McGill University
Department of Mathematics and Statistics
MATH 254 Analysis 1, Fall 2015
Assignment 1

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 2, 5.**

This assignment is due **Monday, September 21, at the end of the class** in class. **Late assignments will not be accepted.**

1. Conjecture a formula for the sum
$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)},$$
and prove your conjecture using Mathematical Induction.

2. Prove that the collection $\mathcal{F}(\mathbb{N})$ of all *finite* subsets of $\mathbb{N}$ is countable.

3. Let E_n, $n=1,2,\cdots$ be an infinite sequence of sets. Let
$$\overline{E}=\bigcap_{n=1}^{\infty}\bigcup_{m=n}^{\infty}E_m,\qquad \underline{E}=\bigcup_{n=1}^{\infty}\bigcap_{m=n}^{\infty}E_m.$$
Prove that
$$\bigcap_{n=1}^{\infty}E_n\subseteq\underline{E}\subseteq\overline{E}\subseteq\bigcup_{n=1}^{\infty}E_n.$$

4.

5. Let $f:D\to E$ be a function and let $A\subseteq D$, $B\subseteq E$. Prove the following:
   (a) $f(f^{-1}(B))\subseteq B$.
   (b) If f is surjective then $f(f^{-1}(B))=B$.
   (c) $f^{-1}(f(A))\supseteq A$.
   (d) If f is injective then $f^{-1}(f(A))=A$.

6. Prove by induction that
$$\underbrace{\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}}_{n\text{ nested square roots}}=2\cos\left(\frac{\pi}{2^{n+1}}\right)$$
for all $n\in\mathbb{N}$.

<u>Hint</u>: The half-angle formula $\cos^2(x)=\frac{1}{2}(1+\cos(2x))$ might be useful.

7. Recall that the binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. Prove by induction on n that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for all $n \in \mathbb{N}_0$. You may use, without proof, the well-known identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ for all $n \in \mathbb{N}_0$ and $1 \le k \le n$.

8. Let A be a countably infinite set and let $B \subseteq A$. Prove that B is countable.