MATH 248 PROBLEM SET 1

DUE TUESDAY SEPTEMBER 27

1. In each of the following cases, verify if the value $f(0)$ can be defined so that the resulting function f is continuous in $\mathbb{R}$, and if so, check if f is differentiable at 0.
   (a) $f(x) = \sin(1/x)$.
   (b) $f(x) = x\sin(1/x)$.
   (c) $f(x) = x^2\sin(1/x)$.
2. Prove that differentiability is a local property, in the sense that $f : (a,b) \to \mathbb{R}$ is differentiable at $y \in (a,b)$ if and only if $g = f|_{(y-\varepsilon,y+\varepsilon)}$ is differentiable at y, where $\varepsilon > 0$ is small. Here $g = f|_{(y-\varepsilon,y+\varepsilon)}$ means that $g : (y-\varepsilon, y+\varepsilon) \to \mathbb{R}$ is defined by $g(x) = f(x)$ for $x \in (y-\varepsilon, y+\varepsilon)$. We say that g is the *restriction* of f to the interval $(y-\varepsilon, y+\varepsilon)$.
3. Let $f : (a,b) \to \mathbb{R}^n$, and let $y \in (a,b)$. Suppose that $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}^n$ satisfy
$$f(x) = \lambda_1 + \mu_1(x-y) + o(x-y) \qquad \text{as} \quad x \to y,$$
   and
$$f(x) = \lambda_2 + \mu_2(x-y) + o(x-y) \qquad \text{as} \quad x \to y.$$
   Show that $\lambda_1 = \lambda_2 = f(y)$ and $\mu_1 = \mu_2$. What is the value of μ_1?
4. For $0 < p < \infty$, define the *p-quasinorm* of a vector $x \in \mathbb{R}^n$ by
$$|x|_p = \left(\sum_{k=1}^{n} |x_k|^p\right)^{\frac{1}{p}}.$$
   This is also called the *p-norm* when $p \geq 1$. Recall that the $p = \infty$ case is defined by
$$|x|_\infty = \max_{k\in\{1,\ldots,n\}} |x_k|.$$
   For $0 < p \leq \infty$, prove the following.
   (a) $|x|_p = 0$ implies $x = 0$.
   (b) $|\lambda x|_p = |\lambda| \cdot |x|_p$ for $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.
   (c) $|x+y|_p^\alpha \leq |x|_p^\alpha + |y|_p^\alpha$ for $x, y \in \mathbb{R}^n$, where $\alpha = \min\{1, p\}$.
   (d) $|x|_\infty \leq |x|_p \leq n^{1/p}|x|_\infty$ for $x \in \mathbb{R}^n$.
5. Let $f : K \to \mathbb{R}^n$ and let $h : K \to \mathbb{R}$, with $K \subset \mathbb{R}$. Suppose that $0 < p < \infty$, and $y \in \mathbb{R}$. Show that the following are equivalent.
   - $|f(x)|_\infty = o(h(x))$ as $K \ni x \to y$.
   - $|f(x)|_p = o(h(x))$ as $K \ni x \to y$.

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Introduce definitions of continuity and differentiability that are based on the p-quasinorm. Note that your definitions must be equivalent to the corresponding definitions given in the lecture notes.

6. Let $A : (a,b) \to \mathbb{R}^{m\times n}$ and $x : (a,b) \to \mathbb{R}^n$ be both differentiable at $t \in (a,b)$. Show that the product $Ax : (a,b) \to \mathbb{R}^m$, defined by $(Ax)(s) = A(s)x(s)$, is differentiable at t, with
$$(Ax)'(t) = A'(t)x(t) + A(t)x'(t).$$
Note: The space $\mathbb{R}^{m\times n}$ is the space of $m \times n$ matrices (with real entries), and hence $A(s)$ is a matrix for each $s \in (a,b)$. Since an $m \times n$ matrix has mn entries, $\mathbb{R}^{m\times n}$ can be identified with $\mathbb{R}^{mn}$. In particular, matrix valued functions such as A are simply a special case of vector valued functions.
7. Does there exist a function $u : \mathbb{R}^2 \to \mathbb{R}$ that is not continuous at $0 \in \mathbb{R}^2$, but whose restriction to every polynomial curve going through $0 \in \mathbb{R}^2$ is continuous? By a polynomial curve we mean the parameterized curve $(t, p(t))$ where p is some polynomial, or a rotated version of $(t, p(t))$.
8. (a) Let $K \subset \mathbb{R}^n$, and let $f : K \to \mathbb{R}$ satisfy $|f(x) - f(y)| \leq h(|x-y|_\infty)$ for all $x, y \in K$, where $h : [0,\infty) \to \mathbb{R}$ is some (fixed) function satisfying $h(t) = o(1)$ as $t \to 0$. Show that f is continuous in K.
(b) Show that $f(x) = |x|_p$ is continuous in $\mathbb{R}^n$, for any $0 < p \leq \infty$.
9. (a) Determine the values of the parameter $\alpha \in \mathbb{R}$ for which the function
$$f(x,y) = \begin{cases} |xy|^\alpha & \text{for } xy \neq 0 \\ 0 & \text{for } xy = 0 \end{cases}$$
is differentiable at every point $(x,y) \in \mathbb{R}^2$.
(b) Let $A \in \mathbb{R}^{n\times n}$, let $b \in \mathbb{R}^n$, and let
$$f(x) = x^T A x + b^T x = \sum_{i,k=1}^n A_{ik} x_i x_k + \sum_{i=1}^n b_i x_i.$$
Show that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable in $\mathbb{R}^n$, and compute the derivative.

Homework policy

You are welcome to consult each other provided (1) you list all people and sources who aided you, or whom you aided and (2) you write-up the solutions independently, in your own language. If you seek help from other people, you should be seeking general advice, not specific solutions, and must disclose this help. This applies especially to internet fora such as `MathStackExchange`.

Similarly, if you consult books and papers outside your notes, you should be looking for better understanding of or different points of view on the material, not solutions to the problems.