COMP 527 - Homework 2 - Question 1 Model Solution

March 1, 2016

Exercise 1

Task 1

$$\dfrac{\dfrac{}{x : A}\,u \qquad \dfrac{}{y : B}\,v}{\dfrac{\begin{array}{cc} \vdots & \vdots \\ M : B & N : A \end{array}}{(\!|x : A.M \otimes y : B.N|\!) : A \equiv B} \equiv I^{u,v}}$$

$$\dfrac{M : A \equiv B \qquad N : A}{M \hookleftarrow N : B} \equiv E_L$$

$$\dfrac{M : A \equiv B \qquad N : B}{M \mapsto N : A} \equiv E_R$$

Task 2

$$\dfrac{\dfrac{\dfrac{}{x : A}\,u \qquad \dfrac{}{y : B}\,v}{\dfrac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_2 \\ M : B & N : A \end{array}}{(\!|x : A.M \otimes y : B.N|\!) : A \equiv B} \equiv I^{u,v} \qquad \mathcal{E} \atop R : A}{(\!|x : A.M \otimes y : B.N|\!) \hookleftarrow R : B} \equiv E_L \quad \Longrightarrow_R \quad \begin{array}{c} \mathcal{E} \\ R : A \\ \mathcal{D}_1 \\ M : B \end{array}$$

$$\dfrac{\dfrac{\dfrac{}{x : A}\,u \qquad \dfrac{}{y : B}\,v}{\dfrac{\begin{array}{cc} \mathcal{D}_1 & \mathcal{D}_2 \\ M : B & N : A \end{array}}{(\!|x : A.M \otimes y : B.N|\!) : A \equiv B} \equiv I^{u,v} \qquad \mathcal{E} \atop S : B}{(\!|x : A.M \otimes y : B.N|\!) \mapsto S : A} \equiv E_R \quad \Longrightarrow_R \quad \begin{array}{c} [\mathcal{E}/v]\mathcal{D}_2 \\ N : A \end{array}$$

$$\begin{array}{c} \mathcal{E} \\ M : A \equiv B \end{array} \Longrightarrow_E \quad \dfrac{\dfrac{\begin{array}{cc} \mathcal{E} & \\ M : A \equiv B & \dfrac{}{r : A}\,u \end{array}}{M \hookleftarrow r : B} \equiv E_L \qquad \dfrac{\begin{array}{cc} \mathcal{E} & \\ M : A \equiv B & \dfrac{}{s : B}\,v \end{array}}{M \mapsto s : A} \equiv E_R}{(\!|x : A.M \hookleftarrow r \otimes y : B.M \mapsto s|\!) : A \equiv B} \equiv I^{u,v}$$

Task 3

We want to prove the substitution lemma, that if $\dfrac{\dfrac{}{x : A}\,u}{\vdots\ M : C}$ and $N : A$ then $[N/x]M : C$. To do this, we must cover three more cases than were covered in lecture:

Case: $\mathcal{D} = \dfrac{\dfrac{}{z : C}\,w \quad \dfrac{\vdots\ \mathcal{D}_1}{x : A}\,u \quad \dfrac{}{z : C}\,w \quad \dfrac{\vdots\ \mathcal{D}_2}{y : B}\,v \quad \dfrac{}{z : C}\,w}{(\!|x : A.M \otimes y : B.N|\!) : A \equiv B} \equiv I^{u,v}$

$L : C$	by assumption
$[L/z]M : B$	by IH $\mathcal{D}_1$
$[L/z]N : A$	by IH $\mathcal{D}_2$
$(\!|x : A.[L/z]M \otimes y : B.[L/z]N|\!) : A \equiv B$	by $\equiv I^{u,v}$
$[L/z](\!|x : A.M \otimes y : B.N|\!) : A \equiv B$	by substitution definition (since $x, y \notin \mathcal{FV}(L)$)

Case: $\mathcal{D} = \dfrac{\dfrac{}{z : C}\,u \quad \dfrac{\vdots\ \mathcal{D}_1}{M : A \equiv B} \quad \dfrac{}{z : C}\,u \quad \dfrac{\vdots\ \mathcal{D}_2}{N : A} \quad \dfrac{}{z : C}\,u}{M \leftarrowtail N : B} \equiv E_L$

$L : C$	by assumption
$[L/z]M : A \equiv B$	by IH $\mathcal{D}_1$
$[L/z]N : A$	by IH $\mathcal{D}_2$
$([L/z]M) \leftarrowtail ([L/z]N) : B$	by $\equiv E_L$
$[L/z](M \leftarrowtail N) : B$	by substitution definition

Case: $\mathcal{D} = \dfrac{\dfrac{}{z : C}\,u \quad \dfrac{\vdots\ \mathcal{D}_1}{M : A \equiv B} \quad \dfrac{}{z : C}\,u \quad \dfrac{\vdots\ \mathcal{D}_2}{N : B} \quad \dfrac{}{z : C}\,u}{M \mapsto N : A} \equiv E_R$

$L : C$	by assumption
$[L/z]M : A \equiv B$	by IH $\mathcal{D}_1$
$[L/z]N : B$	by IH $\mathcal{D}_2$
$([L/z]M) \mapsto ([L/z]N) : A$	by $\equiv E_R$
$[L/z](M \mapsto N) : A$	by substitution definition

Task 4

$$\dfrac{\dfrac{}{A\downarrow}\,u \qquad \dfrac{}{B\downarrow}\,v}{\begin{array}{c}\vdots\\ B\uparrow\end{array} \qquad \begin{array}{c}\vdots\\ A\uparrow\end{array}}{A \equiv B\uparrow} \equiv I^{u,v}$$

$$\dfrac{A \equiv B\downarrow \qquad A\uparrow}{B\downarrow} \equiv E_L$$

$$\dfrac{A \equiv B\downarrow \qquad B\uparrow}{A\downarrow} \equiv E_R$$