COMP 525 Winter 2017
Assignment 2
Due Date: 7th February 2017

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Please turn in questions 1 through 7. There are **7** questions for credit and three additional questions for those who know topology or would like to learn it. If you do questions 8, 9 and 10 you must still do all the other questions as there is no credit for doing the additional questions.

Question 1[15 points] Exercise 3.5 parts (e)-(h) from the book.

Question 2[15 points] Exercise 3.6 from the book.

Question 3[15 points] Exercise 3.11 from the book.

Question 4[15 points] Exercise 3.17 from the book.

Question 5[15 points] Exercise 4.4 from the book.

Question 6[10 points] Exercise 4.5 from the book.

Question 7[15 points] Exercise 4.7 from the book.

We view infinite words over an alphabet Σ as functions from $\mathbb{N}$, the natural numbers, to Σ. This means that we do not have transfinite words and the words have a starting point, they are not infinite in both directions[1]. If x is a word we can write $x(n)$ for the letter that occurs at the nth position. We define a metric on these words as follows:

$$d(x, y) = 2^{-\min n.[x(n) \neq y(n)]}$$

where we set $d(x, y) = 0$ if there is no such n.

[1]These should be obvious but some of you take delight in coming up with perverse interpretations and then insisting that these are the "natural" interpretations.

Question 8[0 points] Show that this is a metric and in fact an *ultrametric*, i.e. for any triple x, y, z of words we have

$$d(x, y) \leq \max\{d(x, z), d(y, z)\}.$$

This has the amusing consequence that all triangles are isosceles.

Show that the closed balls are actually open.

Show that the closed sets defined in the book are exactly the closed sets in the topology induced by this metric. It follows immediately that closed sets are safety properties and dense sets are liveness properties.

Question 9[0 points]

We consider the space $\Sigma^\infty := \Sigma^\omega \cup \Sigma^*$ of finite and infinite words. We define this to be a poset with the prefix order, written $\leq$. We write, for any $\sigma \in \Sigma^*$

$$\sigma \uparrow = \{x \in \Sigma^\infty \mid \sigma \leq x\}.$$

Now we define a topology, called the *Scott topology*, by defining sets of the form $\sigma \uparrow$ to be basic open sets. If you don't believe that this is the base for a topology please convince yourself, but spare me the details. Show that this topology is T_0 but not T_1 hence cannot possibly arise from a metric. Show, nevertheless that if we look at Σ^ω with the subspace topology we get exactly the metric topology defined above.

Question 10[0 points] There is a natural convergence notion associated with the partial order. We observe that every chain has a lub. So we define the limit of a chain to be this lub. We can extend the metric in a natural way to all of Σ^∞. So we can compare the order-theoretic convergence concept with the metric one. Show first that the lub of a chain is in fact the limit in the Scott topology as well as the metric limit. The usual proof that the limit of a convergent sequence is unique uses the Hausdorff condition. Here the Scott topology is not Hausdorff but we have just shown that chains have unique Scott limits.

There are sequences that metrically converge but are not chains so they do not converge in the order-theoretic sense. An example is the sequence of ω-words:

$$ab^\omega, a^2b^\omega, a^3b^\omega, \ldots, a^nb^\omega, \ldots,$$

which metrically converges to a^ω. Note how the b's get "pushed off to infinity." Show that if $\{x_i | i \in I\}$ metrically converges to x there is a sequence of *finite words* w_i such that for each i we have $w_i \leq x_i$ and the least upper bound of the w_is is x.