McGill University

MATH 323, Winter 2017

Midterm examination 2

Date: 22 February, 2017
Time: 18:10–19:20

- Write your name and Student ID in the answer booklet.
- Answer all questions. The problems carry a total of 26 marks. Maximum you can score is 25.
- Explain your argument clearly. Solutions without proper justification may not receive full credit.
- Calculators are not allowed (or necessary for this test). If the final answer contains terms like e^2 or $\log_e 6$, leave it like that.
- Simplify your answer as much as possible. For example, the expression for the final answer should not be an infinite series.

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1. Suppose $n \geq 1$, and you are given $2n$ distinguishable balls and n distinguishable boxes. The balls are to be put in the boxes at random, and assume that all possible ways of doing so are equally likely.

(a) Find the probability that at least one box contains three or more balls.

(b) Conditional on the event that the first box is empty, find the probability that the second box is empty.

(3+3)

2. A coin is tossed independently infinitely many times, and the outcome is represented by a sequence consisting of the letters H and T. Assume that probability of landing heads up in each toss is p.

A run of length k is a maximal sequence of k consecutive heads or tails. For example, if the outcome is **H-H-T-H-T-H-H-H-H-T-H-H-T**..., then the length of the run containing the third letter is one, and the length of the run containing the seventh letter is four.

For $n \geq 1$, let X_n denote the length of the run containing the n-th letter.

(a) Find the probability distribution of X_1.

(b) Find $\mathbb{E}(X_n)$ for $n \geq 1$. (Hint: One way of doing it is to use linearity of expectation.)

(2+4)

3. Mary plays the following game: She tosses a fair coin 10 times. For every head, she is awarded one point, and for every tail, she loses one point.

Let X denote her score at the end. For example, if the 10 tosses result in 2 heads and 8 tails, then $X = -6$.

Mary wins the game if $X \geq 8$, and loses the game otherwise.

(a) Find $\mathbb{P}(X = 2)$.

(b) If Mary keeps playing the game, let Y denote the number of games she loses before winning for the fourth time. Find $\mathbb{E}(Y)$.

(3+3)

4. The number of errors made by a typist while preparing any manuscript follows a Poisson distribution. Assume that the expected number of typographical errors is the same for all manuscripts. Among every 100 manuscripts she prepares, on an average, 50 are free of any typographical errors.

(a) Let X be the number of typographical errors she makes while preparing a new manuscript. Find $\mathrm{V}(X)$.

(b) If she is to pay a fine of amount 2^k for making k typographical errors, $k = 0, 1, 2, \ldots$, then find the expected fine she has to pay.

(3+3)

5. Let X denote the number on the uppermost face when a biased die is rolled. If

$$\mathbb{P}(X = 1) = \mathbb{P}(X = 2) = \mathbb{P}(X = 3) = \mathbb{P}(X = 4) = \mathbb{P}(X = 5), \quad \text{and} \quad \mathbb{E}(X) = 4,$$

find $\mathbb{P}(X = 6)$.

(2)