3. INVESTIGATIONS INTO LOGICAL DEDUCTION

SYNOPSIS

The investigations that follow concern the domain of *predicate logic* (H.-A.[20] call it the 'restricted predicate calculus'). It comprises the types of inference that are continually used in all parts of mathematics. What remains to be added to these are axioms and forms of inference that may be considered as being proper to the particular branches of mathematics, e.g., in elementary number theory the axioms of the natural numbers, of addition, multiplication, and exponentiation, as well as the inference of complete induction; in geometry the geometric axioms.

In addition to *classical logic* I shall also deal with *intuitionist logic* as formalized, for example, by Heyting[21].

The present investigations into classical and intuitionist predicate logic fall essentially into two only loosely connected parts.

1. My starting point was this: The formalization of logical deduction, especially as it has been developed by Frege, Russell, and Hilbert, is rather far removed from the forms of deduction used in practice in mathematical proofs. Considerable formal advantages are achieved in return.

In contrast, I intended first to set up a formal system which comes as close as possible to actual reasoning. The result was a '*calculus of natural deduction*' ('*NJ*' for intuitionist, '*NK*' for classical predicate logic). This calculus then turned out to have certain special properties; in particular, the 'law of the excluded middle', which the intuitionists reject, occupies a special position.

I shall develop the calculus of natural deduction in section II of this paper together with some remarks concerning it.

2. A closer investigation of the specific properties of the natural calculus finally led me to a very general theorem which will be referred to below as the '*Hauptsatz*'.

The *Hauptsatz*[22] says that every purely logical proof can be reduced to a definite, though not unique, normal form. Perhaps we may express the essential properties of such a normal proof by saying: it is not roundabout. No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result.

The *Hauptsatz* holds both for classical and for intuitionist predicate logic.

In order to be able to enunciate and prove the *Hauptsatz* in a convenient form, I had to provide a logical calculus especially suited to the purpose. For this the natural calculus proved unsuitable. For, although it already contains the properties essential to the validity of the *Hauptsatz*, it does so only with respect to its intuitionist form, in view of the fact that the law of excluded middle, as pointed out earlier, occupies a special position in relation to these properties.

In section III of this paper, therefore, I shall develop a new calculus of logical deduction possessing all the desired properties in both their intuitionist and their classical forms ('*LJ*' for intuitionist, '*LK*' for classical predicate logic). The *Hauptsatz* will then be enunciated and proved by means of that calculus.

The *Hauptsatz* permits of a variety of *applications*. To illustrate this I shall develop a decision procedure (IV, § 1) for intuitionist propositional logic in section IV, and shall in addition give a new proof of the consistency of classical arithmetic without complete induction (IV, § 3).

Sections III and IV may be read independently of section II.

3. Section I contains the terminology and notations used in this paper.

In section V, I prove the *equivalence* of the logical calculi *NJ*, *NK*, and *LJ*, *LK*, developed in this paper, by means of a calculus modelled on the formalisms of Russell, Hilbert, and Heyting (and which may easily be compared with them). ('*LHJ*' for intuitionist, '*LHK*' for classical predicate logic.)

SECTION I. TERMINOLOGY AND NOTATIONS

To the concepts 'object', 'function', 'predicate', 'proposition', 'theorem', 'axiom', 'proof', 'inference', etc., in logic and mathematics there correspond, in the formalization of these disciplines, certain symbols or combinations of symbols. We divide these into:

1. *Symbols.*

2. *Expressions*, i.e., finite sequences of symbols.
3. *Figures*, i.e., finite sets of symbols, with some ordering.

Symbols count as special cases of expressions and figures, expressions as special cases of figures.

In this paper we shall consider symbols, expressions, and figures of the following kind:

1. *Symbols*.

These divide into constant symbols and variables.

1.1. Constant symbols:

Symbols for definite objects: 1, 2, 3, . . .

Symbols for definite functions: $+$, $-$, $\cdot$.

Symbols for definite propositions: $\curlyvee$ ('the true proposition'), $\curlywedge$ ('the false proposition').

Symbols for definite predicates: $=$, $<$.

Logical symbols:[23] & 'and', $\vee$ 'or', $\supset$ 'if . . . then', $\supset\subset$ 'is equivalent to', $\neg$ 'not', $\forall$ 'for all', $\exists$ 'there is'.

We shall also use the terms: conjunction symbol, disjunction symbol, implication symbol, equivalence symbol, negation symbol, universal quantifier, existential quantifier.

Auxiliary symbols:) , (, $\rightarrow$.

1.2. Variables:

Object variables. These we divide into *free* object variables: $a, b, c, \ldots, m$ and *bound* object variables: $n, \ldots, x, y, z$.

Propositional variables: $A, B, C, \ldots$.

An arbitrary number of variables will be assumed to be available; if the alphabet is insufficient, we adjoin numerical subscripts, e.g., a_7, C_3.

1.3. German and Greek letters serve as 'syntactic variables', i.e., not as symbols of the logic formalized, but as variables of our deliberations *about* that logic. Their meanings are explained as they are used.

2. *Expressions*.

2.1. The concept of a propositional expression, called a 'formula' for short (defined inductively):

(The concept of a formula is ordinarily used in a more general sense; the special case defined below might thus perhaps be described as a 'purely logical formula'.)

2.11. A symbol for a definite proposition (i.e., the symbols $\curlyvee$ and $\curlywedge$) is a formula.

A propositional variable followed by a number (possibly zero) of free object variables is a formula, e.g., *Abab*.

The object variables are called the *arguments* of the propositional variables.

Formulae of the two kinds mentioned are also called *elementary formulae*.

2.12. If $\mathfrak{A}$ is a formula, then $\neg \mathfrak{A}$ is also a formula.

If $\mathfrak{A}$ and $\mathfrak{B}$ are formulae, then $\mathfrak{A} \,\&\, \mathfrak{B}$, $\mathfrak{A} \vee \mathfrak{B}$, $\mathfrak{A} \supset \mathfrak{B}$ are formulae.

(We shall not introduce the symbol $\supset\subset$ into our presentation; it is in fact superfluous, since $\mathfrak{A} \supset\subset \mathfrak{B}$ may be regarded as an abbreviation for $(\mathfrak{A} \supset \mathfrak{B}) \,\&\, (\mathfrak{B} \supset \mathfrak{A})$.

2.13. A formula not containing the bound object variable $\mathfrak{x}$ yields another formula, if we prefix either $\forall \mathfrak{x}$ or $\exists \mathfrak{x}$. At the same time we may substitute $\mathfrak{x}$ in a number of places for a free object variable occurring in the formula.

2.14. Brackets (or parentheses) are to be used to show the structure of a formula unambiguously. Example of a formula:

$$\exists x\, (((\neg\, Abxa) \vee Bx) \supset (\forall z\, (A \,\&\, B)))$$

By special convention the number of brackets may be reduced, but (with one exception, *vide* 2.4) no use will be made of this, since we do not have to write down many formulae.

2.2. The number of logical symbols occurring in a formula is called the *degree of the formula*. (Thus an elementary formula is of degree 0.)

The logical symbol of a nonelementary formula that has been adjoined last in the construction of the formula according to 2.12 and 2.13, is called the *terminal symbol of the formula*.

Formulae that may have arisen in the course of the construction of a formula according to 2.12 and 2.13, including the formula itself, are called *subformulae*.

Example: the subformulae of $A \,\&\, \forall x\, Bxa$ are A, $\forall x\, Bxa$, $A \,\&\, \forall x\, Bxa$ as well as all formulae of the form $B\mathfrak{a}a$, where $\mathfrak{a}$ represents any free object variable (this variable may also be a, for example). The degree of $A \,\&\, \forall x\, Bxa$ is 2, the terminal symbol is &.

2.3. The concept of a *sequent*:

(This concept will not be used until section III, and it is only then that the purpose of its introduction becomes clear.)

A sequent is an expression of the form

$$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \rightarrow \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu,$$

where $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu, \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$ may represent any formula whatever. (The $\rightarrow$, like commas, is an auxiliary symbol and not a logical symbol.)

The formulae $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu$ form the *antecedent*, and the formulae

$\mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$, the *succedent* of the sequent. Both expressions may be empty.

2.4. The sequent $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \rightarrow \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$ has exactly the same informal meaning as the formula

$$(\mathfrak{A}_1 \,\&\, \ldots \,\&\, \mathfrak{A}_\mu) \supset (\mathfrak{B}_1 \vee \ldots \vee \mathfrak{B}_\nu).$$

(By $\mathfrak{A}_1 \,\&\, \mathfrak{A}_2 \,\&\, \mathfrak{A}_3$ we mean $(\mathfrak{A}_1 \,\&\, \mathfrak{A}_2) \,\&\, \mathfrak{A}_3$, likewise for $\vee$.)

If the antecedent is empty, the sequent reduces to the formula $\mathfrak{B}_1 \vee \ldots \vee \mathfrak{B}_\nu$.

If the succedent is empty, the sequent means the same as the formula $\neg (\mathfrak{A}_1 \,\&\, \ldots \,\&\, \mathfrak{A}_\mu)$ or $(\mathfrak{A}_1 \,\&\, \ldots \,\&\, \mathfrak{A}_\mu) \supset \curlywedge$.

If both the antecedent and the succedent of the formula are empty, the sequent means the same as $\curlywedge$, i.e., a false proposition.

Conversely, to every formula there corresponds an equivalent sequent, e.g., the sequent whose antecedent is empty and whose succedent consists precisely of that formula.

The formulae making up a sequent are called *S-formulae* (i.e., sequent formulae). By this we intend to indicate that we are not considering the formula by itself, but as it appears in the sequent. Thus we say, for example:

'A formula occurs in several places in a sequent as an *S*-formula', which may also be expressed as follows:

'Several distinct *S*-formulae (which shall simply mean: having distinct occurrences in the sequent) are formally identical'.

3. *Figures*

We require inference figures and proof figures.

Such figures consist of formulae or sequents, as the case may be. In what follows (3.1 to 3.3, 3.5) we shall be speaking only of formulae, but whatever is said applies analogously to sequents; all we need to do is to replace the word 'formula', wherever it occurs, by the word 'sequent'.

3.1. An *inference figure* may be written in the following way:

$$\frac{\mathfrak{A}_1, \ldots, \mathfrak{A}_\nu}{\mathfrak{B}} \qquad (\nu \geqq 1),$$

where $\mathfrak{A}_1, \ldots, \mathfrak{A}_\nu$, $\mathfrak{B}$ are formulae. $\mathfrak{A}_1, \ldots, \mathfrak{A}_\nu$ are then called the *upper formulae* and $\mathfrak{B}$ the *lower formula* of the inference figure. (The concepts of the upper sequents and of the lower sequent of an inference figure consisting of sequents are to be understood correspondingly.)

We shall have to consider only particular inference figures and they will be stated for each calculus as they arise.

3.2. A *proof figure*, called a *derivation* for short, consists of a number of formulae (at least one), which combine to form inference figures in the following way: Each formula is a lower formula of at most one inference figure; each formula (with the exception of exactly one: the *endformula*) is an upper formula of at least one inference figure; and the system of inference figures is noncircular, i.e., there is in the derivation no cycle (no sequence whose last member is again succeeded by its first member) of formulae such that each member is an upper formula of an inference figure whose lower formula is the next formula in the sequence.

3.3. The formulae of a derivation that are not lower formulae of an inference figure are called *initial formulae* of the derivation.

A derivation is in 'tree form' if each one of its formulae is an upper formula of *at most one* inference figure.

Thus all formulae except the endformula are upper formulae of *exactly one* inference figure.

We shall have to treat only of derivations in tree form.

The formulae which compose a derivation so defined are called *D-formulae* (i.e., derivation formulae). By this we wish to indicate that we are not considering merely the formula as such, but also its position in the derivation. In this sense we shall be using, for example, expressions such as: 'A formula occurs in a derivation as a *D-formula*'. 'Two distinct *D*-formulae (i.e., formulae occurring merely in distinct places in the derivation) are formally identical, viz., identical to the same formula'.

Thus by '$\mathfrak{A}$ is the *same* D-formula as $\mathfrak{B}$' we mean that $\mathfrak{A}$ and $\mathfrak{B}$ are not only formally identical, but occur also in the same place in the derivation. We shall use the words 'formally identical' to indicate identity of form regardless of place.

For object variables, however, we shall not introduce a special term that would associate the variable with a specific place of occurrence in the formula. Thus we say, e.g.: 'The *same* object variable occurs in two distinct D-formulae.'

3.4. The inference figures of the derivation are called *D-inference figures* (i.e., derivation inference figures).

In a derivation consisting of sequents the S-formulae of the D-sequents are called *D-S-formulae* (i.e., derivation sequent formulae).

3.5. A *path* in a derivation is (following Hilbert) a sequence of D-formulae whose first formula is an initial formula and whose last formula is the endformula, and of which each formula except the last is an upper formula of a D-inference figure whose lower formula is the next formula in the path.

We say that 'a D-formula stands *above* (*below*) another D-formula' if there exists a path in which the former occurs before (after) the latter.

We are here thinking of the fact that a derivation is written in tree form with the initial formulae above and the endformula below. (Examples may be found in II, § 4.)

Furthermore, we say that 'a D-inference figure occurs above (below) a D-formula', if all formulae of the inference figure occur above (below) that D-formula.

A derivation with the endformula $\mathfrak{A}$ is also called a 'derivation of $\mathfrak{A}$'.

The initial formulae of a derivation may be *basic formulae* or *assumption formulae*; more about their nature will have to be said as we reach the different calculi.

SECTION II. THE CALCULUS OF NATURAL DEDUCTION

§ 1. Examples of natural deduction

We wish to set up a formalism that reflects as accurately as possible the actual logical reasoning involved in mathematical proofs.

By means of a number of examples we shall first of all show what form deductions tend to take in practice and shall examine, for this purpose, three 'true formulae' and try to see their truth in the most natural way possible.

1.1. *First example*:

$(X \vee (Y \& Z)) \supset ((X \vee Y) \& (X \vee Z))$ is to be established as a true formula (H.-A., p. 28, formula 19).

The argument runs as follows: Suppose that either X or $Y \& Z$ holds. We distinguish the two cases: 1. X holds, 2. $Y \& Z$ holds. In the first case it follows that $X \vee Y$ holds, and also $X \vee Z$; hence $(X \vee Y) \& (X \vee Z)$ also holds. In the second case $Y \& Z$ holds, which means that both Y and Z hold. From Y follows $X \vee Y$; from Z follows $X \vee Z$. Thus $(X \vee Y) \& (X \vee Z)$ again holds. The latter formula has thus been derived, generally, from $X \vee (Y \& Z)$, i.e., $(X \vee (Y \& Z)) \supset ((X \vee Y) \& (X \vee Z))$ holds.

1.2. *Second example*:

$(\exists x \forall y\, Fxy) \supset (\forall y \exists x\, Fxy)$.

(H.-A., formula 36, p. 60). The argument runs as follows: Suppose there is an x such that for all y Fxy holds. Let a be such an x. Then for all y:

Fay holds. Now let b be an arbitrary object. Then Fab holds. Thus there is an x, viz., a, such that Fxb holds. Since b was arbitrary, our result therefore holds for all objects, i.e., for all y there is an x such that Fxy holds. This yields our assertion.

1.3. *Third example*:

$(\neg \exists x\, Fx) \supset (\forall y \neg Fy)$ is to be established as intuitionistically true. We reason as follows: Assume there is no x for which Fx holds. From this we wish to infer: For all y, $\neg Fy$ holds. Now suppose a is some object for which Fa holds. It then follows that there is an x for which Fx holds, viz., a is such an object. This contradicts our hypothesis that $\neg \exists x\, Fx$. We have therefore a contradiction, i.e., Fa cannot hold. But since a was completely arbitrary, it follows that for all y, $\neg Fy$ holds. Q.E.D.

We intend now to integrate proofs of the kind carried out in these three examples into an exactly defined calculus (in § 4, we shall show how these examples are presented in that calculus).

§ 2. Construction of the Calculus NJ

2.1. We intend now to present a calculus for 'natural' intuitionist derivations of true formulae. The restriction to intuitionist reasoning is only provisional; we shall explain below (cf. § 5) our reasons for doing so and shall show in what way the calculus has to be extended for classical reasoning (by including the law of the excluded middle).

Externally, the essential difference between 'NJ-derivations' and derivations in the systems of Russell, Hilbert, and Heyting is the following: In the latter systems true formulae are derived from a sequence of 'basic logical formulae' by means of a few forms of inference. Natural deduction, however, does not, in general, start from basic logical propositions, but rather from *assumptions* (cf. examples in § 1) to which logical deductions are applied. By means of a later inference the result is then again made independent of the assumption.

Calculi of the former kind will be referred to as *logistic* calculi.

2.2. After this preliminary remark we define the concept of an *NJ-derivation* as follows:

(Examples in § 4.)

An NJ-derivation consists of formulae arranged in tree form (I3.3).

(By demanding that the formulae are arranged in tree form we are deviating somewhat from the analogy with actual reasoning. This is so, since in actual reasoning we necessarily have (1) a linear sequence of propositions due to the linear ordering of our utterances, and (2) we are accustomed to applying repeatedly a result once it has been obtained, whereas the tree form permits only of a single use of a derived formula. These two deviations permit us to define the concept of a derivation in a more convenient form and are not essential.)

The initial formulae of the derivation are *assumption formulae*. Each of these is adjoined to precisely one D-inference figure (and in fact occurs 'above' (I.3.5) the lower formula of that figure, as will be explained more fully below).

All formulae occurring below an assumption formula, but still above the lower formula of the D-inference figure to which that assumption formula was adjoined, the assumption formula itself included, are said to *depend* on that assumption formula. (Thus the inference makes all succeeding propositions independent of the assumption which is correlated with it.)

According to what we have said the endformula of the derivation depends on no assumption formula.

2.21. We shall now state the permissible *inference figures*.

The inference figure schemata below are to be understood in the following way:

We obtain an NJ-inference figure from one of the schemata by replacing $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ by arbitrary formulae; and $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ ($\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$) by an arbitrary formula containing $\forall(\exists)$ for its terminal symbol, where $\mathfrak{x}$ designates the bound object variable belonging to that terminal symbol; and $\mathfrak{F}a$ by the formula obtained from $\mathfrak{F}\mathfrak{x}$ by replacing the bound variable $\mathfrak{x}$, wherever it occurs, by the free object variable a.

(For a we may, for instance, take a variable already occurring in $\mathfrak{F}\mathfrak{x}$. For the inference figures $\forall$–I and $\exists$–E, this possibility will, however, be excluded by the restrictions on variables which follow below, but it remains for $\forall$–E and $\exists$–I. Nor need $\mathfrak{x}$ occur at all in $\mathfrak{F}\mathfrak{x}$, in which case $\mathfrak{F}a$ is, of course, identical with $\mathfrak{F}\mathfrak{x}$. – $\mathfrak{F}a$ is obviously always a subformula of $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ ($\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$), according to the definition of a subformula in I.2.2.)

Symbols written in square brackets have the following meaning: An arbitrary number (possibly zero) of formulae of this form, all formally identical, may be adjoined to the inference figure as assumption formulae. They must then be initial formulae of the derivation and occur, moreover, in those paths of the proof to which the particular upper formula of the inference figure belongs. (I.e., that upper formula above which the square bracket occurs in the scheme. This formula may itself be an assumption formula.)

The adjunction of the respective assumption formulae to a *D*-inference figure in a derivation must in some way be made explicit such as by appropriately numbering these assumption formulae (cf. the examples in § 4).

The designations of the various inference figure schemata: &–*I*, &–*E*, etc., stand for the following: An inference figure formed according to a particular schema is an 'introduction' (*I*) or an 'elimination' (*E*) of the conjunction (&), the disjunction (∨), the universal quantifier (∀), the existential quantifier (∃), the implication (⊃), or of the negation (¬). More about this in § 5.

The inference figure schemata:

&–*I*	&–*E*	∨–*I*	∨–*E*
$\dfrac{\mathfrak{A} \quad \mathfrak{B}}{\mathfrak{A} \,\&\, \mathfrak{B}}$	$\dfrac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{A}} \quad \dfrac{\mathfrak{A} \,\&\, \mathfrak{B}}{\mathfrak{B}}$	$\dfrac{\mathfrak{A}}{\mathfrak{A} \lor \mathfrak{B}} \quad \dfrac{\mathfrak{B}}{\mathfrak{A} \lor \mathfrak{B}}$	$\dfrac{\mathfrak{A} \lor \mathfrak{B} \quad \genfrac{}{}{0pt}{}{[\mathfrak{A}]}{\mathfrak{C}} \quad \genfrac{}{}{0pt}{}{[\mathfrak{B}]}{\mathfrak{C}}}{\mathfrak{C}}$

∀–*I*	∀–*E*	∃–*I*	∃–*E*
$\dfrac{\mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$	$\dfrac{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}{\mathfrak{F}\mathfrak{a}}$	$\dfrac{\mathfrak{F}\mathfrak{a}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}$	$\dfrac{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \quad \genfrac{}{}{0pt}{}{[\mathfrak{F}\mathfrak{a}]}{\mathfrak{C}}}{\mathfrak{C}}$

⊃–*I*	⊃–*E*	¬–*I*	¬–*E*
$\dfrac{\genfrac{}{}{0pt}{}{[\mathfrak{A}]}{\mathfrak{B}}}{\mathfrak{A} \supset \mathfrak{B}}$	$\dfrac{\mathfrak{A} \quad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}}$	$\dfrac{\genfrac{}{}{0pt}{}{[\mathfrak{A}]}{\curlywedge}}{\neg \mathfrak{A}}$	$\dfrac{\mathfrak{A} \quad \neg \mathfrak{A}}{\curlywedge} \quad \dfrac{\curlywedge}{\mathfrak{D}}$.

The free object variable of a ∀–*I* or ∃–*E*, designated by $\mathfrak{a}$ in the respective schema, is called the *eigenvariable*. (This, of course, presupposes that there is such a variable, i.e., that the bound object variable designated by $\mathfrak{x}$ occurs in the formula designated by $\mathfrak{F}\mathfrak{x}$.)

Restrictions on variables:

An *NJ*-derivation is subject to the following restriction (for the significance of this restriction cf. § 3):

The eigenvariable of an ∀–*I* must not occur in the formula designated in the schema by $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$; nor in any assumption formula upon which that formula depends.

The eigenvariable of an $\exists$–E must not occur in the formula designated in the schema by $\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$; nor in an upper formula designated by $\mathfrak{C}$; nor in any assumption formula upon which that formula depends, with the exception of the assumption formulae designated by $\mathfrak{F}\mathfrak{a}$ in the schema of the $\exists$–E.

This concludes the definition of the '*NJ*-derivation'.

§ 3. Informal sense of *NJ*-inference figures

We shall explain the informal sense of a number of inference figure schemata and thus try to show how the calculus in fact reflects 'actual reasoning'.

$\supset$–I: Expressed in words, this schema corresponds to the following inference: If $\mathfrak{B}$ has been proved by means of assumption $\mathfrak{A}$, we have (this time without the assumption): from $\mathfrak{A}$ follows $\mathfrak{B}$. (Further assumptions may, of course, have been made and the result still continues to depend on them.)

$\vee$–E ('Distinction of cases'): If $\mathfrak{A} \vee \mathfrak{B}$ has been proved, we can distinguish two cases: What we first assume is that $\mathfrak{A}$ holds and derive, let us say, $\mathfrak{C}$ from it. If it is then possible to derive $\mathfrak{C}$ also by assuming that $\mathfrak{B}$ holds, then $\mathfrak{C}$ holds generally, i.e., it is now independent of both assumptions (cf.1.1).

$\forall$–I: If $\mathfrak{F}\mathfrak{a}$ has been proved for an 'arbitrary $\mathfrak{a}$', then $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ holds. The presupposition that $\mathfrak{a}$ is 'completely arbitrary' can be expressed more precisely as: $\mathfrak{F}\mathfrak{a}$ must not depend on any assumption in which the object variable $\mathfrak{a}$ occurs. And this, together with the obvious requirement that *every occurrence* of $\mathfrak{a}$ in $\mathfrak{F}\mathfrak{a}$ must be replaced by an $\mathfrak{x}$ in $\mathfrak{F}\mathfrak{x}$, constitutes precisely that part of the 'restrictions on variables' which applies to the schema of the $\forall$–I.

$\exists$–E: We have $\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$. We say: Suppose $\mathfrak{a}$ is an object for which $\mathfrak{F}$ holds, i.e., we assume that $\mathfrak{F}\mathfrak{a}$ holds. (It is, of course, obvious that for $\mathfrak{a}$ we must take an object variable which does not yet occur in $\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$.) If, on this assumption, we then prove a proposition $\mathfrak{C}$ which no longer contains $\mathfrak{a}$ and does not depend on any other assumption containing $\mathfrak{a}$, we have proved $\mathfrak{C}$ independently of the assumption $\mathfrak{F}\mathfrak{a}$. We have here stated the part of the 'restrictions on variables' that concerns the $\exists$–E. (A certain analogy exists between the $\exists$–E and the $\vee$–E since the existential quantifier is indeed the generalization of $\vee$, and the universal quantifier the generalization of &.)

$\neg$–E: $\mathfrak{A}$ and $\neg\, \mathfrak{A}$ signifies a contradiction and as such cannot hold true (law of contradiction). This is formally expressed by the inference figure $\neg$–E, where $\curlywedge$ designates 'the contradiction', 'the false'.

$\neg$–I: (*Reductio ad absurdum.*) If we can derive any false proposition ($\curlywedge$) on an assumption $\mathfrak{A}$, then $\mathfrak{A}$ is not true, i.e., $\neg\,\mathfrak{A}$ holds.

The schema $\dfrac{\curlywedge}{\mathfrak{D}}$ expresses the fact that if a false proposition holds, any arbitrary proposition also holds.

The interpretation of the remaining inference figure schemata should be straightforward.

§ 4. The three examples of § 1 written as NJ-derivations

First example (1.1):

$$\dfrac{\dfrac{\overset{2}{X\vee(Y\,\&\,Z)} \quad \dfrac{\dfrac{\overset{1}{X}}{X\vee Y}\,\vee\text{–}I \quad \dfrac{\overset{1}{X}}{X\vee Z}\,\vee\text{–}I}{(X\vee Y)\,\&\,(X\vee Z)}\,\&\text{–}I \quad \dfrac{\dfrac{\dfrac{\overset{1}{Y\,\&\,Z}}{Y}\,\&\text{–}E}{X\vee Y}\,\vee\text{–}I \quad \dfrac{\dfrac{\overset{1}{Y\,\&\,Z}}{Z}\,\&\text{–}E}{X\vee Z}\,\vee\text{–}I}{(X\vee Y)\,\&\,(X\vee Z)}\,\&\text{–}I}{(X\vee Y)\,\&\,(X\vee Z)}\,\vee\text{–}E_1}{(X\vee(Y\,\&\,Z))\supset((X\vee Y)\,\&\,(X\vee Z))}\,\supset\text{–}I_2\,.$$

In this example the tree form must appear somewhat artificial since it does not bring out the fact that it is *after* the enunciation of $X\vee(Y\,\&\,Z)$ that we distinguish the cases X, $Y\,\&\,Z$.

Second example (1.2):

$$\dfrac{\dfrac{\overset{2}{\exists x\,\forall y\,Fxy} \quad \dfrac{\dfrac{\dfrac{\overset{1}{\forall y Fay}}{Fab}\,\forall\text{–}E}{\exists x\,Fxb}\,\exists\text{–}I}{\forall y\,\exists x\,Fxy}\,\forall\text{–}I}{\forall y\,\exists x\,Fxy}\,\exists\text{–}E_1}{(\exists x\,\forall y\,Fxy)\supset(\forall y\,\exists x\,Fxy)}\,\supset\text{–}I_2\,.$$

If we were using a *linear* arrangement, then the assumption of the $\exists$–E would here also follow naturally behind the upper formula on the left, as was the case in our treatment of that example in § 1.

Third example (1.3):

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overset{2}{Fa}}{\exists x\, Fx}\,\exists\text{-}I \qquad \overset{1}{\neg \exists x\, Fx}}{\curlywedge}\,\neg\text{-}E}{\neg\, Fa}\,\neg\text{-}I_2}{\forall y \neg\, Fy}\,\forall\text{-}I}{(\neg \exists x\, Fx) \supset (\forall y \neg\, Fy)}\,\supset\text{-}I_1 .$$

§ 5. Some remarks concerning the calculus *NJ*. The calculus *NK*

5.1. The calculus *NJ* lacks a certain formal elegance. This has to be put against the following advantages:

5.11. A close affinity to actual reasoning, which had been our fundamental aim in setting up the calculus. The calculus lends itself in particular to the formalization of mathematical proofs.

5.12. In most cases the derivations for true formulae are *shorter* in our calculus than their counterparts in the logistic calculi. This is so primarily because in logistic derivations one and the same formula usually occurs a number of times (as part of other formulae), whereas this happens only very rarely in the case of *NJ*-derivations.

5.13. The designations given to the various inference figures (2.21) make it plain that our calculus is remarkably *systematic*. To every logical symbol &, $\vee$, $\forall$, $\exists$, $\supset$, $\neg$, belongs precisely one inference figure which 'introduces' the symbol – as the terminal symbol of a formula – and one which 'eliminates' it. The fact that the inference figures &–*E* and $\vee$–*I* each have two forms constitutes a trivial, purely external deviation and is of no interest. The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'. An example may clarify what is meant: We were able to introduce the formula $\mathfrak{A} \supset \mathfrak{B}$ when there existed a derivation of $\mathfrak{B}$ from the assumption formula $\mathfrak{A}$. If we then wished to use that formula by eliminating the $\supset$-symbol (we could, of course, also use it to form longer formulae, e.g., $(\mathfrak{A} \supset \mathfrak{B}) \vee \mathfrak{C}$, $\vee$–*I*), we could do this precisely by inferring $\mathfrak{B}$ directly, once $\mathfrak{A}$ has been proved, for what $\mathfrak{A} \supset \mathfrak{B}$ attests is just the existence of a derivation of $\mathfrak{B}$ from $\mathfrak{A}$. Note that in saying this we need not go into the 'informal sense' of the $\supset$-symbol.

By making these ideas more precise it should be possible to display the *E*-inferences as unique functions of their corresponding *I*-inferences, on the basis of certain requirements.

5.2. It is possible to eliminate the *negation* from our calculus by regarding $\neg\,\mathfrak{A}$ as an abbreviation for $\mathfrak{A} \supset \curlywedge$. This is permissible, since by replacing every $\neg\,\mathfrak{A}$ by $\mathfrak{A} \supset \curlywedge$, and thus removing all $\neg$-symbols from an *NJ*-derivation, we obtain another *NJ*-derivation (the inference figures $\neg$–*I* and $\neg$–*E* then become special cases of the $\supset$–*I* and the $\supset$–*E*) and vice versa: If, in an *NJ*-derivation, we replace every occurrence of $\mathfrak{A} \supset \curlywedge$ by $\neg\,\mathfrak{A}$, another *NJ*-derivation results.

The inference figure schema $\dfrac{\curlywedge}{\mathfrak{D}}$ occupies a special place among the schemata: It does not belong to a logical symbol, but to the propositional symbol $\curlywedge$.

5.3. The '*law of the excluded middle*' and the *calculus NK*.

From the calculus *NJ* we obtain a complete classical calculus *NK* by including the 'law of the excluded middle' (*tertium non datur*), i.e.: In addition to the assumption formulae we now also allow 'basic formulae' of the form $\mathfrak{A} \vee \neg\,\mathfrak{A}$, where $\mathfrak{A}$ stands for any arbitrary formula.

We have thus granted to the law of the excluded middle, in a purely external way, a special position, and we have done this because we considered that formulation the 'most natural'. It would be perfectly feasible to introduce a new inference figure schema, say $\dfrac{\neg\,\neg\,\mathfrak{A}}{\mathfrak{A}}$ (a schema analogous to the one formed by Hilbert and Heyting), in place of the basic formula schema $\mathfrak{A} \vee \neg\,\mathfrak{A}$. However, such a schema still falls outside the framework of the *NJ*-inference figures, because it represents a new elimination of the negation whose admissibility does not follow at all from our method of introducing the $\neg$-symbol by the $\neg$–*I*.

SECTION III. THE DEDUCTIVE CALCULI *LJ*, *LK* AND THE HAUPTSATZ

§ 1. The calculi *LJ* and *LK* (logistic intuitionist and classical calculi)

1.1. Preliminary remarks concerning the construction of the calculi *LJ* and *LK*.

What we want to do is to formulate a deductive calculus (for predicate logic) which is 'logistic' on the one hand, i.e., in which the derivations do not, as in the calculus *NJ*, contain assumption formulae, but which, on the other hand, takes over from the calculus *NJ* the division of the forms of inference into introductions and eliminations of the various logical symbols.

The most obvious method of converting an *NJ*-derivation into a logistic one is this: We replace a *D*-formula $\mathfrak{B}$, which depends on the assumption formulae $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu$, by the new formula $(\mathfrak{A}_1 \,\&\, \ldots \,\&\, \mathfrak{A}_\mu) \supset \mathfrak{B}$. This we do with all *D*-formulae.

We thus obtain formulae which are already true *in themselves*, i.e., whose truth is no longer *conditional* on the truth of certain assumption formulae. This procedure, however, introduces new logical symbols & and $\supset$, necessitating additional inference figures for & and $\supset$, and thus upsets the systematic character of our method of introducing and eliminating symbols. For this reason we have introduced the concept of a *sequent* (I.2.3). Instead of a formula $(\mathfrak{A}_1 \,\&\, \ldots \,\&\, \mathfrak{A}_\mu) \supset \mathfrak{B}$, e.g., we therefore write the sequent

$$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \rightarrow \mathfrak{B}.$$

The informal meaning of this sequent is no different from that of the above formula; the expressions differ merely in their formal structure (cf. I. 2.4).

Even now new inference figures are required that cannot be integrated into our system of introductions and eliminations; but we have the advantage of being able to reserve them special places within our system, since they no longer refer to logical symbols, but merely to the structure of the sequents. We therefore call these 'structural inference figures', and the others 'operational inference figures'.

In the *classical* calculus *NK* the law of the excluded middle occupied a special place among the forms of inference (II.5.3), because it could not be integrated into our system of introductions and eliminations. In the classical logistic calculus *LK* about to be presented, this characteristic is removed. This is made possible by the admission of sequents with *several* formulae in the succedent, whereas the transition from the calculus *NJ* just described led only to sequents with *one* formula in the succedent. (For the informal meaning of sequents in general cf. I.2.4.) The symmetry thus obtained is more suited to classical logic. On the other hand, the restriction to at most one formula in the succedent will be retained for the intuitionist calculus *LJ*. (Cf. below. – An empty succedent means the same as if $\curlywedge$ stood in the succedent.)

We have thus outlined a number of points that underlie the construction of the calculi that follow. Their form is largely determined, however, by considerations connected with the '*Hauptsatz*' (§ 2) whose proof follows later. That form cannot therefore be justified more fully at this stage.

1.2. We now define the concepts of an 'LK-derivation' and an 'LJ-derivation' as follows:

An LJ- or LK-derivation consists of sequents arranged in tree form (I.3.3).

The *initial sequents* of the derivation are basic sequents of the form $\mathfrak{D} \to \mathfrak{D}$, where $\mathfrak{D}$ may be an arbitrary formula.

Each *inference figure* of the derivation results from one of the schemata below by a substitution of the following kind (cf. II.2.21):

Replace $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{D}$, $\mathfrak{E}$ by an arbitrary formula; for $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ ($\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$) put an arbitrary formula having $\forall$($\exists$) for its terminal symbol, ~~where~~ $\mathfrak{x}$ designates the associated bound object variable; for $\mathfrak{F}\mathfrak{a}$ put that formula which is obtained from $\mathfrak{F}\mathfrak{x}$ by replacing every occurrence of the bound object variable $\mathfrak{x}$ by the free object variable $\mathfrak{a}$.

For Γ, Δ, Θ, Λ put arbitrary (possibly empty) sequences of formulae separated by commas.

The following restriction is furthermore placed on *LJ-inference figures* (this is the *only* respect in which the concepts of an LJ- and an LK-derivation differ):

'In the succedent of each D-sequent no more than one S-formula may occur'.

The designations of the various schemata for operational inference figures &–IS, &–IA, etc., are intended to mean: An inference figure formed according to the schema is an introduction (I) in the succedent (S) or antecedent (A) of the conjunction (&), the disjunction ($\vee$), the universal quantifier ($\forall$), the existential quantifier ($\exists$), the negation ($\neg$), or the implication ($\supset$).

The inference figure schemata

1.21. Schemata for structural inference figures:

Thinning:

in the antecedent $\qquad$ in the succedent

$$\frac{\Gamma \to \Theta}{\mathfrak{D}, \Gamma \to \Theta}, \qquad \frac{\Gamma \to \Theta}{\Gamma \to \Theta, \mathfrak{D}};$$

Contraction:

in the antecedent in the succedent

$$\frac{\mathfrak{D}, \mathfrak{D}, \Gamma \to \Theta}{\mathfrak{D}, \Gamma \to \Theta}, \qquad \frac{\Gamma \to \Theta, \mathfrak{D}, \mathfrak{D}}{\Gamma \to \Theta, \mathfrak{D}};$$

Interchange:

in the antecedent in the succedent

$$\frac{\Delta, \mathfrak{D}, \mathfrak{E}, \Gamma \to \Theta}{\Delta, \mathfrak{E}, \mathfrak{D}, \Gamma \to \Theta}, \qquad \frac{\Gamma \to \Theta, \mathfrak{E}, \mathfrak{D}, \Lambda}{\Gamma \to \Theta, \mathfrak{D}, \mathfrak{E}, \Lambda};$$

Cut:

$$\frac{\Gamma \to \Theta, \mathfrak{D} \qquad \mathfrak{D}, \Delta \to \Lambda}{\Gamma, \Delta \to \Theta, \Lambda}.$$

1.22. Schemata for operational inference figures:

$$\&\text{–}IS: \frac{\Gamma \to \Theta, \mathfrak{A} \qquad \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \,\&\, \mathfrak{B}},$$

$$\&\text{–}IA: \frac{\mathfrak{A}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta} \qquad \frac{\mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IA: \frac{\mathfrak{A}, \Gamma \to \Theta \qquad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta},$$

$$\vee\text{–}IS: \frac{\Gamma \to \Theta, \mathfrak{A}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}} \qquad \frac{\Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}},$$

$$\forall\text{–}IS: \frac{\Gamma \to \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \to \Theta, \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}},$$

$$\exists\text{–}IA: \frac{\mathfrak{F}\mathfrak{a}, \Gamma \to \Theta}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}, \Gamma \to \Theta}.$$

Restrictions on variables: The object variable in the last two schemata, which is designated by $\mathfrak{a}$ and is called the *eigenvariable* of the $\forall$–*IS* ($\exists$–*IA*), must not occur in the lower sequent of the inference figure (i.e., not in Γ, Θ, and $\mathfrak{F}\mathfrak{x}$).

$$\forall\text{–}IA: \frac{\mathfrak{F}\mathfrak{a}, \Gamma \to \Theta}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}, \Gamma \to \Theta},$$

$$\neg\text{–}IS: \frac{\mathfrak{A}, \Gamma \to \Theta}{\Gamma \to \Theta, \neg \mathfrak{A}},$$

$$\exists\text{-}IS\colon\quad \frac{\Gamma \to \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \to \Theta, \exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}},$$

$$\neg\text{-}IA\colon\quad \frac{\Gamma \to \Theta, \mathfrak{A}}{\neg \mathfrak{A}, \Gamma \to \Theta},$$

$$\supset\text{-}IS\colon\quad \frac{\mathfrak{A}, \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \supset \mathfrak{B}},$$

$$\supset\text{-}IA\colon\quad \frac{\Gamma \to \Theta, \mathfrak{A} \qquad \mathfrak{B}\ \Delta \to \Lambda}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Delta \to \Theta, \Lambda}.$$

1.3. Example of an *LJ*-derivation (using II.1.3):

$$\dfrac{\dfrac{\dfrac{Fa \to Fa}{Fa \to \exists x\, Fx}\,\exists\text{-}IS \qquad \dfrac{\dfrac{\exists x\, Fx \to \exists x\, Fx}{\neg \exists x\, Fx, \exists x\, Fx \to}\,\neg\text{-}IA}{\exists x\, Fx, \neg \exists x\, Fx \to}\,\text{Interchange}}{\dfrac{\dfrac{Fa, \neg \exists x\, Fx \to}{\neg \exists x\, Fx \to \neg Fa}\,\neg\text{-}IS}{\neg \exists x\, Fx \to \forall y \neg Fy}\,\forall\text{-}IS}\,\text{Cut}}{\to (\neg \exists x\, Fx) \supset (\forall \neg Fy)}\,\supset\text{-}IS.$$

1.4. Example of an *LK*-derivation (derivation of the 'law of the excluded middle'):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\to A, \neg A}\,\neg\text{-}IS}{\to A, A \vee \neg A}\,\vee\text{-}IS}{\to A \vee \neg A, A}\,\text{Interchange}}{\to A \vee \neg A, A \vee \neg A}\,\vee\text{-}IS}{\to A \vee \neg A}\,\text{Contraction}.$$

§ 2. Some remarks concerning the calculi *LJ* and *LK*. The Hauptsatz

(We shall make no further use, in this paper, of remarks 2.1 to 2.3.)

2.1. The schemata are not all mutually independent, i.e., certain schemata could be eliminated with the help of the remaining ones. Yet if they were left out, the '*Hauptsatz*' would no longer be valid.

2.2. In general, we could *simplify* the calculi in various respects if we attached no importance to the *Hauptsatz*. To indicate this briefly: the inference figures &–*IS*, ∨–*IA*, &–*IA*, ∨–*IS*, ∀–*IA*, ∃–*IS*, ¬–*IS*, ¬–*IA*, and ⊃–*IA* in the calculus *LK* could be replaced by basic sequents according to the following schemata:

$$\mathfrak{A}, \mathfrak{B} \to \mathfrak{A} \,\&\, \mathfrak{B} \qquad \mathfrak{A} \vee \mathfrak{B} \to \mathfrak{A}, \mathfrak{B} \qquad \mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{A} \qquad \mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{B}$$

$$\mathfrak{A} \to \mathfrak{A} \vee \mathfrak{B} \qquad \mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B} \qquad \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \to \mathfrak{F}\mathfrak{a} \qquad \mathfrak{F}\mathfrak{a} \to \exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$$

$\to \mathfrak{A}, \neg \mathfrak{A}$ (law of the excluded middle)

$\neg \mathfrak{A}, \mathfrak{A} \to$ (law of contradiction)

$\mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \to \mathfrak{B}.$

These basic sequents and our inference figures may easily be shown to be equivalent.

The same possibility exists for the calculus *LJ*, with the exception of the inference figures ∨–*IA* and ¬–*IS*, since *LJ*-*D*-sequents may not in fact contain two *S*-formulae in the succedent (cf. V. § 5).

2.3. The distinction between *intuitionist* and *classical* logic is, externally, of a quite different type in the calculi *LJ* and *LK* from that in the calculi *NJ* and *NK*. In the case of the latter, the distinction is based on the inclusion or exclusion of the law of the excluded middle, whereas for the calculi *LJ* and *LK* the difference is characterized by the restriction on the succedent. (The fact that both distinctions are equivalent will become evident as a result of the equivalence proofs in section V for all calculi discussed in this paper.)

2.4. If ⊃–*IS* and the ⊃–*IA* are excluded, the calculus *LK* is *dual* in the following sense: If we reverse all sequents of an *LK*-derivation (in which the ⊃-symbol does not occur), i.e., if for $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \to \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$ we put $\mathfrak{B}_\nu, \ldots, \mathfrak{B}_1 \to \mathfrak{A}_\mu, \ldots, \mathfrak{A}_1$, and if we exchange, in inference figures with two upper sequents, the right- and left-hand upper sequents, including their derivations, and also replace every occurrence of & by ∨, ∀ by ∃, ∨ by &, and ∃ by ∀ (in the case of & and ∨ we also have to interchange the respective scopes of the symbols, e.g., for $\mathfrak{B} \vee \mathfrak{A}$ we have to put $\mathfrak{A} \,\&\, \mathfrak{B}$), then another *LK*-derivation results.

This can be seen at once from the schemata. (Special care was taken to arrange them in such a way as to bring out their symmetry.) (Cf. H.-A.'s duality principle, p. 62.)

2.41. In any case, the ⊃-symbol may, in a well-known manner, be eliminated from the calculus *NK*, by regarding $\mathfrak{A} \supset \mathfrak{B}$ as an abbreviation for $(\neg \mathfrak{A}) \vee \mathfrak{B}$. It may easily be shown that the schemata for the ⊃–*IS* and the ⊃–*IA* may then be replaced by the schemata for ∨ and ¬.

The calculus *NJ* has no corresponding property.

2.5. The most important fact for us with regard to the calculi *LJ* and *LK* is the following:

HAUPTSATZ: Every *LJ*- or *LK*-derivation can be transformed into an *LJ*- or *LK*-derivation with the same endsequent and in which the inference figure called a 'cut' does not occur.

The proof follows in § 3.

2.51. In order to give greater clarity to the meaning of the *Hauptsatz*, we shall prove a simple corollary (2.513).

For this purpose we introduce a number of expressions (which will be needed frequently later on) relating to the operational inference figures:

2.511. That *S*-formula which contains the logical symbol in its schema will be called the *principal formula* of an inference figure.

For the &–*IS* and the &–*IA* this is simply the *S*-formula of the form $\mathfrak{A} \,\&\, \mathfrak{B}$; for the $\vee$–*IS* and the $\vee$–*IA* it is $\mathfrak{A} \vee \mathfrak{B}$; for the $\forall$–*IS* and the $\forall$–*IA* it is $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$; for the $\exists$–*IS* and the $\exists$–*IA* it is $\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$; for the $\neg$–*IS* and the $\neg$–*IA* it is $\neg\, \mathfrak{A}$; and for the $\supset$–*IS* and the $\supset$–*IA* it is $\mathfrak{A} \supset \mathfrak{B}$.

The *S*-formulae designated by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{F}\mathfrak{a}$ in the schemata will be called the *side formulae* of the respective inference figures.

They are always subformulae of the principal formula (according to the definition of a subformula in I.2.2).

2.512. We can now easily read off the following facts from the inference figure schemata:

The principal formula occurs always in the lower sequent and the side formulae always in the upper sequents of an operational inference figure.

If a formula occurs as an *S*-formula in an upper sequent of a given inference figure, and if it is here neither a side formula nor the $\mathfrak{D}$ of a cut, then it occurs also as an *S*-formula in the lower sequent.

These two facts entail the following:

If anywhere in an *LJ*- or *LK*-derivation a formula occurs as an *S*-formula, and if we trace the path of the derivation from the formula concerned up to the endsequent, the formula can only vanish from that path if it is the $\mathfrak{D}$ of a cut or the side formula of an operational inference figure. In the latter case, however, there appears, in the next sequent, the *principal formula* of the inference figure of which our side formula is a subformula. To that principal formula we can then, continuing downwards, apply the same consideration, and so on. Thus we obtain the following corollary:

2.513. COROLLARY OF THE HAUPTSATZ (SUBFORMULA PROPERTY): In an *LJ*- or

LK-derivation without cuts, all occurring *D-S*-formulae are *subformulae* of the *S*-formulae that occur in the endsequent.

2.514. Intuitively speaking, these properties of derivations without cuts may be expressed as follows: The *S*-formulae become longer as we descend lower down in the derivation, never shorter. The final result is, as it were, gradually built up from its constituent elements. The proof represented by the derivation is not roundabout in that it contains only concepts which recur in the final result (cf. the synopsis at the beginning of this paper).

Example: The derivation given above (1.3) for $\rightarrow (\neg \exists \mathfrak{x} \mathfrak{F}\mathfrak{x}) \supset (\forall \mathfrak{y} \neg \mathfrak{F}\mathfrak{y})$ may be written without a cut as follows:

$$\dfrac{\dfrac{\dfrac{Fa \rightarrow Fa}{Fa \rightarrow \exists x\, Fx}\ \exists\text{-}IS}{\neg\, \exists x\, Fx, Fa \rightarrow}\ \neg\text{-}IA}{Fa, \neg\, \exists x\, Fx \rightarrow}\ \text{Interchange,}$$

etc., as above.

§ 3. Proof of the Hauptsatz

The *Hauptsatz* runs as follows:

Every *LJ*- or *LK*-derivation can be transformed into another *LJ*- or *LK*-derivation with the same endsequent, in which no cuts occur.

3.1. *Proof of the Hauptsatz for LK-derivations.*

We introduce a new inference figure (in order to facilitate the proof) which constitutes a modified form of the cut, and which we call a *mix*.

The schema of that figure runs as follows:

$$\frac{\Gamma \rightarrow \Theta \qquad \Delta \rightarrow \Lambda}{\Gamma, \Delta^* \rightarrow \Theta^*, \Lambda},$$

In order to obtain an inference figure from this schema, Θ and Δ must be replaced by sequences of formulae, separated by commas, in each of which occurs at least once (as a member of the sequence) a formula of the form $\mathfrak{M}$, called the 'mix formula'; and Θ^* and Δ^* must be replaced by the same sequences of formulae, save that all formulae of the form $\mathfrak{M}$ occurring as members of the sequence are omitted. ($\mathfrak{M}$ may be any arbitrary formula.) Γ and Λ must be replaced, as in the other schemata, by arbitrary (possibly empty) sequences of formulae, separated by commas.

Example of a mix:

$$\frac{A \to B, \neg A \qquad B \vee C, B, B, D, B \to}{A, B \vee C, D \to \neg A}.$$

B is the mix formula.

We notice at once that every cut may be transformed into a mix by means of a number of thinnings and interchanges. (Conversely, every mix may be transformed into a cut by means of a certain number of preceding interchanges and contractions, though we do not use this fact.)

In the following we shall consider only derivations in which no cuts occur, but which may contain mixes instead.

Since derivations in the old sense may be transformed into derivations of the new kind, it suffices, for the proof of the *Hauptsatz*, to show that a derivation of the new type may be transformed into a derivation with no mix.

Furthermore, the following lemma is already sufficient:

LEMMA: A derivation with a mix for its lowest inference figure, and not containing any other mix, may be transformed into a derivation (with the same endsequent) in which no mix occurs.

From this the theorem as a whole easily follows:

In an arbitrary derivation consider a mix above whose lower sequent no further mix occurs. The derivation for this lower sequent is then of the kind mentioned in the lemma, i.e., it may be transformed in such a way that it no longer contains a mix. In doing so, the rest of the derivation remains unchanged. This operation is then repeated until every mix has systematically been eliminated.

It now remains for us to establish the *proof of the lemma*. (This proof extends into 3.2 incl.)

We have to consider a derivation whose lowest inference figure is a mix and which contains no other mix.

The degree of the mix formula will be called the 'degree of the derivation' (defined in I.2.2).

We shall call the *rank* of the derivation the sum of its rank on the left and its rank on the right. These two terms are defined as follows:

The *left rank* is the largest number of consecutive sequents in a path so that the lowest of these sequents is the *left-hand* upper sequent of the mix and each of the sequents contains the mix formula in the *succedent*.

The *right rank* is (correspondingly) the largest number of consecutive sequents in a path so that the lowest of these sequents is the *right-hand* upper sequent of the mix and each of the sequents contains the mix formula in the *antecedent*.

The lowest possible rank is evidently 2.

To prove the lemma we carry out two complete inductions, one on the degree γ, the other on the rank ρ, of the derivation, i.e., we prove the theorem for a derivation of degree γ, assuming it to hold for derivations of a lower degree (in so far as there are such derivations, i.e., as long as γ is not equal to zero), supposing, therefore, that derivations of lower degree can already be transformed into derivations with no mix. Furthermore, we shall begin by considering the case where the rank ρ of the derivation equals 2 (3.11), and after that the case of $\rho > 2$ (3.12), where we assume that the theorem already holds for derivations of the same degree, but of a lower rank.

In the following German capital letters will generally serve as syntactic variables for *formulae*, and Greek capital letters as syntactic variables for (possibly empty) *sequences of formulae*.

In transforming derivations, we shall occasionally meet 'identical inference figures', i.e., inference figures with identical upper and lower sequents. Since we have not admitted such figures in our calculus, they must be eliminated as soon as they occur; we can do this trivially by omitting one of the two sequents.

The mix formula of the mix that occurs at the end of the derivation is designated by $\mathfrak{M}$. It is of degree γ.

3.10. *Redesignating of free object variables* in preparation for the transformation of derivations.

We wish to obtain a derivation that has the following properties:

3.101. For every $\forall$–*IS* ($\exists$–*IA*) it holds that: Its eigenvariable occurs in the derivation only in sequents *above* the lower sequent of the $\forall$–*IS* ($\exists$–*IA*) and does not occur as an eigenvariable in any other $\forall$–*IS* ($\exists$–*IA*).

3.102. This is achieved by redesignating free object variables in the following way:

We take a $\forall$–*IS* ($\exists$–*IA*) above whose lower sequent either no further inference figures of this kind occur, or if they do, they have already been dealt with in a way still to be described.

In all sequents above the lower sequent of this inference figure we replace the eigenvariable by one and the same free object variable which, so far, has not yet occurred in the derivation. This obviously leaves the $\forall$–*IS*

($\exists$–*IA*) itself correct, as is easily seen. (The eigenvariable did not in fact occur in its lower sequent.) Furthermore, rest of the derivation remains correct, as is shown by the lemma to follow shortly.

A systematic application of this method to every single $\forall$–*IS* and $\exists$–*IA*, thus leaves the derivation correct throughout and the conclusion obviously has the desired property (3.101). Furthermore, as was essential, the degree and rank of the derivation, as well as its endsequent, have remained unaltered.

3.103. Now we give the still outstanding proof of the following lemma. (It is enunciated in a somewhat more general form than is immediately necessary, since we shall have to apply it again later on (3.113.33).)

An *LK*-basic sequent or inference figure becomes a basic sequent or inference figure of the same kind, if we replace a free object variable which is *not the eigenvariable* of the inference figure in all its occurrences in the basic sequent or inference figure, by one and the same free object variable, provided again that this is *not the eigenvariable* of the inference figure.

This holds trivially except for the $\forall$–*IS*, $\forall$–*IA*, $\exists$–*IS* and $\exists$–*IA*. Even here, however, there is no cause for concern: the restrictions on variables are not violated, since we may neither substitute nor replace the eigenvariable. (This is the reason why both restrictions on variables are necessary.) Furthermore, the formula resulting from $\mathfrak{F}\mathfrak{a}$ is again obtained by substituting $\mathfrak{a}$ for $\mathfrak{x}$ in the formula resulting from $\mathfrak{F}\mathfrak{x}$.

Having prepared the way (3.10), we now proceed to the actual transformation of the derivation which serves to eliminate the mix occurring in it.

As already mentioned, we distinguish the two cases: $\rho = 2$ (3.11) and $\rho > 2$ (3.12).

3.11. Suppose $\rho = 2$.

We distinguish between several individual cases, of which the cases 3.111, 3.112, 3.113.1, 3.113.2 are especially simple in that they allow the mix to be immediately eliminated. The other cases (3.113.3) are the most important since their consideration brings out the basic idea behind the whole transformation. Here we use the induction hypothesis with respect to γ, i.e., we reduce each one of the cases to transformed derivations of a lower degree.

3.111. Suppose the left-hand upper sequent of the mix at the end of the derivation is a basic sequent. The mix then reads:

$$\frac{\mathfrak{M} \to \mathfrak{M} \qquad \Delta \to \Lambda}{\mathfrak{M}, \Delta^* \to \Lambda},$$

which is transformed into:

$$\frac{\Delta \to \Lambda}{\mathfrak{M}, \Delta^* \to \Lambda} \text{ possibly several interchanges and contractions.}$$

That part of the derivation which is above $\Delta \to \Lambda$ remains the same, and we thus have a derivation without a mix.

3.112. Suppose the right-hand upper sequent of the mix is a basic sequent. The treatment of this case is symmetric to that of the previous one. We have only to regard the two schemata as 'duals' (cf. 2.4).

3.113. Suppose that neither the left- nor the right-hand upper sequent of the mix is a basic sequent. Then both are *lower sequents of inference figures* since $\rho = 2$, and the right and left rank both equal 1, i.e.: In the sequents directly above the *left-hand* upper sequent of the mix, the mix formula $\mathfrak{M}$ does not occur in the *succedent*; in the sequents directly above the *right-hand* upper sequent $\mathfrak{M}$ does not occur in the *antecedent.*

Now the following holds generally: If a formula occurs in the antecedent (succedent) of the lower sequent of an inference figure, it is either a principal formula or the $\mathfrak{D}$ of a thinning, or else it also occurs in the antecedent (succedent) in at least one upper sequent of the inference figure.

This can be seen immediately by looking at the inference figure schemata (1.21, 1.22).

If we now consider the hypotheses in the following three cases, we see at once that they exhaust all the possibilities that exist within case 3.113.

3.113.1. Suppose the left-hand upper sequent of the mix is the lower sequent of a thinning. Then the conclusion of the derivation runs:

$$\frac{\dfrac{\Gamma \to \Theta}{\Gamma \to \Theta, \mathfrak{M}} \qquad \Delta \to \Lambda}{\Gamma, \Delta^* \to \Theta, \Lambda}.$$

This is transformed into:

$$\frac{\Gamma \to \Theta}{\Gamma, \Delta^* \to \Theta, \Lambda} \text{ possibly several thinnings and interchanges.}$$

That part of the derivation which occurs above $\Delta \to \Lambda$ disappears.

3.113.2. Suppose the right-hand upper sequent of the mix is the lower sequent of a thinning. This case is dealt with symmetrically to the previous one.

3.113.3. The mix formula $\mathfrak{M}$ occurs both in the succedent of the left-hand upper sequent and in the antecedent of the right-hand upper sequent solely as the *principal formula* of one of the operational inference figures.

Depending on whether the terminal symbol of $\mathfrak{M}$ is &, $\vee$, $\forall$, $\exists$, $\neg$, $\supset$, we distinguish the cases 3.113.31 to 3.113.36 (a formula without logical symbols cannot be a principal formula).

3.113.31. Suppose the terminal symbol of $\mathfrak{M}$ is &. In that case the end of the derivation runs:

$$\dfrac{\dfrac{\Gamma_1 \to \Theta_1, \mathfrak{A} \qquad \Gamma_1 \to \Theta_1, \mathfrak{B}}{\Gamma_1 \to \Theta_1, \mathfrak{A} \,\&\, \mathfrak{B}}\;\&\text{-}IS \qquad \dfrac{\mathfrak{A}, \Gamma_2 \to \Theta_2}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma_2 \to \Theta_2}\;\&\text{-}IA}{\Gamma_1, \Gamma_2 \to \Theta_1, \Theta_2}\;\text{mix}$$

(the other form of the &–*IA* is treated analogously).

We transform it into:

$$\dfrac{\Gamma_1 \to \Theta_1, \mathfrak{A} \qquad \mathfrak{A}, \Gamma_2 \to \Theta_2}{\dfrac{\Gamma_1, \Gamma_2^* \to \Theta_1^*, \Theta_2}{\overline{\Gamma_1, \Gamma_2 \to \Theta_1, \Theta_2}}}\;\text{mix}$$

possibly several thinnings and interchanges.

We can now apply the induction hypothesis with respect to γ to that part of the derivation whose lowest sequent is $\Gamma_1, \Gamma_2^* \to \Theta_1^*, \Theta_2$, because it has a lower degree than γ. ($\mathfrak{A}$ obviously contains fewer logical symbols than $\mathfrak{A} \,\&\, \mathfrak{B}$.) This means that the whole derivation may be transformed into one with no mix.

3.113.32. Suppose the terminal symbol of $\mathfrak{M}$ is $\vee$. This case is dealt with symmetrically to the previous one.

3.113.33. Suppose the terminal symbol of $\mathfrak{M}$ is $\forall$. Then the end of the derivation runs:

$$\dfrac{\dfrac{\Gamma_1 \to \Theta_1, \mathfrak{F}\mathfrak{a}}{\Gamma_1 \to \Theta_1, \forall \mathfrak{x}\mathfrak{F}\mathfrak{x}}\;\forall\text{-}IS \qquad \dfrac{\mathfrak{F}\mathfrak{b}, \Gamma_2 \to \Theta_2}{\forall \mathfrak{x}\mathfrak{F}\mathfrak{x}, \Gamma_2 \to \Theta_2}\;\forall\text{-}IA}{\Gamma_1, \Gamma_2 \to \Theta_1, \Theta_2}\;\text{mix.}$$

This is transformed into:

$$\dfrac{\Gamma_1 \to \Theta_1, \mathfrak{F}\mathfrak{b} \qquad \mathfrak{F}\mathfrak{b}, \Gamma_2 \to \Theta_2}{\dfrac{\Gamma_1, \Gamma_2^* \to \Theta_1^*, \Theta_2}{\overline{\Gamma_1, \Gamma_2 \to \Theta_1, \Theta_2}}}\;\text{mix}$$

possibly several thinnings and interchanges.

Above the left-hand upper sequent of the mix, $\Gamma_1 \to \Theta_1, \mathfrak{F}\mathfrak{b}$, we write the same part of the derivation which previously occurred above $\Gamma_1 \to \Theta_1, \mathfrak{F}\mathfrak{a}$, yet having replaced every occurrence of the free object variable $\mathfrak{a}$ by $\mathfrak{b}$. It now follows from lemma 3.103, together with 3.101, that in performing this operation the part of the derivation *above* $\Gamma_1 \rightarrow \Theta_1, \mathfrak{F}\mathfrak{b}$ has again become a correct part of the derivation. (By virtue of 3.101 neither $\mathfrak{a}$ nor $\mathfrak{b}$ can be the eigenvariable of an inference figure occurring in that part of the derivation.) The same consideration may be applied to that part of the derivation which *includes* the sequent $\Gamma_1 \rightarrow \Theta_1, \mathfrak{F}\mathfrak{b}$, since it too results from $\Gamma_1 \rightarrow \Theta_1, \mathfrak{F}\mathfrak{a}$ by the substitution of $\mathfrak{b}$ for $\mathfrak{a}$. It is now in fact clear that by virtue of the restriction on variables for $\forall$–*IS*, $\mathfrak{a}$ could have occurred neither in Γ_1 and Θ_1, nor in $\mathfrak{F}\mathfrak{x}$. Furthermore, $\mathfrak{F}\mathfrak{a}$ results from $\mathfrak{F}\mathfrak{x}$ by the substitution $\mathfrak{a}$ for $\mathfrak{x}$, and $\mathfrak{F}\mathfrak{b}$ from $\mathfrak{F}\mathfrak{x}$ by the substitution $\mathfrak{b}$ for $\mathfrak{x}$. This is why $\mathfrak{F}\mathfrak{b}$ results from $\mathfrak{F}\mathfrak{a}$ by the substitution $\mathfrak{b}$ for $\mathfrak{a}$.

The mix formula $\mathfrak{F}\mathfrak{b}$ in the new derivation has a lower degree than γ. Therefore, according to the induction hypothesis, the mix may be eliminated.

3.113.34. Suppose the terminal symbol of $\mathfrak{M}$ is $\exists$. This case is dealt with symmetrically to the previous one.

3.113.35. Suppose the terminal symbol of $\mathfrak{M}$ is $\neg$. Then the end of the derivation runs:

$$\dfrac{\dfrac{\mathfrak{A}, \Gamma_1 \rightarrow \Theta_1}{\Gamma_1 \rightarrow \Theta_1, \neg \mathfrak{A}}\ \neg\text{–}IS \qquad \dfrac{\Gamma_2 \rightarrow \Theta_2, \mathfrak{A}}{\neg \mathfrak{A}, \Gamma_2 \rightarrow \Theta_2}\ \neg\text{–}IA}{\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \Theta_2}\ \text{mix.}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma_2 \rightarrow \Theta_2, \mathfrak{A} \qquad \mathfrak{A}, \Gamma_1 \rightarrow \Theta_1}{\Gamma_2, \Gamma_1^* \rightarrow \Theta_2^*, \Theta_1}\ \text{mix}}{\Gamma_1, \Gamma_2 \rightarrow \Theta_1, \Theta_2}\ \text{possibly several interchanges and thinnings.}$$

The new mix may be eliminated by virtue of the induction hypothesis.

3.113.36. Suppose the terminal symbol of $\mathfrak{M}$ is $\supset$. Then the end of the derivation runs:

$$\dfrac{\dfrac{\mathfrak{A}, \Gamma_1 \rightarrow \Theta_1, \mathfrak{B}}{\Gamma_1 \rightarrow \Theta_1, \mathfrak{A} \supset \mathfrak{B}}\ \supset\text{–}IS \qquad \dfrac{\Gamma \rightarrow \Theta, \mathfrak{A} \qquad \mathfrak{B}, \Delta \rightarrow \Lambda}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Delta \rightarrow \Theta, \Lambda}\ \supset\text{–}IA}{\Gamma_1, \Gamma, \Delta \rightarrow \Theta_1, \Theta, \Lambda}\ \text{mix.}$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \rightarrow \Theta, \mathfrak{A} \qquad \dfrac{\mathfrak{A}, \Gamma_1 \rightarrow \Theta_1, \mathfrak{B} \qquad \mathfrak{B}, \Delta \rightarrow \Lambda}{\mathfrak{A}, \Gamma_1, \Delta^* \rightarrow \Theta_1^*, \Lambda}\ \text{mix}}{\Gamma, \Gamma_1^*, \Delta^{**} \rightarrow \Theta^*, \Theta_1^*, \Lambda}\ \text{mix}}{\Gamma_1, \Gamma, \Delta \rightarrow \Theta_1, \Theta, \Lambda}\ \text{possibly several interchanges and thinnings.}$$

(The asterisks are, of course, intended as follows: Δ^* and Θ^* result from Δ and Θ_1 by the omission of all S-formulae of the form $\mathfrak{B}$; Γ^*, Δ^{**} and Θ^* result from Γ_1, Δ^* and Θ by the omission of all S-formulae of the form $\mathfrak{A}$.)

Now we have two mixes, but both mix formulae are of a lower degree than γ. We first apply the induction hypothesis to the upper mix (i.e., to that part of the derivation whose lowest figure it is). Thus the upper mix may be eliminated. We can then also eliminate the lower mix.

3.12. Suppose $\rho > 2$.

To begin with, we distinguish two main cases: First case: The right rank is greater than 1 (3.121). Second case: The right rank is equal to 1 and the left rank is therefore greater than 1 (3.122).

The second case may essentially be dealt with symmetrically to the first.

3.121. *Suppose the right rank is greater than* 1.

I.e.: The right-hand upper sequent of the mix is the lower sequent of an inference figure, let us call it $\mathfrak{I}\mathfrak{f}$, and $\mathfrak{M}$ occurs in the antecedent of at least one upper sequent of $\mathfrak{I}\mathfrak{f}$.

The basic idea behind the transformation procedure is the following:

In the case of $\rho = 2$, we generally reduced the derivation to one of a lower *degree*. Now, however, we shall proceed to reduce the derivation to one of the same degree, but of a lower *rank*, in order to be able to use the induction hypothesis with respect to ρ.

The only exception is the first case, 3.121.1, where the mix may be eliminated immediately.

In the remaining cases the reduction to derivations of a lower rank is achieved in the following way: The mix is, as it were, moved up one level within the derivation, beyond the inference figure $\mathfrak{I}\mathfrak{f}$. (Case 3.121.231, for example, illustrates this point particularly well.) To speak more precisely, the left-hand upper sequent of the mix (which from now on will be designated by $\Pi \rightarrow \Sigma$), at present occurring *beside the lower sequent* of $\mathfrak{I}\mathfrak{f}$, is instead written *next to the upper sequents* of $\mathfrak{I}\mathfrak{f}$. These now become upper sequents of new mixes. The lower sequents of these mixes are now used as upper sequents of a new inference figure that takes the place of $\mathfrak{I}\mathfrak{f}$. This new inference figure takes us back either directly, or after having added further inference figures, to the original endsequent. Each new mix obviously has a rank smaller than ρ, since the left rank remains unchanged and the right rank is diminished by at least 1.

In the strict application of this basic idea special circumstances still arise which make it necessary to distinguish the corresponding cases and to deal with them separately.

3.121.1. Suppose $\mathfrak{M}$ occurs in the antecedent of the left-hand upper sequent of the mix. The end of the derivation runs:

$$\frac{\Pi \to \Sigma \qquad \Delta \to \Lambda}{\Pi, \Delta^* \to \Sigma^*, \Lambda}\text{, thus } \mathfrak{M} \text{ occurs in } \Pi.$$

This is transformed into:

$$\frac{\Delta \to \Lambda}{\overline{\Pi, \Delta^* \to \Sigma^*, \Lambda}} \text{ possibly several thinnings, contractions and interchanges.}$$

3.121.2. Suppose $\mathfrak{M}$ does not occur in the antecedent of the left-hand upper sequent of the mix. (This hypothesis will be used for the first time in 3.121.222.)

3.121.21. Suppose $\mathfrak{J}\mathfrak{f}$ is a thinning, contraction, or interchange in the *antecedent*. Then the end of the derivation runs:

$$\frac{\Pi \to \Sigma \qquad \dfrac{\Psi \to \Theta}{\Xi \to \Theta}\,\mathfrak{J}\mathfrak{f}}{\Pi, \Xi^* \to \Sigma^*, \Theta}\text{ mix.}$$

This is transformed into:

$$\cfrac{\cfrac{\cfrac{\cfrac{\Pi \to \Sigma \qquad \Psi \to \Theta}{\Pi, \Psi^* \to \Sigma^*, \Theta}\text{ mix}}{\overline{\Psi^*, \Pi \to \Sigma^*, \Theta}}\text{ possibly several interchanges}}{\Xi^*, \Pi \to \Sigma^*, \Theta}\ \S}{\overline{\Pi, \Xi^* \to \Sigma^*, \Theta}}\text{ possibly several interchanges.}$$

The inference figure marked § is of the same kind as $\mathfrak{J}\mathfrak{f}$, in so far as the S-formulae designated in the schema of $\mathfrak{J}\mathfrak{f}$ (in 1.21) by $\mathfrak{D}$ and $\mathfrak{E}$, were not equal to $\mathfrak{M}$. If $\mathfrak{D}$ or $\mathfrak{E}$ is equal to $\mathfrak{M}$, we have an identical inference figure (Ψ^* equals Ξ^*).

The derivation for the lower sequent of the new mix has the same left rank as the old derivation, whereas its right rank is lower by 1. Thus the mix may be completely eliminated by virtue of the induction hypothesis.

3.121.22. Suppose $\mathfrak{J}\mathfrak{f}$ is an inference figure with *one* upper sequent, but not containing a thinning, contraction, or interchange in the antecedent. Then the end of the derivation runs:

$$\frac{\Pi \to \Sigma \qquad \dfrac{\Psi, \Gamma \to \Omega_1}{\Xi, \Gamma \to \Omega_2}\,\mathfrak{J}\mathfrak{f}}{\Pi, \Xi^*, \Gamma^* \to \Sigma^*, \Omega_2}\text{ mix.}$$

Here we have collected in Γ the same S-formulae that are designated by Γ in the schema of the inference figure (1.21, 1.22). Hence Ψ may be empty or consist of a side formula of the inference figure, and Ξ may be empty or consist of the principal formula of the inference figure.

First of all, the end of the derivation is transformed into:

$$\cfrac{\cfrac{\cfrac{\Pi \to \Sigma \qquad \Psi, \Gamma \to \Omega_1}{\Pi, \Psi^*, \Gamma^* \to \Sigma^*, \Omega_1}\text{ mix}}{\Psi, \Gamma^*, \Pi \to \Sigma^*, \Omega_1}\text{ possibly several interchanges and thinnings.}}{\Xi, \Gamma^*, \Pi \to \Sigma^*, \Omega_2}$$

The lowest inference is obviously an inference figure of the same kind as $\mathfrak{J}\mathfrak{f}$ (taking Γ^*, Π as the Γ of the inference figure and including Σ^* in the Θ of the inference figure).

We must only be careful not to violate the restrictions on variables (if $\mathfrak{J}\mathfrak{f}$ is a $\forall$–*IS* or $\exists$–*IA*): Any such violation is precluded by 3.101, which entails that an eigenvariable that may have occurred in $\mathfrak{J}\mathfrak{f}$ cannot have occurred in Π and Σ.

The mix may be eliminated from the new derivation by virtue of the induction hypothesis.

We therefore obtain a derivation with no mix and which is terminated by the following inference figure:

$$\frac{\Psi, \Gamma^*, \Pi \to \Sigma^*, \Omega_1}{\Xi, \Gamma^*, \Pi \to \Sigma^*, \Omega_2}.$$

In general, the endsequent is not yet of the form aimed at. Hence we proceed as follows:

3.121.221. Suppose Ξ does not contain $\mathfrak{M}$.

In that case we perform a number of interchanges, if necessary, and obtain the endsequent of the original derivation.

3.121.222. Suppose Ξ contains $\mathfrak{M}$. Then Ξ is the principal formula of $\mathfrak{J}\mathfrak{f}$ and is identical with $\mathfrak{M}$. We then adjoin:

$$\cfrac{\cfrac{\Pi \to \Sigma \qquad \mathfrak{M}, \Gamma^*, \Pi \to \Sigma^*, \Omega_2}{\Pi, \Gamma^*, \Pi^* \to \Sigma^*, \Sigma^*, \Omega_2}\text{ mix}}{\Pi, \Gamma^* \to \Sigma^*, \Omega_2}\text{ possibly several contractions and interchanges.}$$

Once again, this is the endsequent of the original derivation. (Above $\Pi \to \Sigma$ we once more write the derivation associated with it.) Thus we have another mix in the derivation. The left rank of our derivation is the same as that of the original derivation. The right rank is now equal to 1. This is so because directly above the right-hand upper sequent occurs the sequent

$$\Psi, \Gamma^*, \Pi \to \Sigma^*, \Omega_1 .$$

$\mathfrak{M}$ no longer occurs in its antecedent, for Γ^* does not contain $\mathfrak{M}$, nor does Π, because of 3.121.2; and Ψ contains at most one *side formula* of $\mathfrak{J}\mathfrak{f}$, which cannot be equal to $\mathfrak{M}$, since the *principal formula* of $\mathfrak{J}\mathfrak{f}$ is equal to $\mathfrak{M}$.

Hence this mix, too, may be eliminated by virtue of the induction hypothesis.

3.121.23. Suppose $\mathfrak{J}\mathfrak{f}$ is an inference figure with *two* upper sequents, i.e., a &–*IS*, ∨–*IA*, or a ⊃–*IA*.

(In view of the application to intuitionist logic (3.2) we shall deal with each possibility in greater detail than would be necessary for the classical case.)

3.121.231. Suppose $\mathfrak{J}\mathfrak{f}$ is a &–*IS*.

Then the end of the derivation runs:

$$\dfrac{\Pi \to \Sigma \qquad \dfrac{\Gamma \to \Theta, \mathfrak{A} \qquad \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \,\&\, \mathfrak{B}}\,\&\text{–}IS}{\Pi, \Gamma^* \to \Sigma^*, \Theta, \mathfrak{A} \,\&\, \mathfrak{B}}\,\text{mix}.$$

($\mathfrak{M}$ occurs in Γ.) This is transformed into:

$$\dfrac{\dfrac{\Pi \to \Sigma \qquad \Gamma \to \Theta, \mathfrak{A}}{\Pi, \Gamma^* \to \Sigma^*, \Theta, \mathfrak{A}}\,\text{mix} \qquad \dfrac{\Pi \to \Sigma \qquad \Gamma \to \Theta, \mathfrak{B}}{\Pi, \Gamma^* \to \Sigma^*, \Theta, \mathfrak{B}}\,\text{mix}}{\Pi, \Gamma^* \to \Sigma^*, \Theta, \mathfrak{A} \,\&\, \mathfrak{B}}\,\&\text{–}IS.$$

Both mixes may be eliminated by virtue of the induction hypothesis.

3.121.232. Suppose $\mathfrak{J}\mathfrak{f}$ is a ∨–*IA*.

Then the end of the derivation runs:

$$\dfrac{\Pi \to \Sigma \qquad \dfrac{\mathfrak{A}, \Gamma \to \Theta \qquad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta}\,\vee\text{–}IA}{\Pi, (\mathfrak{A} \vee \mathfrak{B})^*, \Gamma^* \to \Sigma^*, \Theta}\,\text{mix}.$$

($(\mathfrak{A} \vee \mathfrak{B})^*$ stands either for $\mathfrak{A} \vee \mathfrak{B}$ or for nothing according as $\mathfrak{A} \vee \mathfrak{B}$ is unequal or equal to $\mathfrak{M}$.)

$\mathfrak{M}$ certainly occurs in Γ. (For otherwise $\mathfrak{M}$ would be equal to $\mathfrak{A} \vee \mathfrak{B}$, and the right rank would be equal to 1 contrary to 3.121.)

To begin with, we transform the end of the derivation into:

$$\dfrac{\dfrac{\dfrac{\Pi \to \Sigma \quad \mathfrak{A}, \Gamma \to \Theta}{\Pi, \mathfrak{A}^*, \Gamma^* \to \Sigma^*, \Theta}\text{ mix}}{\mathfrak{A}, \Pi, \Gamma^* \to \Sigma^*, \Theta}\text{ possibly several interchanges and thinnings} \qquad \dfrac{\dfrac{\Pi \to \Sigma \quad \mathfrak{B}, \Gamma' \to \Theta}{\Pi, \mathfrak{B}^*, \Gamma^* \to \Sigma^*, \Theta}\text{ mix}}{\mathfrak{B}, \Pi, \Gamma^* \to \Sigma^*, \Theta}\text{ possibly several interchanges and thinnings}}{\mathfrak{A} \vee \mathfrak{B}, \Pi, \Gamma'^* \to \Sigma^*, \Theta}\ \vee\text{-}IA.$$

Both mixes may be eliminated by virtue of the induction hypothesis.

From here on the procedure is the same as that in 3.121.221 and 3.121.222, i.e., we distinguish two cases according as $\mathfrak{A} \vee \mathfrak{B}$ is unequal or equal to $\mathfrak{M}$. In the first case we may have to add several interchanges to obtain the endsequent of the original derivation; in the second case we add a mix with $\Pi \to \Sigma$ for its left-hand upper sequent, and thus once again obtain the endsequent of the original derivation by going on to perform a number of contractions and interchanges, if necessary. The mix concerned may be eliminated, since the associated right rank equals 1. (All this as in 3.121.222.)

3.121.233. Suppose $\mathfrak{Jf}$ is a $\supset$–IA.

Then the end of the derivation runs:

$$\dfrac{\Pi \to \Sigma \qquad \dfrac{\Gamma \to \Theta, \mathfrak{A} \qquad \mathfrak{B}, \Delta \to \Lambda}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Delta \to \Theta, \Lambda}\ \supset\text{-}IA}{\Pi, (\mathfrak{A} \supset \mathfrak{B})^*, \Gamma^*, \Delta^* \to \Sigma^*, \Theta, \Lambda}\text{ mix.}$$

3.121.233.1. Suppose $\mathfrak{M}$ occurs in Γ and Δ.

In that case we begin by transforming the derivation into:

$$\dfrac{\dfrac{\Pi \to \Sigma \qquad \Gamma \to \Theta, \mathfrak{A}}{\Pi, \Gamma^* \to \Sigma^*, \Theta, \mathfrak{A}}\text{ mix} \qquad \dfrac{\dfrac{\Pi \to \Sigma \qquad \mathfrak{B}, \Delta \to \Lambda}{\Pi, \mathfrak{B}^*, \Delta^* \to \Sigma^*, \Lambda}\text{ mix}}{\mathfrak{B}, \Pi, \Delta^* \to \Sigma^*, \Lambda}\text{ possibly several interchanges and thinnings}}{\mathfrak{A} \supset \mathfrak{B}, \Pi, \Gamma^*, \Pi, \Delta^* \to \Sigma^*, \Theta, \Sigma^*, \Lambda}\ \supset\text{-}IA.$$

Both mixes may be eliminated by virtue of the induction hypothesis. Then we proceed as in 3.121.221 and 3.121.222. (All that may happen in the first case is that beside interchanges a number of contractions become necessary.)

3.121.233.2. Suppose $\mathfrak{M}$ does not occur in both Γ and Δ simultaneously. $\mathfrak{M}$ must occur in either Γ or Δ because of 3.121. Consider the case of $\mathfrak{M}$ occurring in Δ but not in Γ. The second case is treated analogously.

The end of the derivation is transformed into:

$$\cfrac{\Gamma \to \Theta, \mathfrak{A} \qquad \cfrac{\cfrac{\Pi \to \Sigma \qquad \mathfrak{B}, \Delta \to \Lambda}{\Pi, \mathfrak{B}^*, \Delta^* \to \Sigma^*, \Lambda}\text{ mix}}{\mathfrak{B}, \Pi, \Delta^* \to \Sigma^*, \Lambda}\text{ possibly several interchanges and thinnings}}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Pi, \Delta^* \to \Theta, \Sigma^*, \Lambda}\ \supset\text{-}IA.$$

The mix may be eliminated by virtue of the induction hypothesis. We then proceed as in 3.121.221 and 3.121.222. (In the second case, where $\mathfrak{A} \supset \mathfrak{B}$ is equal to $\mathfrak{M}$, the right rank belonging to the new mix equals 1 as always, since $\mathfrak{M}$ does not occur in $\mathfrak{B}$, Π, Δ^* for the usual reasons, nor does it occur in Γ according to the assumption in the case under consideration.)

3.122. *Suppose the right rank is equal to* 1. In that case the *left rank is greater than* 1.

This case is, in essence, treated dually to 3.121. Special attention is required only for those inference figures with no symmetric counterpart, viz., the $\supset$–IS and the $\supset$–IA.

The inference figures $\mathfrak{J}\mathfrak{f}$ with *one* upper sequent were incorporated, in 3.121.22, in the general schema:

$$\frac{\Psi, \Gamma \to \Omega_1}{\Xi, \Gamma \to \Omega_2}.$$

The dual schema runs:

$$\frac{\Omega_1 \to \Gamma, \Psi}{\Omega_2 \to \Gamma, \Xi},$$

which also covers the $\supset$–IS without any further change. (Γ here represents the formulae designated by Θ in the schemata 1.21, 1.22.)

3.122.1. On the other hand, the case where the inference figure $\mathfrak{J}\mathfrak{f}$ is a $\supset$–IA, must be treated separately. Although this treatment will seem very similar to that in 3.121.233, it is not entirely dual.

Thus the end of the derivation runs:

$$\cfrac{\cfrac{\Gamma \to \Theta, \mathfrak{A} \qquad \mathfrak{B}, \Delta \to \Lambda}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Delta \to \Theta, \Lambda}\ \supset\text{-}IA \qquad \Sigma \to \Pi}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Delta, \Sigma^* \to \Theta^*, \Lambda^*, \Pi}\text{ mix.}$$

3.122.11. Suppose $\mathfrak{M}$ occurs both in Θ and Λ. In that case we transform the end of the derivation into:

$$\dfrac{\dfrac{\dfrac{\to \Theta, \mathfrak{A} \qquad \Sigma \to \Pi}{\Gamma, \Sigma^* \to \Theta^*, \mathfrak{A}^*, \Pi}\ \text{mix}}{\Gamma, \Sigma^* \to \Theta^*, \Pi, \mathfrak{A}^*}\ \text{possibly several interchanges and thinnings} \qquad \dfrac{\mathfrak{B}, \Delta \to \Lambda \qquad \Sigma \to \Pi}{\mathfrak{B}, \Delta, \Sigma^* \to \Lambda^*, \Pi}\ \text{mix}}{\dfrac{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Sigma^*, \Delta, \Sigma^* \to \Theta^*, \Pi, \Lambda^*, \Pi}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Delta, \Sigma^* \to \Theta^*, \Lambda^*, \Pi}\ \text{possibly several contractions and interchanges.}}\ \supset\text{-}IA$$

Both mixes may be eliminated by virtue of the induction hypothesis.

3.122.12. Suppose $\mathfrak{M}$ does not occur in both Θ and Λ simultaneously. It must occur in one of them. We consider the case of $\mathfrak{M}$ occurring in Λ, but not in Θ; the alternative case is completely analogous.

We transform the end of the derivation into:

$$\dfrac{\Gamma \to \Theta, \mathfrak{A} \qquad \dfrac{\mathfrak{B}, \Delta \to \Lambda \qquad \Sigma \to \Pi}{\mathfrak{B}, \Delta, \Sigma^* \to \Lambda^*, \Pi}\ \text{mix}}{\mathfrak{A} \supset \mathfrak{B}, \Gamma, \Delta, \Sigma^* \to \Theta, \Lambda^*, \Pi}\ \supset\text{-}IA\,.$$

The mix may be eliminated by virtue of the induction hypothesis.

3.2. *Proof of the Hauptsatz for LJ-derivations.*

In order to transform an *LJ*-derivation into an *LJ*-derivation *without cuts*, we apply *exactly the same procedure* as for *LK*-derivations.

Since an *LJ*-derivation is a special case of an *LK*-derivation, it is clear that the transformation can be carried out. We have only to convince ourselves that with every transformation step an *LJ*-derivation becomes another *LJ*-derivation, i.e., that the *D*-sequents of the transformed derivation do not contain more than one *S*-formula in the succedent, given that this was the case before.

We therefore examine each step of the transformation from that point of view.

3.21. Replacement of cuts by mixes. An *LJ*-cut runs:

$$\dfrac{\Gamma \to \mathfrak{D} \qquad \mathfrak{D}, \Delta \to \Lambda}{\Gamma, \Delta \to \Lambda}\,,$$

where Λ contains at most one *S*-formula. We transform this cut into:

$$\dfrac{\dfrac{\Gamma \to \mathfrak{D} \qquad \mathfrak{D}, \Delta \to \Lambda}{\Gamma, \Delta^* \to \Lambda}\ \text{mix}}{\Gamma, \Delta \to \Lambda}\ \text{possibly several interchanges and thinnings in the antecedent.}$$

This replacement gives us a new *LJ*-derivation.

3.22. By relabelling free object variables (3.10) we trivially get another LJ-derivation from a previous one.

3.23. The transformation proper (3.11 and 3.12).

We have to show for each of the cases 3.111 to 3.122.12 that the specified transformations do not introduce any sequents with more than one S-formula in the succedent.

3.231. Let us begin with the cases 3.11:

In the cases 3.111, 3.113.1, 3.113.31, 3.113.35 and 3.113.36, only such formulae occur in each succedent of the sequent of a new derivation as had already occurred in the succedent of the sequent of the original derivation.

Essentially the same applies in 3.113.33. The only difference is an additional replacement of free object variables, which does not, of course, alter the number of succedent formulae of a sequent.

Cases 3.112, 3.113.2, 3.113.32, and 3.113.34 were dealt with symmetrically to cases 3.111, 3.113.1, 3.113.31, and 3.113.33, i.e., in order to get one case from another, we read the schemata from right to left instead of from left to right (as well as changing logical symbols, a process which is here of no consequence). Hence in the antecedent of one case we get precisely the same as in the succedent of another. For the antecedents of cases 3.111, 3.113.1, 3.113.31 and 3.113.33, the same applies as for the succedents, viz., in every antecedent of a sequent of the new derivation only such formulae occur as had already occurred in an antecedent of a sequent of the original derivation.

This disposes of all dual cases: 3.112, 3.113.2, 3.113.32 and 3.113.34.

3.232. Now let us look at the cases, 3.12:

3.232.1. For the cases 3.121 it holds generally that Σ^* is empty, since in $\Pi \rightarrow \Sigma$, Σ must contain only *one* formula, and that formula must be equal to $\mathfrak{M}$.

It is now obvious that in every succedent of a sequent only such formulae occur as had already occurred in the succedent of a sequent of the original derivation.

3.232.2. In the cases 3.122 it is somewhat more difficult to see that from an LJ-derivation we always get another LJ-derivation. We must direct our attention, as was done in our earlier consideration of dual cases, to the *antecedents* in the schemata 3.121.

At this point we distinguish two further subcases:

3.232.21. The case which is dual to 3.121.1 is trivial, since in every antecedent of a sequent of a new derivation (in case 3.121.1) only such formulae occur as had already occurred in an antecedent of a sequent of the original derivation.

3.232.22. In the cases that are dual to 3.121.2, the mix in the end of the derivation runs:

$$\frac{\Omega \rightarrow \mathfrak{M} \qquad \Sigma \rightarrow \Pi}{\Omega, \Sigma^* \rightarrow \Pi},$$

where Π contains at most *one* S-formula, and where $\Omega \rightarrow \mathfrak{M}$ is the lower sequent of an LJ-inference figure in which at least one upper sequent contains $\mathfrak{M}$ as a succedent formula.

If we now look at the inference figure schemata 1.21, 1.22, it becomes easily apparent that such an inference figure can only be a thinning, contraction, or interchange in the antecedent, or a $\vee$–IA, a &–IA, a $\exists$–IA, a $\forall$–IA, and a $\supset$–IA. Let us disregard for the moment the $\vee$–IA and the $\supset$–IA. Then all the possibilities enumerated above fall within the case dual to 3.121.22, where both Ψ and Ξ always remain empty. (Γ corresponds to the Θ of the inference figure.) Thus we have the case which is dual to 3.121.221. Furthermore, Γ is equal to $\mathfrak{M}$, i.e., Γ^* is empty, and Π contains at most one formula. Hence in the new derivation there never in fact occurs more than one formula in the succedent of a sequent.

The case of a $\vee$–IA is dual to 3.121.231. Again, Γ is equal to $\mathfrak{M}$, Γ^* is empty, and Π contains at most one formula; all is thus in order.

There now remains the case of a $\supset$–IA, i.e., 3.122.1. In an LJ- $\supset$–IA, the Θ of the schema (1.22) is empty. Thus we have the case set out under 3.122.12. Λ^* is also empty, and Π contains at most one formula, which means that here, too, we again obtain an LJ-derivation from an LJ-derivation.

SECTION IV. SOME APPLICATIONS OF THE HAUPTSATZ

§ 1. Applications of the *Hauptsatz* in propositional logic

1.1. A trivial consequence of the *Hauptsatz* is the already known *consistency* of classical (and intuitionist) *predicate logic* (cf., e.g., D. Hilbert and W. Ackermann, *Grundzüge der theoretischen Logik* (Berlin, 1928, 1st edition), p. 65): the sequent $\rightarrow$ (which is derivable from every contradictory sequent $\rightarrow \mathfrak{A}\ \&\ \neg\, \mathfrak{A}$, cf. 3.21) cannot be the lower sequent of any inference figure other than of a cut and is therefore not derivable.

1.2. *Solution of the decision problem for intuitionist propositional logic.* On the basis of the *Hauptsatz* we can state a simple procedure for deciding of a formula of propositional logic – i.e., a formula without object variables –

whether or not it is classically or intuitionistically true. (For classical propositional logic a simple solution has actually been known for some time, cf., e.g., p. 11 of Hilbert-Ackermann.)

First we prove the following *lemma*:

A sequent in whose antecedent one and the same formula does not occur more than three times as an S-formula, and in whose succedent, furthermore, one and the same formula occurs no more than three times as an S-formula, will be called a 'reduced sequent'. The following lemma now holds:

1.21. Every *LJ*- or *LK*-derivation whose endsequent is reduced, may be transformed into an *LJ*- or *LK*-derivation with the same endsequent, in which all sequents are reduced (and in which no cuts occur if the original derivation did not contain any).

Proof of this lemma: If we eliminate from the antecedent of a sequent, in any places whatever (possibly none), all S-formulae occurring more than once, and if we do the same independently in the succedent, so that eventually these formulae occur only once, twice, or three times, we obtain a sequent that will be called a 'reduction instance of the given sequent'.

From a reduction instance of a sequent we may obviously derive all other reduction instances of the same sequent by means of thinnings, contractions, and interchanges such that in the course of these operations only reduced sequents occur.

After these preliminary remarks we now transform the *LJ*- or *LK*-derivation at hand in the following way:

All *basic sequents* as well as the *endsequent* are left intact; they are already reduced sequents.

The D-sequents which belong to an *inference figure* are transformed into reduction instances of these sequents in a way about to be indicated. By virtue of our preliminary remark it does not matter if a sequent belonging to two different D-inference figures is in each case replaced by a *different reduction instance*, since one sequent is derived very simply from the other by thinnings, contractions, and interchanges so that eventually another complete derivation results. (The same holds for a sequent which, while belonging to an inference figure, is also a basic sequent or an endsequent, since it is of course a reduction instance of itself.)

The transformations of the inference figures are now carried out in the following way:

If a formula occurs more than once within Γ, it is eliminated from Γ, both from the upper sequents and the lower sequent, as many times (from the appropriate places) as is necessary to ensure that finally it occurs in Γ no more than once. The same procedure is used for Δ, Θ, and Λ (i.e., those sequences of formulae that are designated by these letters in the schema III.1.21 and 1.22, of the inference figure concerned).

Having carried out the transformations described, we have now a derivation consisting only of reduced sequents. (An interchange where $\mathfrak{D}$ is identical with $\mathfrak{E}$ may form an exception, yet this figure would be an identical inference figure and could have been avoided.)

The lemma is thus proved.

Given the *Hauptsatz*, together with corollary III. 2.513, and the preceding lemma (1.21), it now holds that:

1.22. For every correctly reduced sequent, both intuitionist and classical, there exists an *LJ*- or *LK*-derivation resp. without cuts consisting only of reduced sequents, and whose *D-S*-formulae are subformulae of the *S*-formula of that sequent.

1.23. Consider now a sequent not containing an object variable. We wish to decide whether or not it is intuitionistically or classically true. We can begin by taking in its place an equivalent reduced sequent $\mathfrak{S}\mathrm{q}$.

The number of all reduced sequents whose *S*-formulae are subformulae of the *S*-formulae of $\mathfrak{S}\mathrm{q}$ is obviously finite. The decision procedure may therefore be carried out without further complications in the following way:

We consider the finite system of sequents in question and investigate first of all, which of these sequents are basic sequents. Then we examine each of the remaining sequents to determine whether there occurs an inference figure in which the sequent in question is the lower sequent and in which there occur as upper sequents one or two of those sequents that have already been found to be derivable. If this is the case, the sequent is added to the derivable sequents. (All this is obviously decidable.) We continue in this way until either the sequent $\mathfrak{S}\mathrm{q}$ itself turns out to be derivable, or until the procedure yields no new derivable sequents. In the latter case the sequent $\mathfrak{S}\mathrm{q}$ (by virtue of 1.22) is not derivable at all in the calculus under consideration (*LJ* or *LK*). We have therefore succeeded in establishing the validity of that sequent.

1.3. A new proof of the *nonderivability of the law of the excluded middle in intuitionist logic.*

Our decision procedure could have been formulated in a way better suited to the needs of practical application; yet the above presentation (1.2) was intended only to indicate a possibility in principle.

As an example, we shall prove the nonderivability of the law of the excluded middle in intuitionist logic by a method independent of the decision procedure described (although this procedure would have to yield the same result). (This nonderivability has already been proved by Heyting[24] in a completely different way.)

The sequent in question is of the form $\rightarrow A \vee \neg A$. Suppose there exists an *LJ*-derivation for it. According to the *Hauptsatz* there then exists such a derivation without cuts. Its lowest inference figure must be a $\vee$–*IS*, for in all other *LJ*-inference figures either the antecedent of the lower sequent is not empty, or a formula occurs in the succedent whose terminal symbol is not $\vee$; there might still be the case of a thinning in the succedent, but the upper sequent would then be a $\rightarrow$, which, by virtue of 1.1, is not derivable.

Hence either $\rightarrow A$ or $\rightarrow \neg A$ would have to be already derivable (without cuts).

(From the same considerations, incidentally, it follows in general: If $\mathfrak{A} \vee \mathfrak{B}$ is an intuitionistically true formula, then either $\mathfrak{A}$ or $\mathfrak{B}$ is an intuitionistically true formula. In classical logic this does not hold, as the example of $A \vee \neg A$ already shows.)

Now $\rightarrow A$ cannot be the lower sequent of any *LJ*-inference figure whatever (if it is not a cut), unless that figure is another thinning with $\rightarrow$ for its upper sequent. Furthermore, since $\rightarrow A$ is not a basic sequent, it is thus not derivable.

The same considerations show that $\rightarrow \neg A$ is derivable only from $A \rightarrow$ by a $\neg$–*IS* figure, and $A \rightarrow$ is in turn derivable only from $A, A \rightarrow$, since A contains no terminal symbol. Continuing in this way, we always reach only sequents of the type $A, A, \ldots, A \rightarrow$, but never a basic sequent.

Hence $A \vee \neg A$ is not derivable in intuitionist predicate logic.

§ 2. A sharpened form of the *Hauptsatz* for classical predicate logic

2.1. We are here concerned with the following SHARPENING OF THE HAUPTSATZ:

Suppose that we have an *LK*-derivation whose endsequent is of the following kind:

Each *S*-formula of this sequent contains $\forall$ and $\exists$-symbols at most at the beginning, and their scope extends over the whole of the remaining formula.

In that case, the given derivation may be transformed into an *LK*-derivation with the same endsequent and having the following properties:

1. It contains no cuts.

2. It contains a *D*-sequent, let us call it the '*midsequent*', which is such that its derivation (and hence the midsequent itself) contains no $\forall$ and

$\exists$-symbols, and where the only inference figures occurring in the *remaining* part of the derivation, the midsequent included, are $\forall$–*IS*, $\forall$–*IA*, $\exists$–*IS*, $\exists$–*IA*, and structural inference figures.

2.11. The midsequent divides the derivation, as it were, into an upper part beolnging to propositional logic, and a lower part containing only $\forall$ and $\exists$-introductions.

Concerning the *form of the transformed derivation*, the following may still be readily concluded: The lower part, from the midsequent to the endsequent, belongs to only one path since only inference figures with *one* upper sequent occur in it. The S-formulae of the midsequent are of the following kind:

Every S-formula in the antecedent of the midsequent results from an S-formula in the antecedent of the endsequent by the elimination of the $\forall$ and $\exists$-symbols (together with the bound object variables beside them), and by the replacement of the bound object variables in the rest of the formula by certain free object variables. The same procedure is followed in the case of succedents.

This follows from the same consideration as in III.2.512.

2.2. PROOF OF THE THEOREM (2.1)[25].

The transformation of the derivation is carried out in several steps.

2.21. We begin by applying the *Hauptsatz* (III.2.5): The derivation may accordingly be transformed into a derivation without cuts.

2.22. Transformation of basic sequents containing a $\forall$- or $\exists$-symbol:

By virtue of the properties of subformulae III.2.513, such sequents can only have the form $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \rightarrow \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ or $\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \rightarrow \exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$. They are transformed into (suppose $\mathfrak{a}$ to be a free object variable not yet occurring in the derivation):

$$\dfrac{\dfrac{\mathfrak{F}\mathfrak{a} \rightarrow \mathfrak{F}\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \rightarrow \mathfrak{F}\mathfrak{a}}\ \forall\text{–}IA}{\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \rightarrow \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}\ \forall\text{–}IS \qquad \text{or} \qquad \dfrac{\dfrac{\mathfrak{F}\mathfrak{a} \rightarrow \mathfrak{F}\mathfrak{a}}{\mathfrak{F}\mathfrak{a} \rightarrow \exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}\ \exists\text{–}IS}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \rightarrow \exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}}\ \exists\text{–}IA.$$

By repeating this procedure sufficiently often we can obviously eliminate all $\forall$- and $\exists$-symbols from every basic sequent of the derivation.

2.23. We now perform a complete induction on the 'order' of the derivation, which is defined as follows:

Of the operational inference figures we call those belonging to the symbols $\&$, $\vee$, $\neg$, and $\supset$ 'propositional inference figures', and the rest, i.e., $\forall$–*IS*, $\forall$–*IA*, $\exists$–*IS*, $\exists$–*IA*, 'predicate inference figures'. To each predicate inference figure in the derivation we assign the following ordinal number:

We consider that path of the derivation that extends from the lower sequent of the inference figure up to the endsequent of the derivation (including the endsequent) and count the number of lower sequents of the propositional inference figures occurring in it. Their number is the ordinal number.

The sum of the ordinal numbers of all predicate inference figures in the derivation is the *order of the derivation.*

We intend to reduce that order step by step until it becomes zero.

Note that once this has been achieved the rest of the proof of the theorem (2.1) is easily carried out: (The steps involved (2.232) will be such as to preserve the properties that were established in 2.21 and 2.22.)

2.231. In order to do so we assume that the derivation has already been reduced to order zero. From the endsequent we now proceed to the upper sequent of the inference figure above it. We stop as soon as we encounter the lower sequent of a propositional inference figure or a basic sequent; that sequent we call $\mathfrak{S}q$. (It will serve us as 'midsequent', once it has been transformed in a way about to be indicated.)

The derivation of $\mathfrak{S}q$ is now transformed as follows:

We simply omit all *D-S*-formulae which still contain the symbols $\forall$ and $\exists$. The above derivation remains correct after the described operation since, by virtue of 2.22, its basic sequents are not affected. Furthermore, no principal or side formula of an inference figure has been eliminated, for if such a formula had contained a symbol $\forall$ or $\exists$, the principal formula would certainly have contained that symbol. But no predicate inference figures occur (if they did, the ordinal number of the inference figure would be greater than zero), and by virtue of the subformula property (III.2.513) and the hypothesis of theorem 2.1, the principal formulae of the propositional inference figures cannot contain a $\forall$ or $\exists$. Now every inference figure remains correct if we eliminate, wherever it occurs as an *S*-formula in the figure, a formula which occurs neither as a principal nor as a side formula. This is easily seen from the schemata III.1.21 and III.1.22. (At worst, an identical inference figure may result, which is then eliminated in the usual manner.)

The sequent $\mathfrak{S}q^*$, which has resulted from $\mathfrak{S}q$ by this transformation, differs from $\mathfrak{S}q$ in that certain *S*-formulae may possibly have been eliminated. We follow the transformation up with several thinnings and interchanges such that in the end the sequent $\mathfrak{S}q$ reappears, and to it we attach the unaltered lower part of the derivation.

We have now reached our goal: $\mathfrak{S}q^*$ is the 'midsequent', and it obviously satisfies all conditions imposed on the latter by theorem (2.1).

2.232. It now remains for us to carry out the *induction step* of our proof, i.e., the order of the derivation is assumed to be greater than zero, and our task is to diminish it.

2.232.1. We begin by *redesignating the free object variables* in the same way as in III.3.10. As a result of this, the derivation has the following property (III.3.101):

For every $\forall$–*IS* (or $\exists$–*IA*) it holds that the eigenvariable in the derivation occurs only in the sequent above the lower sequent of the $\forall$–*IS* (or $\exists$–*IA*) and does furthermore not occur in any other $\forall$–*IS* or $\exists$–*IA* as an eigenvariable.

The order of the derivation is hereby obviously left unchanged.

2.232.2. We now come to the transformation proper.

To begin with, we observe that in the derivation there occurs a predicate inference figure – let us call it $\mathfrak{J}\mathfrak{f}1$ – with the following property: If we follow that path of the inference figure which extends from the lower sequent to the endsequent, then the first lower sequent of an operational inference figure reached is the lower sequent of a *propositional* inference figure (that inference figure we call $\mathfrak{J}\mathfrak{f}2$). If there were no such instance, the order of the derivation would be equal to zero.

Now our aim is to slide the inference figure $\mathfrak{J}\mathfrak{f}1$ lower down in the derivation beyond $\mathfrak{J}\mathfrak{f}2$. This is easily done by means of the following schemata:

2.232.21. Suppose that $\mathfrak{J}\mathfrak{f}2$ has *one* upper sequent.

2.232.211. Suppose that $\mathfrak{J}\mathfrak{f}1$ is a $\forall$–*IS*. Then that part of the derivation on which the operation is to be carried out runs as follows:

$$\begin{array}{ll}
\Gamma \to \Theta, \mathfrak{F}\mathfrak{a} & \\ \hline
\Gamma \to \Theta, \forall \mathfrak{x}\,\mathfrak{F}\mathfrak{x} & \forall\text{–}IS \\ \hline\hline
\Delta \to \Lambda & \mathfrak{J}\mathfrak{f}2, \text{ possibly preceded by structural inference figures.}
\end{array}$$

This we transform into:

$$\begin{array}{ll}
\Gamma \to \Theta, \mathfrak{F}\mathfrak{a} & \\ \hline\hline
\Gamma \to \mathfrak{F}\mathfrak{a}, \Theta, \forall \mathfrak{x}\,\mathfrak{F}\mathfrak{x} & \text{possibly several interchanges, as well as a thinning} \\ \hline\hline
\Delta \to \mathfrak{F}\mathfrak{a}, \Lambda & \text{inference figures of exactly the same kind as above, i.e., } \mathfrak{J}\mathfrak{f}2\text{, possibly preceded by structural inference figures} \\ \hline\hline
\Delta \to \Lambda, \mathfrak{F}\mathfrak{a} & \text{possibly several interchanges} \\ \hline
\Delta \to \Lambda, \forall \mathfrak{x}\,\mathfrak{F}\mathfrak{x} & \forall\text{–}IS \\ \hline\hline
\Delta \to \Lambda & \text{possibly several interchanges and contractions.}
\end{array}$$

The elimination of $\forall \mathfrak{x}\,\mathfrak{F}\mathfrak{x}$ by contraction in the last step of the transformation is made possible by the fact that in Λ, $\forall \mathfrak{x}\,\mathfrak{F}\mathfrak{x}$ must occur as an

S-formula. (For the S-formula $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ could not, in the original derivation, have been eliminated from the succedent by means of $\mathfrak{J}\mathfrak{f}2$ and the preceding structural inference figures, since it can obviously not be a side formula of $\mathfrak{J}\mathfrak{f}2$, by virtue of the subformula property III.2.513 and the hypothesis of theorem 2.1.)

The restriction on variables is satisfied by the above $\forall$–*IS* ($\mathfrak{J}\mathfrak{f}1$) by virtue of 2.232.1.

The order of the derivation has obviously been diminished by 1.

2.232.212. The case where $\mathfrak{J}\mathfrak{f}1$ is a $\exists$–*IS* is dealt with analogously; all we need do is to replace $\forall$ by $\exists$.

2.232.213. The cases where $\mathfrak{J}\mathfrak{f}1$ is a $\forall$–*IA* or $\exists$–*IA* are treated dually to the two preceding cases.

2.232.22. The case where $\mathfrak{J}\mathfrak{f}2$ has *two* upper sequents, i.e., &–*IS*, $\vee$–*IA*, or $\supset$–*IA*, can be dealt with quite correspondingly. At most a number of additional structural inference figures may be required.

2.3. Analogously to theorem 2.1 there are several ways in which the *Hauptsatz* may be further strengthened in the sense that certain restrictions can be placed on the order of occurrence of the operational inference figures in a derivation. For we can permute the inference figures to a large extent by sliding them above and beyond each other as was done above (2.232.2).

We shall not pursue this question further.

§ 3. Application of the *sharpened Hauptsatz* (2.1) to a new[26] consistency proof for arithmetic without complete induction

By *arithmetic* we mean the (elementary, i.e., employing no analytic techniques) theory of the natural numbers. Arithmetic may be formalized by means of our logical calculus *LK* in the following way:

3.1. In arithmetic it is customary to employ 'functions', e.g., x' (equals $x+1$), $x+y$, $x \cdot y$. Since we have not introduced function symbols into our logical formalism, we shall, in order to be able to apply it to arithmetic nevertheless, formalize the propositions of arithmetic in such a way that predicates take the place of functions. In place of the function x', for example, we shall use the predicate $x\mathrm{Pr}y$, which reads: x is the predecessor of y, i.e., $y = x+1$. Furthermore, $[x+y = z]$ will be considered a predicate with three argument places. Thus the symbols $+$ and $=$ have here no independent meaning. A different predicate is $x = y$; the equality symbol here has thus no formal connection at all with the equality symbol in the previous predicate.

The number 1, furthermore, will not be written as a symbol for a definite object, since we have only object *variables* in our logical formalism and no symbols for *definite* objects. We shall overcome this difficulty by saying that the predicate 'One x' means informally the same as 'x is the number 1'.

The sentence '$x+1$ is the successor of x', for example, could be rendered thus in our formalism:

$$\forall x\, \forall y\, \forall z\, \big((\text{One } y\ \&\ (x+y = z)) \supset x\text{Pr}z\big).$$

All other natural numbers can be respresented by the predicates One x & xPry; One x & xPry & yPrz, etc.

How are we now to integrate into our calculus the *predicate symbols* just introduced, having admitted only propositional variables? To do so we simply stipulate that the predicate symbols are to be treated in exactly the same way as propositional variables. More precisely: We regard expressions of the form

$$\text{One } \mathfrak{x},\ \mathfrak{x}\text{Pr}\mathfrak{y},\ \mathfrak{x} = \mathfrak{y},\ (\mathfrak{x}+\mathfrak{y} = \mathfrak{z}),$$

where any object variables stand for $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, merely as more easily intelligible ways of writing the formulae

$$A\mathfrak{x},\ B\mathfrak{x}\mathfrak{y},\ C\mathfrak{x}\mathfrak{y},\ D\mathfrak{x}\mathfrak{y}\mathfrak{z}.$$

In this sense the axiom formulae that follow are indeed *formulae* in accordance with our definition.

(We cannot, of course, regard the number 1 as a way of writing an object variable, since in our calculus the object variables really function as *variables*, which is not so in the case of propositional variables.)

As '*axiom formulae*' *of our arithmetic* we shall initially take the following, and shall later, once the consistency proof has been carried out (cf. 3.3), state general criteria for the formation of further admissible axiom formulae:

Equality:

$\forall x\, (x = x)$	(reflexivity)
$\forall x\, \forall y\, (x = y \supset y = x)$	(symmetry)
$\forall x\, \forall y\, \forall z\, ((x = y\ \&\ y = z) \supset x = z)$	(transitivity)

One:

$\exists x\, (\text{One } x)$	(existence of 1)
$\forall x\, \forall y\, ((\text{One } x\ \&\ \text{One } y) \supset x = y)$	(uniqueness of 1)

Predecessor:

$\forall x\, \exists y\, (x\text{Pr}y)$	(existence of successor)

$\forall x\, \forall y\, (x\mathrm{Pr}y \supset \neg\, \mathrm{One}\; y)$ (1 has no predecessor)
$\forall x\, \forall y\, \forall z\, \forall u\, ((x\mathrm{Pr}y \mathbin{\&} z\mathrm{Pr}u \mathbin{\&} x = z) \supset y = u)$ (uniqueness of successor)
$\forall x\, \forall y\, \forall z\, \forall u\, ((x\mathrm{Pr}y \mathbin{\&} z\mathrm{Pr}u \mathbin{\&} y = u) \supset x = u)$ (uniqueness of predecessor).

A formula $\mathfrak{B}$ is called *derivable* in arithmetic without complete induction, if there is an *LK*-derivation for a sequent

$$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \to \mathfrak{B}$$

in which $\mathfrak{A}_1, \ldots, \mathfrak{A}\mu$ are axiom formulae of arithmetic.

The fact that this formal system does actually allow us to represent the types of proof customary in informal arithmetic (as long as they do not use complete induction) cannot be *proved*, since for considerations of an informal character no precisely delimited framework exists. We can merely verify this in the case of individual informal proofs by testing them.

3.2. We shall now prove the *consistency of the formal system just presented.* With the help of the sharpened *Hauptsatz* (2.1) our task is in fact quite simple.

3.21. A 'contradiction' $\mathfrak{A} \mathbin{\&} \neg\, \mathfrak{A}$ is derivable in our system if and only if there exists an *LK*-derivation for a sequent with an *empty succedent* and with arithmetic axiom formulae in the antecedent, viz.:

From $\Gamma \to \mathfrak{A} \mathbin{\&} \neg\, \mathfrak{A}$ we obtain $\Gamma \to$ in the following way:

$$
\dfrac{\Gamma \to \mathfrak{A} \mathbin{\&} \neg\,\mathfrak{A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathfrak{A} \to \mathfrak{A}}{\neg\,\mathfrak{A}, \mathfrak{A} \to}\;\neg\text{-}IA}{\mathfrak{A} \mathbin{\&} \neg\,\mathfrak{A}\;\; \mathfrak{A} \to}\;\&\text{-}IA}{\mathfrak{A}, \mathfrak{A} \mathbin{\&} \neg\,\mathfrak{A} \to}\;\text{interchange}}{\mathfrak{A} \mathbin{\&} \neg\,\mathfrak{A}, \mathfrak{A} \mathbin{\&} \neg\,\mathfrak{A} \to}\;\&\text{-}IA}{\mathfrak{A} \mathbin{\&} \neg\,\mathfrak{A} \to}\;\text{contraction}}{\Gamma \to}\;\text{cut.}
$$

The converse is obtained by carrying out a thinning in the succedent.

Thus, if our arithmetic is inconsistent, there exists an *LK*-derivation with the endsequent

$$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \to,$$

where $\mathfrak{A}_1, \ldots, \mathfrak{A}\mu$ are arithmetic axiom formulae.

3.22. We now apply the *sharpened Hauptsatz* (2.1). The arithmetic axiom formulae fulfil the requirement laid down for the *S*-formulae of the endsequent. Hence there exists an *LK*-derivation with the same endsequent which has the following properties:

1. It contains no cuts.

2. It contains a D-sequent. the 'midsequent', whose derivation contains no $\forall$ and $\exists$-symbols, and whose endsequent results from a number of inference figures $\forall$–IA, $\exists$–IA, thinnings, contractions and interchanges in the antecedent. The midsequent has an empty succedent (2.11).

3.23. We then proceed to redesignate the free object variables as in III.3.10. All mentioned properties remain unchanged, and the following property is added (III.3.101): The eigenvariable of each $\exists$–IA in the derivation occurs only in sequents *above* the lower sequent of the $\exists$–IA.

3.24. Then we replace every occurrence of a free object variable by one and the same natural number in a way to be described presently. In doing so we are left with a figure which we can no longer call an LK-derivation. We shall see later to what extent it nevertheless has an informal sense.

The replacement of the free object variables by numbers is carried out in the following order:

3.241. First we replace all free object variables which do not occur as the eigenvariable of a $\exists$–IA by the number 1 throughout. (We could also take another number.)

3.242. Then we take every $\exists$–IA inference figure in the derivation, beginning with the lowest and taking each figure in turn, and replace each eigenvariable (wherever it occurs in the 'derivation') by a number. That number is determined as follows:

The $\exists$–IA can only run:

$$\frac{\text{One}\ \mathfrak{a}, \Gamma \rightarrow \Theta}{\exists x\ \text{One}\ x, \Gamma \rightarrow \Theta} \quad \text{or} \quad \frac{v\text{Pr}\mathfrak{a}, \Gamma \rightarrow \Theta}{\exists y\ v\text{Pr}y, \Gamma \rightarrow \Theta}$$

(by virtue of the subformula property III.2.513; v can be only a number, by virtue of 3.241 and 3.23). In the first case we replace $\mathfrak{a}$ by 1, in the second case by the number that is one greater than v.

3.25. Now we examine the figure which has resulted from the derivation. We are particularly interested in what the (former) midsequent now looks like. We can say this about it:

Its succedent is empty, and each of the antecedent S-formulae either has the form One 1 or $v\text{Pr}v'$, where a number stands for v, and where a number one greater than the previous one stands for v'; or it results from an arithmetic axiom formula that has only $\forall$-symbols at the beginning, by the elimination of the $\forall$-symbols (and the bound object variables next to them) and the substitution of *numbers* for the bound object variables in the remaining part of the formula. (All this follows from the same consideration as in III.2.512, also cf. 2.11.)

Thus, the S-formulae in the antecedent of the midsequent represent *informally true numerical propositions.* It further holds for the '*derivation*' of the midsequent that it has resulted from a derivation containing no $\forall$- or $\exists$-symbols, by having all its occurrences of free object variables replaced by numbers. Informally, such a 'derivation' constitutes in effect a *proof in arithmetic using only forms of inference from propositional logic.*

This leads us to the following result:

If our arithmetic is inconsistent, we can derive a contradiction from true numerical propositions through the mere application of inferences from propositional logic.

Here 'true numerical propositions' are propositions of the form One 1, $v\mathrm{Pr}v'$, as well as all numerical special cases of general propositions occurring among the axioms such as, e.g., $3 = 3$, $4 = 5 \supset 5 = 4$, $3\mathrm{Pr}4 \supset \neg$ One 4.

It is almost self-evident that from such propositions no contradictions are derivable by means of propositional logic. A proof for this would hardly be more than a formal paraphrasing of an informally clear situation of fact. Such a proof will therefore not be carried out save for indicating briefly the customary procedure for it:

We determine generally for which numerical values the formulae One μ, $\mu = \nu$, $\mu\mathrm{Pr}\nu$, $\mu+\nu = \rho$, etc., are true and for which values they are false; furthermore, we explain in the customary way (cf., e.g., Hilbert-Ackermann p. 3) the truth or falsity of $\mathfrak{A} \,\&\, \mathfrak{B}$, $\mathfrak{A} \vee \mathfrak{B}$, $\neg \mathfrak{A}$, and $\mathfrak{A} \supset \mathfrak{B}$, as functions of the truth or falsity of the subformulae; we then show that all numerical special cases of axiom formulae are 'true'; and finally, that inference figures of propositional logic always lead from true formulae to other true formulae. A contradiction, however, is not a true formula.

3.3. It is easy to see from the remarks made in 3.25 in what way the system of *arithmetic axiom formulae* may be extended without making a contradiction derivable in it: Quite generally, we can allow the introduction of axiom formulae that begin with $\forall$-symbols spanning the whole formula, which do not contain any $\exists$-symbols, and of which every numerical special case is informally true. (We could also admit certain formulae containing $\exists$-symbols, as long as they can be dealt with in the consistency proof in a way analogous to that of the two cases occurring above.)

E.g., the following axiom formulae for addition are admissible:

$$\forall x\, \forall y\, (x\mathrm{Pr}y \supset [x+1 = y])$$

$\forall x \, \forall y \, \forall z \, \forall u \, \forall v \, ((x\text{Pr}y \,\&\, [z+x = u] \,\&\, [z+y = v]) \supset u\text{Pr}v)$
$\forall x \, \forall y \, \forall z \, \forall u \, (([x+y = z] \,\&\, [x+y = u]) \supset z = u)$
$\forall x \, \forall y \, \forall z \, ([x+y = z] \supset [y+x = z])$
etc.

3.4. Arithmetic without complete induction is, however, of little practical significance, since complete induction is constantly required in number theory. Yet the consistency of arithmetic with complete induction has not been conclusively proved to date.

SECTION V. THE EQUIVALENCE OF THE NEW CALCULI *NJ*, *NK*, AND *LJ*, *LK* WITH A CALCULUS MODELLED ON THE FORMALISM OF HILBERT

§ 1. The concept of equivalence

1.1. We shall introduce the following concept of equivalence between *formulae* and *sequents* (which is in harmony with what was said in I.1.1 and I.2.4, concerning the informal sense of the symbol $\curlywedge$ and of sequents:

Identical formulae are equivalent.

Identical sequents are equivalent.

Two formulae are equivalent if the replacement of every occurrence of the symbol $\curlywedge$ in one of them by the formula $A \,\&\, \neg A$ yields the other formula.

The sequents $\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \rightarrow \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$ is equivalent to the following formula:

If the $\mathfrak{A}$'s and $\mathfrak{B}$'s are not empty:

$$(\mathfrak{A}_1 \,\&\, \ldots \,\&\, \mathfrak{A}_\mu) \supset (\mathfrak{B}_\nu \vee \ldots \vee \mathfrak{B}_1);$$

(this version is more convenient for the equivalence proof than that with $\mathfrak{B}_1 \vee \ldots \vee \mathfrak{B}_\nu$); if the $\mathfrak{A}$'s are empty, but the $\mathfrak{B}$'s are not:

$$\mathfrak{B}_\nu \vee \ldots \vee \mathfrak{B}_1;$$

if the $\mathfrak{B}$'s are empty, but the $\mathfrak{A}$'s are not:

$$(\mathfrak{A}_1 \,\&\, \ldots \,\&\, \mathfrak{A}_\mu) \supset (A \,\&\, \neg A);$$

if the $\mathfrak{A}$'s and the $\mathfrak{B}$'s are empty:

$$A \,\&\, \neg A.$$

The equivalence is transitive.

1.2. (We could of course give a substantially wider definition of equivalence, e.g., two formulae are usually called equivalent if one is derivable from the other. Here we shall content ourselves with the particular definition given, which is adequate for our proofs of equivalence.)

Two *derivations* will be called equivalent if the endformula (endsequent) of one is equivalent to that of the other.

Two *calculi* will be called equivalent if every derivation in one calculus can be transformed into an equivalent derivation in the other calculus.

In § 2 of this section we shall present a calculus (*LHJ* for intuitionist, *LHK* for classical predicate logic) modelled on Hilbert's formalism. In the remaining paragraphs of this section we shall then demonstrate the equivalence of the calculi *LHJ*, *NJ*, and *LJ* (§§ 3–5) as well as the equivalence of the calculi *LHK*, *NK*, and *LK* (§ 6) in the sense just explained. We shall thus successively prove the following:

Every *LHJ*-derivation can be transformed into an equivalent *NJ*-derivation (§ 3); every *NJ*-derivation can be transformed into an equivalent *LJ*-derivation (§ 4); and every *LJ*-derivation can be transformed into an equivalent *LHJ*-derivation (§ 5). This obviously proves the equivalence of all three calculi. The three classical calculi are dealt with analogously in § 6 (6.1–6.3).

§ 2. A logistic calculus according to Hilbert[27] and Glivenko[28]

We shall begin by explaining the *intuitionist* form of the calculus:

An *LHJ*-derivation consists of formulae arranged in tree form, where the initial formulae are basic formulae.

The basic formulae and the inference figures are obtained from the following schemata by the same rule of replacement as in II.2.21, i.e.: For $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, put any arbitrary formula; for $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ or $\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ put any arbitrary formula with $\forall$ or $\exists$ for its terminal symbol, where $\mathfrak{x}$ designates the associated bound object variable; for $\mathfrak{F}\mathfrak{a}$ put that formula which results from $\mathfrak{F}\mathfrak{x}$ by the replacement of every occurrence of the bound object variable $\mathfrak{x}$ by the free object variable $\mathfrak{a}$.

Schemata for basic formulae:

2.11. $\mathfrak{A} \supset \mathfrak{A}$

2.12. $\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{A})$

2.13. $(\mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B})) \supset (\mathfrak{A} \supset \mathfrak{B})$

2.14. $(\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{C})) \supset (\mathfrak{B} \supset (\mathfrak{A} \supset \mathfrak{C}))$

2.15. $(\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{B} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset \mathfrak{C}))$
2.21. $(\mathfrak{A} \,\&\, \mathfrak{B}) \supset \mathfrak{A}$
2.22. $(\mathfrak{A} \,\&\, \mathfrak{B}) \supset \mathfrak{B}$
2.23. $(\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{A} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset (\mathfrak{B} \,\&\, \mathfrak{C})))$
2.31. $\mathfrak{A} \supset (\mathfrak{A} \vee \mathfrak{B})$
2.32. $\mathfrak{B} \supset (\mathfrak{A} \vee \mathfrak{B})$
2.33. $(\mathfrak{A} \supset \mathfrak{C}) \supset ((\mathfrak{B} \supset \mathfrak{C}) \supset ((\mathfrak{A} \vee \mathfrak{B}) \supset \mathfrak{C}))$
2.41. $(\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{A} \supset \neg \mathfrak{B}) \supset \neg \mathfrak{A})$
2.42. $(\neg \mathfrak{A}) \supset (\mathfrak{A} \supset \mathfrak{B})$
2.51. $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \supset \mathfrak{F}\mathfrak{a}$
2.52. $\mathfrak{F}\mathfrak{a} \supset \exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$.

(Several of the schemata are dispensable, but independence does not concern us here.)

Schemata for inference figures:

$$\frac{\mathfrak{A} \qquad \mathfrak{A} \supset \mathfrak{B}}{\mathfrak{B}} \qquad \frac{\mathfrak{A} \supset \mathfrak{F}\mathfrak{a}}{\mathfrak{A} \supset \forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}} \qquad \frac{\mathfrak{F}\mathfrak{a} \supset \mathfrak{A}}{(\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}) \supset \mathfrak{A}}.$$

Restriction on variables: In the inference figures obtained from the last two schemata, the object variable, designated by $\mathfrak{a}$ in the schema, must not occur in the lower formula (hence not in $\mathfrak{A}$ and $\mathfrak{F}\mathfrak{x}$).

(The calculus *LHJ* is essentially equivalent to that of Heyting[29].)

By including the basic formula schema $\mathfrak{A} \vee \neg \mathfrak{A}$, the *calculus LHK* (classical predicate calculus) results.

(This latter calculus is essentially equivalent to the calculus presented in Hilbert-Ackermann, p. 53.)

§ 3. Transformation of an *LHJ*-derivation into an equivalent *NJ*-derivation

From an *LHJ*-derivation (V.2) we obtain an *NJ*-derivation (II.2) with the same endformula by transforming the *LHJ*-derivation in the following way: (In this transformation all *D*-formulae of this derivation will reappear as *D*-formulae of the *NJ*-derivation, and they will not depend on any assumption formula. Included further will be other *D*-formulae dependent on assumption formulae.)

3.1. The *LHJ-basic formulae* are replaced by *NJ*-derivations according to the following schemata:

(2.11)
$$\dfrac{\overset{1}{\mathfrak{A}}}{\mathfrak{A} \supset \mathfrak{A}}\ \supset\text{-}I_1$$

(2.12)
$$\dfrac{\dfrac{\overset{1}{\mathfrak{A}}}{\mathfrak{B} \supset \mathfrak{A}}\ \supset\text{-}I}{\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{A})}\ \supset\text{-}I_1$$

(2.13)
$$\dfrac{\dfrac{\dfrac{\overset{1}{\mathfrak{A}} \quad \dfrac{\overset{1}{\mathfrak{A}} \quad \overset{2}{\mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B})}}{\mathfrak{A} \supset \mathfrak{B}}\ \supset\text{-}E}{\mathfrak{B}}\ \supset\text{-}E}{\mathfrak{A} \supset \mathfrak{B}}\ \supset\text{-}I_1}{(\mathfrak{A} \supset (\mathfrak{A} \supset \mathfrak{B})) \supset (\mathfrak{A} \supset \mathfrak{B})}\ \supset\text{-}I_2$$

(2.14)
$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{\mathfrak{B}} \quad \dfrac{\overset{1}{\mathfrak{A}} \quad \overset{3}{\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{C})}}{\mathfrak{B} \supset \mathfrak{C}}\ \supset\text{-}E}{\mathfrak{C}}\ \supset\text{-}E}{\mathfrak{A} \supset \mathfrak{C}}\ \supset\text{-}I_1}{\mathfrak{B} \supset (\mathfrak{A} \supset \mathfrak{C})}\ \supset\text{-}I_2}{(\mathfrak{A} \supset (\mathfrak{B} \supset \mathfrak{C})) \supset (\mathfrak{B} \supset (\mathfrak{A} \supset \mathfrak{C}))}\ \supset\text{-}I_3$$

(2.15)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\mathfrak{A}} \quad \overset{3}{\mathfrak{A} \supset \mathfrak{B}}}{\mathfrak{B}}\ \supset\text{-}E \qquad \overset{2}{\mathfrak{B} \supset \mathfrak{C}}}{\mathfrak{C}}\ \supset\text{-}E}{\mathfrak{A} \supset \mathfrak{C}}\ \supset\text{-}I_1}{(\mathfrak{B} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset \mathfrak{C})}\ \supset\text{-}I_2}{(\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{B} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset \mathfrak{C}))}\ \supset\text{-}I_3$$

(2.21)
$$\dfrac{\dfrac{\overset{1}{\mathfrak{A} \,\&\, \mathfrak{B}}}{\mathfrak{A}}\ \&\text{-}E}{(\mathfrak{A} \,\&\, \mathfrak{B}) \supset \mathfrak{A}}\ \supset\text{-}I_1$$

2.22, 2.31, 2.32, 2.51 and 2.52 are dealt with analogously to 2.21.

(2.23)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\mathfrak{A}} \quad \overset{3}{\mathfrak{A} \supset \mathfrak{B}}}{\mathfrak{B}}\ \supset\text{-}E \qquad \dfrac{\overset{1}{\mathfrak{A}} \quad \overset{2}{\mathfrak{A} \supset \mathfrak{C}}}{\mathfrak{C}}\ \supset\text{-}E}{\mathfrak{B} \,\&\, \mathfrak{C}}\ \&\text{-}I}{\mathfrak{A} \supset (\mathfrak{B} \,\&\, \mathfrak{C})}\ \supset\text{-}I_1}{(\mathfrak{A} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset (\mathfrak{B} \,\&\, \mathfrak{C}))}\ \supset\text{-}I_2}{(\mathfrak{A} \supset \mathfrak{B}) \supset ((\mathfrak{A} \supset \mathfrak{C}) \supset (\mathfrak{A} \supset (\mathfrak{B} \,\&\, \mathfrak{C})))}\ \supset\text{-}I_3$$

(2.33)
$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{2}{\mathfrak{A}\lor\mathfrak{B}} \qquad \dfrac{\overset{1}{\mathfrak{A}}\quad \overset{4}{\mathfrak{A}\supset\mathfrak{C}}}{\mathfrak{C}}\supset\text{-}E \qquad \dfrac{\overset{1}{\mathfrak{B}}\quad\overset{3}{\mathfrak{B}\supset\mathfrak{C}}}{\mathfrak{C}}\supset\text{-}E}{\mathfrak{C}}\lor\text{-}E_1}{(\mathfrak{A}\lor\mathfrak{B})\supset\mathfrak{C}}\supset\text{-}E_2}{(\mathfrak{B}\supset\mathfrak{C})\supset((\mathfrak{A}\lor\mathfrak{B})\supset\mathfrak{C})}\supset\text{-}E_3}{(\mathfrak{A}\supset\mathfrak{C})\supset((\mathfrak{B}\supset\mathfrak{C})\supset((\mathfrak{A}\lor\mathfrak{B})\supset\mathfrak{C}))}\supset\text{-}E_1$$

(2.41)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\mathfrak{A}}\quad\overset{3}{\mathfrak{A}\supset\mathfrak{B}}}{\mathfrak{B}}\supset\text{-}E \qquad \dfrac{\overset{1}{\mathfrak{A}}\quad\overset{2}{\mathfrak{A}\supset\neg\,\mathfrak{B}}}{\neg\,\mathfrak{B}}\supset\text{-}E}{\text{⅄}}\neg\text{-}E}{\neg\,\mathfrak{A}}\neg\text{-}I_1}{(\mathfrak{A}\supset\neg\,\mathfrak{B})\supset\neg\,\mathfrak{A}}\supset\text{-}I_2}{(\mathfrak{A}\supset\mathfrak{B})\supset((\mathfrak{A}\supset\neg\,\mathfrak{B})\supset\neg\,\mathfrak{A})}\supset\text{-}I_3$$

(2.42)
$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{1}{\mathfrak{A}}\quad\overset{2}{\neg\,\mathfrak{A}}}{\text{⅄}}\neg\text{-}E}{\mathfrak{B}}}{\mathfrak{A}\supset\mathfrak{B}}\supset\text{-}I_1}{(\neg\,\mathfrak{A})\supset(\mathfrak{A}\supset\mathfrak{B})}\supset\text{-}I_2\,.$$

3.2. The *LHJ-inference figures* are replaced by sections of an *NJ*-derivation according to the following schemata:

$\dfrac{\mathfrak{A}\quad\mathfrak{A}\supset\mathfrak{B}}{\mathfrak{B}}$ remains as it is, since it has already the form of a $\supset$–E.

$\dfrac{\mathfrak{A}\supset\mathfrak{F}a}{\mathfrak{A}\supset\forall\mathfrak{x}\,\mathfrak{F}\mathfrak{x}}$ becomes: $\dfrac{\dfrac{\dfrac{\overset{1}{\mathfrak{A}}\quad\mathfrak{A}\supset\mathfrak{F}a}{\mathfrak{F}a}\supset\text{-}E}{\forall\mathfrak{x}\,\mathfrak{F}\mathfrak{x}}\forall\text{-}I}{\mathfrak{A}\supset\forall\mathfrak{x}\,\mathfrak{F}\mathfrak{x}}\supset\text{-}I_1\,.$

$$\frac{\mathfrak{F}a \supset \mathfrak{A}}{(\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}) \supset \mathfrak{A}} \quad \text{becomes:} \quad \cfrac{\cfrac{\overset{1}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}} \qquad \cfrac{\overset{2}{\mathfrak{F}a} \quad \mathfrak{F}a \supset \mathfrak{A}}{\mathfrak{A}}\ {\supset}\text{-}E}{\mathfrak{A}}\ \exists\text{-}E_2}{(\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}) \supset \mathfrak{A}}\ {\supset}\text{-}I_1 .$$

The restriction on variables for $\forall$–I and $\exists$–E is satisfied, as is easily seen, by virtue of the restriction on variables existing for *LHJ*-inference figures.

This completes the transformation of an *LHJ*-derivation into an equivalent *NJ*-derivation.

§ 4. Transformation of an *NJ*-derivation into an equivalent *LJ*-derivation

4.1. We proceed as follows: First we *replace* every *D*-formula of the *NJ*-derivation by the following sequent (cf. III.1.1): In its succedent only the formula itself occurs; in its antecedent occur the assumption formulae upon which the sequent depended, and they occur in the same order from left to right as they did in the *NJ*-derivation. (It is presumably clear what is meant by the order from left to right of the initial formulae of a figure in tree form.)

We then replace every occurrence of the symbol $\curlywedge$ by $A \,\&\, \neg A$. (The formula resulting from A in this way will be designated by A^*.)

4.2. We thus already have a system of sequents in tree form. The antecedent of the endsequent is empty (II.2.2); it is obviously equivalent to the endsequent of the *NJ*-derivation. The initial sequents all have the form $\mathfrak{D}^* \rightarrow \mathfrak{D}^*$ (II.2.2) and are thus already basic sequents of an *LJ*-derivation.

The figures formed from *NJ-inference figures* are transformed into sections of an *LJ*-derivation according to the following schemata:

4.21. The inference figures $\vee$–I, $\forall$–I, and $\exists$–I have become *LJ*-inference figures as a result of the substitution performed. (In the case of a $\forall$–I, the *LJ*-restriction on variables is satisfied by virtue of the *NJ*-restriction on variables.)

4.22. A &–I became:

$$\frac{\Gamma \rightarrow \mathfrak{A}^* \qquad \Delta \rightarrow \mathfrak{B}^*}{\Gamma, \Delta \rightarrow \mathfrak{A}^* \,\&\, \mathfrak{B}^*}.$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \to \mathfrak{A}^*}{\overline{\Gamma, \Delta \to \mathfrak{A}^*}}\ \text{possibly several interchanges and contractions} \quad \dfrac{\Delta \to \mathfrak{B}^*}{\overline{\Gamma, \Delta \to \mathfrak{B}^*}}\ \text{possibly several thinnings}}{\Gamma, \Delta \to \mathfrak{A}^* \,\&\, \mathfrak{B}^*}\ \&\text{-}IS.$$

4.23. A $\supset$–I became:

$$\frac{\Gamma_1, \mathfrak{A}^*, \Gamma_2, \ldots, \mathfrak{A}^*, \Gamma_\rho \to \mathfrak{B}^*}{\Gamma_1, \Gamma_2, \ldots, \Gamma_\rho \to \mathfrak{A}^* \supset \mathfrak{B}^*}.$$

This we transform into:

$$\dfrac{\dfrac{\Gamma_1, \mathfrak{A}^*, \Gamma_2, \ldots, \mathfrak{A}^*, \Gamma_\rho \to \mathfrak{B}^*}{\overline{\mathfrak{A}^*, \Gamma_1, \Gamma_2, \ldots, \Gamma_\rho \to \mathfrak{B}^*}}\ \text{possibly several interchanges and contractions, sometimes a thinning}}{\Gamma_1, \Gamma_2, \ldots, \Gamma_\rho \to \mathfrak{A}^* \supset \mathfrak{B}^*}\ \supset\text{-}IS.$$

4.24. The same procedure applies to a $\neg$–I. Finally, we still have to consider the figure

$$\frac{\mathfrak{A}^*, \Gamma \to A \,\&\, \neg A}{\Gamma \to \neg \mathfrak{A}^*}.$$

First we derive $A \,\&\, \neg A \to$ in the calculus LJ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{\neg A, A \to}\ \neg\text{-}IA}{A \,\&\, \neg A, A \to}\ \&\text{-}IA}{A, A \,\&\, \neg A \to}\ \text{interchange}}{A \,\&\, \neg A, A \,\&\, \neg A \to}\ \&\text{-}IA}{A \,\&\, \neg A \to}\ \text{contraction.}$$

By including this sequent, the figure in question is transformed as follows:

$$\dfrac{\dfrac{\mathfrak{A}^*, \Gamma \to A \,\&\, \neg A \qquad A \,\&\, \neg A \to}{\mathfrak{A}^*, \Gamma \to}\ \text{cut}}{\Gamma \to \neg \mathfrak{A}^*}\ \neg\text{-}IS.$$

4.25. By substitution (4.1) the NJ-inference figure $\dfrac{\curlywedge}{\mathfrak{D}}$ became:

$$\frac{\Gamma \to A \,\&\, \neg A}{\Gamma \to \mathfrak{D}^*}.$$

This is transformed into:

$$\cfrac{\cfrac{\Gamma \rightarrow A \,\&\, \neg A \qquad A \,\&\, \neg A \rightarrow}{\Gamma \rightarrow}\ \text{cut}}{\Gamma \rightarrow \mathfrak{D}^*}\ \text{thinning.}$$

The derivation for $A \,\&\, \neg A \rightarrow$, as presented in 4.24, should here still be written above that sequent.

4.26. A $\forall$–E became:

$$\frac{\Gamma \rightarrow \forall \mathfrak{x}\, \mathfrak{F}^*\mathfrak{x}}{\Gamma \rightarrow \mathfrak{F}^*\mathfrak{a}}.$$

This is transformed into:

$$\cfrac{\Gamma \rightarrow \forall \mathfrak{x}\, \mathfrak{F}^*\mathfrak{x} \qquad \cfrac{\mathfrak{F}^*\mathfrak{a} \rightarrow \mathfrak{F}^*\mathfrak{a}}{\forall \mathfrak{x}\, \mathfrak{F}^*\mathfrak{x} \rightarrow \mathfrak{F}^*\mathfrak{a}}\ \forall\text{–}IA}{\Gamma \rightarrow \mathfrak{F}^*\mathfrak{a}}\ \text{cut.}$$

4.27. The same method is used for &–E.

4.28. A $\supset$–E became:

$$\frac{\Gamma \rightarrow \mathfrak{A}^* \qquad \Delta \rightarrow \mathfrak{A}^* \supset \mathfrak{B}^*}{\Gamma, \Delta \rightarrow \mathfrak{B}^*}.$$

This is transformed into:

$$\cfrac{\cfrac{\Delta \rightarrow \mathfrak{A}^* \supset \mathfrak{B}^* \qquad \cfrac{\Gamma \rightarrow \mathfrak{A}^* \qquad \mathfrak{B}^* \rightarrow \mathfrak{B}^*}{\mathfrak{A}^* \supset \mathfrak{B}^*, \Gamma \rightarrow \mathfrak{B}^*}\ \supset\text{–}IA}{\Delta, \Gamma \rightarrow \mathfrak{B}^*}\ \text{cut}}{\Gamma, \Delta \rightarrow \mathfrak{B}^*}\ \text{possibly several interchanges.}$$

4.29. A $\neg$–E became:

$$\frac{\Gamma \rightarrow \mathfrak{A}^* \qquad \Delta \rightarrow \neg \mathfrak{A}^*}{\Gamma, \Delta \rightarrow A \,\&\, \neg A}.$$

This is transformed into:

$$\cfrac{\cfrac{\cfrac{\Delta \rightarrow \neg \mathfrak{A}^* \qquad \cfrac{\Gamma \rightarrow \mathfrak{A}^*}{\neg \mathfrak{A}^*, \Gamma \rightarrow}\ \neg\text{–}IA}{\Delta, \Gamma \rightarrow}\ \text{cut}}{\Gamma, \Delta \rightarrow}\ \text{possibly several interchanges}}{\Gamma, \Delta \rightarrow A \,\&\, \neg A}\ \text{thinning.}$$

4.2.10. $\vee$–*E*. Both right-hand upper sequents are followed up, as in the case of a $\supset$–*I* and $\neg$–*I* (4.23) above, by interchanges, contractions, and thinnings (wherever necessary) so that in each case the result is a sequent in whose antecedent occurs a formula of the form $\mathfrak{A}^*$ or $\mathfrak{B}^*$ at the beginning (whereas the original assumption formulae involved have been absorbed into the rest of the antecedent). Then follows:

$$\dfrac{\Xi \to \mathfrak{A}^* \vee \mathfrak{B}^* \qquad \dfrac{\dfrac{\mathfrak{A}^*, \Gamma \to \mathfrak{C}^*}{\mathfrak{A}^*, \Gamma, \Delta \to \mathfrak{C}^*}\text{ possibly several thinnings and interchanges} \qquad \dfrac{\mathfrak{B}^*, \Delta \to \mathfrak{C}^*}{\mathfrak{B}^*, \Gamma, \Delta \to \mathfrak{C}^*}\text{ possibly several thinnings and interchanges}}{\mathfrak{A}^* \vee \mathfrak{B}^*, \Gamma, \Delta \to \mathfrak{C}^*}\vee\text{–}IA}{\Xi, \Gamma, \Delta \to \mathfrak{C}^*}\text{ cut.}$$

4.2.11. A $\exists$–*E* is treated quite similarly: First we move $\mathfrak{F}^*\mathfrak{a}$ in the right-hand upper sequent to the beginning of the antecedent (cf. 4.23); then follows:

$$\dfrac{\Delta \to \exists\mathfrak{x}\,\mathfrak{F}^*\mathfrak{x} \qquad \dfrac{\mathfrak{F}^*\mathfrak{a}, \Gamma \to \mathfrak{C}^*}{\exists\mathfrak{x}\,\mathfrak{F}^*\mathfrak{x}, \Gamma \to \mathfrak{C}^*}\exists\text{–}IA}{\Delta, \Gamma \to \mathfrak{C}^*}\text{ cut.}$$

The *LJ*-restriction on variables for $\exists$–*IA* is satisfied by virtue of the *NJ*-restriction on variables for $\exists$–*E*.

This completes the transformation of an *NJ*-derivation into an equivalent *LJ*-derivation.

§ 5. Transformation of an *LJ*-derivation into an equivalent *LHJ*-derivation

This transformation is a little more difficult than the two previous ones. We shall carry it out in a number of separate steps.

Preliminary remark: Contractions and interchanges in the succedent do not occur in the calculus *LJ*, since they require the occurrence of at least two *S*-formulae in the succedent.

5.1. We first introduce *new basic sequents* in place of the figures &–*IA*, $\vee$–*IS*, $\forall$–*IA*, $\exists$–*IS*, $\neg$–*IA*, and $\supset$–*IA*; these are to be formed according to the following schemata (rule of replacement as in III.1.2 – the same rule will always apply below; in addition to the letters $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{D}$, and $\mathfrak{E}$ we shall also, incidentally, use the letters, $\mathfrak{C}$, $\mathfrak{H}$, and $\mathfrak{F}$):

$\mathfrak{B}\mathfrak{s}1$: $\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{A}$	$\mathfrak{B}\mathfrak{s}2$: $\mathfrak{A} \,\&\, \mathfrak{B} \to \mathfrak{B}$
$\mathfrak{B}\mathfrak{s}3$: $\mathfrak{A} \to \mathfrak{A} \vee \mathfrak{B}$	$\mathfrak{B}\mathfrak{s}4$: $\mathfrak{B} \to \mathfrak{A} \vee \mathfrak{B}$
$\mathfrak{B}\mathfrak{s}5$: $\forall\mathfrak{x}\,\mathfrak{F}\mathfrak{x} \to \mathfrak{F}\mathfrak{a}$	$\mathfrak{B}\mathfrak{s}6$: $\mathfrak{F}\mathfrak{a} \to \exists\mathfrak{x}\,\mathfrak{F}\mathfrak{x}$
$\mathfrak{B}\mathfrak{s}7$: $\neg\,\mathfrak{A}, \mathfrak{A} \to$	$\mathfrak{B}\mathfrak{s}8$: $\mathfrak{A} \supset \mathfrak{B}, \mathfrak{A} \to \mathfrak{B}$.

Thus in the LJ-derivation to be considered, we transform the inference figures concerned in the following way:

A &–IA becomes:

$$\dfrac{\begin{array}{c}\mathfrak{Bz}1\\ \mathfrak{A}\,\&\,\mathfrak{B}\to\mathfrak{A}\end{array}\qquad \mathfrak{A},\Gamma\to\Theta}{\mathfrak{A}\,\&\,\mathfrak{B},\Gamma\to\Theta}\text{ cut.}$$

The other form of the &–IA is transformed correspondingly, so is every &–IA.

$\vee$–IS and $\exists$–IS are dealt with symmetrically.

A $\neg$–IA becomes:

$$\begin{array}{c}\dfrac{\Gamma\to\mathfrak{A}\qquad \dfrac{\begin{array}{c}\mathfrak{Bz}7\\ \neg\mathfrak{A},\mathfrak{A}\to\end{array}}{\mathfrak{A},\neg\mathfrak{A}\to}\text{ interchange}}{\Gamma,\neg\mathfrak{A}\to}\text{ cut}\\ \hline\hline \neg\mathfrak{A},\Gamma\to\end{array}\text{ possibly several interchanges.}$$

(The Θ in the schema of $\neg$–IA (III.1.22) must be empty by virtue of the LJ-restrictions on succedents; the same holds for the $\supset$–IA.)

A $\supset$–IA becomes:

$$\begin{array}{c}\dfrac{\dfrac{\Gamma\to\mathfrak{A}\qquad \dfrac{\begin{array}{c}\mathfrak{Bz}8\\ \mathfrak{A}\supset\mathfrak{B},\mathfrak{A}\to\mathfrak{B}\end{array}}{\mathfrak{A},\mathfrak{A}\supset\mathfrak{B}\to\mathfrak{B}}\text{ interchange}}{\Gamma,\mathfrak{A}\supset\mathfrak{B}\to\mathfrak{B}}\text{ cut}\qquad \mathfrak{B},\Delta\to\Lambda}{\Gamma,\mathfrak{A}\supset\mathfrak{B},\Delta\to\Lambda}\text{ cut}\\ \hline\hline \mathfrak{A}\supset\mathfrak{B},\Gamma,\Delta\to\Lambda\end{array}\text{ possibly several interchanges.}$$

5.2. We now write the formula $A\,\&\,\neg A$ in the succedent of all D-formulae whose *succedent is empty*.

In doing so the basic sequents of the form $\mathfrak{D}\to\mathfrak{D}$, as well as $\mathfrak{Bz}1$ to $\mathfrak{Bz}6$ and $\mathfrak{Bz}8$, also the figures &–IS, $\forall$–IS, and $\supset$–IS, remain unchanged. The other basic sequents and inference figures are transformed into new basic sequents and inference figures according to the following schemata:

$\mathfrak{Bz}9$: $\mathfrak{A},\neg\mathfrak{A}\to\mathfrak{H}$

$$\mathfrak{Jf}1:\ \frac{\Gamma\to\mathfrak{H}}{\mathfrak{D},\Gamma\to\mathfrak{H}}\qquad\qquad \mathfrak{Jf}2:\ \frac{\mathfrak{D},\mathfrak{D},\Gamma\to\mathfrak{H}}{\mathfrak{D},\Gamma\to\mathfrak{H}}$$

$$\mathfrak{Jf}3: \frac{\varDelta, \mathfrak{D}, \mathfrak{E}, \varGamma \to \mathfrak{H}}{\varDelta, \mathfrak{E}, \mathfrak{D}, \varGamma \to \mathfrak{H}} \qquad \mathfrak{Jf}4: \frac{\varGamma \to A \mathbin{\&} \neg A}{\varGamma \to \mathfrak{D}}$$

$$\mathfrak{Jf}5: \frac{\varGamma \to \mathfrak{D} \qquad \mathfrak{D}, \varDelta \to \mathfrak{H}}{\varGamma, \varDelta \to \mathfrak{H}} \qquad \mathfrak{Jf}6: \frac{\mathfrak{A}, \varGamma \to \mathfrak{H} \qquad \mathfrak{B}, \varGamma \to \mathfrak{H}}{\mathfrak{A} \lor \mathfrak{B}, \varGamma \to \mathfrak{H}}$$

$$\mathfrak{Jf}7: \frac{\mathfrak{F}\mathfrak{a}, \varGamma \to \mathfrak{H}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x}, \varGamma \to \mathfrak{H}} \qquad \mathfrak{Jf}8: \frac{\mathfrak{A}, \varGamma \to A \mathbin{\&} \neg A}{\varGamma \to \neg \mathfrak{A}}\,.$$

(For $\mathfrak{Jf}7$ the reexists the following restriction on variables: The free object variable designated by $\mathfrak{a}$ must not occur in the lower sequent.)

5.3. The inference figure $\mathfrak{Jf}4$ is now replaceable by other figures as follows (this is mainly due to our having kept general the form of the schema $\mathfrak{Bz}9$):

$$\begin{array}{c}
\dfrac{\dfrac{\varGamma \to A \mathbin{\&} \neg A \qquad \overset{\mathfrak{Bz}1}{A \mathbin{\&} \neg A \to A}}{\varGamma \to A}\,\mathfrak{Jf}5 \qquad \begin{array}{c} \dfrac{\dfrac{\varGamma \to A \mathbin{\&} \neg A \qquad \overset{\mathfrak{Bz}2}{A \mathbin{\&} \neg A \to \neg A}}{\varGamma \to \neg A}\,\mathfrak{Jf}5 \qquad \overset{\mathfrak{Bz}9}{\neg A, A \to \mathfrak{D}}}{\varGamma, A \to \mathfrak{D}}\,\mathfrak{Jf}5 \\ \hline\hline A, \varGamma \to \mathfrak{D} \end{array} \text{possibly several } \mathfrak{Jf}3\text{'s}}{\begin{array}{c} \varGamma, \varGamma \to \mathfrak{D} \\ \hline\hline \varGamma \to \mathfrak{D} \end{array} \text{possibly several } \mathfrak{Jf}2\text{'s and } \mathfrak{Jf}3\text{'s.}}\,\mathfrak{Jf}5
\end{array}$$

In a similar way we replace the inference figure $\mathfrak{Jf}8$ (wherever it occurs in the derivation), only this time we use a new inference figure according to the following schema:

$$\mathfrak{Jf}9: \frac{\varGamma, \mathfrak{A} \to A \qquad \varGamma, \mathfrak{A} \to \neg A}{\varGamma \to \neg A}$$

We substitute as follows (in place of $\mathfrak{Jf}8$):

$$\dfrac{\begin{array}{c} \dfrac{\mathfrak{A}, \varGamma \to A \mathbin{\&} \neg A \qquad \overset{\mathfrak{Bz}1}{A \mathbin{\&} \neg A \to A}}{\mathfrak{A}, \varGamma \to A}\,\mathfrak{Jf}5 \\ \hline\hline \varGamma, \mathfrak{A} \to A \end{array} \text{possibly several } \mathfrak{Jf}3\text{'s} \quad \begin{array}{c} \dfrac{\mathfrak{A}, \varGamma \to A \mathbin{\&} \neg A \qquad \overset{\mathfrak{Bz}2}{A \mathbin{\&} \neg A \to \neg A}}{\mathfrak{A}, \varGamma \to \neg A}\,\mathfrak{Jf}5 \\ \hline\hline \varGamma, \mathfrak{A} \to \neg A \end{array} \text{possibly several } \mathfrak{Jf}3\text{'s}}{\varGamma \to \neg \mathfrak{A}}\,\mathfrak{Jf}9\,.$$

5.4. Now we still introduce two *new inference figures schemata*, viz.:

$$\mathfrak{Jf}10: \frac{\varGamma, \mathfrak{A} \to \mathfrak{B}}{\varGamma \to \mathfrak{A} \supset \mathfrak{B}}$$

and its converse:

$$\mathfrak{If}11\colon \frac{\Gamma \to \mathfrak{A} \supset \mathfrak{B}}{\Gamma, \mathfrak{A} \to \mathfrak{B}}.$$

The two types of inference figures are introduced into the derivation in order to enable us to replace a number of other inference figures by more specialized ones (in 5.42 and 5.43).

5.41. To begin with, $\supset$–*IS* inference figures are now replaceable by means of $\mathfrak{If}10$:

A $\supset$–*IS* is transformed into:

$$\begin{array}{cl}
\mathfrak{A}, \Gamma \to \mathfrak{B} & \\
\hline\hline
\Gamma, \mathfrak{A} \to \mathfrak{B} & \text{possibly several } \mathfrak{If}3\text{'s} \\
\hline
\Gamma \to \mathfrak{A} \supset \mathfrak{B} & \mathfrak{If}10.
\end{array}$$

5.42. The inference figures $\mathfrak{If}1$, $\mathfrak{If}2$, $\mathfrak{If}3$, $\mathfrak{If}5$, $\mathfrak{If}6$, and $\mathfrak{If}7$ are then transformed in the following way:

As an example we take an $\mathfrak{If}2$, which is transformed into the following figure (suppose Γ equals $\mathfrak{J}_1, \ldots, \mathfrak{J}_\rho$):

$$\begin{array}{cl}
\mathfrak{D}, \mathfrak{D}, \mathfrak{J}_1, \ldots, \mathfrak{J}_\rho \to \mathfrak{H} & \\
\hline
\mathfrak{D}, \mathfrak{D}, \mathfrak{J}_1, \ldots, \mathfrak{J}_{\rho-1} \to \mathfrak{J}_\rho \supset \mathfrak{H} & \mathfrak{If}10 \\
\hline\hline
\mathfrak{D}, \mathfrak{D} \to \mathfrak{J}_1 \supset (\mathfrak{J}_2 \supset \ldots (\mathfrak{J}_\rho \supset \mathfrak{H}).) & \text{several } \mathfrak{If}10\text{'s} \\
\hline
\mathfrak{D} \to \mathfrak{J}_1 \supset (\mathfrak{J}_2 \supset \ldots (\mathfrak{J}_\rho \supset \mathfrak{H}).) & \mathfrak{If}13 \\
\hline\hline
\mathfrak{D}, \mathfrak{J}_1, \ldots, \mathfrak{J}_\rho \to \mathfrak{H} & \text{several } \mathfrak{If}11\text{'s.}
\end{array}$$

We proceed quite analogously with all other figures mentioned, i.e., using $\mathfrak{If}10$ and $\mathfrak{If}11$, we replace them by inference figures according to these schemata:

$$\mathfrak{If}12\colon \frac{\to \mathfrak{C}}{\mathfrak{D} \to \mathfrak{C}} \qquad \mathfrak{If}13\colon \frac{\mathfrak{D}, \mathfrak{D} \to \mathfrak{C}}{\mathfrak{D} \to \mathfrak{C}} \qquad \mathfrak{If}14\colon \frac{\Delta, \mathfrak{D}, \mathfrak{E} \to \mathfrak{C}}{\Delta, \mathfrak{E}, \mathfrak{D} \to \mathfrak{C}}$$

$$\mathfrak{If}15\colon \frac{\Gamma \to \mathfrak{D} \qquad \mathfrak{D} \to \mathfrak{C}}{\Gamma \to \mathfrak{C}} \qquad \mathfrak{If}16\colon \frac{\mathfrak{A} \to \mathfrak{C} \qquad \mathfrak{B} \to \mathfrak{C}}{\mathfrak{A} \vee \mathfrak{B} \to \mathfrak{C}} \qquad \mathfrak{If}17\colon \frac{\mathfrak{F}a \to \mathfrak{C}}{\exists \mathfrak{x}\, \mathfrak{F}\mathfrak{x} \to \mathfrak{C}}.$$

(For $\mathfrak{If}17$ there exists a restriction on variables: The free object variable designated by a must not occur in the lower sequent.)

5.43. In a similar way we also replace the inference figures $\mathfrak{If}9$, $\mathfrak{If}13$, and $\mathfrak{If}14$ by the following (using $\mathfrak{If}10$ and $\mathfrak{If}11$):

$$\mathfrak{If}18: \frac{\Gamma \to \mathfrak{A} \supset A \qquad \Gamma \to \mathfrak{A} \supset \neg A}{\Gamma \to \neg \mathfrak{A}} \qquad \mathfrak{If}19: \frac{\to \mathfrak{D} \supset (\mathfrak{D} \supset \mathfrak{C})}{\to \mathfrak{D} \supset \mathfrak{C}}$$

$$\mathfrak{If}20: \frac{\Delta \to \mathfrak{D} \supset (\mathfrak{E} \supset \mathfrak{C})}{\Delta \to \mathfrak{E} \supset (\mathfrak{D} \supset \mathfrak{C})}.$$

The basic sequents $\mathfrak{Bs}8$ and $\mathfrak{Bs}9$ may be replaced in the same way by: $\mathfrak{A} \supset \mathfrak{B} \to \mathfrak{A} \supset \mathfrak{B}$, this form falls under the schema $\mathfrak{D} \to \mathfrak{D}$; as well as $\mathfrak{Bs}10$: $\neg \mathfrak{A} \to \mathfrak{A} \supset \mathfrak{H}$.

5.5. Now comes the *final step*:

Every D-sequent

$$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \to \mathfrak{B}$$

is replaced by the formula $(\mathfrak{A}_1 \,\&\, \ldots \,\&\, \mathfrak{A}_\mu) \supset \mathfrak{B}$.

(If the $\mathfrak{A}$'s are empty, we mean $\mathfrak{B}$. An empty succedent no longer occurs, according to 5.2.)

All basic sequents (viz. $\mathfrak{D} \to \mathfrak{D}$, $\mathfrak{Bs}1$ to $\mathfrak{Bs}6$, $\mathfrak{Bs}10$) are thus transformed into LHJ-basic sequents.

Of the inference figures, $\forall$–IS and $\mathfrak{If}17$ are also transformed into LHJ-inference figures. ($\forall$–IS, however, forms an exception if Γ is empty. In that case we first derive (in the LHJ-calculus) $(A \supset A) \supset \mathfrak{F}\mathfrak{a}$ from $\mathfrak{F}\mathfrak{a}$ by means of 2.12, and by then applying the LHJ-inference figure, we finally obtain $\forall \mathfrak{x}\, \mathfrak{F}\mathfrak{x}$ once again by means of 2.11.)

The figures obtained from the remaining inference figures (which are &–IS, $\mathfrak{If}10$, 11, 12, 15, 16, 18, 19, 20) by substitution, are turned into sections of an LHJ-derivation in the following way:

An &–IS has become (suppose first that Γ is not empty):

$$\frac{\mathfrak{C} \supset \mathfrak{A} \qquad \mathfrak{C} \supset \mathfrak{B}}{\mathfrak{C} \supset (\mathfrak{A} \,\&\, \mathfrak{B})}.$$

This is transformed into:

$$\frac{\mathfrak{C} \supset \mathfrak{B} \qquad \dfrac{\mathfrak{C} \supset \mathfrak{A} \qquad (\mathfrak{C} \supset \mathfrak{A}) \supset ((\mathfrak{C} \supset \mathfrak{B}) \supset (\mathfrak{C} \supset (\mathfrak{A} \,\&\, \mathfrak{B})))}{(\mathfrak{C} \supset \mathfrak{B}) \supset (\mathfrak{C} \supset (\mathfrak{A} \,\&\, \mathfrak{B}))}}{\mathfrak{C} \supset (\mathfrak{A} \,\&\, \mathfrak{B})}.$$

If Γ is empty, we proceed as in the case of $\forall$–IS.

The figures obtained from $\mathfrak{If}12$, 15, 16, and 19 by substitution are dealt with quite analogously using basic formulae according to the schemata 2.12, 2.15, 2.33, and 2.13.

In a similar way $\mathfrak{I}\mathfrak{f}18$ and $\mathfrak{I}\mathfrak{f}20$ are dealt with by means of 2.41 and 2.14 and by the application of 2.15 and 2.14, 2.13.

The only figures now left are those having resulted from $\mathfrak{I}\mathfrak{f}10$ and $\mathfrak{I}\mathfrak{f}11$. Both are trivial for an empty Γ, hence suppose that Γ is not empty. In that case we transform these figures into sections of *LHJ*-derivations as follows:

($\mathfrak{I}\mathfrak{f}10$): From $(\mathfrak{C} \,\&\, \mathfrak{A}) \supset \mathfrak{B}$ we have to derive $\mathfrak{C} \supset (\mathfrak{A} \supset \mathfrak{B})$. Now 2.23 together with 2.11 yields: $(\mathfrak{C} \supset \mathfrak{A}) \supset (\mathfrak{C} \supset (\mathfrak{C} \,\&\, \mathfrak{A}))$. This together with $(\mathfrak{C} \,\&\, \mathfrak{A}) \supset \mathfrak{B}$ and 2.15, 2.14 yields $(\mathfrak{C} \supset \mathfrak{A}) \supset (\mathfrak{C} \supset \mathfrak{B})$, and from this formula together with 2.12, 2.15 yields $\mathfrak{A} \supset (\mathfrak{C} \supset \mathfrak{B})$, and by 2.14 $\mathfrak{C} \supset (\mathfrak{A} \supset \mathfrak{B})$ results.

($\mathfrak{I}\mathfrak{f}11$): From $\mathfrak{C} \supset (\mathfrak{A} \supset \mathfrak{B})$ we derive $(\mathfrak{C} \,\&\, \mathfrak{A}) \supset \mathfrak{B}$ in the *LHJ*-calculus as follows: 2.21 and 2.22 yield $(\mathfrak{C} \,\&\, \mathfrak{A}) \supset \mathfrak{C}$ and $(\mathfrak{C} \,\&\, \mathfrak{A}) \supset \mathfrak{A}$; and from this together with $\mathfrak{C} \supset (\mathfrak{A} \supset \mathfrak{B})$, we obtain $(\mathfrak{C} \,\&\, \mathfrak{A}) \supset \mathfrak{B}$ (by using 2.15, 2.14, 2.15, 2.13).

This completes the transformation of the *LJ*-derivation into an *LHJ*-derivation. Furthermore, the two derivations really are equivalent, since the endsequent of the *LJ*-derivation was affected only by the transformations 5.2 and 5.5, and has thus obviously been transformed into a formula equivalent with it (according to 1.1).

If the results of §§ 3–5 are taken together, the *equivalence of the three calculi LHJ*, *NJ*, and *LJ* is now fully proved.

§ 6. The equivalence of the calculi *LHK*, *NK*, and *LK*

Now that the equivalence of the different *intuitionist* calculi has been proved, it is fairly easy to deduce that of the *classical* calculi.

6.1. In order to transform an *LHK*-derivation into an equivalent *NK*-derivation we proceed exactly as in § 3. The additional basic formulae according to the schema $\mathfrak{A} \vee \neg\, \mathfrak{A}$ remain unchanged, and are thus at once basic formulae of the *NK*-derivation.

6.2. In order to transform an *NK*-derivation into an equivalent *LK*-derivation we proceed initially as in § 4. In this way the additional basic formulae according to the schema $\mathfrak{A} \vee \neg\, \mathfrak{A}$ are transformed into sequents of the form $\rightarrow \mathfrak{A}^* \,\&\, \neg\, \mathfrak{A}^*$. These we then replace by their *LK*-derivations (according to III.1.4). The transformation of an *NK*-derivation into an equivalent *LK*-derivation is thus complete.

6.3. *Transformation of an LK-derivation into an LHK-derivation.*

We introduce an *auxiliary calculus* differing from the *LK*-calculus in the following respect:

Inference figures may be formed according to the schemata III.1.21, III.1.22, but with the following restrictions: Contractions and interchanges in the succedent are not permissible; in the remaining schemata no substitution may be performed on Θ and Λ; these places thus remain empty.

Furthermore, the following two schemata for inference figures are added (rule of replacement as usual: III.1.2):

$$\mathfrak{J}\mathfrak{f}1: \frac{\Gamma \to \mathfrak{A}, \Theta}{\Gamma, \neg \mathfrak{A} \to \Theta}$$

and its converse:

$$\mathfrak{J}\mathfrak{f}2: \frac{\Gamma, \neg \mathfrak{A} \to \Theta}{\Gamma \to \mathfrak{A}, \Theta}.$$

(Thus, here Θ need not be empty.)

6.31. Transformation of an *LK*-derivation into a derivation of the auxiliary calculus:

(The procedure is similar to that in 5.4.)

All inference figures, with the exception of contractions and interchanges in the succedent, are transformed according to the following rule: The upper sequents are followed by inference figures $\mathfrak{J}\mathfrak{f}1$, until all formulae of Θ or Λ have been negated and brought into the antecedent (to the right of Γ or Δ). Then follows an inference figure of the same kind as the one just transformed, which is now actually a permissible inference figure in the auxiliary calculus. (The formulae that have been brought into the antecedent are treated as part of Γ or Δ.) Then follow $\mathfrak{J}\mathfrak{f}2$ inference figures, and Θ and Λ are thus brought back into the succedent. (In the case of the $\supset$–*IA* and the cut, we may first have to carry out interchanges in the antecedent, but these are also permissible inference figures in the auxiliary calculus.)

Now we still have to consider contractions – or interchanges – in the succedent. Here, as in the previous case. the *whole* succedent is negated and brought forward into the antecedent. We then carry out interchanges, a contraction, and further interchanges – or one interchange – in the antecedent, and then the negated formulae are brought back into the succedent (by means of the inference figures $\mathfrak{J}\mathfrak{f}2$).

6.32. Transformation of a derivation of the auxiliary calculus into a derivation of the calculus *LJ* augmented by the inclusion of the basic sequent schema $\to \mathfrak{A} \vee \neg \mathfrak{A}$:

We begin by transforming all *D*-sequents as follows:

$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \to \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$ becomes

$\mathfrak{A}_1, \ldots, \mathfrak{A}_\mu \rightarrow \mathfrak{B}_\nu \vee \ldots \vee \mathfrak{B}_1$. If the succedent was empty, it remains empty.

Now all basic sequents or inference figures of the auxiliary calculus with the exception of the figures $\mathfrak{I}\mathfrak{f}1$ and $\mathfrak{I}\mathfrak{f}2$, have thus already become basic sequents or inference figures of the calculus *LJ*. This is so since these inference figures have resulted from the schemata III.1.21, III.1.22 (with the exception of the schemata for contraction and interchange in the succedent) by Θ and Λ always having remained empty. At most one formula could therefore occur in the succedent.

Hence we still have to transform the figures which have resulted from the inference figures $\mathfrak{I}\mathfrak{f}1$ and $\mathfrak{I}\mathfrak{f}2$ in the course of the above modification.

6.321. First $\mathfrak{I}\mathfrak{f}1$: If Θ is empty, we replace the inference figure by a $\neg$–*IA*, followed by interchanges in the antecedent. Suppose, therefore, that Θ is not empty, where Θ^* designates the formulae belonging to Θ, in reverse order and connected by $\vee$.

After the transformation of the succedents, the inference figure in that case runs as follows:

$$\frac{\Gamma \rightarrow \Theta^* \vee \mathfrak{A}}{\Gamma, \neg \mathfrak{A} \rightarrow \Theta^*}.$$

This is transformed into the following section of an *LJ*-derivation:

$$\dfrac{\Gamma \rightarrow \Theta^* \vee \mathfrak{A} \qquad \dfrac{\dfrac{\dfrac{\Theta^* \rightarrow \Theta^*}{\neg \mathfrak{A}, \Theta^* \rightarrow \Theta^*}\text{ thinning}}{\Theta^*, \neg \mathfrak{A} \rightarrow \Theta^*}\text{ interchange} \qquad \dfrac{\dfrac{\dfrac{\mathfrak{A} \rightarrow \mathfrak{A}}{\neg \mathfrak{A}, \mathfrak{A} \rightarrow}\ \neg\text{–}IA}{\mathfrak{A}, \neg \mathfrak{A} \rightarrow}\text{ interchange}}{\mathfrak{A}, \neg \mathfrak{A} \rightarrow \Theta^*}\text{ thinning}}{\Theta^* \vee \mathfrak{A}, \neg \mathfrak{A} \rightarrow \Theta^*}\ \vee\text{–}IA}{\Gamma, \neg \mathfrak{A} \rightarrow \Theta^*}\text{ cut.}$$

6.322. After the transformation of its succedents, an inference figure $\mathfrak{I}\mathfrak{f}2$ runs as follows:

$$\frac{\Gamma, \neg \mathfrak{A} \rightarrow \Theta^*}{\Gamma \rightarrow \Theta^* \vee \mathfrak{A}},$$

where Θ^* has the same meaning as in the previous case. If Θ is empty, assume Θ^* to be empty too, and let $\Theta^* \vee \mathfrak{A}$ mean $\mathfrak{A}$.

It is transformed into the following section of a derivation:

$$
\cfrac{\to \mathfrak{A} \lor \neg\, \mathfrak{A} \qquad \cfrac{\cfrac{\cfrac{\mathfrak{A} \to \mathfrak{A}}{\mathfrak{A}, \Gamma \to \mathfrak{A}}\,\text{possibly several thinnings and interchanges}}{\mathfrak{A}, \Gamma \to \Theta^* \lor \mathfrak{A}}\,\lor\text{-}IS \qquad \cfrac{\cfrac{\Gamma, \neg\, \mathfrak{A} \to \Theta^*}{\neg\, \mathfrak{A}, \Gamma \to \Theta^*}\,\text{possibly several interchanges}}{\neg\, \mathfrak{A}, \Gamma \to \Theta^* \lor \mathfrak{A}}\,\lor\text{-}IS}{\mathfrak{A} \lor \neg\, \mathfrak{A}, \Gamma \to \Theta^* \lor \mathfrak{A}}\,\lor\text{-}IA}{\Gamma \to \Theta^* \lor \mathfrak{A}}\,\text{cut.}
$$

It is easy to see that in the case of an empty Θ all is in order.

6.33. The *LJ*-derivation now obtained, together with the additional basic sequents of the form $\to \mathfrak{A} \lor \neg\, \mathfrak{A}$, may be transformed, as in § 5, into an *LHJ*-derivation with the inclusion of additional basic formulae of the form $\mathfrak{A} \lor \neg\, \mathfrak{A}$ (cf. 5.5), i.e., into an *LHK*-derivation. This completes the transformation of the *LK*-derivation into an *LHK*-derivation. At the same time, the endsequent has been transformed (in accordance with 6.32, 5.2, and 5.5) into an equivalent formula (according to 1.1).

By combining the results of 6.1, 6.2, and 6.3, we have now also proved *the equivalence of the three classical calculi of predicate logic*: *LHK*, *NK*, and *LK*.

4. THE CONSISTENCY OF ELEMENTARY NUMBER THEORY

By '*elementary number theory*' I mean the theory of the natural numbers that does not make use of techniques from analysis such as, e.g., irrational numbers or infinite series.

The aim of the present paper is to prove the *consistency* of elementary number theory or, rather, to reduce the question of consistency to certain general fundamental principles.

How such a consistency proof can be carried out at all and for what reasons it is necessary or at least very desirable to do so will be discussed in section I.

The entire paper can be read without any specialized knowledge.

SECTION I. REFLECTIONS ON THE PURPOSE AND POSSIBILITY OF CONSISTENCY PROOFS

In § 1, I shall consider the question why consistency proofs are *necessary* and, in § 2, how such proofs are *possible*[30]. In doing so, I shall briefly restate those aspects of the problem, already familiar to many readers, which are of particular relevance to the rest of this paper.

§ 1. The antinomies of set theory and their significance for mathematics as a whole[31]

1.1. Mathematics is regarded as the most certain of all the sciences. That it could lead to results which contradict one another seems impossible. This faith in the indubitable certainty of mathematical proofs was sadly shaken around 1900 by the discovery of the '*antinomies* (or 'paradoxes') of set theory'. It turned out that in this specialized branch of mathematics contradictions arise without our being able to recognize any specific error in our reasoning.

Particularly instructive is '*Russell's antinomy*', which I shall now discuss in detail.

1.2. A set is a collection of arbitrary objects ('elements of the set'). An 'empty set', which has no elements at all, is also admitted. We now divide the sets into 'sets of the first kind', i.e., sets which contain *themselves* as an element, and 'sets of the second kind', i.e., sets which do *not* contain themselves as an element.

We consider the set $\mathfrak{m}$ which has for its elements the entire collection of the sets of the second kind. Does this set itself belong to the first or the second kind? Both alternatives are absurd: For if the set $\mathfrak{m}$ belongs to the first kind, i.e., if it contains itself as an element, then this contradicts its definition by which all of its elements were supposed to be sets of the second kind. Suppose, therefore, that the set $\mathfrak{m}$ belongs to the second kind, i.e., that it does not contains itself as an element. Since, by definition, it has all sets of the second kind as elements, it must also contain itself as an element and we have thus once again arrived at a contradiction.

1.3. The result is *Russell's antinomy* which shows how easily an obvious contradiction can result from a small number of admittedly somewhat subtle inferences.

What is the actual *significance* of this fact for *mathematics as a whole*? We may be inclined, at first, to dismiss the entire argument as *unmathematical* by claiming that the concept of a 'set of arbitrary objects' is too vague to count as a mathematical concept.

This objection becomes void if we restrict ourselves to quite specific purely *mathematical* objects by making the following stipulation, for example: The only objects admitted as elements of a 'set' are, first: arbitrary natural numbers (1, 2, 3, 4 etc.); second: arbitrary sets consisting of admissible elements.

Example: The following three elements form an admissible set: First, the number 4; second, the set of all natural numbers; third, the set whose two elements are the number 3 and the set of all natural numbers.

Using this purely *mathematical* concept of a set, we can repeat the above (1.2) argument and obtain the *same contradiction*.

1.4. The fact that we happen to have chosen the natural numbers for our initial objects has obviously no bearing on the emergence of the antinomy. It cannot, therefore, be said that a contradiction has been revealed in the domain of the *natural numbers*; the fault must be sought rather in the *logical inferences* employed.

1.5. It is thus natural to go back to look for a *definite error* in the reasoning that has led to the antinomy. We might, for example, argue that the set $\mathfrak{m}$ was defined by referring to the totality of *all* sets (which was indeed subdivided into sets of the first and second kinds, and where $\mathfrak{m}$ was formed with sets of the second kind). The set was then itself regarded as belonging to this totality, which raised the question of whether it belongs to the first or second kind. Such a procedure is *circular*; it is illicit to define an object by means of a totality and to regard it then as belonging to that totality, so that in some sense it contributes to its own definition ('circulus vitiosus').

We might feel that the *correct* interpretation of the set $\mathfrak{m}$ should rather be the following:

If a *definite totality* of sets is given, then this totality may be subdivided into sets of the first and second kinds. Yet if the sets of the second kind (or alternatively, the first kind) are combined into a new set $\mathfrak{m}$, then that set constitutes something *completely new* and cannot itself be regarded as belonging to that totality.

1.6. The impression, at first sight, that the forms of inference leading to the antinomy seem correct derives from the conception of the concept of a 'set' as something '*actualistically*' ((*an sich*)) determined (and the totality of all sets, therefore, constitutes a predetermined closed totality); the critique advanced against this view implies that new sets can be formed only '*constructively*' so that a new set depends in its construction on already existing sets.

1.7. If we were to think that the antinomy has thus been explained away quite satisfactorily, we must at once face up to a new difficulty: The form of reasoning (the circulus vitiosus) which we have just declared to be inadmissible is being used *in analysis* in a quite similar form in the usual proofs of some rather simple theorems, e.g., the theorem: 'A function which is continuous on a closed interval and is of different sign at the endpoints has a zero in the interval.'

The proof of this result is essentially carried out in the following way: The totality of points in the interval is divided into points of the first and second kinds, so that a point is of the first kind if the function has the same sign for all points to the right of it, up to the end of the interval, and it belongs to the second kind if this is not the case. The limit point defined by this subdivision is then the required zero. It belongs itself to the points of the interval. Hence we have the 'circulus vitiosus': The real number concerned is defined by referring to the *totality* of the real numbers (in an interval) and is then itself regarded as belonging to that totality.

This form of inference is nevertheless considered correct in analysis on the following grounds: The number concerned is, after all, not *newly created* by the given definition, it already *actually* exists within the totality of the real numbers and is merely *singled out* from this totality by its definition.

Yet exactly the same could be said about the antinomy mentioned above: The set $\mathfrak{m}$ is already *actually* present in the totality of all sets (defined at 1.3) and is merely singled out by its definition (at 1.2) from this totality.

Considerable *differences* certainly exist between the forms of inference used to derive the antinomy and those customary in proofs in analysis. Yet we must ask ourselves whether these differences are radical enough to justify the further use of these inferences in analysis – since no contradictions have so far arisen – or whether their similarity with the inferences that have led to the antinomies should not prompt us to eliminate these inferences also from analysis. Here the *opinions* of mathematicians concerned with these questions *diverge*.

1.8. We can indeed *challenge* the correctness of *other* forms of inference customary in mathematics because of certain remote analogies that may be drawn between them and inferences leading to the antinomies. Especially radical in this respect are the '*intuitionists*' (Brouwer), who even object to forms of inference customary in *number theory*, not only because these inferences might possibly lead to *contradictions*, but because the theorems to which they lead have no actual *sense* and are therefore worthless. I shall come back to this point later in greater detail (§§ 9–11 and 17.3).

Less radical are the '*logicists*' (Russell). They draw a line between permissible and non-permissible forms of inference, and the antinomies turn out to be a consequence of a nonpermissible circulus vitiosus. At one time the logicists had also disallowed the inference applied in the example from analysis cited above ('ramified theory of types'), but this inference was later readmitted.

1.9. Altogether we are left with the following picture:

The contradictions (antinomies) which had occurred in set theory, a specialized branch of mathematics, had given rise to further doubts about the correctness of certain forms of inference customary in the rest of mathematics. Various attempts to draw a line between permissible and nonpermissible forms of inference have led to different approaches to the subject.

In order to end this unsatisfactory state of affairs, Hilbert drew up the following *programme*:

The *consistency* of the whole of mathematics, in so far as it actually is consistent, is to be *proved* along exact mathematical lines. This proof is to be carried out by means of forms of inference that are completely unimpeachable ('finitist' forms of inference).

How such a consistency proof is conceivable at all will be discussed more fully in § 2.

In the remainder of this paper, I shall then carry out such a consistency proof for *elementary number theory*. Yet even here we shall meet forms of inference whose closer inspection will give us cause for concern. More about this in section III. One point should however be made clear from the outset: *those* forms of inference which might possibly be considered disputable *hardly ever* occur in actual number-theoretical proofs (11.4); we must therefore not be mislead and, because of the great self-evidence of these proofs, consider a consistency proof as superfluous.

§ 2. How are consistency proofs possible?

2.1. *General remarks about consistency proofs.*

2.11. The consistency of geometries is usually proved by appealing to an *arithmetic* model. Here the consistency of *arithmetic* is therefore *presupposed.* In a similar way we can also effect a reduction of some parts of arithmetic to others, e.g., the theory of the complex numbers to that of the real numbers.

What remains to be proved ultimately is the consistency of the theory of the *natural* numbers (elementary number theory) and the theory of the *real* numbers (analysis) of which the former forms a part; and finally the consistency of set theory as far as that theory is consistent.

2.12. This task is *basically different* and more difficult than that of reducing the consistency of one theory to that of another by mapping the objects of the former theory onto the objects of the latter. Let us look more closely at the situation in the case of the *natural* numbers:

These numbers can obviously not be mapped onto a *simpler domain of objects*. Nor are we indeed concerned with the consistency of the *domain of numbers* itself, i.e., with the consistency of the basic relationships between the numbers as determined by the 'axioms' (e.g., the 'Peano axioms' of number theory). To prove the consistency of these axioms without invoking other equivalent assumptions seems inconceivable. We are concerned rather with the consistency of our *logical reasoning* about the natural numbers (starting from their axioms) as it occurs in the *proofs* of number theory. For it is precisely our logical reasoning which in its unrestricted application leads to the antinomy (1.4). We do not of course consider such general constructions as those of arbitrary sets of sets (1.3) as part of *number theory*. Elementary number theory comprises merely *finite* sets (of natural