Assignment #4
Honours set theory – MATH 488

Jacob Thomas Errington

10 March 2017

Proposition 1. *Suppose X is a set and $A \subseteq X$. Then, the closure of A in βX is equal to $\hat{A}$.*

Proof. Denote by $\mathcal{A}$ the set A interpreted as the image of A under the principal ultrafilter map. First, note that $\mathcal{A} \subseteq \hat{A}$. To see this, take $p \in \mathcal{A}$ a principal ultrafilter generated by some $a \in A$, so $\{a\} \in \mathcal{A}$. Since $\{a\} \subseteq A$, we have that $A \in \mathcal{A}$ by the upwards closure property of filters. Then by definition of $\hat{A}$, we have $\mathcal{A} \in \hat{A}$. Next, we know that $\hat{A}$ is closed, and that a closed set contains another set if and only if it contains the closure of that set. Hence, $\mathrm{Cl}\,(A) \subseteq \hat{A}$.

Next, take arbitrary $p \in \hat{A}$. We want to show that $p \in \mathrm{Cl}\,(A)$. To do so, we will show that p is a point of closure of A, i.e. that every neighbourhood of p contains a point of A. Take an arbitrary neighbourhood of p and restrict it to a basic neighbourhood $\hat{B}$ such that $B \subseteq X$. Then we have that $A \in P$ and $B \in p$, so $A \cap B \in p$. Hence, for any $x \in A \cap B$, we have $u(x) \in u(A) \cap \hat{B}$, where $u : X \hookrightarrow \beta X$ is the principal ultrafilter map. In particular, $u(A) \cap B \neq \emptyset$ for any neighbourhood of p. This shows that every neighbourhood of p contains a point of A. Hence $\hat{A} \subseteq \mathrm{Cl}\,(A)$. □

Proposition 2. *Let X be a set, C a compact Hausdorff space, and $f : X \to C$ a function. Suppose p is an ultrafilter on X. Then, there exists a unique point $z \in C$ such that for every neighbourhood U of z we have*

$$\{x \in X : f(x) \in U\} \in p$$

Proof. Suppose not. Then for all $z \in C$, there is a neighbourhood U_z of z such that $\{x \in X : f(x) \in U_z\} \notin p$.

The collection of all such U_z forms a cover of C. Hence there exists finite $Z_0 \subseteq C$ such that $\{U_z\}_{z \in Z_0}$ is a subcover of C. Let $A_z = f^{-1}(U_z)$. Then,

$$\bigcup_{z \in Z_0} A_z = \bigcup_{z \in Z_0} f^{-1}(U_z) = f^{-1}\left(\bigcup_{z \in Z_0} U_z\right) = f^{-1}(C) = X$$

which shows that X can be finitely partitioned. We take the complement of both sides. Since each piece of the partition is not in the ultrafilter, each piece's complement is. Hence, the intersection of all the complements, which is finite, is also in the ultrafilter, but this is empty set.

$$\overline{\bigcup_{z \in Z_0} A_z} = \bigcap_{z \in Z_0} \overline{A_z} = \overline{X} = \emptyset \in p$$

This is a contradiction.

Next, we look at uniqueness. Suppose $z_1, z_2 \in C$ are such that $z_1 \neq z_2$. Since C is Hausdorff, we have disjoint neighbourhoods $U_{z_1} \ni z_1$ and $U_{z_2} \ni z_2$. By definition of the limit,

$$f^{-1}(U_{z_1}) \cap f^{-1}(U_{z_2} = f^{-1}(U_{z_1} \cap U_{z_2}) = \emptyset \in p$$

which is a contradiction. □

Proposition 3. *Suppose X is a set and C is a compact Hausdorff space. Let $f : X \to C$ be a function. Then f has a unique continuous extension $\beta f : \beta X \to C$.*

Proof. Define $\beta f(p) = \lim_p f$. To see that this is an extension, look at $g = \beta f \circ u$ where $u : X \to \beta X$ is the principal ultrafilter map. We would like that $z = g(a) = f(a)$ for all a. Since $g(a)$ is a limit, we have that for every neighbourhood U of $g(a)$,

$$\begin{aligned} \{x \in X : f(x) \in U\} \in \{A \subseteq X : a \in A\} \\ a \in f^{-1}(U) \\ f(a) \in U \end{aligned}$$

We have thus that $f(a)$ is in every neighbourhood of z. Suppose $f(a) \neq z$. Then by the Hausdorffness of C, we can separate $f(a)$ and z by neighbourhoods, which is a contradiction.

Next, we want to see that βf is continuous. For simplicity, we assume that C is zero-dimensional, so it has a basis consisting of clopen sets. To show that βf is continuous, it suffices to verify that the preimage of any basic open set is open. Let $A \subseteq C$ be a basic open set, $f^* = \beta f$, and $A' = \widehat{f^{-1}(A)}$.

We claim that $A' = f^{*-1}(A)$.

First, take arbitrary $p \in A'$, so $f^{-1}(A) \in p$. We want to show that

$$p \in f^{*-1}(A) = \left\{ q \in \beta X : f^*(q) = \lim_q f \in A \right\}$$

i.e. that $z = \lim_p f \in A$. Suppose not, so $z \notin A$. Since C is compact Hausdorff, it is regular. Then as A is closed, we can find open sets $A^* \supseteq A$ and $Z \ni z$ such that $Z \cap A^* = \emptyset$. Notice that since A is also open, we can identify A^* and A. Since z is a limit over an ultrafilter, we have that the preimage of the neighbourhood $Z \ni z$ belongs to p. We also have that $f^{-1}(A) \in p$, so $f^{-1}(Z) \cap f^{-1}(A) \in p$. Since $Z \cap A = \emptyset$, the preimages under f must also have empty intersection, so $\emptyset \in p$, which is a contradiction.

The reverse direction is simpler. Take arbitrary $p \in f^{*-1}(A)$, so $z = \lim_p f \in A$. We want to show that $p \in A' = \widehat{f^{-1}(A)}$. Since z is a limit over the ultrafilter p, we have for any neighbourhood $U_z \ni z$ that $f^{-1}(U_z) \in p$, i.e. $p \in \widehat{f^{-1}(U_z)}$. Since A is an open neighbourhood of z, we're done. □

Proposition 4. *Suppose $(S, \cdot)$ is a semigroup. Then, $(\beta S, \cdot)$ is a left topological semigroup.*

Proof. We need to verify two properties.

Associativity. Take arbitrary $p, q, r \in \beta S$. We want to see that for any $A \subseteq S$,

$$\begin{aligned} A \in p \cdot (q \cdot r) &\iff \left\{ s \in S : l_s^{-1}(A) \in q \cdot r \right\} \in p \\ &\iff \left\{ s \in S : \left\{ s' \in S : l_{s'}^{-1}(l_s^{-1}(A)) \in r \right\} \in q \right\} \in p \\ &\iff \left\{ s \in S : \left\{ s' \in S : l_s^{-1}(l_{s'}^{-1}(A)) \in r \right\} \in q \right\} \in p \\ &\iff \left\{ s \in S : l_s^{-1} \left(\left\{ s' \in S : l_{s'}(A) \in r \right\} \right) \in q \right\} \in p \\ &\iff \left\{ s \in S : l_s^{-1}(A) \in r \right\} \in p \cdot q \\ &\iff A \in (p \cdot q) \cdot r \end{aligned}$$

Continuity. We want to check that for any $s \in S$, the left multiplication map $L_s : \beta S \to \beta S$ is continuous.

In general, for any $f : X \to Y$, we can obtain a continuous extension to $\beta f : \beta X \to \beta Y$. First, by postcomposing with the principal ultrafilter map, the codomain becomes a compact Hausdorff space. Then, we can apply proposition 2 to continuously extend the domain of the map to βX.

Recall that $L_s = u \circ \beta l_s$, where $u : S \to \beta S$ is the principal ultrafilter map, so it is continuous.

□

Recall that the right multiplication map $R_p = \beta r_p : \beta S \to \beta S$ is defined as the continuous extension of $r_p(s) = L_s(p)$. Recall also that the

Proposition 5. *For any $p, q \in \beta S$, we have that $p \cdot q = R_q(p)$*

Proof. Take arbitrary $A \in p \cdot q$, so $\left\{s \in S : l_s^{-1}(A) \in q\right\} \in p$. We want to show that $A \in R_q(p)$.

Let $z = R_q(p) = \lim_p r_q$. By the definition of the limit over an ultrafilter, we have for every neighbourhood $U_z \ni z$ that

$$\begin{aligned} r_q^{-1}(U_z) &\in p \\ \{s \in S : r_q(s) \in U_z\} &\in p \\ \{s \in S : L_s(q) \in U_z\} &\in p \\ \left\{s \in S : u(\lim_q l_s) \in U_z\right\} &\in p \end{aligned}$$

where $u : S \to \beta S$ is the principal ultrafilter map. Notice that $y = \lim_q l_s$ means that for every neighbourhood $U_y \ni y$, we have $l_s^{-1}(U_y) \in q$.

Suppose $A \notin z$, so $z \notin \hat{A}$. By regularity, we can find $Z \ni z$ such that $Z \cap \hat{A} = \emptyset$. Instantiating $Z = U_z$, we have

$$\begin{aligned} \left\{s \in S : u(\lim_q l_s) \in Z\right\} &\in p \\ \left\{s \in S : \exists y \in S\ .\ \left(\forall U_y \ni y\ .\ l_s^{-1}(U_y) \in q\right) \wedge \{B \subseteq S : y \in B\} \in Z\right\} &\in p \end{aligned}$$

From here I'm not sure how to proceed. □

Proposition 6. *If S is a cancellative semigroup, then for any $p, q \in \beta S$, if $p \cdot q \in S$, then $p \in u(S)$ and $q \in u(S)$, where $u : S \to \beta S$ is the principal ultrafilter map.*

Proof. Take arbitrary $p, q \in \beta S$ and suppose that $p \cdot q \in u(S)$. Hence we have $t \in S$ such that $p \cdot q = u(t)$ is the principal ultrafilter generated by t, i.e. $p \cdot q = \{A \subseteq S : t \in A\}$. We want to show that $p \in u(S)$ and that $q \in u(S)$, i.e. that there are $p = u(t_p)$ and that $q = u(t_q)$ for some $t_p, t_q \in S$.

By definition of the product of ultrafilters in the Čech-Stone compactification, we have

$$p \cdot q = \left\{A \subseteq S : \left\{s \in S : l_s^{-1}(A) \in q\right\} \in p\right\}$$

Since $p \cdot q$ is principal, $t \in A$ for each $A \in p \cdot q$. Hence, $\left\{s \in S : l_s^{-1}(\{t\}) \in q\right\} \in p$.

Since S is cancellative, its left product maps are injective, so all the preimages $l_s^{-1}(\{t\})$ have at most one element.

Now we look more closely at $B = \left\{s \in S : l_s^{-1}(\{t\}) \in q\right\}$. Of course, this set must be nonempty, else the empty set would belong to p. Suppose two arbitrary elements in B, say s_1 and s_2. Then we have

$S_1 = l_{s_1}^{-1}(\{t\}) \in q$ and $S_2 = l_{s_2}^{-1}(\{t\}) \in q$. Hence, $\emptyset \neq S_1 \cap S_2 \in q$. Let $s \in S_1 \cap S_2$. Then $s_1 \cdot s = t$ and $s_2 \cdot s = t$, so $s_1 = s_2$ by the cancellation property of S. Hence $B = \{t_p\}$ is a singleton, and t_p is a generator for p.

Let s now be the unique element of B. Then $\{t_q\} = l_s^{-1}(A) \in q$ is a singleton as well, and is a generator for q.

Hence, p and q are principal ultrafilters. $\square$