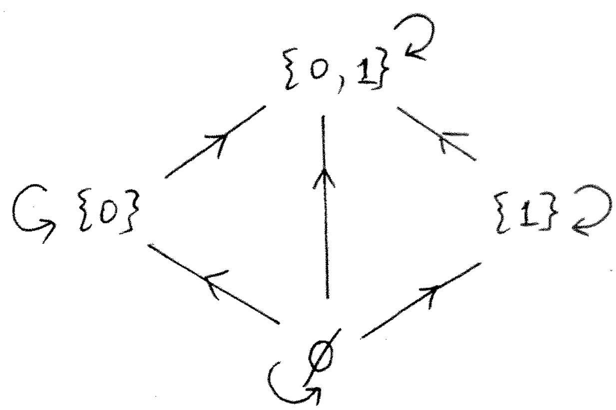
Chapter 3 Orders and lattices

Section 3.1 Orders

Recall from Section 2.1 that an *order* is a reflexive, transitive and antisymmetric relation. We also pointed out that possibly the most fundamental example for order is the subset-relation on the powerset $\mathcal{P}(A)$ of any set A. Let us emphasize that what we call an order is also often called *partial order.*

For $A = \{0, 1\}$, $(\mathcal{P}(\{0, 1\}), \subseteq)$ is the order



Often, an arbitrary order is denoted by the symbol $\leq$, which is read "less than or equal to", or more briefly, "below". It is to be remembered that $\leq$ may not mean the standard order of real numbers.

Recall from section 2.1 (pages 34 and 35; especially Exercise 4) that we have the notion of *irreflexive order*; and that each order R (that we may, for emphasis, call a *reflexive order*) has its *irreflexive version* $R^{\#}$; and that each irreflexive order S has its *reflexive version* S^{*}. When R is the usual order $\leq$ ("less-than-or-equal") of the reals, say, then $R^{\#}$ is $<$, the usual (strict) "less-than" relation; and if S is $<$, then S^{*} is $\leq$. Moreover, when R is $\subseteq$, the subset-relation, then $R^{\#}$ is $\subset$, the proper-subset relation; and when S is $\subset$, then S^{*} is $\subseteq$. In brief: it is a matter of taste whether we talk about a reflexive order, or the corresponding irreflexive one; we can always pass from one to the other. We will see, however, that, depending on the situation, one or the other of the reflexive and irreflexive versions will be preferable to deal with.

Orders appear in mathematics and "real life" very often. We will see several mathematical examples. As for real life, consider a complex manufacturing process; let A be a set of jobs to be done; let $a \prec b$ mean that job a has to be done before b. Then $\prec$ is an irreflexive order, the *precedence* order of the process. Another example is a glossary of technical terms in which the definition of one term may use other terms; now $a \prec b$ means that the definition of b depends directly or indirectly on that of term a. The irreflexivity of this definition is the condition that there are no *circular* definitions.

Note that in both examples, we may have *partial* orders: trichotomy (for the irreflexive versions) is not assumed: it is not necessary that for two different jobs a and b, it be decided that one of them has to precede the other; their "order" may be immaterial, and not recorded in the precedence order $\prec$. A similar remark holds in the case of dependence of terms in a glossary.

Let us make the idea of *circularity* a mathematical one. Let (A, R) be any relation. Recall from Section 2.3 that an R-path, or simply a path, from x to y is a sequence $\langle a_0, a_1, \ldots, a_n\rangle$ of elements such that $a_0=x$, $a_n=y$ and $a_i R a_{i+1}$ for $i<n$; n is the length of the path. We will now insist that the length of a path be positive.

A *circuit* is a path $\langle a_0, a_1, \ldots, a_n\rangle$ with $a_0=a_n$. A relation is *circular* if there is at least one circuit (of positive length) in it. Here is the connection with orders:

The transitive closure of a relation is an irreflexive order if and only if the relation is not circular.

Indeed, recall from Section 2.3 that $xR^{\mathrm{tr}}y$ iff there is an R-path from x to y. To say that R^{tr} is irreflexive is to say that $xR^{\mathrm{tr}}x$ never holds; this means that there cannot be a path from any x to itself, which is the same thing as to say that there are no circuits in R, that is,

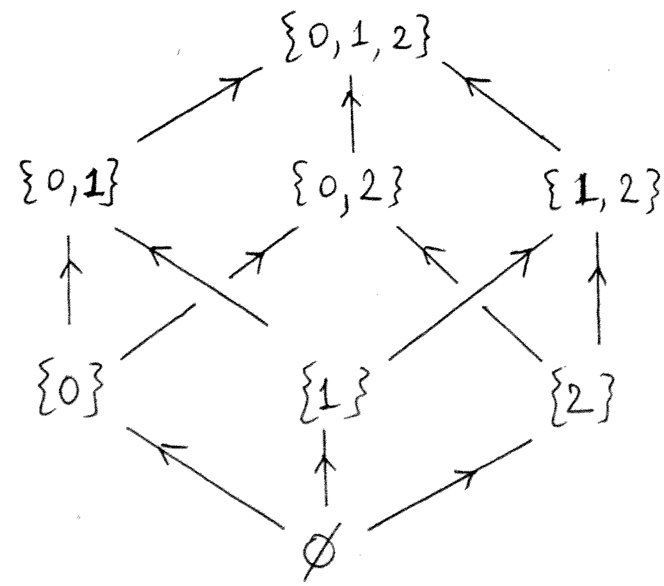
R is not circular. R^{tr} is, of course, always transitive; so it is an irreflexive order just in case R is not circular.

To return to the example of the glossary, let R be the relation of "direct dependence": xRy iff the definition of y directly refers to the term x. Then the relation $\prec$ of "direct or indirect dependence" is nothing but the transitive closure of R. This will be an irreflexive order iff R is not circular, that is, if there is no sequence of definitions exhibiting circularity.

The relation U defined in Section 2.3 (p. 53) is circular; $\langle 0, 1, 2, 0 \rangle$ is a circuit in it. The transitive closure is the relation on top of p. 55 in Chapter 2; the failure of irreflexivity is in the encircled set. If we imagine that U depicts the direct dependence in a proposed glossary, from the transitive closure we can see that one or more of the definitions for terms $0, 1, 2$ have to be changed to ensure non-circularity.

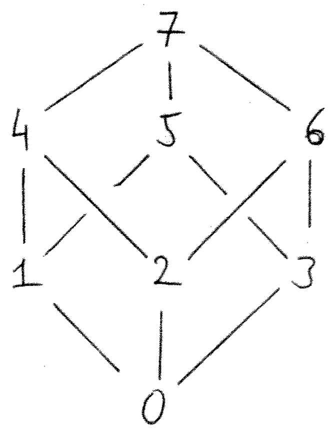
The concept of transitive closure is used to give an abbreviated representation of an (irreflexive) order. The idea is simply that if we know of a relation that it is transitive, and we know a couple of particular pairs in the relation, then transitivity will give us others, which we therefore do not have to include in the data bank. To put it more succinctly, an (irreflexive) order may be represented by any subrelation(!; see Section 2.1)) whose transitive closure is the order itself.

Let us take an example. Consider the irreflexive order $(A, R) = (\mathcal{P}(\{0, 1, 2\}, \subset)$; the relation $\subset$ is "proper subset": $X \subset Y$ iff $X \subseteq Y$ and $X \neq Y$. The relation



is not the same as our (A, R) , but its transitive closure is. For instance, there is no arrow from $\{1\}$ to $\{0, 1, 2\}$ in the picture, but one can get from $\{1\}$ to $\{0, 1, 2\}$ along two arrows.

The usual graphic representation of any order follows the pattern of the example. As an additional convention, arrows are often replaced by non-directed edges; this can be done if there is a global direction such as upward in which all arrows are supposed to point. Thus, the picture



(3)

represents an order isomorphic to the last one. In this, x is less than y just in case one can reach y from x going always upward along edges. Since this is not possible for $x=3$ and $y=4$, 3 is not less than 4 , although 4 is higher up than 3 .

The "minimal" graphic representation of an order is called its *Hasse diagram*. Actually, the Hasse diagram of any order is uniquely determined, and can be described mathematically in an elegant way. We proceed to explain this.

Let $(A, \leq)$ be an order; of course, $<$ denotes the irreflexive version. Let $x, y \in A$. We say that y *covers* x in the given order if $x<y$, but there is no z such that $x<z<y$; that is, if y is strictly above x , but there is no element strictly between the two.

As we said, any subrelation R of $<$ whose transitive closure equals the given order,

$$R^{r/tr} = < \qquad (3')$$

can be used to "represent" $<$. Of course, $R = <$ is a possible choice; however, of course, we want to have R as *small* as possible: in the graphic representation, the number of arcs, which is the cardinality of R, should be made as small as possible. Let us call a subrelation R of $<$ for which (3') holds *representative*. Now, the fact is this.

Assume that $(A,\leq)$ *is a finite order. Then*

there is a unique minimal element H *among the representative subrelation of* $<$; H *is the "cover" relation derived from* $<$;

$$xHy \iff y \text{ covers } x \text{ in } (A,<) .$$

By the minimality of H we mean this: H is representative (read 3' for H as R); and if R is representative, then $H \subseteq R$: R has to contain all the pairs that H contains, and possibly more.

To prove the assertion, first, let us see that H is representative, $H^{tr} = <$ indeed. Since H is a subrelation of $<$, and $<$ is transitive, we have that $H^{tr} \subseteq <$. To show the opposite inclusion, let a, b be arbitrary elements of A such that $a<b$, to show that $aH^{tr}b$.

Let

$$a = c_1 < c_2 < \ldots < c_{n-1} = c_n = b \tag{3''}$$

be a chain "connecting" a and b of *maximal-possible length*; the number n the largest possible. Chains connecting a to b certainly exist, since the pair $a<b$ is such a chain. Since $<$ is irreflexive and transitive, $c_i \neq c_j$ for $i \neq j$; thus, $n \leq |A|$, the number of elements in A. Therefore, there are altogether finitely many such chains, thus, we can select one that has the largest possible length [this argument is actually an application of the proposition, stated and proved later, according to which any non-empty order has at least one maximal element]. I claim that $c_i H c_{i+1}$ (that is, c_{i+1} covers c_i) for all $i=1, \ldots, n-1$. Indeed, if there were d such that $c_i<d<c_{i+1}$, then we could insert d into the given chain (3''), and get

finite

line-6

$$a = c_1 < c_2 < \ldots c_i < d < c_{i+1} < \ldots < c_{n-1} = c_n = b$$

a chain from a to b which is one longer than (3'') -- a contradiction to the maximal choice of (3"). But now we have that (3") is an *H-path* from a to b, showing that $aH^{tr}b$.

We have proved that H is a representative relation; it remains to show that if R is any representative relation, then $H \subseteq R$. Suppose that R is representative, $R^{tr} = <$; assume that xHy, that is, y covers x in $<$; we want to show that xRy. Since, in particular, $x<y$, we have $xR^{tr}y$; therefore, there exists an R-path

$$x = u_1 , u_2 , \ldots , u_{n-1} , u_n = y$$

from x to y: $u_i R u_{i+1}$ for all $i=1, \ldots, n-1$; and $n \geq 2$. Since $R \subseteq <$, it follows that

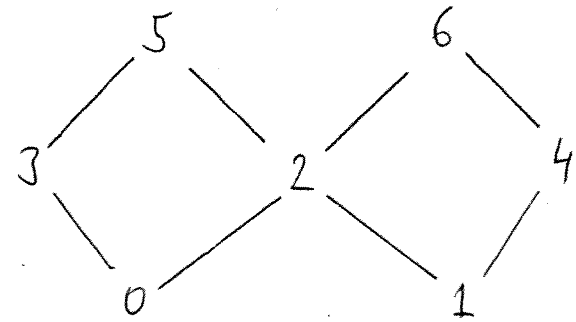
$$x = u_1 < u_2 < \ldots < u_{n-1} < u_n = y .$$

If we had $n>2$, then we would have some u (namely, $u=u_2$) such that $x<u<y$, which would contradict xHy. Therefore, we must have $n=2$; which means $x=u_1 R u_2=y$; we have shown xRy as promised.

Let us note in passing that $R^{tr}=<$ implies $R^{r/tr}=\leq$ (why?).

The Hasse diagram of a finite order is the network displaying the cover relation for the order. It is the most economical way of graphically representing the order.

Let $(A, \leq)$ be a reflexive order; let $<$ denote the corresponding irreflexive order. A *minimal* element of $(A, \leq)$ is any $a \varepsilon A$ such that for all $x \varepsilon A$, $x \leq a$ only if $x=a$. This is the same thing as to say that $x<a$ never happens for x in A. A *maximal* element is any a for which $a<x$ never happens for an element x of A. The order represented by he Hasse diagram



has two minimal elements, 0 and 1, and two maximal elements, 5 and 6. $(\mathcal{P}(A), \subseteq)$ has a unique minimal element, the empty set $\emptyset$, and a unique maximal element, A itself. The order $(\mathbb{R}, \leq)$ does not have either a minimal, or a maximal element. $(\mathbb{N}, \leq)$ has a unique minimal element, 0, but no maximal element.

An order is *finite* if its underlying set is finite; the *cardinality* of an order is the cardinality of its underlying set (and *not* the cardinality of the relation as a set of ordered pairs)

Any finite non-empty order has at least one minimal and at least one maximal element.

The proof is by induction of the cardinality of the order [those who have not seen induction should skip this proof; induction will be discussed towards the end of the course]. If the cardinality is 1 (the least possible for a non-empty set), the assertion is clearly true. Suppose $(A, <)$ is an (irreflexive) order, and $|A|=n+1$, $n\geq 1$. Pick any $a\in A$. The restriction $(A-\{a\}, <\restriction(A-\{a\}))$ is again an order (see Section 2.1); it is of cardinality n. By the induction hypothesis, it has a minimal element, say b. **Either** b is a minimal element for the whole $(A, <)$; **or else**, b is not minimal in $(A, <)$; but this second case can take place only if $a<b$, since b is minimal in $A-\{a\}$. In the second case, a *is minimal in* $(A, <)$: if, on the contrary, we had $c<a$, then, first of all, $c\neq a$, and so $c\in A-\{a\}$, and secondly, $c<a<b$ implies that $c<b$; $c\in A-\{a\}$ and $c<b$ together say that b is not minimal in $A-\{a\}$; contradiction.

The proof for "maximal" is similar.

In Section 2.1, we defined a (*reflexive*) *total order* R as an order in which the dichotomy law (either xRy or yRx) holds. An *irreflexive total* order is a transitive, irreflexive (equivalently, strictly antisymmetric) relation satisfying trichotomy (either xRy , or $x=y$, or yRx). We noted that the standard example for total order is the "less than or equal" relation $\leq$ for numbers, and that for irreflexive total order is the "less than" relation $<$ for numbers.

Total orders, at least in the finite case, are structurally very simple. We have that

any two total orders with the same finite cardinality of their underlying sets are isomorphic.

The proof is by induction on the size of the total orders. If the size is $\emptyset$ (the empty order), the assertion is obvious. Assume $(A, <)$, $(B, <')$ are total orders, $|A| = |B| = n+1$. Let, by the previously proved assertion, a be a minimal element of $(A, <)$, b a minimal element of $(B, <')$. Consider the restrictions $(A-\{a\}, <|(A-\{a\}))$ and $(B-\{b\}, <'|(B-\{b\}))$. These are total orders of cardinality n ; hence, by the induction hypothesis, they are isomorphic. Let

$$f : (A-\{a\}, <\restriction(A-\{a\})) \xrightarrow{\cong} (B-\{b\}, <'\restriction(B-\{b\}))$$

be an isomorphism. Then the function

$$g: A \longrightarrow B$$

$$x \longmapsto \begin{cases} f(x) & \text{if } x \neq a \\ b & \text{if } x = a \end{cases}$$

is an isomorphism $g: (A, <) \xrightarrow{\cong} (B, <')$ (why?).

The last-proved assertion says that for any finite cardinality, there is exactly one total order up to isomorphism; a concrete representation of the total order of size n is $([n), <\restriction[n))$; here, the set $[n)$ is $\{0, 1, ..., n-1\}$, the set of natural numbers less than n ; the irreflexive order $<\restriction[n)$ is the usual order relation $<$ among integers restricted to the set $[n)$.

In contrast, there are many non-isomorphic partial orders of the same cardinality (if that cardinality is greater than 1). If A is any set, then there is a minimal order on A in which $x \leq y$ only if $x = y$; for the irreflexive formulation, this means that $<$ is the empty set of pairs, $x < y$ never happens. Such a trivial order is called *discrete*; discrete orders are at an opposite extreme to total orders.

Let $(A, \leq)$ be any order, B a subset of A ; consider the restriction relation $(B, \leq \restriction B)$, written more simply as $(B, \leq)$. As we mentioned in Section 2.1, and as it is obvious, $(B, \leq)$ is also an order. If $(B, \leq)$ is a total order, we say B is a *chain* in $(A, \leq)$; if $(B, \leq)$ is a discrete order, we say B is an *antichain* in $(A, \leq)$. The set $\{\emptyset, \{1\}, \{1, 2\}, \{0, 1, 2\}\}$ is a chain in $\mathcal{P}(\{0, 1, 2\})$; $\{\{0\}, \{1\}, \{2\}\}$ is an antichain in the same (see the picture of $\mathcal{P}(\{0, 1, 2\})$ above).

Insert from p. 73 →

Chains and antichains in an order are in some sense "orthogonal" to each other. One precise way of putting this is that

the intersection of a chain and an antichain can have at most one element.

In fact, this is obvious when one thinks about it (right?).

For a finite order, the *length* of the order is the largest possible cardinality of any chain in the order; the *width* of the order is the cardinality of the largest antichain in the order. The length of $\mathcal{P}(C)$ for $|C| = 3$ is 4 , the width of it is 3 (see the picture of $\mathcal{P}(\{0, 1, 2\})$ above); the chain and the antichain pointed out above are in fact of the maximal possible sizes in the example.

The graphical display of orders suggests the notion of *level* in an order. The minimal elements are on the lowest level; the ones that are "immediately above" minimal ones are on the next level, etc. To introduce the notion of level precisely, we first define a few other simple concepts.

With any element x of an order $(A, \leq)$, let $x\downarrow$ denote the *downsegment of* x , that is, the set of elements y for which $y \leq x$. (Of course, the *upsegment* $x\uparrow$ *of* x is the set $\{y \in A \mid x \leq y\}$.) The *height* of x is, by definition, the length of the downsegment of x . In other words, the height of an element x is the same as the length of the longest chain ending in x. For instance, the height of the element 5 in the order under (3) is 3, because the two longest chains in $5\downarrow$ are $\{0, 3, 5\}$ and $\{0, 1, 5\}$, both of length 3.

The elements of height 1 are exactly the minimal elements of the order. The one with height 2 are those x that are not minimal, but which are such that every $y<x$ is minimal. In general, if the height of x is k, then for every $1 \leq i < k$, there is at least one $y<x$ such that the height of y is i: just take a maximal-length chain C ending in x; its length is k; if

$$C=\{y_1<y_2<...<y_{k-1}<y_k=x\}$$

then the height of y_i is i: it is obviously at least i, but it cannot be larger than i, since that would make a chain ending in x that is longer than k.

The n^{th} *level in* $(A, \leq)$ is the subset of A consisting of the elements of height equal to n. The levels in $\mathcal{P}(\{0, 1, 2, \})$ are $\{\emptyset\}$ (first level), $\{\{0\}, \{1\}, \{2\}\}$ (second level), $\{\{0, 1\}, \{0, 2\}, \{1, 2\}\}$ (third level), $\{\{0, 1, 2\}\}$ (fourth level).

Clearly, the number of non-empty levels of an order is the same as the length of the order; this is 4 in the example $\mathcal{P}([3])$ or in the order whose Hasse diagram is (3).

Each level is an antichain. Indeed, if $x < y$, then the height of y is necessarily larger than that of x, since for any chain ending in x we can make one ending in y which is 1 longer than the chain ending in x. Therefore, for elements x and y on the same level, $x < y$ is impossible, which says that a level is an antichain. We can now easily see that

every finite order can be extended to a total order on the same underlying set.

Indeed, let the order be $(A, <)$. Let us define the relation $\prec$ on A as follows. For any level L of $(A, \leq)$, choose an arbitrary total order $\prec_L$ on L. Given any x and y in A, let us define

[illegible] $\prec$ [illegible] alternatively. No problem.

$$x \prec y \underset{\text{def}}{\iff} \quad \begin{array}{l} \textbf{either } x \text{ and } y \text{ are on the same level } L \text{ and } x \prec_L y, \\ \textbf{or } x \text{ is on a lower level than } y. \end{array}$$

What this says is that in the relation $\prec$ everything on a lower level precedes everything on a higher level; and for two things on the same level, what precedes what is determined by the individual total order chosen on that level. It is practically obvious, and certainly it is not hard to see, that $\prec$ so defined is a total order. Also, if $x<y$, then x is on a lower level than y (as we said above), hence, $x \prec y$. This shows that $< \subseteq \prec$, $\prec$ extends $<$.

In the case of the order under (3) (which is isomorphic to $\mathcal{P}(\{3\})$), the total order $\prec$ is the natural order of the integers < 8, provided we choose the individual orders $\prec_L$ of the second and third levels appropriately.

When the order under investigation is a precedence order for a set of jobs, then a total order extending the given order solves a *scheduling problem*, namely how to schedule the jobs one after the other so that every time we do a job all others that must have been done before are indeed done.

Returning to the levels of an order, let us repeat that each of them is an antichain. Also, the underlying set of the order is the disjoint union of the levels. We also noted that the number of levels is the same as the length (the length of the longest chain) of the order. Thus

we have succeeded writing the underlying set of an order as the union of as many antichains as the length of the order.

We could not have done the same with a smaller number of antichains. If the underlying set is the union of some antichains, and C is a chain, then no two elements of C may be in the same antichain (the intersection of a chain and an antichain may have at most one element), hence, there must be at least as many antichains to cover C as there are elements of C. Now if we take C to be the longest possible chain, that is, C is the length of the whole order itself, we get that the family of antichains covering the order must have at least as many members as the length of the order.

Now, let us consider the "dual" question of covering (the underlying set of) an order with chains rather than antichains. The same argument as the one we just gave shows that

it is not possible to cover an order with fewer chains than the width of the order.

For instance, the order under (3) has width $=3$, as we said above. It is not possible to cover this order by two chains, since the antichain $\{1, 2, 3\}$ can not be part of just two chains. However, it can be covered by three chains:

$$A = \{0, 1, 4, 7\} \cup \{2, 6\} \cup \{3, 5\} .$$

Indeed, this is a general fact.

The underlying set of any finite order may be written as the union of exactly as many chains as the width of the order.

This is **R. P. Dilworth's theorem.** The proof is not as simple as it was for the case of covering with antichains. It proceeds by induction on the cardinality of the order. If the order has cardinality 0, that is, we are talking about the order on the empty set, the width is 0, and the order is covered as the union of 0 many chains, so the assertion is true in this case. Now, let the size of the order $(A, \leq)$ be $|A| = n+1$, and let the width of $(A, \leq)$ be w. The induction hypothesis is that for any order of size at most n, the order can be covered by as many chains as the width of the order.

We start by taking a maximal-size chain in $(A, \leq)$, say C. Consider $B = A - C$, and the order induced on B, $\leq\restriction B$. Certainly, $|B| < |A| = n+1$, that is $|B| \leq n$, so the induction hypothesis applies to $(B, \leq\restriction B)$. Now, there are two cases.

Case 1. The width of $(B, \leq\restriction|B)$ is less than w. Then, B may be covered by less than w chains (in $(B, \leq\restriction|B)$, which are the same as chains in $(A, \leq)$ entirely within B). Together with the chain C, this means a covering of A with at most w chains, and in this case we are done.

Case 2. The width of $(B, \leq\restriction|B)$ is equal to w. Let X be an antichain in B of size w. Consider the following two sets

$$U = \{u \in A \mid u \geq x \text{ for some } x \in X\},$$

$$L = \{\ell \in A \mid \ell \leq x \text{ for some } \ell \in X\}.$$

U is the set of elements of A that are above some element of X; L is the set of those below some element of X.

First note that

$$U \cup L = A.$$

The reason is that any $a \in A$ is comparable with some $x \in X$, since otherwise one could add a to the antichain X, making it an antichain of size $w+1$, contradicting the fact that w is the maximal size of any antichain in $(A, \leq)$. Since a is comparable with some element of X, it must be either in U or in L.

Note that X itself is a part of both U and L; in fact, $U \cap L = X$. The reason is that if $y \in U \cap L$, then there are $x_1, x_2 \in X$ such that $x_1 \leq y \leq x_2$; thus, $x_1 \leq x_2$, and since X is an antichain, $x_1 = x_2$. But then $x_1 \leq y \leq x_2$ means that $x_1 = y = x_2$, and so $y \in X$.

Next note that

neither U nor L is the whole set A.

Indeed, the *unique* minimal element c of the chain C is not in U, since if it were, there would be $x \in X$ with $c \geq x$, but C and X are disjoint, X being a subset of $B = A - C$, so $c > x$, and this would mean that x can be attached to the chain C as its least element, making it longer, in contradiction to the choice of C as the longest chain. Similarly, the maximal element of the chain C is not in L. The upshot is that both U and L are of smaller sizes than A, and thus the induction hypothesis can be applied to them.

Now, we forget about C entirely, and concentrate on the antichain X, as well its *upper and* *lower shadows* U and L. Note that the width of both U and L is w, since the maximal-size antichain X is a part of both. Using the induction hypothesis, we write U as a union of w chains U_i (for $i=0, \ldots, w-1$; in short, $i < w$), and L as the union of w chains L_j ($j < w$).

Now, looking at U, and in particular an element $x \in X$, x must belong to *at least one* of the U_i's, and of course, distinct x's must belong to distinct U_i's. Since there are as many U_i's as x's, namely w, each U_i contains exactly one $x \in X$; call that x x_i.

Now, let us fix $i < w$. Let the unique minimal element of the chain U_i' be u; we have $u \le x$ since $x_i \in U_i$; since $u \in U$, there is $y \in X$ with $y \le u$; it follows that $y \le x_i$; but X is an antichain, and both x_i and y are in X; therefore $x_i = y$, and so $x_i \le u \le x_i$, $x_i = u$. We conclude that

the minimal element x_i of each U_i is in X, and the x_i's are all the distinct elements of X;

What we said about the U_i's, after switching "up" and "down", we can say about the L_j's as well. We get that

the maximal element x'_j of each L_j is in X, and the x'_j's are all the distinct elements of X.

We can now finish the proof by producing w chains covering A. Take a chain U_i; find the chain L_j for which $x'_j = x_i$; then $L_j \cup U_i$ is a chain, since everything in L_j is $\le x$, and everything in U_i is $\ge x$, where $x = x'_j = x_i$. Do this for each $i<w$, and get i chains; these will cover A since the U_i's cover U and the L_j cover L, and $U \cup L = A$.

---

nsert to p. 68:

(We mean the relation $\subseteq$ (subset relation) when we refer to $\mathcal{P}(B)$ as an order, unless otherwise indicated.)

In what follows, we put ourselves into a fixed order $(A; \le)$:

Section 3.2 Lattices

Let $(A, \leq)$ be a (reflexive) order, considered fixed throughout this section. Thus, the symbol $\leq$ now means an arbitrary order on the arbitrary set A, rather than the usual less-than-or-equal relation on numbers.

As usual, $x \geq y$ means the same as $y \leq x$.

When, as sometimes happens, we also want to refer to the ordinary meaning of $\leq$ in the same context with an "arbitrary" order, we have to use a letter like R for the "arbitrary" order. Thus, you should be able to see what follows also with R replacing $\leq$.

Let $X \subseteq A$, a subset of A, and $y \in A$, an element of A.

Definition y is an *upper bound* of X if for all $x \in X$, we have $x \leq y$. In symbols

$$y \text{ is an upper bound of } X \iff \forall x.\, x \in X \longrightarrow x \leq y .$$

Note the following obvious facts:

If y is an upper bound of X, and $y \leq z$, then z is also an upper bound of X.

x is an upper bound of $\{x\}$.

Any $y \in A$ is an upper bound of $\emptyset$. (Note: the empty set $\emptyset$ is a subset of A; thus, we can take X to be $\emptyset$.)

Exersise 1. Prove the assertions just made.

The set of all upper bounds of X is denoted by $X\uparrow$. Thus, to say that y is an upper bound of X is the same as to write $y \in X\uparrow$.

The concept of *lower bound* is similar; it is obtained by replacing $\leq$ with $\geq$:

$$y \text{ is an lower bound of } X \iff \forall x.\, x\in X \longrightarrow x\geq y\ .$$

The set of all lower bounds of X is denoted by $X\!\downarrow$.

It is important to keep in mind that the expressions $X\!\uparrow$, $X\!\downarrow$ make an implicit reference to the order in which they are evaluated.

Example 1. Let $A=\mathbb{R}$, and $\leq$ the usual less-than-or-equal relation on $\mathbb{R}$. Let $X=\{\frac{n}{n+1}: n\in\mathbb{N}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$. Then $y=1$ is an upper bound of X since $\frac{n}{n+1}\leq 1$ for all $n\in\mathbb{N}$ (in fact, $\frac{n}{n+1}<1$). In fact, $y\in\mathbb{R}$ is an upper bound of X if and only if $y\geq 1$ (why?) . In other words,

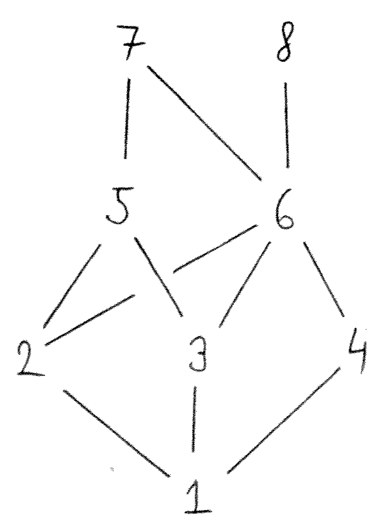
$$\{\frac{n}{n+1}: n\in\mathbb{N}\}\!\uparrow\ = \{y\in\mathbb{R}:\ y\geq 1\}\ .$$

Also,

$$\{\frac{n}{n+1}: n\in\mathbb{N}\}\!\downarrow\ = \{y\in\mathbb{R}:\ y\leq 0\}$$

(why?).

Example 2. Let $(A, \leq)$ be represented by the Hasse diagram



Now, we have

$$\{3, 4\}\uparrow = \{6, 7, 8\}, \quad \{2, 3\}\uparrow = \{5, 6, 7, 8\}, \quad \emptyset\uparrow = A, \quad \{5, 8\}\uparrow=\emptyset,$$
$$\{5, 6\}\downarrow = \{1, 2, 3\}, \quad \{7, 8\}\downarrow = \{1, 2, 3, 4, 6\}, \quad \emptyset\downarrow = A.$$

Definition For any subset Y of A, we say that $b\in Y$ is *the least element of* Y if for all $y\in Y$, we have $b\leq y$. Similarly, $b\in Y$ is the *greatest element* of Y if for all $y\in Y$, we have $b\geq y$.

A subset Y of A may or may not have a least element; but if it has one, the least element is uniquely determined (why is that?). Similar statements can be made for the "greatest element".

Example 1 (continued) The set $\{y\in\mathbb{R}: y\geq 1\}$ has a least element; it is 1. The set $\{y\in\mathbb{R}: y>1\}$ has no least element (why?). The subset $\mathbb{R}$ of $\mathbb{R}$ has no least element, and no greatest element (why?). The set $\{y\in\mathbb{R}: y\leq 0\}$ has a greatest element, 0; but $\{y\in\mathbb{R}: y<0\}$ has no greatest element.

Example 2 (continued) The set $\{6, 7, 8\}$ has a least element, 6: $6\leq 6$, $6\leq 7$, and $6\leq 8$. The set $\{5, 6, 7, 8\}$ has no least element. $\emptyset$ has no least, or greatest, element, since it has no element at all.

The concept of "least element" has to be compared to that of "minimal element" carefully.

By a *minimal element* of a subset X of A we mean any element $a\in X$ such that for all $x\in X$, $x\nless a$: it is *not* the case that $x<a$. The least element of X is certainly a minimal element of X, but not the other way around. For instance, in Example 2, $\{5, 6, 7, 8\}$ has two minimal elements, 5 and 6, but no least element. On the other hand, if a subset X of A does have a least element, that element is necessarily the *unique* minimal element of X.

The concept of *unique minimal* element agrees with the least element in case the set A is finite; but not in general. Consider the following example. Let $A=\mathbb{Z}\times\mathbb{Z}$, and let R be the relation on A defined by

$$(a, b)\,R\,(c, d) \iff a \le c \text{ and } b \le d$$

(here, we used $\le$ in its usual sense as the less-than-or-equal relation on numbers). Let $X = \{(0, b) : b \in \mathbb{Z}\} \cup \{(1, c) : c \in \mathbb{N}\}$. The element $(1, 0)$ is a minimal element of X in the present order (A, R) . In fact, $(1, 0)$ is the *only* minimal element in X : $(1, 0)$ is the *unique minimal* element. However, X has *no least* element in the given order (A, R) .

Definition Let X be a subset of A .

The *join of* X , written as $\bigvee X$, is the least element of $X\uparrow$, if it exists.

The *meet of* X , written as $\bigwedge X$, is the greatest element of $X\downarrow$, if it exists.

The join of X is also called the *supremum*, or more briefly the *sup* of X ; for "meet" we also say "*infimum*" or "*inf*".

Example 1 (continued) We have $\bigvee \{\frac{n}{n+1} : n \in \mathbb{N}\} = 1$. This is because $\{\frac{n}{n+1} : n \in \mathbb{N}\}\uparrow = \{y \in \mathbb{R} : y \ge 1\}$, and the least element of $\{y \in \mathbb{R} : y \ge 1\}$ is 1 . On the other hand, $\bigvee \mathbb{R}$ does not exist. This is because $\mathbb{R}\uparrow = \emptyset$, and $\emptyset$ has no least element (no element at all).

We have $\bigwedge \{\frac{n}{n+1} : n \in \mathbb{N}\} = 0$, since $\{\frac{n}{n+1} : n \in \mathbb{N}\}\downarrow = \{y \in \mathbb{R} : y \le 0\}$, and the ~~least~~ greatest element of $\{y \in \mathbb{R} : y \le 0\}$ is 0 .

Example 2 (continued) We have $\bigvee \{3, 4\} = 6$, since $\{3, 4\}\uparrow = \{6, 7, 8\}$, and the least element of $\{6, 7, 8\}$ is 6 . On the other hand, $\bigvee \{2, 3\}$ does not exist, since $\{2, 3\}\uparrow = \{5, 6, 7, 8\}$, and $\{5, 6, 7, 8\}$ has no least element. $\cancel{\bigvee}\ \bigwedge\ \emptyset$ does not exist, since $\cancel{\emptyset\uparrow}\ \emptyset\downarrow = A$, the total set, and A has no greatest element.

We have $\bigwedge \{7, 8\} = 6$, since $\{7, 8\}\downarrow = \{1, 2, 3, 4, 6\}$, and the greatest element of $\{1, 2, 3, 4, 6\}$ is 6 . On the other hand, $\bigwedge \{5, 6\}$ does not exist, since $\{5, 6\}\downarrow = \{1, 2, 3\}$, and $\{1, 2, 3\}$ has no greatest element. $\cancel{\bigwedge}\ \bigvee\ \emptyset = 1$, since $\cancel{\emptyset\downarrow}\ \emptyset\uparrow = A$, and A has a least element, 1 .

Two reformulations of the concepts of "join" and "meet":

$$\bigvee X = a \quad \text{if and only if} \quad X\uparrow = \{a\}\uparrow .$$

(*) $\bigvee X = a$ if and only if the following holds:

$$\text{for all } u\in A , \quad u\geq a \iff \forall x.\, x\in X \longrightarrow u\geq x .$$

Exercise 2. Prove the two assertions above.

We always have the following facts:

$$\bigvee \{a\} = a , \qquad \bigwedge \{a\} = a .$$

$\bigvee \emptyset$, if it exists, is the *bottom element* of the order; it is denoted by $\bot$. Note that $\emptyset\uparrow$ is the whole of A (every element of A is an upper bound of $\emptyset$); thus, $\bot = \bigvee \emptyset$, being the least element of $\emptyset\uparrow$, it is the least element of A (if it exists). $\bot$ is characterized by the fact that for all $a\in A$, we have $\bot\leq a$.

Similarly, $\bigwedge \emptyset$, if it exists, is the *top element*, or *greatest* element of the order; it is denoted by $\top$; we have $a\leq\top$ for all $a\in A$.

$$\bigwedge A = \bigvee \emptyset = \bot , \qquad \bigvee A = \bigwedge \emptyset = \top .$$

If the set X has a least element a , then $\bigwedge X=a$;
if the set X has a greatest element a , then $\bigvee X=a$.

Of course, the converses of the last statements are, in general, false. For instance, in Example 2, we have $\bigwedge \{7, 8\}=6$, but 6 is not the least element of $\{7, 8\}$; 6 is not an element of $\{7, 8\}$ at all.

Exercise 3. Prove that

$$\bigvee X = \bigwedge (X\uparrow)$$
$$\bigwedge X = \bigvee (X\downarrow) \ ;$$

more precisely, in each case, the (value of the) left-hand-side expression exists if and only if the right-hand-side does, and when they exist, they are equal. (Hint: first prove that, for any subset $X \subseteq A$, if $a = \bigwedge (X\uparrow)$ exists, then a must be the least element of $X\uparrow$; and a similar statement for $\bigvee (X\downarrow)$.)

When the set X is given in the form $X = \{x_i : i \in I\}$, where x_i is some expression of the variable i ranging over some set I , then we may write $\bigvee_{i\in I} x_i$ for $\bigvee X$, and $\bigwedge_{i\in I} x_i$ for $\bigwedge X$. For instance, if $\mathcal{X}$ is a set of some subsets of A , in other words, every X in $\mathcal{X}$ is a subset of A , then the expression $\bigvee_{X\in\mathcal{X}} (\bigvee X)$ means the same as $\bigvee \{\bigvee X : X \in \mathcal{X}\}$. In turn, this means taking the join $\bigvee X$ of each set X in the collection $\mathcal{X}$, and then taking the join of the set of all the joins so obtained.

Let us recall that $\bigcup \mathcal{X}$ stands for the union of all the sets that are elements of $\mathcal{X}$. We can say the same by the formula

$$x \in \bigcup \mathcal{X} \iff \exists X \in \mathcal{X} . x \in X \ .$$

We may also write

$$x \in \bigcup_{i\in I} X_i \iff \exists i \in I . x \in X_i \ .$$

Exercise 4. Prove that

$$\bigvee (\bigcup \mathcal{X}) = \bigvee_{X\in\mathcal{X}} (\bigvee X) \ ,$$

meaning that if one side exists, so does the other, and they are equal.

Examples for the last equality:

$$\bigvee \{x_1, x_2, x_3\} = \bigvee \{ \bigvee \{x_1, x_2\}, x_3\} ,$$

$$\bigvee \{x_1, x_2, x_3, x_4, x_5\} = \bigvee \{ \bigvee \{x_1, x_2\}, \bigvee \{x_3, x_4, x_5\}\}$$

Special notation:

$$x \vee y \overset{=}{\text{def}} \bigvee \{x, y\} .$$
$$x \wedge y \overset{=}{\text{def}} \bigwedge \{x, y\} .$$

Note the obvious facts:

if $x \leq y$, then $x \vee y = y$ and $x \wedge y = x$.

Item (*) above becomes, for $X = \{x, y\}$, the following characterizations of $x \vee y$ and $x \wedge y$:

$$\frac{u \geq x \vee y}{u \geq x \quad \text{and} \quad u \geq y} \qquad \frac{u \leq x \wedge y}{u \leq x \quad \text{and} \quad u \leq y}$$

These abbreviated statements use the horizontal lines to mean "if and only if". It is understood that the statements hold true for all u in A .

Definition A *lattice* is an order $(A, \leq)$ in which $\bigvee X$, $\bigwedge X$ exist for all *finite* subsets X of A .

A *complete lattice* is an order $(A, \leq)$ in which $\bigvee X$, $\bigwedge X$ exist for all subsets X of A .

Facts:

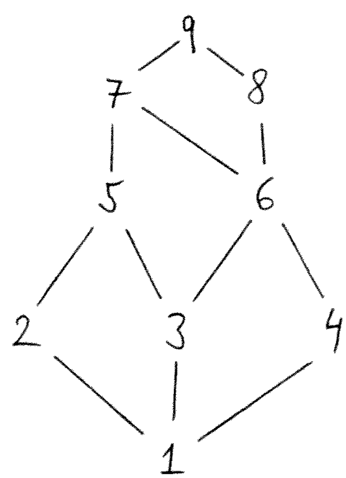
An order $(A, \leq)$ is a lattice if and only if $\top\ (= \bigwedge \emptyset)$, $\bot\ (= \bigvee \emptyset)$, $x \wedge y$, $x \vee y$ all exist in it, the latter two for all $x, y \in A$.

If A is a ~~non-empty~~ finite set, then the order $(A, \leq)$ is a lattice iff there is a top element $\top$ in it and $x \wedge y$ exists in it, for all $x, y \in A$.

In the previous statement, $x \wedge y$ can be replaced with $x \vee y$.

Exercise 5. Prove the facts (**Hints**: For the first, use the assertion of Exercise 4, to obtain the join or meet of a set of finitely many elements to the successive evaluation of joins of two elements. For the second and third, first of all notice that for a finite set A , (A, R) is a lattice if and only if it is a complete lattice, since all subsets of A are finite; then, use Exercise 3.)

Example 3. We modify the Hasse diagram (A, H) given in Example 2 by removing the element $(2, 6)$ from H , adding the element 9 to A , and adding the pairs $(7, 9)$, $(8, 9)$ to H :



This represents a lattice (that is, its transitive closure is the irreflexive version of a lattice) which we call $(A, \leq)$; now, $A=\{i \in \mathbb{N} : 1 \leq i \leq 9\}$.

We can display this fact as follows. First of all, the Hasse diagram shows that we have a unique maximal element and a unique minimal element:

$\top = 9 \qquad \bot = 1$.

Secondly, for any x and y in A that are *comparable* in the order, that is, either $x \le y$ or $y \le x$, we know that $x \wedge y$, $x \vee y$ both exist (in this case, these values are all equal to either x or y). Thus, in checking joins $x \vee y$ and meets $x \wedge y$, we may confine our attention to *incomparable* x and y. But also, it is not necessary to consider both $x \vee y$ and $y \vee x$, since one of these exists if the other does, and they are equal: they are equal to $\bigvee\{x, y\} = \bigvee\{y, x\}$. Therefore, in our example, we make a list of all pairs (x, y) of incomparable elements x, y for which, also, $x<y$ in the usual ordering $<$ of the integers; the latter condition is to make sure that we do not list both (x, y) and (y, x).

Here is the list of incomparable pairs:

(2, 3), (2, 4), (2, 6), (2, 8),
(3, 4),
(4, 5),
(5, 6), (5, 8),
(7, 8) .

We have

8 $\wedge$

$(2, 3)\uparrow = \{5, 7, 9\}$,	$2 \vee 3 = 5$;	$(2, 3)\downarrow = \{1\}$,	$2 \wedge 3 = 1$;
$(2, 4)\uparrow = \{7, 9\}$,	$2 \vee 4 = 7$;	$(2, 4)\downarrow = \{1\}$,	$2 \vee 4 = 1$;
$(2, 6)\uparrow = \{7, 9\}$,	$2 \vee 6 = 7$;	$(2, 6)\downarrow = \{1\}$,	$2 \vee 6 = 1$;
$(2, 8)\uparrow = \{9\}$,	$2 \vee 8 = 9$;	$(2, 8)\downarrow = \{1\}$,	$2 \vee 6 = 1$;
$(3, 4)\uparrow = \{6, 7, 8, 9\}$,	$3 \vee 4 = 6$;	$(3, 4)\downarrow = \{1\}$,	$3 \vee 4 = 1$;
$(4, 5)\uparrow = \{7, 9\}$,	$2 \vee 6 = 7$;	$(4, 5)\downarrow = \{1\}$,	$4 \vee 5 = 1$;
$(5, 6)\uparrow = \{7, 9\}$,	$5 \vee 6 = 7$;	$(5, 6)\downarrow = \{1, 3\}$,	$5 \vee 6 = 3$;
$(5, 8)\uparrow = \{9\}$,	$2 \vee 6 = 9$;	$(5, 8)\downarrow = \{1, 3\}$,	$2 \vee 6 = 3$;
$(7, 8)\uparrow = \{9\}$,	$2 \vee 6 = 9$;	$(7, 8)\downarrow = \{1, 3, 4, 6\}$,	$7 \vee 8 = 6$;

$4 \vee 5$ $5 \vee 8$ $7 \vee 8$ $5 \wedge 8$

It may be noted that it would have been sufficient to verify the existence of all joins $x \vee y$, or alternatively, of all meets $x \wedge y$, because now the underlying set A is finite (see Exercise 5 above).

Examples for lattices:

1. For any set B, the order $(\mathcal{P}(B), \subseteq)$ of all subsets of B is a lattice. In fact, in this case, we have

$$\begin{aligned} \top &= B \\ \bot &= \emptyset \\ X\wedge Y &= X\cap Y \\ X\vee Y &= X\cup Y\ . \end{aligned}$$

$(\mathcal{P}(B), \subseteq)$ is in fact a complete lattice. We call any lattice of the form $(\mathcal{P}(B), \subseteq)$ a *power-set lattice*.

Particularly important is the case when $B = \{0\}$. Now, $A=\mathcal{P}(\{0\}) = \{\emptyset, B\}$. Now, $\emptyset=\bot$ and $B=\{0\}=\top$; thus, $A=\{\top, \bot\}$. This lattice has two elements; it is frequently denoted by **2**.

The meet and join tables for **2** are those for "and" (conjunction) and "inclusive or" (disjunction). This interpretation depends on reading $\top$ as "true", and $\bot$ as "false". Here are the tables:

$\vee$	$\top$	$\bot$
$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$

$\wedge$	$\top$	$\bot$
$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$

2. Here is a particular infinite lattice: $(\mathbb{N}, \mid)$. The relation $\mid$ is "divides":

$$a\mid b \iff \exists c\in\mathbb{N}.\ a\cdot c = b$$

In this case, we have:

$$\top = 0$$
$$\bot = 1$$
$$a \wedge b = \mathrm{gcd}(a, b)$$
$$a \vee b = \mathrm{lcm}(a, b)$$

gcd means "greatest common divisor"; lcm means "least common multiple". $(\mathbb{N}, |)$ is not a complete lattice. Can you say why?

3. There are many lattices formed by certain special subsets, as opposed to all subsets, of a given set. For instance, let B be any set, and let us consider the set of all equivalence relations on the set B; this set is denoted by $\mathcal{E}(B)$. Since every equivalence relation is, in particular, a subset of $B \times B$, we have that $\mathcal{E}(B) \subseteq \mathcal{P}(B \times B)$. We may consider the subset relation $\subseteq$ restricted to $\mathcal{E}(B)$; this is the order induced by $\subseteq$ on $\mathcal{E}(B)$. It turns out that $(\mathcal{E}(B), \subseteq)$ is a lattice; in fact, a complete lattice.

Exercise 6. Prove the last assertion. (**Hint**: show that the intersection of any non-empty collection of equivalence relations is again an equivalence. Conclude that in $(\mathcal{E}(B), \subseteq)$, the meet of any non-empty set of elements exists, and is equal to the intersection of those elements. The meet of the empty set of elements also exists (obviously). Finally, use Exercise 3.)

4. Let V be a vector space (over any scalar field). Let $\mathrm{Sub}(V)$ be the set of all subspaces of V. $(\mathrm{Sub}(V), \subseteq)$ is a lattice. We have that, in this case,

$$\top = V$$
$$\bot = \{0\}$$
$$X \wedge Y = X \cap Y$$
$$X \vee Y = X + Y = \{x + y : x \in X, y \in Y\}$$

5. Here is an important, but somewhat special, construction of a lattice.

Let us start with two fixed sets M and N, and a relation T between them: $T \subseteq M \times N$ (thus,

T is somewhat more general then our relations so far, since not one, but two "underlying sets" are involved). Let us use the variable U to denote subsets of M, V for subsets of N, a elements of M, x elements of N. Given any $U \subseteq M$, $V \subseteq N$, we define $U^* \subseteq N$, $V^{\#} \subseteq M$ by

$$x \in U^* \iff \forall a \in U.\, aTx$$
$$a \in V^{\#} \iff \forall x \in V.\, aTx$$

We let $A = \{U \subseteq M: U^{*\#} = U\}$. A is a subset of $\mathcal{P}(M)$. The induced order $(A, \subseteq)$ is a lattice, called the *concept lattice* derived from $T \subseteq M \times N$.

Exercise 7* Prove the last-stated assertion.

Laws holding in all lattices:

$x \wedge y = y \wedge x$	$x \vee y = y \vee x$	**(commutative laws)**
$(x \wedge y) \wedge z = x \wedge (y \wedge z)$	$(x \vee y) \vee z = x \vee (y \vee z)$	**(associative laws)**
$x \vee (x \wedge y) = x$	$x \wedge (x \vee y) = x$	**(absorption laws)**
$x \wedge x = x$	$x \vee x = x$	**(idempotent laws)**
$\top \wedge x = x$ $\quad$ $\top \vee x = \top$	$\bot \wedge x = \bot$ $\quad$ $\bot \vee x = x$	**(identity laws)**

$$x \leq y \iff x \wedge y = x \iff x \vee y = y$$

$x \leq y$ and $u \leq v$ imply that $x \wedge u \leq y \wedge v$ and $x \vee u \leq y \vee v$

$$(x \vee y) \wedge z \geq (x \wedge z) \vee (y \wedge z)$$

$$(x \wedge y) \vee z \leq (x \vee z) \wedge (y \vee z)$$

$x \leq \top \qquad\qquad x \geq \bot$

$x \leq \bot \Rightarrow x = \bot \qquad\qquad x \geq \top \Rightarrow x = \top$

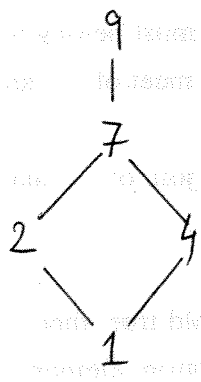
Exercise 8. Prove the laws.

A *sublattice* of lattice (A, R) is a lattice (B, S) for which B is a subset of A, and for which the meanings of $\top$ (top), $\bot$ (bottom), and of $x \wedge y$, $x \vee y$ for elements x and y in the (smaller) set B, are the same in the two lattices. When (B, S) is a sublattice of (A, R), then, for $x, y \in B$, xRy iff xSy; but, this condition is not enough to ensure that (B, S) is a sublattice of (A, R).

At any rate, a sublattice (B, S) of (A, R) is completely determined by its underlying set B. However, if we take a subset B of A, define S by $xSy \Longleftrightarrow xRy$ for all $x, y \in B$ (this order S on B is called the *order induced by* R *on* B), then (B, S) is not necessarily a sublattice of (A, R) *even if* (B, S) *is a lattice on its own right.*

Example 3 (continued) Now, (A, R) as in Example 3 above.

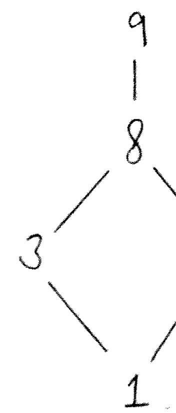
Let $B = \{1, 2, 4, 7, 9\}$. Then the order induced by R on B is given by the Hasse diagram



Notice that in the Hasse diagram of (A, R), there is no arc from 2 to 7; although, of course, we have $2R7$. However, in the induced order S, it is not only the case that $2S7$, but we also have that 7 *covers* 2, since there is no element *in* B between 2 and 7. This is why we do have an arc from 2 to 7 in the Hasse diagram of (B, S).

Next, we verify that (B, S) is a lattice, and in fact, a sublattice of (A, R). There is only one incomparable pair, up to the order of mention: $(2, 4)$; and $2 \vee 4 = 7$, $2 \wedge 4 = 1$ in (B, S). When we check what the values of $2 \vee 4$, $2 \wedge 4$ were in (A, R), we see that these are 7 and 1, respectively. We can conclude that (B, S) is a sublattice of (A, R).

Next, let B be the subset $B=\{1, 3, 4, 8, 9\}$ of A. The induced order (B, S) now is given by the Hasse diagram



This is also a *lattice*. However, it is *not a sublattice of* (A, R), since in (B, S), $3\vee 4=8$, but in (A, S), $3\vee 4=6$.

Let B be a subset of A. When does B determine a sublattice of $(A; \leq)$; when is $(B, \leq\upharpoonright B)$ a sublattice of $(A, \leq)$? For this,

> $\top$, the top element of $(A, \leq)$, must belong to B;
> $\bot$, the bottom element of $(A, \leq)$, must belong to B;
> if x and y in B, then $x\wedge y$, the meet of x and y in the sense of the lattice $(A, \leq)$, must belong to B;
> if x and y in B, then $x\vee y$, the join of x and y in the sense of the lattice $(A, \leq)$, must belong to B.

These conditions are also enough: if they hold true, then $(B, \leq\upharpoonright B)$ is a lattice; in fact, $\top$ is the top element of $(B, \leq\upharpoonright B)$; $\bot$ is the bottom element of $(B, \leq\upharpoonright B)$; and if $x, y\in B$, then $x\wedge y$, $x\vee y$ are the meet and join, respectively, of $\{x,y\}$ in $(B, \leq\upharpoonright B)$ as well.

Given any lattice $(A, \leq)$, and *any* subset X of A, we can form the sublattice $\langle X\rangle$ *of* $(A, \leq)$ *generated* by X. The underlying set B of $\langle X\rangle$ is the *least subset* B of A for which

> X is a subset of B;
> $\top$, the top element of (A, R), belongs to B;

$\perp$, the bottom element of (A, R) , belongs to B ;

for any x and y in B , both $x \wedge y$, $x \vee y$ computed in the given lattice (A, R) belong to B again.

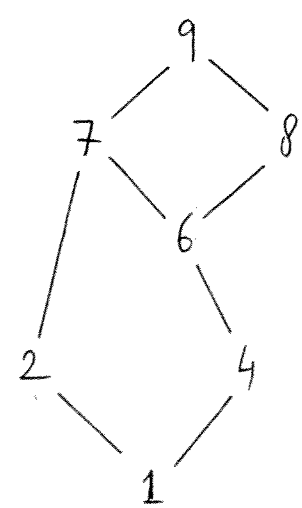
The way to obtain $\langle X \rangle$ is to build B by (i) throwing in all the elements of X into B , (ii) throwing in $\top$ and $\perp$ into B , and (iii) every time we have x and y already in B for which $x \wedge y$ and/or $x \vee y$ is not yet in B , throwing $x \wedge y$ and/or $x \vee y$ into B -- until B is *closed under* said operations.

Example 3 (continued) We take (A, R) as in Example 3 again. Let $X = \{2, 4, 8\}$. We construct the underlying set B of

$$\langle X \rangle = \langle \{2, 4, 8\} \rangle = \langle 2, 4, 8 \rangle$$

by starting with the elements $2, \ 4, \ 8$ and $1 = \perp$, $9 = \top$. We have $2 \vee 4 = 7$, therefore $7 \in B$. But then, since 7 and 8 are both in B , we must have that $7 \wedge 8 = 6$ belongs to B . When we take the set of the elements listed so far, $B = \{2, 4, 8, 1, 9, 7, 6\}$, we see that for each of the incomparable pairs of elements of B , which are $(2, 4)$, $(2, 6)$, $(2, 8)$ and $(7, 8)$, the join and the meet of the two elements of the pair is again an element of B ; this can be checked by the listing for (A, R) given above. (Of course, when x and y are comparable elements of B , then $x \wedge y$, $x \vee y$, being elements of $\{x, y\}$, do already belong to B .) This ensures that B is closed under the lattice operations in (A, R) , and we conclude that the underlying set of $\langle 2, 4, 8 \rangle$ is $\{2, 4, 8, 1, 9, 7, 6\}$.

The Hasse diagram of the lattice $\langle 2, 4, 8 \rangle$ is



Let us return to the general situation. When $\langle X \rangle$ equals the whole lattice $(A, \leq)$, that is, $B=A$, we say that X *generates* the lattice $(A, \leq)$, or, that X is a *set of generators for* the lattice $(A, \leq)$.

In the example, the only elements of $(A, \leq)$ missing from $\langle 2, 4, 8 \rangle$ are 3 and 5. Adding either of these to the set $\{2, 4, 8\}$ results in a set of generators for the lattice. For instance, $\{2, 3, 4, 8\}$ is a set of generators for $(A, \leq)$, $\langle 2, 3, 4, 8 \rangle = (A, \leq)$. The reason is that $2 \vee 3 = 5$, and thus all elements of A are in the underlying set of $\langle 2, 3, 4, 8 \rangle$.

Consider the **distributive identity**:

$$(x \vee y) \wedge z \overset{?}{=} (x \wedge z) \vee (y \wedge z)$$

This may or may not hold in a lattice. A lattice is said to be *distributive* if the distributive identity holds for all values of the variables x, y, z.

Example 3 (continued) This lattice is distributive. It is not easy to check this directly, since the number of triples (x, y, z) to check is quite large. However, there is a characterization, given below, of distributivity that makes the verification somewhat easier.

The most important examples for distributive lattices are the power-set lattices. $(\mathcal{P}(B), \subseteq)$ is distributive, since the distributive identity holds for union and intersection of sets:

$$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z) .$$

This was proved in Section 1.2.

Exercise 9. Prove that the inequality $(x \vee y) \wedge z \geq (x \wedge z) \vee (y \wedge z)$ always holds in any lattice.

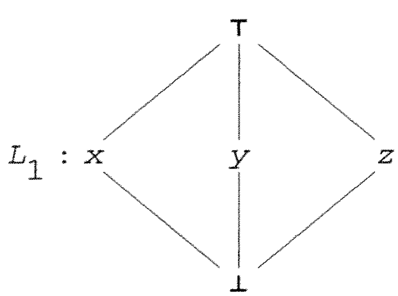
We need two auxiliary concepts.

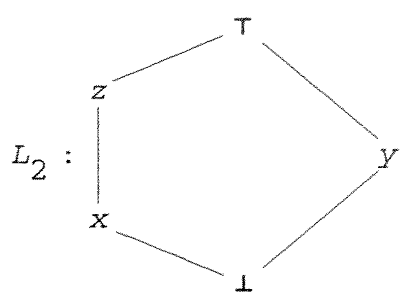
Let us say that the lattice (B, S) is a *weak sublattice* of (A, R) if all the conditions for (B, S) being a sublattice of (A, R) are satisfied except possibly the ones for "top" and "bottom"; that is, it may happen that $\top \notin B$ or $\bot \notin B$.

Example 3 (continued) For instance, $B=\{1, 3, 4, 6\}$ is the underlying set of a weak sublattice (B, S) of the lattice of Example 3, because $3 \wedge 4=1$ and $3 \vee 4=6$ in both (A, R) and (B, S) , and the only pair of incomparables in B is $(3, 4)$. However, (B, S) is not an (ordinary) sublattice of (A, R) , since the top element of (B, S) is $6 \neq 9$.

Recall that isomorphic relations share all "mathematical" properties. For instance, one is a lattice if and only if the other is; one is distributive if and only if the other is.

To return to distributivity, here are two particular non-distributive lattices:





In the first lattice, $(x \vee y) \wedge z = z$ and $(x \wedge z) \vee (y \wedge z) = \bot$; in the second case, $(x \vee y) \wedge z = z$ and $(x \wedge z) \vee (y \wedge z) = x$.

It follows that if either of L_1 , L_2 is *isomorphic* to a weak sublattice of a lattice $(A, \leq)$, the latter cannot be distributive: one or the other of the above counterexamples to distributivity will be present in $(A, \leq)$. It is a **theorem** (not very difficult to prove) that conversely, if $(A, \leq)$ is a non-distributive lattice, then either ~~L_1 or L_2 is~~ *isomorphic* to a weak sublattice

[illegible] !

of $(A, \leq)$.

Example 3 (continued) It is not too difficult to see that in this example, neither L_1 nor L_2 is isomorphic to a weak sublattice of $(A, \leq)$. Therefore, $(A, \leq)$ is distributive. However, when we modify the example by removing the arc $(6, 7)$ from the Hasse diagram, the resulting lattice (A, R') contains the weak sublattice (B, S) with $B=\{3, 5, 6, 7, 9\}$, which is isomorphic to L_2; therefore, the modified lattice is not distributive. In fact, in (A, R'), $(5 \vee 8) \wedge 7 \neq (5 \wedge 7) \vee (8 \wedge 7)$.