Chapter 7

Sequent Calculus

In this chapter, we prove consistency of our formal system described by natural deduction rules. We employ a classic approach already used by Gentzen in his original paper. We observe that in natural deduction there are many derivations which are not "normal" and the reasoning system admits detours. Consider the following two proofs for $A \supset B \supset A$. The proof on the left is normal while the one one the right is not; it contains a detour, i.e. $\wedge I$ followed by $\wedge E_l$, which can be eliminated by using a local reduction.

$$
\dfrac{\dfrac{\overline{A}^{\,v}}{B \supset A}\supset I^v}{A \supset B \supset A}\supset I^u
\qquad\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{\,u} \quad \overline{B}^{\,v}}{A \wedge B}\wedge I}{A}\wedge E_l}{B \supset A}\supset I^v}{A \supset B \supset A}\supset I^u
$$

It is these detours which make it difficult to argue that our system is consistent, i.e. from no assumptions we cannot derive falsehood. Gentzen hence introduced as a technical device another calculus, a sequent calculus, where such detours are not present. In fact, it is fairly obvious that falsehood is not derivable in the sequent calculus when we have no assumptions. Hence, consistency of the sequent calculus is obvious. He then showed that the sequent calculus plus one additional rule, the cut-rule, is equivalent to the natural deduction system. This is fairly easy. The hard part is to show that the cut-rule is *admissible*, i.e. it is not necessary. As a consequence, we know something stronger: all propositions provable in the natural deduction system are also provable in the sequent calculus without cut. Since we know that the sequent calculus is consistent, we hence also know that the natural deduction calculus is.

The sequent calculus is not only of interest because it is a convenient technical device for establishing consistency of natural deduction. It also gives rise to proof search procedures. In fact, we can study its proof-theoretic properties further and arrive at proof systems where proof search is feasible and amendable to efficient implementations.

7.0.1 Sequent calculus

In this section, we develop a closely related calculus to the bi-directional natural deduction system, called the sequent calculus. Studying meta-theoretical properties such as consistency is easier in this system; moreover, it provides a good calculus for proof search.

In the bi-directional natural deduction calculus, we keep the context $\Gamma^{\downarrow}$ for bookkeeping, but all the action happens on the right hand side of $\vdash^{+}$ (or $\vdash$). In particular, we switch from reasoning bottom-up via introduction rules to reasoning top-down via elimination rules; this switch is identified by the $\uparrow\downarrow$ rule.

In the sequent calculus, we only reason from the bottom-up by turning elimination rules into *left* rules that directly manipulate our assumptions in the context Γ. The judgement for a *sequent* is written as:

$$u_1{:}A_1,\ \ldots,\ u_n{:}A_n \quad \Longrightarrow \quad C$$

Note that the proposition C on the right directly corresponds to the proposition whose truth is established by a natural deduction. The propositions on the left however do not directly correspond to hypothesis in natural deduction, since in general they include hypothesis and propositions derived from assumptions by elimination rules. It is important to keep this difference in mind when we relate sequent proofs to natural deduction derivations.

Since the order of the propositions on the left is irrelevant, we write $\Gamma, u{:}A$ instead of the more pedantic $\Gamma, u{:}A, \Gamma'$.

Initial sequent The initial sequent allows us to conclude from our assumptions that A is true. Note that the initial sequent does not correspond to the hypothesis rule in natural deduction; it corresponds to the coercion rule $\uparrow\downarrow$, since our left hand side contains not only assumptions introduced by for example $\supset$ introduction (right), but also additional assumptions we have extracted from it.

$$\frac{}{\Gamma, u{:}A \Longrightarrow A}\ \text{init}$$

Conjunction The right and left rules are straightforward; we turn the introduction rules into *right* rules and the elimination rules are turned upside-down.

$$\frac{\Gamma \Longrightarrow A \qquad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B} \wedge R$$

$$\frac{\Gamma, u{:}A \wedge B, v : A \Longrightarrow C}{\Gamma, u{:}A \wedge B \Longrightarrow C} \wedge L_1 \qquad \frac{\Gamma, u{:}A \wedge B, v : B \Longrightarrow C}{\Gamma, u{:}A \wedge B \Longrightarrow C} \wedge L_2$$

In the introduction rule, read from the bottom-up, we propagate Γ to both premises. This is similar to natural deduction and reflects the fact that we can use assumptions as often as we like. In the elimination rule for $A \wedge B$ (see the rule $\wedge L_1$ and $\wedge L_2$), the assumption $A \wedge B$ persists. This reflects that assumptions may be used more than once. We analyze later which of these hypothesis are actually needed and which can be "garbage collected" if all possible information has been extracted from them. For now, however, leaving the assumptions untouched is useful since it will simplify the translation from sequent proofs to normal natural deduction derivations.

Implication The right rule is again straightforward. The left rule however is a bit more difficult. Given the assumption $A \supset B$, we can extract an additional assumption B, provided we are able to prove A.

$$\frac{\Gamma, u{:}A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B} \supset R^u \qquad \frac{\Gamma, u{:}A \supset B \Longrightarrow A \qquad \Gamma, u{:}A \supset B, v{:}B \Longrightarrow C}{\Gamma, u{:}A \supset B \Longrightarrow C} \supset L$$

Disjunction Considering disjunction does not require any new considerations.

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B} \vee R_2$$

$$\frac{\Gamma, u{:}A \vee B, v{:}A \Longrightarrow C \qquad \Gamma, u{:}A \vee B, w{:}B \Longrightarrow C}{\Gamma, u{:}A \vee B \Longrightarrow C} \vee L^{v,w}$$

Truth Since there is no elimination rule for $\top$, we only have to consider the introduction rule which directly translates to a right rule in the sequent calculus.

$$\frac{}{\Gamma \Longrightarrow \top} \top R$$

Falsehood Since there is no introduction rule for $\perp$, we only consider the elimination rule which turns into a left rule.

$$\frac{}{\Gamma, u{:}\perp \Longrightarrow C}\perp L$$

Note that the left rule has no premise.

Exercise 7.0.1. Derive the rules for universal and existential quantifiers in the sequent calculus.

Exercise 7.0.2. Derive the rules for negation in the sequent calculus form.

7.0.2 Theoretical properties of sequent calculus

We first begin by stating and revisiting some of the structural properties.

Lemma 7.0.1 (Structural properties of sequents)**.**

1. *(Weakening) If* $\Gamma \Longrightarrow C$ *then* $\Gamma, u{:}A \Longrightarrow C$.
2. *(Contraction) If* $\Gamma, u{:}A, v{:}A \Longrightarrow C$ *then* $\Gamma, u{:}A \Longrightarrow C$.

Proof. By structural induction on the first derivation. □

Next, we prove that our sequent calculus indeed characterizes normal natural deductions.

Theorem 7.0.2 (Soundness of sequent calculus)**.**
If $\Gamma \Longrightarrow C$ *then* $\Gamma^{\downarrow} \Longrightarrow C \uparrow$.

Proof. By induction on the structure of the derivation $\Gamma \Longrightarrow C$. Surprisingly, this proof is straightforward and we show a few cases.

Case $\mathcal{D} = \frac{}{\Gamma, u{:}C \Longrightarrow C}\text{init}$

$\Gamma^{\downarrow}, u{:}C \downarrow \vdash C \downarrow$ by hypothesis u
$\Gamma^{\downarrow}, u{:}C \downarrow \vdash C \uparrow$ by rule $\uparrow\downarrow$

This case confirms that initial sequents correspond to coercions $\uparrow\downarrow$.

$$\textbf{Case}\quad \mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma, u{:}A \Longrightarrow B\end{array}}{\Gamma \Longrightarrow A \supset B} \supset R^u$$

$\Gamma^{\downarrow}, u{:}A \downarrow \vdash B \uparrow$ by i.h. on $\mathcal{D}'$
$\Gamma^{\downarrow} \vdash A \supset B \uparrow$ by rule $\supset R^u$

$$\textbf{Case}\quad \mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma, u{:}A \supset B \Longrightarrow A\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Gamma, u{:}A \supset B, v{:}B \Longrightarrow C\end{array}}{\Gamma, u{:}A \supset B \Longrightarrow C} \supset L$$

$\Gamma^{\downarrow}, u{:}A \supset B \downarrow \vdash A \uparrow$ by i.h. $\mathcal{D}_1$
$\Gamma^{\downarrow}, u{:}A \supset B \downarrow \vdash A \supset B \downarrow$ by hypothesis u
$\Gamma^{\downarrow},\ u{:}A \supset \downarrow \vdash B \downarrow$ by rule $\supset E$
$\Gamma^{\downarrow},\ u{:}A \supset B \downarrow,\ v{:}B \downarrow \vdash C \uparrow$ by i.h. $\mathcal{D}_2$
$\Gamma^{\downarrow},\ u{:}A \supset B \downarrow \vdash C \uparrow$ by substitution property 4.0.4 $\square$

We now establish completeness: Every normal natural deduction derivation can be translated into a sequent proof. We cannot prove this statement directly, but we need to generalize is slightly. The readers not familiar with such generalization may want to test their understanding and try proving the completeness statement directly and see where such a direct proof breaks down.

Theorem 7.0.3 (Completeness of sequent calculus)**.**

1. *If* $\Gamma^{\downarrow} \vdash C \uparrow$ *then* $\Gamma \mathrm{der} C$.
2. *If* $\Gamma^{\downarrow} \vdash A \downarrow$ *and* $\Gamma, u{:}A \Longrightarrow C$ *then* $\Gamma \Longrightarrow C$.

Proof. By structural induction on the structure of the given derivation $\Gamma^{\downarrow} \vdash C \uparrow$ and $\Gamma^{\downarrow} \vdash A \downarrow$ respectively.

$$\textbf{Case}\quad \mathcal{D} = \dfrac{}{\Gamma_1^{\downarrow}, u{:}A \downarrow, \Gamma_2^{\downarrow} \vdash A \downarrow} u$$

$\Gamma_1, u{:}A, \Gamma_2, v{:}A \Longrightarrow C$ by assumption
$\Gamma_1, u{:}A, \Gamma_2 \Longrightarrow C$ by contraction (Lemma 7.0.1)

Case $\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma^{\downarrow} \vdash C \downarrow\end{array}}{\Gamma^{\downarrow} \vdash C \uparrow} \uparrow\downarrow$

$\Gamma, u{:}C \Longrightarrow C$ by init
$\Gamma \Longrightarrow C$ by i.h. $\mathcal{D}'$

Case $\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma^{\downarrow}, u{:}A \downarrow \vdash B \uparrow\end{array}}{\Gamma^{\downarrow} \vdash A \supset B \uparrow} \supset I^u$

$\Gamma, u{:}A \Longrightarrow B$ by i.h. $\mathcal{D}'$
$\Gamma \Longrightarrow A \supset B$ by $\supset R^u$

Case $\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma^{\downarrow} \vdash A \supset B \downarrow\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Gamma^{\downarrow} \vdash A \uparrow\end{array}}{\Gamma^{\downarrow} \vdash B \downarrow} \supset E$

$\Gamma, u{:}B \Longrightarrow C$ by assumption
$\Gamma, w{:}A \supset B, u{:}B \Longrightarrow C$ by weakening (Lemma 7.0.1)
$\Gamma \Longrightarrow A$ by i.h. $\mathcal{D}_2$
$\Gamma, w{:}A \supset B \Longrightarrow C$ by $\supset L$
$\Gamma \Longrightarrow C$ by i.h. $\mathcal{D}_1$
□

In order to establish soundness and completeness with respect to arbitrary natural deductions, we need to establish a connection to the bi-directional natural deduction system extended with $\downarrow\uparrow$ rule. We will now extend the sequent calculus with one additional rule which will correspond to the coercion $\downarrow\uparrow$ and is called the *cut* rule.

$$\frac{\Gamma \overset{+}{\Longrightarrow} A \qquad \Gamma, u{:}A \overset{+}{\Longrightarrow} C}{\Gamma \overset{+}{\Longrightarrow} C}\,\text{cut}$$

The overall picture of the argument is depicted in Fig below. We can so far conclude that normal natural deduction derivations correspond to sequent derivations. However, what we still need is the link in bold black between the extended

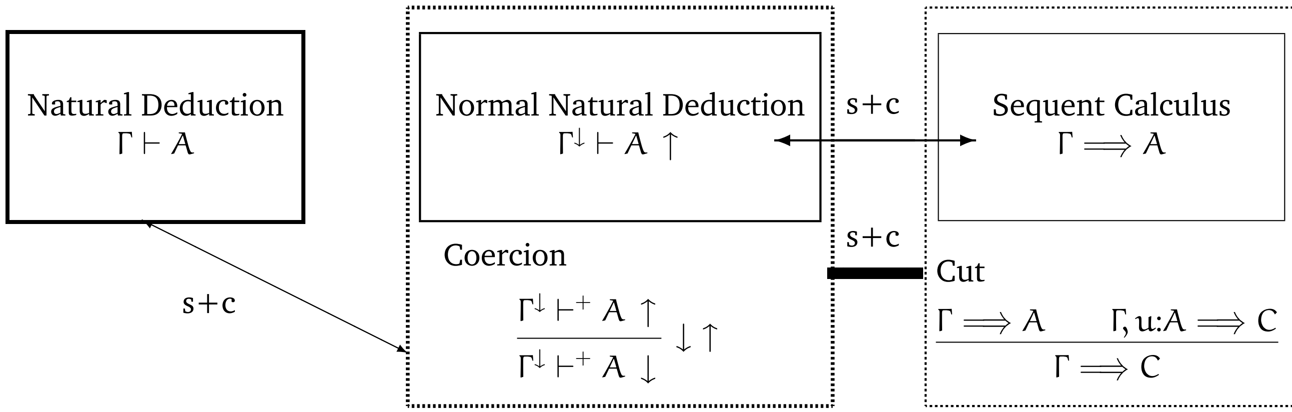

Figure 7.1: Proof outline

bi-directional natural deduction system and the sequent calculus with cut. If we have that link, we can show that *for all propositions which are provable in natural deduction, there exists a normal proof*.

Soundness of the extended sequent calculus with cut is straightforward.

Theorem 7.0.4 (Soundness of sequent calculus with cut)**.** *If* $\Gamma \overset{+}{\Longrightarrow} C$ *then* $\Gamma^{\downarrow} \vdash^{+} C \uparrow$.

Proof. Induction on the structure of the given derivation as in Soundness Theorem 7.0.2 with one additional case for handling the cut rule.

Case $\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Gamma \overset{+}{\Longrightarrow} A\end{array} \qquad \begin{array}{c}\mathcal{D}_2 \\ \Gamma, u{:}A \overset{+}{\Longrightarrow} C\end{array}}{\Gamma \overset{+}{\Longrightarrow} C}\,\text{cut}$

$\Gamma^{\downarrow} \vdash^{+} A \uparrow$	by i.h. $\mathcal{D}_1$
$\Gamma^{\downarrow} \vdash^{+} A \downarrow$	by $\downarrow\uparrow$
$\Gamma^{\downarrow}, v{:}A \vdash^{+} C \uparrow$	by i.h. $\mathcal{D}_2$
$\Gamma^{\downarrow} \vdash^{+} C \uparrow$	By substitution (Lemma 4.0.4) generalized)

□

Indeed this highlights the fact that the cut rule corresponds to the coercion $\downarrow\uparrow$ from neutral to normal (read from bottom-up).

We can now establish completeness of the sequent calculus with cut.

Theorem 7.0.5 (Completeness of sequent calculus with cut)**.**

1. *If* $\Gamma^{\downarrow} \vdash^{+} C \uparrow$ *then* $\Gamma \overset{+}{\Longrightarrow} C$.
2. *If* $\Gamma^{\downarrow} \vdash^{+} C \downarrow$ *and* $\Gamma, u{:}A \overset{+}{\Longrightarrow} C$ *then* $\Gamma \overset{+}{\Longrightarrow} C$.

Proof. As in the previous proof for completeness between the normal natural deduction and sequent calculus without cut with one additional case.

Case $\mathcal{D} = \dfrac{\begin{array}{c}\mathcal{D}' \\ \Gamma^{\downarrow} \vdash^{+} A \uparrow\end{array}}{\Gamma^{\downarrow} \vdash^{+} A \downarrow} \downarrow\uparrow$

$\Gamma \Longrightarrow A$	by i.h. $\mathcal{D}'$
$\Gamma, u{:}A \overset{+}{\Longrightarrow} C$	by assumption
$\Gamma \overset{+}{\Longrightarrow} C$	by cut

□

We are almost finished. The main theorem still missing is that the cut-rule is not necessary. This will establish that indeed if $\Gamma \overset{+}{\Longrightarrow} C$ then $\Gamma \Longrightarrow C$. In fact, we went through this detour simply because it is easier to show that the cut-rule is not necessary rather than showing directly that all natural deduction derivations can be translated to normal derivations.

Proving that the cut-rule is admissible, is called the cut-elimination theorem (Gentzen's Hauptsatz) and it is one of the central theorems of logic. As an immediate consequence, we have that not every proposition has a proof, since no rule is applicable to derive $\cdot \Longrightarrow \bot$, i.e. given no assumptions we cannot derive falsehood. Hence, it shows that our system is (weak) consistent.

7.0.3 Cut-elimination

In this section, we show one of the fundamental main theorems in logic, i.e. that the rule of cut is redundant in the sequent calculus. First, we prove that cut is *admissible*, i.e. whenever the premise of the cut rules are derivable in the sequent calculus *without cut*, then the conclusion is. Intuitively, it should be clear that adding an admissible rule to a deductive system does not change what can be derived. Formally, we can prove by induction over the structure of derivations that may contain cuts, i.e. $\Gamma \overset{+}{\Longrightarrow} C$ then $\Gamma \Longrightarrow C$.

To prove that cut is admissible, we prove the following theorem:

$$\textit{If } \mathcal{D} : \Gamma \Longrightarrow A \textit{ and } \mathcal{E} : \Gamma, A \Longrightarrow C \textit{ then } \Gamma \Longrightarrow C$$

We call A the *cut formula*. Moreover, recall that each left or right rule in the sequent calculus focuses on an occurrence of a proposition in the conclusion, called the *prinipal formula* of the inference.

In the proof, we reason by induction on the structure of the cut formula and on the structure of the given derivations $\mathcal{D}$ and $\mathcal{E}$. Either the cut formula is strictly smaller or with an identical cut formula, we either have $\mathcal{D}$ is strictly smaller while $\mathcal{E}$ remains the same or $\mathcal{E}$ is strictly smaller while $\mathcal{D}$ remains the same. The proof will first proceed by an outer induction on the structure of the cut-formula and then on an inner induction over the structure of the derivations.

The proof is constructive, which means we show hat to transform a proof $\mathcal{E} : \Gamma, A \Longrightarrow C$ and a proof $\mathcal{D} : \Gamma \Longrightarrow A$ into a proof $\Gamma \Longrightarrow C$. The proof is divided into several classes of cases. More than one case may be applicable which just means that the algorithm for constructing derivations of $\Gamma \Longrightarrow C$ is non-deterministic.

Theorem 7.0.6 (Admissibility of Cut).
If $\mathcal{D} : \Gamma \Longrightarrow A$ and $\mathcal{E} : \Gamma, A \Longrightarrow C$ then $\Gamma \Longrightarrow C$.

Proof. By nested induction on the structure of A, the derivation $\mathcal{D}$ of $\Gamma \Longrightarrow A$ and $\mathcal{E}$ of $\Gamma, A \Longrightarrow C$.

Case $\mathcal{D}$ is an initial sequent.

$$\mathcal{D} = \frac{}{\Gamma', A \Longrightarrow A}\,\text{init}$$

$\Gamma = \Gamma', A$	by assumption
$\Gamma', A, A \Longrightarrow C$	by assumption $\mathcal{E}$
$\Gamma', A \Longrightarrow C$	by contraction
$\Gamma \Longrightarrow C$	

Case $\mathcal{E}$ is an initial sequent and uses the cut formula

$$\mathcal{E} = \frac{}{\Gamma, A \Longrightarrow A}\,\text{init}$$

$C = A$	by assumption
$\Gamma \Longrightarrow A$	by derivation $\mathcal{D}$

Case $\mathcal{E}$ is an initial sequent and does not use the cut formula

$$\mathcal{E} = \frac{}{\Gamma', C, A \Longrightarrow C}\ \text{init}$$

$\Gamma = \Gamma', C$	by assumption
$\Gamma', C \Longrightarrow C$	by rule init
$\Gamma \Longrightarrow C$	by using the fact that $\Gamma = \Gamma', C$

Case A is the principal formula of the final inference in both $\mathcal{D}$ and $\mathcal{E}$. We show here some of the cases.

Subcase $A = A_1 \wedge A_2$.

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Longrightarrow A_1\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Gamma \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \wedge A_2}\ \wedge R \quad \text{and} \quad \mathcal{E} = \frac{\begin{array}{c}\mathcal{E}_1\\ \Gamma, A_1 \wedge A_2, A_1 \Longrightarrow C\end{array}}{\Gamma, A_1 \wedge A_2 \Longrightarrow C}\ \wedge L_1$$

$\mathcal{D}' : \Gamma, A_1 \Longrightarrow A_1 \wedge A_2$	by weakening
$\mathcal{F}_1 : \Gamma, A_1 \Longrightarrow C$	by i.h. $A_1 \wedge A_2$, $\mathcal{D}'$ and $\mathcal{E}_1$
$\mathcal{F}\ : \Gamma \Longrightarrow C$	by i.h. A_1, $\mathcal{D}_1$ and $\mathcal{F}_1$

We note that weakening $\mathcal{D}$ to $\mathcal{D}'$ does not alter the size of the derivation. Hence, the appeal to the induction hypothesis using $\mathcal{D}'$ and $\mathcal{E}_1$ is valid, because $\mathcal{E}_1$ is smaller than $\mathcal{E}$. We will not be explicit about such weakening steps subsequently.

We also note that the second appeal to the induction hypothesis using $\mathcal{D}_1$ and $\mathcal{F}_1$ is valid, since the cut formula A_1 is smaller than the original cut-formula $A_1 \wedge A_2$; hence it did not matter that we do know nothing about the size of $\mathcal{F}_1$.

Subcase $A = A_1 \supset A_2$.

$$\mathcal{D} = \frac{\begin{array}{c}\mathcal{D}_1\\ \Gamma, A_1 \Longrightarrow A_2\end{array}}{\Gamma \Longrightarrow A_1 \supset A_2}\ \supset R \quad \text{and} \quad \mathcal{E} = \frac{\begin{array}{c}E_1\\ \Gamma, A_1 \supset A_2 \Longrightarrow A_1\end{array} \qquad \begin{array}{c}E_2\\ \Gamma, A_1 \supset A_2, A_2 \Longrightarrow C\end{array}}{\Gamma, A_1 \supset A_2 \Longrightarrow C}\ \supset L$$

$\mathcal{F}_1 : \Gamma \Longrightarrow A_1$	by i.h. $A_1 \supset A_2$, $\mathcal{D}$ and $\mathcal{E}_1$
$\mathcal{F}_2 : \Gamma \Longrightarrow A_2$	by i.h. A_1, $\mathcal{F}_1$ and $\mathcal{D}_1$

$\mathcal{E}_2' : \Gamma, A_2 \Longrightarrow C$ by i.h. $A_1 \supset A_2$, $\mathcal{D}$, and $\mathcal{E}_2$
$\mathcal{F} \ : \Gamma \Longrightarrow C$ by i.h. A_2, $\mathcal{F}_2$ and $\mathcal{E}_2'$

Case A is not the principal formula of the last inference in $\mathcal{D}$. In that case $\mathcal{D}$ must end in a left rule and we can directly appeal to the induction hypothesis on one of the premises.

Subcase $$\mathcal{D} = \dfrac{\begin{matrix} D_1 \\ \Gamma', B_1 \wedge B_2, B_1 \Longrightarrow A \end{matrix}}{\Gamma', B_1 \wedge B_2 \Longrightarrow A} \wedge L_1$$

$\Gamma = \Gamma', B_1 \wedge B_2$ by assumption
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$ by i.h. A, $\mathcal{D}_1$, and $\mathcal{E}$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$ by $\wedge L_1$

Subcase $$\mathcal{D} = \dfrac{\begin{matrix} \mathcal{D}_1 \\ \Gamma', B_1 \supset B_2 \Longrightarrow B_1 \end{matrix} \qquad \begin{matrix} \mathcal{D}_2 \\ \Gamma', B_1 \supset B_2, B_2 \Longrightarrow A \end{matrix}}{\Gamma', B_1 \supset B_2 \Longrightarrow A} \supset L$$

$\Gamma = \Gamma', B_1 \supset B_2$ by assumption
$\Gamma', B_1 \supset B_2, B_2 \Longrightarrow C$ by i.h. A, $\mathcal{D}_2$ and $\mathcal{E}$
$\Gamma', B_1 \supset B_2 \Longrightarrow C$ by $\supset L$ using $\mathcal{D}_1$ and the above
$\Gamma \Longrightarrow C$

Case A is not the principal formula of the last inference in $\mathcal{E}$.

Subcase $$\mathcal{E} = \dfrac{\begin{matrix} \mathcal{E}_1 \\ \Gamma, A \Longrightarrow C_1 \end{matrix} \qquad \begin{matrix} \mathcal{E}_2 \\ \Gamma, A \Longrightarrow C_2 \end{matrix}}{\Gamma, A \Longrightarrow C_1 \wedge C_2} \wedge R$$

$C = C_1 \wedge C_2$ by assumption
$\Gamma \Longrightarrow C_1$ by i.h. A, $\mathcal{D}$, and $\mathcal{E}_1$
$\Gamma \Longrightarrow C_2$ by i.h. A, $\mathcal{D}$, and $\mathcal{E}_2$

$\Gamma der C_1 \wedge C_2$ by $\wedge$R on the above

Subcase $$\mathcal{E} = \dfrac{\begin{array}{c}\mathcal{E}_1\\ \Gamma', B_1 \wedge B_2, B_1, A \Longrightarrow C\end{array}}{\Gamma', B_1 \wedge B_2, A \Longrightarrow C}\wedge L_1$$

$\Gamma = \Gamma', B_1 \wedge B_2$ by assumption
$\Gamma', B_1 \wedge B_2, B_1 \Longrightarrow C$ by i.h. A, $\mathcal{D}$, $\mathcal{E}_1$
$\Gamma', B_1 \wedge B_2 \Longrightarrow C$ by $\wedge L_1$
$\Gamma \Longrightarrow C$ by $\Gamma = \Gamma', B_1 \wedge B_2$
□

As mentioned above, adding an admissible rule does not change the judgements which are derivable.

Theorem 7.0.7 (Cut Elimination)**.**
If $\Gamma \stackrel{+}{\Longrightarrow} C$ *then* $\Gamma \Longrightarrow C$

Proof. By structural induction on the given derivation $\Gamma \stackrel{+}{\Longrightarrow} C$. The proof is straightforward, and we only write out the case for cut.

Case $$\mathcal{D}^+ = \dfrac{\begin{array}{c}\mathcal{D}_1^+\\ \Gamma \stackrel{+}{\Longrightarrow} A\end{array} \qquad \begin{array}{c}\mathcal{D}_2^+\\ \Gamma, A \stackrel{+}{\Longrightarrow} c\end{array}}{\Gamma \stackrel{+}{\Longrightarrow} C}\text{cut}$$

$\Gamma \Longrightarrow A$ by i.h. on $\mathcal{D}^+ + 1$
$\Gamma, A \Longrightarrow C$ by i.h. on $\mathcal{D}^+ + 2$
$\Gamma \Longrightarrow C$ by admissibility of cut (Theorem 7.0.6)
□

7.1 Consequences of Cut Elimination

The cut elimination theorem is the central piece to complete our proof that it suffices to concentrate on normal natural deduction derivations to find a proof for a given proposition, i.e. if $\Gamma \vdash A$ then $\Gamma^{\downarrow} \vdash A \uparrow$.

Theorem 7.1.1 (Normalization for Natural Deduction).
If $\Gamma \vdash A$ *then* $\Gamma^{\downarrow} \vdash A \uparrow$.

Proof. Direct from the previous lemmas and theorems.

$\Gamma \vdash A$	by assumption
$\Gamma^{\downarrow} \vdash^{+} A \uparrow$	by completeness of natural deduction (Theorem 4.0.3)
$\Gamma \overset{+}{\Longrightarrow} A$	by completeness of sequent calculus with cut (Theorem 7.0.3)
$\Gamma \Longrightarrow A$	by completeness of sequ. calc. without cut (Cut-elimination Thm. 7.0.7)
$\Gamma^{\downarrow} \vdash A \uparrow$	by soundness of sequent calculus (Theorem 7.0.2)

□

Our normalization theorem justifies that for every proof $\Gamma \vdash A$, there exists some cut-free proof of the same theorem. This is often referred to as *weak normalization*: it suffices to provide some strategy of eliminating the cut.

Another important consequence of cut-elimination is that to find a proof for A in the natural deduction calculus, it suffices to show that there exists a proof in the sequent calculus without cut. As a consequence, if we want a proof for $\perp$, it suffices to show that there exists a proof $\cdot \Longrightarrow \perp$. Since $\Gamma = \cdot$, it is empty, we could not have used a left rule to derive $\perp$. However, there is no right rule which ends with $\perp$. Therefore, it is impossible to derive $\cdot \Longrightarrow \perp$.

Corollary 7.1.2 (Consistency). *There is no derivation for* $\cdot \vdash \perp$.

Proof. Assume there is a proof for $\cdot \vdash \perp$. Then by completeness of annotated deduction (Theorem 4.0.3) and completeness of seq. calculus with cut (Theorem 7.0.3) and cut-elimination (Thm. 7.0.7), it suffice to show that there is a proof for $\cdot \Longrightarrow \perp$. Since $\Gamma = \cdot$, there is no principal formula on the left and no left rule is applicable. There is also no right rule which ends in $\perp$. Therefore $\cdot \Longrightarrow \perp$ cannot be derived and hence $\cdot \vdash \perp$ is not derivable. □

Another consequence, is that we can show that the excluded middle $A \vee \neg A$ is not derivable. We also say that $A \vee \neg A$ is independent for arbitrary A.

Corollary 7.1.3 (Independence of Excluded Middle).
There is no deduction of $\vdash A \vee \neg A$ *for arbitrary* A.

Proof. Assume there is a derivation $\vdash A \vee \neg A$. By the completeness results and cut-elimination, it suffices to show $\cdot \Longrightarrow A \vee \neg A$. By inversion, we must have either $\cdot \Longrightarrow A$ or $\cdot \Longrightarrow \neg A$. The former judgement $\cdot \Longrightarrow A$ has no derivation. The latter judgement can only be inferred from $A \Longrightarrow \perp$. But there is no right rule with the conclusion $\perp$ and we cannot prove given an arbitrary A we can derive a contradiction. Hence, there cannot be a deduction of $\vdash A \vee \neg A$. □

Cut-elimination justifies that we can concentrate on finding a normal proof for a proposition A. We can also observe that proofs in the sequent calculus without cut are already much more restricted. Hence they are more amendable to proof search. The sequent calculus is hence an excellent foundation for proof search strategies. However, some non-determinism is still present. Should we apply a right rules or a left rule? And if we choose to apply a left rule, which formula from Γ should we pick as our principal formula?

Without techniques to restrict some of these choices, proof search is still infeasible. However, the sequent calculus lends itself to study these choices by considering two important properties: *inversion properties* allow us apply rules eagerly, i.e. their order does not matter (don't care non-determinism), and *focusing properties* allow us to chain rules which involve a choice and order does matter (do-care non-determinism), i.e. we make a choice we might as well continue to make choices and not postpone them.

7.2 Towards a focused sequent calculus

The simplest way to avoid non-determinism is to consider those propositions on the left or right for which a unique way to apply a rule. The following property essentially justifies that if we have the conclusion, then we must have had a proof of the premises. For example, to prove $A \wedge B$, we can immediately apply the right rule without loosing completeness. On the other hand, to prove $A \vee B$, we cannot immediately apply the right rule. As a counter example, consider $B \vee A \Longrightarrow A \vee B$; we need to apply first the left rule for splitting the derivation and then apply the right rule.

Inversion property:
The premises of an inference rule are derivable, if and only if the conclusion.

Given a sequent, a number of invertible rules may be applicable. However, the order of this choice, i.e. when to apply an invertible rule, does not matter. In other words, we are replacing *don't know non-determinism* by *don't care non-determinism*.

For controlling and restricting the search space, we can refine the inversion property as stated above further. In particular, in left rules, the principal formula is still present in the premises which means we can continue to apply the same left rule over and over again leading to non-termination. So we require in addition that the principal formula of a left rule is no longer needed, thereby guaranteeing the termination of the inversion phase.

Theorem 7.2.1 (Inversion)**.**

1. *If* $\Gamma \Longrightarrow A \wedge B$ *then* $\Gamma \Longrightarrow A$ *and* $\Gamma \Longrightarrow B$
2. *If* $\Gamma \Longrightarrow A \supset B$ *then* $\Gamma, A \Longrightarrow B$.
3. *If* $\Gamma, A \wedge B \Longrightarrow C$ *then* $\Gamma, A, B \Longrightarrow C$.
4. *If* $\Gamma, A \vee B \Longrightarrow C$ *then* $\Gamma, A \Longrightarrow C$ *and* $\Gamma, B \Longrightarrow C$.

Proof. Proof by structural induction on the given derivation; or, simply taking advantage of cut.

$\Gamma \Longrightarrow A \wedge B$	by assumption
$\Gamma, A \wedge B, A \Longrightarrow A$	by init
$\Gamma, A \wedge B \Longrightarrow A$	by $\wedge L_1$
$\Gamma \Longrightarrow A$	by cut

$\square$

We observe that $\top R$ and $\bot L$ are special; they can be applied eagerly, but they have no premises and therefore do not admit an inversion theorem.

There is a remarkable symmetry between the rules which are invertible and which are not. This picture is slightly spoiled by the left rule for conjunction. This can be corrected in moving to linear logic which we will not pursue here.

Chaining all invertible rules together, already gives us a good proof search strategy. However, it still leaves us with many possible choices once we have applied all invertible rules. How should one proceed to handle such choices? For example, if we pick a rule $\vee R_1$ for $(A \vee B) \vee C$, we now need to prove $A \vee B$ again facing a choice. Shall we stick to our committed choice and further split $A \vee B$ or shall we revisit other possible choices we have? - It turns out that *focusing* properties justify that we can stick to a choice we have made and continue to make further choices.

Focusing properties are dual to inversion properties; while inversion properties allow us to chain together invertible rules, focusing properties allow us to chain together non-invertible rules committing to a choice.

The focusing property and the duality between invertible / non-invertible rules was first observed by Andreoli in linear logic which did not show the anomaly for conjunction. Following Andreoli, we can classify formulas into positive and negative propositions where *positive propositions* are non-invertible on the right, but invertible on the left and *negative propositions* are non-invertible on the left, but invertible on the right.

$$
\begin{array}{llll}
\text{Formula} & A, B & ::= & R \mid L \\
\text{Positive} & R & ::= & P^{+} \mid A_1 \vee A_2 \mid \exists x{:}\tau.A(x) \mid \bot \mid A_1 \wedge A_2 \\
\text{Negative} & L & ::= & P^{-} \mid A_1 \supset A_2 \mid \forall x{:}\tau.A(x) \mid A_1 \wedge A_2 \\
\text{Stable Context} & \Delta & ::= & \cdot \mid \Delta, L
\end{array}
$$

Moreover, we will describe a stable context Δ which only consists of negative formulas. Intuitively, we will first consider first a formula A and apply invertible rules on the right until we obtain a *positive proposition*; at this point, we shift our attention to the context of assumptions and apply invertible left rules until our context is *stable*, i.e. it contains only negative propositions. At this point we have to make a choice. This phase is called the *asynchronous phase*.

From the asynchronous phase, we transition to the synchronous phase. We either commit to work on the right hand side of the sequent ($\Delta > R$) or we commit to work on the left hand side choosing an assumption from Δ, i.e. $\Delta > L \Longrightarrow R$. Let us summarize the four judgements:

$$
\begin{array}{ll}
\Delta; \Gamma \Longrightarrow [L] & \text{Asynchronous phase (right)} \\
\Delta; [\Gamma] \Longrightarrow R & \text{Asynchronous phase (left)} \\
\Delta > R & \text{Synchronous phase (right)} \\
\Delta > L \Longrightarrow R & \text{Synchronous phase (left)}
\end{array}
$$

The notation $\Delta > R$ and $\Delta > L \Longrightarrow R$ is chosen to draw attention to the part we focus on via $>$; the left hand side of $>$ describes the narrowing the focus of our attention.

Synchronous phase (left)

$$\frac{\Delta > A \qquad \Delta > B \Longrightarrow R}{\Delta > A \supset B \Longrightarrow R}$$

$$\frac{\Delta > A(t) \Longrightarrow R}{\Delta > \forall x.A(x) \Longrightarrow R}$$

$$\frac{\Delta > A_i \Longrightarrow R}{\Delta > A_1 \wedge A_2 \Longrightarrow R}$$

Asynchronous phase (right)

$$\frac{\Delta; \Gamma, A \Longrightarrow [\ B\ ]}{\Delta; \Gamma \Longrightarrow [\ A \supset B\ ]}$$

$$\frac{\Delta; \Gamma \Longrightarrow [\ A(a)\ ]}{\Delta; \Gamma \Longrightarrow [\ \forall x.A(x)\ ]}$$

$$\frac{\Delta; \Gamma \Longrightarrow [\ A\ ] \qquad \Delta; \Gamma \Longrightarrow [\ B\ ]}{\Delta; \Gamma \Longrightarrow [\ A \wedge B\ ]}$$

Asynchronous phase (left)

$$\frac{\Delta; [\ \Gamma, A(a)\ ] \Longrightarrow R}{\Delta; [\ \Gamma, \exists x.A(x)\ ] \Longrightarrow R}$$

$$\frac{\Delta; [\ \Gamma, A\ ] \Longrightarrow R \qquad \Delta; [\ \Gamma, B\ ] \Longrightarrow R}{\Delta; [\ \Gamma, A \vee B\ ] \Longrightarrow R}$$

$$\frac{\Delta; [\ \Gamma, A, B\ ] \Longrightarrow R}{\Delta; [\ \Gamma, A \wedge B\ ] \Longrightarrow R}$$

Synchronous phase (right)

$$\frac{\Delta > A(t)}{\Delta > \exists x.A(x)}$$

$$\frac{\Delta > A_i}{\Delta > A_1 \vee A_2}$$

$$\frac{\Delta > A_1 \qquad \Delta > A_2}{\Delta > A_1 \wedge A_2}$$

Identity (positive)

$$\frac{}{\Delta, P > P^+}$$

Identity (negative)

$$\frac{}{\Delta > P \Longrightarrow P^-}$$

Transition rules

$$\frac{L \in \Delta \quad \Delta > L \Longrightarrow R}{\Delta; [\ \cdot\ ] \Longrightarrow R}\ \text{choice}_L$$

$$\frac{\Delta > R}{\Delta; [\ \cdot\ ] \Longrightarrow R}\ \text{choice}_R$$

$$\frac{\Delta; \cdot \Longrightarrow [\ L\ ]}{\Delta > L}\ \text{Blur}_R$$

$$\frac{\Delta; [\ R\ ] \Longrightarrow R'}{\Delta > R \Longrightarrow R'}\ \text{Blur}_L$$

$$\frac{\Delta, L; [\ \Gamma\ ] \Longrightarrow R}{\Delta; [\ \Gamma, L\ ] \Longrightarrow R}\ \text{Move-to-stable-context}$$

$$\frac{\Delta; [\ \Gamma\ ] \Longrightarrow R}{\Delta; \Gamma \Longrightarrow [\ R\ ]}\ \text{Transition-to-left}$$

Let's work through an example. We first specify a predicate $\text{fib}(n, x)$ which reads x is the fibonacci number corresponding to n.

$$\begin{array}{rl} \mathcal{P} = & \text{fib}(0,0),\ \text{fib}(1,1), \\ & \forall n \forall x \forall y.\text{fib}(n,x) \supset \text{fib}(s\ n,\ y) \supset \text{fib}(s\ s\ n,\ x+y) \end{array}$$

We now want to prove $\mathcal{P} \Longrightarrow \text{fib}(3,2)$. We will consider here the derivation beginning with the focusing phase. Moreover, we will treat the predicate $\text{fib}(m, r)$ as negative. Note that focusing essentially leaves us no choice in the derivation shown below.

$$\dfrac{\dfrac{\mathcal{P} \Longrightarrow \text{fib}(1,1)}{\mathcal{P} > \text{fib}(1,1)} \qquad \dfrac{\dfrac{\mathcal{P} \Longrightarrow \text{fib}(2,1)}{\mathcal{P} > \text{fib}(2,1)} \qquad \dfrac{}{\mathcal{P} > \text{fib}(3,2) \Longrightarrow \text{fib}(3,2)}\,\text{init}}{\mathcal{P} > \text{fib}(s\ 1,\ 1) \supset \text{fib}(s\,s\,1,\ 1+1) \Longrightarrow \text{fib}(3,2)}}{\dfrac{\mathcal{P} > \text{fib}(1,1) \supset \text{fib}(s\ 1,\ 1) \supset \text{fib}(s\,s\,1,\ 1+1) \Longrightarrow \text{fib}(3,2)}{\dfrac{\mathcal{P} > \forall y.\text{fib}(1,1) \supset \text{fib}(s\ 1,\ y) \supset \text{fib}(s\,s\,1,\ 1+y) \Longrightarrow \text{fib}(3,2)}{\dfrac{\mathcal{P} > \forall x \forall y.\text{fib}(1,x) \supset \text{fib}(s\ 1,\ y) \supset \text{fib}(s\,s\,1,\ x+y) \Longrightarrow \text{fib}(3,2)}{\dfrac{\mathcal{P} > \forall n \forall x \forall y.\text{fib}(n,x) \supset \text{fib}(s\,n,\ y) \supset \text{fib}(s\,s\,n,\ x+y) \Longrightarrow \text{fib}(3,2)}{\mathcal{P}; [\ \cdot\ ] \Longrightarrow \text{fib}(3,3)}}}}}$$

Note that because we have chosen the predicate fib to be negative, we must blur our focus in the two open derivations and also the application of the init rule is determined. We can now in fact collapse this derivation into a "big-step" derived rule:

$$\dfrac{\mathcal{P} \Longrightarrow \text{fib}(1,1) \qquad \mathcal{P} \Longrightarrow \text{fib}(2,1)}{\mathcal{P} \Longrightarrow \text{fib}(3,2)}$$

This rule exactly captures our intuition. Another "big-step" rule which can be derived is:

$$\dfrac{\mathcal{P} \Longrightarrow \text{fib}(0,0) \qquad \mathcal{P} \Longrightarrow \text{fib}(1,1)}{\mathcal{P} \Longrightarrow \text{fib}(2,1)}$$

Proof search then amounts to searching over "big-step" rules - this makes proof search more efficient and easier to interact with.

We can prove that our given focused calculus is sound and complete. It is also interesting to note that by giving different polarities to atoms, we obtain different proof search strategies - assigning negative polarities to atoms allows us to model backwards proof search as we have illustrated; assigning positive polarities to atoms in fact leads to forward proof search. Hence the system is general enough to model different proof search strategies. This was observed by Chaudhuri and Pfenning.

Moreover, it is worth noting that we can give it a computational interpretations; focusing calculi provide a type system for languages supporting pattern matching and different proof strategies correspond to different evaluation strategies modelling call-by-value or call-by-name evaluation (see for example work by Noam Zeilberger).