MATH 254
Fall 2016

Solutions to Assignment 2

Exercise 1.

For any $n \in \mathbb{N}$, let A_n be the set of all subsets of $\mathbb{N}_n = \{1, \dots, n\}$.

First prove the following

$$F(\mathbb{N}) = \bigcup_{n=0}^{\infty} A_n \cup \{\emptyset\}.$$

The inclusion $F(\mathbb{N}) \supseteq \bigcup_{n=0}^{\infty} A_n \cup \{\emptyset\}$ follows from the fact that $\mathbb{N}_n \subset \mathbb{N}$, $\emptyset \subset \mathbb{N}$, and any subset of a finite set is finite (proved in class).

Conversely, the inclusion $F(\mathbb{N}) \subseteq \bigcup_{n=0}^{\infty} A_n \cup \{\emptyset\}$ follows from the fact that any finite, nonempty subset of $\mathbb{N}$ has a maximal element (prove this by induction).

Now we prove that A_n is finite by induction on n.

If $n=1$, then $A_1 = \{\{1\}, \emptyset\}$ which is finite with 2 elements.

Assume that A_n is finite.

Remark that we can write

$$A_{n+1} = \underbrace{\{A \in A_{n+1} : n+1 \in A\}}_{\text{This set has same cardinality as } A_n} \cup \overbrace{\{A \in A_{n+1} : n+1 \notin A\}}^{\text{This is the set } A_n}.$$

(to prove this, consider the bijection $\{A \in A_{n+1} : n+1 \in A\} \to A_n$
$f : A \mapsto f(A) = A \setminus \{n+1\}$.

Since A_{n+1} is the union of two finite sets, it follows that A_{n+1} is finite.

We have proven that A_n is finite for all $n \in \mathbb{N}$.

Since $F(\mathbb{N})$ is the countable union of finite sets, it follows that $F(\mathbb{N})$ is countable.

Exercise 2

Assume by contradiction that there exists a bijection $\varphi : \mathbb{N} \longrightarrow P(\mathbb{N})$

Define $A = \{ n \in \mathbb{N};\ n \notin \varphi(n) \} \in P(\mathbb{N})$

Since φ is surjective, $\exists n_A \in \mathbb{N},\ \varphi(n_A) = A$.

Then by definition of A, we have

$$n_A \in \varphi(n_A) \Longrightarrow n_A \notin \varphi(n_A)$$

and

$$n_A \notin \varphi(n_A) \Longrightarrow n_A \in \varphi(n_A)$$

which is a contradiction.

Exercise 3. For any $n \in \mathbb{N}$ and $a_n, \cdots, a_0 \in \mathbb{Z}$ such that $a_n \neq 0$, we let $A_{a_n, \cdots, a_0}$ be the set of solutions of the equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0.$$

From algebra, we know that $A_{a_n, \cdots, a_0}$ is finite with $\mathrm{Card}(A_{a_n, \cdots, a_0}) \leq n$.

By definition the set A of all algebraic numbers can be written as

$$A = \bigcup_{n \in \mathbb{N}} \ \bigcup_{a_n \in \mathbb{Z} \setminus \{0\}} B_{n, a_n}$$

where

$$B_{n, a_n} = \bigcup_{a_{n-1} \in \mathbb{Z}} \ \bigcup_{a_{n-2} \in \mathbb{Z}} \cdots \bigcup_{a_0 \in \mathbb{Z}} A_{a_n, \cdots, a_0}.$$

By induction we prove that for any $n, k \in \mathbb{N}$, if $k \leq n$, then for any $a_n, \cdots, a_k \in \mathbb{Z}$ such that $a_n \neq 0$, the set

$$B_{n, a_n, \cdots, a_k} = \bigcup_{a_{k-1} \in \mathbb{Z}} \ \bigcup_{a_{k-2} \in \mathbb{Z}} \cdots \bigcup_{a_0 \in \mathbb{Z}} A_{a_n, \cdots, a_0}$$

is countable

If $k=1$, then $B_{n,a_n,\dots,a_1} = \bigcup_{a_0 \in \mathbb{Z}} A_{a_n,\dots,a_0}$. Since $\mathbb{Z}$ is countable (proven in class), we obtain that $B_{n,a_n,\dots,a_1}$ is countable as a countable union of finite sets.

Assume that for some $k \in \mathbb{N}$ we have that if $k \leq n$, then for all $a_n,\dots,a_k \in \mathbb{Z}$ such that $a_n \neq 0$, the set $B_{n,a_n,\dots,a_k}$ is countable.

We want to prove that this is true for $k+1$.

If $k+1 \leq n$, i.e. $k<n$, by definition $B_{n,a_n,\dots,a_{k+1}} = \bigcup_{a_k \in \mathbb{Z}} B_{n,a_n,\dots,a_k}$.

Since by our induction assumption the sets $B_{n,a_n,\dots,a_k}$ are countable and since $\mathbb{Z}$ is countable, we obtain that $B_{n,a_n,\dots,a_{k+1}}$ is countable as a countable union of countable sets.

We have proven that $B_{n,a_n,\dots,a_k}$ is countable for all $k,n \in \mathbb{N}$ such that $k \leq n$ and $a_n,\dots,a_0 \in \mathbb{Z}$ such that $a_n \neq 0$.

In particular this is true for $k=n$, namely B_{n,a_n} is countable.

Since $\mathbb{Z}\setminus\{0\}$ is countable as a subset of the countable set $\mathbb{Z}$, it follows that $\bigcup_{a_n \in \mathbb{Z}\setminus\{0\}} B_{n,a_n}$ is countable. Since moreover $\mathbb{N}$ is countable, it follows that $A = \bigcup_{n \in \mathbb{N}} \bigcup_{a_n \in \mathbb{Z}\setminus\{0\}} B_{n,a_n}$ is countable as a countable union of countable sets.

Exercise 4

(i) First we write

$$(-x)(-y)+(-x)y = (-x)((-y)+y) \quad \text{by using (D)}$$
$$= (-x)\cdot 0 \quad \text{—— (A4)}$$

By using this result, we obtain

$$(-x)\cdot 0 + xy = ((-x)(-y)+(-x)y)+xy$$
$$= (-x)(-y)+((-x)y+xy) \quad \text{by using (A2)}$$
$$= (-x)(-y)+((-x)+x)y \quad \text{—— (D)}$$
$$= (-x)(-y)+0\cdot y \quad \text{—— (A4)}$$

Now we prove that $a\cdot 0 = 0$ for all $a \in \mathbb{R}$

$$a\cdot 0 = a\cdot 0 + 0 \quad \text{by using (A3)}$$
$$= a\cdot 0 + (a+(-a)) \quad \text{—— (A4)}$$
$$= (a\cdot 0 + a)+(-a) \quad \text{—— (A2)}$$
$$= (a\cdot 0 + a\cdot 1)+(-a) \quad \text{—— (M3)}$$
$$= a(0+1)+(-a) \quad \text{—— (D)}$$
$$= a\cdot 1 + (-a) \quad \text{—— (A3)}$$
$$= a+(-a) \quad \text{—— (M3)}$$
$$= 0 \quad \text{—— (A4)}$$

It follows from the above results that

$$(-x)(-y) = (-x)(-y)+0\cdot y = (-x)\cdot 0 + xy = \underbrace{0+xy = xy}_{\text{by using (A3)}}$$

(ii) Assume that $x < y < 0$, namely $y + (-x) > 0$ and $-y > 0$.

Then

$$-x = 0 + (-x) \quad \text{by using (A3)}$$
$$= ((-y) + y) + (-x) \quad \text{(A4)}$$
$$= (-y) + (y + (-x)) \quad \text{(A2)}$$
$$> 0 \quad \text{since } y + (-x) > 0 \text{ and } -y > 0$$

Since $-x > 0$ and $-y > 0$, it follows from the trichotomy condition that $x \neq 0$ and $y \neq 0$, hence by using (M4), there exist $\frac{1}{x}, \frac{1}{y} \in \mathbb{R}$ such that $x\frac{1}{x} = \frac{1}{x}x = 1$ and $y\frac{1}{y} = \frac{1}{y}y = 1$.

We want to prove that $\frac{1}{y} < \frac{1}{x} < 0$, namely $\frac{1}{x} + (-\frac{1}{y}) > 0$ and $-\frac{1}{y} > 0$.

We begin with proving that $-\frac{1}{y} > 0$ and $-\frac{1}{x} > 0$

Assume by contradiction $-\frac{1}{y} \leq 0$. Then it follows from the trichotomy condition that either $\frac{1}{y} = 0$ or $\frac{1}{y} > 0$.

We write

$$y = 1 \cdot y \quad \text{by using (M3)}$$
$$= (y\tfrac{1}{y})y \quad \text{(M4)}$$
$$= ((-y)(-\tfrac{1}{y}))y \quad \text{by using the result proven in (i)}$$
$$= (-y)((-\tfrac{1}{y})y) \quad \text{by using (M2)}$$
$$= (-y)(\tfrac{1}{y}(-y)) \quad \text{by using the result proven in (i)}$$

If $\frac{1}{y} > 0$, then $y = (-y)(\frac{1}{y}(-y)) > 0$ since $-y > 0$. This is a contradiction

If $\frac{1}{y} = 0$, then $1 = y\frac{1}{y} = y \cdot 0 = 0$ by using (M4) and since we proved in (i) that $a \cdot 0 = 0$ $\forall a \in \mathbb{R}$. Hence we also obtain a contradiction.

This proves that $-\frac{1}{y} > 0$.

Similarly we obtain that $-x>0$ implies $-\frac{1}{x}>0$

Now we prove that $\frac{1}{x}+\left(-\frac{1}{y}\right)>0$

We write

$$\begin{aligned}
\frac{1}{x}+\left(-\frac{1}{y}\right) &= 1\cdot\frac{1}{x}+\left(-\frac{1}{y}\right)\cdot 1 && \text{by using (M3)}\\
&= \left(\frac{1}{y}\cdot y\right)\cdot\frac{1}{x}+\left(-\frac{1}{y}\right)\left(x\cdot\frac{1}{x}\right) && \text{——— (M4)}\\
&= \left(\left(-\frac{1}{y}\right)(-y)\right)\frac{1}{x}+\left(-\frac{1}{y}\right)\left((-x)\left(-\frac{1}{x}\right)\right) && \text{by using the result proven in (i)}\\
&= \left(-\frac{1}{y}\right)\left((-y)\frac{1}{x}\right)+\left(-\frac{1}{y}\right)\left((-x)\left(-\frac{1}{x}\right)\right) && \text{by using (M2)}\\
&= \left(-\frac{1}{y}\right)\left((-y)\frac{1}{x}+(-x)\left(-\frac{1}{x}\right)\right) && \text{——— (D)}\\
&= \left(-\frac{1}{y}\right)\left(y\left(-\frac{1}{x}\right)+(-x)\left(-\frac{1}{x}\right)\right) && \text{by using the result proven in (i)}\\
&= \left(-\frac{1}{y}\right)\left((y+(-x))\left(-\frac{1}{x}\right)\right) && \text{by using (D)}\\
&&& \text{since } -\tfrac{1}{y}>0,\ y+(-x)>0,\ -\tfrac{1}{x}>0\\
&>0
\end{aligned}$$

Exercise 4.

(i) By applying the triangular inequality, we obtain

$$2|x|=|2x|=|(x+y)+(x-y)|\leq|x+y|+|x-y|$$

and

$$2|y|=|2y|=|(x+y)+(y-x)|\leq|x+y|+|y-x|=|x+y|+|x-y|$$

By summing these two inequalities and dividing by 2, we obtain

$$|x|+|y|\leq|x+y|+|x-y|.$$

(ii) Let $u = x-1$ and $v = y-1$.

Observe that
$$xy - 1 = (x-1+1)(y-1+1) - 1$$
$$= (x-1)(y-1) + (x-1) + (y-1) + \cancel{1} - \cancel{1}$$
$$= uv + u + v$$

Apply the triangle inequality
$$1 + |xy-1| \leq 1 + |uv| + |u| + |v| = 1 + |u|\cdot|v| + |u| + |v| = (1+|u|)(1+|v|)$$
$$= (1+|x-1|)(1+|y-1|)$$

Exercise 6.

(i) $\forall x \in \mathbb{R}$, $\frac{1}{4} - x(1-x) = \frac{1}{4} - x + x^2 = \left(\frac{1}{2} - x\right)^2 \geq 0$

$$\Rightarrow x(1-x) \leq 1/4$$

(ii) Let $x, y \in \mathbb{R}$ be such that $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Assume by contradiction that $xy > 1/4$ and $(1-x)(1-y) > 1/4$.

By multiplying these two numbers, we obtain
$$xy(1-x)(1-y) > \frac{1}{16}$$
On the other hand, it follows from (i) that $x(1-x) \leq 1/4$ and $y(1-y) \leq 1/4$. Since moreover $x \geq 0$, $1-x \geq 0$, $y \geq 0$, and $1-y \geq 0$, we obtain
$$0 \leq x(1-x) \leq 1/4 \quad \text{and} \quad 0 \leq y(1-y) \leq 1/4.$$
By multiplying these two numbers, we obtain
$$0 \leq xy(1-x)(1-y) \leq 1/16.$$
This is in contradiction with what we proved above

(iii) Let $x, y, z \in \mathbb{R}$ be such that $x, y, z > 0$.

Assume by contradiction that $x(1-y) > 1/4$, $y(1-z) > 1/4$ and $z(1-x) > 1/4$.

By multiplying these three numbers, we obtain

$$xyz(1-x)(1-y)(1-z) > \frac{1}{64}.$$

On the other hand, since $x > 0$ and $x(1-y) > 1/4 > 0$, we obtain that $1-y > 0$. Similarly, $1-x > 0$ and $1-z > 0$.

By using (i), we then obtain

$$0 < x(1-x) \leq 1/4, \quad 0 < y(1-y) \leq 1/4, \text{ and } 0 < z(1-z) \leq 1/4.$$

By multiplying these three numbers, we obtain

$$xyz(1-x)(1-y)(1-z) \leq \frac{1}{64}$$

This is in contradiction with what we proved above.