McGill University
Department of Mathematics and Statistics
MATH 254 Analysis 1, Fall 2015

Assignment 6: Solutions

1. Let (x_n) be a sequence such that $x_n > 0$ for $n \in \mathbb{N}$. Set

$$y_n = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}, \quad n \in \mathbb{N}.$$

(a) Suppose that (x_n) is a convergent sequence. Prove that

$$\lim y_n = \lim x_n.$$

Hint: You may use Problem 3 on the Assignment 4.

(b) Suppose that $\lim x_n = +\infty$. Prove that $\lim y_n = +\infty$.

Solution:

(a) We will use problem 3 of assignment 5. Set

$$P_{nk} = \frac{1}{x_k \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)} \geq 0$$

Then,

$$\sum_{k=1}^{n} P_{nk} x_k = \sum_{k=1}^{n} \frac{1}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} = \frac{n}{\frac{1}{x_1} + \cdots + \frac{1}{x_n}} = y_n$$

Furthermore,

$$\sum_{k=1}^{n} P_{nk} = \sum_{k=1}^{n} \frac{1}{x_k \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)} = \sum_{k=1}^{n} \frac{x_1 \cdots x_{k-1} x_{k+1} \cdots x_n}{x_1 \cdots x_n \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)}$$
$$= \frac{\sum_{k=1}^{n} x_1 \cdots x_{k-1} x_{k+1} \cdots x_n}{x_1 \cdots x_n \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)} = \frac{\sum_{k=1}^{n} x_1 \cdots x_{k-1} x_{k+1} \cdots x_n}{\sum_{k=1}^{n} x_1 \cdots x_{k-1} x_{k+1} \cdots x_n} = 1$$

Lastly, we need to check $\lim_{n \to \infty} P_{nk} = 0$. Fix k. Since x_n is convergent, $\frac{1}{x_n}$ cannot converge to zero. You may recall that this implies the series $\sum_{n=1}^{\infty} \frac{1}{x_n}$ diverges, and which implies the result. However, it's easy enough to show directly that P_{nk} converges to zero: let $\epsilon > 0$. Since (x_n) converges, it is bounded; say by $M > 0$. Since $x_n > 0$, $\frac{1}{x_n} > \frac{1}{M}$. Choose $N > \frac{M}{\epsilon x_k}$. Then for $n \geq N$,

$$P_{nk} = \frac{1}{x_k \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n} \right)} \leq \frac{1}{x_k} \frac{1}{\underbrace{\frac{1}{M} + \cdots + \frac{1}{M}}_{n\text{times}}} = \frac{M}{n x_k} < \epsilon$$

Note that we allowed N to depend on ϵ and x_k, but not n, since we are showing for each fixed k that $P_{nk} \to 0$.

Our P_{nk} satisfy the criteria of problem 3 assignment 5, so since x_n converges, $y_n = \sum_{k=1}^{n} P_{nk} x_k$ converges to the same limit.

Notice we were careful to use an approach that does not assume $\lim x_n > 0$; we only know $\lim x_n \geq 0$. If we ever needed to work with $\lim_{n\to\infty} \frac{1}{x_n}$, we would need to be careful in applying limit theorems; for most approaches to this problem it was necessary to treat the $x_n \to 0$ case separately.

(b) It is easy to show for positive sequence x_n, $\lim_{n\to\infty} x_n = +\infty \iff \lim_{n\to\infty} \frac{1}{x_n} = 0$. Then $\frac{1}{y_n} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{x_k}$; since $\frac{1}{x_n}$ converges to zero, we may apply problem 4 of assignment 5 (a special case of problem 3) to conclude that $\frac{1}{y_n} \to 0$ as well. Then $\lim_{n\to\infty} y_n = +\infty$, as we wanted.

2. Let $a > 0$ and let (x_n) be a sequence defined recursively as $x_1 = \sqrt{a}, x_{n+1} = \sqrt{a + x_n}, n \geq 1$. Prove that (x_n) is convergent and find $\lim x_n$.

Solution:

We show (x_n) is increasing by induction. First, $x_1 = \sqrt{a} \leq \sqrt{2a} = x_2$. Next, if $x_n \leq x_{n+1}$, $x_n + a \leq x_{n+1} + a$, and so $x_{n+1} = \sqrt{x_n + a} \leq \sqrt{x_{n+1} + a} = x_{n+2}$, so (x_n) is increasing.

We wish to show (x_n) is bounded and so convergent. Let's try to guess a bound. If (x_n) were convergent, then it's limit would be a bound (since (x_n)) is increasing; furthermore, if we showed (x_n) were convergent, we could write $\lim x_n = \lim x_{n+1} = x$; that is, $x = \sqrt{x + a}$. Then we would have

$$x^2 - x - a = 0 \implies x = \frac{1 \pm \sqrt{1 + 4a}}{2}$$

We know the limit would have to be non-negative, were it to exist, so we would choose $x = \frac{1+\sqrt{1+4a}}{2}$. Now let us forget how we obtained this particular real number and just show that $x = \frac{1+\sqrt{1+4a}}{2}$ is a bound for the sequence (x_n). This is easy to do by induction: $\sqrt{a} \leq x$. If $x_n \leq x$, then $x_{n+1} = \sqrt{a + x_n} \leq \sqrt{a + x} = x$. So (x_n) is bounded by x; then by the monotonce convergence theorem, (x_n) converges. By the logic above then, we can use $\lim x_n = \lim x_{n+1}$ to show that $x_n \to x$.

3. Let $x_1 \in \mathbb{R} \setminus \{0\}$ and let

$$x_{n+1} = x_n + \frac{1}{x_n} \qquad \forall n \in \mathbb{N}$$

(a) Prove that $\lim (x_n) = +\infty$ if $x_1 > 0$.

(b) Prove that $\lim (x_n) = -\infty$ if $x_1 < 0$.

Solution:

We show first that (x_n) diverges. Assume the opposite i.e. that (x_n) converges; let $x = \lim (x_n)$. Then

$$x_{n+1} = x_n + \frac{1}{x_n} \implies x_{n+1} x_n = (x_n)^2 + 1$$

Taking limits yields $x^2 = x^2 + 1$ which is impossible. Our assumption was thus wrong; this proves that (x_n) diverges.

(a) We will show by induction that $x_1 > 0$ implies $x_n > 0$ for all $n \in \mathbb{N}$. The base step is trivial. Inductive step: assume that $x_n > 0$. Then $\frac{1}{x_n} > 0$ and $x_{n+1} = x_n + \frac{1}{x_n} > 0$.

This proves $x_n > 0$ for all $n \in \mathbb{N}$. Consequently, $x_{n+1} = x_n + \frac{1}{x_n} > x_n$ for all $n \in \mathbb{N}$ i.e. (x_n) is increasing.

Summarizing, we know that (x_n) is increasing and divergent. From this we conclude that (x_n) is unbounded, since a bounded and increasing sequence is convergent by the monotone convergence theorem. And since an unbounded and increasing sequence diverges to infinity, as shown in class, we now have that $\lim(x_n) = +\infty$.

(b) This part can be proved in exactly the same way as part (a). We present instead a different solution. We'll make use of the (rather obvious) lemma below.

Lemma: Let (x_n) be a sequence with $\lim(x_n) = +\infty$ and let $C < 0$. Then $\lim(C\,x_n) = -\infty$.

Proof: Let $M \in \mathbb{R}$. Then there exists an $N \in \mathbb{N}$ such that $x_n \geq \frac{M}{C}$ for all $n \geq N$. Then $C\,x_n \leq M$ for all $n \geq N$ which shows that $\lim(C\,x_n) = -\infty$. q.e.d.

Now consider the sequence (y_n) with $y_1 := -x_1 > 0$ and $y_{n+1} = y_n + \frac{1}{y_n}$. We will show by induction that $y_n = -x_n$ for all $n \in \mathbb{N}$. Base step: $y_1 = -x_1$ by definition. Inductive step: Assume that $y_n = -x_n$. Then $y_{n+1} = y_n + \frac{1}{y_n} = -x_n + \frac{1}{-x_n} = -\left(x_n + \frac{1}{x_n}\right) = -x_{n+1}$. q.e.d.

Thus $y_n = -x_n$ for all $n \in \mathbb{N}$. By part (a), $\lim(y_n) = +\infty$. Applying the lemma for $C = -1$ yields that $\lim(-y_n) = \lim(x_n) = -\infty$. This is what we had to show.

4. Find

$$\lim \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right)$$

Solution:

We have

$$\begin{aligned}
x_n :&= \prod_{k=2}^{n} 1 - \frac{1}{k^2} \\
&= \prod_{k=2}^{n} \frac{k^2-1}{k^2} = \prod_{k=2}^{n} \frac{(k+1)(k-1)}{k^2} \prod_{k=2}^{n} \frac{k+1}{k} \prod_{k=2}^{n} \frac{k-1}{k} \\
&= \left(\prod_{k=2}^{n} \frac{k+1}{k}\right)\left(\prod_{k=2}^{n} \frac{k-1}{k}\right) \\
&= \left(\prod_{k=3}^{n+1} \frac{k}{k-1}\right)\left(\prod_{k=2}^{n} \frac{k-1}{k}\right) \\
&= \frac{n+1}{n} \prod_{k=3}^{n} \frac{k}{k-1} \prod_{k=3}^{n} \frac{k-1}{k} \frac{1}{2} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n} \xrightarrow{n\to\infty} \frac{1}{2}
\end{aligned}$$

5. Find

$$\lim \left(\frac{2^3-1}{2^3+1}\right)\left(\frac{3^3-1}{3^3+1}\right)\cdots\left(\frac{n^3-1}{n^3+1}\right)$$

Solution:

Set

$$x_n = \prod_{k=2}^{n} \frac{k^3-1}{k^3+1}$$

for $n \geq 2$. We notice that x_n is (strictly) decreasing: $x_{n+1} = \frac{(n+1)^3-1}{(n+1)^3+1} x_n < x_n$. Furthermore, each $x_n \geq 0$ for $n \geq 2$. So (x_n) is a decreasing sequence bounded below, and so converges. But to find the limit, we will need to some cancellation. So we compute. We factor the sum and difference of cubes, to find

$$\begin{aligned} x_n = \prod_{k=2}^{n} \frac{k^3-1}{k^3+1} &= \prod_{k=2}^{n} \frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)} \\ &= \prod_{k=2}^{n} \frac{k-1}{k+1} \prod_{k=2}^{n} \frac{k^2+k+1}{k^2-k+1}. \end{aligned}$$

Let's try to find cancellation within each product:

$$\prod_{k=2}^{n} \frac{k-1}{k+1} = \prod_{k=2}^{n} \frac{1}{k+1} \prod_{k=2}^{n} (k-1) = \prod_{k=4}^{n+2} \frac{1}{k-1} \prod_{k=2}^{n} (k-1)$$

by shifting the index of the first product without changing the value of the product. (If this seems confusing, write out $\prod_{k=2}^{n} \frac{1}{k+1}$ and $\prod_{k=4}^{n+2} \frac{1}{k-1}$). Then we pull out explicitly the last two terms of the first product and the first two terms of the second product and find

$$= \frac{1}{n(n+1)} \prod_{k=4}^{n} \frac{1}{k-1} \prod_{k=4}^{n} (k-1) \cdot (2) = \frac{2}{n(n+1)}.$$

We perform similar manipulations for the second product:

$$\begin{aligned} \prod_{k=2}^{n} \frac{k^2+k+1}{k^2-k+1} &= \prod_{k=2}^{n} \frac{1}{k^2-k+1} \prod_{k=2}^{n} (k^2+k+1) \\ &= \prod_{k=2}^{n} \frac{1}{k^2-k+1} \prod_{k=3}^{n+1} ((k-1)^2+(k-1)+1) \\ &= \prod_{k=2}^{n} \frac{1}{k^2-k+1} \prod_{k=3}^{n+1} (k^2-2k+1+k) \\ &= \frac{1}{2^2-2+1} \prod_{k=3}^{n} \frac{1}{k^2-k+1} \prod_{k=3}^{n} (k^2-k+1)((n+1)^2-(n+1)+1) \\ &= \frac{n^2+1+2n-n-1+1}{3} = \frac{n^2+n+1}{3} \end{aligned}$$

Then,

$$x_n = \frac{2}{3} \frac{n^2+n+1}{n^2+n} = \frac{2}{3} \left(1 + \frac{1}{n^2+n}\right) \xrightarrow{n \to \infty} \frac{2}{3}$$

6. Prove that for $a > 0$,

$$\lim n\left(a^{\frac{1}{n}} - 1\right) = \ln a$$

Solution:

We will do this by first assuming the result for the case $a = e$ and $a = \frac{1}{e}$, and use this to generalize to arbitrary $a > 0$. Then we will prove those two cases by hand.

Assumptions:

$$\lim_{n\to\infty} n\left(e^{1/n} - 1\right) = \ln e = 1,$$
$$\lim_{n\to\infty} n\left(\left(\frac{1}{e}\right)^{1/n} - 1\right) = \ln\frac{1}{e} = -1.$$

We will use the first assumption to prove the statement for $a > 1$.

Claim. If $c > 0$, if $\lim_{n\to\infty} f(n)$ exists, then $\lim_{n\to\infty} f(cn) = \lim_{n\to\infty} f(n)$.

Proof. Let $\lim_{n\to\infty} f(n) = L$ and $\epsilon > 0$. We know there exists $N \in \mathbb{N}$ with $n \geq N \implies |f(n) - L| < \epsilon$. Let $N' = N/c$. Then for $n \geq N'$, $nc \geq N$, so $|f(nc) - L| < \epsilon$. That is, $\lim_{n\to\infty} f(nc) = L$. □

Let us use this claim; set $f(n) = n\left(e^{1/n} - 1\right)$. We assumed $\lim_{n\to\infty} f(n) = 1$. Consider the case $a > 1$. Then $\ln a > 0$; set $c = 1/\ln a$. Then

$$1 = \lim_{n\to\infty} f(nc) = \lim_{n\to\infty} nc\left(e^{1/(nc)} - 1\right) = c\lim_{n\to\infty} n\left(\left(e^{\ln a}\right)^{1/n} - 1\right) = c\lim_{n\to\infty} n\left(a^{1/n} - 1\right)$$
$$\implies \ln a = \frac{1}{c} = \lim_{n\to\infty} n\left(a^{1/n} - 1\right)$$

So that proves the exercise in the case $a > 1$.

Now we will use the second assumption to prove the statment for $0 < a < 1$.

Claim. If $c < 0$ and $\lim_{n\to-\infty} f(n)$ exists, then $\lim_{n\to\infty} f(nc) = \lim_{n\to-\infty} f(n)$.

Proof. Let $\epsilon > 0$. If $\lim_{n\to-\infty} f(n) = L$, then there exists $M < 0$ such that for all $n \leq M$, $|f(n) - L| < \epsilon$. Then $n \geq M/c \implies nc \leq M$, so for $n \geq M/c$, $|f(nc) - L| < \epsilon$ too, and we have our result. □

Again choose $f(n) = n\left(e^{1/n} - 1\right)$. We know $\lim_{n\to\infty} f(n) = 1$. We can easily see $\lim_{n\to-\infty} f(n) = \lim_{n\to\infty} f(-n) = \lim_{n\to\infty} -n\left(e^{-1/n} - 1\right) = -\lim_{n\to\infty} n\left(\left(\frac{1}{e}\right)^{1/n} - 1\right) = -(-1) = 1$, using our assumption.

Consider $0 < a < 1$, so $\ln a < 0$. Set $c = 1/\ln a < 0$ as well.

$$1 = \lim_{n\to\infty} f(nc) = \lim_{n\to\infty} nc\left(e^{1/(nc)} - 1\right) = c\lim_{n\to\infty}\left(\left(e^{\ln a}\right)^{1/n} - 1\right) = c\lim_{n\to\infty} n\left(a^{1/n} - 1\right)$$
$$\implies \ln a = \frac{1}{c} = \lim_{n\to\infty} n\left(a^{1/n} - 1\right)$$

as we wanted. All that remains is the case $a = 1$; then $a^{1/n} = 1$, so $(a^{1/n} - 1) = 0$, and the sequence is constant zero, and thus the limit equals $\ln a = \ln 1 = 0$, as we desire.

Now we need to prove our assumptions: the statement for $a = e$ and $a = \frac{1}{e}$.

Case: $a = e$**:** We will assume the inequality

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1} \qquad \forall n \in \mathbb{N}.$$

We will prove the inequality

$$\left(1+\frac{1}{n}\right)^{\frac{1}{n}} \leq 1+\frac{1}{n^2} \qquad \forall n \in \mathbb{N}$$

<u>Proof</u>:

$$\left(1+\frac{1}{n^2}\right)^n = 1 + n \cdot \frac{1}{n^2} + \sum_{k=2}^{n} \binom{n}{k} \frac{1}{n^{2k}} \geq 1 + \frac{1}{n}$$

Thus

$$1 + \frac{1}{n^2} \geq \left(1+\frac{1}{n}\right)^{\frac{1}{n}}$$

q.e.d.

Now we will construct lower and upper bounds for $n\left(e^{\frac{1}{n}} - 1\right)$. For all $n \in \mathbb{N}$ we have

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

Thus

$$1+\frac{1}{n} < e^{\frac{1}{n}} < \left(1+\frac{1}{n}\right)^{\frac{n+1}{n}} = \left(1+\frac{1}{n}\right) \cdot \left(1+\frac{1}{n}\right)^{\frac{1}{n}} \leq \left(1+\frac{1}{n}\right) \cdot \left(1+\frac{1}{n^2}\right) = 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^3}$$

which implies

$$1 < n\left(e^{\frac{1}{n}} - 1\right) < 1 + \frac{1}{n} + \frac{1}{n^2} \qquad \forall n \in \mathbb{N}$$

Since $\lim \left(1 + \frac{1}{n} + \frac{1}{n^2}\right) = 1$ it follows from the squeeze theorem that $\lim n\left(e^{\frac{1}{n}} - 1\right) = 1$.

Case: $a = \frac{1}{e}$**:** We will show

$$\lim n\left(\frac{1}{e}^{\frac{1}{n}} - 1\right) = -1$$

using the same methods as the case for e.

We have

$$\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}} < \frac{1}{e} < \frac{1}{\left(1+\frac{1}{n}\right)^{n}}$$

$$\frac{1}{\left(1+\frac{1}{n^2}\right)\left(1+\frac{1}{n}\right)} \leq \frac{1}{\left(1+\frac{1}{n}\right)^{1/n}\left(1+\frac{1}{n}\right)} = \left(\frac{1}{\left(1+\frac{1}{n}\right)^{n+1}}\right)^{1/n} < \left(\frac{1}{e}\right)^{1/n} < \frac{1}{1+\frac{1}{n}}$$

$$\frac{1}{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}} \leq \left(\frac{1}{e}\right)^{1/n} < \frac{n}{1+n}$$

$$\frac{n^3}{1+n+n^2+n^3} \leq \left(\frac{1}{e}\right)^{1/n} < \frac{n}{1+n}$$

$$\frac{n^3}{1+n+n^2+n^3} - 1 \leq \left(\frac{1}{e}\right)^{1/n} - 1 < \frac{n}{1+n} - 1$$

$$\frac{n^3-1-n-n^2-n^3}{1+n+n^2+n^3} \leq \left(\frac{1}{e}\right)^{1/n} - 1 < \frac{n-1-n}{1+n}$$

$$\frac{-\frac{1}{n^2}-\frac{1}{n}-1}{\frac{1}{n^2}+\frac{1}{n}+1+n} \leq \left(\frac{1}{e}\right)^{1/n} - 1 < \frac{-1}{1+n}$$

$$\frac{-\frac{1}{n}-1-n}{\frac{1}{n^2}+\frac{1}{n}+1+n} \leq n\left(\left(\frac{1}{e}\right)^{1/n} - 1\right) < \frac{-n}{1+n}$$

$$\frac{-\frac{1}{n^2}-\frac{1}{n}-1}{\frac{1}{n^3}+\frac{1}{n^2}+\frac{1}{n}+1} \leq n\left(\left(\frac{1}{e}\right)^{1/n} - 1\right) < \frac{-1}{\frac{1}{n}+1}$$

Applying limit theorems, the LHS and RHS converge to -1, and so by squeeze theorem,

$$\lim_{n\to\infty} n\left(\left(\frac{1}{e}\right)^{1/n} - 1\right) = -1.$$

7. Let x_n be a convergent sequence with $\lim(x_n) = x$. Prove that $(\sin x_n)$ converges and that

$$\lim(\sin x_n) = \sin x.$$

Hint: Prove this result first in the special case that $\lim(x_n) = 0$.

Solution:

Assume for now that $\lim(x_n) = 0$. Then $\lim(|x_n|) = 0$ by assignment 4, question 3. And since $|\sin x_n| \leq |x_n|$ it follows that

$$-|x_n| \leq \sin x_n \leq |x_n|$$

where $\lim(|x_n|) = 0$ and $\lim(-|x_n|) = 0$. Thus it follows from the squeeze theorem that $\lim(\sin x_n) = 0$.

Now we treat the general case. We will use the trigonometric identity

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \qquad \forall \alpha,\beta \in \mathbb{R}$$

Let $\lim(x_n) = x$. Then

$$\sin x_n - \sin x = 2\sin\left(\frac{x_n - x}{2}\right)\cos\left(\frac{x_n + x}{2}\right)$$

And since $-1 \leq \cos\left(\frac{x_n + x}{2}\right) \leq 1$ it follows that

$$-2\sin\left(\frac{x_n - x}{2}\right) \leq \sin x_n - \sin x \leq 2\sin\left(\frac{x_n - x}{2}\right)$$

or

$$\sin x - 2\sin\left(\frac{x_n - x}{2}\right) \leq \sin x_n \leq \sin x + 2\sin\left(\frac{x_n - x}{2}\right)$$

Since $\lim\left(\frac{x_n - x}{2}\right) = 0$ it follows from the special case above that

$$\lim\left(\sin x \pm 2\sin\left(\frac{x_n - x}{2}\right)\right) = \sin x$$

Another application of the squeeze theorem yields now $\lim(\sin x_n) = \sin x$.

8. Let the sequence (x_n) be recursively defined by $x_1 = 1$ and $x_{n+1} = \sin x_n$ for all $n \in \mathbb{N}$. Prove that the sequence (x_n) is convergent and find $\lim (x_n)$.

Solution:

We will first show by induction that $0 < x_n < \frac{\pi}{2}$ for all $n \in \mathbb{N}$. Base case: $0 < x_1 = 1 < \frac{\pi}{2}$. Inductive step: Assume that $0 < x_n < \frac{\pi}{2}$. Then $x_{n+1} = \sin x_n > 0$. And since $x_n > 0$ we have $x_{n+1} = \sin x_n < x_n < \frac{\pi}{2}$. Thus $0 < x_{n+1} < \frac{\pi}{2}$. q.e.d.

We thus have $0 < x_n < \frac{\pi}{2}$ for all $n \in \mathbb{N}$. Consequently, $x_{n+1} = \sin x_n < x_n$ for all $n \in \mathbb{N}$ i.e. (x_n) is decreasing. Furthermore, (x_n) is bounded below by 0. Hence (x_n) converges; let $x = \lim (x_n)$. Since $x_n > 0$ for all $n \in \mathbb{N}$, we have $x \geq 0$.

Now consider the equation $x_{n+1} = \sin x_n$. Using question 4, we can take limits of both sides:

$$x = \lim (x_n) = \lim (x_{n+1}) = \lim (\sin x_n) = \sin x$$

Thus $\sin x = x$. Since $x \geq 0$ and $\sin x < x$ for all $x > 0$ this implies that $x = 0$. Thus $\lim (x_n) = 0$.

9. Let

$$x_n = \sin\left(\pi\sqrt{n^2+1}\right) \qquad \forall n \in \mathbb{N}$$

Prove that (x_n) is convergent and find $\lim (x_n)$.

Solution:

The idea of the solution is to use that fact that $\sqrt{n^2+1}$ is "close to" n and that $\sin(\pi n) = 0$. We will also use the sum-to-product formula

$$\sin\alpha - \sin\beta = 2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right) \qquad \forall \alpha, \beta \in \mathbb{R}$$

Now we can estimate x_n:

$$\begin{aligned} |x_n| &= \left|\sin\left(\pi\sqrt{n^2+1}\right)\right| = \left|\sin\left(\pi\sqrt{n^2+1}\right) - \sin(\pi n)\right| \\ &= 2\left|\sin\left(\frac{\pi}{2}(\sqrt{n^2+1}-n)\right)\right| \cdot \left|\cos\left(\frac{\pi}{2}(\sqrt{n^2+1}+n)\right)\right| \end{aligned}$$

where

$$\left|\cos\left(\frac{\pi}{2}(\sqrt{n^2+1}+n)\right)\right| \leq 1$$

and

$$\left|\sin\left(\frac{\pi}{2}(\sqrt{n^2+1}-n)\right)\right| \leq \left|\frac{\pi}{2}(\sqrt{n^2+1}-n)\right|$$

Thus

$$\begin{aligned} |x_n| &\leq \pi\left|\sqrt{n^2+1}-n\right| = \pi\left|\frac{(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n)}{\sqrt{n^2+1}+n}\right| = \frac{\pi}{\sqrt{n^2+1}+n} \\ &< \frac{\pi}{n} \qquad \forall n \in \mathbb{N} \end{aligned}$$

and $\lim (x_n) = 0$.