McGill University
Department of Mathematics and Statistics
MATH 254 Analysis 1, Fall 2015
Practice Assignment

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies the functional equation

$$f(x + y) = f(x) + f(y) \qquad \forall x, y \in \mathbb{R}$$

Assume furthermore that f is continuous at 0 and that $f(1) = 1$. Prove that $f(x) = x$ for all $x \in \mathbb{R}$.

Hint: Show first that f is continuous at any point $c \in \mathbb{R}$. Prove then that for any rational number r, $f(r) = r$, and deduce the statement using the continuity of f.

2. Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let c be a cluster point of A. Prove that the following statements are equivalent:

   (a) The function f does **not** have a limit at c.

   (b) There exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $f(x_n)$ does **not** converge in $\mathbb{R}$.

   Remark. It is important that you write a detailed proof and understand every step of the argument.

3. Using the ε-δ definition of the limit of a function, prove that

   (a) $\lim\limits_{x \to a} \dfrac{x}{1+x} = \dfrac{a}{1+a}$ for all $a \in \mathbb{R}$, $a \neq -1$.

   (b) $\lim\limits_{x \to -1} \dfrac{x}{1+x}$ does not exist.

4. Use the ε-δ definition of the limit of a function to prove that

$$\lim_{x \to a} x^n = a^n$$

for all $n \in \mathbb{N}$ and all $a \in \mathbb{R}$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   (a) Prove that $\lim\limits_{x \to 0} f(x) = 0$.

   (b) Prove that if $a \neq 0$ then $\lim\limits_{x \to a} f(x)$ does not exist.

6. Let $A \subseteq \mathbb{R}$, let a be a cluster point of A and let $f : A \to \mathbb{R}$ be a function.

   (a) Prove that $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to a} |f(x) - L| = 0$.

   (b) Prove that $\lim\limits_{x \to a} f(x) = L$ if and only if $\lim\limits_{x \to 0} f(x + a) = L$.

7. Let $A \subseteq \mathbb{R}$, let a be a cluster point of A and let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be functions which satisfy the following conditions:

   - f is bounded on a neighborhood of a i.e. there exists a $\tau > 0$ and an $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in A \cap V_\tau(a)$.
   - $\lim\limits_{x \to a} g(x) = 0$.

   Prove that then $\lim\limits_{x \to a} (f(x)g(x)) = 0$.

8. Prove that

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$$