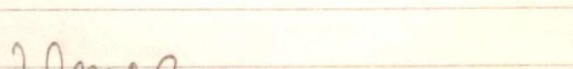
Then $P(A_1) = (n-1)^{2n} / n^{2n}$, and

$P(A_1 \cap A_2) = (n-2)^{2n} / n^{2n}$. Hence

$$P(A_2 | A_1) = \left(\frac{n-2}{n-1} \right)^{2n} \quad \square$$

(2) [a] The event $\{X_1 = n\}$ occurs if

the first n tosses result in a run of heads of length n which is then terminated by a tail at the $(n+1)$ -st toss, or if

the first n tosses  tails of length n  a head at the $(n+1)$ -st toss. Hence,

$$P(X_1 = n) = p^n(1-p) + (1-p)^n p, \quad n = 1, 2, 3, \dots$$

[b] Let $Y_{n,i} = 1$ {the i -th letter and the n -th letter are part of the same run},

for $i, n = 1, 2, 3, \dots$

Then $X_n = \sum_{i=1}^{\infty} Y_{n,i}$. Since $Y_{n,i}$ are nonnegative,

$$E(X_n) = \sum_{i=1}^{\infty} E(Y_{n,i}) = \sum_{i=1}^{\infty} P(Y_{n,i} = 1).$$

$$\text{Now } P(Y_{n,i} = 1) = p^{|n-i|+1} + (1-p)^{|n-i|+1}$$

$$\text{Hence, } E(X_n) = p^n + p^{n-1} + \dots + p + p^2 + p^3 + \dots \\ + q^n + q^{n-1} + \dots + q + q^2 + q^3 + \dots$$

(where $q = 1-p$)

$$= \frac{p - p^{n+1} + p^2}{q} + \frac{q - q^{n+1} + q^2}{p}$$

$$= \frac{p^2 - p^{n+2} + p^3 + q^2 - q^{n+2} + q^3}{pq}$$

(3) [a]

$\{X=2\} = \{10 \text{ tosses result in 6 heads and 4 tails}\}$.

Since the number of heads in ¹⁰ tosses $\sim \text{Bin}(10, 1/2)$,

$$P(X=2) = \binom{10}{6} \left(\frac{1}{2}\right)^{10} = \frac{210}{2^{10}} = \frac{105}{512}.$$

[b] Let $p = \text{prob. of winning a single game}$.

$$\text{Then } p = P(X \geq 8) = P(X=8) + P(X=10)$$

$$= \left[\binom{10}{9} + \binom{10}{10} \right] \frac{1}{2^{10}}$$

$$= 11 / 1024.$$

Now,

$$Y+4 \sim \text{NBin}(4, p)$$

$$\Rightarrow E(Y) = \frac{4}{p} - 4 = 4 \left(\frac{1024}{11} - 1 \right)$$

$$= \frac{4 \times 1013}{11} = \frac{4052}{11}.$$

(4) [a] Assume that $X \sim \text{Poi}(\lambda)$.

To find the value of λ , note that the probability that a manuscript is free of errors is $P(X=0) = e^{-\lambda}$.

Hence, among 100 manuscripts, expected number of manuscripts that are free of errors is

$$100 e^{-\lambda}.$$

$$\Rightarrow 100 e^{-\lambda} = 50 \Rightarrow \lambda = \ln 2.$$

$$\Rightarrow V(X) = \ln 2.$$

[b] Expected value is

$$E(2^X) = \sum_{k=0}^{\infty} 2^k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \cdot e^{2\lambda}.$$

$$= e^{\lambda} = 2.$$

(5) Let $p = P(X=6)$.

Then $P(X=i) = \frac{1-p}{5}$, for $i=1, \dots, 5$.

$$\Rightarrow E(X) = \left(\frac{1-p}{5}\right) \sum_{i=1}^5 i + 6p = 4$$

$$\Rightarrow 3(1-p) + 6p = 4$$

$$\Rightarrow 3 - 3p + 6p = 4$$

$$\Rightarrow p = 1/3 \quad \blacksquare$$