Math 340 - Jan. 20

Assignment 1 is out, due Jan. 27
Midterm in class: Friday February 10th

Applications of Hall's Theorem:

Matching Markets

n potential buyers interested in n different houses. Our goal is to clear the market by selling all the houses.
We can think of buyers as set B, houses set H, and eventual trades is a perfect matching in a graph with bipartition (B, H).

Each buyer b_i assigns a value v_{ij} to the j^{th} house.

Given a price p_j of house H_j, buyer i^{th} satisfaction from buying it is $s_{ij}(p) = v_{ij} - p_j$
$p = (p_1, \dots, p_n)$.

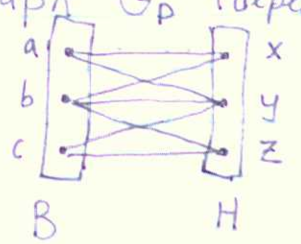
Each buyer wants to buy a house which maximizes satisfaction.

houses

v	x	y	z
a	4	2	4
b	6	4	7
c	6	5	8
prices	3	1	4

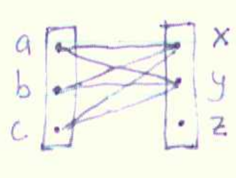
buyers

Graph G_p (depends on prices)



b_i and h_j are joined by an edge in G_p if $s_{ij}(p)$ maximizes satisfaction of b_i

for prices $p = (3, 1, 5)$:



has no perfect matching

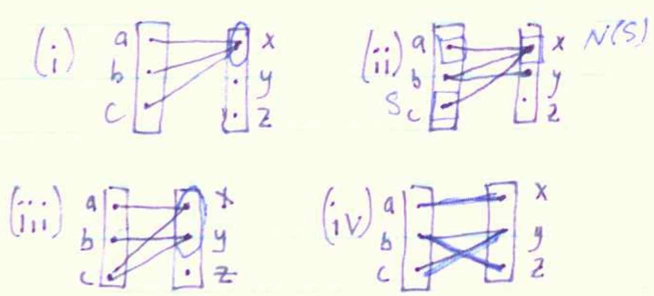
So if the set of market clearing prices do not exist then there exists a set S of buyers s.t. $N(S)$ in the graph G_p satisfies $|N(S)| < |S|$.
We say that $N(S)$ is a <u>constructing set</u> and S is <u>unsuppliable set of buyers</u>.
We want to find a set of prices p given the valuation s.t. the market clears

<u>Algorithm:</u>

- We start with $\bar{p}$ (collection of prices) all zeros.
- At each step, consider the graph G_p, if it has a perfect matching, we are done (output p).
- Otherwise, find a constructing set of houses C and raise the price of all houses in C by one
- Finally, if all prices are at least one, reduce them all by one

repeat

v	x	y	z	
a	12	4	2	
b	8	7	6	
c	7	5	2	
prices	0	0	0	(i)
	1	0	0	(ii)
	2	0	0	(iii)
	3	1	0	(iv) ← market clearing prices



<u>Theorem:</u>
Algorithm terminates at a set of market clearing prices.

Proof:
The potential energy of the set of prices.

$$\varphi(\bar{p}) = \sum_{b_i} s_i(\bar{p}) + \sum_j p_j$$

where $s_i(\bar{p}) = \max s_{ij}(\bar{p})$

We will show that $\varphi(\bar{p})$ always decreases in each step of the algorithm.

Assume we found constricting set $N(S)$ corresponding to set of buyers S.

We increase the sum of prices by $|N(S)|$, but $s_i(\bar{p})$ decreased by one for all buyers in S.
$|S| > |N(S)|$, so $\varphi(\bar{p})$ decreased.

In the final step, $\varphi(\bar{p})$ doesn't change.

If $v_{ij} \geq 0$, then $\varphi(\bar{p})$ is always non-negative.
$s_i(\bar{p}) \geq v_{ii} - p_i$

$$\varphi(\bar{p}) \geq \sum_i (v_{ii} - p_i) + \sum_i p_i = \sum_i v_{ii} \geq 0$$

So we conclude that the algorithm terminates in at most $\varphi(0)$ steps. □

"Communist Way"
- find a matching M^* s.t. $\sum_{i,j \in M^*} v_{ij}$ is maximum.

Theorem:
The matching produced by our algorithm also maximizes the sum of valuations. (eg. is optimum in "communist" setting.

Proof:
Let M^* be the matching maximizing the sum of valuations.

Let M be the matching clearing the market with prices p.

$$\begin{aligned}
\sum_{i,j \in M^*} v_{ij} &= \sum_{i,j \in M^*} v_{ij} - \sum_j p_j + \sum_j p_j \\
&= \sum_{i,j \in M^*} s_{ij}(p) + \sum_j p_j \\
&\leq \sum_{i,j \in M} s_{ij}(p) + \sum_j p_j \\
&= \sum_{i,j \in M} v_{ij} - \sum_j p_j + \sum_j p_j \\
&= \sum_{i,j \in M} v_{ij}
\end{aligned}$$

Thus $\sum_{i,j \in M^*} v_{ij} \leq \sum_{i,j \in M} v_{ij}$.

Since M^* maximizes the sum of valuations, $M^* = M$ and $\sum_{i,j \in M^*} v_{ij} = \sum_{i,j \in M} v_{ij}$.

Completing Latin Squares

Latin square: $n \times n$ table filled in with n symbols such that every symbol occurs exactly once in each row & each column

A	B	C	D	E
B	C	D	E	A
C	D	E	A	B
D	E	A	B	C
E	A	B	C	D

- examples: Sudoku, group multiplication table

Suppose the first k rows of a table are given such that every symbol occurs exactly once in each row and at most once in each column.

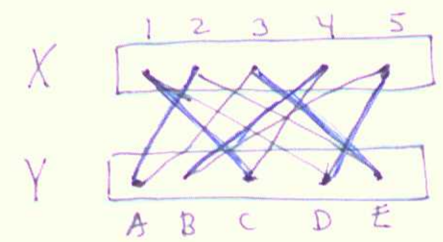
Can we complete this table to a latin square?

1	2	3	4	5
A	B	C	D	E
B	C	D	E	A
E	D	B	A	C
C	A	E	B	D

The answer is always yes.

We can always complete a latin square.
Proof:
It is enough to fill in $(k+1)^{st}$ row such that the conditions are still satisfied. This corresponds to a perfect matching in a bipartite graph with bipartition (X, Y).
X - set of columns
Y - set of symbols
A symbol and a column are adjacent if this symbol wasn't used in the column yet.

→



The resulting graph is $(n-k)$-regular and so we know it has a perfect matching. $\square$