MATH 254 Tutorial 8 (limsup, Cauchy and Contractive Sequences):

Problem 1: Let (x_n) be a bounded sequence and $A = \{x_n : n \in \mathbb{N}\}$. Prove that if $supA$ is not in A, then we have $limsup(x_n) = supA$. (Similar statement for $liminf$.)

Problem 2: Let (x_n) and (y_n) be two bounded sequences.

a) Prove that if $x_n \leq y_n$ for all but finitely many $n \in \mathbb{N}$, then we have $limsup(x_n) \leq limsup(y_n)$.

b) Prove or disprove the converse of part a.

c) Prove that if $x_n \leq y_m$ for all but finitely many pairs $m, n \in \mathbb{N}$, then we have $limsup(x_n) \leq liminf(y_n)$.

d) Prove or disprove the converse of part c.

Problem 3: Let (x_n) and (y_n) be two bounded sequences. Prove the following inequality:

$$limsup(x_n + y_n) \leq limsup(x_n) + limsup(y_n)$$

Give an example for strict inequality. (We also have $liminf(x_n) + liminf(y_n) \leq liminf(x_n + y_n)$.)

Problem 4: Let (x_n) be a sequence such that

$$|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n| \qquad n \in \mathbb{N},$$

(being contractive with the constant 1) and $lim_{n\to\infty}|x_{n+1} - x_n| = 0$. Can we conclude that the sequence is convergent?

Problem 5: Consider the following sequence:

$$x_1 = 1, \qquad x_{n+1} := \frac{1}{1 + x_n} \qquad n \in \mathbb{N}$$

Prove that it is convergent and find its limit.