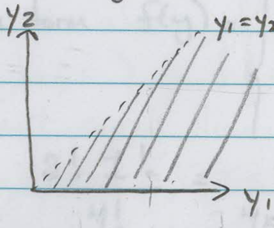
Tutorial 11 (March 29th /2017)

1. (Exercise 5.108)

Let $f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \le y_2 \le y_1 \le \infty \\ 0, & \text{elsewhere} \end{cases}$

be the joint density of Y_1 and Y_2. Find $\mathbb{E}(Y_1 - Y_2)$ and $V(Y_1 - Y_2)$.

Solution: The region defined by $0 \le y_2 \le y_1 \le \infty$ has the following form:



By definition, $\mathbb{E}(Y_1 - Y_2) = \int_0^\infty \int_0^{y_1} (y_1 - y_2) e^{-y_1} \, dy_2 \, dy_1$

$$= \int_0^\infty \left\{ y_2 y_1 e^{-y_1} - \frac{1}{2} y_2^2 e^{-y_1} \right\} \Big|_0^{y_1} dy_1$$

$$= \int_0^\infty y_1^2 e^{-y_1} - \frac{1}{2} y_1^2 e^{-y_1} \, dy_1$$

$$= \frac{1}{2} \int_0^\infty y_1^2 e^{-y_1} \, dy_1 = \frac{1}{2} \left\{ -y_1^2 e^{-y_1} - 2 y_1 e^{-y_1} - 2e^{-y_1} \right\} \Big|_0^\infty$$

$$= \frac{1}{2} \left\{ 0 - (-2) \right\} = \frac{1}{2}(2) = 1$$

By definition, $V(U) = \mathbb{E}\left[(U - \mathbb{E}(U))^2\right]$, in our case $U = Y_1 - Y_2$ and $\mathbb{E}(U) = 1$ from our calculation above.

$$V(U) = \mathbb{E}\left[U^2 - 2U\mathbb{E}(U) + (\mathbb{E}(U))^2\right]$$
$$= \mathbb{E}\left[U^2 - 2U + 1\right]$$
$$= \mathbb{E}[U^2] - 2\mathbb{E}[U] + 1$$
$$= \mathbb{E}[U^2] - 2 + 1$$
$$= \mathbb{E}[U^2] - 1$$

$$\mathbb{E}(U^2) = \int_0^\infty \int_0^{y_1} (y_1 - y_2)^2 e^{-y_1} \, dy_2 dy_1$$

$$= \int_0^\infty \int_0^{y_1} (y_1^2 - 2y_1y_2 + y_2^2) e^{-y_1} \, dy_2 dy_1$$

$$= \int_0^\infty \left\{ y_2 y_1^2 e^{-y_1} - y_2^2 y_1 e^{-y_1} + \frac{1}{3} y_2^3 e^{-y_1} \right\} \Big|_0^{y_1} dy_1$$

$$= \int_0^\infty \frac{1}{3} y_1^3 e^{-y_1} \, dy_1 = \frac{1}{3} \int_0^\infty y_1^3 e^{-y_1} \, dy_1$$

$$= \frac{1}{3} \left\{ -y_1^3 e^{-y_1} - 3y_1^2 e^{-y_1} - 6y_1 e^{-y_1} - 6e^{-y_1} \right\} \Big|_0^\infty$$

$$= \frac{1}{3} \left\{ 6 \right\} = 2$$

$$\Rightarrow V(U) = E[U^2] - 1 = 2 - 1 = 1. \blacksquare$$

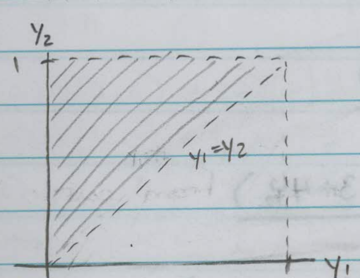
2. (Exercise 5.92).

Suppose $f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

Find $Cov(Y_1, Y_2)$. Are Y_1 and Y_2 independent?

<u>Solution</u>:

$$Cov(Y_1, Y_2) = \mathbb{E}(Y_1 Y_2) - \mathbb{E}(Y_1)\mathbb{E}(Y_2)$$



$$= \int_0^1 \int_{y_1}^1 y_1 y_2 \, 6(1-y_2) \, dy_2 \, dy_1 - \left(\int_0^1 \int_{y_1}^1 y_1 \, 6(1-y_2) dy_2 dy_1 \right) \times$$

$$\left(\int_0^1 \int_{y_1}^1 y_2 \, 6(1-y_2) dy_2 dy_1 \right)$$

$$= \int_0^1 \int_{y_1}^1 6y_1 y_2(1-y_2)\, dy_2 dy_1 - \left(\int_0^1 \int_{y_1}^1 6y_1(1-y_2) dy_2 dy_1 \right) \left(\int_0^1 \int_{y_1}^1 6y_2(1-y_2)\, dy_2 dy_1 \right)$$

$$= \int_0^1 \left\{ 3y_1 y_2^2 - 2y_1 y_2^3 \right\}\Big|_{y_1}^1 dy_1 - \left(\int_0^1 \left\{ 6y_1 y_2 - 3y_1 y_2^2 \right\}\Big|_{y_1}^1 dy_1 \right) \times$$

$$\left(\int_0^1 \left\{ 3y_2^2 - 2y_2^3 \right\}\Big|_{y_1}^1 dy_1 \right)$$

$$= \int_0^1 y_1 - 3y_1^3 + 2y_1^4\, dy_1 - \left(\int_0^1 3y_1 - 6y_1^2 + 3y_1^3\, dy_1 \right) \left(\int_0^1 1 - 3y_1^2 + 2y_1^3\, dy_1 \right)$$

$$= \left(\frac{1}{2}y_1^2 - \frac{3}{4}y_1^4 + \frac{2}{5}y_1^5 \right)\Big|_0^1 - \left\{ \left(\frac{3}{2}y_1^2 - 2y_1^3 + \frac{3}{4}y_1^4 \right)\Big|_0^1 \right\} \left\{ \left(y_1 - y_1^3 + \frac{2}{4}y_1^4 \right)\Big|_0^1 \right\}$$

$$= \left(\frac{1}{2} - \frac{3}{4} + \frac{2}{5} \right) - \left(\frac{3}{2} - 2 + \frac{3}{4} \right) \left(1 - 1 + \frac{1}{2} \right)$$

$$= \frac{10 - 12 + 8}{20} - \left(\frac{6 - 8 + 3}{4} \right) \left(\frac{1}{2} \right) = \frac{3}{10} - \left(\frac{1}{4} \right) \left(\frac{1}{2} \right) = \frac{12 - 5}{40} = \frac{7}{40} \neq 0.$$

Since $Cov(Y_1, Y_2) \neq 0 \Rightarrow Y_1 \not\perp\!\!\!\perp Y_2$.

3. Recall: $\rho = \dfrac{Cov(Y_1, Y_2)}{\sigma_1 \sigma_2}$ where $\sigma_i = \sqrt{V(Y_i)}$

(Exercise 5.110) Suppose Y_1 and Y_2 have correlation coefficient $\rho = 0.2$. What is the value of the correlation coefficient between
a) $1 + 2Y_1$ and $3 + 4Y_2$
b) $1 + 2Y_1$ and $3 - 4Y_2$
c) $1 - 2Y_1$ and $3 - 4Y_2$

<u>Solution</u>:

(a) $$\frac{Cov(1 + 2Y_1, 3 + 4Y_2)}{\sqrt{V(1+2Y_1)}\sqrt{V(3+4Y_2)}}$$

$$= \frac{E\{(1+2Y_1)(3+4Y_2)\} - E(1+2Y_1)E(3+4Y_2)}{\sqrt{V(2Y_1)}\sqrt{V(4Y_2)}}$$

$$= \frac{E\{3 + 4Y_2 + 6Y_1 + 8Y_1Y_2\} - (1 + 2E(Y_1))(3 + 4E(Y_2))}{2 \times 4\sqrt{V(Y_1)}\sqrt{V(Y_2)}}$$

$$= \frac{3 + 4E(Y_2) + 6E(Y_1) + 8E(Y_1Y_2) - 3 - 4E(Y_2) - 6E(Y_1) - 8E(Y_1)E(Y_2)}{8\sigma_1\sigma_2}$$

$$= \frac{8(E(Y_1Y_2) - E(Y_1)E(Y_2))}{8\sigma_1\sigma_2} = \frac{8}{8}\frac{Cov(Y_1, Y_2)}{\sigma_1\sigma_2} = \rho = 0.2$$

(b) $\dfrac{Cov(1+2Y_1, 3-4Y_2)}{\sqrt{V(1+2Y_1)}\sqrt{V(3-4Y_2)}}$

$$= \frac{\mathbb{E}\{(1+2Y_1)(3-4Y_2)\} - \mathbb{E}(1+2Y_1)\mathbb{E}(3-4Y_2)}{\sqrt{V(2Y_1)}\sqrt{V(-4Y_2)}}$$

$$= \frac{\mathbb{E}\{3-4Y_2+6Y_1-8Y_1Y_2\} - (1+2\mathbb{E}(Y_1))(3-4\mathbb{E}(Y_2))}{2\times 4\sqrt{V(Y_1)}\sqrt{V(Y_2)}}$$

$$= \frac{3-4\mathbb{E}(Y_2)+6\mathbb{E}(Y_1)-8\mathbb{E}(Y_1Y_2)-3+4\mathbb{E}(Y_2)-6\mathbb{E}(Y_1)+8\mathbb{E}(Y_1)\mathbb{E}(Y_2)}{8\sigma_1\sigma_2}$$

$$= -\frac{8}{8}\left(\frac{\mathbb{E}(Y_1Y_2)-\mathbb{E}(Y_1)\mathbb{E}(Y_2)}{\sigma_1\sigma_2}\right) = -\frac{8}{8}\,\frac{Cov(Y_1,Y_2)}{\sigma_1\sigma_2} = -\rho = -0.2$$

(c) $\dfrac{Cov(1-2Y_1, 3-4Y_2)}{\sqrt{V(1-2Y_1)}\sqrt{V(3-4Y_2)}}$

$$= \frac{\mathbb{E}\{(1-2Y_1)(3-4Y_2)\} - \mathbb{E}(1-2Y_1)\mathbb{E}(3-4Y_2)}{\sqrt{V(-2Y_1)}\sqrt{V(-4Y_2)}}$$

$$= \frac{\mathbb{E}\{3-4Y_2-6Y_1+8Y_1Y_2\} - (1-2\mathbb{E}(Y_1))(3-4\mathbb{E}(Y_2))}{2\times 4\sqrt{V(Y_1)}\sqrt{V(Y_2)}}$$

$$= \frac{3-4\mathbb{E}(Y_2)-6\mathbb{E}(Y_1)+8\mathbb{E}(Y_1Y_2)-3+4\mathbb{E}(Y_2)+6\mathbb{E}(Y_1)-8\mathbb{E}(Y_1)\mathbb{E}(Y_2)}{8\sigma_1\sigma_2}$$

$$= \frac{8}{8}\,\frac{(\mathbb{E}(Y_1Y_2)-\mathbb{E}(Y_1)\mathbb{E}(Y_2))}{\sigma_1\sigma_2} = \frac{8}{8}\,\frac{Cov(Y_1,Y_2)}{\sigma_1\sigma_2} = \rho = 0.2 \quad \blacksquare$$