CHAPTER 10

Metric vector spaces

43 Geometry of metric spaces

4301. What vector spaces with bilinear forms in Exercise 3701 are metric?

4302. Prove that the real part $f(x, y)$ and the imaginary part $g(x, y)$ of an Hermitian function on a complex vector space V are invariant under multiplication by i; i.e. for any vectors $x, y \in V$

$$f(ix, iy) = f(x, y), \quad g(ix, iy) = g(x, y).$$

4303. Prove that a metric vector space is a direct sum of a subspace L and its orthocomplement $L^{\perp}$ if and only if the scalar product on L is nondegenerate and that in this case the scalar product on $L^{\perp}$ is also nondegenerate.

4304. Let $\mathbf{M}_n(\mathbb{C})$ be a space with an Hermitian scalar product

$$(X, Y) = \mathrm{tr} X^t \bar{Y}.$$

Find the orthocomplement of the subspace:

a) of all matrices with zero trace;

b) of all Hermitian matrices;

c) of all skew-Hermitian matrices;

d) of all upper-triangular matrices.

4305. Show that Hermitian and Euclidean spaces are normed.

4306. What norms on the spaces $\mathbb{R}^n$, $\mathbb{C}^n$ in Exercise 4201 are induced by Euclidean or Hermitian metric?

4307. Complete the system of vectors in Euclidean and Hermitian spaces to an orthogonal basis:

a) $((1, -2, 2, -3), \quad (2, -3, 2, 4))$;

b) $((1, 1, 1, 2), \quad (1, 2, 3, -3))$;

c) $\left(\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right), \quad \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)\right)$;

d) $\left(\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)\right)$;

e) $((1, 1-i, 2), \quad (2, -1+3i, 3-i))$;

f) $((-i, 2, -4+i), \quad (4-i, -1, i))$.

4308. Find the orthogonal projection of a vector x in Euclidean (Hermitian) space into the linear span of the orthonormal system of vectors $(e_1, \dots, e_k)$.

4309. Prove that it is possible to choose orthonormal bases $(e_1, \dots, e_k)$ and $(f_1, \dots, f_l)$ of any two subspaces of Euclidean (Hermitian) space such that $(e_i, f_j) = 0$ if $i \neq j$ and $(e_i, f_i) \geq 0$.

4310. Let $(e_1, \dots, e_k)$ and $(f_1, \dots, f_l)$ be orthonormal bases of subspaces L and M of Euclidean (Hermitian) space, and let $A = \big((e_i, f_j)\big)$ be the matrix of size $k \times l$. Prove that all characteristic numbers of the matrix ${}^tA \cdot A$ belong to the segment $[0, 1]$ and do not depend on the choice of bases of the subspaces L and M.

4311. Prove that any real symmetric matrix of rank $\leq n$ with non-negative (positive) principal minors is the Gram matrix of some (linearly independent, respectively) system of vectors of n-dimensional Euclidean space.

Prove the similar statement for a Hermitian matrix and Hermitian space.

4312. Prove that the sum of squares of lengths of projections of vectors of any orthonormal basis of Euclidean (Hermitian) space into k-dimensional subspace is equal to k.

4313. Let G be the matrix of the scalar product in a basis $(e_1, \dots, e_n)$ of Euclidean space V. Find the matrix of the change of the base to the dual one $(f_1, \dots, f_n)$ and the matrix of the scalar product in the dual basis.

4314. Let S be the matrix of a change of the basis $\mathbf{e}$ to the basis $\mathbf{e}'$. Find the matrix of the change of the basis $\mathbf{e}'$ dual to $\mathbf{e}$ to the basis $\mathbf{f}'$ dual to $\mathbf{f}$:

a) in Euclidean space;

b) in Hermitian space.

4315. Construct, with the help of the orthonormalization process, an orthogonal basis of the linear span of the system of vectors in Euclidean (Hermitian) space:

a) $((1, 2, 2, -1), (1, 1, -5, 3), (3, 2, 8, -7))$;

b) $((1, 1, -1, -2), (5, 8, -2, -3), (3, 9, 3, 8))$;

c) $((2, 1, 3, -1), (7, 4, 3, -3), (1, 1, -6, 0), (5, 7, 7, 8))$;

d) $((2, 1, -i), (1-i, 2, 0), (-i, 0, 1-i))$;

e) $((0, 1-i, 2), (-i, 2+3i, i), (0, 0, 2i))$.

4316. Find a basis of the orthocomplement of the linear span of the system of vectors in Euclidean (Hermitian) space:

a) $((1, 0, 2, 1), (2, 1, 2, 3), (0, 1, -2, 1))$;

b) $((1, 1, 1, 1), (-1, 1, -1, 1), (2, 0, 2, 0))$;

c) $((0, 1+2i, -i), (1, -1, 2-i))$.

4317. Prove that the systems of linear equations determining a linear subspace of $\mathbb{R}^n$ and its orthocomplement are connected as follows: the coefficients of the linearly independent system determining one of these subspaces are coordinates of vectors of a basis of the other subspace.

4318. Find the equations determining the orthocomplement of the subspaces given by the system of equations:

a)
$$\begin{aligned} 2x_1 + x_2 + 3x_3 - x_4 &= 0, \\ 3x_1 + 2x_2 \phantom{{}+3x_3} - 2x_4 &= 0, \\ 3x_1 + x_2 + 4x_3 - x_4 &= 0; \end{aligned}$$

b)
$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 - 3x_4 &= 0, \\ 3x_1 - x_2 + 11x_3 - 13x_4 &= 0, \\ 4x_1 + x_2 + 18x_3 - 23x_4 &= 0; \end{aligned}$$

c)
$$\begin{aligned} x_1 + (1-i)x_2 - ix_3 &= 0, \\ -ix_1 \phantom{{}+(1-i)} + 4x_2 \phantom{{}-ix_3} &= 0. \end{aligned}$$

4319. Find the projection of the vector x on the subspace L and the orthogonal component of x where:

a) $L = \langle (1, 1, 1, 1), (1, 2, 2, -1), (1, 0, 0, 3) \rangle$,
$x = (4, -1, -3, 4)$;

b) $L = \langle (2, 1, 1, -1), (1, 1, 3, 0), (1, 2, 8, 1) \rangle$,
$x = (5, 2, -2, 2)$;

c) L is given by the system of equations

$$\begin{aligned} 2x_1 + x_2 + x_3 + 3x_4 &= 0, \\ 3x_1 + 2x_2 + 2x_3 + x_4 &= 0, \\ x_1 + 2x_2 + 2x_3 - 4x_4 &= 0, \end{aligned}$$

$x = (7, -4, -1, 2)$;

d) $L = \langle(-i, 2+i, 0), (3, -i+1, i)\rangle$,
$x = (0, 1+i, -i)$;

e) L is given by the system of equations

$$\begin{aligned} (2+i)x_1 \quad & -ix_2 + 2x_3 + ix_4 = 0, \\ (2+i)x_1 \quad & -ix_2 + 2x_3 + ix_4 = 0, \\ 5x_1 + (-1+i)x_2 & + x_3 \qquad = 0, \end{aligned}$$

$x = (i, 2-i, 0)$.

4320. Let the orthogonalization process transform the system of vectors $a_1, \ldots, a_n$ into the system $b_1, \ldots, b_n$. Prove that the vector b_k is the orthogonal component of the vector a_k with respect to the linear span of the system $a_1, \ldots, a_{k-1}$ $(k > 1)$.

4321. Find the distance between the vector x and the subspace given by the system of equations:

a) $x = (2, 4, 0, -1)$;
$2x_1 + 2x_2 + x_3 + x_4 = 0$,
$2x_1 + 4x_2 + 2x_3 + 4x_4 = 0$;

b) $x = (3, 3, -4, 2)$;
$x_1 + 2x_2 + x_3 - x_4 = 0$,
$x_1 + 3x_2 + x_3 - 3x_4 = 0$;

c) $x = (3, 3, -1, 1, -1)$;
$2x_1 - 2x_2 + 3x_3 - 2x_4 + 2x_5 = 0$;

d) $x = (3, 3, -1, 1, -1)$;
$x_1 - 3x_2 + 2x_4 - x_5 = 0$;

e) $x = (0, -i, 1+i)$;
$x_1 + ix_2 - (2-i)x_3 = 0$;

f) $x = (1, -1, i)$;
$x_1 + (5+4i)x_2 - ix_3 = 0$.

4322. *Bessel inequality. Parceval equality.* Let $(e_1, \ldots, e_k)$ be an orthonormal system of vectors in n-dimensional Euclidean (Hermitian) space V. Prove that for any vector x the inequality holds in

$$\sum_{i=1}^{k} |(x, e_i)|^2 \leq \|x\|^2.$$

Prove that the equality holds for any x if and only if $k = n$, i.e. the given syste of vectors is an orthonormal basis of V (the *Parceval equality*).

4323. Applying the Cauchy inequality prove that

$$\left|\sum_{i=1}^{k} a_i b_i\right|^2 \leq \sum_{i,j=1}^{k} |a_i|^2 |b_i|^2$$

for any complex numbers $a_1, \ldots, a_k, b_1, \ldots, b_k$.

4324. Prove that the square of the distance between a vector x in Euclidea (Hermitian) space and a subspace with a basis $(e_1, \ldots, e_k)$ is equal to the ratio o the Gram determinants of the systems of vectors $(e_1, \ldots, e_k, x)$ and $(e_1, \ldots, e_k$

4325. Prove that the Gram determinant of any system of vectors

a) does not change under the orthogonalization process;

b) is non-negative;

c) is equal to zero if and only if the system is linearly dependent;

d) does not surpass the product of squares of lengths of vectors of the system and equality takes place if and only if either the vectors are pairwis orthogonal or one of them is equal to zero.

4326. Prove that the determinant of the matrix of a positively definite quadrati form does not surpass the product of entries of its principal diagonal.

4327. *Hadamard inequality.* Prove that for any real square matrix $A = (a_{ij})$ o size n the inequality

$$(\det A)^2 \leq \prod_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij}^2\right),$$

holds and the equality takes place if and only if either

$$\sum_{k=1}^{n} a_{ik} a_{jk} = 0 \quad (i, j = 1, \ldots, n; \quad i \neq j),$$

or the matrix A has a zero row.

Formulate and prove the similar statement for a complex matrix A.

4328. Find the lengths of the sides and the internal angles in the triangle abc i the space $\mathbb{R}^5$:

a) $a = (2, 4, 2, 4, 2)$,
$b = (6, 4, 4, 4, 6)$,
$c = (5, 7, 5, 7, 2)$;

b) $a = (1, 2, 3, 2, 1)$,
$b = (3, 4, 0, 4, 3)$,
$c = \left(1 + \frac{5}{26}\sqrt{78}, 2 + \frac{5}{13}\sqrt{78}, 3 + \frac{10}{13}\sqrt{78}, 2 + \frac{5}{13}\sqrt{78}, 1 + \frac{5}{26}\sqrt{78}\right)$.

4329. Prove with the help of the scalar product of vectors that

a) the sum of squares of the diagonals in a parallelogram is equal to the sum of the squares of its sides;

b) the square of a side of a triangle is equal to the sum of the squares of two other sides minus the double product of these sides by the cosine of the angle between them.

4330. Solve the system of linear equations by the method of least squares:

a)
$$\begin{aligned} x_1 + x_2 - 3x_3 &= -1, \\ 2x_1 + x_2 - 2x_3 &= 1, \\ x_1 + x_2 + x_3 &= 3, \\ x_1 + 2x_2 - 3x_3 &= 1; \end{aligned}$$

b)
$$\begin{aligned} 2x_1 - 5x_2 + 3x_3 + x_4 &= 5, \\ 3x_1 - 7x_2 + 3x_3 - x_4 &= -1, \\ 5x_1 - 9x_2 + 6x_3 + 2x_4 &= 7, \\ 4x_1 - 6x_2 + 3x_3 + x_4 &= 8. \end{aligned}$$

4331. *n-dimensional Pythagorean theorem.* Prove that the square of a diagonal of an n-dimensional rectangular parallelepiped is equal to the sum of squares of its edges outgoing from one vertex.

4332. Find the number of the diagonals in an n-dimensional cube which are orthogonal to a given diagonal.

4333. Find the length of a diagonal and the angles between the diagonals of the cube and its edges in an n-dimensional cube with an edge a.

4334. Find the radius R of a sphere circumscribed around an n-dimensional cube with an edge a, and solve the inequality $R < a$.

4335. Prove that the length of an orthogonal projection of an edge in an n-dimensional cube onto any of its diagonals is equal to $1/n$ of the length of the diagonal.

4336. Calculate the volume of an n-dimensional parallelepiped with sides:

a) $(1, -1, 1, -1)$, $(1, 1, 1, 1)$, $(1, 0, -1, 0)$, $(0, 1, 0, -1)$;

b) $(1, 1, 1, 1)$, $(1, -1, -1, 1)$, $(2, 1, 1, 3)$, $(0, 1, -1, 0)$;

c) $(1, 1, 1, 2, 1)$, $(1, 0, 0, 1, -2)$, $(2, 1, -1, 0, 2)$, $(0, 7, 3, -4, -2)$, $(39, -37, 51, -29, 5)$;

d) $(1, 0, 0, 2, 5)$, $(0, 1, 0, 3, 4)$, $(0, 0, 1, 4, 7)$, $(2, -3, 4, 11, 12)$, $(0, 0, 0, 0, 1)$.

4337. Prove, for the volume of a parallelepiped, the inequality

$$V(a_1, \dots, a_k, b_1, \dots, b_l) \le V(a_1, \dots, a_k) \cdot V(b_1, \dots, b_l),$$

and show that the equality takes place if and only if $(a_i, b_j) = 0$ for all i and j.

4338. Find the angle between the vector x and the subspace L:

a) $L = \langle (3, 4, -4, -1), (0, 1, -1, 2) \rangle$, $x = (2, 2, 1, 1)$;

b) $L = \langle (5, 3, 4, -3), (1, 1, 4, 5), (2, -1, 1, 2) \rangle$, $x = (1, 0, 3, 0)$;

c) $L = \langle (1, 1, 1, 1), (1, 2, 0, 0), (1, 3, 1, 1) \rangle$, $x = (1, 1, 0, 0)$;

d) $L = \langle (0, 0, 0, 1), (1, -1, -1, 1), (-3, 3, 3, 0) \rangle$, $x = (1, 2, 3, 0)$.

4339. Prove that if the angle between any two of k distinct vectors in Euclidean space V is equal to $\pi/3$, then $k \le \dim V$.

4340. Prove that if the angle between any two of k distinct vectors in Euclidean space V is obtuse, then $k \le 1 + \dim V$.

4341. Find the angle between a diagonal of an n-dimensional cube and its k-dimensional face.

4342. Find the angle between two-dimensional sides $a_0a_1a_2$ and $a_0a_3a_4$ in the regular four-dimensional simplex $a_0a_1a_2a_3a_4$.

4343. Find the angle between the subspaces

$$\langle (1, 0, 0, 0), (0, 1, 0, 0) \rangle \quad \text{and} \quad \langle (1, 1, 1, 1), (1, -1, 1, -1) \rangle.$$

4344. Polynomials of the type

$$P_0(x) = 1, \quad P_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} \left[(x^2 - 1)^k \right] \quad (k = 1, 2, \dots, n)$$

are called *Legendre polynomials.*

a) Prove that Legendre polynomials form an orthogonal basis of Euclidean space $\mathbb{R}[x]_n$ with the scalar product $\int_{-1}^{1} f(x)g(x)\,dx$.

b) Find the explicit form of the polynomials $P_k(x)$ for $k \le 4$.

c) Prove that $\deg P_k(x) = k$ and find the expansion of $P_k(x)$ for all k.

d) Calculate the length of the Legendre polynomial $P_k(x)$.

e) Calculate the value of $P_k(1)$.

f) Prove that the orthogonalization process applied to the basis $(1, x, x^2, \dots, x^n)$ of the space $\mathbb{R}[x]_n$ gives us a basis which differs only by

3924. Prove that if $f(t) = f_1(t)f_2(t)$ is a factorization of a polynomial $f(t)$ into coprime factors and a linear operator $\mathcal{A}$ satisfies the condition $f(\mathcal{A}) = 0$, then the matrix of $\mathcal{A}$ with some basis has the form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where $f_1(A_1) = 0$, $f_2(A_2) = 0$.

40 Eigenvectors, invariant subspaces, root subspaces

4001. Find the eigenvectors and eigenvalues

a) of the operator of differentiation on the space $\mathbb{R}[x]_n$;

b) of the operator $X \mapsto {}^tX$ on the space $\mathbf{M}_n(\mathbb{R})$;

c) of the operator $X\dfrac{d}{dx}$ on the space $\mathbb{R}[X]_n$;

d) of the operator $\dfrac{1}{X}\displaystyle\int_0^X f(t)dt$ on the space $\mathbb{R}[X]_n$;

e) of the operator $f \mapsto \dfrac{d^n f}{dx^n}$ on the linear span

$$\langle 1, \cos x, \sin x, \ldots, \cos mx, \sin mx\rangle;$$

f) of the operator $f \to \int_0^x f(t)dt$ on the linear span

$$\langle 1, \cos x, \sin x, \ldots, \cos mx, \sin mx\rangle.$$

4002. Prove that the linear operator $f \mapsto f(ax+b)$ on the space $\mathbb{R}[x]_n$ has the set of eigenvalues $1, a, \ldots, a^n$.

4003. Prove that an eigenvector of a linear operator $\mathcal{A}$ with an eigenvalue λ is an eigenvector of an operator $f(\mathcal{A})$, where $f(t)$ is a polynomial, with an eigenvalue $f(\lambda)$.

4004. Prove that if an operator $\mathcal{A}$ is nonsingular then operators $\mathcal{A}$ and $\mathcal{A}^{-1}$ have the same eigenvectors.

4005. Prove that all nonzero vectors of a space are eigenvectors of a linear operator $\mathcal{A}$ if and only if $\mathcal{A}$ is a homothety $x \mapsto \alpha x$, where α is some fixed scalar.

4006. Prove that if a linear operator $\mathcal{A}$ on a n-dimensional space has n distinct eigenvalues then any linear operator commuting with $\mathcal{A}$ has a basis consisting of its eigenvectors.

4007. Prove that the subspace $V_\lambda(\mathcal{A})$, consisting of all eigenvectors of an operator $\mathcal{A}$ with an eigenvalue λ and a zero vector, is invariant under any linear operator $\mathcal{B}$ commuting with $\mathcal{A}$.

4008. Prove that for any (possibly infinite) set of commuting linear operators on a finite-dimensional complex space

a) there exists a common eigenvector;

b) there exists a basis in which the matrices of all these operators are upper-triangular.

4009. Prove that if an operator $\mathcal{A}^2$ has an eigenvalue λ^2 then one of the numbers λ and $-\lambda$ is an eigenvalue of $\mathcal{A}$.

4010. Prove that

a) coefficients $c_1, \ldots, c_n$ of the polynomial

$$|A - \lambda E| = (-\lambda)^n + c_1(-\lambda)^{n-1} + \cdots + c_n$$

are sums of principal minors of corresponding sizes of the matrix A;

b) the sum and the product of eigenvalues of the matrix A are equal to its trace and its determinant, respectively.

4011. Prove that any polynomial of degree n with the leading coefficient $(-1)^n$ is a characteristic polynomial of some matrix of size n.

4012. Prove that if A and B are square matrices of the same size then matrices AB and BA have the same characteristic polynomials.

4013. Find eigenvalues of a matrix ${}^tA \cdot A$, where A is a row $(a_1, \ldots, a_n)$.

4014. Prove that all eigenvalues of a matrix are different from zero if and only if this matrix is nonsingular.

4015. Find the eigenvalues and eigenvectors of the linear operators given in some bases by the matrices:

a) $\begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$; b) $\begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{pmatrix}$; c) $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$;

d) $\begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}$; e) $\begin{pmatrix} 4 & -5 & 7 \\ 1 & -4 & 9 \\ -4 & 0 & 5 \end{pmatrix}$; f) $\begin{pmatrix} 3 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 0 & 5 & -3 \\ 4 & -1 & 3 & -1 \end{pmatrix}$.

4016. Find out if the following matrices can be reduced to diagonal ones by a change of basis over the field $\mathbb{R}$ and over the field $\mathbb{C}$:

a) $\begin{pmatrix} -1 & 3 & -1 \\ -3 & 5 & -1 \\ -3 & 3 & 1 \end{pmatrix}$; b) $\begin{pmatrix} 4 & 7 & -5 \\ -4 & 5 & 0 \\ 1 & 9 & -4 \end{pmatrix}$; c) $\begin{pmatrix} 4 & 2 & -5 \\ 6 & 4 & -9 \\ 5 & 3 & -7 \end{pmatrix}$;

d) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$.

Find this basis and the corresponding form of the matrix.

4017. Given a matrix with the entries $\alpha_1, \ldots, \alpha_n$ on the secondary diagonal, and all other entries equal to zero, under what conditions is the matrix similar to a diagonal matrix?

4018. Given a matrix A of size n, whose entries on the secondary diagonal are equal to 1, and all other of whose entries are equal to zero, find a matrix T such that $B = T^{-1}AT$ is a diagonal matrix. Calculate the matrix B.

4019. Prove that the number of linearly independent eigenvectors of a linear operator $\mathcal{A}$, with an eigenvalue λ, is less or equal to the multiplicity of λ as a root of the characteristic polynomial of $\mathcal{A}$.

4020. Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues of a linear operator $\mathcal{A}$ on an n-dimensional complex space. Find the eigenvalues of $\mathcal{A}$ as an operator on the corresponding $2n$-dimensional real space.

4021. Let $\lambda_1, \ldots, \lambda_n$ be roots of the characteristic polynomial of a matrix A. Find the eigenvalues:

a) of the linear operator $X \mapsto AX^tA$ on the space $\mathbf{M}_n(\mathbb{R})$;

b) of the linear operator $X \mapsto AXA^{-1}$ on the space $\mathbf{M}_n(\mathbb{R})$ (the matrix A is nonsingular).

4022. Find all invariant subspaces of the operator of differentiation on the space $\mathbb{R}[x]_n$.

4023. Prove that a linear span of any system of eigenvectors of a linear operator $\mathcal{A}$ is invariant under $\mathcal{A}$.

4024. Prove that

a) the kernel and the image of a linear operator $\mathcal{A}$ are invariant under $\mathcal{A}$;

b) any subspace containing the image of an operator $\mathcal{A}$ is invariant under $\mathcal{A}$;

c) if a subspace L is invariant under $\mathcal{A}$ then $\mathcal{A}(L)$ and $\mathcal{A}^{-1}(L)$ are invariant under $\mathcal{A}$;

d) if a linear operator $\mathcal{A}$ is nonsingular then any subspace invariant under $\mathcal{A}$ is invariant under $\mathcal{A}^{-1}$.

4025. Prove that any linear operator on n-dimensional complex space has an invariant subspace of dimension $n-1$.

4026. Let a linear operator on a vector space over a field K have the matrix

$$\begin{pmatrix} a_1 & 1 & 0 & \dots & 0 \\ a_2 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n-1} & 0 & 0 & \dots & 1 \\ a_n & 0 & 0 & \dots & 0 \end{pmatrix}$$

with some basis, where the polynomial $x^n - a_1x^{n-1} - \dots - a_{n-1}x - a_n$ is irreducible over K. Prove that the operator has no nontrivial invariant subspaces.

4027. Let a linear operator $\mathcal{A}$ on an n-dimensional space have, with some basis, a diagonal matrix with distinct entries on the diagonal. Find all subspaces invariant under $\mathcal{A}$.

4028. Find all invariant subspaces for a linear operator having, with some basis, the matrix consisting of a Jordan box.

4029. Find in three-dimensional vector space all subspaces invariant under the linear operator with the matrix

$$\begin{pmatrix} 4 & -2 & 2 \\ 2 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}.$$

4030. Find in three-dimensional vector space all subspaces invariant under two linear operators given by the matrices

$$\begin{pmatrix} 5 & -1 & -1 \\ -1 & 5 & -1 \\ -1 & -1 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -6 & 2 & 3 \\ 2 & -3 & 6 \\ 3 & 6 & 2 \end{pmatrix}.$$

4031. Find in $\mathbb{R}[X]_n$ and $\mathbb{C}[X]_n$ all subspaces invariant under operators

a) $\mathcal{A}(f) = X\dfrac{df}{dX}$; b) $\mathcal{A}(f) = \dfrac{1}{X}\displaystyle\int_0^X f(t)dt$.

4032. Find in the linear span of functions

$$\langle \cos x, \sin x, \dots, \cos nx, \sin nx \rangle$$

all subspaces invariant under the operator

a) $\mathcal{A}(f) = \dfrac{df}{dx}$; b) $\mathcal{A}(f) = \displaystyle\int_0^x f(t)dt$.

4033. Let $\mathcal{A}, \mathcal{B}$ be linear operators on a finite-dimensional vector space V over the field $\mathbb{C}$ such that $\mathcal{A}^2 = \mathcal{B}^2 = \mathcal{E}$. Prove that there exists either a one-dimensional or two-dimensional subspace of V which is invariant under $\mathcal{A}$ and $\mathcal{B}$.

4034. Prove that a complex vector space with only one line invariant under a linear operator $\mathcal{A}$ is indecomposable into a direct sum of nonzero subspaces invariant under $\mathcal{A}$.

4035. Find the eigenvalues and the root subspaces of the linear operator given in some bases by the matrix:

a) $\begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix}$; b) $\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}$; c) $\begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix}$;

d) $\begin{pmatrix} 0 & -2 & 3 & 2 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$.

4036. Prove that a linear operator on a complex vector space has a diagonal matrix in some basis if and only if all its root vectors are eigenvectors.

4037. Prove that if a linear operator on a complex vector space has a diagonal matrix in some basis then its restriction to any invariant subspace L also has a diagonal matrix in some basis of L.

4038. Prove that any root subspace of a linear operator $\mathcal{A}$ is invariant under any linear operator $\mathcal{B}$ commuting with $\mathcal{A}$.

4039. Prove that if a matrix of a linear operator $\mathcal{A}$ is reduced to the Jordan form then any invariant subspace L is a direct sum of intersections of L with the root subspaces of $\mathcal{A}$.

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4040. Let $A \in \mathbf{M}_n(\mathbb{C})$. Consider an operator L_A on the space $\mathbf{M}_{n\times m}(\mathbb{C})$ where $L_A(X) = AX$. Find the eigenvalues of L_A. Find the root subspaces of L_A where A is an upper-triangular matrix.

4041. Let $A \in \mathbf{M}_n(\mathbb{C})$, $B \in \mathbf{M}_m(\mathbb{C})$ and A, B have no common eigenvalues. Prove that