MATH323 - Tutorial 6 (Feb. 15th /17)

Linearity of the Expectation

1. Suppose a weighted coin (Heads occurs with probability $0 \le p \le 1$, tails with probability $1-p$) is tossed n times such that each toss is independent of all the others. Let X_n the number of heads that are observed out of the n tosses. Without using the Binomial Probability Distribution, find $\mathbb{E}(X_n)$.

Solution:

Without using the fact that $X_n \sim \text{Binomial}(n,p)$, we will break down X_n in terms of "smaller" random variables.

Let $Y_i = \begin{cases} 1, & \text{if flip } i \text{ is heads} \\ 0, & \text{if flip } i \text{ is tails} \end{cases}$ $1 \le i \le n$, where

$p(Y_i = 1) = p$ for all $1 \le i \le n$. Thus, we observe that

$X_n = \sum_{i=1}^{n} Y_i$ (i.e. we are adding up all the "smaller" successes to obtain the total number of successes).

$$\Rightarrow \quad \mathbb{E}(X_n) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} \mathbb{E}(Y_i)$$

since $Y_i \sim \text{Bernouilli}(p) \Rightarrow \mathbb{E}(Y_i) = p$

$$\Rightarrow \sum_{i=1}^{n} \mathbb{E}(Y_i) = \sum_{i=1}^{n} p = np.$$

Thus, we find that this answer agrees with the expectation of a Binomial (n,p) random variable. ∎

This problem was somewhat simplistic as we have the nice properties of the Binomial probability distribution. In the next problem, we allow the probability of the coin to change from flip to flip. Thus in problem 2, we cannot rely on the Binomial probability distribution.

2. Suppose a weighted coin is flipped n times. On the first flip, the probability of heads is $0 \le p \le 1$, and tails is $1-p$. On the i^{th} flip, $1 \le i \le n$, the probability of heads is $\left(\frac{1}{2}\right)^{i-1} p$ and the probability of tails is $1-\left(\frac{1}{2}\right)^{i-1} p$. Let X_n be the number of heads that are observed out of the n tosses. Find $\mathbb{E}(X_n)$.

Solution:

Observe that in this problem, the properties of the Binomial distribution break down. The probability of obtaining heads does not stay constant from trial to trial. We will use the same technique from problem 1:

$$\text{Let } Y_i = \begin{cases} 1, & \text{if trial } i \text{ is heads} \\ 0, & \text{if trial } i \text{ is tails} \end{cases}$$

$$\mathbb{P}(Y_i) = \left(\tfrac{1}{2}\right)^{i-1} p, \quad \text{where } 1 \le i \le n.$$

$$\Rightarrow \mathbb{E}(Y_i) = \left(\tfrac{1}{2}\right)^{i-1} p$$

Observe that $X_n = \sum_{i=1}^{n} Y_i$

$$\Rightarrow \mathbb{E}(X_n) = \mathbb{E}\left(\sum_{i=1}^{n} Y_i\right) \underset{\text{Linearity}}{=} \sum_{i=1}^{n} \mathbb{E}(Y_i) = \sum_{i=1}^{n} \left(\tfrac{1}{2}\right)^{i-1} p$$

$$\sum_{i=1}^{n} \left(\tfrac{1}{2}\right)^{i-1} p = \left(\tfrac{1}{2}\right)^{-1} p \sum_{i=1}^{n} \left(\tfrac{1}{2}\right)^{i} = 2p \sum_{i=1}^{n} \left(\tfrac{1}{2}\right)^{i}$$

$$= 2p\left(1 - 2^{-n}\right)$$

where we used the fact that $\sum_{i=1}^{n} \left(\frac{1}{2}\right)^{i} = 1 - 2^{-n}$.

Therefore, $\mathbb{E}(X_n) = 2p(1-2^{-n})$. ■

3. (Problem from Lecture 02-07-2017) We alter this question slightly to illustrate a trick in solving difficult expected values. Let $Y \sim \text{Poisson}(2)$, $C_1 = 100(1/3)^Y$ and $C_2 = 100u^Y$ where $0 < u < 1$ is a fixed constant. Calculate $\mathbb{E}(C_1)$, and $\mathbb{E}(C_2)$.

Solution: We will apply the definition of the expected value:

$$\mathbb{E}(C_1) = \mathbb{E}(100(1/3)^Y) = 100\,\mathbb{E}((1/3)^Y)$$
$$= 100 \sum_{k=0}^{\infty} (1/3)^k \frac{e^{-2} \cdot 2^k}{k!}$$
$$= 100 \sum_{k=0}^{\infty} \frac{\left(\frac{2}{3}\right)^k e^{-2}}{k!}$$

This sum is difficult to solve. We will use a trick by factoring out the e^{-2} and multiplying and dividing by $e^{-2/3}$. Thus, we obtain:

$$= 100e^{-2}\left(\frac{e^{-2/3}}{e^{-2/3}}\right)\sum_{k=0}^{\infty} \frac{\left(\frac{2}{3}\right)^k}{k!} = \frac{100e^{-2}}{e^{-2/3}}\left(\sum_{k=0}^{\infty} \frac{e^{-2/3}\left(\frac{2}{3}\right)^k}{k!}\right)$$

Observe that if $X \sim \text{Poisson}(2/3)$, $\mathbb{P}(X=x) = \frac{e^{-2/3}(2/3)^x}{x!}$

$\Rightarrow \sum_{x=0}^{\infty} \mathbb{P}(X=x) = 1$ since it is a probability distribution.

$$\Rightarrow \frac{100e^{-2}}{e^{-2/3}}\left(\sum_{k=0}^{\infty} \frac{e^{-2/3}(2/3)^k}{k!}\right) = 100e^{-2}e^{2/3}(1) = 100e^{-4/3}.$$

Similarly, we calculate $\mathbb{E}(C_2)$.

$$\mathbb{E}(C_2) = \mathbb{E}(100u^Y) = 100\,\mathbb{E}(u^Y) = 100\sum_{k=0}^{\infty} u^k \frac{e^{-2} \cdot 2^k}{k!}$$

$$= 100\sum_{k=0}^{\infty} \frac{(2u)^k e^{-2}}{k!} = \frac{100e^{-2}}{e^{-2u}}\left(\sum_{k=0}^{\infty} \frac{e^{-2u}(2u)^k}{k!}\right) = 100e^{-2+2u}(1)$$
$$= 100e^{2(u-1)}$$