MATH254

Solutions to assignment 1

Exercise 1 We calculate the first terms

$$\frac{1}{2!} = \frac{1}{2} \qquad \frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{1}{2} + \frac{1}{3} + \frac{1}{2\times 4} = \frac{23}{24}$$

In view of these calculations we can conjecture that

$$\forall n \geq 1, \quad \frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} = \frac{(n+1)!-1}{(n+1)!} \qquad (E_n)$$

We prove the formula (E_n) by induction.

For $n=1$, (E_1) is true (see above).

Assume that (E_n) is true for some $n \geq 1$. We want to prove that (E_{n+1}) is true

By using our induction hypothesis we obtain

$$\frac{1}{2!} + \frac{2}{3!} + \cdots + \frac{n}{(n+1)!} + \frac{n+1}{(n+2)!} = \frac{(n+1)!-1}{(n+1)!} + \frac{n+1}{(n+2)!}$$

$$= \frac{(n+2)((n+1)!-1) + n+1}{(n+2)!} = \frac{(n+2)! - \cancel{n} - 2 + \cancel{n} + 1}{(n+2)!} = \frac{(n+2)!-1}{(n+2)!}$$

hence (E_{n+1}) is true.

Conclusion: we have proven that (E_n) is true for all $n \geq 1$.

Exercise 2. We want to prove the inequality

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) \qquad (I_n)$$

For $n=1$,

$$\left(\sum_{k=1}^{1} a_k b_k\right)^2 = (a_1 b_1)^2 = a_1^2 b_1^2 = \left(\sum_{k=1}^{1} a_k^2\right)\left(\sum_{k=1}^{1} b_k^2\right)$$

hence (I_1) is true.

Assume that (I_n) is true for some $n \geq 1$. We want to prove that (I_{n+1}) is true.

We have

$$\left(\sum_{k=1}^{n+1} a_k b_k\right)^2 = \left(\sum_{k=1}^{n} a_k b_k + a_{n+1} b_{n+1}\right)^2$$

$$= \left(\sum_{k=1}^{n} a_k b_k\right)^2 + (a_{n+1} b_{n+1})^2 + 2 a_{n+1} b_{n+1} \sum_{k=1}^{n} a_k b_k$$

By using our induction hypothesis, it follows that

$$\left(\sum_{k=1}^{n+1} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + (a_{n+1} b_{n+1})^2 + 2 a_{n+1} b_{n+1} \sum_{k=1}^{n} a_k b_k$$

For any $k \in \{1, \dots, n\}$, we have

$$2 a_{n+1} b_{n+1} a_k b_k = 2 (a_{n+1} a_k)(b_{n+1} b_k) \leq (a_{n+1} a_k)^2 + (b_{n+1} b_k)^2.$$

It follows that

$$\left(\sum_{k=1}^{n+1} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right) + (a_{n+1} b_{n+1})^2 + \sum_{k=1}^{n} (a_{n+1} a_k)^2 + \sum_{k=1}^{n} (b_{n+1} b_k)^2$$

$$= \left(\sum_{k=1}^{n} a_k^2 + a_{n+1}^2\right)\left(\sum_{k=1}^{n} b_k^2 + b_{n+1}^2\right) = \left(\sum_{k=1}^{n+1} a_k^2\right)\left(\sum_{k=1}^{n+1} b_k^2\right)$$

which proves that (I_{n+1}) is true.

Conclusion: we have proven that (I_n) is true for all $n \geq 1$.

Exercise 3.

We prove by induction on n that $\exists i \in \{1, \dots, n\}\ \forall j \in \{1, \dots, n\} \setminus \{i\}\ A_j \not\subseteq A_i$ $\quad (P_n)$

For $n = 2$, since $A_1 \neq A_2$, we have either $A_1 \not\subseteq A_2$ or $A_2 \not\subseteq A_1$ which implies that (P_2) is true.

Assume that (P_n) is true for some $n \geq 2$. We want to prove that (P_{n+1}) is true.

By applying our induction hypothesis, we know that there exists $i \in \{1, \dots, n\}$ such that $\forall j \in \{1, \dots, n\} \setminus \{i\}\ A_j \not\subseteq A_i$

In case $A_{n+1} \not\subseteq A_i$, we obtain that A_i contains none of the sets A_j for $j \in \{1, \dots, n+1\} \setminus \{i\}$ hence (P_{n+1}) is true

In case $A_{n+1} \subseteq A_i$, we are going to prove that $\forall j \in \{1, \dots, n\}\ A_j \not\subseteq A_{n+1}$

Since $A_{n+1} \subseteq A_i$ and $A_{n+1} \neq A_i$, we have $A_i \not\subseteq A_{n+1}$.

Moreover for any $j \in \{1, \dots, n\} \setminus \{i\}$, since $A_j \not\subseteq A_i$, we obtain that $\exists x \in A_j, x \notin A_i$. Since $A_{n+1} \subseteq A_i$, it follows that $x \notin A_{n+1}$ which proves that $A_j \not\subseteq A_{n+1}$.

In both case we have proven that (P_{n+1}) is true

Conclusion: (P_n) is true for all $n \geq 2$

Remark: It is also possible to do a proof by contradiction here.

Exercise 4

(i) Let $G \in \mathcal{P}(A)$. Let $x \in G$. Then $f(x) \in f(G)$ namely $\{f(x)\} \subseteq f(G)$. Then $f^{-1}(\{f(x)\}) \subseteq f^{-1}(f(G))$. But $x \in f^{-1}(\{f(x)\})$ hence $x \in f^{-1}(f(G))$. This proves that $G \subseteq f^{-1}(f(G))$.

(ii) Assume that f is injective. Let $G \in \mathcal{P}(A)$. It follows from (i) that $G \subseteq f^{-1}(f(G))$. Let $x \in f^{-1}(f(G))$. Then $f(x) \in f(G)$, namely $\exists \bar{x} \in G, f(x) = f(\bar{x})$. Since f is injective, this implies $x = \bar{x} \in G$. This proves that $G \supseteq f^{-1}(f(G))$. Thus $G = f^{-1}(f(G))$.

Conversely, assume that $\forall G \in \mathcal{P}(A)$ $G = f^{-1}(f(G))$
Let $x_1, x_2 \in A$ be such that $f(x_1) = f(x_2)$. Consider $G = \{x_1\}$.
We have $f(G) = \{f(x_1)\} = \{f(x_2)\}$ hence $x_2 \in f^{-1}(f(G)) = G$ namely $x_1 = x_2$. This proves that f is injective

(iii) Let $H \in \mathcal{P}(B)$. Let $y \in f(f^{-1}(H))$. Then $\exists x \in f^{-1}(H), f(x) = y$. Since $x \in f^{-1}(H)$, we obtain that $y = f(x) \in H$. This proves that $f(f^{-1}(H)) \subseteq H$.

(iv) Assume that f is surjective. Let $H \in \mathcal{P}(B)$. It follows from (iii) that $f(f^{-1}(H)) \subseteq H$. Let $y \in H$. Since f is surjective, $\exists x \in f^{-1}(H), f(x) = y$. Since $x \in f^{-1}(H)$, it follows that $y = f(x) \in f(f^{-1}(H))$. This proves that $H \subseteq f(f^{-1}(H))$. Thus $H = f(f^{-1}(H))$

Conversely, assume that $\forall H \in \mathcal{P}(B)$ $H = f(f^{-1}(H))$.
Let $y \in B$. By considering $H = \{y\}$, we have $\{y\} = f(f^{-1}(\{y\}))$
hence $f^{-1}(\{y\}) \neq \emptyset$. This proves that f is surjective.

Exercise 5.

Assume that f is bijective. Let $G \in \mathcal{P}(A)$.
Let $y \in f(A \setminus G)$. Then $\exists x \in A \setminus G, f(x) = y$. Assume by contradiction that $y \in f(G)$. Then $\exists \bar{x} \in G, f(\bar{x}) = y$.
Since f is injective, it follows that $x = \bar{x}$, which is not possible since $x \in A \setminus G$ and $\bar{x} \in G$. This proves that
$f(A \setminus G) \subseteq B \setminus f(G)$.
Let $y \in B \setminus f(G)$. Since f is surjective, $\exists x \in A, y = f(x)$.
But $y \notin f(G)$ hence $x \notin G$ which implies $y \in f(A \setminus G)$.
This proves that $B \setminus f(G) \subseteq f(A \setminus G)$. Thus $B \setminus f(G) = f(A \setminus G)$

Conversely, assume that $\forall G \in \mathcal{P}(A)$ $f(A \setminus G) = B \setminus f(G)$.
Let $x_1, x_2 \in A$ be such that $x_1 \neq x_2$. Then by taking $G = \{x_1\}$, we obtain $x_2 \in A \setminus G$. It follows that $f(x_2) \in f(A \setminus G) = B \setminus f(G) = B \setminus \{f(x_1)\}$
namely $f(x_1) \neq f(x_2)$. This proves that f is injective.
Now we prove that f is surjective. By taking $G = B$, we obtain
$f(A) = B \setminus (B \setminus f(A)) = B \setminus f(A \setminus A) = B \setminus f(\emptyset) = B \setminus \emptyset = B$. This proves that f is surjective. Thus we have proven that f is bijective

Exercise 6

(i) If f is injective, then $\forall x \in A$, $f(f(x)) = f(x) \Rightarrow f(x) = x \Rightarrow x \in f(A)$ hence f is surjective

If f is surjective, then $\forall y \in A$ $\exists x \in A$, $y = f(x)$ which gives $f(y) = f(f(x)) = f(x) = y$. It follows that $\forall x_1, x_2 \in A$ $x_1 \neq x_2 \Rightarrow f(x_1) = x_1 \neq x_2 = f(x_2)$ hence f is injective.

Thus we have proven that f is injective $\Leftrightarrow$ f is surjective.

(ii) Assume that f is injective.

Let $y \in A$. We have $f((f \circ f)(y)) = f(y)$ hence $f \circ f(y) = y$ since f is injective. By taking $x = f(y)$, we obtain $f(x) = y$.

This proves that f is surjective.

Conversely, assume that f is surjective.

Let $x_1, x_2 \in A$ be such that $f(x_1) = f(x_2)$. Since f is surjective, $f \circ f$ is surjective hence $\exists y_1, y_2 \in A$, $f \circ f(y_1) = x_1$ and $f \circ f(y_2) = x_2$.

It follows that $f(y_1) = f \circ f \circ f(y_1) = f(x_1) = f(x_2) = f \circ f \circ f(y_2) = f(y_2)$ and then $x_1 = f(f(y_1)) = f(f(y_2)) = x_2$. This proves that f is injective.

Thus we have proven that

$$f \text{ is injective} \Leftrightarrow f \text{ is surjective}$$