McGill University
Department of Mathematics and Statistics
MATH 254 Analysis 1, Fall 2015

Assignment 5: Solutions

1. Let (y_n) be an unbounded sequence of positive numbers satisfying $y_{n+1} > y_n$ for all $n \in \mathbb{N}$. Let (x_n) be another sequence and suppose that the limit

$$\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

exists. Prove that

$$\lim \frac{x_n}{y_n} = \lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}.$$

Hint: You may use the Problem 3 on Assignment 4.

Using the above result, prove that for any $p \in \mathbb{N}$ the following holds:

(a)
$$\lim \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} = \frac{1}{p+1}$$

(b)
$$\lim \left(\frac{1^p + 2^p + \cdots + n^p}{n^p} - \frac{n}{p+1} \right) = \frac{1}{2}$$

(c)
$$\lim \frac{1^p + 3^p \cdots + (2n+1)^p}{n^{p+1}} = \frac{2^p}{p+1}$$

Solution:

Let $\lim \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = L$. Then, for any $\epsilon > 0, \exists\, N \in \mathbb{N}$ such that for $n \geq N$,

$$\left| \frac{x_{n+1} - x_n}{y_{n+1} - y_n} - L \right| < \epsilon \iff L - \epsilon < \frac{x_{n+1} - x_n}{y_{n+1} - y_n} < L + \epsilon.$$

Using that (y_n) is a strictly increasing sequence,

$$(L - \epsilon)(y_{n+1} - y_n) < x_{n+1} - x_n < (L + \epsilon)(y_{n+1} - y_n), \quad \forall n \geq N$$

Summing from some $n_0 \geq N$ to some $m \geq n_0 + 1$,

$$(L - \epsilon) \sum_{n=n_0}^{m-1} (y_{n+1} - y_n) < \sum_{n=n_0}^{m-1} x_{n+1} - x_n < (L + \epsilon) \sum_{n=n_0}^{m-1} (y_{n+1} - y_n).$$

Since we have a telescoping sum, the previous equation is equivalent to:

$$(L - \epsilon)(y_m - y_{n_0}) < x_m - x_{n_0} < (L + \epsilon)(y_m - y_{n_0})$$

Since (y_n) is unbounded, we can choose m large enough so that y_m is positive; then we may divide by y_m without changing the sides of the inequalities:

$$(L-\epsilon)\left(1-\frac{y_{n_0}}{y_m}\right) < \frac{x_m}{y_m} - \frac{x_{n_0}}{y_m} < (L+\epsilon)\left(1-\frac{y_{n_0}}{y_m}\right)$$

Since (y_n) is unbounded, taking the limit as $m \to \infty$ gives us:

$$L-\epsilon \leq \lim_{m\to\infty} \frac{x_m}{y_m} \leq L+\epsilon.$$

Since ϵ is an arbitrary positive number, we have (after replacing the dummy m by n):

$$\lim \frac{x_n}{y_n} = L = \lim \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$$

as we wanted to show.

(a) We want to compute

$$\lim \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}$$

if it exists. Expanding,

$$\frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}} = \frac{n^p + \sum_{k=1}^{p}\binom{p}{k} n^{p-k}}{(p+1)n^p + \sum_{k=2}^{p+1}\binom{p+1}{k} n^{p-k}} = \frac{1 + \sum_{k=1}^{p}\binom{p}{k} n^{-k}}{(p+1) + \sum_{k=2}^{p+1}\binom{p+1}{k} n^{-k}}$$

Taking the limit as $n \to \infty$,

$$\lim \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}} = \frac{1}{p+1}$$

By the theorem above,

$$\lim \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} = \frac{1}{p+1}$$

(b) Let

$$\frac{1^p + 2^p + \cdots + n^p}{n^p} - \frac{n}{p+1} = \frac{\sum_{k=1}^{n} k^p}{n^p} - \frac{n}{p+1} = \frac{(p+1)\sum_{k=1}^{n} k^p - n^{p+1}}{n^p(p+1)} = \frac{x_n}{y_n}.$$

Then

$$\frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \frac{(p+1)(n+1)^p - (n+1)^{p+1} + n^{p+1}}{(p+1)\cdot[(n+1)^p - n^p]}.$$

Using binomial theorem, the above expands into

$$\frac{(p+1)\cdot\left[n^p + pn^{p-1} + \sum_{k=2}^{p}\binom{p}{k} n^{p-k}\right] - \left[(p+1)n^p + \binom{p+1}{2} n^{p-1} + \sum_{k=3}^{p}\binom{p+1}{k} n^{p+1-k}\right]}{(p+1)\cdot\left[pn^{p-1} + \sum_{k=2}^{p}\binom{p}{k} n^{p-k}\right]}$$

Cancelling the n^p terms and regrouping the n^{p-1} terms, we get

$$\frac{\left[(p+1)p - \frac{(p+1)!}{(p-1)!2!}\right] n^{p-1} + (p+1)\cdot\sum_{k=2}^{p}\binom{p}{k} n^{p-k} - \sum_{k=3}^{p}\binom{p+1}{k} n^{p+1-k}}{(p+1)\cdot\left[pn^{p-1} + \sum_{k=2}^{p}\binom{p}{k} n^{p-k}\right]}.$$

Dividing top and bottom by n^{p-1}, we get

$$\frac{(p+1)p - \frac{(p+1)!}{(p-1)!2!} + (p+1)\cdot\sum_{k=2}^{p}\binom{p}{k} n^{-k+1} - \sum_{k=3}^{p}\binom{p+1}{k} n^{-k+2}}{(p+1)\cdot\left[p + \sum_{k=2}^{p}\binom{p}{k} n^{-k+1}\right]}.$$

Since each n-dependent term has a negative exponent, we will get in the limit as $n \to \infty$,

$$\frac{(p+1)p - \frac{(p+1)!}{(p-1)!2!}}{(p+1)p} = \frac{(p+1)p - \frac{(p+1)(p)(p-1)!}{(p-1)!2!}}{(p+1)p} = 1 - \frac{1}{2} = \frac{1}{2}.$$

Since

$$\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = \frac{1}{2},$$

it follows from our theorem above that

$$\lim\left(\frac{1^p + 2^p + \cdots + n^p}{n^p} - \frac{n}{p+1}\right) = \lim \frac{x_n}{y_n} = \frac{1}{2}.$$

(c) Want to compute

$$\lim \frac{(2(n+1)+1)^p}{(n+1)^{p+1} - n^{p+1}} = \lim \frac{(2n+3)^p}{(n+1)^{p+1} - n^{p+1}}.$$

Expanding,

$$\frac{(2n+3)^p}{(n+1)^{p+1} - n^{p+1}} = \frac{2^p n^p + \sum_{k=1}^{p}\binom{p}{k} 3^k (2n)^{p-k}}{(p+1)n^p + \sum_{k=2}^{p+1}\binom{p+1}{k} n^{p-k}} = \frac{2^p + \sum_{k=1}^{p}\binom{p}{k} 3^k (2n)^{-k}}{(p+1) + \sum_{k=2}^{p+1}\binom{p+1}{k} n^{-k}}.$$

Taking the limit as $n \to \infty$,

$$\lim \frac{(2n+3)^p}{(n+1)^{p+1} - n^{p+1}} = \frac{2^p}{p+1}.$$

By the theorem above,

$$\lim \frac{1^p + 3^p \cdots + (2n+1)^p}{n^{p+1}} = \frac{2^p}{p+1}.$$

2. Let (x_n) and (y_n) be two sequences defined recursively as follows: $x_1 = a \geq 0$, $y_1 = b \geq 0$,

$$x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2}, \quad n \geq 1$$

Prove that the sequences (x_n) and (y_n) are convergent and that

$$\lim x_n = \lim y_n.$$

Solution:

Lemma. *For any $s, t \in \mathbb{R}$,*

$$st \leq \frac{s^2 + t^2}{2}.$$

Proof. It is clear that

$$(s - t)^2 \geq 0$$

Expanding,

$$s^2 - 2st + t^2 \geq 0 \Rightarrow \frac{s^2 + t^2}{2} \geq st \quad \square$$

In particular, if $s = \sqrt{u}$ and $t = \sqrt{v}$, then for any $u, v \geq 0$,

$$\sqrt{uv} \leq \frac{u + v}{2}.$$

Then, for any $n \geq 1$,

$$x_{n+1} = \sqrt{x_n y_n}, \quad y_{n+1} = \frac{x_n + y_n}{2}$$

Since $x_1, y_1 \geq 0$, $x_n, y_n \geq 0$ for $n \geq 1$ by definition. We can therefore use our lemma to conclude that for $n \geq 2$, $x_n \leq y_n$. Then for $n \geq 2$,

$$x_{n+1} = \sqrt{x_n y_n} \geq \sqrt{x_n^2} = x_n,$$

$$y_{n+1} = \frac{x_n + y_n}{2} \leq \frac{2y_n}{2} = y_n.$$

Thus, (x_{n+1}) is an increasing sequence and (y_{n+1}) is a decreasing sequence. Since $x_{n+1} \leq y_{n+1} \leq y_n$ for $n \geq 3$, we inductively conclude $x_{n+1} \leq y_3$. Thus, (x_{n+1}) is a monotonically increasing sequence bounded above, and converges by the monotone convergence theorem. Since $y_{n+1} \geq 0$ for $n \geq 0$, (y_{n+1}) is a monotonically decreasing sequence bounded below and converges by the monotone convergence theorem. Letting L be the limit of (x_{n+1}), and thus of (x_n), and noticing:

$$y_n = \frac{(\sqrt{x_n y_n})^2}{x_n},$$

we have:

$$\lim y_n = \frac{(\lim x_{n+1})^2}{\lim x_n} = \frac{L^2}{L} = L = \lim x_n.$$

3. Prove that

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

for all $n \in \mathbb{N}$.

Solution:

It was shown in class that

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

for all $n \in \mathbb{N}$. Taking logarithms yields

$$\ln\left[\left(1+\frac{1}{n}\right)^n\right] = n\ln\left(1+\frac{1}{n}\right) < 1 < \ln\left[\left(1+\frac{1}{n}\right)^{n+1}\right] = (n+1)\ln\left(1+\frac{1}{n}\right)$$

The left part of this inequality implies that $\ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$ whereas the right part implies that $\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right)$ for all $n \in \mathbb{N}$. Combining both inequalities yields

$$\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$$

which is what we had to show.

4. Prove that the sequence

$$x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} - \ln n, \qquad n \in \mathbb{N},$$

converges.

Remark: The limit of this sequence is called the *Euler-Mascheroni constant*; its numerical value is 0.5772156649.... It is currently unknown whether this constant is rational or irrational.

Solution:

We start by showing that (x_n) is decreasing; we will use the estimates from question 3.

$$\begin{aligned} x_n - x_{n+1} &= \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\frac{1}{n+1}\right) - \ln n + \ln(n+1) \\ &= -\frac{1}{n+1} - \ln n + \ln(n+1) = -\frac{1}{n+1} + \ln\left(\frac{n+1}{n}\right) = -\frac{1}{n+1} + \ln\left(1+\frac{1}{n}\right) > 0 \end{aligned}$$

by question 4 i.e. (x_n) is decreasing. We will show next that (x_n) is bounded below by 0 i.e. that $x_n \geq 0$ for all $n \in \mathbb{N}$.

$$\begin{aligned} \ln n &= \ln n - \ln 1 = (\ln n - \ln(n-1)) + (\ln(n-1) - \ln(n-2)) + \cdots + (\ln 2 - \ln 1) \\ &= \ln\left(\frac{n}{n-1}\right) + \ln\left(\frac{n-1}{n-2}\right) + \cdots + \ln\left(\frac{2}{1}\right) \\ &= \ln\left(1+\frac{1}{n-1}\right) + \ln\left(1+\frac{1}{n-2}\right) + \cdots + \ln\left(1+\frac{1}{1}\right) < \frac{1}{n-1} + \frac{1}{n-2} + \cdots + 1 \end{aligned}$$

by question 3. Thus

$$x_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} - \ln n > \left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \left(\frac{1}{n-1}+\frac{1}{n-2}+\cdots+1\right) = \frac{1}{n} > 0$$

Thus (x_n) is decreasing and bounded below and is thus convergent.

5. Prove that
$$\lim\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right)=\ln 2.$$

Solution:

Let

$$\begin{aligned}x_n &:= \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\\ y_n &:= \ln\left(1+\frac{1}{n}\right)+\ln\left(1+\frac{1}{n+1}\right)+\cdots+\ln\left(1+\frac{1}{2n-1}\right)\end{aligned}$$

and

$$z_n := \frac{1}{n}+\frac{1}{n+1}+\cdots+\frac{1}{2n-1}$$

Then, by question 3, we have that $x_n < y_n < z_n$ for all $n \in \mathbb{N}$.

For y_n we obtain:

$$\begin{aligned}y_n &= \ln\left[\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right)\cdots\left(1+\frac{1}{2n}\right)\right]\\ &= \ln\left[\frac{n+1}{n}\cdot\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdots\frac{2n}{2n-1}\right]\end{aligned}$$

(Note that this a telescoping product!)

$$= \ln\frac{2n}{n} = \ln 2$$

Furthermore, $z_n = x_n + \dfrac{1}{n} - \dfrac{1}{2n} = x_n + \dfrac{1}{2n}$.

Combining these results yields $x_n < \ln 2 < x_n + \dfrac{1}{2n}$ and thus $\ln 2 - \dfrac{1}{2n} < x_n < \ln 2$. Since $\lim\left(\ln 2 - \dfrac{1}{2n}\right) = \lim(\ln 2) = \ln 2$, it follows from the squeeze theorem that (x_n) converges to $\ln 2$.

6. Let
$$x_n = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2^n}\right), \qquad n \in \mathbb{N}.$$

Prove that the sequence (x_n) converges.

Solution:

$$x_{n+1} = \left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{2^{n+1}}\right) = x_n\left(1+\frac{1}{2^{n+1}}\right) > x_n$$

The sequence (x_n) is thus increasing. We show next that (x_n) is bounded above by e, using estimates obtained in question 3 as well as the formula for the sum of a finite geometric series (recall that $a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} a\, r^k = a\,\frac{1-r^n}{1-r}$).

$$\begin{aligned}\ln(x_n) &= \ln\left[\left(1+\frac{1}{2}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{2^n}\right)\right]\\ &= \ln\left(1+\frac{1}{2}\right) + \ln\left(1+\frac{1}{4}\right) + \cdots + \ln\left(1+\frac{1}{2^n}\right)\end{aligned}$$

By question 3, $\ln\left(1+\frac{1}{k}\right) < \frac{1}{k}$. Thus

$$\ln(x_n) < \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{2} + \cdots + \frac{1}{2}\cdot\frac{1}{2^{n-1}}$$

where the right-hand side is a finite geometric series with $a = \frac{1}{2}$ and $r = \frac{1}{2}$. Thus

$$\ln(x_n) < \frac{1}{2}\,\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 1 - \frac{1}{2^n}$$

(The formula $\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$ can also be proved by induction.)

Consequently $\ln(x_n) < 1$ which means that $x_n < e$ for all $n \in \mathbb{N}$.

The sequence (x_n) is thus increasing and bounded above and thus converges.

7. Let (x_n), $x_n > 0$ be a convergent sequence, then

$$\lim_{n\to\infty} \sqrt[n]{x_1 x_2 x_3 \cdots x_n} = \lim_{n\to\infty} x_n$$

Solution:

Let $L \equiv \lim_{n\to\infty} x_n$ and let $y_n \equiv \sqrt[n]{x_1 x_2 x_3 \cdots x_n}$ There are two possibilities.

First scenario is that $L = 0$. In this case, for a given $\varepsilon > 0$, there exists $N' \in \mathbb{N}$ such that for all $n > N'$,

$$0 < x_n < \frac{\varepsilon}{2}$$

Thus,

$$\begin{aligned}y_n = \sqrt[n]{x_1 \cdots x_{N'} \cdots x_n} &= \sqrt[n]{c}\,\sqrt[n]{x_{N'+1} \cdots x_n}\\ &\le c^{\frac{1}{n}} \cdot \left(\frac{\varepsilon}{2}\right)^{\frac{n-N'}{n}}\\ &= \left(\frac{2^{N'} c}{\varepsilon^{N'}}\right)^{\frac{1}{n}} \frac{\varepsilon}{2}\\ &= (a)^{\frac{1}{n}}\frac{\varepsilon}{2}\end{aligned}$$

where $c \equiv \sqrt[n]{x_1 \cdots x_{N'}}$ and $a \equiv \frac{2^{N'} c}{\varepsilon^{N'}}$. Since $\lim_{n\to\infty} a^{\frac{1}{n}} = 1$, there exists $N'' \in \mathbb{N}$ such that for all $n \geq N''$,

$$a^{\frac{1}{n}} < 2$$

Thus, choosing $N > \max\{N', N''\}$, we have

$$y_n < (2)\frac{\varepsilon}{2} = \varepsilon$$

for all $n \geq N$. Thus, $\lim y_n = 0$.

In the second scenario, $L > 0$. In this case, by Theorem 3.2.3 (b) in Bartle and Sherbert, we can equivalently show that

$$\lim_{n\to\infty} \frac{y_n}{L} = 1$$

Thus, defining $z_n = \frac{y_n}{L}$, our goal becomes to show that $\lim z_n = 1$. We now examine the expression for z_n:

$$\begin{aligned} z_n &= \frac{y_n}{x_n} \\ &= \frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{L} \\ &= \sqrt[n]{\frac{x_1 x_2 \cdots x_n}{L^n}} \\ &= \sqrt[n]{\left(\frac{x_1}{L}\right)\left(\frac{x_2}{L}\right)\cdots\left(\frac{x_n}{L}\right)} \end{aligned}$$

Fixing $\varepsilon > 0$ (Without loss of generality, $\varepsilon < 1$) we may choose $N' \in \mathbb{N}$ large enough so that $|x_n - L| < \frac{\varepsilon}{2}L$ for all $n \geq N'$. In particular, we have $x_n > L - L\frac{\varepsilon}{2}$ for $n \geq N'$. Thus,

$$\begin{aligned} z_n &> \sqrt[n]{\left(\frac{x_1}{L}\right)\left(\frac{x_2}{L}\right)\cdots\left(\frac{x_{N'}}{L}\right)} \left(\frac{L\left(1 - \frac{\varepsilon}{2}\right)}{L}\right)^{\frac{n-N'}{n}} \\ &= c^{\frac{1}{n}}\left(1 - \frac{\varepsilon}{2}\right) \end{aligned}$$

where

$$c = \left(\frac{2^{N'} \prod_{i=1}^{N'} \frac{x_i}{L}}{(2-\varepsilon)^{N'}}\right) > 0$$

Once again, we shall use the fact that $\lim c^{\frac{1}{n}} = 1$ to choose N'' large enough so that

$$c^{\frac{1}{n}} > 1 - \frac{\varepsilon}{2}$$

Thus, choosing $N > \max\{N', N''\}$, we have that

$$z_n > \left(1 - \frac{\varepsilon}{2}\right)^2 = 1 - \varepsilon + \frac{\varepsilon^2}{4} > 1 - \varepsilon$$

for $n \geq N$.

In an analogous manner, one can ensure that N is suitably large enough so that for $n \geq N$,

$$z_n < 1 + \varepsilon$$

(it is an exercise to show this rigorously). Summarizing, for a given $\varepsilon > 0$, we have found $N \in \mathbb{N}$, such that

$$-\varepsilon < z_n - 1 < \varepsilon$$

for all $n \geq N$. Equivalently stated,

$$|z_n - 1| < \varepsilon \quad \forall n \geq N$$

This proves the claim.

8. Let (x_n), $x_n > 0$ be a sequence such that the limit

$$L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$$

exists. Prove that

$$\lim_{n\to\infty} \sqrt[n]{x_n} = L.$$

Using this result, prove that

$$\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$$

Solution:

Let y_n be a sequence defined by

$$\begin{aligned} y_1 &= x_1 \\ y_n &= \frac{x_n}{x_{n-1}} \text{ for } n \geq 2 \end{aligned}$$

Then, one can easily verify that $y_n > 0$ for all $n \in \mathbb{N}$ and that

$$\lim_{n\to\infty} y_n = L$$

We may thus use the result from Exercise 7 to conclude that

$$\lim_{n\to\infty} \sqrt[n]{y_1 y_2 \cdots y_n} = L$$

However, we see that

$$y_1 y_2 \cdots y_n = x_1 \left(\frac{x_2}{x_1}\right) \left(\frac{x_3}{x_2}\right) \cdots \left(\frac{x_n}{x_{n-1}}\right) = x_n$$

and thus

$$\lim_{n\to\infty} \sqrt[n]{x_n} = L.$$

To prove the second statement in the problem, we define the sequence (x_n) by

$$x_n \equiv \frac{n^n}{n!}.$$

We then examine the ratio of consecutive terms in the sequence,

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \frac{(n+1)^{n+1}n!}{n^n(n+1)!} \\
&= \frac{(n+1)(n+1)^n n!}{(n+1)n!n^n} \\
&= \left(\frac{n+1}{n}\right)^n \\
&= \left(1+\frac{1}{n}\right)^n
\end{aligned}$$

Thus, by our definition of the number e, we have

$$\lim_{n\to\infty} \frac{x_{n+1}}{n} = e$$

which, using the first result of this problem, implies that

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \frac{n}{\sqrt[n]{n!}} = e.$$

9. Let $(x_n)_{n=1}^{\infty}$ be a positive *subadditive* sequence. That is, for any $n, m \in \mathbb{N}$, we have

$$0 \le x_{n+m} \le x_n + x_m$$

Show that $\lim_{n\to\infty} \frac{x_n}{n}$ exists and

$$\lim_{n\to\infty} \frac{x_n}{n} = \inf\left\{\frac{x_n}{n} : n \in \mathbb{N}\right\}$$

Solution:

We let $L \equiv \inf\left\{\frac{x_n}{n} : n \in \mathbb{N}\right\}$. We must clearly have

$$L \le \frac{x_n}{n} \quad \forall n \in \mathbb{N} \tag{1}$$

To show convergence, we let $\varepsilon > 0$ be given and choose $k \in \mathbb{N}$, such that

$$\frac{a_k}{k} < L + \frac{\varepsilon}{2}$$

Note that this is always possible by the definition of L. Now for each $n \geq k$, we have $n = p_n \cdot k + r_n$ for some $p_n \in \mathbb{N}$ and $r_n \in \{0, \ldots, k-1\}$. By inductively applying the subadditivity condition, we find that

$$x_n = x_{p_n \cdot k + r_n} \leq x_{p_n \cdot k} + x_{r_n} \leq p_n x_k + x_{r_n}$$

dividing by n, we find that

$$\frac{x_n}{n} \leq \frac{p_n x_k}{p_n \cdot k + r_k} + \frac{x_{r_n}}{n} \leq \frac{x_k}{k} + \frac{x_{r_n}}{n} < L + \frac{\varepsilon}{2} + \frac{x_{r_n}}{n} \tag{2}$$

for all $n \geq k$.

Now, since $r_n \in \{0, \ldots, k-1\}$, we have that

$$x_{r_n} \leq \max\{x_1, \ldots, x_{k-1}\} \equiv M$$

Thus, choosing $N \in \mathbb{N}$ such that $N > \max\left\{k, \frac{2M}{\varepsilon}\right\}$, equations (**??**) and (**??**) imply that for all $n \geq N$, we have

$$L \leq \frac{x_n}{n} \leq L + \varepsilon$$

and since ε was an arbitrary positive number, we have completed the proof.

10. Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n = \sup\{x_k : k \geq n\}$ and $S = \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.

Solution:

First note that for $n \in \mathbb{N}$

$$\begin{aligned} s_{n+1} &= \sup\{x_k : k \geq n+1\} = \sup\{x_{n+1}, x_n, x_{n-1}, \ldots\} \\ &= \sup(\{x_{n+1}\} \cup \{x_n, x_{n-1}, \ldots\}) \\ &= \max\{x_{n+1}, \sup\{x_n, x_{n-1}, \ldots\}\} \\ &\leq \sup\{x_n, x_{n-1}, \ldots\} = s_n. \end{aligned}$$

Since (x_n) is bounded, $\exists M \in \mathbb{R}$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. By definition, we have that $s_n \geq x_n \geq -M$. Therefore, since $n \in \mathbb{N}$ was arbitrary, $(s_n)_{n \in \mathbb{N}}$ is bounded below by $-M$ and decreasing and we conclude that $S := \inf s_n = \lim_{n\to\infty} s_n$. We will inductively construct a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ which converges to S.

The first element Simply take $x_{n_1} = x_1$.

The *k*-th element (*k* >1) Assume we constructed $x_{n_{k-1}}$. We want to find x_{n_k} such that $|x_{n_k} - S| < 1/k$. Since, $S = \lim_{m\to\infty} s_m$, $\exists M'_k \in \mathbb{N}$ such that $|s_m - S| < 1/(2k)$ for any $m \geq M'_k$. In particular, we can find $M_k > n_{k-1}$ such that $|s_{M_k} - S| < 1/(2k)$. Now since $s_{M_k} = \sup_{n \geq M_k} x_n$, $\exists n_k \geq M_k > n_{k-1}$ such that $s_{M_k} \leq x_{n_k} < s_{M_k} + 1/(2k)$. Then,

$$|x_{n_k} - S| \leq |x_{n_k} - s_{M_k}| + |s_{M_k} - S| < 1/(2k) + 1/(2k) = 1/k.$$

Since the sequence $(x_{n_k})_{k\in\mathbb{N}}$ satisfies $|x_{n_k} - S| < 1/k$ for all k, by the Archimedean property,

$$\forall \epsilon > 0, \exists K(\epsilon) \in \mathbb{N} \text{ such that } 1/K(\epsilon) < \epsilon \Rightarrow |x_{n_k} - S| < \epsilon \quad \forall k \geq K(\epsilon)$$

and we conclude $x_{n_k} \to S$ as $k \to \infty$.

11. Let $L \subset \mathbb{R}$. The set L is said to be open if for any $x \in L$ there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset L$. The set L is said to be closed if its complement $L^c = \{x \in \mathbb{R} : x \notin L\}$ is open.

   (a) Prove that L is closed if and only if for any convergent sequence (x_n) with $x_n \in L$, the limit $x = \lim_{n\to\infty} x_n = x$ is also an element of L.

   (b) Let (x_n) be a bounded sequence. A point $x \in \mathbb{R}$ is called an accumulation point of (x_n) if there exists a subsequence (x_{n_k}) of (x_n) such that $\lim_{k\to\infty} x_{n_k} = x$. We denote by L the set of all accumulation points of (x_n). By the Bolzano-Weierstraß Theorem, the set L is non-empty. Prove that L is a bounded closed set.

   (c) Let (x_n) be a bounded sequence, let L be as in part (b) and let S be as in problem 1. Prove that $S = \sup L$.

Solution:

(a) ($\Rightarrow$) Let L be a closed set. Let $(x_n)_{n\in\mathbb{N}}$ be converging sequence with $x_n \in L$ and $\lim_{n\to\infty} x_n = x$. Then, in particular, $\forall \epsilon > 0$, $\exists n \in \mathbb{N}$ such that $|x_n - x| < \epsilon$. Therefore, $\forall \epsilon > 0$, can find $x_n \in L$ such that $x_n \in (x-\epsilon, x+\epsilon)$. Therefore, $\forall \epsilon > 0$, $(x-\epsilon, x+\epsilon) \not\subseteq L^c$. Equivalently, $\nexists \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq L^c$. Since L^c is open, we must conclude that $x \notin L^c$ and therefore that $x \in L$.

($\Leftarrow$) Let L be a set that is not closed. Then L^c is not open, which implies that $\exists x \in L^c$ such that $\forall n \in \mathbb{N}$, $(x - n^{-1}, x + n^{-1}) \not\subseteq L^c \Leftrightarrow (x - n^{-1}, x + n^{-1}) \cap L \neq \emptyset$. Hence, $\forall n \in \mathbb{N}$, $\exists x_n \in L$ such that $x_n \in (x - n^{-1}, x + n^{-1})$. We have constructed a sequence $(x_n)_{n\in\mathbb{N}}$ in L with the property that $\forall n \in \mathbb{N}, |x_n - x| < n^{-1}$, which implies that $\lim_{n\to\infty} x_n = x$ where $x \notin L$. Therefore, it is not true that for any converging sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \in L$, the limit $x = \lim_{n\to\infty} x_n$ is also an element of L.

Remark: A lot of people did well on the first implication. For the second, a lot of people have confused "L is not closed" and "L is open": they do not mean the same thing. Some sets are both open and closed (clopen) and some sets are neither closed nor open.

Also note that closed [resp. open] sets are not necessarily closed [resp. open] intervals and that in general, $(x-\epsilon, x+\epsilon) \not\subset L^c$ does not imply $(x-\epsilon, x+\epsilon) \subset L$.

(b) Let $\ell \in L$ be an accumulation point. Then there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{n_k} = \ell$. In particular, $\exists K \in \mathbb{N}$ such that $|x_{n_K} - \ell| < 1$. This implies (write the details) $|\ell| < 1 + |x_{n_K}|$. Since $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence, $\exists M \in \mathbb{R}$ such that $M \geq |x_n|$ for all $n \in \mathbb{N}$ and in particular, $M \geq |x_{n_K}|$. Therefore, $|\ell| < 1 + M$. Since $\ell \in L$ was arbitrary, we conclude L is bounded (by $M+1$).

Let $(\ell_m)_{m\in\mathbb{N}}$ be a convergent sequence with $\ell_m \in L$ and $\ell = \lim_{m\to\infty} \ell_m$. We want to construct a subsequence $(x_{n_{M(j),K(M(j))}})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ that converges to ℓ. Let $x_{n_{M(1),K(M(1))}} = x_1$. For $j > 1$, since $\ell = \lim_{m\to\infty} \ell_m$, $\exists M(j) \in \mathbb{N}$ such that $|\ell_{M(j)} - \ell| < 1/(2j)$. Since $\ell_{M(j)} \in L$, we can find a subsequence $(x_{n_{M(j),k}})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{n_{M(j),k}} = \ell_{M(j)}$. In particular, $\exists K(M(j)) \in \mathbb{N}$ such that $|x_{n_{M(j),K(M(j))}} - \ell_{M(j)}| < 1/(2j)$ and $n_{M(j),K(M(j))} > n_{M(j-1),K(M(j-1))}$. This defines a subsequence $(x_{n_{M(j),K(M(j))}})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ satisfying

$$
\begin{aligned}
\left|x_{n_{M(j),K(M(j))}} - \ell\right| &\leq \left|x_{n_{M(j),K(M(j))}} - \ell_{M(j)}\right| + \left|\ell_{M(j)} - \ell\right| \\
&< 1/(2j) + 1/(2j) \\
&= 1/j.
\end{aligned}
$$

Using the Archimedean property as in part (a), we get that $\lim_{j\to\infty} x_{n_{M(j),K(M(j))}} = \ell$ and we conclude $\ell \in L$.

Remark: One has to be careful with notation in order not to give new meaning to objects that were already defined. This might require a lot of indices.

(c) We first need to show that S is an upper bound of L. Let $\ell \in L$. Then, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\lim_{k\to\infty} x_{n_k} = \ell$. Since $n_k \geq k$, we have $x_{n_k} \leq s_k$, which implies that $\lim_{k\to\infty} x_{n_k} \leq \lim_{k\to\infty} s_k$, that is $\ell \leq S$ (using Problem 1). Since $\ell \in L$ was arbitrary, we conclude L is bounded above by S and $S \geq \sup L$.

By Problem 1, S is an accumulation point of (x_n), *i.e.* $S \in L$, so that $S \leq \sup L$. We conclude $S = \sup L$.

12. Using the Cauchy Convergence Criterion, prove that the sequence

$$x_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

Solution:

Let $\epsilon > 0$. By the Archimedean property, $\exists N \in \mathbb{N}$ such that $N > \epsilon^{-1}$. Let $n, m \in \mathbb{N}$ satisfy $n \geq N$ and $m \geq N$. Without loss of generality, $m > n$. Then,

$$\begin{aligned}
|x_m - x_n| &= \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots + \frac{1}{m^2} \\
&< \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \cdots + \frac{1}{(m-1)m} \\
&= \left(\frac{1}{n} - \frac{1}{n+1}\right) + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \cdots + \left(\frac{1}{m-1} - \frac{1}{m}\right) \\
&= \frac{1}{n} - \frac{1}{m} \\
&< \frac{1}{N} < \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude (x_n) is a Cauchy sequence and therefore convergent.

13. Definition: a sequence (x_n) has bounded variation if there exists $c > 0$ such that for all $n \in \mathbb{N}$,

$$|x_2 - x_1| + |x_2 - x_2| + \cdots + |x_n - x_{n-1}| < c.$$

Show that if a sequence has bounded variation, then the sequence is converging. Find an example of a convergent sequence which does not have bounded variation.

Solution:

We show the contrapositive: if a sequence is not converging, then it does not have bounded variation. Assume (x_n) does not converge. Then, (x_n) is not Cauchy, *i.e.*

$$\exists \epsilon_0 \text{ such that } \forall N \in \mathbb{N},\ \exists n(N), m(N) \geq N \text{ such that } |x_{m(N)} - x_{n(N)}| \geq \epsilon_0. \tag{3}$$

Let $c > 0$ be arbitrary. Then, by the Archimedean property, $\exists K \in \mathbb{N}$ such that $K\epsilon_0 \geq c$.

1. By (**??**) with $N = 1$, there exists $n(1), m(1) \in \mathbb{N}$ such that $|x_{m(1)} - x_{n(1)}| \geq \epsilon_0$ without loss of genreality $m(1) > n(1)$.
2. By (**??**) with $N = m(1) + 1$, there exists $m(2) > n(2) \geq m(1) + 1 > m(1) > n(1)$ such that $|x_{m(2)} - x_{n(2)}| \geq \epsilon_0$.

   $\vdots$

K. By (**??**) with $N = m(K-1) + 1$, we get $m(K) > m(K) > m(K-1) > n(K-1)$ such that $|x_{m(K)} - x_{n(K)}| \geq \epsilon_0$.

Then,

$$\begin{aligned} c &\leq K\epsilon_0 \\ &\leq |x_{m(1)} - x_{n(1)}| + |x_{m(2)} - x_{n(2)}| + \cdots + |x_{m(K)} - x_{n(K)}| \\ &\leq |x_2 - x_1| + |x_2 - x_3| + \cdots |x_{m(K)-1} - x_{m(K)}|. \end{aligned}$$

(Write the details). Since $c > 0$ was arbitrary, we conclude (x_n) does not have bounded variation.

Consider the sequence $x_n = (-1)^n/n$. It is easy to show, using the Archimedean property, that (x_n) converges to 0. Note that (write the details)

$$|x_2 - x_1| + |x_3 - x_2| + \cdots + |x_n - x_{n-1}| \geq 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

and that there exists no $c > 0$ such that $1 + \frac{1}{2} + \cdots + \frac{1}{n} < c$ for all $n \in \mathbb{N}$ (see Example 3.3.3(b) in Bartle and Sherbert, fourth edition). We conclude (x_n) does not have bounded variation.

14. Let $x_1 < x_2$ be arbitrary real numbers and

$$x_n = \frac{x_{n-1}}{3} + \frac{2x_{n-2}}{3}, \quad n > 2.$$

Find the formula for x_n and $\lim x_n$.

Solution:

The characteristic equation is $\lambda^2 = \lambda/3 + 2/3$, which has roots $\lambda_1 = -2/3$ and $\lambda_2 = 1$. The nth term is therefore given by

$$x_n = (-2/3)^n c_1 + 1^n c_2.$$

We have $x_1 = (-2/3)c_1 + c_2$ and $x_2 = (4/9)c_1 + c_2$. Solving the linear system (write the details) yields $c_1 = -9(x_1 - x_2)/10$ and $c_2 = (2x_1 + 3x_2)/5$, so that

$$x_n = (-2/3)^n(-9(x_1 - x_2)/10) + (2x_1 + 3x_2)/5.$$

Since $(-2/3)^n \to 0$ as $n \to \infty$, we have $(-2/3)^n(-9(x_1 - x_2)/10) \to 0$ as $n \to \infty$ and $x_n = (-2/3)^n(-9(x_1 - x_2)/10) + (2x_1 + 3x_2)/5 \to (2x_1 + 3x_2)/5$ as $n \to \infty$.

15. Let $x_1 > 0$ and
$$x_{n+1} = \frac{1}{2+x_n}, \quad n \geq 1.$$
Show that (x_n) is a contractive sequence and find $\lim x_n$.

Solution:

We first show by induction that $x_n > 0$ for all $n \in \mathbb{N}$. Base case $n = 1$ is given. Assume $x_n > 0$. Then, $2 + x_n > 0$ so that $x_{n+1} = \frac{1}{2+x_n} > 0$.

Now, we have
$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left| \frac{1}{2+x_{n+1}} - \frac{1}{2+x_n} \right| = \left| \frac{2 + x_n - 2 - x_{n+1}}{(2+x_{n+1})(2+x_n)} \right| \\ &= \frac{|x_n - x_{n+1}|}{|(2+x_{n+1})(2+x_n)|} < \frac{|x_n - x_{n+1}|}{4} \end{aligned}$$
since $x_{n+1} > 0$ and $x_n > 0$. This shows (x_n) is contractive. Therefore, it is Cauchy and it converges. Let $x = \lim x_n$. Then,
$$\begin{aligned} 2 + \lim x_n = 2 + x \quad &\Rightarrow \quad \lim \frac{1}{2+x_n} = \frac{1}{2+\lim x_n} = \frac{1}{2+x} \\ &\Rightarrow \quad x = \lim x_{n+1} = \lim \frac{1}{2+x_n} = \frac{1}{2+x} \\ &\Rightarrow \quad x^2 + 2x = 1 \\ &\Rightarrow \quad x = -1 \pm \sqrt{2}. \end{aligned}$$
However, since $x_n > 0$ for all $n \in \mathbb{N}$, we must have $\lim x_n \geq 0$ and we conclude $\lim x_n = -1 + \sqrt{2}$.