McGill University
Department of Mathematics and Statistics

MATH 254 Analysis 1, Fall 2015
Practice Assignment

1. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for each $x \in I$. Prove that there exists a number $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in I$.

2. Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function such that for each $x \in I$ there exists $y \in I$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove that there exists a point $c \in I$ such that $f(c) = 0$.

3. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1) = 0$. Prove that there exists a point $c \in [0, \frac{1}{2}]$ such that $f(c) = f(c + \frac{1}{2})$.

4. Let $I = [a, b]$, let $f : I \to \mathbb{R}$ be continuous on I, and assume that $f(a) < 0$, $f(b) > 0$. Let $W = \{x \in I : f(x) < 0\}$ and let $c = \sup W$. Prove that $f(c) = 0$. (This provides an alternative proof of **Location of Roots Theorem**.)

5. Prove that for any $x \in \mathbb{R}$ the double limit

$$\chi(x) = \lim_{m\to\infty} \left(\lim_{n\to\infty} \cos^n(\pi m! x) \right)$$

   exists. Prove that the function $\chi(x)$ is discontinuous at any point of $\mathbb{R}$.

6. Let $I = (0, \infty)$ and let $f : I \to \mathbb{R}$ be a continuous and bounded function. Show that for any real number T there exists a sequence (x_n) such that $\lim x_n = \infty$ and

$$\lim \left(f(x_n + T) - f(x_n) \right) = 0.$$

7. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that the functions

$$m(x) := \inf\{f(y) : a \leq y \leq x\}, \qquad M(x) := \sup\{f(y) : a \leq y \leq x\}$$

   are also continuous on I.

8. Let $I = [0, \infty)$ and let $f : I \to \mathbb{R}$ be a continuous function. Suppose that $\lim_{x\to\infty} f(x) = L$, where L is a real number. Prove that the function f is uniformly continuous on I.

9. Prove that the function $\sin x^2$ is not uniformly continuous on $\mathbb{R}$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Prove that there exist positive constants A and B such that

$$|f(x)| \leq A|x| + B$$

    for any $x \in \mathbb{R}$.