Homework 2
COMP 527 Logic and Computations

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Exercise 1 (**30 pts total**) In this problem, you will give a direct definition of "A iff B", which means "A implies B and B implies A". Here are the introduction and elimation rules:

$$\frac{\begin{array}{cc} \overline{A\,\mathsf{true}}^{\,u} & \overline{B\,\mathsf{true}}^{\,v} \\ \vdots & \vdots \\ B\,\mathsf{true} & A\,\mathsf{true} \end{array}}{A \equiv B\,\mathsf{true}} \equiv I^{u,v} \qquad \frac{A \equiv B\,\mathsf{true} \quad A\,\mathsf{true}}{B\,\mathsf{true}} \equiv E_L \qquad \frac{A \equiv B\,\mathsf{true} \quad B\,\mathsf{true}}{A\,\mathsf{true}} \equiv E_R$$

Task 1 (8 pts). Annotate the introduction and elimination rules with proof terms.

Task 2 (8 pts). Using these proof terms, give the local reduction $\Rightarrow_R$ and local expansion $\Rightarrow_E$ rules.

Task 3 (8 pts). Show the cases for the substitution lemma for the given extension:

If $\Gamma, x : A, \Gamma' \vdash M : B$ and $\Gamma \vdash N : A$ then $\Gamma, \Gamma' \vdash [N/x]M : B$

Task 4 (6 pts). Annotate the introduction and elimination rules such that they only describe normal proofs, i.e. restate the rules using the judgment $A \uparrow$ (for normal proofs) and $A \downarrow$ (for neutral proofs).

Exercise 2 Type uniqueness (**25 pts total**)

$$\frac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash \langle M,\, N \rangle : A \wedge B} \wedge I \qquad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \mathsf{fst}\, M : A} \wedge E_l \qquad \frac{\Gamma \vdash M : A \wedge B}{\Gamma \vdash \mathsf{snd}\, M : B} \wedge E_r$$

$$\frac{\Gamma, u{:}A \vdash M : B}{\Gamma \vdash \lambda u{:}A.M : A \supset B} \supset I^u \qquad \frac{\Gamma \vdash N : A \supset B \quad \Gamma \vdash A}{\Gamma \vdash M\, N : B} \supset E$$

$$\frac{\Gamma \vdash M : A}{\Gamma \vdash \mathsf{inl}^B\, M : A \vee B} \vee I_l \quad \frac{\Gamma \vdash M : B}{\Gamma \vdash \mathsf{inr}^A\, M : A \vee B} \vee I_r \quad \frac{\Gamma \vdash M : A \vee B \quad \Gamma, x : A \vdash N_l : C \quad \Gamma, y : B \vdash N_r : C}{\Gamma \vdash \mathsf{case}\, M\, \mathsf{of}\, \mathsf{inl}^A\, x \to N_l \mid \mathsf{inr}^B\, y \to N_r : C} \vee E^{u,v}$$

$$\frac{}{\Gamma \vdash () : \top} \top I \qquad \frac{\Gamma \vdash M : \bot}{\Gamma \vdash \mathsf{abort}^C C : C} \bot E \qquad \frac{x : A \in \Gamma}{\Gamma \vdash x : A} u$$

Prove that every expression has a unique type, i.e. if $\Gamma \vdash M : A$ and $\Gamma \vdash M : B$ then $A = B$.

Exercise 3 (30 pts) : Constructively prove the following conjectures using natural deduction or indicate if a conjecture is not true. For the following conjectures, you can assume that x does not occur in A. If it is provable, provide the actual proof and the proof term. If it is not provable, briefly explain how the proof breaks down.(30 pts)

1. $(\exists x \in \tau . A \supset B(x)) \supset A \supset \exists x \in \tau . B(x)$
2. $(\exists x \in \tau . (A \supset P(x))) \supset A \supset \forall x \in \tau . P(x)$
3. $((\forall x \in \tau . P(x)) \supset A) \supset \exists x \in \tau . (P(x) \supset A)$
4. $(\exists x \in \tau . (P(x) \supset A)) \supset (\forall x \in \tau . P(x)) \supset A$
5. $\neg(\forall x \in \tau . B(x)) \supset \exists x \in \tau . \neg B(x)$
6. $\neg(\exists x \in \tau . B(x)) \supset \forall x \in \tau . \neg B(x))$

Exercise 4 (15 pts) : Give proofs and proof terms for the following propositions.

1. $(A \wedge \exists x \in \tau . B(x)) \supset (\exists x \in \tau . A \wedge B(x))$
2. $(\exists x \in \tau . A \wedge B(x)) \supset (A \wedge \exists x \in \tau . B(x))$
3. $(A \vee \forall x \in \tau . B(x)) \supset (\forall x \in \tau . A \vee B(x))$