MATH 254 Assignment 4 McGill University
Fall 2016

You should work carefully on all problems. However you only have to hand in solutions to problems 6, 9, and 12. This assignment is due on <u>*Tuesday, November 1st*</u> *in class.*

Exercise 1. Determine the limits, when these exist, of the sequences (u_n), (v_n), (w_n), (x_n) defined by

$$u_n = \frac{n - \sqrt{n^2+1}}{n+\sqrt{n^2-1}} \qquad v_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$$

$$w_n = \sum_{k=1}^{n} \frac{n}{n^2+k} \qquad x_n = \sum_{k=0}^{n} (-1)^{n-k} k!$$

Exercise 2. Compare

$$\lim_{m\to+\infty} \lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^m, \quad \lim_{n\to+\infty} \lim_{m\to+\infty} \left(1-\frac{1}{n}\right)^m, \quad \text{and} \quad \lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^n.$$

Exercise 3. Let (x_n) be a sequence of nonzero real numbers such that $(x_{n+1}/x_n)_n$ converges to 0. Determine the limit of (x_n).

Exercise 4. Let (x_n) be a decreasing sequence in $\mathbb{R}$ which converges to 0 and (S_n) be the sequence defined by

$$S_n = \sum_{k=0}^{n} (-1)^k x_k.$$

Show that (S_{2n}) and (S_{2n+1}) are monotone and converge to the same limit and deduce that (S_n) converges.

Exercise 5. Let (x_n) be a sequence in $\mathbb{R}$ such that (x_{2n}), (x_{2n+1}) and (x_{3n}) converge. Show that (x_n) converges.

Exercise 6. [10 points] Let (x_n) be a sequence of real numbers and A be the set of all limits of converging subsequences of (x_n). Let (y_n) be a sequence of real numbers such that $y_n \in A$ for all $n \in \mathbb{N}$. Show that if (y_n) converges, then $\lim\limits_{n\to\infty} y_n \in A$.

Exercise 7. Determine the limit superior and limit inferior of the sequence (x_n) defined by

$$x_n = \frac{n\,(-1)^n}{2n-1}.$$

Exercise 8. Let (x_n) and (y_n) be two bounded sequences. Show that

$$\limsup_{n\to\infty} (x_n + y_n) \le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.$$

Give an example where this inequality is strict.

Exercise 9. [10 points] Let (x_n) be an sequence in $\mathbb{R}$ and (y_n) be the sequence defined by

$$y_n = \frac{x_1 + \cdots + x_n}{n}\,.$$

Show that if (x_n) is bounded, then (y_n) is bounded and

$$\liminf_{n\to\infty} x_n \le \liminf_{n\to\infty} y_n \le \limsup_{n\to\infty} y_n \le \limsup_{n\to\infty} x_n\,.$$

Note that this implies that if (x_n) converges, then (y_n) converges.

Exercise 10. Let $a \in (0,1)$ and (x_n) be a sequence of real numbers such that

$$|x_{n+1} - x_n| < a^n \qquad \forall n \in \mathbb{N}.$$

Show that (x_n) is a Cauchy sequence (and thus (x_n) converges).

Exercise 11. Let (x_n) be a sequence in $\mathbb{R}$ such that $x_1 > 0$ and

$$x_{n+1} = (2 + x_n)^{-1} \qquad \forall n \in \mathbb{N}.$$

Show that (x_n) is a contractive sequence. What is the limit of (x_n)?

Exercise 12. [10 points] Show that

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots \sqrt{1 + \sqrt{1 + 1}}}}} = \frac{1 + \sqrt{5}}{2}\,.$$

Hint: This equality can be written as a limit of a contractive sequence.