McGill University
Department of Mathematics and Statistics
MATH 254 Analysis 1, Fall 2015

Assignment 1: Solutions

Some solutions will be only sketched and your are expected to fill in the details.

1. Conjecture a formula for the sum

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\cdots+\frac{1}{(2n-1)(2n+1)},$$

and prove your conjecture using Mathematical Induction.

Solution

Set

$$S(n)=\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}.$$

By computing $S(n)$ for $n=1,2,3$ one can conjecture that

$$S(n)=\frac{n}{2n+1}.$$

An alternative approach to this conjecture (and its proof) is to note that

$$\frac{1}{(2k-1)(2k+1)}=\frac{1}{2}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right),$$

and so

$$\begin{aligned}S(n)&=\frac{1}{2}\left(\left(1-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)\right)\\&=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)\\&=\frac{n}{2n+1}.\end{aligned}$$

We now prove the formula for $S(n)$ be induction.

Base case $n=1$: $\frac{1}{3}=\frac{1}{1\cdot 3}$. This is what we had to show.

Inductive step $n \to n+1$: We assume that $S(n) = \frac{n}{2n+1}$. Then

$$\begin{aligned} S(n+1) &= S(n) + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\ &= \frac{n(2n+3)+1}{(2n+1)2n+3} \\ &= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} \\ &= \frac{n+1}{2(n+1)+1}. \end{aligned}$$

In the second line we have used the induction hypothesis and in the fourth line the identity

$$n(2n+3)+1 = n(2n+1)+2n+1 = (n+1)(2n+1).$$

2. Prove that the collection $\mathcal{F}(\mathbb{N})$ of all *finite* subsets of $\mathbb{N}$ is countable.

   Solution(Sketch) In the tutorial (see Yariv's notes, Proposition 1.12) it was shown that if A is a finite set with n-elements, then $\mathcal{P}(A)$ has 2^n elements. In particular, $\mathcal{P}(A)$ is a finite set. Let $\mathcal{F}_n$ be the collection of all subsets of $\{1, \cdots, n\}$. Then

$$\mathcal{F}(\mathbb{N}) = \bigcup_{n=1}^{\infty} \mathcal{F}_n.$$

   You should write a detailed proof of this identity. Each $\mathcal{F}_n$ is a finite set and hence countable. Since $\mathcal{F}(\mathbb{N})$ is a countable union of countable sets, $\mathcal{F}(\mathbb{N})$ is countable (quote the precise result proven in class).

3. Let E_n, $n = 1, 2, \cdots$ be an infinite sequence of sets. Let

$$\overline{E} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m, \qquad \underline{E} = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m.$$

   Prove that

$$\bigcap_{n=1}^{\infty} E_n \subseteq \underline{E} \subseteq \overline{E} \subseteq \bigcup_{n=1}^{\infty} E_n.$$

   Solution(Sketch) For each n, $\bigcap_{k=1}^{\infty} E_k \subseteq \bigcap_{m=n}^{\infty} E_m$. Hence,

$$\bigcap_{k=1}^{\infty} E_k \subseteq \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} E_m = \underline{E}.$$

   Suppose that $x \in \underline{E}$. Then $x \in \bigcap_{m=n_0}^{\infty} E_m$ for some n_0, that is, $x \in E_m$ for all $m \geq n_0$. Hence, $x \in \bigcup_{m=n}^{\infty} E_m$ for all n, and we deduce that $x \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m$. It follows that $x \in \overline{E}$, and so $\underline{E} \subseteq \overline{E}$.

Finally, $\bigcup_{m=n}^{\infty} E_m \subseteq \bigcup_{k=1}^{\infty} E_k$ for all n. Hence,

$$\overline{E} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m \subseteq \bigcup_{k=1}^{\infty} E_k.$$

This completes the proof.

4. Let $f : D \to E$ be a function and let $A \subseteq D$, $B \subseteq E$. Prove the following:

   (a) $f(f^{-1}(B)) \subseteq B$.

   (b) If f is surjective then $f(f^{-1}(B)) = B$.

   (c) $f^{-1}(f(A)) \supseteq A$.

   (d) If f is injective then $f^{-1}(f(A)) = A$.

   Solution

   (a) Let $y \in f(f^{-1}(B))$. Then there exists $x \in f^{-1}(B)$ with $f(x) = y$. Since $x \in f^{-1}(B)$ we have that $f(x) \in B$. This means that $y = f(x) \in B$ i.e. $f(f^{-1}(B)) \subseteq B$.

   (b) Let f be surjective. By part (a) we just need to show that $B \subseteq f(f^{-1}(B))$. Let $y \in B$. Since f is surjective there exists $x \in f^{-1}(B)$ with $f(x) = y$. Since $x \in f^{-1}(B)$ we have $y = f(x) \in f(f^{-1}(B))$ i.e. $B \subseteq f(f^{-1}(B))$ which is what we had to show.

   (c) Let $x \in A$. Then $f(x) \in f(A)$ i.e. $\{f(x)\} \subseteq f(A)$. Then $f^{-1}(\{f(x)\}) \subseteq f^{-1}(f(A))$. But $x \in f^{-1}(\{f(x)\})$ hence $x \in f^{-1}(f(A))$. This proves $f^{-1}(f(A)) \supseteq A$.

   (d) Let f be injective. By part (c) we just need to show that $f^{-1}(f(A)) \subseteq A$. Let $x \in f^{-1}(f(A))$. Then $f(x) \in f(A)$. Thus there exists $\tilde{x} \in A$ with $f(\tilde{x}) = f(x)$. But since f is injective this implies $\tilde{x} = x$ i.e. $x = \tilde{x} \in A$. This proves $f^{-1}(f(A)) \subseteq A$ which is what we had to show.

5. Prove by induction that

$$\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{n \text{ nested square roots}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

   for all $n \in \mathbb{N}$.

   Hint: The half-angle formula $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ might be useful.

   Solution

   Base case $n = 1$: we have to show that $2\cos\frac{\pi}{4} = \sqrt{2}$. This is true since $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

   Inductive step $n \to n+1$: We assume that

$$\underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}}}_{n \text{ nested square roots}} = 2\cos\left(\frac{\pi}{2^{n+1}}\right)$$

Then

$$\underbrace{\sqrt{2+\underbrace{\sqrt{2+\sqrt{2+\cdots+\sqrt{2}}}}_{n\text{ nested square roots}}}}_{n+1\text{ nested square roots}} \overset{\text{Ind. Hyp}}{=} \sqrt{2+2\cos\left(\frac{\pi}{2^{n+1}}\right)} = \sqrt{4\,\frac{1}{2}\left(1+\cos\left(\frac{\pi}{2^{n+1}}\right)\right)}$$

By the half-angle formula for cosine this equals

$$= \sqrt{4\cos^2\left(\frac{1}{2}\cdot\frac{\pi}{2^{n+1}}\right)} = 2\cos\left(\frac{\pi}{2^{n+2}}\right)$$

since $\dfrac{\pi}{2^{n+2}}$ is an angle in the first quadrant. This is what we had to show in the inductive step.

Finally, this proves the formula for all $n \in \mathbb{N}$.

6. Recall that the binomial coefficient $\binom{n}{k}$ is defined as $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. Prove by induction on n that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ for all $n \in \mathbb{N}_0$. You may use, without proof, the well-known identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ for all $n \in \mathbb{N}_0$ and $1 \le k \le n$.

Solution

<u>Base case $n = 0$</u>: $\sum_{k=0}^{0}\binom{n}{k} = \binom{0}{0} = 1 = 2^0$. This is what we had to show.

<u>Inductive step $n \to n+1$</u>: We assume that $\sum_{k=0}^{n}\binom{n}{k} = 2^n$. Then

$$\begin{aligned}
\sum_{k=0}^{n+1}\binom{n+1}{k} &= \binom{n+1}{0} + \sum_{k=1}^{n}\binom{n+1}{k} + \binom{n+1}{n+1} = 1 + \sum_{k=1}^{n}\left[\binom{n}{k} + \binom{n}{k-1}\right] + 1 \\
&= 1 + \sum_{k=1}^{n}\binom{n}{k} + \sum_{k=1}^{n}\binom{n}{k-1} + 1 \\
&= 1 + \sum_{k=1}^{n}\binom{n}{k} + \sum_{k=0}^{n-1}\binom{n}{k} + 1 \quad \text{(Index shift in the second sum)} \\
&= \left[\binom{n}{0} + \sum_{k=1}^{n}\binom{n}{k}\right] + \left[\sum_{k=0}^{n-1}\binom{n}{k} + \binom{n}{n}\right] \\
&= \sum_{k=0}^{n}\binom{n}{k} + \sum_{k=0}^{n}\binom{n}{k} = 2\sum_{k=0}^{n}\binom{n}{k} \overset{\text{Ind. Hyp}}{=} 2\cdot 2^n = 2^{n+1}
\end{aligned}$$

This completes the inductive step.

Finally, this proves the formula for all $n \in \mathbb{N}_0$.

7. Let A be a countably infinite set and let $B \subseteq A$. Prove that B is countable.

Solution

If $B = \emptyset$ there is nothing to prove. Let $B \neq \emptyset$ and let $b \in B$ be arbitrary. Pick an enumeration $a_1, a_2, a_3, \dots$ of A. We define a function $f : \mathbb{N} \to B$ as follows:

$$f(n) = \begin{cases} b & \text{if } a_n \notin B \\ a_n & \text{if } a_n \in B \end{cases}$$

Then f is surjective (but, in general, not injective). As seen in class this means that B is finite or countably infinite i.e. countable.