CHAPTER VII

FORCING

The method of constructibility discussed in VI produced one model—**L**—and hence established only the consistency of statements true in **L**, such as GCH or $\diamondsuit^+$. Forcing, on the other hand, is a general technique for producing a wide variety of models satisfying diverse mathematical properties.

§1. General remarks

There are two obstacles to understanding forcing—one mathematical and the other metamathematical.

The mathematical difficulty is that one must become proficient in handling partial orders, dense sets, and filters. The reader who is familiar with Martin's Axiom (see II §2) has already come a long way towards mastering this difficulty, although it will now become necessary to consider the relativization of these concepts to various models of set theory.

The metamathematical difficulty is that to prove the consistency of ZF + $\mathbf{V} \neq \mathbf{L}$ (or of any stronger theory, such as ZFC + $\neg$CH), we cannot, as we are used to, simply work in ZF or ZFC and define a transitive model for the desired axioms.

To appreciate the difficulty, suppose we were able to work within ZFC, define a transitive proper class **N**, and prove that each axiom of ZF + $\mathbf{V} \neq \mathbf{L}$ is true **N**. Then, by minimality of **L**, we would have $\mathbf{L} \subset \mathbf{N}$; but also $\mathbf{L} \neq \mathbf{N}$, since $\mathbf{V} = \mathbf{L}$ is true in **L** and false in **N**. Thus, arguing in ZFC, we could prove that there is a proper extension of **L**, so ZFC $\vdash \mathbf{V} \neq \mathbf{L}$, which is impossible (assuming Con(ZFC)), since ZFC + $\mathbf{V} = \mathbf{L}$ is consistent (by VI 3.4).

The naive way to sidestep this difficulty is simply to produce a transitive *set* model N for ZF + $\mathbf{V} \neq \mathbf{L}$. The above argument applied to N, using minimality of **L** for set models (VI 3.8) yields only $L(\mathrm{o}(N)) \subset N$ and $L(\mathrm{o}(N)) \neq N$, but that does not contradict $\mathbf{V} = \mathbf{L}$; if $x \in N \setminus L(\mathrm{o}(N))$, then x can still be in **L**, in which case $\rho(x) > \mathrm{o}(N)$.

Naively still, our general procedure will be as follows. We start with any countable transitive model M for ZFC. M is called the *ground model*. We shall describe a general procedure for finding countable transitive models N for ZFC such that $M \subset N$ and $o(M) = o(N)$. An N obtained by our procedure will be called a *generic extension* of M. As long as we succeed in making $M \neq N$, N will satisfy $\mathbf{V} \neq \mathbf{L}$, since, by VI 3.8

$$\mathbf{L}^N = L(o(N)) = L(o(M)) = \mathbf{L}^M \subset M.$$

However, we shall in fact be able to make N satisfy $\neg$CH, or CH + $2^{\omega_1} = \omega_5$, or a wide variety of other statements by varying certain details in our construction.

Unfortunately, this naive approach is not quite correct. By results related to the Gödel Incompleteness Theorem, one cannot argue within ZFC and produce any set models at all for ZFC; see IV §§7 and 10, as well as §9 of this chapter for a further discussion of this point. Fortunately, however, we can produce in ZFC countable transitive models M for any desired finite list of axioms of ZFC, or even of ZFC + $\mathbf{V} = \mathbf{L}$ (see IV 7.11 and VI 4.10).

The method of forcing will then be used to show how to produce models N for any given finite list of axioms of ZFC + $\mathbf{V} \neq \mathbf{L}$ (or ZFC + $\neg$CH, etc.); such N will be generic extensions of models M for suitably many axioms of ZFC.

The formal structure of our proof of

$$\mathrm{Con(ZFC)} \to \mathrm{Con(ZFC} + \mathbf{V} \neq \mathbf{L})$$

will be as follows. Assume we can derive a contradiction from ZFC + $\mathbf{V} \neq \mathbf{L}$. Then there is a finite list of axioms, $\phi_1, \ldots, \phi_n$ of ZFC + $\mathbf{V} \neq \mathbf{L}$ such that

$$\phi_1 \cdots \phi_n \vdash \psi \wedge \neg\psi$$

for some (or any) ψ. But, by the method of forcing, we shall show that

$$\mathrm{ZFC} \vdash \exists N\,(\phi_1^N \wedge \cdots \wedge \phi_n^N),$$

so

$$\mathrm{ZFC} \vdash \exists N\,(\psi^N \wedge \neg\psi^N),$$

whence ZFC is inconsistent. This method in fact produces a completely finitistic relative consistency proof, since we define explicitly how to construct an inconsistency in ZFC given one in ZFC + $\mathbf{V} \neq \mathbf{L}$.

The advantage of this approach is that when studying forcing we may temporarily forget the metamathematical niceties in the previous paragraph, and just assume naively that we have a countable transitive M satisfying all of ZFC (or even ZFC + $\mathbf{V} = \mathbf{L}$). We may then concentrate on the mathematical problems involved in constructing the generic extension N of M, satisfying, e.g., ZFC + $\neg$CH. Once the construction is understood, we may check that the previous paragraph applies to yield a finistic proof of

$$\mathrm{Con(ZFC)} \to (\mathrm{ZFC} + \neg\mathrm{CH}).$$

In §9 we shall return to the metamathematics of forcing, and shall discuss several other approaches for doing the same thing. In the meantime, when we say "let M be a countable transitive model for ZFC", the reader may consider this to be an abbreviation for "let M be a countable transitive model for enough axioms of ZFC to carry out the argument at hand."

§2. Generic extensions

Let M be a countable transitive model for ZFC. If $\langle \mathbb{P}, \leq \rangle$ is a partial order (in the sense of our discussion of MA in II §2) and $\langle \mathbb{P}, \leq \rangle \in M$, then $\langle \mathbb{P}, \leq \rangle$ will yield a method of obtaining a generic extension, N, of M, which is also a model of ZFC. By varying $\langle \mathbb{P}, \leq \rangle$, we shall be able to produce a wide variety of relative consistency results.

For technical reasons, it will be convenient to restrict our attention to partial orders with a largest element. One can do forcing without this restriction (Exercise B1), but it is slightly more cumbersome, and it produces no more consistency results (Exercise B2). Most partial orders occurring "naturally", such as in applications of MA in II §2, have a largest element anyway. To avoid excess verbiage, we define the following.

2.1. Definition. A p.o. is a triple, $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ such that $\leq$ partially orders $\mathbb{P}$ and $\mathbb{1}$ is a largest element of $\mathbb{P}$ (i.e., $\forall p \in \mathbb{P}\, (p \leq \mathbb{1})$). c.t.m. abbreviates "countable transitive model." □

Following standard abuses of notation, we shall often write $\mathbb{P}$ when we mean $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$. Thus, $\mathbb{P} \in M$ means $\mathbb{P} \in M$, $\leq \in M$, and $\mathbb{1} \in M$ (although $\mathbb{1} \in M$ follows from $\mathbb{P} \in M$ by transitivity of M). If two p.o.'s are under discussion, we may refer to them as, e.g., $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}} \rangle$. Formally, of course, the set $\mathbb{P}$ does not determine its ordering, $\leq_{\mathbb{P}}$; and $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ may not determine $\mathbb{1}_{\mathbb{P}}$, since the fact that we do not require $\leq_{\mathbb{P}}$ to be a partial order in the strict sense means that there could be many largest elements (see II 2.1 and following discussion).

2.2. Definition. Let $\mathbb{P}$ be a p.o. G is $\mathbb{P}$-*generic* (i.e., $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$-generic) over M iff G is a filter on $\mathbb{P}$ and for all dense $D \subset \mathbb{P}$, $D \in M \to G \cap D \neq 0$. □

2.3. Lemma. *If M is countable and $p \in \mathbb{P}$, then there is a G which is $\mathbb{P}$-generic over M such that $p \in G$.*

PROOF. Exactly like the proof of MA(ω) (see II 2.6(c)). Let $D_n (n \in \omega)$ enumerate all dense subsets of $\mathbb{P}$ which are in M. Inductively choose a sequence $q_n (n \in \omega)$ so that

$$p = q_0 \geq q_1 \geq \cdots$$

and $q_{n+1} \in D_n$. Let G be the filter generated by $\{q_n : n \in \omega\}$. $\square$

It is important to keep track of what is absolute for M and what is not. In our intended applications, M will be a c.t.m. for ZFC and $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ will be in M. It is then easily seen by the methods of IV §3 that notions like "p.o." or "dense" are absolute for M. However, the enumeration of the D_n takes place outside of M. By absoluteness,

$$\{D \in M : D \text{ is dense in } \mathbb{P}\} = \{D : D \text{ is dense in } \mathbb{P}\}^M,$$

but this set will not usually be countable in M (countable is *not* absolute).

2.2 and 2.3 did not require that M is a model for anything. But it will become important as we go along that M satisfy at least some of ZFC, to ensure that various dense sets we construct actually lie in M. This occurs, for example, in the proof of the next lemma, which says that in most cases $G \notin M$.

2.4. LEMMA. *If M is a transitive model of* ZF − P, $\mathbb{P} \in M$ *is a* p.o. *such that*

$$\forall p \in \mathbb{P}\, \exists q, r \in \mathbb{P}\, (q \leq p \wedge r \leq p \wedge q \perp r), \tag{1}$$

and G is $\mathbb{P}$-generic over M, then $G \notin M$.

PROOF. If $G \in M$, then $D = \mathbb{P} \setminus G \in M$, since set-theoretic difference is absolute. Also, D is dense: if $p \in \mathbb{P}$ and q, r are as in (1), then q, r cannot both be in G (since G is a filter); thus, p has an extension in D.

However, $G \cap D = 0$, contradicting the definition of generic. $\square$

The proof of Lemma 2.4 only required M to satisfy a very weak fragment of ZF − P, but there is no reason to try to keep track of precisely which finite set of axioms of ZFC are required for M at each step.

We remark that if condition (1) fails for $\mathbb{P}$, then there is a filter G on $\mathbb{P}$ which intersects *all* dense subsets of $\mathbb{P}$, and if $\mathbb{P} \in M$, then $G \in M$ (see Exercise A1). Any application of MA or forcing to such a $\mathbb{P}$ will be trivial. Thus, almost all partial orders considered in II, VII, or VIII satisfy (1), although (1) is never needed in the abstract treatment of MA or forcing.

We may now unveil slightly more about generic extensions. Let M be a c.t.m. for ZFC, with $\mathbb{P}$ a p.o. in M and G $\mathbb{P}$-generic over M. We shall show how to construct another c.t.m. for ZFC, called $M[G]$, which will satisfy

$M \subset M[G]$, $o(M) = o(M[G])$, and $G \in M[G]$. $M[G]$ will be the least extension of M to a c.t.m. for ZFC containing G. The fact that $G \in M[G]$ will imply, by Lemma 2.4, that in most cases $M \neq M[G]$.

The particular axioms of set theory that $M[G]$ satisfies beyond ZFC will be very sensitive to the combinatorial properties satisfied by $\mathbb{P}$ *in* M; most of these properties are *not* absolute. For example, consider the c.c.c. (Def. II 2.3). If M is a c.t.m. and $\mathbb{P} \in M$, then in $\mathbf{V}$, $\mathbb{P}$ is countable and thus trivially has the c.c.c. But $\mathbb{P}$ may well fail to have the c.c.c. in M.

Working within M, one may construct the various c.c.c. p.o.'s considered in II, plus many more which are not c.c.c. (in M). These all can be used for generic extensions.

We now return to the basic theory, which works equally well with any p.o. in M. The whole procedure of constructing $M[G]$ may seem rather complicated at first, but once over this hurdle, the techniques of cooking up a $\mathbb{P}$ to produce a desired consistency result will be reduced to (sometimes very difficult) problems in the combinatorics of partial orders.

The first step is to define $M[G]$. Roughly, this will be the set of all sets which can be constructed from G by applying set-theoretic processes definable in M. Each element of $M[G]$ will have a *name* in M, which tells how it has been constructed from G. We use letters τ, σ, π to range over names.

People living within M will be able to comprehend a name, τ, for an object in $M[G]$, but they will not in general be able to decide the object, τ_G, that τ names, since this will require a knowledge of G.

2.5. Definition. τ is a $\mathbb{P}$-name iff τ is a relation and

$$\forall \langle \sigma, p \rangle \in \tau\, [\sigma \text{ is a } \mathbb{P}\text{-name} \wedge p \in \mathbb{P}]. \quad \square$$

This definition does not mention models or any order on $\mathbb{P}$. The collection of $\mathbb{P}$-names will be a proper class if $\mathbb{P} \neq 0$.

Definition 2.5 must be understood as a definition by transfinite recursion. Formally, one defines the characteristic function of the $\mathbb{P}$-names, $\mathbf{H}(\mathbb{P}, \tau)$, by

$$\mathbf{H}(\mathbb{P}, \tau) = 1 \text{ iff } \tau \text{ is a relation} \wedge \forall \langle \sigma, p \rangle \in \tau\, [\mathbf{H}(\mathbb{P}, \sigma) = 1 \wedge p \in \mathbb{P}].$$

$$\mathbf{H}(\mathbb{P}, \tau) = 0 \text{ otherwise.}$$

Then, τ is defined to be a $\mathbb{P}$-name iff $\mathbf{H}(\mathbb{P}, \tau) = 1$. For a fixed $\mathbb{P}$, $\mathbf{H}(\mathbb{P}, \tau)$ is defined from $\mathbf{H} \restriction \operatorname{tr\,cl}(\tau)$ using concepts absolute for transitive models of ZF $-$ P, so $\mathbf{H}$ is absolute for transitive models of ZF $-$ P. (We are using IV 5.6, where $x \mathbf{R} y$ iff $x \in \operatorname{tr\,cl}(y)$). Thus also, the concept "τ is a $\mathbb{P}$-name" is absolute for transitive models of ZF $-$ P. For more practice in such recursions, see III, Exercises 13 and 14.

2.6. DEFINITION. $\mathbf{V}^{\mathbb{P}}$ is the class of $\mathbb{P}$-names. If M is a transitive model of ZFC and $\mathbb{P} \in M$, $M^{\mathbb{P}} = \mathbf{V}^{\mathbb{P}} \cap M$. Or, by absoluteness,

$$M^{\mathbb{P}} = \{\tau \in M : (\tau \text{ is a } \mathbb{P}\text{-name})^M\}. \quad \square$$

When forcing over M, use is made only of the $\mathbb{P}$-names in $M^{\mathbb{P}}$, which we may think of as being defined within M.

2.7. DEFINITION. $\mathrm{val}(\tau, G) = \{\mathrm{val}(\sigma, G) : \exists p \in G\,(\langle \sigma, p\rangle \in \tau)\}$. We also write τ_G for $\mathrm{val}(\tau, G)$. $\square$

$\mathrm{val}(\tau, G)$ is defined by transfinite recursion on τ, as was "τ is a $\mathbb{P}$-name".

2.8. DEFINITION. If M is a transitive model of ZFC, $\mathbb{P} \in M$, and $G \subset \mathbb{P}$, then

$$M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}. \quad \square$$

$\mathrm{dom}(\tau) = \{\sigma : \exists p\,(\langle \sigma, p\rangle \in \tau)\}$, the usual definition of domain (although τ is usually not a function). By absoluteness, the M-people know $\mathrm{dom}(\tau)$, and they may think of $\mathrm{dom}(\tau)$ as a set of names for objects which may possibly be in τ_G.

$\mathrm{val}(\tau, G)$ was defined by transfinite recursion using absolute concepts, and is thus absolute for transitive models of $\mathrm{ZF} - \mathrm{P}$ for the same reason "τ is a $\mathbb{P}$-name" was. Of course, the absoluteness of $\mathrm{val}(\tau, G)$ says nothing for M unless $G \in M$, which will usually be false. It does yield the following.

2.9. LEMMA. *Under the notation of Definition* 2.8, *if N is a transitive model of* ZFC *with $M \subset N$ and $G \in N$, then $M[G] \subset N$.*

PROOF. For each $\tau \in M^{\mathbb{P}}$, $\tau \in N$ and $G \in N$, so $\mathrm{val}(\tau, G) = (\mathrm{val}(\tau, G))^N \in N$. $\square$

Thus, once we check that $M[G]$ is indeed a transitive extension of M containing G and satisfying ZFC, it will be the least such extension.

We pause for some examples in our intended framework where M is a c.t.m. for ZFC and $\mathbb{P}$ is a p.o. in M. 0 is a $\mathbb{P}$-name, since it trivially satisfies Definition 2.5, and $0_G = 0$ for any G by Definition 2.7. If $p \in \mathbb{P}$, then $\{\langle 0, p\rangle\} \in M^{\mathbb{P}}$, and

$$\mathrm{val}(\{\langle 0, p\rangle\}, G) = \begin{cases} \{0\} & \text{if } p \in G, \\ 0 & \text{if } p \notin G. \end{cases}$$

There will always be generic G with $p \in G$ (by Lemma 2.3), and, assuming $\exists q \in \mathbb{P}\,(q \perp p)$, there will be generic G with $p \notin G$. Thus, τ_G can depend on

G. However, in some special cases, τ_G is independent of G. We saw $0_G = 0$, and, $\mathrm{val}(\{\langle 0, \mathbb{1}\rangle\}, G) = \{0\}$ for all generic G since any non-empty filter contains $\mathbb{1}$. More generally,

$$\mathrm{val}(\{\langle \sigma_i, \mathbb{1}\rangle \colon i \in I\}, G) = \{\mathrm{val}(\sigma_i, G) \colon i \in I\}.$$

This observation enables us to see that any element $x \in M$ is represented in a canonical way by a name, called $\check{x}$.

2.10. Definition. If $\mathbb{P}$ is a p.o., define the $\mathbb{P}$-name $\check{x}$ recursively by: $\check{x} = \{\langle \check{y}, \mathbb{1}_{\mathbb{P}}\rangle \colon y \in x\}$. □

Formally, $\check{x}$ depends on $\mathbb{1}_{\mathbb{P}}$ as well as x but the p.o. $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}\rangle$ will always be clear from context. Definition 2.10 is another definition by recursion and is easily seen to be absolute for transitive models of ZFC, so if $x \in M$, then $\check{x} \in M$. As examples $\check{0} = 0$, $\check{1} = \{0\}\check{} = \{\langle 0, \mathbb{1}\rangle\}$, $\check{2} = \{\langle \check{0}, \mathbb{1}\rangle, \langle \check{1}, \mathbb{1}\rangle\}$, etc. We just saw that $\mathrm{val}(\check{0}, G) = 0$ and $\mathrm{val}(\check{1}, G) = 1$.

2.11. Lemma. *If M is a transitive model of* ZFC, $\mathbb{P}$ *is a* p.o. *in M, and G is a non-empty filter on* $\mathbb{P}$, *then*

(a) $\forall x \in M\, [\check{x} \in M^{\mathbb{P}} \wedge \mathrm{val}(\check{x}, G) = x]$.

(b) $M \subset M[G]$.

Proof. For (a), absoluteness of $\check{}$ implies $\check{x} \in M^{\mathbb{P}}$. $\mathrm{val}(\check{x}, G) = x$ is proved by induction on x, using

$$\mathrm{val}(\check{x}, G) = \{\mathrm{val}(\check{y}, G) \colon y \in x\}.$$

(b) is immediate from (a). □

We may now see that $G \in M[G]$ by cooking up a name that represents it.

2.12. Definition. If $\mathbb{P}$ is a p.o., $\Gamma = \{\langle \check{p}, p\rangle \colon p \in \mathbb{P}\}$. □

Γ of course depends on $\mathbb{P}$, which will always be clear from context. Unlike names of the form $\check{x}$, the object named by Γ depends on G. By absoluteness, Γ is in M if $\mathbb{P}$ is.

2.13. Lemma. *Under the hypotheses of Lemma* 2.11, $\Gamma_G = G$. *Hence*, $G \in M[G]$.

Proof. $\Gamma_G = \{(\check{p})_G \colon p \in G\} = \{p \colon p \in G\} = G$. □

Two more easy facts about $M[G]$ are the following.

2.14. Lemma. *Under the hypotheses of Lemma* 2.11, $M[G]$ *is transitive.*

Proof. Immediate from Definitions 2.7 and 2.8. $\square$

2.15. Lemma. *Under the hypotheses of Lemma* 2.11,
(a) $\forall \tau \in M^{\mathbb{P}}\,(\mathrm{rank}(\tau_G) \leq \mathrm{rank}(\tau))$.
(b) $o(M[G]) = o(M)$.

Proof. (a) is by induction on τ. For (b), we have $M[G] \cap \mathbf{ON} \subset M \cap \mathbf{ON}$ by (a) and the fact that $\mathrm{rank}(\tau) \in M$ for all $\tau \in M$. Thus, $M[G] \cap \mathbf{ON} = M \cap \mathbf{ON}$ since $M \subset M[G]$. $\square$

As a further example, of building names, we check that $M[G]$ satisfies some of the easier axioms of ZFC. Thus, pairing holds because given $\sigma, \tau \in M^{\mathbb{P}}$, we can define a name $\mathrm{up}(\sigma, \tau) \in M^{\mathbb{P}}$ which always names $\{\sigma_G, \tau_G\}$.

2.16. Definition. (a) $\mathrm{up}(\sigma, \tau) = \{\langle \sigma, \mathbb{1}\rangle, \langle \tau, \mathbb{1}\rangle\}$.
(b) $\mathrm{op}(\sigma, \tau) = \mathrm{up}(\mathrm{up}(\sigma, \sigma), \mathrm{up}(\sigma, \tau))$. $\square$

2.17. Lemma. *Under the hypotheses of Lemma* 2.11, *if* $\sigma, \tau \in M^{\mathbb{P}}$, *then*
(a) $\mathrm{up}(\sigma, \tau) \in M^{\mathbb{P}}$ *and* $\mathrm{val}(\mathrm{up}(\sigma, \tau), G) = \{\sigma_G, \tau_G\}$.
(b) $\mathrm{op}(\sigma, \tau) \in M^{\mathbb{P}}$ *and* $\mathrm{val}(\mathrm{op}(\sigma, \tau), G) = \langle \sigma_G, \tau_G\rangle$. $\square$

2.18. Lemma. *Under the hypotheses of Lemma* 2.11, *the Axioms of Extensionality, Foundation, Pairing, and Union are true in* $M[G]$.

Proof. Extensionality holds because $M[G]$ is transitive, Foundation is true relativized to any class, and Pairing is immediate from Lemma 2.17(a). For Union, it is sufficient to show that if $a \in M[G]$, then there is a $b \in M[G]$ such that $\bigcup a \subset b$ (see IV 2.10). Fix $\tau \in M^{\mathbb{P}}$ such that $a = \tau_G$; let $\pi = \bigcup \mathrm{dom}(\tau)$; then $\pi \in M^{\mathbb{P}}$, so $b = \pi_G \in M[G]$. If c is any element of a, $c = \sigma_G$ for some $\sigma \in \mathrm{dom}(\tau)$. Since $\sigma \subset \pi$, $c = \sigma_G \subset \pi_G = b$ (by applying Definition 2.7 to σ and π). Thus, $\bigcup a \subset b$. $\square$

Observe that in proving Lemma 2.18, we did not show that $\bigcup a \in M[G]$, although this will follow once we show that $M[G]$ satisfies the Comprehension Axiom. For a direct proof, see Exercise A6. Observe also that we have not yet used the notion of generic in a non-trivial way; our last six lemmas are true for any $G \subset \mathbb{P}$, such that $\mathbb{1} \in G$. The fact that G intersects the dense sets of M becomes important in the development of the concept of forcing in §3, which is then used to show $M[G]$ satisfies ZFC in §4.

We conclude this section with some additional technical facts which will be useful later when we wish to show that a generic filter intersects some sets which are not dense.

2.19. DEFINITION. If $E \subset \mathbb{P}$ and $p \in \mathbb{P}$, then E is *dense below* p iff

$$\forall q \leq p \, \exists r \leq q \, (r \in E). \quad \square$$

2.20. LEMMA. *Assume that M is a transitive model of* ZFC, $\mathbb{P} \in M$, $E \subset \mathbb{P}$, *and $E \in M$. Let G be $\mathbb{P}$-generic over M; then*

(a) *Either $G \cap E \neq 0$, or $\exists q \in G \;\; \forall r \in E \, (r \perp q)$.*

(b) *If $p \in G$ and E is dense below p, then $G \cap E \neq 0$.*

PROOF. For (a), let

$$D = \{p\colon \exists r \in E \, (p \leq r)\} \cup \{q\colon \forall r \in E \, (r \perp q)\}.$$

D is dense, since if $q \in \mathbb{P}$, and $q \notin D$, then fix $r \in E$ with r and q compatible; if $p \leq r$ and $p \leq q$, then p is an extension of q in D. Thus, $G \cap D \neq 0$, which implies (a).

For (b), if $G \cap E = 0$, then, by (a), fix $q \in G$ with $\forall r \in E \, (r \perp q)$. Let $q' \in G$ with $q' \leq q$ and $q' \leq p$, and then, since E is dense below p, let $r \in E$ with $r \leq q'$; then $r \leq q$, contradicting $r \perp q$. $\square$

§3. Forcing

Let us consider first a specific example. Fix a c.t.m. M for ZFC, and let $\mathbb{P}$ be the set of finite partial functions from ω to 2 ordered by reverse inclusion (as in II §2, Example 5). $\mathbb{1}_{\mathbb{P}}$ is the empty function. $\langle \mathbb{P}, \leq, \mathbb{1} \rangle \in M$, since its definition is absolute for transitive models of ZFC (or ZF $-$ P).

If G is a filter on $\mathbb{P}$, $f_G = \bigcup G$ is a function with $\mathrm{dom}(f_G) \subset \omega$. For each n, we let, as in II §2, $D_n = \{p \in \mathbb{P}\colon n \in \mathrm{dom}(p)\}$; then D_n is dense, and $D_n \in M$ (again by absoluteness of its definition). Thus, if G is $\mathbb{P}$-generic over M, $G \cap D_n \neq 0$ for all n, so $\mathrm{dom}(f_G) = \omega$.

We now show $f_G \in M[G]$. Since $G \in M[G]$ and $f_G = \bigcup G$, $f_G \in M[G]$ will follow immediately from the absoluteness of $\bigcup$ for transitive models of ZF, once we have shown $M[G]$ satisfies ZFC (or enough of ZF $-$ P to obtain absoluteness of $\bigcup$). However, we may check $f_G \in M[G]$ directly. Let

$$\Phi = \{\langle (\langle n, m \rangle)\check{}, p \rangle\colon p \in \mathbb{P} \wedge n \in \mathrm{dom}(p) \wedge p(n) = m\}.$$

Since $\mathrm{val}((\langle n, m \rangle)\check{}, G) = \langle n, m \rangle$ (see Lemma 2.11),

$$\Phi_G = \{\langle n, m \rangle\colon \exists p \in G \, (n \in \mathrm{dom}(p) \wedge p(n) = m)\} = f_G.$$

Thus, $f_G \in M[G]$.

If G is $\mathbb{P}$-generic over M, then $G \notin M$ by Lemma 2.4. Also, $f_G \notin M$, for let $E = \{p\colon p \not\subset f\}$; then E is dense and $G \cap E = 0$. If $f_G \in M$, then also $E \in M$, contradicting the definition of generic. Note the similarity of this argument with the one for $\neg \mathrm{MA}(2^{\omega})$ (II 2.6).

We now bring in the idea of *forcing*. In II §2, we had the intuitive idea that elements $p \in \mathbb{P}$ were conditions which say something about G or some object (such as f_G above), which we plan to construct from G. We continue with this motivation, but now in the context of models, rather than of MA.

People living in M cannot construct a G which is $\mathbb{P}$-generic over M. They may believe on faith that there exists a being to whom their universe, M, is countable. Such a being will have a generic G and an $f_G = \bigcup G$. The people in M do not know what G and f_G are but they have names for them, Γ and Φ. They may also read the preceding few paragraphs and thus figure out certain properties of G and f_G; for example, f_G is a function from ω to 2. They do not know what $f_G(0)$ is, since that depends on the particular G chosen. But they can see that $f_G(0)$ will be 0 if $\{\langle 0,0\rangle\} \in G$ and 1 if $\{\langle 0,1\rangle\} \in G$. More generally, they can construct a *forcing language*, where a sentence ψ of the forcing language uses the names in $M^{\mathbb{P}}$ to assert something about $M[G]$; an example of such a ψ is $\Phi(\check{0}) = \check{1}$. The person in M may not know whether a given ψ is true in $M[G]$. The truth or falsity of ψ in $M[G]$ will in general depend on G. We write $p \Vdash \psi$ (*p forces ψ*) to mean that for all G which are $\mathbb{P}$-generic over M, if $p \in G$, then ψ is true in $M[G]$. For example,

$$\{\langle 0,0\rangle\} \Vdash \Phi(\check{0}) = \check{0}, \quad \text{and} \quad \{\langle 0,1\rangle\} \Vdash \Phi(\check{0}) = \check{1}.$$

Also,

$$\mathbb{1} \Vdash \Phi \text{ is a function from } \check{\omega} \text{ into } \check{2}, \quad \text{and} \quad \mathbb{1} \Vdash \Phi = \bigcup \Gamma;$$

i.e., these last two sentences are true for all generic G. Now, people living in M can figure out all the above forcing facts without even seeing a generic G. This illustrates the following.

Fact 1. It may be decided within M whether or not $p \Vdash \psi$.

This will be very important not only for proving that $M[G]$ satisfies ZFC, but for applying forcing later, since the people of M will have to be able to apply their combinatorial techniques to construct various complicated $\mathbb{P}$ for which the desired axioms of set theory (beyond ZFC) are forced to be true in $M[G]$.

Fact 1 is at first surprising, since the notion $p \Vdash \psi$ seems to require a knowledge of all generic G, but a person in M may always decide whether $p \Vdash \psi$ by going through the kind of analysis we have used in our examples.

It is immediate from the definition of $\Vdash$ that if G is $\mathbb{P}$-generic over M

and $p \Vdash \psi$ for some $p \in G$, then ψ is true in $M[G]$. As a converse of this observation, we shall show the following.

Fact 2. If G is $\mathbb{P}$-generic over M and ψ is true in $M[G]$, then for some $p \in G$, $p \Vdash \psi$.

For example, if ψ is $\Phi(\check{0}) = \check{0}$ and ψ is true (i.e., $f_G(0) = 0$), then $p(0) = 0$ for some $p \in G$. If this $p \in H$, where H is another generic filter, then $f_H(0) = 0$ also; i.e., ψ will be true in $M[H]$. Thus, $p \Vdash \psi$.

We now leave our specific example and turn to a more rigorous discussion of forcing with an arbitrary $\mathbb{P}$. The actual theorem expressing Facts 1 and 2 (Theorem 3.6) will form the backbone of our forcing technique.

3.1. Definition. Let $\phi(x_1, \dots, x_n)$ be a formula with all free variables shown; let M be a c.t.m. for ZFC, $\mathbb{P}$ a p.o. in M, $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$, and $p \in \mathbb{P}$; then $p \Vdash_{\mathbb{P},M} \phi(\tau_1, \dots, \tau_n)$ iff

$$\forall G\,[(G \text{ is } \mathbb{P}\text{-generic over } M \;\wedge\; p \in G) \rightarrow \phi^{M[G]}(\mathrm{val}(\tau_1, G), \dots, \mathrm{val}(\tau_n, G))]. \qquad \square \quad (1)$$

The subscript $\mathbb{P}$ on $\Vdash_{\mathbb{P},M}$ should really be $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$. We shall usually just write $\Vdash$ when there is only one partial order and one ground model M under discussion.

Intuitively, the $\phi(\tau_1, \dots, \tau_n)$ in Definition 3.1 is a sentence of the *forcing language*; this idea could be made rigorous by formalizing logic within set theory and defining the forcing language to be the first-order language whose one binary relation symbol is $\in$, and whose constant symbols are the elements of $M^{\mathbb{P}}$. Instead, our approach is not actually to define a forcing language. Formally, 3.1 is a definition schema in the metatheory. For each formula $\phi(x_1, \dots, x_n)$, with free variables among $x_1, \dots, x_n$, we can define another formula $\mathrm{Force}_\phi(\tau_1, \dots, \tau_n, \mathbb{P}, \leq, \mathbb{1}, M, p)$, which asserts (1), along with $\langle \mathbb{P}, \leq, \mathbb{1} \rangle \in M$, $p \in \mathbb{P}$, and $\tau_1, \dots, \tau_n \in M^{\mathbb{P}}$.

As an exercise in understanding Definition 3.1, one may verify the following.

3.2. Lemma. *In the notation of Definition* 3.1,

(a) $(p \Vdash \phi(\tau_1, \dots, \tau_n) \wedge q \leq p) \rightarrow q \Vdash \phi(\tau_1, \dots, \tau_n)$.

(b) $(p \Vdash \phi(\tau_1, \dots, \tau_n)) \wedge (p \Vdash \psi(\tau_1, \dots, \tau_n))$ *iff*
$p \Vdash (\phi(\tau_1, \dots, \tau_n) \wedge \psi(\tau_1, \dots, \tau_n))$. $\square$

Now, the notion "$p \Vdash \phi(\tau_1, \dots, \tau_n)$" has been defined in $\mathbf{V}$, not M, and involves a knowledge of all possible generic G. By Fact 1, we should be able to decide within M whether $p \Vdash \phi(\tau_1, \dots, \tau_n)$; we translate this rigorously by defining another relation, $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$ and showing that for all ϕ,

$$p \Vdash \phi(\tau_1, \ldots, \tau_n) \leftrightarrow (p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M.$$

Thus, $p \Vdash \phi(\tau_1, \ldots, \tau_n)$ will be equivalent to some statement relativized to M.

After this section, we shall rarely refer back to the details of the definition of $\Vdash^*$, although we shall frequently use Facts 1 and 2 (Theorem 3.6) and their Corollary, 3.7. Thus, the reader who is bored by these details may simply skip directly to Theorem 3.6. There are as many different (equivalent) definitions of $\Vdash^*$ as there are texts in set theory; see, e.g., Exercises B3 and B4 for a somewhat slicker approach.

The most difficult part of our definition of $\Vdash^*$ will be when $\phi(\tau_1, \tau_2)$ is $\tau_1 = \tau_2$. As a simple example of what to expect, suppose $\tau_1 = \{\langle \pi_1, s \rangle\}$ and $\tau_2 = \{\langle \pi_2, s \rangle\}$, and we are trying to tell a person in M which p force $\tau_1 = \tau_2$. If $p \perp s$, then $p \Vdash \tau_1 = \tau_2$, since whenever $p \in G$, $s \notin G$, so $\tau_{1_G} = 0 = \tau_{2_G}$. If $p \leq s$, then whenever $p \in G$, $\tau_{1_G} = \{\pi_{1_G}\}$ and $\tau_{2_G} = \{\pi_{2_G}\}$, so $p \Vdash \tau_1 = \tau_2$ iff $p \Vdash \pi_1 = \pi_2$. It is instructive to check that for any p, $p \Vdash \tau_1 = \tau_2$ iff

$$\forall q (q \leq p \wedge q \leq s \rightarrow q \Vdash \pi_1 = \pi_2),$$

but we mainly wish to emphasize that in the definition of $\Vdash^*$, the question of whether $p \Vdash^* \tau_1 = \tau_2$ must depend on whether $q \Vdash^* \pi_1 = \pi_2$ for various $q \in \mathbb{P}$, $\pi_1 \in \mathrm{dom}(\tau_1)$, and $\pi_2 \in \mathrm{dom}(\tau_2)$.

We begin by defining, in $\mathbf{V}$, the notion $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$; so this definition does not mention any model. However, in our intended application, we shall consider only the relativized notion $(p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M$, where M is the ground model.

3.3. Definition. Fix a p.o. $\mathbb{P}$. The following clauses define the notion $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$ where $\phi(x_1, \ldots, x_n)$ is a formula with all free variables shown, $p \in \mathbb{P}$, and $\tau_1, \ldots, \tau_n \in \mathbf{V}^{\mathbb{P}}$.

(a) $p \Vdash^* \tau_1 = \tau_2$ iff

(α) for all $\langle \pi_1, s_1 \rangle \in \tau_1$,

$$\{q \leq p \colon q \leq s_1 \rightarrow \exists \langle \pi_2, s_2 \rangle \in \tau_2 (q \leq s_2 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

is dense below p, and

(β) for all $\langle \pi_2, s_2 \rangle \in \tau_2$,

$$\{q \leq p \colon q \leq s_2 \rightarrow \exists \langle \pi_1, s_1 \rangle \in \tau_1 (q \leq s_1 \wedge q \Vdash^* \pi_1 = \pi_2)\}$$

is dense below p.

(b) $p \Vdash^* \tau_1 \in \tau_2$ iff

$$\{q \colon \exists \langle \pi, s \rangle \in \tau_2 (q \leq s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p.

(c) $p \Vdash^* (\phi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n))$ iff

$$p \Vdash^* \phi(\tau_1, \ldots, \tau_n) \quad \text{and} \quad p \Vdash^* \psi(\tau_1, \ldots, \tau_n).$$

(d) $p \Vdash^* \neg \phi(\tau_1, \ldots, \tau_n)$ iff there is no $q \leq p$ such that $q \Vdash^* \phi(\tau_1, \ldots, \tau_n)$.

(e) $p \Vdash^* \exists x\, \phi(x, \tau_1, \ldots, \tau_n)$ iff

$$\{r \colon \exists \sigma \in V^{\mathbb{P}}\, (r \Vdash^* \phi(\sigma, \tau_1, \ldots, \tau_n))\}$$

is dense below p. □

A casual inspection of Definition 3.3 will reveal that the definition is circular, and must thus be a recursion, but the exact nature of this recursion deserves some additional comment. It is intended that clause (a) be applied first to define the notion $p \Vdash^* \tau_1 = \tau_2$. Formally, we are defining a function $\mathbf{F} : V^{\mathbb{P}} \times V^{\mathbb{P}} \to \mathscr{P}(\mathbb{P})$, where $\mathbf{F}(\langle \tau_1, \tau_2 \rangle)$ is intended to be

$$\{p \in \mathbb{P} \colon p \Vdash^* \tau_1 = \tau_2\}.$$

$\mathbf{F}$ is defined by transfinite recursion on the relation $\mathbf{R}$, where

$$\langle \pi_1, \pi_2 \rangle\, \mathbf{R}\, \langle \tau_1, \tau_2 \rangle$$

iff $\pi_1 \in \mathrm{dom}(\tau_1)$ and $\pi_2 \in \mathrm{dom}(\tau_2)$. $\mathbf{R}$ is clearly set-like, and $\mathbf{R}$ is well-founded because $\langle \pi_1, \pi_2 \rangle\, \mathbf{R}\, \langle \tau_1, \tau_2 \rangle$ implies $\mathrm{rank}(\pi_1) < \mathrm{rank}(\tau_1)$.

Once the notion $p \Vdash^* \tau_1 = \tau_2$ is defined, clause (b) defines the notion $p \Vdash^* \tau_1 \in \tau_2$ explicitly. Now that $\Vdash^*$ is defined for atomic formulas, clauses (c)–(e) define $\Vdash^*$ for all formulas by a straightforward induction on length. Formally, the induction takes place in the metatheory. As with $\Vdash$, for each formula $\phi(x_1, \ldots, x_n)$ we are defining a formula

$$\mathrm{Force}^*_\phi(\tau_1, \ldots, \tau_n, \mathbb{P}, \leq, p).$$

For atomic formulas, the recursion used in defining $\Vdash^*$ involves only absolute concepts and is thus absolute for transitive models of ZF − P. More precisely, we are using the absoluteness of $\mathbf{R}$ (see above), plus the absoluteness of $\{\langle \pi_1, \pi_2 \rangle \colon \langle \pi_1, \pi_2 \rangle\, \mathbf{R}\, \langle \tau_1, \tau_2 \rangle\}$ to conclude the absoluteness of $\mathbf{F}$ (see IV 5.6). However, $\Vdash^*$ for arbitrary ϕ is not absolute; the $\exists \sigma \in \mathbf{V}^{\mathbb{P}}$ in clause (e) becomes $\exists \sigma \in M^{\mathbb{P}}$ when relativized to a model M. In any case, in checking Fact 1, we are only interested in looking at $\Vdash^*$ relativized to M.

As motivation for the specific details of clauses (a)–(e) of the definition of $\Vdash^*$, we may think of $(p \Vdash^* \phi)^M$ as an attempt by a person living in M to decide $\Vdash$. We shall eventually prove Fact 1, that $\Vdash$ is definable in M, by showing that $p \Vdash \phi$ iff $(p \Vdash^* \phi)^M$. Thus, we use, as the inductive clauses in the definition of $\Vdash^*$, relations which $\Vdash$ itself satisfies. We can then try to prove Fact 1 by induction on ϕ.

To see that $\Vdash$ indeed satisfies (a)–(e) sometimes requires some argument. For (c), it is immediate from the definition of $\Vdash$ that $p \Vdash (\phi \wedge \psi)$ iff $p \Vdash \phi$ and $p \Vdash \psi$ (see Lemma 3.2). Regarding (d), assume that $\neg \exists q \leq p (q \Vdash \phi)$; to show $p \Vdash \neg \phi$, assume not. Then, there is a generic G with $p \in G$ and ϕ true in M. By Fact 2, there is an $r \in G$ such that $r \Vdash \phi$. Let $q \in G$ with $q \leq r$ and $q \leq p$; then $q \Vdash \phi$ (by Lemma 3.2), contradicting $\neg \exists q \leq p (q \Vdash \phi)$.

Clause (e), relativized to M, says $(p \Vdash^* \exists x \, \phi(x))^M$ iff

$$\{r \leq p \colon \exists \sigma \in M^{\mathbb{P}} \, (r \Vdash^* \phi(\sigma))^M\}$$

is dense below p. To check this (in one direction) for $\Vdash$, suppose $D = \{r \leq p \colon \exists \sigma \in M^{\mathbb{P}} \, (r \Vdash \phi(\sigma))\}$ is dense below p. By Fact 1, $D \in M$. Thus, whenever G is generic over M and $p \in G$, $G \cap D \neq 0$, so there is a $\sigma \in M^{\mathbb{P}}$ and $r \in G$ with $r \Vdash \phi(\sigma)$; the $(\phi(\sigma_G))^{M[G]}$, so $(\exists x \, \phi(x))^{M[G]}$. Thus, $p \Vdash \exists x \, \phi(x)$.

Of course, the arguments in the preceding two paragraphs are useful only for motivation, since to verify that $\Vdash$ satisfies clauses (d) and (e), we are appealing to Facts 1 and 2, which have not yet been proved. The reader may find it a useful exercise to complete our (circular) justification of Definition 3.3.

We now proceed on a somewhat different tack to obtain a rigorous proof of Facts 1 and 2. As a preliminary lemma, we prove the following.

3.4. LEMMA. *For p and $\phi(\tau_1, \ldots, \tau_n)$ as in Definition 3.3, the following are equivalent*:

(1) $p \Vdash^* \phi(\tau_1, \ldots, \tau_n)$.
(2) $\forall r \leq p \, (r \Vdash^* \phi(\tau_1, \ldots, \tau_n))$.
(3) $\{r \colon r \Vdash^* \phi(\tau_1, \ldots, \tau_n)\}$ *is dense below p.*

PROOF. Observe first that (2) → (1) and (2) → (3) are trivial. Next, assume $\phi(\tau_1, \tau_2)$ is either $\tau_1 = \tau_2$ or $\tau_1 \in \tau_2$. (1) → (2) follows from the fact that if D is dense below p and $r \leq p$, then D is dense below r. (3) → (1) follows from the fact that if $\{r \colon D \text{ is dense below } r\}$ is dense below p, then D is dense below p. Note that in both cases, we do not refer to the precise details of the definition of $p \Vdash^* \phi(\tau_1, \tau_2)$; rather, we need only that the definition involves certain sets being dense below p.

Now that the equivalence of (1)–(3) has been checked for atomic ϕ, it is easily checked for all ϕ by induction, using clauses (c)–(e) of Definition 3.3. The only place the inductive hypothesis is used in this argument is in the step for $\wedge$. □

We now express the relationship between $\Vdash^*$ and truth in $M[G]$. This is the key to relating $\Vdash^*$ to $\Vdash$.

3.5. THEOREM. *Let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown.*

Let M be a transitive model for ZFC, $\mathbb{P}$ *a* p.o. *in M, and $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$. Let G be $\mathbb{P}$-generic over M; then*

(1) *If $p \in G$ and $(p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M$, then $(\phi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G)))^{M[G]}$.*

(2) *If $\phi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G))^{M[G]}$, then $\exists p \in G((p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M)$.*

PROOF. When $\phi(\tau_1, \tau_2)$ is $\tau_1 = \tau_2$, the proof proceeds by transfinite induction, using clause (a) of Definition 3.3. The fact that this is indeed an induction on a well-founded relation is seen in precisely the same way that we justified the definition of $\Vdash^*$ for such ϕ. Since $\Vdash^*$ for atomic formulas is absolute for M, we may drop the relativizations to M.

To check (1), we assume $p \in G$ and $p \Vdash^* \tau_1 = \tau_2$. We must show $\tau_{1_G} = \tau_{2_G}$. We shall show $\tau_{1_G} \subset \tau_{2_G}$ using (α) of Definition 3.3(a); the proof of $\tau_{2_G} \subset \tau_{1_G}$ using (β) is the same. Every element of τ_{1_G} is of the form π_{1_G}, where $\langle \pi_1, s_1 \rangle \in \tau_1$ for some $s_1 \in G$. We must show that $\pi_{1_G} \in \tau_{2_G}$. Fix $r \in G$ with $r \le p$ and $r \le s_1$. Then $r \Vdash^* \tau_1 = \tau_2$ (by Lemma 3.4), so (by Lemma 2.20(b)), there is $q \in G$ such that $q \le r$ and such that $q \le s_1$ implies

$$\exists \langle \pi_2, s_2 \rangle \in \tau_2 (q \le s_2 \ \wedge \ q \Vdash^* \pi_1 = \pi_2). \qquad (*)$$

But $q \le s_1$, so fix $\langle \pi_2, s_2 \rangle$ as in ($*$); then $s_2 \in G$, so $\pi_{2_G} \in \tau_{2_G}$. Also, by (1) for $\pi_1 = \pi_2$, $q \Vdash^* \pi_1 = \pi_2$ implies $\pi_{1_G} = \pi_{2_G}$, so $\pi_{1_G} \in \tau_{2_G}$.

To check (2), assume $\tau_{1_G} = \tau_{2_G}$. Let D be the set of all $r \in \mathbb{P}$ such that either $r \Vdash^* \tau_1 = \tau_2$, or

(α') $\exists \langle \pi_1, s_1 \rangle \in \tau_1 (r \le s_1 \ \wedge \ \forall \langle \pi_2, s_2 \rangle \in \tau_2 \ \forall q \in \mathbb{P}$

$$((q \le s_2 \ \wedge \ q \Vdash^* \pi_1 = \pi_2) \to q \perp r)),$$

or

(β') $\exists \langle \pi_2, s_2 \rangle \in \tau_2 (r \le s_2 \ \wedge \ \forall \langle \pi_1, s_1 \rangle \in \tau_1 \ \forall q \in \mathbb{P}$

$$((q \le s_1 \ \wedge \ q \Vdash^* \pi_1 = \pi_2) \to q \perp r)).$$

First note that no $r \in G$ can satisfy (α') or (β'), for suppose $r \in G$ and $\langle \pi_1, s_1 \rangle \in \tau_1$ as in (α'); then $s_1 \in G$ so $\pi_{1_G} \in \tau_{1_G} = \tau_{2_G}$, so fix $\langle \pi_2, s_2 \rangle \in \tau_2$ with $s_2 \in G$ and $\pi_{1_G} = \pi_{2_G}$; then, by (2) for $\pi_1 = \pi_2$, fix $q_0 \in G$ with $q_0 \Vdash^* \pi_1 = \pi_2$; now fix $q \in G$ with $q \le q_0$ and $q \le s_2$; since $q \Vdash^* \pi_1 = \pi_2$ (see Lemma 3.4), we have $q \perp r$ (by (α')), $q \in G$, and $r \in G$, a contradiction. If $\neg \exists r \in G (r \Vdash^* \tau_1 = \tau_2)$, then $D \cap G = 0$. Since $D \in M$ by absoluteness, we shall be done if we can check that D is dense. Fix $p \in \mathbb{P}$. Either $p \Vdash^* \tau_1 = \tau_2$ or (α) or (β) of Definition 3.3 (a) fails. If (α) fails, then, applying the definition of "dense below p," fix $\langle \pi_1, s_1 \rangle \in \tau_1$ and $r \le p$ such that

$$\forall q \le r (q \le s_1 \ \wedge \ \forall \langle \pi_2, s_2 \rangle \in \tau_2 (\neg (q \Vdash^* \pi_1 = \pi_2 \ \wedge \ q \le s_2))). \qquad (\dagger)$$

In particular, $r \le s_1$. If $\langle \pi_2, s_2 \rangle \in \tau_2$, $q \le s_2$, and $q \Vdash^* \pi_1 = \pi_2$ then $q \perp r$, since a common extension q' of q and r would contradict ($\dagger$). Thus, $r \le p$

and r satisfies (α'). Likewise, if (β) fails, there is an $r \leq p$ satisfying (β').

Now, assume $\phi(\tau_1, \tau_2)$ is $\tau_1 \in \tau_2$. To check (1), assume $p \in G$ and $p \Vdash^* \tau_1 \in \tau_2$; then

$$D = \{q\colon \exists \langle \pi, s\rangle \in \tau_2 (q \leq s \wedge q \Vdash^* \pi = \tau_1)\}$$

is dense below p, so fix $q \in G \cap D$, and fix $\langle \pi, s\rangle \in \tau_2$ so that $q \leq s$ and $q \Vdash^* \pi = \tau_1$. Since $s \in G$ and $\langle \pi, s\rangle \in \tau_2$, $\pi_G \in \tau_{2_G}$ by definition of τ_{2_G}. Since $q \in G$ and $q \Vdash^* \pi = \tau_1$, $\pi_G = \tau_{1_G}$ by (1) applied to $\pi = \tau_1$. Thus, $\tau_{1_G} \in \tau_{2_G}$.

To check (2) for $\tau_1 \in \tau_2$, assume $\tau_{1_G} \in \tau_{2_G}$. By definition of τ_{2_G}, there is a $\langle \pi, s\rangle \in \tau_2$ such that $s \in G$ and $\pi_G = \tau_{1_G}$. By (2) for $\pi = \tau_1$, there is an $r \in G$ such that $r \Vdash^* \pi = \tau_1$. Let $p \in G$ be such that $p \leq s$ and $p \leq r$. Then $\forall q \leq p (q \leq s \wedge q \Vdash^* \pi = \tau_1)$, so $p \Vdash^* \tau_1 \in \tau_2$ (we have verified a statement stronger than that required by Definition 3.3(b)).

This concludes the proof of (1) and (2) for atomic ϕ. We now prove (1) and (2) simultaneously for all ϕ by induction on ϕ; formally, this induction takes place in the metatheory. There are six parts to this, since the induction steps must be done for $\neg$, $\wedge$, and $\exists$, and (1) and (2) must be checked. Since $\Vdash^*$ is not absolute when ϕ has quantifiers, it is now important that we relativize $\Vdash^*$ to M.

In the following, we shall, for brevity, drop explicit mention of the $\tau_1, \ldots, \tau_n$, since they may easily be filled in.

(1) $\neg$: We assume (1) and (2) for ϕ, and we conclude (1) for $\neg\phi$. Assume $p \in G$ and $(p \Vdash^* \neg\phi)^M$; we must show $\neg\phi^{M[G]}$. But if $\phi^{M[G]}$, then by (2) for ϕ, there is a $q \in G$ with $(q \Vdash^* \phi)^M$. Let $r \in G$ with $r \leq p$ and $r \leq q$; then $(r \Vdash^* \phi)^M$, contradicting the definition of $p \Vdash^* \neg\phi$.

(2) $\neg$: Assume $(\neg\phi)^{M[G]}$, and let

$$D = \{p\colon (p \Vdash^* \phi)^M \vee (p \Vdash^* \neg\phi)^M\}.$$

$D \in M$ and D is dense by the definition of $\Vdash^*$ applied within M, so fix $p \in D \cap G$. If $(p \Vdash^* \neg\phi)^M$, we are done. If $(p \Vdash^* \phi)^M$, then by (1) for ϕ, we have $\phi^{M[G]}$, a contradiction.

(1) $\wedge$: We assume (1) and (2) for ϕ and ψ, and we conclude (1) for $\phi \wedge \psi$. Assume $p \in G$ and $(p \Vdash^* (\phi \wedge \psi))^M$; then $(p \Vdash^* \phi)^M$ and $(p \Vdash^* \psi)^M$, so $\phi^{M[G]}$ and $\psi^{M[G]}$, so $(\phi \wedge \psi)^{M[G]}$.

(2) $\wedge$: Assume $(\phi \wedge \psi)^{M[G]}$. By (2) for ϕ and ψ, there are $p, q \in G$ such that $(p \Vdash^* \phi)^M$ and $(q \Vdash^* \psi)^M$. Let $r \in G$ be such that $r \leq p$ and $r \leq q$; then $(r \Vdash^* \phi)^M$ and $(r \Vdash^* \psi)^M$, so $(r \Vdash^* \phi \wedge \psi)^M$.

(1) $\exists$: Assume $p \in G$ and $(p \Vdash^* \exists x\, \phi(x))^M$; then

$$\{r\colon \exists \sigma \in M^{\mathbb{P}} (r \Vdash^* \phi(\sigma))^M\}$$

is dense below p and in M, so fix $r \in G$ and $\sigma \in M^{\mathbb{P}}$ with $(r \Vdash^* \phi(\sigma))^M$. By (1) for ϕ, $(\phi(\sigma_G))^{M[G]}$, so $(\exists x\, \phi(x))^{M[G]}$.

(2) $\exists$: Assume $(\exists x\, \phi(x))^{M[G]}$ and fix $\sigma \in M^{\mathbb{P}}$ with $(\phi(\sigma_G))^{M[G]}$. By (2) for ϕ, fix $p \in G$ so that $(p \Vdash^* \phi(\sigma))^M$; then $\forall r \le p\, ((r \Vdash^* \phi(\sigma))^M)$, so $(p \Vdash^* \exists x\, \phi(x))^M$ (we have verified a statement stronger than that required by Definition 3.3(e)). $\square$

Finally, we may state and prove Facts 1 and 2 formally.

3.6. THEOREM. *Let M be a* c.t.m. *for* ZFC *and* $\mathbb{P}$ *a* p.o. *in M; let $\phi(x_1, \ldots, x_n)$ be a formula with all free variables shown; let $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$.*

(1) *For all $p \in \mathbb{P}$,*

$$p \Vdash \phi(\tau_1, \ldots, \tau_n) \leftrightarrow (p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M.$$

(2) *For all G which are $\mathbb{P}$-generic over M,*

$$\phi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]} \leftrightarrow \exists p \in G\, (p \Vdash \phi(\tau_1, \ldots, \tau_n)).$$

PROOF. In (1), the implication from right to left is immediate from Theorem 3.5(1) and the definition of $\Vdash$. For the implication from left to right, assume $p \Vdash \phi(\tau_1, \ldots, \tau_n)$. To show $(p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M$, it is sufficient (by Lemma 3.4) to show that $D = \{r\colon (r \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M$ is dense below p. If not, let $q \le p$ be such that $\neg \exists r \le q\, (r \in D)$. Then, by definition of $\Vdash^*$,

$$(q \Vdash^* \neg \phi(\tau_1, \ldots, \tau_n))^M,$$

whence, by (1) from right to left, $q \Vdash \neg\phi(\tau_1, \ldots, \tau_n)$. Let G be $\mathbb{P}$-generic over M with $q \in G$; then $(\neg \phi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G)))^{M[G]}$, but also $p \in G$, since $p \ge q$, so $(\phi(\mathrm{val}(\tau_1, G), \ldots, \mathrm{val}(\tau_n, G)))^{M[G]}$, a contradiction.

For (2), the implication from left to right follows from (1) and from Theorem 3.5(2), which asserts the same thing about $\Vdash^*$. The implication from right to left is immediate from the definition of $\Vdash$. $\square$

In practice, Theorem 3.6(1) will be used to show that various sets defined using $\Vdash$ actually lie in M. For example for fixed $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$

$$\{p \in \mathbb{P}\colon p \Vdash \phi(\tau_1, \ldots, \tau_n)\}$$

in in M, since this set is equal to

$$\{p \in \mathbb{P}\colon (p \Vdash^* \phi(\tau_1, \ldots, \tau_n))^M\},$$

which lies in M by Comprehension in M. Likewise, e.g., for fixed σ, $\tau_2, \ldots, \tau_n \in M^{\mathbb{P}}$,

$$\{\langle p, \tau_1\rangle \in \mathbb{P} \times \mathrm{dom}(\sigma)\colon p \Vdash \phi(\tau_1, \ldots, \tau_n)\} \in M.$$

Theorem 3.6(2) will be important because it relates truth in $M[G]$ to $\Vdash$. The following additional facts about $\Vdash$ will also be useful.

3.7. COROLLARY. *Let M be a* c.t.m. *for* ZFC, $\mathbb{P}$ *a* p.o. *in M, and* $\sigma, \tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$; *then*

(a) $\{p \in \mathbb{P}: (p \Vdash \phi(\tau_1, \ldots, \tau_n)) \vee (p \Vdash \neg\phi(\tau_1, \ldots, \tau_n))\}$ *is dense.*

(b) $p \Vdash \neg\phi(\tau_1, \ldots, \tau_n)$ *iff* $\neg\exists q \leq p\,(q \Vdash \phi(\tau_1, \ldots, \tau_n))$.

(c) $p \Vdash \exists x\, \phi(x, \tau_1, \ldots, \tau_n)$ *iff*

$$\{r \leq p\colon \exists \sigma \in M^{\mathbb{P}}\,(r \Vdash \phi(\sigma, \tau_1, \ldots, \tau_n))\}$$

is dense below p.

(d) *If* $p \Vdash \exists x\,(x \in \sigma \wedge \phi(x, \tau_1, \ldots, \tau_n))$, *then*

$$\exists q \leq p\, \exists \pi \in \mathrm{dom}(\sigma)\,(q \Vdash \phi(\pi, \tau_1, \ldots, \tau_n)).$$

PROOF. (a)–(c) are true of $\Vdash^*$ by definition, and thus hold for $\Vdash$ by Theorem 3.6(1). For (d), fix a generic G with $p \in G$. By definition of $\Vdash$, there is an $a \in \sigma_G$ such that $(\phi(a, \tau_1, \ldots, \tau_n))^{M[G]}$. $a = \pi_G$ for some $\pi \in \mathrm{dom}(\sigma)$. By Theorem 3.6(2), there is an $r \in G$ such that $r \Vdash \phi(\pi, \tau_1, \ldots, \tau_n)$. If q is a common extension of p and r, then $q \leq p$ and $q \Vdash \phi(\pi, \tau_1, \ldots, \tau_n)$. □

§4. ZFC in $M[G]$

We now apply the results of §3 to show that our generic extension is a model of ZFC. It will be convenient to verify AC in a form slightly different from the usual one.

4.1. LEMMA (ZF). AC *holds iff*

$$\forall x\, \exists \alpha \in \mathbf{ON}\, \exists f\,(f \textit{ is a function} \wedge \mathrm{dom}(f) = \alpha \wedge x \subset \mathrm{ran}(f)). \qquad (*)$$

PROOF. If x, α, and f are as in $(*)$, we may define a well-order of x as follows. Let $g(z) = \min(f^{-1}\{z\})$ for $z \in x$; then g maps x 1–1 into α. If we let $y\,R\,z \leftrightarrow g(y) < g(z)$, then R well-orders x. □

4.2. THEOREM. *Let M be a* c.t.m. *for* ZFC, $\langle \mathbb{P}, \leq, \mathbb{1} \rangle$ *a* p.o. *in M, and G $\mathbb{P}$-generic over M; then $M[G]$ satisfies* ZFC.

PROOF. We have already verified Extensionality, Foundation, Pairing, and Union (see Lemma 2.18). Let us check Comprehension. To do this we must see that whenever $\sigma, \tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$ and $\phi(x, v, y_1, \ldots, y_n)$ is any formula,

$$\{a \in \sigma_G\colon (\phi(a, \sigma_G, \tau_{1G}, \ldots, \tau_{nG}))^{M[G]}\} \in M[G].$$

Let

$$\rho = \{\langle \pi, p \rangle \in \mathrm{dom}(\sigma) \times \mathbb{P}\colon p \Vdash (\pi \in \sigma \wedge \phi(\pi, \sigma, \tau_1, \ldots, \tau_n))\}.$$

$\rho \in M^{\mathbb{P}}$ by definability of forcing (Theorem 3.6(1)). We now verify that $\rho_G = \{a \in \sigma_G\colon \phi(a)^{M[G]}\}$; for brevity, we suppress mention of $\tau_1, \ldots, \tau_n$ in the rest of this argument. First, any element of ρ_G is of the form π_G where $\langle \pi, p\rangle \in \rho$ for some $p \in G$. By definition of ρ, $p \Vdash (\pi \in \sigma \wedge \phi(\pi))$ so, by the definition of $\Vdash$, $\pi_G \in \sigma_G$ and $\phi(\pi_G)^{M[G]}$. Thus $\rho_G \subset \{a \in \sigma_G\colon \phi(a)^{M[G]}\}$. To show equality, assume $a \in \sigma_G$ and $\phi(a)^{M[G]}$. $a = \pi_G$ for some $\pi \in \mathrm{dom}(\sigma)$. Then $(\pi_G \in \sigma_G \wedge \phi(\pi_G))^{M[G]}$. Since any statement true in $M[G]$ is forced (Theorem 3.6(2)), there is a $p \in G$ such that $p \Vdash (\pi \in \sigma \wedge \phi(\pi))$; then $\langle \pi, p\rangle \in \rho$, so $\pi_G \in \rho_G$.

Next, we verify Replacement. For this we must check that for each formula $\phi(x, v, r, z_1, \ldots, z_n)$ and each $\sigma_G, \tau_{1_G}, \ldots, \tau_{n_G} \in M[G]$, if

$$(\forall x \in \sigma_G\, \exists! y\, \phi(x, y, \sigma_G, \tau_{1_G}, \ldots, \tau_{n_G}))^{M[G]},$$

then there is a $\rho \in M^{\mathbb{P}}$ such that

$$\forall x \in \sigma_G\, \exists y \in \rho_G\, (\phi(x, y, \sigma_G, \tau_{1_G}, \ldots, \tau_{n_G}))^{M[G]}.$$

Again, suppress mention of $\tau_1, \ldots, \tau_n$. Let $S \in M$ be such that $S \subset M^{\mathbb{P}}$ and

$$\forall \pi \in \mathrm{dom}(\sigma)\, \forall p \in \mathbb{P}\, [\exists \mu \in M^{\mathbb{P}}\, (p \Vdash \phi(\pi, \mu)) \to \exists \mu \in S\, (p \Vdash \phi(\pi, \mu))];$$

S exists because by Theorem 3.6(1), $p \Vdash \phi(\pi, \mu)$ is defined by a formula relativized to M, so by reflection in M we may take $S = R(\alpha)^{(M)} \cap M^{\mathbb{P}}$ for a suitable α (see IV 7.4). Let $\rho = S \times \{\mathbb{1}\}$; then $\rho_G = \{\mu_G\colon \mu \in S\}$. Fix $x \in \sigma_G$. We show $\exists y \in \rho_G\, (\phi(x, y))^{M[G]}$. $x = \pi_G$ for some $\pi \in \mathrm{dom}(\sigma)$. By assumption, $(\exists y\, \phi(\pi_G, y))^{M[G]}$, so for some $v \in M^{\mathbb{P}}$, $\phi(\pi_G, v_G)^{M[G]}$, and by Theorem 3.6(2), there is a $p \in G$ such that $p \Vdash \phi(\pi, v)$. There is then a $\mu \in S$ such that $p \Vdash \phi(\pi, \mu)$, so we have $\mu_G \in \rho_G$ and $(\phi(\pi_G, \mu_G))^{M[G]}$.

We remark that it looks like we have proved a stronger form of Replacement which weakens the $\exists! y$ in the hypothesis to $\exists y$. But this "stronger" axiom is in fact a version of reflection and is derivable in ZF (see III Exercise 15).

We have now checked all axioms of ZF $-$ P in $M[G]$ except Infinity. But now Infinity holds also, since $\omega (= (\check{\omega})_G)$ is in $M[G]$. Thus, $M[G]$ satisfies ZF $-$ P.

For the Power Set Axiom, fix $\sigma_G \in M[G]$. We shall produce a $\rho \in M^{\mathbb{P}}$ such that $\forall x \in M[G]\, (x \subset \sigma_G \to x \in \rho_G)$. Let $\rho = S \times \{\mathbb{1}\}$, where

$$S = \{\tau \in M^{\mathbb{P}}\colon \mathrm{dom}(\tau) \subset \mathrm{dom}(\sigma)\} = (\mathscr{P}(\mathrm{dom}(\sigma) \times \mathbb{P}))^M.$$

Fix any $\mu \in M^{\mathbb{P}}$ such that $\mu_G \subset \sigma_G$. We show $\mu_G \in \rho_G$. Let

$$\tau = \{\langle \pi, p\rangle\colon \pi \in \mathrm{dom}(\sigma) \wedge p \Vdash \pi \in \mu\};$$

then $\tau \in S$ so $\tau_G \in \rho_G$, so we shall be done if we can show $\mu_G = \tau_G$. To see that $\mu_G \subset \tau_G$, note that since $\mu_G \subset \sigma_G$, any element of μ_G is of the form π_G

for some $\pi \in \text{dom}(\sigma)$; since $\pi_G \in \mu_G$, there is a $p \in G$ such that $p \Vdash \pi \in \mu$, whence $\langle \pi, p \rangle \in \tau$, so $\pi_G \in \tau_G$. To see that $\tau_G \subset \mu_G$, note that any element of τ_G is of the form π_G where $\langle \pi, p \rangle \in \tau$ for some $p \in G$; then $p \Vdash \pi \in \mu$, so $\pi_G \in \mu_G$.

The key to the proof of the Power Set Axiom in $M[G]$ is that in M there is a set of names which contains representatives for any possible subset of σ_G, even though the collection of all μ such that $\mu_G \subset \sigma_G$ (or even $\mu_G = 0$) is usually not contained in a set of M (see Exercise A9).

We now know that ZF holds in $M[G]$. To check that AC holds in $M[G]$, we shall verify the equivalent of AC presented in Lemma 4.1. Fix $x = \sigma_G \in M[G]$. By AC^M, let $\text{dom}(\sigma) = \{\pi_\gamma : \gamma < \alpha\}$, where the function which takes γ to π_γ is in M. Let

$$\tau = \{\text{op}(\check{\gamma}, \pi_\gamma) : \gamma < \alpha\} \times \{\mathbb{1}\}$$

(see Definition 2.16). Then $\tau \in M$ and $\tau_G = \{\langle \gamma, \pi_{\gamma G} \rangle : \gamma < \alpha\}$, so τ_G is a function with $\text{dom}(\tau_G) = \alpha$ and $\sigma_G \subset \text{ran}(\tau_G)$.

Thus, all axioms of ZFC hold in $M[G]$. $\square$

Our next task is to show how to design $\mathbb{P}$ to produce $M[G]$ satisfying desired additional axioms, but we mention now one immediate consequence of our results so far.

4.3. COROLLARY. *Let M be* c.t.m. *for* ZFC, *then there is a* c.t.m. $N \supset M$ *such that N satisfies* ZFC + $\mathbf{V} \neq \mathbf{L}$.

PROOF. With the notation of Theorem 4.2, just choose $\mathbb{P}$ such that $G \notin M$. This is true whenever $\mathbb{P}$ satisfies the condition of Lemma 2.4; for example, let $\mathbb{P}$ be finite partial functions from ω to 2. Let $N = M[G]$, then, since $o(N) = o(M)$ (see Lemma 2.15), $\mathbf{L}^N = \mathbf{L}^M \subset M$, so N satisfies $\mathbf{V} \neq \mathbf{L}$. $\square$

As pointed out in §1, Corollary 4.3 yields the following.

4.4. COROLLARY. Con(ZFC) $\to$ Con(ZFC + $\mathbf{V} \neq \mathbf{L}$). $\square$

As we now proceed with the development of forcing, we shall often be discussing the relation $p \Vdash \phi$ where ϕ is a statement of some mathematical complexity. Then, as usual, ϕ will not be explicitly displayed as a formula in the official language of set theory; rather, we shall express ϕ using standard mathematical notation, which we consider to be an abbreviation for a formula of set theory. It is then worth noting that we do not have to worry about the exact way we write the unabbreviated formula, since two formulas which are equivalent in ZFC are forced by the same conditions. More precisely, the following holds.

4.5. LEMMA. (a) *Let* $\phi(x_1, \ldots, x_n)$ *and* $\psi(x_1, \ldots, x_n)$ *be formulas, and assume*

$$\mathrm{ZFC} \vdash \forall x_1, \ldots, x_n \, (\phi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n));$$

then for any c.t.m. M *for* ZFC, p.o. $\mathbb{P} \in M$, $p \in \mathbb{P}$, *and* $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$,

$$(p \Vdash \phi(\tau_1, \ldots, \tau_n)) \to (p \Vdash \psi(\tau_1, \ldots, \tau_n)).$$

(b) *If we assume also that*

$$\mathrm{ZFC} \vdash \forall x_1, \ldots, x_n \, (\phi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)),$$

then we may conclude

$$(p \Vdash \phi(\tau_1, \ldots, \tau_n)) \leftrightarrow (p \Vdash \psi(\tau_1, \ldots, \tau_n)).$$

PROOF. In (a), for any G which is $\mathbb{P}$-generic over M, $M[G]$ satisfies ZFC, so

$$\phi(\tau_{1_G}, \ldots, \tau_{n_G})^{M[G]} \to \psi(\tau_{1G}, \ldots, \tau_{nG})^{M[G]}.$$

(a) thus follows from the definition of $\Vdash$. (b) follows from (a). □

§5. Forcing with finite partial functions

The most famous relative consistency proof produced by forcing is that of Con(ZFC + $\neg$CH). The methods of this section allow us to construct models in which 2^{ω} is ω_2, ω_5, ω_{ω_1} or anything else not obviously contradictory.

Throughout this section, M is a fixed c.t.m. for ZFC. We consider forcing over M with *finite partial functions* from one set I into anto another set J.

5.1. DEFINITION.

$$\mathrm{Fn}(I, J) = \{p\colon |p| < \omega \wedge p \text{ is a function } \wedge \mathrm{dom}(p) \subset I \wedge \mathrm{ran}(p) \subsetneq J\}.$$

Order $\mathrm{Fn}(I, J)$ by: $p \leq q \leftrightarrow p \supset q$. □

$\mathrm{Fn}(I, J)$ is a p.o., with largest element $\mathbb{1} = 0$ (the empty function). Since "finite" is absolute, so is $\mathrm{Fn}(I, J)$, so if $I, J \in M$, then $\mathrm{Fn}(I, J) = \mathrm{Fn}(I, J)^M \in M$. $\mathrm{Fn}(\omega, 2)$ was discussed briefly at the beginning of §3, and the elementary discussion of $\mathrm{Fn}(I, J)$ in general is similar.

If G is a filter in $\mathrm{Fn}(I, J)$, $\bigcup G$ is a function with $\mathrm{dom}(\bigcup G) \subset I$ and $\mathrm{ran}(\bigcup G) \subset J$. If $J \neq 0$, $D_i = \{p \in \mathrm{Fn}(I, J)\colon i \in \mathrm{dom}(p)\}$ is dense for all $i \in I$; furthermore, by absoluteness, $D_i \in M$ if $I, J \in M$; thus, if G is generic over M, $G \cap D_i \neq 0$ for each $i \in I$, whence $\mathrm{dom}(\bigcup G) = I$. Likewise, if I is infinite, $\{p \in \mathrm{Fn}(I, J)\colon j \in \mathrm{ran}(p)\}$ is dense and in M, so $\mathrm{ran}(\bigcup G) = J$. We have thus proved the following.

5.2. LEMMA. *If $I, J \in M$, I is infinite, $J \neq 0$, and G is* $\mathrm{Fn}(I, J)$*-generic over M, then $\bigcup G$ is a function from I onto J.* $\square$

A simple application of this kind of partial order is that the notion of cardinal need not be absolute for M, $M[G]$. Thus, let κ be an uncountable cardinal of M; i.e., $\kappa \in M$ and (κ is an uncountable cardinal)^M. Let $\mathbb{P} = \mathrm{Fn}(\omega, \kappa)$, and let G be $\mathbb{P}$-generic over M. Then $\bigcup G \in M[G]$ by absoluteness of $\bigcup$), and G is a function from ω onto κ, so in $M[G]$, κ is a countable ordinal. We say that $\mathbb{P}$ *collapses* κ.

With a different I, J, we can use $\mathrm{Fn}(I, J)$ to obtain a model in which CH is false. Again let κ be an uncountable cardinal of M, but now let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, so, if G is $\mathbb{P}$-generic over M, then $\bigcup G : \kappa \times \omega \to 2$. We may think of G as coding a κ-sequence of functions from ω into 2; namely, let $f_\alpha(n) = (\bigcup G)(\alpha, n)$ for $\alpha < \kappa$, $n < \omega$. By absoluteness, the sequence $\langle f_\alpha \colon \alpha < \kappa \rangle$ (i.e., the function which assigns, to each α, f_α) is in $M[G]$. Furthermore, the f_α are all distinct; to see this, if $\alpha \neq \beta$, let

$$D_{\alpha\beta} = \{p \in \mathbb{P} \colon \exists n \in \omega (\langle \alpha, n \rangle \in \mathrm{dom}(p) \wedge \langle \beta, n \rangle \in \mathrm{dom}(p) \wedge p(\alpha, n) \neq p(\beta, n))\};$$

$D_{\alpha\beta}$ is dense and in M, so $G \cap D_{\alpha\beta} \neq 0$, which implies $f_\alpha \neq f_\beta$. Thus, $M[G]$ contains a κ-sequence of distinct functions from ω into 2, so the following is obtained.

5.3. LEMMA. *If $\kappa \in M$ and G is* $\mathrm{Fn}(\kappa \times \omega, 2)$*-generic over M, then* $(2^\omega \geq |\kappa|)^{M[G]}$. $\square$

Taking $\kappa = (\omega_2)^M$, this would seem to imply that $2^\omega \geq \omega_2$ in $M[G]$, i.e., CH fails in $M[G]$. *But*, we must first check that κ is also $(\omega_2)^{M[G]}$; this is not immediate since we have just seen, with a slightly different partial order, that an uncountable cardinal of M could become a countable ordinal in $M[G]$. That this does not happen with $\mathrm{Fn}(\kappa \times \omega, 2)$ involves the fact that, as we shall show, this partial order has the countable chain condition (c.c.c.) *in M* (it trivially has c.c.c. in **V**, since M is countable, but that is irrelevant).

The fact that $(\mathrm{Fn}(\kappa \times \omega, 2)$ has c.c.c.$)^M$ follows from the following slightly more general result, relativized to M.

5.4. LEMMA. *If I is arbitrary and J is countable, then* $\mathrm{Fn}(I, J)$ *has* c.c.c.

PROOF. Let $p_\alpha \in \mathrm{Fn}(I, J)$ for $\alpha < \omega_1$ and let $a_\alpha = \mathrm{dom}(p_\alpha)$. By the Δ-system lemma (II 1.5), there is an uncountable $X \subset \omega_1$ such that $\{a_\alpha \colon \alpha \in X\}$ forms a Δ-system, with some root r. Since J is countable, so is J^r, so there are only countably many possibilities for $p_\alpha \restriction r$. It follows that there is an uncountable $Y \subset X$ such that the $p_\alpha \restriction r$ for $\alpha \in Y$ are all the same. But then the p_α for $\alpha \in Y$ are all compatible. Thus, there can never be a family $\{p_\alpha\colon \alpha < \omega_1\}$ of incompatible conditions. $\square$

There are many more examples of c.c.c. p.o's (see the discussion of MA in II §2). The importance of c.c.c. in forcing is the following lemma, which gives us a way of approximating, within M, any function which appears in $M[G]$.

5.5. LEMMA. *Assume* $\mathbb{P} \in M$, ($\mathbb{P}$ *is* c.c.c.)^M, *and* $A, B \in M$; *let* G *be* $\mathbb{P}$*-generic over* M, *and let* $f \in M[G]$, *with* $f\colon A \to B$. *Then there is a map* $F\colon A \to \mathscr{P}(B)$ *with* $F \in M$, $\forall a \in A\,(f(a) \in F(a))$ *and* $\forall a \in A\,((|F(a)| \le \omega)^M)$.

PROOF. Fix $\tau \in M^{\mathbb{P}}$ with $f = \tau_G$. Since any statement true in $M[G]$ is forced, there is a $p \in G$ such that

$$p \Vdash \tau \text{ is a function from } \check{A} \text{ into } \check{B}.$$

Formally, we are applying here Theorem 3.6(2) to a formula $\phi(x, y, z)$ which asserts that x is a function from y into z; exactly which ϕ we use is irrelevant by Lemma 4.5.

Define

$$F(a) = \{b \in B\colon \exists q \le p\,(q \Vdash \tau(\check{a}) = \check{b})\}.$$

$F \in M$ by definability of $\Vdash$ (see Theorem 3.6 and following discussion).

Fix $a \in A$. To see that $f(a) \in F(a)$, let $b = f(a)$. Then there is an $r \in G$ such that $r \Vdash \tau(\check{a}) = \check{b}$, and r and p have a common extension, q. Then $q \Vdash \tau(\check{a}) = \check{b}$, so $b \in F(a)$.

To see that $(|F(a)| \le \omega)^M$, apply AC in M to find a function $Q \in M$ such that $Q\colon F(a) \to \mathbb{P}$ and, for $b \in F(a)$, $Q(b) \le p$ and $Q(b) \Vdash \tau(\check{a}) = \check{b}$. If $b, b' \in F(a)$ and $b \ne b'$, then $Q(b) \perp Q(b')$, since they force inconsistent statements; more precisely, if $Q(b)$ and $Q(b')$ were compatible, there would be a generic H containing both of them, and in $M(H)$, $\tau_H\colon A \to B$, $\tau_H(a) = b$, and $\tau_H(a) = b'$. Thus, $\{Q(b)\colon b \in F(a)\}$ is an antichain in $\mathbb{P}$, so, since $Q \in M$ and ($\mathbb{P}$ is c.c.c.)^M, $(|F(a)| \le \omega)^M$. $\square$

We now discuss the relevance of the c.c.c. to absoluteness of cardinals.

5.6. DEFINITION. If $\mathbb{P} \in M$, $\mathbb{P}$ *preserves cardinals* iff whenever G is $\mathbb{P}$-generic over M,

$$\forall \beta \in \mathrm{o}(M)\,((\beta \text{ is a cardinal})^M \leftrightarrow (\beta \text{ is a cardinal})^{M[G]}). \quad \square$$

Note that since ω is absolute, preservation of cardinals is only problematic for $\beta > \omega$. Also, if β is a cardinal of $M[G]$, it is automatically a cardinal of M since any function in M from a smaller ordinal onto β would be in $M[G]$ also. Thus, $\mathbb{P}$ preserved cardinals iff

$$\forall \beta \in o(M)\,[(\beta > \omega \wedge (\beta \text{ is a cardinal})^M) \rightarrow (\beta \text{ is a cardinal})^{M[G]}].$$

It is now easily seen from Lemma 5.5 that if $(\mathbb{P}$ is c.c.c.$)^M$, then $\mathbb{P}$ preserves cardinals (take $B = \beta$ and A an ordinal $< \beta$). In fact, $\mathbb{P}$ preserves cofinalities as well, which is a slightly stronger assertion.

5.7. DEFINITION. If $\mathbb{P} \in M$, $\mathbb{P}$ *preserves cofinalities* iff whenever G is $\mathbb{P}$-generic over M and γ is a limit ordinal in M,

$$\text{cf}(\gamma)^M = \text{cf}(\gamma)^{M[G]}. \quad \square$$

5.8. LEMMA. *If* $\mathbb{P}$ *preserves cofinalities, then* $\mathbb{P}$ *preserves cardinals.*

PROOF. Assume $\mathbb{P}$ preserves cofinalities. If $\alpha \geq \omega$ is a regular cardinal of M, then $\text{cf}(\alpha)^{M[G]} = \text{cf}(\alpha)^M = \alpha$, so α is a regular cardinal of $M[G]$. If $\beta > \omega$ is a limit cardinal of M, then the regular (in fact successor) cardinals of M are unbounded in β; since these remain regular in $M[G]$, β is a limit cardinal in $M[G]$ as well. Since every infinite cardinal is either regular or a limit cardinal (or both), every infinite cardinal of M is a cardinal of $M[G]$. $\square$

There are examples of $\mathbb{P}$ which preserve cardinals without preserving cofinalities (see [Prikry 1970]), but we shall not discuss them in this book.

The following simplifies what needs to be checked for preservation of cofinalities.

5.9. LEMMA. *Assume* $\mathbb{P} \in M$ *and whenever* G *is* $\mathbb{P}$*-generic over* M *and* κ *is a regular uncountable cardinal of* M, $(\kappa$ *is regular*$)^{M[G]}$. *Then* $\mathbb{P}$ *preserves cofinalities.*

PROOF. Let γ be a limit ordinal in M, and let $(\kappa = \text{cf}(\gamma))^M$; then there is an $f \in M$ such that f maps κ into γ cofinally and f is strictly increasing (applying I 10.31 within M). Since $(\kappa$ is regular$)^M$, $(\kappa$ is regular$)^{M[G]}$ (applying absoluteness of ω if $\kappa = \omega$). Since $f \in M[G]$, $(\kappa = \text{cf}(\gamma))^{M[G]}$ (applying I 10.32 within $M[G]$). $\square$

5.10. THEOREM. *If* $\mathbb{P} \in M$ *and* $(\mathbb{P}$ *has* c.c.c.$)^M$, *then* $\mathbb{P}$ *preserves cofinalities (and hence cardinals).*

PROOF. If not, then by Lemma 5.9, there is a $\kappa \in M$ with $\kappa > \omega$, $(\kappa$ regular$)^M$, and $(\kappa$ not regular$)^{M[G]}$. Thus, there is an $\alpha < \kappa$ and an $f \in M[G]$ such that f maps α cofinally into κ. By Lemma 5.5, let F be in M, with $F : \alpha \rightarrow \mathscr{P}(\kappa)$,

$\forall\xi < \alpha(f(\xi) \in F(\xi))$, and $\forall\xi < \alpha(|F(\xi)| \le \omega)^M$. Let $S = \bigcup_{\xi<\alpha} F(\xi)$. Then, $S \in M$ and S is an unbounded subset of κ. Applying in M the fact that the union of $|\alpha|$ countable sets has cardinality $|\alpha|$, $(|S| = |\alpha| < \kappa)^M$, contradicting that ($\kappa$ is regular)^M. □

This completes everything needed to produce a model of $\neg$CH. Let $\mathbb{P} = \mathrm{Fn}(\omega_2^M \times \omega, 2)$; then $\mathbb{P}$ has c.c.c. in M and thus preserves cardinals, so $\omega_2^M = \omega_2^{M[G]}$. It follows by Lemma 5.3 that $(2^\omega \ge \omega_2)^{M[G]}$.

The next question is whether 2^ω can be exactly ω_2. It is easy to see that if $(2^\omega \ge \omega_3)^M$, the same would be true in any cardinal-preserving extension of M. However, we shall show that if M is a model for GCH, forcing with $\mathrm{Fn}(\omega_2^M \times \omega, 2)$ makes 2^ω exactly ω_2 in $M[G]$. More generally, we shall use the values of cardinal exponents in M to put an upper bound on cardinal exponents in $M[G]$.

We obtain upper bounds by doing the proof of the Power Set Axiom in $M[G]$ slightly more carefully. For Power Set it was sufficient, given a $\sigma \in M^{\mathbb{P}}$, to obtain in M some set S of names which represented all possible subsets of σ. Now, we try to obtain such an S of small cardinality.

5.11. Definition. If $\sigma \in \mathbf{V}^{\mathbb{P}}$, a *nice name* for a subset of σ is $\tau \in \mathbf{V}^{\mathbb{P}}$ of the form $\bigcup\{\{\pi\} \times A_\pi \colon \pi \in \mathrm{dom}(\sigma)\}$, where each A_π is an antichain in $\mathbb{P}$. □

As usual, we plan to use this notion within M, but the property of being a nice name is absolute.

5.12. Lemma. *If $\mathbb{P} \in M$ and $\sigma, \mu \in M^{\mathbb{P}}$, then there is a nice name $\tau \in M^{\mathbb{P}}$ for a subset of σ such that*

$$\mathbb{1} \Vdash (\mu \subset \sigma \to \mu = \tau).$$

Proof. For each $\pi \in \mathrm{dom}(\sigma)$, let $A_\pi \subset \mathbb{P}$ be such that:

(1) $\forall p \in A_\pi (p \Vdash \pi \in \mu)$,

(2) A_π is an antichain in $\mathbb{P}$, and

(3) A_π is maximal with respect to (1) and (2).

We may assume $\langle A_\pi \colon \pi \in \mathrm{dom}(\sigma)\rangle \in M$ by definability of $\Vdash$ and Zorn's Lemma applied within M. Let

$$\tau = \bigcup\{\{\pi\} \times A_\pi \colon \pi \in \mathrm{dom}(\sigma)\}.$$

To show that $\mathbb{1} \Vdash (\mu \subset \sigma \to \mu = \tau)$, we show that whenever G is $\mathbb{P}$-generic over M, $\mu_G \subset \sigma_G \to \mu_G = \tau_G$. Assume $\mu_G \subset \sigma_G$.

To show $\mu_G \subset \tau_G$, fix $a \in \mu_G$. Since $\mu_G \subset \sigma_G$, $a = \pi_G$ for some $\pi \in \mathrm{dom}(\sigma)$. If $A_\pi \cap G \ne 0$, fix $p \in A_\pi \cap G$; then $\langle \pi, p\rangle \in \tau$ and $p \in G$, so $a = \pi_G \in \tau_G$. However, if $A_\pi \cap G = 0$, let $q \in G$ be such that $\forall p \in A\,(p \perp q)$ (see Lemma

2.20(a)). Let $q' \in G$ be such that $q' \Vdash \pi \in \mu$, and let r be a common extension of q and q'; then $A_\pi \cup \{r\}$ satisfies (1) and (2) above, contradicting maximality of A_π.

To show $\tau_G \subset \mu_G$, fix $a \in \tau_G$; then $a = \pi_G$, where $\langle \pi, p \rangle \in \tau$ for some $p \in G$. By definition of τ, $p \Vdash \pi \in \mu$, so $a = \pi_G \in \mu_G$. □

If τ is a nice name for a subset of σ, it need not in general be true that $\tau_G \subset \sigma_G$, but that is irrelevant. The important fact is that every subset of σ does get represented by a nice name.

5.13. Lemma. *Assume that* $\mathbb{P} \in M$ *and that in* M, $\mathbb{P}$ *is* c.c.c., $|\mathbb{P}| = \kappa \geq \omega$, λ *is an infinite cardinal, and* $\theta = \kappa^\lambda$ *(i.e., the preceding holds relativized to* M*). Let* G *be* $\mathbb{P}$*-generic over* M*. Then in* $M[G]$, $2^\lambda \leq \theta$.

Proof In M, every antichain in $\mathbb{P}$ is countable, so there are at most κ^ω such antichains. Since $\mathrm{dom}(\check{\lambda}) = \{\check{\xi} : \xi < \lambda\}$ has cardinality λ, there are at most $(\kappa^\omega)^\lambda = \kappa^\lambda = \theta$ nice names for subsets of $\check{\lambda}$. Let $\tau_\alpha (\alpha < \theta)$ enumerate, in M, all nice names for subsets of $\check{\lambda}$.

In $M[G]$, there is a function f with domain θ such that $f(\alpha) = \mathrm{val}(\tau_\alpha, G)$ for each $\alpha < \theta$; namely, $f = \pi_G$, where $\pi = \{\langle \mathrm{op}(\check{\alpha}, \tau_\alpha), \mathbb{1} \rangle : \alpha < \theta\}$. But by Lemma 5.12, $\mathscr{P}(\lambda)^{M[G]} \subset \mathrm{ran}(f)$, so $(2^\lambda \leq \theta)^{M[G]}$. □

This may be applied to show that it is consistent that 2^ω can be almost anything.

5.14. Lemma. *Let* κ *be an infinite cardinal of* M *such that* $(\kappa^\omega = \kappa)^M$, *and let* $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. *Let* G *be* $\mathbb{P}$*-generic over* M. *Then* $(2^\omega = \kappa)^{M[G]}$.

Proof. Applying Lemma 5.13 with $\lambda = \omega$ yields $2^\omega \leq \kappa$ in M. But by Lemma 5.3, $2^\omega \geq \kappa$ in M; κ is still a cardinal in $M[G]$ since $\mathbb{P}$ has c.c.c. in M. □

In particular, if M satisfies GCH, then in M, $\kappa^\omega = \kappa$ whenever $\mathrm{cf}(\kappa) > \omega$ (see I 10.42). It follows that it is consistent for the continuum to be anything not cofinal with ω (by König's Lemma (I 10.41), $\mathrm{cf}(2^\omega) > \omega$). E.g., we may prove the following.

5.15. Corollary. Con(ZFC) *implies*

(a) Con(ZFC + $2^\omega = \omega_2$),

(b) Con(ZFC + $2^\omega = \omega_{\omega_1}$), *etc.*

Proof. The fact that our method of generic extensions yields relative consistency proofs was discussed in §1. We may start with M satisfying

ZFC + GCH since in ZFC we can prove the existence of a c.t.m. for any finite number of axioms of ZFC + $\mathbf{V} = \mathbf{L}$ (see VI 4.10), and $\mathbf{V} = \mathbf{L}$ implies GCH.

Thus, to obtain (b), start with M satisfying GCH and apply Lemma 5.14 with $(\kappa = \omega_{\omega_1})^M$. Then in $M[G]$, $2^\omega = \kappa$. Since $\mathbb{P}$ preserves cardinals $\kappa = \omega_{\omega_1}$ in $M[G]$. □

The continuum can also be weakly inaccessible.

5.16. COROLLARY. *The following four theories are equiconsistent*; *i.e.*,

$$\mathrm{Con}(T_1) \leftrightarrow \mathrm{Con}(T_2) \leftrightarrow \mathrm{Con}(T_3) \leftrightarrow \mathrm{Con}(T_4),$$

where

T_1 *is* ZFC + GCH + $\exists\kappa$ (κ *is strongly inaccessible*).

T_2 *is* ZFC + $\exists\kappa$ (κ *is weakly inaccessible*).

T_3 *is* ZFC + 2^ω *is weakly inaccessible.*

T_4 *is* ZFC + $\exists\kappa < 2^\omega$(κ *is weakly inaccessible*).

PROOF. Both $\mathrm{Con}(T_3)$ and $\mathrm{Con}(T_4)$ obviously imply $\mathrm{Con}(T_2)$. To see that $\mathrm{Con}(T_2) \to \mathrm{Con}(T_1)$, observe that as a theorem of ZFC, if κ is weakly inaccessible, then κ is weakly inaccessible in $\mathbf{L}$ and hence (by GCH in $\mathbf{L}$), strongly inaccessible in $\mathbf{L}$. Thus, within T_2 we can prove that $\mathbf{L}$ is an inner model for T_1.

To prove that $\mathrm{Con}(T_1)$ implies $\mathrm{Con}(T_3)$ and $\mathrm{Con}(T_4)$, let M be a c.t.m. for T_1. If $\mathbb{P}$ is c.c.c. in M and κ is weakly inaccessible in M, then, by preservation of cofinalities, κ will be both regular and a limit cardinal in $M[G]$, and hence κ will remain weakly inaccessible in $M[G]$. Thus, if $\lambda > \kappa$ and (λ is a cardinal)^M, then forcing with $\mathbb{P} = \mathrm{Fn}(\lambda \times \omega, 2)$ makes $M[G]$ a model for T_4. If κ is strongly inaccessible in M, then $(\kappa^\omega = \kappa)^M$, so forcing with $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$ makes $(2^\omega = \kappa)^{M[G]}$, whence $M[G]$ satisfies T_3.

Formally, to see that the previous paragraph yields a finitistic relative consistency proof of $\mathrm{Con}(T_1) \to \mathrm{Con}(T_3)$ (or of $\mathrm{Con}(T_1) \to \mathrm{Con}(T_4)$), we apply the discussion in §1 with T_1 as the basic theory instead of ZFC. Thus, within T_1, we may prove the existence of a c.t.m., M, for any desired finite list of axioms of T_1 (see IV 7.11), and then, by forcing, produce a c.t.m. $M[G]$ for any finite list of axioms of T_3. □

By the Gödel Incompleteness Theorem, we cannot expect to produce relative consistency proofs of the form $\mathrm{Con(ZFC)} \to \mathrm{Con}(T_1)$; see IV §10.

It is also possible to calculate powers of uncountable cardinals in extensions by $\mathrm{Fn}(\kappa \times \omega, 2)$; see Exercise G1. A special case of this, when $\kappa = 1$, yields a quotable relative consistency result.

5.17. COROLLARY. $\mathrm{Con(ZFC)} \to \mathrm{Con(ZFC + GCH + V \neq L)}$.

PROOF. Start with M satisfying GCH. Let $\mathbb{P} = \mathrm{Fn}(\omega, 2)$. As pointed out in the proof of Corollary 4.3, $M[G]$ satisfies $\mathbf{V} \neq \mathbf{L}$. If λ is an infinite cardinal of M, let $\theta = (\lambda^+)^M = (\omega^\lambda)^M$. By Lemma 5.13, $(2^\lambda \leq \theta)^{M[G]}$. Thus, $\forall \lambda \geq \omega (2^\lambda \leq \lambda^+)^{M[G]}$ so GCH holds in $M[G]$. □

Finally, we remark that extensions by $\mathrm{Fn}(I, 2)$ cannot be used to get a model of MA + $\neg$CH (Exercise G7); this will require a much more complicated p.o. (see VIII §6).

§6. Forcing with partial functions of larger cardinality

The p.o.'s considered here enable us to violate GCH at larger cardinals without violating CH. Again M is always a fixed c.t.m. for ZFC.

6.1. DEFINITION. For any infinite cardinal λ,

$\mathrm{Fn}(I, J, \lambda) = \{p\colon |p| < \lambda \wedge p \text{ is a function} \wedge \mathrm{dom}(p) \subset I \wedge \mathrm{ran}(p) \subset J\}$.

Order $\mathrm{Fn}(I, J, \lambda)$ by: $p \leq q \leftrightarrow q \subset p$. □

Thus, $\mathrm{Fn}(I, J) = \mathrm{Fn}(I, J, \omega)$. As with $\lambda = \omega$, $\mathrm{Fn}(I, J, \lambda)$ is a p.o. with largest element $\mathbb{1} = 0$.

When $\lambda > \omega$, $\mathrm{Fn}(I, J, \lambda)$ is *not* absolute for M. In forcing, we always use $\mathrm{Fn}(I, J, \lambda)^M$ where (λ is a cardinal)^M. Interesting results are only obtained when also (λ is regular)^M, but this restriction does not appear in the elementary discussion.

Analogously to Lemma 5.2 we have the following.

6.2. LEMMA. *If $I, J, \lambda \in M$, (λ is a cardinal)^M, $J \neq 0$ $(|I| \geq \lambda)^M$, and G is $\mathrm{Fn}(I, J, \lambda)^M$-generic over M, then $\bigcup G$ is a function from I onto J.* □

Continuing to replace ω by λ in the discussion of §5, we see that when $I = \kappa \times \lambda$ and $J = 2$, we may think of $\bigcup G$ as coding a κ-sequence of distinct functions from λ into 2, so, analogously to Lemma 5.3, we have the following.

6.3. LEMMA. *If (λ is a cardinal)^M; $\kappa \in M$, and G is $\mathrm{Fn}(\kappa \times \lambda, 2, \lambda)^M$-generic over M, then $(2^{|\lambda|} \geq |\kappa|)^{M[G]}$.* □

As in §5, the difficult part of the discussion involves checking that cardinals are preserved. Some new ideas will be needed here, since if $\lambda > \omega$, then $\mathrm{Fn}(I, J, \lambda)$ has the c.c.c. only in trivial cases (namely $|I| < \omega$ or $|J| \leq 1$).

Our argument will split into two parts. First, we modify the c.c.c. argument to check that cardinals $> \lambda$ are preserved. Next, we introduce a new idea to check that cardinals $\leq \lambda$ in M remain cardinals in $M[G]$; this fact was trivial when λ was ω. For our arguments to work, we shall eventually need that λ is regular and $2^{<\lambda} = \lambda$ in M.

As in §5, we shall verify preservation of cardinals by verifying preservation of cofinalities. In analogy with 5.6–5.9,

6.4. Definition. Assume that $\mathbb{P} \in M$ and θ is an infinite cardinal of M.

(1) $\mathbb{P}$ *preserves cardinals* $\geq \theta$ (or $\leq \theta$) iff whenever G is $\mathbb{P}$-generic over M, $\beta \in o(M)$, and $\beta \geq \theta$ (resp., $\beta \leq \theta$),

$$(\beta \text{ is a cardinal})^M \leftrightarrow (\beta \text{ is a cardinal})^{M[G]}.$$

(2) $\mathbb{P}$ *preserves cofinalities* $\geq \theta$ (or $\leq \theta$) iff whenever G is $\mathbb{P}$-generic over M, γ is a limit ordinal in M, and $\mathrm{cf}(\gamma)^M \geq \theta$ (resp., $\mathrm{cf}(\gamma)^M \leq \theta$), then

$$\mathrm{cf}(\gamma)^M = \mathrm{cf}(\gamma)^{M[G]}. \quad \square$$

6.5. Lemma. *Under the assumptions of Definition* 6.4, *if* $\mathbb{P}$ *preserves cofinalities* $\leq \theta$, *then* $\mathbb{P}$ *preserves cardinals* $\leq \theta$. *If* $\mathbb{P}$ *preserves cofinalities* $\geq \theta$ *and* (θ *is regular*)^M, *then* $\mathbb{P}$ *preserves cardinals* $\geq \theta$. $\square$

6.6. Lemma. *Under the assumptions of Definition* 6.4, *assume also that whenever* κ *is a regular cardinal of* M, $\kappa \geq \theta$, *and* G *is* $\mathbb{P}$*-generic over* M, *then* (κ *is regular*)$^{M[G]}$. *Then* $\mathbb{P}$ *preserves cofinalities* $\geq \theta$. *Likewise with* $\leq \theta$ *replacing* $\geq \theta$. $\square$

If, in the definition of c.c.c., we weaken "countable" to "$<\theta$", then we preserve cofinalities $\geq \theta$.

6.7. Definition. $\mathbb{P}$ has the θ-chain condition (θ-c.c.) iff every antichain in $\mathbb{P}$ has cardinality $< \theta$. $\square$

Thus, the c.c.c. is the ω_1-c.c. Exactly as in 5.5 and 5.10, we have the following.

6.8. Lemma. *Assume* $\mathbb{P} \in M$, $A, B \in M$, *and, in* M, θ *is a cardinal and* $\mathbb{P}$ *is* θ-c.c. *Let* G *be* $\mathbb{P}$*-generic over* M, *and let* $f \in M[G]$, *with* $f : A \to B$. *Then there is a map* $F : A \to \mathscr{P}(B)$ *with* $F \in M$, $\forall a \in A\, (f(a) \in F(a))$, *and* $\forall a \in A\, (|F(a)| < \theta)^M$. $\square$

6.9. LEMMA. *Assume* $\mathbb{P} \in M$, θ *is a cardinal of* M, *and* ($\mathbb{P}$ *has the* θ-c.c.)^M. *Then* $\mathbb{P}$ *preserves cofinalities* $\geq \theta$. *Hence, if also* (θ *is regular*)^M, $\mathbb{P}$ *preserves cardinals* $\geq \theta$. $\square$

We remark on which chain conditions will occur in practice. Let c.c.($\mathbb{P}$) be the least θ such that $\mathbb{P}$ has the θ-c.c. By a theorem of Tarski (see Exercise F4), c.c.($\mathbb{P}$) is finite or regular; it follows that the assumption (θ is regular)^M may be dropped in Lemma 6.9. Also c.c.($\mathbb{P}$) cannot be ω (Exercise F1). If c.c.($\mathbb{P}$) $< \omega$, then $\mathbb{P}$ is uninteresting for forcing (Exercise F2); but if $0 < n < \omega$ then there is a $\mathbb{P}$ with c.c.($\mathbb{P}$) $= n$—namely, $\mathrm{Fn}(1, n-1)$. If θ is weakly inaccessible, then there is an important example of a $\mathbb{P}$ with c.c.($\mathbb{P}$) $= \theta$—namely, the Lévy order (see §8).

Finally, assume $\theta = \lambda^+$. $\mathrm{Fn}(1, \lambda)$ is a trivial example of a $\mathbb{P}$ with c.c.($\mathbb{P}$) $= \theta$. More important, if $\mathbb{P} = \mathrm{Fn}(I, 2, \lambda)$, then under GCH, c.c.($\mathbb{P}$) $= \lambda^+$ if $|I| \geq \lambda$; without GCH, c.c.($\mathbb{P}$) $= (2^{<\lambda})^+$. We leave the fact that c.c.($\mathbb{P}$) $\geq (2^{<\lambda})^+$ as an exercise (F5), but we prove that c.c.($\mathbb{P}$) $\leq (2^{<\lambda})^+$ since that is important for preservation of cardinals.

6.10. LEMMA. $\mathrm{Fn}(I, J, \lambda)$ *has the* $(|J|^{<\lambda})^+$-c.c.

PROOF. Let $\theta = (|J|^{<\lambda})^+$, and suppose that $\{p_\xi \colon \xi < \theta\}$ formed an antichain. First, assume λ is regular. Then $(|J|^{<\lambda})^{<\lambda} = |J|^{<\lambda}$, so $\forall \alpha < \theta(|\alpha^{<\lambda}| < \theta)$, so by the Δ-system lemma (see II 1.6) there is an $X \subset \theta$ with $|X| = \theta$ such that $\{\mathrm{dom}(p_\xi) \colon \xi \in X\}$ forms a Δ-system with some root r. Since there are less than θ possibilities for $p_\xi \restriction r$, we have a contradiction as in the proof for $\lambda = \omega$ (see Lemma 5.4).

If λ is singular, then since θ is regular and $> \lambda$, we could find a regular $\lambda' < \lambda$ such that $Y = \{\xi \colon |p_\xi| < \lambda'\}$ has cardinality θ. Then $\{p_\xi \colon \xi \in Y\}$ contradicts the $(|J|^{<\lambda'})^+$-c.c. which we have just proved for regular λ'. $\square$

6.11. COROLLARY. *Assume* $I, J \in M$, *and, in* M, λ *is regular*, $|J| \leq 2^{<\lambda}$, *and* $\theta = (2^{<\lambda})^+$. *Then* $\mathrm{Fn}(I, J, \lambda)^M$ *preserves cofinalities and cardinals* $\geq \theta$.

PROOF. Applying Lemma 6.10 within M, $\mathrm{Fn}(I, J, \lambda)^M$ has the θ-c.c. in M, since $(|J|^{<\lambda} = 2^{<\lambda})^M$. Now apply Lemma 6.9. $\square$

By a completely different argument, we shall now show that if λ is regular in M, then $(\mathrm{Fn}(I, \kappa, \lambda))^M$ preserves cofinalities and cardinals $\leq \lambda$. Under GCH, $2^{<\lambda} = \lambda$, so using Corollary 6.11, all cofinalities and cardinals will be preserved. However, if in M there are cardinals κ with $\lambda^+ \leq \kappa \leq 2^{<\lambda}$, then except in trivial cases such κ will have cardinality λ, and hence cease to be cardinals, in $M[G]$ (see Exercise G3).

6.12. DEFINITION. A p.o. $\mathbb{P}$ is *λ-closed* iff whenever $\gamma < \lambda$ and $\{p_\xi\colon \xi < \gamma\}$ is a decreasing sequence of elements of $\mathbb{P}$ (i.e., $\xi < \eta \to p_\xi \geq p_\eta$), then

$$\exists q \in \mathbb{P}\, \forall \xi < \gamma (q \leq p_\xi). \quad \square$$

6.13. LEMMA. *If λ is regular, then* $\mathrm{Fn}(I, J, \lambda)$ *is λ-closed.*

PROOF. The q of Definition 6.12 is just $\bigcup \{p_\xi\colon \xi < \gamma\}$. $|q| < \lambda$ since each $|p_\xi| < \lambda$ and λ is regular. $\square$

If λ is singular, $\mathrm{Fn}(\lambda, 2, \lambda)$ is not λ-closed. Also, if $(\lambda$ is singular$)^M$, $\mathrm{Fn}(\lambda, 2, \lambda)^M$ collapses λ (Exercise G5).

If λ is regular, then the fact that $\mathrm{Fn}(I, J, \lambda)$ is λ-closed will be used to show that cardinals $\leq \lambda$ are preserved.

The following result should be compared with Lemma 6.8. Lemma 6.8 used a chain condition to approximate, in M, functions from A to B in $M[G]$. Theorem 6.14 shows that functions from A to B are in fact in M if A is small enough.

6.14. THEOREM. *Assume $\mathbb{P} \in M$, $A, B \in M$, and, in M, λ is a cardinal, $\mathbb{P}$ is λ-closed, and $|A| < \lambda$. Let G be $\mathbb{P}$-generic over M and let $f \in M[G]$ with $f : A \to B$. Then $f \in M$.*

PROOF. Observe first that it is sufficient to prove this with A an ordinal, $A = \alpha < \lambda$. For, then, to prove the general result, we let $j \in M$ be a 1–1 map from $\alpha = |A|^M < \lambda$, onto A, and apply the special case with $f \circ j : \alpha \to B$ to show that $f \circ j$, and hence f, is in M.

Now, let $K = ({}^\alpha B)^M = {}^\alpha B \cap M$, and $f \in {}^\alpha B \cap M[G]$. We wish to show $f \in K$. If not, fix $\tau \in M^{\mathbb{P}}$ with $f = \tau_G$, and then fix $p \in G$ such that

$$p \Vdash (\tau \text{ is a function from } \check{\alpha} \text{ into } \check{B} \wedge \tau \notin \check{K}). \qquad (*)$$

We now forget about f and G and derive a contradiction directly from $(*)$.

Within M: use transfinite recursion plus AC to choose sequences $\{p_\eta\colon \eta \leq \alpha\}$ from $\mathbb{P}$ and $\{z_\eta\colon \eta < \alpha\}$ from B so that

(1) $p_0 = p$,

(2) $p_\eta \leq p_\xi$ for all $\xi \leq \eta$, and

(3) $p_{\eta+1} \Vdash \tau(\check{\eta}) = \check{z}_\eta$.

For the successor steps in this recursion, we are given p_η, and we find $p_{\eta+1}$ and z_η as follows: $p_\eta \leq p$, so

$$p_\eta \Vdash \tau \text{ is a function from } \check{\alpha} \text{ into } \check{B},$$

so (since a consequence of a forced statement is forced—see Lemma 4.5(a)),

$$p_\eta \Vdash \exists x \in \check{B}\, (\tau(\check{\eta}) = x).$$

Thus, by Corollary 3.7(d), there is a $z_\eta \in B$ and a $p_{\eta+1} \leq p_\eta$ such that $p_{\eta+1} \Vdash \tau(\check{\eta}) = \check{z}_\eta$. At the limit steps, p_η, for η a limit, may be chosen to satisfy (2) by the definition of λ-closed.

Still in M, let $g = \langle z_\eta : \eta < \alpha \rangle$; i.e., g is the function with domain α such that $g(\eta) = z_\eta$ for each η. Then $g \in K$.

Let H be $\mathbb{P}$-generic over M, with $p_\alpha \in H$, and hence each $p_\eta \in H$. Then $\tau_H(\eta) = z_\eta$ for each $\eta < \alpha$, so $\tau_H = g \in K$. But $p_0 = p \Vdash \tau \notin \check{K}$, so $\tau_H \notin K$, a contradiction. $\square$

6.15. Corollary. *Assume $\mathbb{P} \in M$, (λ is a cardinal)^M, and ($\mathbb{P}$ is λ-closed)^M; then $\mathbb{P}$ preserves cofinalities $\leq \lambda$, and hence cardinals $\leq \lambda$.*

Proof. If not, then, by Lemma 6.6, there is a $\kappa \leq \lambda$ such that (κ is regular)^M but (κ is singular)$^{M[G]}$. Thus, there is an $\alpha < \kappa$ and an $f \in M[G]$ which maps α cofinally into κ. By Theorem 6.14, $f \in M$, contradicting (κ is regular)^M. $\square$

6.16. Theorem. *Let $\lambda, I, J \in M$, and assume that in M, λ is regular, $2^{<\lambda} = \lambda$, and $|J| \leq \lambda$; then $\mathrm{Fn}(I, J, \lambda)^M$ preserves cofinalities (and hence cardinals).*

Proof. By regularity of λ, $\mathrm{Fn}(I, J, \lambda)^M$ is λ-closed in M, and hence preserves cofinalities $\leq \lambda$ (see 6.13–6.15). By $2^{<\lambda} = \lambda$, $\mathrm{Fn}(I, J, \lambda)^M$ has the λ^+-c.c. in M and hence preserves cofinalities $\geq (\lambda^+)^M$ (see 6.8–6.11). $\square$

In particular, we may now force with orders of the form $\mathrm{Fn}(\kappa \times \lambda, 2, \lambda)^M$ to violate GCH as badly as we wish at λ. We may use nice names, as in §5, to obtain a precise computation of 2^λ in $M[G]$. Generalizing Corollary 5.15, we have the following.

6.17. Theorem. *In M, assume that $\lambda < \kappa$, λ is regular, $2^{<\lambda} = \lambda$, and $\kappa^\lambda = \kappa$. Let $\mathbb{P} = \mathrm{Fn}(\kappa \times \lambda, 2, \lambda)^M$. Then $\mathbb{P}$ preserves cardinals and if G is $\mathbb{P}$-generic over M, then $(2^\lambda = \kappa)^{M[G]}$.*

Proof. We just proved preservation of cardinals, and $(2^\lambda \geq \kappa)^{M[G]}$ is easy (see Lemma 6.3). We must show $(2^\lambda \leq \kappa)^{M[G]}$.

In M, $\mathbb{P}$ has cardinality $\kappa^{<\lambda} = \kappa$, and $\mathbb{P}$ has the λ^+-c.c., so there are at most $\kappa^\lambda = \kappa$ antichains in $\mathbb{P}$. Hence, there are at most $\kappa^\lambda = \kappa$ nice names for subsets of λ. Let $\langle \tau_\alpha : \alpha < \kappa \rangle$ enumerate these, and let

$$\pi = \{\langle \mathrm{op}(\check{\alpha}, \tau_\alpha), \mathbb{1} \rangle : \alpha < \kappa\}.$$

Then, as in the proof of Lemma 5.13, in $M[G]$, π_G is a function, $\mathrm{dom}(\pi_G) = \kappa$, and $\mathscr{P}(\lambda) \subset \mathrm{ran}(\pi_G)$, so $2^\lambda \leq \kappa$. $\square$

One can generalize Theorem 6.17 to compute the powers of all cardinals in $M[G]$ (not only λ) in terms of cardinal arithmetic in M, but it is probably better always to refer back to the method of Theorem 6.17 instead of trying to memorize the most general result.

We may use the method of Theorem 6.17 to violate GCH as desired at any regular cardinal, or even at any finite number of regular cardinals. As examples, we prove the following.

6.18. THEOREM. *If* ZFC *is consistent, so are*:

(a) ZFC + CH + $2^{\omega_1} = \omega_2 + 2^{\omega_2} = \omega_{\omega_8}$.

(b) ZFC + CH + $2^{\omega_1} = \omega_5 + 2^{\omega_2} = \omega_7$.

(c) ZFC + $2^{\omega} = \omega_3 + 2^{\omega_1} = \omega_4 + 2^{\omega_2} = \omega_6$.

PROOF. In all cases, start with M satisfying ZFC + GCH.

For (a) Let $\mathbb{P} = (\mathrm{Fn}(\omega_{\omega_8} \times \omega_2, 2, \omega_2))^M$. By Theorem 6.17, $\mathbb{P}$ preserves cardinals and if G is $\mathbb{P}$-generic over M, $(2^{\omega_2} = \omega_{\omega_8})^{M[G]}$. The fact that $2^{\omega_1} = \omega_2$ holds in $M[G]$ follows from the fact that $({}^{(\omega_1)}2)^M = ({}^{(\omega_1)}2)^{M[G]}$ by Theorem 6.14. Thus, if $F \in M$ and (F maps ω_2 onto ${}^{(\omega_1)}2)^M$, then (F maps ω_2 onto ${}^{(\omega_1)}2)^{M[G]}$. Likewise, $(2^{\omega} = \omega_1)^{M[G]}$.

For (b) we force twice. Let $\mathbb{P} = (\mathrm{Fn}(\omega_7 \times \omega_2, 2, \omega_2))^M$, let G be $\mathbb{P}$-generic over M, and let $N = M[G]$; then as in (a),

$$(2^{\omega} = \omega_1 \ \wedge\ 2^{\omega_1} = \omega_2 \ \wedge\ 2^{\omega_2} = \omega_7)^N.$$

Furthermore, $(\kappa^{\omega_1} = \kappa)^N$ whenever $(\kappa \geq \omega_2 \wedge \kappa \text{ is regular})^N$, since this is true in M by $(\mathrm{GCH})^M$, and $({}^{(\omega_1)}\kappa)^M = ({}^{(\omega_1)}\kappa)^N$. We now apply our results on forcing with N as the ground model instead of M. Let

$$\mathbb{Q} = (\mathrm{Fn}(\omega_5 \times \omega_1, 2, \omega_1))^N.$$

By $(2^{<\omega_1} = \omega_1)^N$, $\mathbb{Q}$ preserves cardinals. Let H be $\mathbb{Q}$-generic over N. $(\mathrm{CH})^{N[H]}$ is proved as in (a). $(2^{\omega_2} \geq \omega_7)^{N[H]}$ follows from $(2^{\omega_2} \geq \omega_7)^N$. To see that in fact $(2^{\omega_2} = \omega_7)^{N[H]}$, use the method of Theorem 6.17; namely, in N, $\mathbb{Q}$ has the ω_2-c.c. and $|\mathbb{Q}| = \omega_5^{\omega_1} = \omega_5$, so there are only $((\omega_5)^{\omega_1})^{\omega_2} = \omega_7$ nice names for subsets of ω_2. To see that $(2^{\omega_1} = \omega_5)^{N[H]}$, apply Theorem 6.17 directly, plus the fact that $(\omega_5^{\omega_1} = \omega_5)^N$.

For (c), force three times, and construct $M \subset N_1 \subset N_2 \subset N_3$. N_1 satisfies $2^{\omega} = \omega_1 \wedge 2^{\omega_1} = \omega_2 \wedge 2^{\omega_2} = \omega_6$, N_2 satisfies $2^{\omega} = \omega_1 \wedge 2^{\omega_1} = \omega_4 \wedge 2^{\omega_2} = \omega_6$, and N_3 satisfies (c). □

In proving (b) and (c), it is very important that we proceed *backwards*, dealing with the largest cardinal first. For example, if we tried to prove (b) by letting $\mathbb{P} = (\mathrm{Fn}(\omega_5 \times \omega_1, 2, \omega_1))^M$ and $N = M[G]$, where G is $\mathbb{P}$-generic over M, then N would satisfy $2^{\omega_1} = \omega_5$. Thus, $(2^{<\omega_2} \neq \omega_2)^N$, so if we set $\mathbb{Q} = (\mathrm{Fn}(\omega_7 \times \omega_2, 2, \omega_2))^N$, $\mathbb{Q}$ would not preserve cardinals. In fact, if H is $\mathbb{Q}$-generic over N, then $(\omega_5)^N$ would have cardinality ω_2 in $N[H]$ (see Exercise G3), and $N[H]$ would satisfy $2^{\omega_1} = \omega_2$.

We may easily generalize Theorem 6.18 to deal with the powers of any finite number of regular cardinals (see Exercise G6). The following two questions now suggest themselves.

Question 1. How free are we to monkey with the powers of regular cardinals?

The answer, due to Easton, is: as free as we wish, subject, of course, to monotonicity ($\lambda < \lambda' \to 2^\lambda \leq 2^{\lambda'}$) and König's Lemma ($\mathrm{cf}(2^\lambda) > \lambda$; see I 10.41). We discuss this further in VIII §4. We merely remark here that some new idea is needed. Say, e.g., we start with M satisfying GCH and we want $M[G]$ to satisfy $\forall n \in \omega\, (2^{\omega_n} = \omega_{n+3})$. The method of proof of Theorem 6.18 indicates that we should iterate forcing ω times, starting with the *largest* ω_n, which is clearly nonsense. Even if we could turn this around, one cannot naively repeat the forcing process ω times and expect to produce a model of ZFC (see Exercise B6).

Question 2. What about singular cardinals?

If λ is singular in M, p.o.'s of the form $\mathrm{Fn}(I, \kappa, \lambda)^M$ will always collapse λ (Exercise G5), so questions about powers of singular cardinals are not settled by the methods of this section. In fact, there are restrictions beyond monotonicity and König's Lemma on the powers of singular cardinals; see the end of VIII §4 for more details.

§7. Embeddings, isomorphisms, and Boolean-valued models

As before, M is always a fixed c.t.m. for ZFC.

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are p.o.'s in M, $i \in M$, and $i : \mathbb{P} \to \mathbb{Q}$ embeds $\mathbb{P}$ as a sub-order of $\mathbb{Q}$. We shall show how, under suitable restrictions on i, we may use i to embed the whole $\mathbb{P}$ forcing apparatus into the $\mathbb{Q}$ forcing apparatus. This one idea has many diverse and seemingly unrelated applications. For one, the special case where $\mathbb{P} = \mathbb{Q}$ is the key idea behind constructing models of ZF + $\neg$AC (see Exercises E1–5). For another, if $\mathbb{P} \neq \mathbb{Q}$ but i is an isomorphism, we may show that isomorphic p.o.'s lead to identical generic extensions (provided that the isomorphism is in M). A third application, when $\mathbb{P} \subset \mathbb{Q}$ and i is inclusion, is that we may regard larger p.o.'s as yielding larger generic extensions; this will be of vital importance for iterated forcing in VIII. Finally, we shall use these ideas to relate forcing to Boolean-valued models.

For Boolean-valued models, we consider the special case where $\mathbb{Q} = \{b \in \mathscr{B}: b > 0\}$, where ($\mathscr{B}$ is a complete Boolean algebra)^M. For this discussion we shall assume that the reader is familiar with the relationship between p.o.'s and Boolean algebras discussed in II §3. However, the reader who is not interested in Boolean-valued models may simply skip all references to Boolean algebras in this section without loss of continuity.

We begin by examining the third application above. Suppose that $\mathbb{P}$ is a sub-order of $\mathbb{Q}$; i.e., $\mathbb{P} \subset \mathbb{Q}$ and $\leq_{\mathbb{P}} = \leq_{\mathbb{Q}} \cap \mathbb{P} \times \mathbb{P}$, then $\mathbb{Q}$ "should" yield a bigger extension than does $\mathbb{P}$; if H is $\mathbb{Q}$-generic over M, then $H \cap \mathbb{P}$ "should" be $\mathbb{P}$-generic over M, with $M[H \cap \mathbb{P}] \subset M[H]$. However, this is false without some further restrictions on $\mathbb{P}$ and $\mathbb{Q}$.

To appreciate one necessary restriction, suppose that $p_1, p_2 \in \mathbb{P}$ and that p_1 and p_2 are incompatible in $\mathbb{P}$ but are compatible in $\mathbb{Q}$; say $q \in \mathbb{Q}$, $q \leq p_1$, and $q \leq p_2$. If H is $\mathbb{Q}$-generic and $q \in H$, then $p_1, p_2 \in H$, so $H \cap \mathbb{P}$ is not even a filter in $\mathbb{P}$. We must thus require that if p_1 and p_2 are incompatible in $\mathbb{P}$, then they are incompatible in $\mathbb{Q}$ also.

To obtain a second restriction, fix $q \in \mathbb{Q}$, and let $D = \{p \in \mathbb{P}: p \perp q\}$. If H is $\mathbb{Q}$-generic over M and $q \in H$, then $H \cap D = 0$, so if we wish $H \cap \mathbb{P}$ to be $\mathbb{P}$-generic, we had better require that D not be dense in $\mathbb{P}$. Thus, there must be a $p \in \mathbb{P}$ such that

$$\forall p' \in \mathbb{P}\,(p' \leq p \rightarrow p' \text{ and } q \text{ are compatible in } \mathbb{Q}).$$

If these two restrictions hold, we shall say that $\mathbb{P}$ is *completely contained* in $\mathbb{Q}$, or $\mathbb{P} \subset_c \mathbb{Q}$. These restrictions are sufficient, as we shall see in Theorem 7.5. We now present the formal development in the somewhat more general framework of an embedding from one p.o. into another.

7.1. DEFINITION. Let $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle$ and $\langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}} \rangle$ be p.o.'s and $i : \mathbb{P} \rightarrow \mathbb{Q}$. i is a *complete embedding* iff

(1) $\forall p, p' \in \mathbb{P}\,(p' \leq p \rightarrow i(p') \leq i(p))$.

(2) $\forall p_1, p_2 \in \mathbb{P}\,(p_1 \perp p_2 \leftrightarrow i(p_1) \perp i(p_2))$.

(3) $\forall q \in \mathbb{Q}\, \exists p \in \mathbb{P}\, \forall p' \in \mathbb{P}\,(p' \leq p \rightarrow (i(p') \text{ and } q \text{ are compatible in } \mathbb{Q}))$.

In (3), we call p *a reduction* of q to $\mathbb{P}$. □

In Definition 7.1, we very quickly dropped the subscripts for $\leq_{\mathbb{P}}$ and $\leq_{\mathbb{Q}}$, since it is clear from context which order is being referenced. Likewise, there should, formally, be subscripts on the $\perp$. Observe in (3) that the reduction, p, of q to $\mathbb{P}$ is not unique; if $p_1 \leq p$, then p_1 is another such reduction.

7.2. DEFINITION. $\langle \mathbb{P}, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}} \rangle \subset_c \langle \mathbb{Q}, \leq_{\mathbb{Q}}, \mathbb{1}_{\mathbb{Q}} \rangle$, or $\mathbb{P} \subset_c \mathbb{Q}$, iff $\leq_{\mathbb{P}} = \leq_{\mathbb{Q}} \cap \mathbb{P} \times \mathbb{P}$ and the inclusion (identity) map from $\mathbb{P}$ to $\mathbb{Q}$ is a complete embedding. □

If i is an inclusion, as in Definition 7.2, then (1) of Definition 7.1 holds trivially and (2) and (3) of 7.1 are precisely the two restrictions we discussed before stating 7.1. In the more general framework of Definition 7.1, the reader might have expected us to state (1) as

$$\forall p, p' \in \mathbb{P}\,(p' \leq p \leftrightarrow i(p') \leq i(p)),$$

and to add to (1) the requirements that i be 1–1 and that $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$. It is true that in many (but not all) cases of interest, all these additional things hold (see Exercise C2), but the general theory is just as easy to carry out under Definition 7.1 as it stands, and, as we shall see later, this generality is needed in the theory of Boolean-valued models (see also Exercise C9).

We remark that in Definition 7.1(2), the implication from right to left follows from (1), but the implication from left to right says something new.

As a trivial example of Definition 7.1, if all elements of $\mathbb{P}$ are compatible, and $i(p) = \mathbb{1}_{\mathbb{Q}}$ for all $p \in \mathbb{P}$, then i is a complete embedding. More useful examples are given by the following.

7.3. Lemma. (a) *If i is an isomorphism from $\mathbb{P}$ onto $\mathbb{Q}$, then i is a complete embedding.*

(b) *If $I \subset I'$, then* $\mathrm{Fn}(I, J, \kappa) \subset_{\mathrm{c}} \mathrm{Fn}(I', J, \kappa)$.

Proof. For (b), clauses (1) and (2) of Definition 7.1 are clear. For (3), if $q \in \mathrm{Fn}(I', J, \kappa)$, then $q \restriction I$ is a reduction of q to $\mathrm{Fn}(I, J, \kappa)$. $\square$

It should not be presumed that all "naturally occurring" inclusions are complete inclusions. For example, $\mathrm{Fn}(\kappa, 2) \subset \mathrm{Fn}(\kappa, 2, \omega_1)$, but this inclusion is not complete if $\kappa \geq \omega$; clauses (1) and (2) of Definition 7.1 hold, but no $q \in \mathrm{Fn}(\kappa, 2, \omega_1)$ with infinite domain has a reduction to $\mathrm{Fn}(\kappa, 2)$. Furthermore, relativizing to a c.t.m. M, $\mathrm{Fn}(\kappa, 2, \omega_1)^M$ cannot be thought of as corresponding to a larger extension than $\mathrm{Fn}(\kappa, 2)$, since $\mathrm{Fn}(\kappa, 2)$ adds new subsets of ω, whereas $\mathrm{Fn}(\kappa, 2, \omega_1)^M$ does not.

We now proceed to show that if $\mathbb{P} \subset_{\mathrm{c}} \mathbb{Q}$, then $\mathbb{Q}$ does yield a bigger extension than does $\mathbb{P}$. As usual, we would expect to relativize all relevant order-theoretic notions to M, but the definitions of "complete embedding" and "$\subset_{\mathrm{c}}$" are easily seen to be absolute for M, so we do not have to relativize these notions.

As a preliminary, we prove the following.

7.4. Lemma. *Suppose $\mathbb{P} \in M$ and $G \subset \mathbb{P}$; then G is $\mathbb{P}$-generic over M iff*

(1) $\forall p, q \in G\, \exists r \in \mathbb{P}\,(r \leq p \wedge r \leq q)$,

(2) $\forall p \in G\, \forall q \in \mathbb{P}\,(q \geq p \rightarrow q \in G)$, *and*

(3) $\forall D \subset \mathbb{P}\,((D \in M \wedge D \text{ dense in } \mathbb{P}) \rightarrow G \cap D \neq 0)$.

PROOF. The only difference between (1)–(3) and the definition of generic is that we required G to be a filter, which meant that (1) was strengthened to require r to be in G (see II Definition 2.4). Thus, to prove Lemma 7.4, we assume G satisfies (1)–(3), fix $p, q \in G$, and show that $\exists r \in G\,(r \leq p \wedge r \leq q)$. Let

$$D = \{r \in \mathbb{P}\colon r \perp p \vee r \perp q \vee (r \leq p \wedge r \leq q)\}.$$

D is dense and in M, so by (3) fix $r \in G \cap D$. Since, by (1), elements of G are pairwise compatible, $r \leq p \wedge r \leq q$. $\square$

7.5. THEOREM. *Suppose i, $\mathbb{P}$. and $\mathbb{Q}$ are in M, $i : \mathbb{P} \to \mathbb{Q}$, and i is a complete embedding. Let H be $\mathbb{Q}$-generic over M. Then $i^{-1}(H)$ is $\mathbb{P}$-generic over M and $M[i^{-1}(H)] \subset M[H]$.*

PROOF. We check first that $i^{-1}(H)$ is generic. Clauses (1) and (2) of Lemma 7.4 are easily verified using clauses (2) and (1), respectively, of Definition 7.1. For (3), fix $D \in M$ with D dense in $\mathbb{P}$. If $i^{-1}(H) \cap D = 0$, then $H \cap i''D = 0$, so there is a $q \in H$ such that $\forall q' \in i''D\,(q' \perp q)$ (see Lemma 2.20(a)), so $\forall p' \in D\,(i(p') \perp q)$. If p is a reduction of q to $\mathbb{P}$, then for all $p' \leq p$, $\neg(i(p') \perp q)$, so $p' \notin D$, which is impossible if D is dense.

Since $i \in M \subset M[H]$ and $H \in M[H]$, we have $i^{-1}(H) \in M[H]$, whence $M[i^{-1}(H)] \subset M[H]$ by minimality of $M[i^{-1}(H)]$ (see Lemma 2.9). $\square$

Also, if we are given a $G \subset \mathbb{P}$ which is $\mathbb{P}$-generic over M, then we can always find a $\mathbb{Q}$-generic H such that $G = i^{-1}(H)$. Furthermore, there is a p.o. $\mathbb{R} \in M[G]$ such that $M[H] = M[G][K]$ for some K which is $\mathbb{R}$-generic over $M[G]$ (see Exercises D3–D6). The relationship between complete embeddings and iterated forcing extensions will be taken up again in VIII.

If $i : \mathbb{P} \to \mathbb{Q}$ is in fact an isomorphism then we may apply Theorem 7.5 both to i and its inverse to show that p.o.'s isomorphic by an isomorphism in M yield the same extensions.

7.6. COROLLARY. *In Theorem 7.5, suppose also that i is an isomorphism. Let $G \subset \mathbb{P}$. Then G is $\mathbb{P}$-generic over M iff $i''G$ is $\mathbb{Q}$-generic over M, and in that case, $M[G] = M[i''G]$.* $\square$

Thus, for example, if $\kappa \geq \omega$ and $\kappa \in M$, then $\mathrm{Fn}(\kappa, 2)$ and $\mathrm{Fn}(\kappa \times \omega, 2)$ yield the same extensions, since $(|\kappa| = |\kappa \times \omega|)^M$, so that the two p.o.'s are isomorphic in M. We used $\mathrm{Fn}(\kappa \times \omega, 2)$, which made it simpler to describe the κ-sequence of elements of ${}^{\omega}2$ added, but $\mathrm{Fn}(\kappa, 2)$ is quicker to write down and is the order usually referred to in the literature. If $(\kappa \geq \omega_1)^M$, then $\mathrm{Fn}(\kappa, 2)$ and $\mathrm{Fn}(\omega, 2)$ are still isomorphic in $\mathbf{V}$, but *not* in M, and they need *not* yield the same extensions, since CH may be true in one extension but false in another.

The orders $\mathrm{Fn}(\omega, 2)$, $\mathrm{Fn}(\omega, 3)$ and $\mathrm{Fn}(\omega, \omega)$ are *not* isomorphic in M or in $\mathbf{V}$, but they do yield the same extensions. This can be proved (Exercise C3) using the concept of *dense embedding*, which we take up next.

7.7. Definition. Let $\mathbb{P}$, and $\mathbb{Q}$, be p.o.'s and $i : \mathbb{P} \to \mathbb{Q}$. i is a *dense embedding* iff

(1) $\forall p, p' \in \mathbb{P}\,(p' \leq p \to i(p') \leq i(p))$.

(2) $\forall p_1, p_2 \in \mathbb{P}\,(p_1 \perp p_2 \to i(p_1) \perp i(p_2))$.

(3) $i``\mathbb{P}$ is dense in $\mathbb{Q}$. □

7.8. Lemma. *Every dense embedding is a complete embedding.*

Proof. If $q \in \mathbb{Q}$, any $p \in \mathbb{P}$ with $i(p) \leq q$ is a reduction of q to $\mathbb{P}$. □

An important special case is the following.

7.9. Corollary. *If $\mathbb{P}$ is a sub-order of $\mathbb{Q}$ and $\mathbb{P}$ is dense in $\mathbb{Q}$, then the identity on $\mathbb{P}$ is a dense embedding into $\mathbb{Q}$.* □

If $i : \mathbb{P} \to \mathbb{Q}$ is a dense embedding, then $\mathbb{P}$ and $\mathbb{Q}$ yield the same generic extensions, as we show now after a preliminary lemma.

7.10. Lemma. *Suppose $\mathbb{P} \in M$, G_1 and G_2 are $\mathbb{P}$-generic over M, and $G_1 \subset G_2$. Then $G_1 = G_2$.*

Proof. Suppose $p \in G_2$ but $p \notin G_1$. Since $G_1 \cap \{p\} = 0$, there is a $q \in G_1$, with $q \perp p$ (applying Lemma 2.20(a) to G_1 and $\{p\}$), which is impossible since G_2 is a filter. □

7.11. Theorem. *Suppose i, $\mathbb{P}$, and $\mathbb{Q}$ are in M, $i : \mathbb{P} \to \mathbb{Q}$, and i is a dense embedding. If $G \subset \mathbb{P}$, let $\tilde{\imath}(G) = \{q \in \mathbb{Q} \colon \exists p \in G\,(i(p) \leq q)\}$.*

(a) *If $H \subset \mathbb{Q}$ is $\mathbb{Q}$-generic over M, then $i^{-1}(H)$ is $\mathbb{P}$-generic over M and $H = \tilde{\imath}(i^{-1}(H))$.*

(b) *If $G \subset \mathbb{P}$ is $\mathbb{P}$-generic over M, then $\tilde{\imath}(G)$ is $\mathbb{Q}$-generic over M and $G = i^{-1}(\tilde{\imath}(G))$.*

(c) *In (a) or (b), if $G = i^{-1}(H)$ (or, equivalently, if $H = \tilde{\imath}(G)$), then $M[G] = M[H]$.*

Proof. We first verify genericity of $i^{-1}(H)$ in (a) and $\tilde{\imath}(G)$ in (b). Since every dense embedding is a complete embedding genericity of $i^{-1}(H)$ in (a) follows from Theorem 7.5. In (b), $\tilde{\imath}(G)$ is easily seen to be a filter in $\mathbb{Q}$. To see that it is generic, let $D \in M$ be dense in $\mathbb{Q}$. Let

$$D^* = \{p \in \mathbb{P}: \exists q \in D\,(i(p) \leq q)\}.$$

If $D^* \cap G \neq 0$, then $D \cap \tilde{\imath}(G) \neq 0$, and $D^* \cap G \neq 0$ will follow if we can show that D^* is dense in $\mathbb{P}$. To see this fix $p \in \mathbb{P}$; now let $q \in D$ be such that $q \leq i(p)$, and let $p' \in \mathbb{P}$ be such that $i(p') \leq q$; then $i(p') \leq i(p)$, so $i(p')$ and $i(p)$ are compatible, and hence p' and p are compatible. Let $p'' \in \mathbb{P}$ be such that $p'' \leq p$ and $p'' \leq p'$; then $p'' \in D^*$ (since $i(p'') \leq q \in D$), and $p'' \leq p$.

To see that $G = i^{-1}(\tilde{\imath}(G))$ in (b), we have just seen that $\tilde{\imath}(G)$ is $\mathbb{Q}$-generic over M, and hence that $i^{-1}(\tilde{\imath}(G))$ is $\mathbb{P}$-generic over M. Since $G \subset i^{-1}(\tilde{\imath}(G))$ is immediate from the definitions, equality follows by Lemma 7.10. Likewise in (a), $\tilde{\imath}(i^{-1}(H)) \subset H$ follows directly from the definitions, so equality holds by Lemma 7.10.

Finally, for (c), we have seen (Theorem 7.5) that $M[G] \subset M[H]$. The same proof now shows that $M[H] \subset M[G]$; namely $H \in M[G]$ and $M \subset M[G]$, so $M[H] \subset M[G]$ by minimality of $M[H]$ (Lemma 2.9). □

We may also use our $i : \mathbb{P} \to \mathbb{Q}$ to associate to every $\mathbb{P}$-name, a $\mathbb{Q}$-name for the same object.

7.12. Definition. If $i : \mathbb{P} \to \mathbb{Q}$, define, by recursion on $\tau \in V^{\mathbb{P}}$,

$$i_*(\tau) = \{\langle i_*(\sigma), i(p)\rangle : \langle \sigma, p\rangle \in \tau\}. \quad \square$$

It is easily seen that $i_*(\tau) \in V^{\mathbb{Q}}$ and that the definition of i_* is absolute for M, so that, if $\mathbb{P}$, $\mathbb{Q}$ and i are in M, then $i_* : M^{\mathbb{P}} \to M^{\mathbb{Q}}$.

7.13. Lemma. *Suppose i, $\mathbb{P}$, and $\mathbb{Q}$ are in M, $i : \mathbb{P} \to \mathbb{Q}$, and i is a complete embedding, then*:

(a) *If H is $\mathbb{Q}$-generic over M, then* $\mathrm{val}(\tau, i^{-1}(H)) = \mathrm{val}(i_*(\tau), H)$ *for each* $\tau \in M^{\mathbb{P}}$.

(b) *If $\phi(x_1, \ldots, x_n)$ is a formula which is absolute for transitive models of* ZFC, *then*

$$p \Vdash_{\mathbb{P}} \phi(\tau_1, \ldots, \tau_n) \quad \textit{iff} \quad i(p) \Vdash_{\mathbb{Q}} \phi(i_*(\tau_1), \ldots, i_*(\tau_n)).$$

(c) *If i is a dense embedding and $\phi(x_1, \ldots, x_n)$ is any formula, then*

$$p \Vdash_{\mathbb{P}} \phi(\tau_1, \ldots, \tau_n) \quad \textit{iff} \quad i(p) \Vdash_{\mathbb{Q}} \phi(i_*(\tau_1), \ldots, i_*(\tau_n)).$$

Proof. (a) is a straightforward induction on τ, and does not actually require that H be generic or that i be complete.

To prove the implication from left to right in (b) and (c), suppose

$p \Vdash_{\mathbb{P}} \phi(\tau_1, \ldots, \tau_n)$. Fix $H \subset \mathbb{Q}$ with $i(p) \in H$ and H $\mathbb{Q}$-generic over M, then $p \in i^{-1}(H)$, so by definition of $\Vdash_{\mathbb{P}}$,

$$\phi(\mathrm{val}(\tau_1, i^{-1}(H)), \ldots, \mathrm{val}(\tau_n, i^{-1}(H)))^{M[i^{-1}(H)]}.$$

Since $\mathrm{val}(\tau_i, i^{-1}(H)) = \mathrm{val}(i_*(\tau_i), H)$, and $M[i^{-1}(H)] \subset M[H]$, we have

$$\phi(\mathrm{val}(i_*(\tau_1), H), \ldots, \mathrm{val}(i_*(\tau_n), H))^{M[H]}$$

(applying absoluteness in (b) and $M[i^{-1}(H)] = M[H]$ in (c)). Thus, by definition of $\Vdash_{\mathbb{Q}}$,

$$i(p) \Vdash_{\mathbb{Q}} \phi(i_*(\tau_1), \ldots, i_*(\tau_n)).$$

To prove (b) and (c) from right to left, assume $\neg(p \Vdash_{\mathbb{P}} \phi(\tau_1, \ldots, \tau_n))$; then there is a $p' \leq p$ such that $p' \Vdash_{\mathbb{P}} \neg\phi(\tau_1, \ldots, \tau_n)$, whence, as we have just seen, $i(p') \Vdash_{\mathbb{Q}} \neg\phi(i_*(\tau_1), \ldots, i_*(\tau_n))$. Since $i(p') \leq i(p)$,

$$\neg(i(p) \Vdash_{\mathbb{Q}} \phi(i_*(\tau_1), \ldots, i_*(\tau_n))). \quad \square$$

We turn now to Boolean-valued models. Our discussion parallels the treatment in II §3 relating MA to Boolean algebras.

If $\mathscr{B}$ is a Boolean algebra, we shall, as in II §3, abuse notation somewhat and apply our forcing terminology to $\mathscr{B}$ when we really mean $\mathscr{B} \setminus \{\mathbb{0}\}$. Thus, $M^{\mathscr{B}}$ is really $M^{\mathbb{P}}$ where $\mathbb{P} = \mathscr{B} \setminus \{\mathbb{0}\}$, and $p \Vdash \phi$ has been defined only when $p \neq \mathbb{0}$. However, it is consistent with our terminology to take $\mathbb{0} \Vdash \phi$ to be always true (vacuously), since no filter contains $\mathbb{0}$.

Now recall that by II 3.3, for every p.o. $\mathbb{P}$ there is a dense embedding i, of $\mathbb{P}$ into some complete Boolean algebra, $\mathscr{B}$. $\mathscr{B}$ is called *the completion* of $\mathbb{P}$ and is unique up to isomorphism (see II Exercise 18). i is not in general 1–1 (see II §3 and Exercises C8 and D3 of this chapter). Applying II 3.3 relativized to M, together with Theorem 7.11, we see that any generic extension of M can be obtained by forcing over M with a $\mathscr{B} \in M$ such that ($\mathscr{B}$ is a complete Boolean algebra)^M. This fact suggests an alternate approach to the exposition of forcing. One may go through the abstract development in §§2–4 only for the very special case of forcing with complete Boolean algebras of M; in this special case, many of the basic definitions are simpler, and it is easier to grasp intuitively what is going on. Then, when, as in §§5–6, we wish to apply a specific p.o. $\mathbb{P}$, which is probably not a Boolean algebra, we simply force not with $\mathbb{P}$, but with the $\mathscr{B} \in M$ such that ($\mathscr{B}$ is the completion of $\mathbb{P}$)^M.

We did not take this approach in this book because we did not wish to make familiarity with Boolean algebras a prerequisite for understanding forcing. Also, the general theory of forcing can easily be adapted to produce generic extensions of models of ZF $-$ P (Exercise B10) or even weaker theories. This has applications in model theory and recursion theory (see

[Keisler 1973] and [Sacks 1971]). Since constructing the completion requires the Power Set Axiom, one cannot, when forcing over these models, reduce the general theory to forcing with complete Boolean algebras of M.

We now outline more specifically what simplifications take place when forcing with complete Boolean algebras of M.

7.14. Definition. If $\mathscr{B} \in M$, ($\mathscr{B}$ is a complete Boolean algebra)^M, and $\tau_1, \ldots, \tau_n \in M$, then

$$[\![\phi(\tau_1, \ldots, \tau_n)]\!] = \bigvee \{p \in \mathscr{B}\colon p \Vdash \phi(\tau_1, \ldots, \tau_n)\}.$$

$[\![\phi(\tau_1, \ldots, \tau_n)]\!]$ is called the *truth value* of $\phi(\tau_1, \ldots, \tau_n)$. $\square$

Of course, "complete" is not absolute, and $\mathscr{B}$ may well fail to be complete *in* $\mathbf{V}$; *in fact, every countable complete Boolean algebra is finite* (Exercise F6). Nevertheless, the definition of $[\![\phi]\!]$ makes sense by the definability of $\Vdash$; $\{p \in \mathscr{B}\colon p \Vdash \phi\}$ is in M, so its supremum exists.

Intuitively, we may think of the people living in M as defining a *Boolean-valued model of set theory*, where the truth values, $[\![\phi]\!]$ (or $[\![\phi(\tau_1, \ldots, \tau_n)]\!]$) may be $\mathbb{1}$ (true) or may be $\mathbb{0}$ (false), but may also have some value in $\mathscr{B}$ intermediate between $\mathbb{0}$ and $\mathbb{1}$. If ϕ is true in all $\mathscr{B}$-generic extensions, then $\mathbb{1} \Vdash \phi$, so $[\![\phi]\!] = \mathbb{1}$, and if ϕ is false in all $\mathscr{B}$-generic extensions, then only $\mathbb{0} \Vdash \phi$, so $[\![\phi]\!] = \mathbb{0}$, but if ϕ is true in some extensions and false in others, then $\mathbb{0} < [\![\phi]\!] < \mathbb{1}$. By definability of $\Vdash$, the M-people are able to define the Boolean truth value of ϕ without ever being able to construct a real (2-valued) generic filter.

(a) of the next lemma says that $[\![\phi]\!]$ is the largest condition which forces ϕ.

7.15. Lemma. *Under the assumptions of Definition* 7.14,

(a) $\forall p \in \mathscr{B}\,(p \Vdash \phi(\tau_1, \ldots, \tau_n) \leftrightarrow p \le [\![\phi(\tau_1, \ldots, \tau_n)]\!])$.
(b) $[\![\phi(\tau_1, \ldots, \tau_n) \wedge \psi(\tau_1, \ldots, \tau_n)]\!] = [\![\phi(\tau_1, \ldots, \tau_n)]\!] \wedge [\![\psi(\tau_1, \ldots, \tau_n)]\!]$.
(c) $[\![\neg\phi(\tau_1, \ldots, \tau_n)]\!] = [\![\phi(\tau_1, \ldots, \tau_n)]\!]'$.
(d) $[\![\exists x\, \phi(x, \tau_1, \ldots, \tau_n)]\!] = \bigvee \{[\![\phi(\sigma, \tau_1, \ldots, \tau_n)]\!]\colon \sigma \in M^{\mathscr{B}}\}$.

Proof. In all cases, we drop explicit mention of $\tau_1, \ldots, \tau_n$.

For (a), $p \Vdash \phi$ implies $p \le [\![\phi]\!]$ by the definition of $[\![\phi]\!]$. Now, assume $p \le [\![\phi]\!]$. Either $p \Vdash \phi$ or there is a (non-$\mathbb{0}$) $q \le p$ such that $q \Vdash \neg\phi$. But for such a q, $\forall r\,(r \Vdash \phi \rightarrow q \wedge r = \mathbb{0})$, so $q \wedge [\![\phi]\!] = \mathbb{0}$, contradicting $\mathbb{0} < q \le p \le [\![\phi]\!]$.

(b)–(d) are easy exercises using (a) and basic properties of $\Vdash$. We do (b) as an example. $[\![\phi]\!] \Vdash \phi$ by (a), and $[\![\phi]\!] \wedge [\![\psi]\!] \le [\![\phi]\!]$, so

$$[\![\phi]\!] \wedge [\![\psi]\!] \Vdash \phi.$$

Likewise, $[\![\phi]\!] \wedge [\![\psi]\!] \Vdash \psi$, so $[\![\phi]\!] \wedge [\![\psi]\!] \Vdash \phi \wedge \psi$, whence

$$[\![\phi]\!] \wedge [\![\psi]\!] \leq [\![\phi \wedge \psi]\!].$$

Also, $[\![\phi \wedge \psi]\!] \Vdash \phi \wedge \psi$, and ZFC $\vdash \phi \wedge \psi \to \phi$, so $[\![\phi \wedge \psi]\!] \Vdash \phi$ (by Lemma 4.5). Thus, $[\![\phi \wedge \psi]\!] \leq [\![\phi]\!]$. Likewise, $[\![\phi \wedge \psi]\!] \leq [\![\psi]\!]$, so

$$[\![\phi \wedge \psi]\!] \leq [\![\phi]\!] \wedge [\![\psi]\!]. \quad \square$$

Lemma 715 (b)–(d) say that in computing Boolean truth values, the logical operations are mirrored by the corresponding Boolean operations. By Lemma 7.15 (a), $\Vdash$ can be defined in terms of $[\![\ldots]\!]$.

Lemma 7.15 suggests a substantial simplification in the treatment of $\Vdash^*$ in §3. Given a complete Boolean algebra, $\mathscr{B}$, we consider $\Vdash^*$ to be an auxiliary notion, introduced by the definition:

$$p \Vdash^* \phi(\tau_1, \ldots, \tau_n) \leftrightarrow p \leq [\![\phi(\tau_1, \ldots, \tau_n)]\!]^*.$$

Thus, $[\![\ldots]\!]^*$ is considered to be the basic notion, and its definition is similar to that of $\Vdash^*$ in Definition 3.3—but much simpler. In the induction steps for $\wedge$, $\neg$, and $\exists$, we use the clauses of Lemma 7.15(b)–(d) as definitions; thus, $[\![\neg\phi]\!]^*$ is defined to be $([\![\phi]\!]^*)'$. The two inductive clauses for atomic formulas are also simplified (see Exercise D7). In all cases in the definition of $[\![\ldots]\!]^*$, logical operations are reflected by Boolean operations in a natural way, and we avoid the use of auxiliary concepts such as "dense below". As with $\Vdash^*$, we may think of the definition of $[\![\ldots]\!]^*$ as taking place within $\mathbf{V}$, but we relativize it to M when discussing generic extensions, using a $\mathscr{B} \in M$ such that ($\mathscr{B}$ is a complete Boolean algebra)^M.

Boolean-valued models also simplify the treatment of nice names (Definition 5.11). It is convenient here to allow $\mathbb{0}$ as a second co-ordinate in names; i.e., re-define $\tau \in V^{\mathscr{B}} \leftrightarrow \tau \subset V^{\mathscr{B}} \times \mathscr{B}$ (not $V^{\mathscr{B}} \times (\mathscr{B} \setminus \{\mathbb{0}\})$, as we would have if we are forcing with $\mathscr{B} \setminus \{\mathbb{0}\}$). It is easily seen that this entails no essential change in the development of §§2–4, and it is more natural to treat all elements of $\mathscr{B}$ equally. Now, if

$$\tau = \bigcup \{\{\pi\} \times A_\pi : \pi \in \mathrm{dom}(\sigma)\}$$

is a nice name for a subset of σ, let $b_\pi = \bigvee A_\pi$ (so $b_\pi = \mathbb{0}$ if $A_\pi = 0$), and let $\tilde{\tau} = \{\langle \pi, b_\pi \rangle : \pi \in \mathrm{dom}(\tau)\}$. Then $\tilde{\tau}$ is a function from $\mathrm{dom}(\sigma)$ into $\mathscr{B}$. Call such a $\tilde{\tau}$ a *very nice name*. It is easily seen that $[\![\tau = \tilde{\tau}]\!] = \mathbb{1}$, so that corresponding to Lemma 5.12, every subset of σ in $M[G]$ is represented by a very nice name. Beside eliminating the somewhat ad hoc introduction of antichains, the use of such names makes it intuitively clearer what the elements of $M[G]$ "are". For example, a very nice name for a subset of $\check{\omega}$ is a function from $\{\check{n} : n \in \omega\}$ into $\mathscr{B}$, which we may indentify with a function $f : \omega \to \mathscr{B}$. We may think of f as a characteristic function of a Boolean-valued subset of ω.

Another simplification occurs in the notion of complete embedding (Definition 7.1). Instead of another seemingly ad hoc definition, this notion, when restricted to complete Boolean algebras reduces to the natural notion of complete injective homomorphism (Exercise C7).

Thus, we see that restricting our p.o.'s to be complete Boolean algebras of M makes the abstract theory somewhat simpler and more natural, although this approach then requires us to go through the additional step of embedding arbitrary p.o.'s into complete Boolean algebras.

For more on Boolean-valued models, see §9.

§8. Further results

In this section we collect some technical results which did not seem to fit in earlier, together with three more examples of forcing. As always, M is a fixed c.t.m. for ZFC.

8.1. LEMMA. *Suppose that in M: A is an antichain in $\mathbb{P}$ and, for each $q \in A$, σ_q is a $\mathbb{P}$-name. Then there is a $\pi \in M^{\mathbb{P}}$ such that $q \Vdash \pi = \sigma_q$ for each $q \in A$.*

PROOF. In M, let

$$\pi = \bigcup_{q \in A} \{\langle \tau, r\rangle : (r \leq q) \wedge (r \Vdash \tau \in \sigma_q) \wedge (\tau \in \mathrm{dom}(\sigma_q))\}.$$

Fix $q \in A$ and fix a generic G with $q \in G$. We show $\pi_G = \mathrm{val}(\sigma_q, G)$.

Any element of π_G is of the form τ_G where $\langle \tau, r\rangle \in \pi$ for some $r \in G$. Then for some $q' \in A$, $r \leq q'$ and $r \Vdash \tau \in \sigma_{q'}$. But A is an antichain and $q \in G$, so $q' = q$. Thus, $r \Vdash \tau \in \sigma_q$, so $\tau_G \in \mathrm{val}(\sigma_q, G)$.

Any element of $\mathrm{val}(\sigma_q, G)$ is of the form τ_G for some $\tau \in \mathrm{dom}(\sigma_q)$. Fix $p \in G$ with $p \Vdash \tau \in \sigma_q$, and let $r \in G$ be a common extension of p and q. Then $\langle \tau, r\rangle \in \pi$, so $\tau_G \in \pi_G$. $\square$

As motivation for the next result, observe that in proving that $M[G]$ satisfies ZFC, every time that we verified an existential statement we produced a name which witnessed the existence independently of G. For example, to verify the Pairing Axiom, it would have been sufficient to show that for each $\sigma, \tau \in M^{\mathbb{P}}$, $\mathbb{1} \Vdash \exists x\, (\sigma \in x \wedge \tau \in x)$; equivalently for every generic G there is a π such that $\sigma_G \in \pi_G \wedge \tau_G \in \pi_G$. But in fact, we found a π, namely $\mathrm{up}(\sigma, \tau)$, which was independent of G (see Definition 2.16), such that $\mathbb{1} \Vdash (\sigma \in \pi \wedge \tau \in \pi)$. The fact that this can be done is part of a more general fact, known as the *maximal principle*.

8.2. THEOREM. *If $\mathbb{P} \in M$, $\tau_1, \ldots, \tau_n \in M^{\mathbb{P}}$, and $p \Vdash \exists x\, \phi(x, \tau_1, \ldots, \tau_n)$, then there is a $\pi \in M^{\mathbb{P}}$ such that $p \Vdash \phi(\pi, \tau_1, \ldots, \tau_n)$.*

PROOF. We suppress throughout mention of $\tau_1, \ldots, \tau_n$. Using Zorn's Lemma in M, let $A \in M$ be such that

(1) A is an antichain in $\mathbb{P}$.

(2) $\forall q \in A\,(q \leq p \wedge \exists \sigma \in M^{\mathbb{P}}\,(q \Vdash \phi(\sigma)))$.

(3) A is a maximal with respect to (1) and (2).

By AC in M, pick $\sigma_q \in M^{\mathbb{P}}$ for $q \in A$ so that $q \Vdash \phi(\sigma_q)$, and by Lemma 8.1, let $\pi \in M^{\mathbb{P}}$ be such that $q \Vdash \pi = \sigma_q$ for each $q \in A$. So, for $q \in A$,

$$q \Vdash (\phi(\sigma_q) \wedge \pi = \sigma_q),$$

so $q \Vdash \phi(\pi)$.

We now show that $p \Vdash \phi(\pi)$. If not, let $r \leq p$ be such that $r \Vdash \neg\phi(\pi)$. Since $p \Vdash \exists x\, \phi(x)$, $\{q\colon \exists \sigma \in M^{\mathbb{P}}\,(q \Vdash \phi(\sigma))\}$ is dense below p (see Corollary 3.7), so fix $q_0 \leq r$ with $\exists \sigma \in M^{\mathbb{P}}\,(q_0 \Vdash \phi(\sigma))$. For each $q \in A$, we have $q \Vdash \phi(\pi)$ and $q_0 \Vdash \neg\phi(\pi)$, so $q \perp q_0$. Thus, $A \cup \{q_0\}$ satisfies (1) and (2) above, contradicting maximality of A. □

Theorem 8.2 is best appreciated in the framework of Boolean-valued models (see §7). By Lemma 7.15,

$$[\![\exists x\, \phi(x)]\!] = \bigvee \{[\![\phi(\sigma)]\!]\colon \sigma \in M^{\mathbb{P}}\},$$

but Theorem 8.2 says that we can in fact find a $\sigma \in M^{\mathbb{P}}$ such that $[\![\phi(\sigma)]\!]$ is the maximum possible value, $[\![\exists x\, \phi(x)]\!]$.

We now turn to some more applications of forcing.

Many of the combinatorial principles known to be true in **L** can also be proved consistent by forcing. The easiest of these is $\diamondsuit$ (see II 7.1). This argument uses countable partial functions, but has a different flavor from the methods of §6.

8.3. THEOREM. *Let $\mathbb{P} = \mathrm{Fn}(\omega_1, 2, \omega_1)^M$. If G is $\mathbb{P}$-generic over M, then $\mathscr{P}(\omega) \cap M = \mathscr{P}(\omega) \cap M[G]$, $\omega_1^M = \omega_1{}^{M[G]}$, and $\diamondsuit$ holds in $M[G]$.*

PROOF. The first two statements are immediate from $\mathbb{P}$ being ω_1-closed in M (see 6.14 and 6.15). The argument for $\diamondsuit$ will be more transparent if we use a different p.o. isomorhic to $\mathbb{P}$ in M. Let $I = \{\langle \alpha, \xi \rangle\colon \xi < \alpha < \omega_1^M\}$, and let $\mathbb{Q} = \mathrm{Fn}(I, 2, \omega_1)^M$. Since $|I| = \omega_1$ in M, $\mathbb{P}$ and $\mathbb{Q}$ are isomorphic in M, so (by Corollary 7.6) it is sufficient to check that whenever G is $\mathbb{Q}$-generic over M, $\diamondsuit$ holds in $M[G]$.

If A is any function from I into 2, we let $A_\alpha\colon \alpha \to 2$ be defined by $A_\alpha(\xi) = A(\alpha, \xi)$ (for $\alpha < \omega_1^M$). If G is $\mathbb{Q}$-generic over M, then $\bigcup G$ is a function from I into 2. We shall prove $(\diamondsuit)^{M[G]}$ by showing that in $M[G]$, $\langle (\bigcup G)_\alpha\colon \alpha < \omega_1^M \rangle$ is a $\diamondsuit$-sequence if we identify sets with their characteristic functions. Equivalently, we shall show that if $B \in M[G]$ and $B : \omega_1^M \to 2$, then $\{\alpha\colon B \restriction \alpha = (\bigcup G)_\alpha\}$ is stationary in $M[G]$. Assume that this is false. Then there is a name $\tau \in M^{\mathbb{Q}}$ (for B), a name $\sigma \in M^{\mathbb{Q}}$ (for a c.u.b.), and a $p \in G$ such that

$$p \Vdash [(\tau \subset \omega_1) \wedge (\sigma \subset \omega_1) \wedge (\sigma \text{ is c.u.b.}) \wedge \forall \alpha \in \sigma (\tau \restriction \alpha \neq (\bigcup \Gamma)_\alpha)], \quad (*)$$

where $\Gamma = \{\langle \check{q}, q \rangle : q \in \mathbb{Q}\}$ is the $\mathbb{Q}$-name for the $\mathbb{Q}$-generic filter (see Definition 2.12). We now forget about G and B, and derive a contradiction directly from $(*)$.

The following paragraph takes place in M. For any $q \in \mathbb{Q}$, let $\text{supt}(q)$ be the least $\beta < \omega_1$ such that $\text{dom}(q) \subset \{\langle \alpha, \xi \rangle : \xi < \alpha < \beta\}$. Now, define inductively p_n ($n \in \omega$) along with β_n, δ_n, and b_n so that

(1) $p_0 = p$.
(2) $\beta_n = \text{supt}(p_n)$.
(3) $\delta_n > \beta_n$.
(4) $p_{n+1} \leq p_n$.
(5) $p_{n+1} \Vdash \check{\delta}_n \in \sigma$.
(6) $\text{supt}(p_{n+1}) > \delta_n$.
(7) $b_n : \beta_n \to 2$ and $p_{n+1} \Vdash (\tau \restriction \check{\beta}_n = \check{b}_n)$.

Let us check that this induction can be accomplished. Given p_n, β_n is defined. Since $p_n \leq p$, $p_n \Vdash (\sigma$ is c.u.b.$)$, so $p_n \Vdash \exists x \in \check{\omega}_1 (x > \check{\beta}_n \wedge x \in \sigma)$. Thus, applying Corollary 3.7(d), there is a $q \leq p_n$ and a $\delta_n \in \omega_1$ such that $q \Vdash (\check{\delta}_n > \check{\beta}_n \wedge \check{\delta}_n \in \sigma)$, so $\delta_n > \beta_n$ and $q \Vdash \check{\delta}_n \in \sigma$. Let r be any extension of q with $\text{supt}(r) > \delta_n$. Finally, to handle (7), let $F = {}^{(\beta_n)}2$. Then $r \Vdash \tau \restriction \check{\beta}_n \in \check{F}$, since an ω_1-closed p.o. adds no new functions from β_n into 2 (see Theorem 6.14). Thus, there is a $b_n \in F$ and a $p_{n+1} \leq r$ such that $p_{n+1} \Vdash \tau \restriction \check{\beta}_n = \check{b}_n$ (see Corollary 3.7(d)).

Still within M, we have $\beta_0 < \delta_0 < \beta_1 < \delta_1 < \cdots$. Let

$$\gamma = \sup\{\beta_n : n \in \omega\} = \sup\{\delta_n : n \in \omega\}.$$

Let $p_\omega = \bigcup_n p_n$; then $\text{supt}(p_\omega) = \gamma$. For each $n < \omega$, $p_\omega \leq p_{n+1}$ so $p_\omega \Vdash (\tau \restriction \check{\beta}_n = \check{b}_n)$. Thus, the b_n for $n \in \omega$ agree on their common domains, $b_\omega = \bigcup_n b_n$ is a function from γ to 2, and $p_\omega \Vdash (\tau \restriction \check{\gamma} = \check{b}_\omega)$. Now there are no pairs $\langle \gamma, \xi \rangle \in \text{dom}(p_\omega)$, so we may extend p_ω to an s such that $s(\gamma, \xi) = b_\omega(\xi)$ for each $\xi < \gamma$. Then $s \Vdash [(\bigcup \Gamma)_{\check{\gamma}} = \check{b}_\omega]$, so $s \Vdash [\tau \restriction \check{\gamma} = (\bigcup \Gamma)_{\check{\gamma}}]$. But also, $s \Vdash (\check{\gamma} \in \sigma)$ since $s \Vdash (\sigma$ is closed) and $s \Vdash (\check{\delta}_n \in \sigma)$ for each n. Thus,

$$s \Vdash [\exists \alpha \in \sigma (\tau \restriction \alpha = (\bigcup \Gamma)_\alpha)],$$

which, since $s \leq p$, is a contradiction. □

Since $\diamondsuit \to \text{CH}$, CH holds in $M[G]$ regardless of whether it holds in M; what happens is that if CH fails in M, the cardinal $(2^\omega)^{(M)}$ gets collapsed in $M[G]$.

We cannot use the method of Theorem 8.3 to prove the consistency of $\diamondsuit^+$, since $\diamondsuit^+$ may fail in $M[G]$ (see VIII Exercises J5–J7). If M is chosen carefully, $\diamondsuit^+$ will hold in $M[G]$. For example, if M satisfied $\mathbf{V} = \mathbf{L}$, then $M[G]$ would satisfy $\exists X \subset \omega_1(\mathbf{V} = \mathbf{L}(X))$, which implies $\diamondsuit^+$ (see VI Exercise 8). Of course, M also satisfies $\diamondsuit^+$, so this forcing argument establishes nothing new. One can, in the spirit of Theorem 8.3, prove the consistency of $\diamondsuit^+$ by forcing, but the partial order is more complicated (see Exercises H18–H20).

Theorem 8.3 does tell us something that $\mathbf{L}$ did not; namely, that $\diamondsuit$ does not imply $2^{\omega_1} = \omega_2$. To see this, start with M satisfying CH + $2^{\omega_1} > \omega_2$; there is such an M by Theorem 6.18. Then $\mathbb{P}$ will not collapse cardinals, so $M[G]$ will satisfy $\diamondsuit + 2^{\omega_1} > \omega_2$. Likewise, by Exercise H20, $\diamondsuit^+$ does not imply $2^{\omega_1} = \omega_2$.

Since $\diamondsuit$ implies that there is an ω_1-Suslin tree, there is one in the $M[G]$ of Theorem 8.3. Historically, Jech first showed that one can add a Suslin tree in a generic extension, and with the advent of $\diamondsuit$, his argument was easily modified to yield Theorem 8.3. His original argument is still of some interest, since it can be generalized to add a κ-Suslin tree for any regular κ of M (see Exercises H11–H14), even though the existence of such a tree does not follow from the analogue of $\diamondsuit$ on κ.

It is also easy to destroy a Suslin tree; in fact if we stand the tree upside down it will destroy itself.

8.4. THEOREM. *Let $\kappa > \omega$ be a regular cardinal of M and T a κ-Suslin tree of M. Then there is a* p.o. $\mathbb{P} \in M$, *such that* $(\mathbb{P}$ *has* κ-c.c.$)^M$, $\mathbb{P}$ *preserves cofinalities, and, whenever G is $\mathbb{P}$-generic over M,*

(1) *T is not a κ-Suslin tree in $M[G]$, and*

(2) *If $\alpha < \kappa$ and $B \in M$, then ${}^{\alpha}B \cap M[G] = {}^{\alpha}B \cap M$.*

PROOF. We begin by imitating the proof that MA(ω_1) implies that there are no ω_1-Suslin trees (see II 5.14). Within M, let T' be a well-pruned κ-sub-tree of T (see II 5.11). Let $\lhd$ be the tree order on T', and let $\mathbb{P}$ be T' with the reverse order; $p <_{\mathbb{P}} q$ iff $q \lhd p$; then $\mathbb{P}$ is a p.o., with $\mathbb{1}_{\mathbb{P}}$ the least element of the tree T'. Since T' is well-pruned, $D_\alpha = \{p\colon \mathrm{ht}(p, T') > \alpha\}$ is dense in $\mathbb{P}$ for each $\alpha < \kappa$. If G is a filter in $\mathbb{P}$, then G is a chain in T'. If G is $\mathbb{P}$-generic over M, then $G \cap D_\alpha \neq 0$ for all $\alpha < \kappa$, so G contains elements of T' of arbitrarily large height below κ. Thus, T' is not even Aronszajn in $M[G]$.

Since T' is a κ-Suslin tree in M, $(\mathbb{P}$ has the κ-c.c.$)^M$, so $\mathbb{P}$ preserves cofinalities $\geq\kappa$ (see Lemma 6.9). We shall thus be done if we can check (2), since (2) implies that $\mathbb{P}$ preserves cofinalities $\leq\kappa$ (as in the proof of Corollary 6.15 from Theorem 6.14).

To prove (2), suppose $f\colon \alpha \to B$, $\alpha < \kappa$, $B \in M$, and $f \in M[G]$. Say $f = \tau_G$. For each $\xi < \alpha$, there is a $p_\xi \in G$ such that $p_\xi \Vdash \tau(\check{\xi}) = (f(\xi))\check{}$. Furthermore, the sequence $\langle p_\xi : \xi < \alpha \rangle$ may be chosen in $M[G]$ by using AC in $M[G]$ and the fact that

$$\{\langle p, \xi, y \rangle : p \in \mathbb{P} \wedge y \in B \wedge \xi \in \alpha \wedge p \Vdash \tau(\check{\xi}) = \check{y}\}$$

is in M and hence in $M[G]$. Now G is a chain, $\{p_\xi : \xi < \alpha\} \subset G$, and κ is regular in $M[G]$, so there is a $q \in G$ above all the p_ξ in T'; i.e., $\forall \xi < \alpha\, (q \leq p_\xi)$. Then $q \Vdash \tau(\check{\xi}) = (f(\xi))\check{}$ for each $\xi < \alpha$, so

$$f = \{\langle \xi, y \rangle \in \alpha \times B : q \Vdash \tau(\check{\xi}) = \check{y}\} \in M. \quad \square$$

We see in Theorem 8.4 our first application in forcing of a p.o. which did not arise from partial functions (although other such p.o.'s have already been used when applying MA in II). Theorem 8.4 also illustrates that 8.4(2) can hold for $\mathbb{P}$ without $\mathbb{P}$ being κ-closed in M. A well-pruned κ-Suslin tree can never be κ-closed, since if it were, one could inductively pick a path through it. For any $\mathbb{P}$. the conclusion 8.4(2) is equivalent to $\mathbb{P}$ having the property of being κ-Baire in M (see Exercise B4). It is possible to show directly that a κ-Suslin tree is κ-Baire (Exercise H16), without mentioning forcing, but the proof given of Theorem 8.4(2) is somewhat easier.

The ease with which Theorem 8.4 was proved should not mislead one to thinking that it is easy to prove the consistency of SH, since for this one needs to iterate forcing a transfinite number of times to destroy all the ω_1-Suslin trees in M. We shall do such an iteration in VIII to prove the consistency of MA + $\neg$CH, and hence of SH.

Since Theorem 8.4 says that one may destroy an ω_1-Suslin tree without enlarging ${}^{\omega}2$, it might be expected that SH + CH is consistent. It is, but the iterated forcing argument (due to Jensen; see [Devlin–Johnsbråten 1974]) is more difficult than the ones considered in this book.

When $\kappa > \omega_1$, it becomes more difficult to destroy all κ-Suslin trees, and there are still a number of open questions regarding the consistency of the κ-Suslin Hypothesis. Most notably, it is unknown whether GCH + ω_2-SH is consistent; it "should" be in view of Theorem 8.4, but [Gregory 1976] shows that GCH + ω_2-SH implies that ω_2 is Mahlo in L. For more on the κ-SH, see II §5.

To introduce our next example of forcing, we ask: what is $\omega_1^{M[G]}$? $\omega_1^{M[G]}$ must be regular in M, and we shall see that by suitably defining $\mathbb{P}$, we can arrange for $\omega_1^{M[G]}$ to be any regular cardinal of M except ω. We first remark that we have already taken care of successor cardinals.

8.5. Lemma. *Suppose that in M, λ is an infinite cardinal and $\kappa = \lambda^+$. Let $\mathbb{P} = \mathrm{Fn}(\omega, \lambda)$. Let G be $\mathbb{P}$-generic over M. Then $\omega_1^{M[G]} = \kappa$.*

Proof. If $\alpha < \kappa$, then in M there is a map from λ onto α. Since $\bigcup G$ maps ω onto λ, (α is countable)$^{M[G]}$. Thus, $\omega_1^{M[G]} \geq \kappa$. However, $(|\mathbb{P}| = \lambda)^M$, so

($\mathbb{P}$ has the κ-c.c.)^M, so $\mathbb{P}$ preserves cardinals $\geq\kappa$. Thus, κ remains a cardinal in $M[G]$, so $\omega_1^{M[G]} = \kappa$. $\square$

If κ is weakly inaccessible in M, we force with finite functions in a slightly different way, collapsing every ordinal below κ, but preserving κ. We wish to add, for every $\alpha < \kappa$, a map $f_\alpha\colon \omega \to \alpha$. For notational convenience, we code $\langle f_\alpha\colon \alpha < \kappa\rangle$ as on function with domain $\kappa \times \omega$.

8.6. DEFINITION. For any κ, the *Lévy collapsing order* for κ, $\mathrm{Lv}(\kappa)$, is

$$\{p\colon |p| < \omega \ \wedge\ p \text{ is a function } \wedge\ \mathrm{dom}(p) \subset \kappa \times \omega\ \wedge$$
$$\wedge\ \forall \langle \alpha, n\rangle \in \mathrm{dom}(p)\,(p(\alpha, n) \in \alpha)\}.$$

$\mathrm{Lv}(\kappa)$ is ordered by reverse inclusion; $p \leq q$ iff $q \subset p$. $\square$

We shall show now that if κ is a regular uncountable cardinal of M, then forcing with $\mathrm{Lv}(\kappa)$ makes $\kappa = \omega_1^{M[G]}$. This is usually only of interest when κ is weakly inaccessible in M, since successor cardinals were handled by the easier argument of Lemma 8.5.

8.7. LEMMA. *If κ is regular and uncountable, then* $\mathrm{Lv}(\kappa)$ *has the* κ-c.c.

PROOF. Fix $p_\mu \in \mathrm{Lv}(\kappa)$ for $\mu < \kappa$. By the Δ-system lemma (II 1.6), there is a set $B \subset \kappa$ such that $|B| = \kappa$, and $\{\mathrm{dom}(p_\mu)\colon \mu \in B\}$ forms a Δ-system with some root r. Since κ is regular and there are less than κ possibilities for the $p_\mu \restriction r$, there is a $C \subset B$ with $|C| = \kappa$ and the $p_\mu \restriction r$ for $\mu \in C$ all the same; then the p_μ for $\mu \in C$ are pairwise compatible. In particular, the p_μ for $\mu < \kappa$ could not have been pairwise incompatible. $\square$

8.8. THEOREM. *Suppose that in M, κ is regular and uncountable. Let G be* $\mathrm{Lv}(\kappa)$*-generic over M. Then $\kappa = \omega_1^{M[G]}$.*

PROOF. Applying Lemma 8.7 in M, ($\mathrm{Lv}(\kappa)$ has the κ-c.c.)^M, so κ remains a cardinal in $M[G]$. However, if $0 < \alpha < \kappa$, then standard density arguments show that

$$(\textstyle\bigcup G)_\alpha = \{\langle n, \xi\rangle\colon \langle \alpha, n, \xi\rangle \in \bigcup G)$$

is a function from ω onto α, so α is countable in $M[G]$. $\square$

[Solovay 1970] uses the Lévy collapsing order to get a model of ZF in which all subsets of $\mathbb{R}$ are Lebesgue measurable. More precisely, he starts with κ strongly inaccessible in M. In $M[G]$ it is true that if $X \subset \mathbb{R}$ and $X \in \mathbf{HOD}\,(\mathscr{P}(\omega))$ (defined in V Exercise 9), then X is Lebesgue measurable.

Of course, $M[G]$ satisfies AC and thus has a non-measurable set, but the inner model $\mathbf{HOD}(\mathscr{P}(\omega))$ (defined within $M[G]$) satisfies ZF + "all sets of reals are Lebesgue measurable." It is unknown whether one can prove from Con(ZFC) alone the consistency of ZF + "all sets of reals are Lebesgue measurable."[1]

The Lévy partial order is also relevant to Kurepa's Hypothesis, KH (see II 5.15).

8.9. Corollary. *Suppose that in M, κ is strongly inaccessible. Let G be $\mathrm{Lv}(\kappa)$-generic over M. Then* $(\mathrm{KH})^{M[G]}$.

Proof. In M, let T be the complete binary tree of height κ. Since cardinals $\geq\kappa$ do not get collapsed, and since in M T has $\geq\kappa^+$ paths, (T is an ω_1-Kurepa tree$)^{M[G]}$. □

However, one does not need inaccessibles to get models of KH. To prove $\mathrm{Con(ZFC)} \to \mathrm{Con(ZFC + KH)}$, one can either use the fact that KH follows from $\diamondsuit^+$ and thus holds in $\mathbf{L}$ (see II 7.11 and VI 5.2), or one can use a different forcing argument which does not require inaccessibles (see Exercise H19).

Ironically, a minor variant of $\mathrm{Lv}(\kappa)$ can be used to construct a model of $\neg$KH. Again, κ is strongly inaccessible in M, but we force with countable (in M) partial functions to add, for $\alpha < \kappa$, a map from ω_1^M onto α. The argument in Theorem 8.8 is easily modified to show that $\omega_1^M = \omega_1^{M[G]}$ and $\kappa = \omega_2^{M[G]}$. However, to prove $(\neg\mathrm{KH})^{M[G]}$, we need the technique of iterated forcing, so we defer the further discussion of this order until VIII §3.

Unlike KH, $\neg$KH needs an inaccessible, since $\neg$KH implies that ω_2 is inaccessible in $\mathbf{L}$ (Exercise B9).

§9. Appendix: Other approaches and historical remarks

There are several different ways of presenting forcing. They all yield precisely the same consistency proofs, but they differ in their metamathematical conception. We survey here the various approaches.

Approach 1: via countable transitive models. This is usually the approach favored by non-logicians, since we handle models and their extensions in a rather straightforward mathematical way. An analogy is often drawn between generic extensions and field extensions in algebra, where one also uses names (polynomials) for objects in the extension field. One can always skim over, or ignore, the rather tedious details in §§3 and 4 of showing that the procedure really works. One can likewise ignore the logical unpleas-

[1](Added in proof.) One cannot, by a recent result of Shelah.

antries associated with the fact that in ZFC we cannot actually produce a c.t.m. for ZFC. These unpleasantries may be handled in one of the following three ways.

(1a) The approach in this book. We show that, given any finite list, $\phi_1, \ldots, \phi_n$, of axioms of, say, ZFC + $\neg$CH, we can prove in ZFC that there is a c.t.m. for $\phi_1, \ldots, \phi_n$. The procedure involves finding (in the metatheory) another finite list $\psi_1, \ldots, \psi_m$ of axioms of ZFC, and proving in ZFC that given a c.t.m. M for $\psi_1, \ldots, \psi_m$, there is a generic extension, $M[G]$, satisfying $\phi_1, \ldots, \phi_n$. The inelegant part of this argument is that the procedure for finding $\psi_1, \ldots, \psi_m$, although straightforward, completely effective, and finitistically valid, is also very tedious. We must list in $\psi_1, \ldots, \psi_m$ not only the axioms of ZFC "obviously" used in checking that $\phi_1, \ldots, \phi_n$ hold in $M[G]$ (e.g., if ϕ_1 is the Power Set Axiom, then ϕ_1 should be listed among $\psi_1, \ldots, \psi_m$), but also all the axioms needed to verify that various concepts are absolute for M ("finite", "p.o.", etc.), as well as the axioms needed to show that certain mathematical results, such as the Δ-system lemma, hold in M. Of course, for the relative consistency proof,

$$\mathrm{Con(ZFC)} \to \mathrm{Con(ZFC} + \neg\mathrm{CH)},$$

it is not necessary to display explicitly $\psi_1, \ldots, \psi_m$; it is sufficient to convince the reader that $\psi_1, \ldots, \psi_m$ may be found.

(1b) This is a way of avoiding dealing with an unspecified list, $\psi_1, \ldots, \psi_m$. Let $\mathscr{L}$ be the language with basic non-logical symbols $\in$ and c, where c is a constant symbol. Let T be the theory in $\mathscr{L}$ consisting of ZFC (written using just $\in$), plus the sentence "c is countable and transitive," plus the sentence ϕ^{c} for each axiom ϕ of ZFC. T may be seen (finitistically) to be a conservative extension of ZFC, since, by the Reflection Theorem (IV 7.11) any finite sub-theory of T may be interpreted within ZFC. Thus, Con(ZFC) $\to$ Con(T). Within T, one can produce an extension $c[G]$ and prove $\phi^{c[G]}$ for each axiom ϕ of ZFC + $\neg$CH. Thus,

$$\mathrm{Con}(T) \to \mathrm{Con(ZFC} + \neg\mathrm{CH)}.$$

(1c) One can formalize logic within ZFC and then write a predicate, $M \models \ulcorner\mathrm{ZFC}\urcorner$, in the free variable M (see IV §10). It is then a formal theorem of ZFC that

$$\forall M\,(M \text{ is a c.t.m. for } \ulcorner\mathrm{ZFC}\urcorner \to \exists N \supset M\,(N \text{ is a c.t.m. for } \ulcorner\mathrm{ZFC} + \neg\mathrm{CH}\urcorner)).$$

The mathematics of the proof of this theorem is precisely the material in §§2–5, although the logical interpretation is different. This approach does not yield a finitistic proof of Con(ZFC) $\to$ Con(ZFC + $\neg$CH), since in ZFC one cannot prove that

$$\exists M\,(M \text{ is a c.t.m. for } \ulcorner\mathrm{ZFC}\urcorner),$$

but it should convince the confirmed Platonist that Con(ZFC + $\neg$CH) is "true", since the existence of such an M is "true" (see IV §7).

Approach 2: via syntactical models, or forcing over **V**. Here one never discusses set models at all.

(2a) In §3, we defined an auxiliary notion, $\Vdash^*$. The definition of $\Vdash^*$ did not refer to models, although our intent was to relativize it to M to prove that $(p \Vdash \phi) \leftrightarrow (p \Vdash^* \phi)^M$. After §3, we essentially forgot about $\Vdash^*$, although we frequently used the result that $p \Vdash \phi$ was equivalent to some (it did not matter which) formula relativized to M. However, in the syntactical model approach, we forget about $\Vdash$, do $\Vdash^*$ in **V**, and never relativize it to anything. One must check that all the facts that we developed about $\Vdash$ may in fact be proved in **V** about $\Vdash^*$; see Exercises B12–B15. Eventually one checks that $\mathbb{1} \Vdash^* \phi$ whenever ϕ is an axiom for ZFC, and that, with the right $\mathbb{P}$, $\mathbb{1} \Vdash^* \neg$CH. It is also necessary to verify the following.

9.1. Lemma. *If* $\phi_1, \ldots, \phi_n \vdash \psi$ *and* $p \Vdash^* \phi_1, \ldots, \phi_n$, *then* $p \Vdash^* \psi$. $\square$

Using Lemma 9.1, if we succeed in finding a contradiction in ZFC + $\neg$CH, say

$$\phi_1, \ldots, \phi_n \vdash \psi \wedge \neg\psi,$$

where $\phi_1, \ldots, \phi_n$ are axioms of ZFC + $\neg$CH, then in ZFC we could prove $\mathbb{1} \Vdash^* \psi \wedge \neg\psi$, which is easily seen to contradict the definition of $\Vdash^*$. Thus, we have defined a procedure for finding a contradiction in ZFC, given one in ZFC + $\neg$CH, so Con(ZFC) $\rightarrow$ Con(ZFC + $\neg$CH). This approach is unpalatable to some, since a model in the traditional sense for ZFC + $\neg$CH is never constructed. We may think of this approach as putting ourselves (in **V**) in the place of the M-people of the c.t.m. approach; so ware making up names for, and talking about, objects in some generic extension of **V** which does not exist at all (to us).

(2b) The Boolean-valued model approach (over **V**) is the special case of (2a) in which we consider only p.o.'s which are complete Boolean algebras. As in §7, this special case will produce all the independence proofs that can be done using arbitrary p.o.'s. This special case has, perhaps, a clearer intuitive motivation, since we may think that we really are creating a model, $\mathbf{V}^{\mathscr{B}}$, for ZFC + $\neg$CH, except that the model is in many-valued logic, with truth values lying in some complete Boolean algebra. Lemma 9.1 becomes now the following lemma, which asserts that Boolean valued logic, like 2-valued logic, is valid for classical proof theory.

9.2. Lemma. *If* $\phi_1, \ldots, \phi_n \vdash \psi$, *then* $[\![\phi_1]\!]^* \wedge \cdots \wedge [\![\phi_n]\!]^* \leq [\![\psi]\!]^*$. $\square$

Intuitively, we may think of $\mathbf{V}$ as a sub-class of $\mathbf{V}^{\mathscr{B}}$, although formally we are defining an embedding, $\check{}$, from $\mathbf{V}$ into $\mathbf{V}^{\mathscr{B}}$. For a detailed development of this approach, see [Rosser 1969] or [Bell 1977].

(2c) Two-valued class models. This is just a curiosity, but in (2b) if G is *any* ultrafilter on $\mathscr{B}$, we may form a two-valued relational system, $\mathbf{V}^{\mathscr{B}}/G$ as follows: $\mathbf{V}^{\mathscr{B}}/G$ has $\mathbf{V}^{\mathscr{B}}$ as its base class, but interprets $\in$ as (2-valued) binary relation E, and interprets $=$ as another binary relation $\equiv$ (instead of as real equality, as is more common in model theory). We define $\tau \, E \, \sigma$ iff $[\![\tau \in \sigma]\!]^* \in G$ and $\tau \equiv \sigma$ iff $[\![\tau = \sigma]\!]^* \in G$. We then show, by induction on ϕ (in the metatheory), that

$$[\![\phi(\tau_1, \ldots, \tau_n)]\!]^* \in G \leftrightarrow V^{\mathscr{B}}/G \models \phi[\tau_1, \ldots, \tau_n].$$

The induction is straightforward except in the step for $\exists$, where essential use is made of the maximal principle (see Theorem 8.2). Thus, with a suitable $\mathscr{B}$, $V^{\mathscr{B}}/G$ will be a 2-valued model for ZFC + $\neg$CH.

$V^{\mathscr{B}}/G$ is almost never well founded, and thus cannot be identified with a transitive model for set theory. See Exercise B18 for more details.

We now briefly survey the history of forcing. Forcing was invented by Cohen, who used it to establish the consistency of ZFC + $\neg$CH, ZF + $\neg$AC, and ZFC + GCH + $\mathbf{V} \neq \mathbf{L}$. Cohen conceived of forcing via the syntactical model approach (2a), but developed the presentations (1a) and (1c) in his published works (see [Cohen 1963, 1964, 1966]), so as to deal with real models. The modification (1b) is due to Shoenfield.

Cohen's original treatment made forcing seem very much related to the constructible hierarchy. His M was always a model for $\mathbf{V} = \mathbf{L}$, so $M = L(\gamma)$, where $\gamma = o(M)$, and his $M[G]$ was defined as $L(\gamma, G)$ (as in VI Exercise 6); this is in fact an equivalent definition in this case (see Exercise B10). He also did not have the idea of working with an abstract p.o., but thought of his conditions as associated with sets of statements in a formal language. Scott and Solovay developed the approach (2b), using an arbitrary complete Boolean algebra, and they realized that the $L(\alpha)$ construction really had nothing to do with forcing, but that it was the $R(\alpha)$ construction that was relevant; in fact, $\mathbf{V}^{\mathscr{B}}$ may be thought of as constructed by iterating the $\mathscr{B}$-valued power set operation (see Exercise B17). They also saw that one could embed any p.o. densely into a complete Boolean algebra, so that the Boolean algebra approach is completely general.

Modern expositions of forcing owe much to Shoenfield, who realized that one could do the Scott–Solovay construction directly from a p.o., without embedding it first into a Boolean algebra. He also invented our definition of $\Vdash$. Previously, there was only $\Vdash^*$. Expositions defined $\Vdash^*$, and proved the basic fact that

$$\phi^{M[G]} \leftrightarrow \exists p \in G\,((p \Vdash^* \phi)^M)$$

(our 3.5). Once this is done, it follows easily that

$$(p \Vdash^* \phi)^M \leftrightarrow \forall \text{generic } G\,(p \in G \to \phi^{M[G]}),$$

but there is really no need to introduce the notion of $\Vdash$, since we may always refer to $\Vdash^*$. The advantage of introducing $\Vdash$ first by the definition

$$p \Vdash \phi \leftrightarrow \forall \text{generic } G\,(p \in G \to \phi^{M[G]}),$$

as we have done (following [Shoenfield 1971a]), is that the reader may gain some insight into what is going on before plunging into the details of $\Vdash^*$.

In the literature, the notations $\Vdash^*$ and $[\![\phi]\!]^*$ are not used. Thus, $p \Vdash \phi$ may mean, in our notation, $p \Vdash \phi$, $p \Vdash^* \phi$, or $(p \Vdash^* \phi)^M$, depending on context. Once the basics are understood, this ambiguity never causes confusion.

Actually, our forcing, $\Vdash$, was what was once called "weak forcing". Cohen defined strong forcing, $\Vdash_s$, as the basic concept. Unlike $\Vdash$, $\Vdash_s$ did not respect logical equivalence; for example, $p \Vdash_s \neg\neg\phi$ did not imply $p \Vdash \phi$. Weak forcing was defined in terms of $\Vdash_s$ by: $p \Vdash \phi$ iff $p \Vdash_s \neg\neg\phi$. $\Vdash_s$ may now seem like a historical anachronism, but it is still relevant to intuitionistic logic, where $\neg\neg\phi$ is not equivalent to ϕ.

There are two important precursors to the modern theory of forcing: one in recursion theory and one in model theory.

In recursion theory, many classical results may be viewed, in hindsight, as forcing arguments. Consider, for example, the Kleene–Post theorem that there are incomparable Turing degrees. Let $\mathbb{P} = \mathrm{Fn}(2 \times \omega, 2)$, let G be $\mathbb{P}$-generic over M, and think of G as coding $f_0, f_1 \in 2^\omega$, where $f_i(n) = \bigcup G(i,n)$. Then, f_0 and f_1 are recursively incomparable (see Exercise G8). Furthermore, to conclude recursively incomparability of f_0 and f_1, it is not necessary that G be generic over all of M; it is sufficient that G intersect only a few of the arithmetically defined dense sets of M; so few that in fact G, and hence also f_0 and f_1, may be taken to be recursive in $0'$. This forcing argument for producing incomparable degrees below $0'$ is in fact precisely the original Kleene–Post argument, with a slight change in notation. See [Sacks 1971] for some deeper applications of forcing to recursion theory and a comparison of these methods with earlier (pre-forcing) techniques.

In model theory, it was well-known that models with truth values taken in an arbitrary Boolean algebra (instead of $\{\mathbb{0}, \mathbb{1}\}$) were correct for classical logic (in the sense of Lemma 9.2 above). Indeed, one proof of the Gödel Completeness Theorem, due to [Rasiowa–Sikorski 1963], involved first constructing a Boolean valued model for a theory, and then applying the Rasiowa–Sikorski Theorem (Exercise D2) to get a suitably "generic" homomorphism into $\{\mathbb{0}, \mathbb{1}\}$ which produced a 2-valued model. This approach also yields a completeness theorem for logic with *infinite* formulas of countable length. However, using Boolean valued models, the analogous completeness theorem for uncountable languages was known to fail. For example, if ϕ is the infinite distributive law,

$$\bigwedge_{n<\omega} \bigvee_{i<2} P_{n,i} \rightarrow \bigvee_{f \in {}^{\omega}2} \bigwedge_{n<\omega} P_{n,f(n)}$$

(ϕ is a sentence of length 2^{ω}, in proposition letters P_{ni}), then ϕ is valid, but is not derivable using the ordinary infinitary proof rules, since ϕ has truth value $\mathbb{0}$ in a suitable Boolean interpretation—namely, in the regular open algebra of 2^{ω} (which is the completion of $\mathrm{Fn}(\omega, 2)$; see Exercise G9), assign P_{ni} truth value $\{f \in 2^{\omega}: f(n) = i\}$. In modern times, ϕ fails to be valid in the extension $\mathbf{V}^{\mathscr{B}}$ because in the disjunction $\bigvee_{f \in {}^{\omega}2}$, f ranges only over functions in $\mathbf{V}$. For more on Boolean valued methods in infinitary logic, see [Karp 1964].

Ironically, Cohen was not at first aware of these precursors to his work, and the relationships discussed above only became clear as the theory of forcing was developed further.

EXERCISES

In the following, unless we state otherwise: M represents a c.t.m. for ZFC and $\mathbb{P}$ is a p.o.; furthermore, if $\mathbb{P} \in M$, then G is a filter which is $\mathbb{P}$-generic over M.

A. Warming-up exercises

(A1) $p \in \mathbb{P}$ is called an *atom* iff

$$\neg \exists q, r \in \mathbb{P}\, (q \leq p \wedge r \leq p \wedge q \perp r).$$

$\mathbb{P}$ is *non-atomic* iff $\mathbb{P}$ has no atoms. Show that if $\mathbb{P} \in M$ and p is an atom of $\mathbb{P}$, then there is a filter $G \in M$ such that $p \in G$ and G intersects *all* dense subsets of $\mathbb{P}$. *Remark.* This is a converse to Lemma 2.4.

(A2) Assume that $\mathbb{P} \in M$ and $\mathbb{P}$ infinite. Show that there is an $H \subset \mathbb{P}$ such that $M[H]$ is not a model of ZF − P. *Hint.* Fix $f \in M$ such that f maps $\omega \times \omega$ 1–1 into $\mathbb{P}$. Choose H so that $f^{-1}(H)$ is a well-order of ω in type $> \mathrm{o}(M)$.

(A3) In Exercise A2, assume also that $\mathbb{P}$ is non-atomic. Show that H may be chosen to be a filter.

(A4) Suppose that $\mathbb{P} \in M$ and $\mathbb{P}$ is non-atomic. Show that $\{G\colon G$ is $\mathbb{P}$-generic over $M\}$ has cardinality 2^{ω}.

(A5) If $\sigma, \tau \in M^{\mathbb{P}}$, show that $\sigma_G \cup \tau_G = (\sigma \cup \tau)_G$. *Remark.* This does not require G to be generic.

(A6) If $\tau \in M$ $\mathbb{P}$, let

$$\pi = \{\langle \rho, p\rangle\colon \exists \langle \sigma, q\rangle \in \tau\, \exists r\, (\langle \rho, r\rangle \in \sigma \,\wedge\, p \leq r \,\wedge\, p \leq q\}.$$

Show that $\pi_G = \bigcup(\tau_G)$. *Remark.* This requires G to be a filter, but G need not be generic.

(A7) If $\tau, \sigma \in M^{\mathbb{P}}$ and $\operatorname{dom}(\tau), \operatorname{dom}(\sigma) \subset \{\check{n}\colon n \in \omega\}$, let

$$\pi = \{\langle \check{n}, p\rangle\colon \exists q, r(p \leq q \,\wedge\, p \leq r \,\wedge\, \langle \check{n}, q\rangle \in \tau \,\wedge\, \langle \check{n}, r\rangle \in \sigma)\}.$$

Show that $\pi_G = \tau_G \cap \sigma_G$. *Remark.* This requires G to be a filter, but G need not be generic.

(A8) Suppose $\tau \in M^{\mathbb{P}}$ and $\operatorname{dom}(\tau) \subset \{\check{n}\colon n \in \omega\}$. Let

$$\sigma = \{\langle \check{n}, p\rangle\colon \forall q \in \mathbb{P}(\langle \check{n}, q\rangle \in \tau \to p \perp q)\}.$$

Show that $\sigma_G = \omega \setminus \tau_G$. *Hint.*

$$\{r\colon \exists p \geq r(\langle \check{n}, p\rangle \in \sigma \,\vee\, \langle \check{n}, p\rangle \in \tau)\}$$

is dense. *Remark.* To show $\forall \tau \in M^{\mathbb{P}}\, ((\omega \setminus \tau_G) \in M[G])$ requires forcing.

(A9) Assume that $\mathbb{P} \in M$, $p \in \mathbb{P}$, and $\exists q \in \mathbb{P}\, (p \perp q)$. Show that

$$\{\tau \in M^{\mathbb{P}}\colon p \Vdash \tau = \check{0}\}$$

is a proper class in M. *Remark.* Thus, if $p \in G$, $\{\tau \in M^{\mathbb{P}}\colon \tau_G = 0\}$ is not a subset of any set of M.

(A10) Assume $\mathbb{P} \in M$ and that $\mathbb{P}$ is separative (see II Exercise 15). Show that

$$p \Vdash \{\langle\{\langle 0, q\rangle\}, r\rangle\} = \check{1},$$

iff $p \leq r$ and $p \perp q$.

(A11) Assume $\mathbb{P} \in M$ and $p \perp q$ for some $p, q \in \mathbb{P}$. Show that

$$\{\tau \in M^{\mathbb{P}}\colon \mathbb{1} \Vdash \tau = \check{1}\}$$

is a proper class of M. *Hint.* Consider, for any $\sigma \in M^{\mathbb{P}}$,

$$\{\langle\{\langle \sigma, p\rangle\}, q\rangle, \langle \check{0}, \mathbb{1}\rangle\}.$$

(A12) Assume $\mathbb{P} \in M$ and G is a filter in $\mathbb{P}$. Show that the following are equivalent.

(a) $G \cap D \neq 0$, whenever $D \in M$ and D is dense in $\mathbb{P}$.

(b) $G \cap A \neq 0$, whenever $A \in M$ and A is a maximal antichain in $\mathbb{P}$.

(c) $G \cap E \neq 0$, whenever $E \in M$ and $\forall p \in \mathbb{P}\, \exists q \in E(p$ and q are compatible).

Furthermore, show that in the definition of filter, we may weaken the requirement

(1) $\forall p, q \in G\, \exists r \in G\, (r \leq p \wedge r \leq q)$ to

(1′) $\forall p, q \in G\, \exists r \in \mathbb{P}\, (r \leq p \wedge r \leq q)$.

Thus, this Exercise provides $3 \cdot 2 = 6$ equivalent definitions of "generic".

B. Miscellaneous results

(B1) Show that if we redefine $\check{x}$ by

$$\check{x} = \{\langle \check{y}, p \rangle : y \in x \wedge p \in \mathbb{P}\},$$

then we may drop the assumption that $\mathbb{P}$ has a largest element for all the results in this chapter. *Remark.* The usefulness of $\mathbb{1}_{\mathbb{P}}$ will be more apparent in VIII.

(B2) Suppose $\langle \mathbb{P}, \leq \rangle$ is a partial order in M which may or may not have a largest element. In M, fix $\mathbb{1} \notin \mathbb{P}$, and define the p.o. $\langle \mathbb{Q}, \leq, \mathbb{1} \rangle$ by: $\mathbb{Q} = \mathbb{P} \cup \{\mathbb{1}\}$, where $\mathbb{P}$ retains the same order and $\forall p \in \mathbb{P}\, (p < \mathbb{1})$. Show that if $G \subset \mathbb{P}$, G is $\mathbb{P}$-generic over M iff $G \cup \{\mathbb{1}\}$ is $\mathbb{Q}$-generic over M, and $M[G]$ (defined as a $\mathbb{P}$-extension) is the same as $M[G \cup \{\mathbb{1}\}]$ (defined as a $\mathbb{Q}$-extension).

(B3) Define $\Vdash'$ by the following clauses:

(a) $p \Vdash' \tau_1 = \tau_2$ iff

$$\forall \pi \in \mathrm{dom}(\tau_1) \cup \mathrm{dom}(\tau_2)\, \forall q \leq p\, \big((q \Vdash' \pi \in \tau_1) \leftrightarrow (q \Vdash' \pi \in \tau_2)\big).$$

(b)–(e) As in the definition of $\Vdash^*$ (3.3). Show that this definition makes sense. *Remark* and *hint.* Clauses (a) and (b) define $\Vdash'$ for both kinds of atomic formulas by a simultaneous recursion. Say, in (a),

$$\mathrm{rank}(\tau_1) < \mathrm{rank}(\pi) < \mathrm{rank}(\tau_2).$$

Then $\langle \pi, \tau_2 \rangle$ is "bigger" than $\langle \tau_1, \tau_2 \rangle$. However, clause (b) now reduces the forcing of $\pi \in \tau_2$ to the forcing of $\pi = \pi'$ for $\pi' \in \mathrm{dom}(\tau_2)$, and

$$\max\big(\mathrm{rank}(\pi), \mathrm{rank}(\pi')\big) < \max\big(\mathrm{rank}(\tau_1), \mathrm{rank}(\tau_2)\big).$$

(B4) Prove Enunciations 3.4–3.6 using $\Vdash'$ instead of $\Vdash^*$.

(B5) Assume $f : A \to M$ and $f \in M[G]$. Show that there is a $B \in M$ such that $f : A \to B$. *Hint.* Let

$$B = \{b\colon \exists p \in \mathbb{P}\,(p \Vdash \check{b} \in \operatorname{ran}(\tau))\},$$

where $f = \tau_G$.

(B6) Assume $\mathbb{P} \in M$ and α is a cardinal of M. Show that the following are equivalent.

(1) Whenever $B \in M$, ${}^{\alpha}B \cap M = {}^{\alpha}B \cap M[G]$.

(2) ${}^{\alpha}M \cap M = {}^{\alpha}M \cap M[G]$.

(3) In M: The intersection of α dense open subsets of $\mathbb{P}$ is dense.

See the proof of II 3.3 for the topology on $\mathbb{P}$. *Remark.* A p.o. satisfying (3) is called α^+*-dense* or α^+*-Baire*. κ-Baire means that the intersection of *less than* κ dense open sets is dense.

(B7) Show that if $\mathbb{P}$ is λ-closed and λ is singular then $\mathbb{P}$ is λ^+-closed.

(B8) Let $\mathbb{P} \in M$ be non-atomic (see Exercise A1). Let

$$M = M_0 \subset M_1 \subset \cdots \subset M_n \subset \cdots \qquad (n \in \omega).$$

such that $M_{n+1} = M_n[G_n]$ for some G_n which is $\mathbb{P}$-generic over M_n. Show that $\bigcup_n M_n$ cannot satisfy the Power Set Axiom. Furthermore, show that the G_n may be chosen so that there is no c.t.m. N for ZFC with $\langle G_n\colon n \in \omega\rangle \in N$ and $o(N) = o(M)$. *Hint.* $\{n\colon p \in G_n\}$ can code $o(M)$.

(B9) Show that $\neg$KH implies that ω_2 is inaccessible in $\mathbf{L}$. *Hint.* Suppose ω_2 is $(\lambda^+)^{\mathbf{L}}$, where λ is a cardinal of $\mathbf{L}$. Let $X \subset \omega_1$ be such that $\omega_1^{\mathbf{L}(X)} = \omega_1$ and $\omega_2^{\mathbf{L}(X)} = \omega_2$. $\diamondsuit^+$, and hence KH, hold in $\mathbf{L}(X)$ (see VI Exercise 8), and a Kurepa tree of $\mathbf{L}(X)$ remains such in $\mathbf{V}$.

(B10) Suppose M satisfies $\mathbf{V} = \mathbf{L}$, so $M = L(\gamma)$, where $\gamma = o(M)$. Show that $M[G] = L(\gamma, G)$ in the sense of VI Exercise 6.

(B11) Verify that one may do forcing over a c.t.m. M for ZF* $-$ P to produce $M[G]$ satisfying ZF* $-$ P. Show that $\mathrm{AC}^M \to \mathrm{AC}^{M[G]}$ and that $M[G]$ will satisfy the Power Set Axiom if M does. *Remark.* See III Exercise 19 for ZF*. AR^* is needed in M to prove AR (and AR^*) in $M[G]$.

(B12) Express Exercises A9–A11 as exercises about $\Vdash^*$ in $\mathbf{V}$ (without any mention of models), and work them using the definition of $\Vdash^*$. For example, the conclusion of Exercise A9 says that

$$\{\tau \in \mathbf{V}^{\mathbb{P}} : p \Vdash^* \tau = \check{0}\}$$

is a proper class.

(B13) Let $\mathbb{P}$ be a p.o. Show that for any x, y,

$$x = y \to \mathbb{1} \Vdash^* \check{x} = \check{y},$$

$$x \neq y \to \mathbb{1} \Vdash^* \neg(\check{x} = \check{y})$$

$$x \in y \to \mathbb{1} \Vdash^* \check{x} \in \check{y},$$

$$x \notin y \to \mathbb{1} \Vdash^* \neg(\check{x} \in \check{y}).$$

Remark. This exercise does not mention models.

(B14) Show that all the relative consistency results of this chapter can be done in the syntactical model approach ((2a) of §9). As a start, one must show, without reference to models, that $\mathbb{1} \Vdash^* \phi$ for each axiom ϕ of ZFC. Interpret Exercise B6 as a characterization of κ-Baire in $\mathbf{V}$.

(B15) Do Exercise B14 using reflection. Thus, if $\neg(\mathbb{1} \Vdash^* \phi)$, for ϕ an axiom of ZFC, there would be c.t.m.'s, M, for arbitrary finite fragments of ZFC, such that $\neg(\mathbb{1} \Vdash^* \phi)^M$, and hence $\neg(\mathbb{1} \Vdash \phi)$.

(B16) Let $\mathscr{B}$ be a complete Boolean algebra. A *$\mathscr{B}$-valued structure* for the language of set theory is a triple $\langle M, f_=, f_\in \rangle$, where $f_=, f_\in : M \times M \to \mathscr{B}$ and the axioms of predicate calculus with equality are valid; for example, we require

$$[\![\forall x, y, z\,((x = y \wedge y \in z) \to x \in z)]\!] = \mathbb{1},$$

where we evaluate $[\![\phi]\!]$ by setting $[\![x \in y]\!] = f_\in(x, y)$, $[\![x = y]\!] = f_=(x, y)$, and interpreting the logical connectives by their corresponding Boolean operations. Show that if $\vdash \phi$, then $[\![\phi]\!] = \mathbb{1}$ in any such structure.

(B17) Verify that one can construct $\mathbf{V}^{\mathscr{B}}$ by iterating the Boolean power set operation. Thus, for each α, define $\mathscr{B}$-valued structures, $\langle R^{\mathscr{B}}(\alpha), f_=(\alpha), f_\in(\alpha) \rangle$ (see Exercise B16). $R^{\mathscr{B}}(\alpha + 1)$ is $R^{\mathscr{B}}(\alpha)$ plus the set of *extensional* functions, h, from $R^{\mathscr{B}}(\alpha)$ into $\mathscr{B}$; where extensional means

$$\forall x, y \in R^{\mathscr{B}}(\alpha)\,(([\![x = y]\!] \wedge h(y)) \leq h(x)).$$

Let $\mathbf{WF}^{\mathscr{B}} = \bigcup_{\alpha \in \mathbf{ON}} R^{\mathscr{B}}(\alpha)$. There is a Boolean isomorphism, $\mathbf{I}$, between $\mathbf{V}^{\mathscr{B}}$ (in the sense of approach (2b) of §9) and $\mathbf{WF}^{\mathscr{B}}$; thus, $\mathbf{I}$: $\mathbf{V}^{\mathscr{B}} \times \mathbf{WF}^{\mathscr{B}} \to \mathscr{B}$, and satisfies

$$(\llbracket \sigma \in \tau \rrbracket^* \wedge \mathbf{I}(\sigma, x) \wedge \mathbf{I}(\tau, y)) \leq \llbracket x \in y \rrbracket,$$
$$\bigvee \{\mathbf{I}(\sigma, x) \colon x \in \mathbf{WF}^{\mathscr{B}}\} = \mathbb{1}, \text{etc.}$$

(B18) In Approach (2c) of § 9, show that $\mathbf{V}^{\mathscr{B}}/G$ is well founded iff G is countably complete. Furthermore, if $\mathscr{B}$ is non-atomic, G cannot be countably complete unless $|\mathscr{B}|$ is at least as large as the first 2-valued measurable cardinal.

C. Complete embedding and complete Boolean algebras

(C1) Show that a composition of complete embeddings is complete. That is, if $i : \mathbb{P} \to \mathbb{Q}$ and $j : \mathbb{Q} \to \mathbb{R}$ are complete, so is $j \circ i : \mathbb{P} \to \mathbb{R}$.
(C2) Show that if $\mathbb{P}$ and $\mathbb{Q}$ are separative (see II Exercise 15), and $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding, then i is 1–1, $i(\mathbb{1}_{\mathbb{P}}) = \mathbb{1}_{\mathbb{Q}}$, and for all $p, p' \in \mathbb{P}$, $p \leq p'$ iff $i(p) \leq i(p')$.

(C3) Let $i : \mathbb{P} \to \mathbb{Q}$. Show that if i is a complete embedding, then

$$\mathbb{P} \text{ is non-atomic} \to \mathbb{Q} \text{ is non-atomic},$$

and if i is a dense embedding, then

$$\mathbb{P} \text{ is non-atomic} \leftrightarrow \mathbb{Q} \text{ is non-atomic}.$$

(C4) Let $\mathbb{P}$ be a countable non-atomic p.o. Show that there is a dense embedding from $\{p \in \mathrm{Fn}(\omega, \omega) \colon \mathrm{dom}(p) \in \omega\}$ into $\mathbb{P}$. *Hint.* If $\mathbb{P} = \mathrm{Fn}(\omega, \omega)$, inclusion works. In general, map $\{p \colon \mathrm{dom}(p) = 1\}$ onto an infinite antichain in $\mathbb{P}$, now handle $\{p \colon \mathrm{dom}(p) = 2\}$, etc. *Remark.* Hence, $\mathbb{P}$, $\mathrm{Fn}(\omega, 2)$, $\mathrm{Fn}(\omega, \omega)$, and $\mathrm{Fn}(\omega \times \omega, 18)$ all yield the same generic extentions.

(C5) If $\mathbb{P}$ is the non-$\mathbb{0}$ elements of a Boolean algebra $\mathscr{B}$, show that $\mathbb{P}$ is non-atomic by the definition in Exercise A1 iff $\mathscr{B}$ is non-atomic in the usual sense for Boolean algebras:

$$\forall b > \mathbb{0}\, \exists c\, (\mathbb{0} < c < b).$$

(C6) Let $\mathscr{A}$ and $\mathscr{B}$ be non-atomic countable Boolean algebras. Show that $\mathscr{A}$ and $\mathscr{B}$ are isomorphic. *Remark.* Via Stone spaces, this implies that all compact 0-dimensional second-countable Hausdorff spaces with no isolated points are homomorphic to the Cantor set.

(C7) If $\mathscr{A}$ and $\mathscr{B}$ are Boolean algebras, a homomorphism $i : \mathscr{A} \to \mathscr{B}$ is *complete* iff whenever $S \subset \mathscr{A}$ and $\bigvee S$ exists, then $\bigvee (i''S)$ exists in $\mathscr{B}$ and

$\bigvee(i''S) = i(\bigvee S)$. Show that $i : \mathscr{A} \to \mathscr{B}$ is a complete embedding (in the sense of Definition 7.1) iff i is a complete injective homomorphism. Furthermore, show that in this case, if, for $b \in \mathscr{B}$, $h(b) = \bigwedge \{a \in \mathscr{A}: i(a) \geq b\}$, then $h(b)$ is the largest reduction of b to $\mathscr{A}$. $h(b)$ is called the *$\mathscr{A}$-hull* of b.

(C8) Let $i : \mathbb{P} \to \mathbb{Q}$ be a complete embedding, $\mathscr{A}$ = the completion of $\mathbb{P}$, and $\mathscr{B}$ = the completion of $\mathbb{Q}$. Show how i defines a complete embedding, j, from $\mathscr{A}$ into $\mathscr{B}$. If i is a dense embedding, show that j is an isomorphism.

(C9) Let $\mathbb{P}$ be the p.o. used in proving the $<c$ additivity of Lebesgue measure from MA (see II 2.21), and let $i : \mathbb{P} \to \mathscr{B}$ be the completion of $\mathbb{P}$. Show that there are $p, q \in \mathbb{P}$ such that $p \not\leq q$, $q \not\leq p$, and $i(p) = i(q)$.

(C10) Let $\mathscr{B}$ be a complete Boolean algebra. Show that $\mathscr{B}$ is ω_1-Baire (see Exercise B6) iff $\mathscr{B}$ is (ω, ∞) distributive, i.e., for each κ, the equation

$$\bigwedge_{n\in\omega} \bigvee_{\alpha\in\kappa} b_{n,\alpha} = \bigvee_{f\in\kappa^\omega} \bigwedge_{n\in\omega} b_{n,f(n)}$$

holds.

(C11) Show that $\neg$SH is equivalent to the existence of a non-atomic, c.c.c., complete Boolean algebra, $\mathscr{B}$, which is (ω, ∞) distributive. Such a $\mathscr{B}$ is called a *Suslin algebra*.

D. Relations of C to forcing

(D1) Let $\mathscr{B}$ be a complete Boolean algebra of M and $F \subset \mathscr{B}$. Show that F is $\mathscr{B}$-generic over M iff F is an ultrafilter and the associated homomorphism h of $\mathscr{B}$ into the 2-element algebra preserves all sups in M—i.e., for all $S \in M$ with $S \subset \mathscr{B}$, $h(\bigvee S) = \bigvee\{h(b): b \in S\}$.

(D2) ([Rasiowa–Sikorsky 1963]). Let $\mathscr{B}$ be any Boolean algebra, and, for $n \in \omega$, $S_n \subset \mathscr{B}$ such that the supremum of S_n exists and $\bigvee S_n = b_n$. Show that there is a homomorphism h of $\mathscr{B}$ into the 2-element algebra preserving each $\bigvee S_n$ (i.e., $h(b_n) = \bigvee\{h(b): b \in S_n\}$). *Remark.* By Exercise D1, this Rasiowa–Sikorski theorem is the generic filter existence theorem for Boolean-valued models.

(D3) Assume that $i, \mathbb{P}$, and $\mathbb{Q}$ are in M, and $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. For any $G \subset \mathbb{P}$, define

$$\mathbb{Q}/G = \{q \in \mathbb{Q}: \forall p \in G\,(q \text{ is compatible with } i(p))\}.$$

Show that p is a reduction of q to $\mathbb{P}$ iff $p \Vdash \check{q} \in \check{\mathbb{Q}}/\Gamma$.

(D4) Suppose that in M, $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Let G be $\mathbb{P}$-generic over M, and let K be $\mathbb{Q}/G$ generic over $M[G]$. Show that K is $\mathbb{Q}$-generic over M and that $M[K]_{\mathbb{Q}} = M[G][K]_{\mathbb{Q}/G}$. Here $M[K]_{\mathbb{Q}} = \{\tau_K : \tau \in M^{\mathbb{Q}}\}$, while $M[G][K]_{\mathbb{Q}/G} = \{\tau: \tau \in M[G]^{\mathbb{Q}/G}\}$; this notation is needed since K is a filter in two different p.o.'s. *Hint.* If D is dense in $\mathbb{Q}$, then $D \cap \mathbb{Q}/G$ is dense in $\mathbb{Q}/G$. To see this, fix $q_0 \in \mathbb{Q}/G$ and fix $p_0 \in G$ with $p_0 \Vdash \check{q}_0 \in \check{\mathbb{Q}}/\Gamma$. Show that the following is dense below p_0:

$$\{p \in \mathbb{P}: \exists q \in \mathbb{Q}(q \leq q_0 \wedge q \in D \wedge p \Vdash \check{q} \in \check{\mathbb{Q}}/\Gamma)\}.$$

(D5) Suppose that in M, $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. Let H be $\mathbb{Q}$-generic over M and let $G = i^{-1}(H)$. Show that $H \subset \mathbb{Q}/G$, H is $\mathbb{Q}/G$-generic over $M[G]$, and $M[H]_{\mathbb{Q}} = M[G][H]_{\mathbb{Q}/G}$. *Hint* If $D \subset \mathbb{Q}/G$, D is dense, and $D \in M[G]$, let $D = \tau_G$ and let $p_0 \Vdash (\tau$ is dense in $\check{\mathbb{Q}}/\Gamma)$; then

$$\{q: \exists p\, \exists q_1 [(p \Vdash \check{q}_1 \in \tau) \wedge q \leq i(p) \wedge q \leq q_1]\}$$

is dense below $i(p_0)$. *Remark.* Exercises D4 and D5 show that a one-step extension by $\mathbb{Q}$ is equivalent to extending via $\mathbb{P}$ and then $\mathbb{P}/G$.

(D6) Assume that in M, $i : \mathscr{A} \to \mathscr{B}$ is a complete Boolean embedding of complete Boolean algebras. Let G be $\mathscr{A}$-generic over M, and, in $M[G]$, let

$$\mathscr{I} = \{b \in \mathscr{B}: \exists a \in G\,(b \perp i(a))\}.$$

Show that $\mathscr{I}$ is an ideal in $\mathscr{B}$, the quotient algebra $\mathscr{B}/\mathscr{I}$ is complete in $M[G]$, and $\mathscr{I} = \{b: (h(b))' \in G\}$, where h is as in Exercise C7. Furthermore, if $\mathscr{B}/G$ is as in Exercise D3, and $i(b) = [b]_{\mathscr{I}}$, then $i : \mathscr{B}/G \to \mathscr{B}/\mathscr{I}$, is a dense embedding, so that $\mathscr{B}/\mathscr{I}$ is the completion of $\mathscr{B}/G$ in $M[G]$.

(D7) If $\mathscr{B}$ is a complete Boolean algebra of M, show that

(a) $[\![\tau_1 = \tau_2]\!] = \bigwedge\{[\![\pi \in \tau_1]\!] \leftrightarrow [\![\pi \in \tau_2]\!]: \pi \in \operatorname{dom}(\tau_1) \cup \operatorname{dom}(\tau_2)\}$, and

(b) $[\![\tau_1 \in \tau_2]\!] = \bigvee\{[\![\pi = \tau_1]\!] \wedge b: \langle \pi, b\rangle \in \tau_2\}$,

where, for $a, b \in \mathscr{B}$, $(a \leftrightarrow b) = (a' \vee b) \wedge (b' \vee a)$.
Remark. In the spirit of B3, (a) and (b) can be taken as clauses in an inductive definition of $[\![\ldots]\!]^*$.

(D8) In M, let $i : \mathbb{P} \to \mathscr{B}$ be the completion of $\mathbb{P}$. Show that $i(p) = i(q)$ iff for all G which are $\mathbb{P}$-generic over $M, p \in G \leftrightarrow q \in G$.

E. Automorphisms and AC

(E1) If $\mathbb{P}$ (i.e., $\langle \mathbb{P}, \leq, \mathbb{1}\rangle$) is a p.o., an *automorphism* of $\mathbb{P}$ is a 1–1 i from $\mathbb{P}$ onto $\mathbb{P}$ which preserves $\leq$ and satisfies $i(\mathbb{1}) = \mathbb{1}$; thus also $i_*(\check{x}) = \check{x}$ for each x. $\mathbb{P}$ is called *almost homogeneous* iff for all $p, q \in \mathbb{P}$, there is an automorphism i of $\mathbb{P}$ such that $i(p)$ and q are compatible. Suppose that $\mathbb{P} \in M$ and $\mathbb{P}$ is almost homogeneous in M. Show that if $p \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$, then $\mathbb{1} \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$; thus, either $\mathbb{1} \Vdash \phi(\check{x}_1, \ldots, \check{x}_n)$ or $\mathbb{1} \Vdash \neg\phi(\check{x}_1, \ldots, \check{x}_n)$.

(E2) Show that any $\mathrm{Fn}(I, J, \kappa)$ is almost homogeneous.

(E3) (A. Miller). In M, let I and J be uncountable, $\mathbb{P} = \mathrm{Fn}(I, 2)$, and $\mathbb{Q} = \mathrm{Fn}(J, 2)$. Let $\phi(x)$ be a formula. Show that

$$\mathbb{1} \Vdash_{\mathbb{P}} \left(\phi(\check{\alpha})^{\mathbf{L}(\mathscr{P}(\omega))}\right) \quad \text{iff} \quad \mathbb{1} \Vdash_{\mathbb{Q}} \left(\phi(\check{\alpha})^{\mathbf{L}(\mathscr{P}(\omega))}\right).$$

Hint. If $(|I| = |J|)^M$ apply Lemma 7.13. More generally, say $(|I| \leq |J|)^M$ and let H be $\mathrm{Fn}(I, J, \omega_1)^M$-generic over M; then $(|I| = |J|)^{M[H]}$. If G is $\mathbb{P}$-generic over $M[H]$, then G is $\mathbb{P}$-generic over M and $\mathscr{P}(\omega) \cap M[H][G] = \mathscr{P}(\omega) \cap M[G]$. Thus, applying E1,

$$\mathbb{1} \Vdash_{\mathbb{P},M} \left(\phi(\check{\alpha})^{\mathbf{L}(\mathscr{P}(\omega))}\right) \quad \text{iff} \quad \mathbb{1} \Vdash_{\mathbb{P},M[H]} \left(\phi(\check{\alpha})^{\mathbf{L}(\mathscr{P}(\omega))}\right).$$

Likewise with $\mathbb{Q}$.

(E4) Show $\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}(\mathrm{ZF} + \neg\mathrm{AC})$. In fact, show the consistency of $\neg\mathrm{AC}$ with $\mathbf{V} = \mathbf{L}(\mathscr{P}(\omega)) = \mathbf{HOD}(\mathscr{P}(\omega))$. *Hint.* (A. Miller). Let $(|I| \geq \omega_1)^M$, Let G be $\mathrm{Fn}(I, 2)$-generic over M, and let $N = \mathbf{L}(\mathscr{P}(\omega))^{M[G]}$. If AC holds in N, let $(\kappa = |\mathscr{P}(\omega)|)^N$, then $\mathbb{1} \Vdash \left((\check{\kappa} = |\mathscr{P}(\omega)|)^{\mathbf{L}(\mathscr{P}(\omega))}\right)$. Now, taking J such that $(|J| > \kappa)^M$ will contradict E3.

(E5) Show the consistency of ZFC + GCH with $\mathbf{V} \neq \mathbf{OD}$. *Hint.* Use $\mathrm{Fn}(I, 2)$, and apply E1.

F. Chain conditions

(F1) Let c.c.($\mathbb{P}$) be the least θ such that $\mathbb{P}$ has the θ-c.c. Show that c.c.($\mathbb{P}$) $\neq \omega$. *Hint.* Consider a maximal incompatible family of atoms.

(F2) If $\mathbb{P} \in M$ and c.c.($\mathbb{P}$) $< \omega$, show that every G which is $\mathbb{P}$-generic over M is in M. Do likewise when $(\text{c.c.}(\mathbb{P}) = \kappa \wedge \mathbb{P} \text{ is } \kappa\text{-closed})^M$.

(F3) If $\mathbb{P} \in M$, show that c.c.($\mathbb{P}$) $< \omega$ iff $(\text{c.c.}(\mathbb{P}) < \omega)^M$. *Hint.* Resolve $\mathbb{P}$ into atoms in M.

*(F4) (Tarski) Show that if c.c.($\mathbb{P}$) $\geq \omega$ then c.c.($\mathbb{P}$) is regular.

(F5) Show that if $|I| \geq \lambda$ and $|J| \geq 2$, then $\mathrm{Fn}(I, J, \lambda)$ has an antichain of size $2^{<\lambda}$. *Remark.* This is easier when λ is a successor cardinal.

(F6) Show that every countable complete Boolean algebra is finite. *Hint.* See Exercise F1.

G. $\mathbf{Fn}(I, J, \lambda)$

(G1) Let κ, λ, θ be infinite cardinals of M, and let $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$. Show that

$$(\lambda^{\theta})^{M[G]} = ((\max(\kappa, \lambda))^{\theta})^{M}.$$

(G2) Suppose $I \in M$ is infinite, and let $2 \leq \alpha \leq \beta \leq \omega$. Show that $\mathrm{Fn}(I, \alpha)$ and $\mathrm{Fn}(I, \beta)$ yield the same generic extensions. *Hint.* Let $I = \kappa \times \omega$, and apply Exercise C4 co-ordinatewise.

(G3) Assume λ is regular, and let $\mathbb{P}(J) = \{p \in \mathrm{Fn}(\lambda, J, \lambda)\colon \mathrm{dom}(p) \in \lambda\}$. Show that $\mathbb{P}(J)$ is densely included in $\mathrm{Fn}(\lambda, J, \lambda)$. Furthermore, show that if $2 \leq |J| \leq 2^{<\lambda}$, then $\mathbb{P}(2^{<\lambda})$ densely embeds into $\mathrm{Fn}(\lambda, J, \lambda)$. *Remark.* Applying this within M show that $\mathrm{Fn}(\lambda, 2, \lambda)$ adds a function from λ onto $2^{<\lambda}$, since $\mathrm{Fn}(\lambda, 2^{<\lambda}, \lambda)$ obviously does.

(G4) Let $(\mathbb{P} = \mathrm{Fn}(I, 2, \omega_1))^M$, where $(|I| \geq \omega_1)^M$. Show that $M[G]$ satisfies CH, regardless of whether M does.

(G5) Suppose, in M, $\theta = \mathrm{cf}(\lambda) < \lambda$. Show that $\mathrm{Fn}(\lambda, 2, \lambda)$ adds a map from θ onto λ^+. *Hint.* Say $\lambda = (\omega_\omega)^M$. Let $f(n)$ be the ω_n-th element of $\{\alpha\colon \bigcup G(\alpha) = 1\}$. Let $g(n)$ be the $\beta < \omega_n$ such that $f(n)$ is of the form $\omega_n \cdot \delta + \beta$. Then g maps ω onto λ. Now show that $\bigcup G$ codes every function in M from ω into λ.

(G6) Suppose M satisfies GCH. Let $\kappa_1 < \cdots < \kappa_n$ be regular cardinals of M, and let $\lambda_1 \leq \cdots \leq \lambda_n$ be cardinals of M such that $(\mathrm{cf}(\lambda_i) > \kappa_i)^M$. Force n times to construct a c.t.m. $N \supset M$ with the same cardinals such that for each i, $(2^{\kappa_i} = \lambda_i)^N$.

(G7) Suppose M satisfies CH and $\mathbb{P} = \mathrm{Fn}(I, 2)$, where $(|I| \geq \omega_2)^M$. Show that MA fails in $M[G]$. *Hint.* Show that if $f \in \omega^\omega \cap M[G]$, there is a $g \in \omega^\omega \cap M$ such that $\{n\colon g(n) \geq f(n)\}$ is infinite. Now apply II Exercise 8.

(G8) If $\mathbb{P} = \mathrm{Fn}(\kappa \times \omega, 2)$, and $f_\alpha(n) = \bigcup G(\alpha, n)$, show that the f_α for $\alpha < \kappa$ are recursively incomparable.

(G9) Show that the completion of the p.o. $\mathrm{Fn}(I, 2)$ is isomorphic to the regular open algebra of the space 2^I (where $2 = \{0, 1\}$ has the discrete topology and 2^I has the product topology).

H. Specific forcing constructions

(H1) Assume in M that $\kappa > \omega$, κ is regular, and $\mathbb{P}$ has the κ-c.c. In $M[G]$, let $C \subset \kappa$ and C c.u.b. Show that there is a $C' \in M$ such that $C' \subset C$ and C' is c.u.b. in κ. *Hint.* In $M[G]$, let $f\colon \kappa \to \kappa$ be such that

$$\forall \alpha < \kappa\,(\alpha < f(\alpha) \in C),$$

and apply Lemma 6.8.

(H2) Suppose that in M: $\kappa > \omega$, κ is regular, $S \subset \kappa$ is stationary and $\mathbb{P}$ is either κ-c.c. or κ-closed. Show that S remains stationary in $M[G]$.

(H3) A cardinal κ is called *weakly Mahlo* iff κ is regular and $\{\alpha < \kappa\colon \alpha \text{ is regular}\}$ is stationary in κ. Show that if κ is weakly Mahlo, then

$$\{\alpha < \kappa\colon \alpha \text{ is weakly inaccessible}\}$$

is stationary in κ.

(H4) Assume in M that $\kappa > \omega$ and $\mathbb{P}$ is either κ-closed or has the λ-c.c. for some $\lambda < \kappa$. Show that if κ is weakly Mahlo in M, then κ is weakly Mahlo in $M[G]$. Do the same for weakly hyper-Mahlo, where κ is *weakly hyper-Mahlo* iff κ is regular and

$$\{\alpha < \kappa\colon \alpha \text{ is weakly Mahlo}\}$$

is stationary in κ.

(H5) Show that 5.16 remains true if "inaccessible" is everywhere replaced by "Mahlo". Do likewise for "hyper-Mahlo". See II Exercises 48 and 50 for the definitions of strongly Mahlo and strongly hyper-Mahlo. Furthermore, show that it is relatively consistent to have a Mahlo or hyper-Mahlo κ such that $2^\kappa > \kappa^+$ and GCH holds below κ.

(H6) Assume that in M, $\mathscr{A}, \mathscr{C} \subset \mathscr{P}(\omega)$ and for all $y \in \mathscr{C}$ and all finite $F \subset \mathscr{A}$ $|y \setminus \bigcup F| = \omega$. Show that there is a $\mathbb{P} \in M$ such that $(\mathbb{P} \text{ is c.c.c.})^M$ and whenever G is $\mathbb{P}$-generic over M, $M[G]$ contains a $d \subset \omega$ such that

$$\forall x \in \mathscr{A}\,(|d \cap x| < \omega) \quad \text{and} \quad \forall y \in \mathscr{C}\,(|d \cap y| = \omega).$$

Hint. See II 2.15.

(H7) Let $\langle X, <\rangle$ be a total order in M. Show that there is a $\mathbb{P} \in M$ such that ($\mathbb{P}$ has the c.c.c.)^M and whenever G is $\mathbb{P}$-generic over M, there are $a_x \subset \omega\,(x \in X)$ such that $x < y \to a_x \subset^* a_y$. *Hint.* See II Exercise 22.

(H8) Assume ($\mathbb{P}$ is c.c.c. $\wedge$ $|\mathbb{P}| \le \omega_1)^M$ and $\diamondsuit$ holds in M. Show that $\diamondsuit$ holds in $M[G]$. *Hint.* Use $\diamondsuit$ in M to "capture" all nice names for subsets of ω_1.

(H9) Assume ($\mathbb{P}$ is c.c.c.)^M and $\diamondsuit$ holds in $M[G]$. Show that $\diamondsuit$ holds in M. *Hint.* It is sufficient to verify $\diamondsuit^-$ in M; see II 7.13, 7.14.

(H10) Suppose that in M, $\mathbb{P}$ has ω_1 as a precaliber (see II Exercise 26). Show that if T is an ω_1-Suslin tree in M, then T remains a Suslin tree in $M[G]$. Do likewise if $\mathbb{P}$ is ω_1-closed in M.

(H11) Let $\kappa > \omega$ be regular. Let $\mathbb{P}$ be the Jech p.o. for adding a κ-Suslin tree. Elements of $\mathbb{P}$ are either the empty tree, $\mathbb{1}$, or subtrees, p, of $2^{<\kappa}$ such that $\mathrm{ht}(p) = \alpha + 1$ for some limit $\alpha < \kappa$,

$$\forall s \in p\,(\mathrm{dom}(s) < \alpha \to (s^\frown\langle 0\rangle \in p \wedge s^\frown\langle 1\rangle \in p)).$$

$$\forall \xi \le \alpha\,(|\mathrm{Lev}_\xi(p)| < \kappa), \quad \text{and} \quad \forall s \in p\, \exists t \in p\,(\mathrm{dom}(t) = \alpha \wedge s \subset t).$$

$p \le q$ iff $q = \{s \in p\colon \mathrm{dom}(s) < \mathrm{ht}(q)\}$, i.e., q is obtained by sawing off p parallel to the ground. Show that $\mathbb{P}$ is ω_1-closed and not ω_2-closed. *Hint.* If $p_0 > p_1 > \cdots$, $\bigcup_n p_n$ is a tree of some limit height, γ, but since $\mathrm{cf}(\gamma) = \omega$, it is easy to extend $\bigcup_n p_n$ to a tree of height $\gamma + 1$ in $\mathbb{P}$. However, $\bigcup_{\xi<\omega_1} p_\xi$ could be an ω_1-Aronszajn tree.

(H12) Show that if $\mathbb{P}$ is as in Exercise H11 and $\alpha < \kappa$, $D_\alpha = \{p \in \mathbb{P}\colon \mathrm{ht}(p) > \alpha\}$ is dense in $\mathbb{P}$. *Remark.* This is where it is important that conditions have successor height. *Hint.* To see that $D_0 \ne 0$, let

$$p = \{s \in 2^{<\alpha+1}\colon |\{\xi\colon s(\xi) = 1\}| < \omega\}.$$

(H13) Show that the $\mathbb{P}$ of Exercise H11 is κ-Baire (see Exercise B6), even though it is not κ-closed if $\kappa > \omega_1$. *Hint.* Suppose $\lambda < \kappa$ and $D_\xi(\xi < \lambda)$ are dense and open. Find a decreasing sequence, $p_\xi(\xi \le \lambda)$, with $p_\xi \in D_\xi$ for each $\xi < \lambda$ (so then $p_\lambda \in \bigcap_{\xi<\lambda} D_\xi$). $\mathrm{ht}(p_\xi)$ will be $\alpha_\xi + 1$. For limit γ, $\alpha_\gamma = \sup\{\alpha_\xi\colon \xi < \gamma\}$. To ensure $\bigcup_{\xi<\gamma} p_\xi$ can be extended when $\mathrm{cf}(\gamma) > \omega$, choose, along with each p_ξ, an $f(s, \xi) \in \mathrm{Lev}_{\alpha_\xi}(p_\xi)$ for each $s \in p_\xi$ such that $s \subset f(s, \xi)$ and $\xi < \eta \to f(s, \xi) \subset f(s, \eta)$. *Remark.* For $\kappa > \omega_1$, this argument is due to Prikry and Silver.

(H14) Let $\kappa > \omega$ be a regular cardinal of M, and let $\mathbb{P}$ be defined within

M as in Exercise H11. Show that in $M[G]$, $\bigcup G$ is a κ-Suslin tree, and that $\mathbb{P}$ preserves cofinalities if $2^{<\kappa} = \kappa$ in M. *Hint.* If

$$p \Vdash (\tau \text{ is a maximal antichain in } \Gamma),$$

find $p_0 = p > p_1 > p_2 > \cdots$ such that

$$\forall n \, \forall s \in p_n\big((p_{n+1} \Vdash \check{s} \in \tau) \vee (p_{n+1} \Vdash \check{s} \notin \tau)\big).$$

Let $q = \bigcup_n p_n$ (note $q \notin \mathbb{P}$), and let $A = \{s \in q\colon \exists n(p_n \Vdash \check{s} \in \tau)\}$. Make sure that the p_n are defined so that A is a maximal antichain in q. Now, extend q to a q' such that $q' \Vdash \tau = A$. This is like the proof of $\diamondsuit \to \neg\mathrm{SH}$ (see II 7.8).

(H15) Show that $\mathrm{Fn}(\omega_1, 2^\omega, \omega_1)$ and the Jech p.o. for adding an ω_1-Suslin tree have isomorphic completions. *Hint.*

$$\{p \in \mathrm{Fn}(\omega_1, 2^\omega, \omega_1)\colon \mathrm{dom}(p) \text{ is a successor ordinal}\}$$

can be densely embedded in both of them.

(H16) Let T be a κ-Suslin tree; concepts like "dense" refer to the reverse order, $<$, on T, as in Theorem 8.4. Show that if $X \subset T$, then there is an $\alpha < \kappa$ such that for all $p \in \mathrm{Lev}_\alpha(T)$, either X is dense below p or empty below p (i.e. $\neg\exists q < p(q \in X)$). Use this to show that T is κ-Baire.

(H17) Suppose that in M, $\mathbb{P}$ is c.c.c. and T is a κ-Suslin tree for some $\kappa > \omega_1$. Show that T remains Suslin in $M[G]$. *Hint.* In M, if $X \subset T$ and $|X| = \kappa$, then X contains an uncountable chain and an uncountable antichain.

(H18) Jensen's $\diamondsuit^+$ order is the set $\mathbb{P}$ of pairs $\langle p, \mathscr{S}\rangle$, where p is a condition in the Jech p.o. (H11, with $\kappa = \omega_1$), $\mathscr{S} \subset 2^{\omega_1}$, $|\mathscr{S}| \le \omega$, and

$$(f \restriction (\mathrm{ht}(p) - 1)) \in p$$

for each $f \in \mathscr{S}$. $\langle p, \mathscr{S}\rangle \le \langle p', \mathscr{S}'\rangle$ iff $p \le p'$ and $\mathscr{S}' \subset \mathscr{S}$. Assuming CH, show that $\mathbb{P}$ is ω_1-closed and ω_2-c.c.

(H19) Let $\mathbb{P}$ be the $\diamondsuit^+$-order in M (see H18), and assume CH in M. In $M[G]$, let $T = \bigcup\{p\colon \exists \mathscr{S}(\langle p, \mathscr{S}\rangle \in G)\}$, and show that T is a Kurepa tree.

(H20) Let $\mathbb{P}$ be the $\diamondsuit^+$-order in M and assume CH in M. Show that $\diamondsuit^+$ holds in $M[G]$. *Hint.* See VI Exercise 9. In $M[G]$, fix $h \in 2^{\omega_1}$ such that $T \in \mathbf{L}(h)$ and $\mathscr{P}(\omega) \in \mathbf{L}(h)$; then show, in $M[G]$, that

(a) $\forall f \in 2^{\omega_1} \cap M \, \exists g \in 2^{\omega_1} \big(f \in \mathbf{L}(g, h) \wedge g \text{ is a path through } T\big)$, and

(b) $\forall f \in 2^{\omega_1} \, \exists g \in 2^{\omega_1} \cap M\big(f \in \mathbf{L}(g, h)\big)$.

For (b), observe that $G \cap \mathbf{L}(g, h) \in \mathbf{L}(g, h)$.

(H21) In M, let $\mathscr{B}$ be the measure algebra of 2^I; that is, $2 = \{0, 1\}$ is given a measure by defining $\{0\}$ and $\{1\}$ to have measure $\frac{1}{2}$. 2^I has the product measure, and $\mathscr{B}$ is the algebra of measurable sets modulo measure 0 sets. Show that $\mathscr{B}$ has the c.c.c. in M, and, if G is $\mathscr{B}$-generic over M, then $2^\omega \geq |I|$ in $M[G]$. *Hint.* G defines a function $f : I \to 2$. If $I = \kappa \times \omega$, and $f_\alpha(n) = f(\alpha, n)$, then the f_α are distinct.

(H22) In Exercise H21, assume also that in M, $|I| = \kappa$ and $\kappa^\omega = \kappa$. Show that $2^\omega = \kappa$ in $M[G]$. *Remark.* In M, $\mathscr{B}$ is isomorphic to the Baire sets in 2^I modulo measure 0 sets, whereas the completion of $\mathrm{Fn}(I, 2)$ is the regular open algebra of 2^I, which is the same (for 2^I) as the Baire sets modulo first category. For more on the analogy between measure and category in generic extensions, see [Kunen 1900].

(H23) In Exercise H21 (or with $\mathscr{B}$ any measure algebra in M), let $f \in \omega^\omega \cap M[G]$. Show that there is a $g \in \omega^\omega \cap M$ such that $\forall n\,(f(n) \leq g(n))$. Show that this fails in extension by $\mathrm{Fn}(I, 2)$. *Remark.* Thus, in contrast with Exercise H22, not all properties of these extensions are the same. *Hint.* If $\mathbb{1} \Vdash \tau : \check{\omega} \to \check{\omega}$, and $\varepsilon > 0$, let $g(n)$ be an m such that $\mu([\![\tau(\check{n}) \leq \check{m}]\!]) \geq 1 - \varepsilon/2^n$. For $\mathrm{Fn}(I, 2)$, consider instead $\mathrm{Fn}(\omega, \omega)$.

(H24) (Prikry). In M, assume that κ is 2-valued measurable and that $\mathbb{P}$ is c.c.c. Show that $\exists \mathscr{I}\, S(\kappa, \omega_1, \mathscr{I})$ holds in $M[G]$ (see II Exercise 60). *Hint.* If $S(\kappa, 2, \mathscr{I})$ holds in M, define

$$\mathscr{J} = \{X \in \mathscr{P}(\kappa) \cap M[G] \colon \exists Y \in \mathscr{I}\, (X \subset Y)\}.$$

Remark. If $\mathbb{P}$ is a measure algebra then κ is real-valued measurable in $M[G]$; see [Solovay 1971].

*(H25) (Jensen). In M, let $T \subset \omega_1$ be stationary, and let $\mathbb{P}$ be the set of all $p \subset T$ such that p is closed in ω_1 and countable. $p \leq q$ iff $q \subset p$ and $q = p \cap (\max(q) + 1)$. Show that $\mathbb{P}$ is ω_1-Baire and that in $M[G]$, T contains a c.u.b. *Remark.* Thus, $S = \omega_1 \setminus T$ is not stationary in $M[G]$, although it may be stationary in M; compare this with H2.