

# THE COLLECTED PAPERS OF GERHARD GENTZEN

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*Edited by*

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This volume is based on the following papers of Gerhard Gentzen:

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- #2 Über das Verhältnis zwischen intuitionistischer und klassischer Arithmetik, Galley proof, *Mathematische Annalen* (1933) received on 15th March 1933. (Kindly made available by Prof. Paul Bernays.) . . . . . 53
- #3 Untersuchungen über das logische Schliessen, *Mathematische Zeitschrift* 39 (1935) 176–210, 405–431. (Accepted as Inaugural Dissertation by the faculty of mathematics and natural science of the university of Göttingen.) . . . . 68
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- #6 Der Unendlichkeitsbegriff in der Mathematik, *Semester-Berichte, Münster in/W.*, 9th Semester, Winter 1936–37, 65–80. . . . . 223
- #7 Die gegenwärtige Lage in der mathematischen Grundlagenforschung, *Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften*, New Series, No. 4, Leipzig (Hirzel), (1938) 5–18; also: *Deutsche Mathematik* 3 (1939) 255–268. . . . . 234
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## BIOGRAPHICAL SKETCH

Gerhard Gentzen was born in Greifswald, Pomerania, on November 24th, 1909. He spent his childhood in Bergen on the Isle of Rügen in the Baltic Sea, where his father was practicing law. There he attended elementary school and the local *Realgymnasium*. After his father's death in the First World War, his mother decided in 1920 to move to Stralsund, where Gentzen completed his secondary education at the *Humanistische Gymnasium*. On February 29th, 1928, he was granted the *Abitur* with distinction, having attained the highest academic standing in his school and, on the recommendation of his headmaster, he received a university scholarship from the *Deutsche Studentenwerk* enabling him to continue his higher education.

Even as a young boy, Gentzen is said to have displayed exceptional mathematical ability and had declared categorically that the only subject which he would ever be able to study was mathematics. He enrolled at the University of Greifswald for two semesters and there earned Hans Kneser's respect as 'a particularly gifted student'. From Greifswald Gentzen went to Göttingen, where he matriculated for the first time on April 22nd, 1929. After two semesters he went to Munich, studied there for one semester, and after a further semester at Berlin, he finally returned to Göttingen and worked under Hermann Weyl. Five semesters later, in the summer of 1933, Gentzen sat his *Staatsexamen* and, at the age of twenty-three, was granted a doctorate in mathematics. The great mental strain which his studies had involved and his delicate constitution forced him to interrupt his academic career and to return home for an extended period of rest.

The major turning point in Gentzen's academic life came undoubtedly with his appointment in 1934 as Hilbert's assistant in Göttingen, where he continued to work even after Hilbert's retirement. During these years Gentzen published some of his most important papers and was also given the responsible task of reviewing numerous works of eminent researchers

from many countries for the *Zentralblatt für Mathematik*. These reviews attest his extraordinary range of interest and the great extent of his involvement in the international community of scholars. In 1937, he was invited to the Philosophical Congress in Paris and delivered an address on the 'Concept of Infinity and the Consistency of Mathematics'.

At the outbreak of the Second World War, Gentzen was conscripted into the armed forces and was given an assignment in Telecommunications in Braunschweig. Within two years he became seriously ill and spent three months in a military hospital. Upon his release, he was freed from military service for reasons of ill health. After a period of rest, he rejoined the University of Göttingen, where in 1942 he attained the Dr. phil. habil. degree for his papers on the 'Provability and Nonprovability of Restricted Transfinite Induction in Elementary Number Theory'.

Upon the request of the director of the Mathematical Institute of the German University of Prague, Gentzen was subsequently appointed *Dozent* at that University in the autumn of 1943. He taught there until, on May 5th, 1945, he and all other professors at the University were taken into custody by the new local authorities. On August 4th, 1945 amid the turmoil and confusion that must have marked that period, Gentzen died tragically in his cell of malnutrition after several months of extreme physical hardship. One of his friends writes: "I can still see him lying on his wooden bunk thinking all day about the ((mathematical)) problems which preoccupied him. He once confided in me that he was really quite contented since now he had at last time to think about a consistency proof for analysis. He was in fact fully convinced that he would succeed in carrying out such a proof. He also concerned himself with other questions such as that of an artificial language, etc. Now and then he would give a short talk . . . We were continually reassured that the formalities of our release would take only a few days longer . . . He was hoping to be able to return to Göttingen and devote himself fully to the study of mathematical logic and the foundations of mathematics. He was dreaming of an Institute for this purpose, perhaps together with H. Scholz . . ."

M.E.S.

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