MATH 254 Assignment 2 McGill University
Fall 2016

You should work carefully on all problems. However you only have to hand in solutions to problems 3 and 4. This assignment is due on Tuesday, October 4 in class.

Solutions will be posted shortly after the class so that you can look at them before the midterm. No late assignments will be accepted.

Exercise 1. Show that the set $\mathcal{F}(\mathbb{N})$ of all finite subsets of $\mathbb{N}$ is countable.

Exercise 2. We want to prove that the set $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$ is not countable. For this, we proceed by contradiction and assume that there exists a bijection φ from $\mathbb{N}$ to $\mathcal{P}(\mathbb{N})$. Show that there is a contradiction. *Hint: You may use the set* $A = \{n \in \mathbb{N} : n \notin \varphi(n)\}$.

Exercise 3. [10 points] We call algebraic number any real number x such that there exist $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ such that

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \quad \text{and} \quad a_n \neq 0. \tag{$*$}$$

Show that the set of all algebraic numbers is countable. You may assume the following result from algebra: for any $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in \mathbb{Z}$, the equation $(*)$ has at most n solutions.

Exercise 4. [10 points] Let x and y be two real numbers. By using only the field and order properties of $\mathbb{R}$, show that the following statements are true:

(i) (-x)(-y)=xy
(ii) If $x < y < 0$, then $1/y < 1/x < 0$.

Write every step of the proofs carefully indicating which property you are using. If needed, you can find the statements of the field and order properties of $\mathbb{R}$ in a file on the course webpage (below the link to this assignment).

Exercise 5. Let x, y, z be three real numbers. Show that the following inequalities are true:

(i) $|x| + |y| \leq |x - y| + |x + y|$
(ii) $1 + |xy - 1| \leq (1 + |x - 1|)(1 + |y - 1|)$

Exercise 6.

(i) Show that $x(1 - x) \leq 1/4$ for all real numbers x.
(ii) Let x, y be two real numbers such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Show that at least one of the two numbers xy and $(1 - x)(1 - y)$ is less than or equal to $1/4$.
(iii) Let x, y, z be three positive real numbers. Show that at least one of the three numbers $x(1 - y)$, $y(1 - z)$, $z(1 - x)$ is less than or equal to $1/4$.