Math 340 - Jan. 18

Matching & Vertex Cover

Recall: Theorem : Hall
Let G be a bipartite graph with bipartition (A,B) then G contains a matching covering A iff
$|N(S)| \geq |S| \quad \forall S \subseteq A$
└ set of neighbours of vertices in S.

A set of vertices $X \subseteq V(G)$ is a vertex cover if every edge of G has an end in X.

not a vertex cover

is a vertex cover

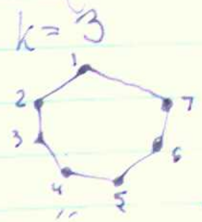
$\tau(G)$ is the minimum size of a vertex cover in G.
$\nu(G)$ is the maximum size of a matching in G.

Ex. Complete graphs:
K_n
$n = 5$

$\tau_n(K_n) = n-1$
$\nu(K_n) = \lfloor \frac{n}{2} \rfloor$

$*$ In general, $\nu(G) \leq \frac{|V(G)|}{2}$.

Ex. Cycles : C_{2k+1} odd number vertices
$k = 3$



$\nu(C_{2k+1}) = k$
$\tau(C_{2k+1}) = k+1$

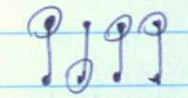
$\tau(C_{2k+1}) \leq k+1$
- If $v_1, \dots, v_{2k+1}$ are vertices of C_{2k+1} in order then $X = \{v_1, v_3, \dots, v_{2k+1}\}$ is a vertex cover

$\tau(C_{2k+1}) \geq k+1$
- because every vertex "covers" only 2 edges, to cover all $2k+1$ edges, we need at least $\lceil \frac{2k+1}{2} \rceil = k+1$ vertices.

Lemma:
For any graph G,
$\nu(G) \leq \tau(G) \leq 2\nu(G)$

Proof:
$\nu(G) \leq \tau(G)$:
Let M be a matching in G s.t. $|M| = \nu(G)$
Let X be a vertex cover in G s.t. $|X| = \tau(G)$

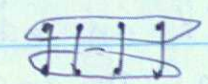
$|X| \geq |M|$



M

Every edge $e \in M$ has an end in X and different edges in M must have different vertices of X corresponding to them.

$\tau(G) \leq 2\nu(G)$:
Let M be a matching in G with $|M| = \nu(G)$
We need to find a vertex cover of G with $\leq 2\nu(G) = 2|M|$ vertices.



Let X be the set of all end of edges in M.
Then X is a vertex cover.
If not, and e is an edge not covered by X. then $M \cup \{e\}$ is a matching in G. This contradicts the maximality of M.
$\square$

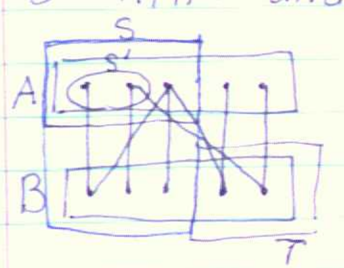
Theorem: König
In a bipartite graph G. $\nu(G) = \tau(G)$.

Proof:
By Lemma, $\nu(G) \le \tau(G)$
Now, we need to show $\nu(G) \ge \tau(G)$.

Let X be a vertex cover in G $|X| = \tau(G)$

Our goal is to show that there exists a matching in G with $|M| \ge |X|$.

Let (A, B) be a bipartition of G, and let $S = X \cap A$ and $T = X \cap B$

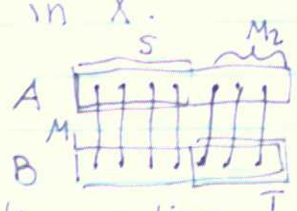


We find a matching M_1 covering S such that the ends of edges of M in B are in $B \setminus T$.

If such a matching does not exist.
Then by Hall's theorem there exists $S' \subseteq S$ s.t. $|S'| > |N(S')|$ where here $N(S')$ denotes neighbours of S' in $B \setminus T$.

Let $X' = (X \setminus S') \cup N(S')$. $|X'| < |X|$. X' is a vertex cover of G.
If an edge does not have an end in X' then it has an end in S' and so its second end is either in $N(S')$ or T, still in X'.

So M_1 exists.



Symmetrically, there exists a matching M_2, covering T matching vertices of $A \setminus S$

$|M_1| = |S|$, $|M_2| = |T|$

$M = M_1 \cup M_2$ is a matching. $|M| = |S| + |T| = |X|$. □

Ex. Given a 0-1 matrix M.

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \\ \\ \end{array}\begin{array}{c}\begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}\end{array}$$

- rank of a matrix is the size of a maximum collection of linearly independent rows (or columns)

$\tau(M)$ (cover number of M) is the size of minimum collection of lines (rows and/or columns) such that deleting them results in all zero matrix.
- $\tau(M) \le 4$ (in this example)

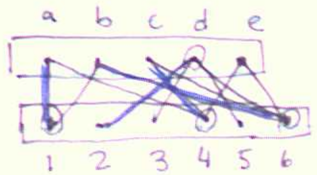
$\mathrm{rank}(M) \le \tau(M)$

term rank: $\nu(M)$ is the size of the maximum collection of 1's such that no two of them share a row or column.
- $\nu(M) \ge 4$ (in this example)

$\nu(M) \le \tau(M)$

$\nu(M)$ is the lower bound on the maximum possible rank of a matrix obtained from M by replacing 1's by some real numbers. And $\tau(M)$ is an upper bound.

Theorem: König - Egervary
$\tau(M) = \nu(M)$ for any 0-1 matrix M



graph of matrix

Proof:
Given M construct a bipartite graph G with bipartition (R, C) where R is the set of rows of M, and C is the set of colums of M, and $r \in R$ is adjacent to $c \in C$ iff the entry of M on the intersection of r & c is 1.

$\tau(G) = \tau(M)$: deleted rows and colums are vertex covers

$\nu(G) = \nu(M)$: "independent" set of ones correspond to matchings

Philogenetic Trees

Given a set of species $s_1, s_2, \dots, s_m$ with characteristics $c_1, c_2, \dots, c_n$.
We want to understand the evolution of these species from a common ancestor.

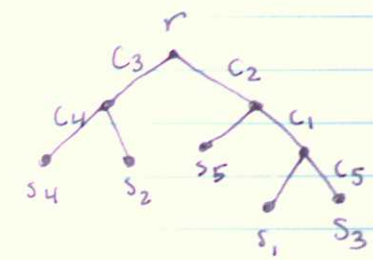
Ex.

$$\begin{array}{c|ccccc} & c_1 & c_2 & c_3 & c_4 & c_5 \\ \hline s_1 & 1 & 1 & 0 & 0 & 0 \\ s_2 & 0 & 0 & 1 & 0 & 0 \\ s_3 & 1 & 1 & 0 & 0 & 1 \\ s_4 & 0 & 0 & 1 & 1 & 0 \\ s_5 & 0 & 1 & 0 & 0 & 0 \end{array}$$

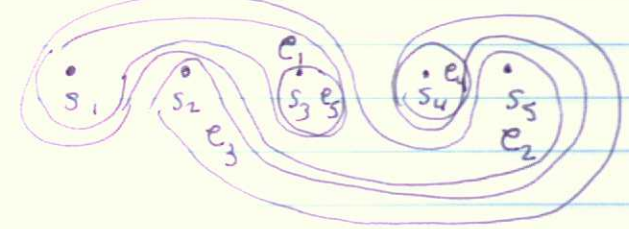
A perfect philogenetic tree T for these species. T has a root r and m leaves corresponding to $s_1, \dots, s_m$.

For each characteristic c_i, there is exactly one edge of T labelled by c_i.

Species s_i has a characteristic c_j iff some edge on a path from r to s_i is labelled c_i



When does a perfect philogenetic tree exist?
For each characteristic c_i let $\mathcal{C}_i$ be the set of all species having characteristic c_i.



For any two sets $\mathcal{C}_i$ and $\mathcal{C}_j$, we will have $\mathcal{C}_i \subseteq \mathcal{C}_j$ or $\mathcal{C}_j \subseteq \mathcal{C}_i$ or $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$.

This condition is necessary for the existence of the tree. It is also sufficient.

Suppose we want the tree fitting the data in the best way possible.

c_i and c_j are <u>conflicting</u> if $\mathcal{C}_i \cap \mathcal{C}_j \neq \emptyset$ but $\mathcal{C}_i \not\subseteq \mathcal{C}_j$, $\mathcal{C}_j \not\subseteq \mathcal{C}_i$.

To eliminate conflicting pairs, we find minimum vertex cover in the graph of conflicting pairs, and ignore the corresponding characteristics.