MATH323 - Tutorial 2 (January 18th)

Counting & Probability Problems

1. (Exercise 2.51) A local fraternity sells 50 tickets (1 per person) with 3 prizes. There are 4 organizers who buy 1 ticket each. What is the probability that the 4 organizers win
   - a) All prizes
   - b) Exactly two of the prizes?
   - c) Exactly one of the prizes?
   - d) None of the prizes?
   - Not in textbook → e) Two or more prizes?

Solution: Note that the question does not distinguish between the prizes (i.e. no ordering). Thus, S = the sample space has

$$\binom{50}{3} = \frac{50!}{(50-3)!\,3!} = \frac{50 \times 49 \times 48 \times \cancel{47!}}{3 \times 2 \times 1 \times \cancel{47!}}$$

$$= 19600 \text{ possible outcomes}$$

a) If all 4 organizers win all 3 prizes then there are

$$\binom{4}{3} = \frac{4!}{1!\,3!} = 4$$

many ways of assigning the 3 prizes to the 4 organizers.

Thus, the probability is calculated to be $\frac{4}{19600}$

b) If the 4 organizers win 2 prizes then there are $\binom{4}{2} = 6$ ways of assigning 2 prizes to 4 organizers. Then the 3rd prize goes to one of the other 46 people $\Rightarrow \binom{46}{1} = 46$ ways.

Using the multiplication rule $\Rightarrow 46 \times 6 = 276$ ways the organizers win exactly 2 prizes.

Thus, the probability is calculated to be $\frac{276}{19600}$

c) Same explanation as in (b). 1 prize goes to 4 organizers, 2 prizes go to other 46 people.

$$\Rightarrow \binom{4}{1} \times \binom{46}{2} = 4 \times \frac{46!}{44!\,2!} = \frac{4 \times 46 \times 45}{2} = 4140$$

d) Let A be the event the organizers win all 3 prizes

B " " exactly 2 prizes.

C " " exactly 1 prize

D " " no prizes.

Observe A, B, C, D are mutually disjoint and $S = A \cup B \cup C \cup D$

$$\Rightarrow \mathbb{P}(S) = \mathbb{P}(A \cup B \cup C \cup D) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) + \mathbb{P}(D)$$

$$1 = \frac{4}{19600} + \frac{276}{19600} + \frac{4140}{19600} + \mathbb{P}(D)$$

$$\Rightarrow \mathbb{P}(D) = \frac{19600 - 4 - 276 - 4140}{19600} = \frac{15180}{19600}$$

(Note: could have solved like part (a), of 46 other people how many ways can the 3 prizes be distributed $= \binom{46}{3}$)

e) Let E be the event that the organizers win 2 or more prizes.

Observe that $E = A \cup B$

$$\Rightarrow \mathbb{P}(E) = \mathbb{P}(A \cup B)$$

$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

But $A \cap B = \emptyset \Rightarrow \mathbb{P}(A \cap B) = 0$

$$= \mathbb{P}(A) + \mathbb{P}(B)$$

$$= \frac{4}{19600} + \frac{276}{19600} = \frac{280}{19600}.$$

2. (Birthday Problem) There are 25 people at a party. Assuming a 365 day year, what is the probability that 2 or more people share the same birthday?

<u>Solution</u>: Let S be the sample space of all birthday / person configurations. Then $|S| = \underbrace{365 \times 365 \times \ldots \times 365}_{25 \text{ times}} = (365)^{25}$ since each person can have any choice of any of the 365 days.

Let A be the event that 2 or more people share the same birthday.
$\Rightarrow \bar{A}$ is the event that no one shares the same birthday.
$\Rightarrow \mathbb{P}(A) = 1 - \mathbb{P}(\bar{A})$

Note that there <u>is</u> an ordering in this question! (i.e. the days)

e.x. Person 1 - Jan 3^{rd} Person 2 - Jan 5^{th}

$\neq$

Person 1 - Jan 5^{th} Person 2 - Jan 3^{rd}

So how many ways to order 25 distinct days from 365 distinct days?

$$|\bar{A}| = {}^{365}P_{25} = \frac{365!}{(365-25)!} = 365 \times 364 \times \ldots \times 341$$

$$\Rightarrow \mathbb{P}(A) = 1 - \frac{\left(\frac{365!}{25!}\right)}{(365)^{25}} = 1 - \frac{365 \times 364 \times \ldots \times 341}{(365)^{25}} \approx 1 - 0.431 = 0.5686997$$

(Note: for 35 people, $\mathbb{P}(A) \approx 0.8143832$)

3. (Exercise 2.68) Show that, for any integer $n \geq 1$,

a) $\binom{n}{n} = 1$. Interpret this result

b) $\binom{n}{0} = 1$. Interpret this result

c) $\binom{n}{r} = \binom{n}{n-r}$. Interpret this result

d) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$ [Consider binomial expansion $(x+y)^n$ with $x=y=1$]

Not in text $\rightarrow$ e) $\sum_{n_1+n_2+n_3+n_4=1} \binom{n}{n_1\ n_2\ n_3\ n_4} = 4^n$ [Consider multinomial expansion with $k=4$]

Solution:

a) $\binom{n}{n} = \frac{n!}{(n-n)!\,n!} = \frac{n!}{0!\,n!} = \frac{n!}{1 \cdot n!} = 1$

There is only 1 way of selecting n objects from n objects. Take all of them!

b) $\binom{n}{0} = \frac{n!}{(n-0)!\,0!} = \frac{n!}{n! \cdot 1} = 1$

There is only 1 way of selecting 0 objects from n objects. Take none of them!

c) $\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-(n-r))!\,(n-r)!} = \binom{n}{n-r}$

The number of ways of selecting r objects from n objects is equal to the number of ways of selecting n-r objects from n objects. For each selection of r objects there is 1 selection of the other n-r objects.

d) Recall: $\sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^{i} = (x+y)^n$

Using the hint: $x = y = 1$

$\Rightarrow \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^{i} = \sum_{i=0}^{n} \binom{n}{i} = (1+1)^n = 2^n$

e) Recall: $\sum_{n_1+n_2+\ldots+n_k=1} \binom{n}{n_1\ n_2 \cdots n_k} y_1^{n_1} y_2^{n_2} \cdots y_k^{n_k} = (y_1+y_2+\ldots+y_k)^n$

Using the hint: $k=4$ & $y_1 = y_2 = y_3 = y_4 = 1$

$\Rightarrow \sum_{n_1+n_2+n_3+n_4=1} \binom{n}{n_1\ n_2\ n_3\ n_4} 1^{n_1} 1^{n_2} 1^{n_3} 1^{n_4} = \sum_{n_1+n_2+n_3+n_4=1} \binom{n}{n_1\ n_2\ n_3\ n_4} = (1+1+1+1)^n = 4^n$

4. (Exercise 2.72) Consider the following table with Sex and Exam Result categories. Let A be the event of pass, M be the event of male.

	Sex		
Outcome	Male (M)	Female (F)	Total
Pass (A)	24	36	60
Fail ($\bar{A}$)	16	24	40
Total	40	60	100

a) Are the events A and M independent?
b) Are the events $\bar{A}$ and F independent?

Solution

a) Recall that if events A and B are independent then
$\mathbb{P}(A \mid B) = \mathbb{P}(A)$ or equivalently $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

We will check the second equality. Does $\mathbb{P}(A \cap M) = \mathbb{P}(A)\mathbb{P}(M)$?

$$\mathbb{P}(A \cap M) = \frac{24}{100}$$

$$\mathbb{P}(A)\mathbb{P}(M) = \left(\frac{24+36}{100}\right)\left(\frac{24+16}{100}\right) = \left(\frac{60}{100}\right)\left(\frac{40}{100}\right) = \frac{24}{100}$$

Yes! They are independent.

Check using first equality. Does $\mathbb{P}(A \mid M) = \mathbb{P}(A)$?

$\mathbb{P}(A \mid M)$ is the probability of A given that the person is male. So our conditional table looks like

Outcome	Male
Pass(A)	24
Fail($\bar{A}$)	16
Total	40

$$\Rightarrow \mathbb{P}(A \mid M) = \frac{24}{40} = \frac{6}{10}$$

$$\mathbb{P}(A) = \frac{24+36}{100} = \frac{60}{100} = \frac{6}{10}$$

Yes! We reach the same conclusion of independence between A and M.

b) Does $\mathbb{P}(\bar{A} \cap F) = \mathbb{P}(\bar{A})\mathbb{P}(F)$?

$$\mathbb{P}(\bar{A} \cap F) = \frac{24}{100} \qquad \mathbb{P}(\bar{A})\mathbb{P}(F) = \left(\frac{16+24}{100}\right)\left(\frac{36+24}{100}\right)$$

$$= \left(\frac{40}{100}\right)\left(\frac{60}{100}\right) = \frac{24}{100}$$

Yes! They are independent!

Equivalent calculation for independence. Does $\mathbb{P}(\bar{A}|F) = \mathbb{P}(\bar{A})$

$\mathbb{P}(\bar{A}|F)$ is the probability of failing given the person is female. So our conditional table looks like

Outcome	Female
Pass (A)	36
Fail ($\bar{A}$)	24
Total	60

$$\Rightarrow \mathbb{P}(\bar{A}|F) = \frac{24}{60} = \frac{4}{10}$$

$$\mathbb{P}(\bar{A}) = \frac{16+24}{100} = \frac{40}{100} = \frac{4}{10}$$

Yes! We reach the same conclusion of independence.

5. Let $A, B \subset S$. If $\mathbb{P}(A|B) = \mathbb{P}(A)$ (i.e. A and B are independent events) then $\mathbb{P}(\bar{A}|B) = \mathbb{P}(\bar{A})$ (i.e. $\bar{A}$ and B are independent events).

<u>Solution</u>:

Recall that $(\bar{A} \cap B) \cup (A \cap B) = B$ and since $(A^c \cap B)$ and $(A \cap B)$ are disjoint then

$$\mathbb{P}(\bar{A} \cap B) + \mathbb{P}(A \cap B) = \mathbb{P}(B)$$

$$\Rightarrow \mathbb{P}(\bar{A} \cap B) = \mathbb{P}(B) - \mathbb{P}(A \cap B) \qquad (*)$$

$\mathbb{P}(\bar{A}|B) = \dfrac{\mathbb{P}(\bar{A} \cap B)}{\mathbb{P}(B)}$ then by $(*)$

$$= \frac{\mathbb{P}(B) - \mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(B)}{\mathbb{P}(B)} - \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$= 1 - \mathbb{P}(A|B)$ (by the conditional prob. formula)

But by the question's assumption that $\mathbb{P}(A|B) = \mathbb{P}(A)$

$$= 1 - \mathbb{P}(A)$$

$$= \mathbb{P}(\bar{A})$$

Thus, we have shown that if A and B are independent then A^c and B are independent.

Exercise: Let A and B be independent events. Using the fact that $\bar{A}$ and B are independent events (established above), prove that $\bar{A}$ and $\bar{B}$ are independent events.

6. (Exercise 2.82) Suppose that $A \subset B$ and $\mathbb{P}(A) > 0$ and $\mathbb{P}(B) > 0$. Show that $\mathbb{P}(B|A) = 1$ and $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A)}{\mathbb{P}(B)}$.

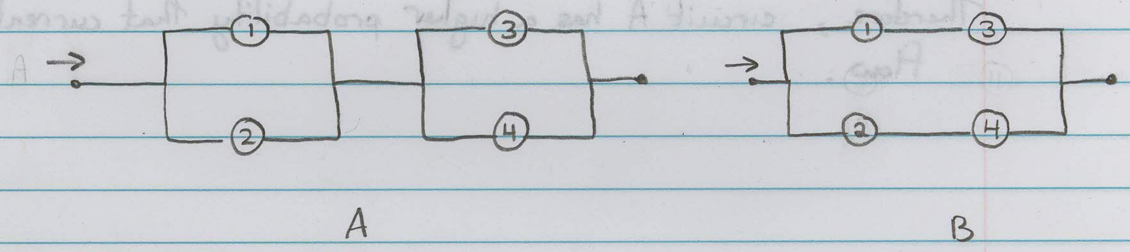
Solution :

Note : $A \subset B \Rightarrow A \cap B = A \Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)$

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)} = \frac{\mathbb{P}(A)}{\mathbb{P}(A)} = 1 \quad \blacksquare$$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{\mathbb{P}(A)}{\mathbb{P}(B)} \quad \blacksquare$$

7. (Exercise 2.163) Relays function properly with probability 0.9. Assume each relay acts independently, which circuit yields a higher probability that current will flow when the relays are activated?



Solution :

Let E_i be the event that relay (i) functions.

$\mathbb{P}(E_i) = 0.9$

If current flows in A, then either or both (1) and (2) function AND
" " (3) and (4) "

Thus, current flowing in the first section of A can be written as $E_1 \cup E_2$
" second " " $E_3 \cup E_4$

$$\mathbb{P}(E_1 \cup E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1 \cap E_2) = \mathbb{P}(E_1) + \mathbb{P}(E_2) - \mathbb{P}(E_1)\mathbb{P}(E_2)$$

(by independence)

$$= 0.9 + 0.9 - 0.81 = 0.99$$

Similarly, $\mathbb{P}(E_3 \cup E_4) = 0.99$.

Since all relays are independent, this implies that the first part of the circuit A is independent of the second part of circuit B.

$$\Rightarrow \mathbb{P}((E_1 \cup E_2) \cap (E_3 \cup E_4))$$
$$= \mathbb{P}(E_1 \cup E_2)\mathbb{P}(E_3 \cup E_4)$$
$$= (0.99)^2 = 0.9801$$

In circuit B, current flows if the top line flows, the bottom line flows or both lines flow.

Symbolically, this corresponds to $(A_1 \cap A_3) \cup (A_2 \cap A_4)$

$$\Rightarrow \mathbb{P}((A_1 \cap A_3) \cup (A_2 \cap A_4))$$
$$= \mathbb{P}(A_1 \cap A_3) + \mathbb{P}(A_2 \cap A_4) - \mathbb{P}(A_1 \cap A_2 \cap A_3 \cap A_4)$$
$$= (0.9)^2 + (0.9)^2 - (0.9)^4$$
$$= 0.9639$$

Therefore, circuit A has a higher probability that current will flow.