MATH323 - Tutorial 5 (Feb 8^{th}/17)

Distributions Review

1. Uniform Discrete : Y takes values $y_1, y_2, \dots, y_n$ each with probability $1/n$
$\mathbb{P}(Y=y_i) = \frac{1}{n}$, $i \in \{1, \dots, n\}$, $\mathbb{E}(Y) = \frac{1}{n}\sum_{i=1}^{n} y_i$, $V(Y) = \frac{1}{n}\sum_{i=1}^{n} y_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} y_i\right)^2$

2. Bernouilli : Y takes values 1 or 0 with probability p and $1-p$ (resp.)
$\mathbb{P}(Y=1) = p$, $\mathbb{E}(Y) = p$, $V(Y) = p(1-p)$

3. Binomial : Y takes values $0, 1, \dots, n$. Y counts the number of 1's out of n independent Bernouilli trials each with value of 1 occurring with probability $= p$
$\mathbb{P}(Y=x) = \binom{n}{x} p^x (1-p)^{n-x}$; $x \in \{0, \dots, n\}$, $\mathbb{E}(Y) = np$, $V(Y) = np(1-p)$

4. Poisson : Y takes values $0, 1, 2, \dots$. Y counts the number of events that occur over a preset period of time. $\mathbb{P}(Y=x) = \frac{\lambda^x}{x!} e^{-\lambda}$, $\lambda > 0$, $x \in \{0, 1, \dots\}$.
$\mathbb{E}(Y) = V(Y) = \lambda$

5. Geometric : Y takes values $1, 2, 3, \dots$. Y counts the number of trials until the first success out of an infinite sequence of independent Bernoulli trials with success occurring with probability $= p$. $\mathbb{P}(Y=x) = (1-p)^{x-1} p$; $x \in 1, 2, \dots$, $0 \le p \le 1$
$\mathbb{E}(Y) = \frac{1}{p}$, $V(Y) = \frac{1-p}{p^2}$

6. Negative Binomial : Y determines the number of the trial where r^{th} success occurs out of an infinite sequence of independent Bernouilli trials with success occurring with probability p. $\mathbb{P}(Y=x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$, $x \in \{r, r+1, r+2, \dots\}$, $0 \le p \le 1$
$\mathbb{E}(Y) = \frac{r}{p}$, $V(Y) = \frac{r(1-p)}{p^2}$

7. Hypergeometric : A population of N elements that possess one of two characteristics. Let r many elements have characteristic 1 and $N-r$ have characteristic 2. Select n elements from the N elements. Y counts the number of elements out of n that possess characteristic 1.
$\mathbb{P}(Y=x) = \frac{\binom{r}{x}\binom{N-r}{n-x}}{\binom{N}{n}}$; $x = 0, 1, \dots, n$ subject to $x \le r$, $n-x \le N-r$, $\mathbb{E}(Y) = \frac{nr}{N}$
$V(Y) = n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$

Problems

1. (Exercise 3.64): If there are n trials in a binomial experiment and we observe y_0 "successes", show $\mathbb{P}(Y=y_0)$ is maximized when $p = y_0/n$.

Solution: $$\mathbb{P}(Y=y_0) = \binom{n}{y_0} p^{y_0}(1-p)^{n-y_0}$$

$$\frac{d}{dp}\mathbb{P}(Y=y_0) = \binom{n}{y_0}\left\{\left(y_0 p^{y_0-1}\right)\left(1-p\right)^{n-y_0} + \left(p^{y_0}\right)\left((1-p)^{n-y_0-1}(n-y_0)(-1)\right)\right.$$

Set equal to 0 to obtain maximum, i.e. $\frac{d}{dp}\mathbb{P}(Y=y_0)=0$.

Since $\binom{n}{y_0} > 0$, we can divide out this term to obtain:

$$y_0 p^{y_0-1}(1-p)^{n-y_0} - p^{y_0}(1-p)^{n-y_0-1}(n-y_0) = 0$$
$$y_0 p^{y_0-1}(1-p)^{n-y_0} = p^{y_0}(1-p)^{n-y_0-1}(n-y_0)$$
$$y_0\left(\frac{p^{y_0-1}}{p^{y_0}}\right)\left(\frac{(1-p)^{n-y_0}}{(1-p)^{n-y_0-1}}\right) = n-y_0$$
$$y_0 p^{-1}(1-p) = n-y_0$$
$$y_0(1-p) = p(n-y_0)$$
$$y_0 = pn$$

$$\Rightarrow p = \frac{y_0}{n}$$

Check $p = \frac{y_0}{n}$ is a maximum by applying the second derivative test from univariate Calculus (Exercise)

2. (Exercise 3.142): Let $p(y)$ denote the probability function associated with a Poisson random variable with mean λ.

a) Show that the ratio of successive probabilities satisfies:

$$\frac{p(y)}{p(y-1)} = \frac{\lambda}{y} \quad \text{for } y = 1, 2, \ldots$$

b) For which values of y is $p(y) > p(y-1)$.

Solution:

a) $$p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$$

$$\Rightarrow \frac{p(y)}{p(y-1)} = \left(\frac{\lambda^y e^{-\lambda}}{y!}\right) \Big/ \left(\frac{\lambda^{y-1} e^{-\lambda}}{(y-1)!}\right)$$

$$= \frac{\lambda^y e^{-\lambda} (y-1)!}{\lambda^{y-1} e^{-\lambda} (y)!}$$

$$= \frac{\lambda}{y} \quad \blacksquare$$

b) $$p(y) > p(y-1)$$

$$\Rightarrow \frac{\lambda^y e^{-\lambda}}{y!} > \frac{\lambda^{y-1} e^{-\lambda}}{(y-1)!}$$

$$\Rightarrow \frac{\lambda^y e^{-\lambda} (y-1)!}{y!} > \lambda^{y-1} e^{-\lambda}$$

$$\Rightarrow \frac{\lambda^y e^{-\lambda}}{y} > \lambda^{y-1} e^{-\lambda}$$

$$\Rightarrow \frac{\lambda^y}{\lambda^{y-1}} \frac{1}{y} > \frac{e^{-\lambda}}{e^{-\lambda}}$$

$$\Rightarrow \frac{\lambda}{y} > 1 \quad \Rightarrow \quad y < \lambda$$

Therefore, the values of y for which $p(y) > p(y-1)$ are those in the set $\{x \in \mathbb{N} : x < \lambda\}$ $\blacksquare$

3. Let Y be a Binomial(n, p) random variable. Let $m(t) = \mathbb{E}(e^{tY})$ be a function of $t \in \mathbb{R}$. Show

$$\frac{d\,m(t)}{dt}\bigg|_{t=0} = \mathbb{E}(Y) \quad \text{and}$$

$$\frac{d^2}{dt^2} m(t)\bigg|_{t=0} - \left(\frac{d}{dt} m(t)\bigg|_{t=0}\right)^2 = V(Y).$$

Solution:

$$\begin{aligned} m(t) &= \mathbb{E}(e^{tY}) \\ &= \sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} e^{ty} \\ &= \sum_{y=0}^{n} \binom{n}{y} (pe^t)^y (1-p)^{n-y} \end{aligned}$$

Then by the Binomial expansion:

$$m(t) = (pe^t + (1-p))^n$$

$$\frac{d}{dt} m(t) = n(pe^t + (1-p))^{n-1} \cdot pe^t$$

$$\frac{d}{dt} m(t)\bigg|_{t=0} = np = \mathbb{E}(Y)$$

$$\frac{d^2}{dt^2} m(t) = np\left\{e^t (pe^t + (1-p))^{n-1} + e^t (n-1)(pe^t + 1 - p)^{n-2} pe^t\right\}$$

$$\frac{d^2}{dt^2} m(t)\bigg|_{t=0} = np(1 + p(n-1)) = np + n^2p^2 - np^2$$

$$\begin{aligned} \frac{d^2}{dt^2} m(t)\bigg|_{t=0} - \left(\frac{d}{dt} m(t)\bigg|_{t=0}\right)^2 &= np + n^2p^2 - np^2 - (np)^2 \\ &= np - np^2 \\ &= np(1-p) = V(Y) \quad \blacksquare \end{aligned}$$

(Remark: we call $m(t)$ the "moment generating function")

4. (Exercise 3.96): The telephone lines serving an airline reservation office are all busy about 60% of the time.

a) If you are calling this office, what is the probability that you will complete your call on the first try? second try? third try? fourth try?

b) If you and a friend must both complete calls to this office, what is the probability that a total of four tries will be necessary for both of you to get through?

<u>Solution</u>:

a) Let $Y \sim NB(r,p) = NB(1, 0.4)$

Probability of success on first try: $\mathbb{P}(Y=1) = \binom{1-1}{1-1} 0.4^1 0.6^{1-1} = 0.4$

" " second try: $\mathbb{P}(Y=2) = \binom{2-1}{1-1} 0.4^1 0.6^{2-1} = (0.4)(0.6)$

" " third try: $\mathbb{P}(Y=3) = \binom{3-1}{1-1} 0.4^1 0.6^{3-1} = (0.4)(0.6)^2$

" " fourth try: $\mathbb{P}(Y=4) = \binom{4-1}{1-1} 0.4^1 0.6^{4-1} = (0.4)(0.6)^3$

b) Let A be the event that you make your call before trial 4.
Let B be the event that your friend " "

$\mathbb{P}(\{\text{both make calls before trial 4}\}) = \mathbb{P}(A \cap B)$

Under the assumption that A and B are independent events

$$= \mathbb{P}(A)\,\mathbb{P}(B)$$
$$= \mathbb{P}(Y \le 4)\,\mathbb{P}(Y \le 4)$$
$$= (\mathbb{P}(Y \le 4))^2$$
$$= (0.4 + (0.4)(0.6) + (0.4)(0.6)^2 + (0.4)(0.6)^3)^2$$
$$\doteq 0.7576$$