McGill University
Department of Mathematics and Statistics
MATH 254 Analysis 1, Fall 2015
Assignment 4

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 3 and 5.**

This assignment is due **Monday, November 2, at the end of the class. Late assignments will not be accepted.**

1. Define a sequence (x_n) recursively by $x_1 = 0$, $x_2 = 1$, $x_{n+2} = \dfrac{1}{2}(x_{n+1} + x_n)$.

   (a) Prove by induction that $x_{2k-1} < x_{2k}$ for all $k \in \mathbb{N}$.

   By part (a) we can define intervals $I_k := [x_{2k-1}, x_{2k}]$ for all $k \in \mathbb{N}$. Prove the following:

   (b) The I_k form a nested sequence of closed and bounded intervals i.e. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$.

   (c) Show that the intervals I_k have exactly one point in common i.e. show that $\displaystyle\bigcap_{k\in\mathbb{N}} I_k = \{x\}$ for some $x \in \mathbb{R}$.

   (d) Show that (x_n) converges and that $\lim (x_n) = x$.

   (It can be shown that $x = \frac{2}{3}$. Proving this is *not* part of this problem.)

2. Use the definition of the limit of a sequence to show that:

   (a) $\lim \left(\dfrac{n^2 - 1}{2n^2 + 3} \right) = \dfrac{1}{2}$ (b) $\lim \left(\dfrac{\sqrt{n}}{n+1} \right) = 0$ (c) $\lim \left(\dfrac{(-1)^n\, n}{n^2 + 1} \right) = 0$

3. Let P_{nk}, $n, k \in \mathbb{N}$, be real numbers satisfying the following:

   (a) $P_{nk} \geq 0$ for all n, k.

   (b) $\sum_{k=1}^{n} P_{nk} = 1$ for all n.

   (c) $\lim_{n\to\infty} P_{nk} = 0$ for all k.

   Let (x_n) be a convergent sequence and let a sequence (y_n) be defined by

$$y_n = \sum_{k=1}^{n} P_{nk} x_k.$$

   Prove that (y_n) is a convergent sequence and that

$$\lim y_n = \lim x_n.$$

4. Let (x_n) be a convergent sequence and let (y_n) be a sequence defined by

$$y_n = \frac{x_1 + \cdots + x_n}{n}.$$

Prove that (y_n) is convergent and that $\lim y_n = \lim x_n$.

5. Let $b \in \mathbb{R}$ with $b > 1$. Prove that $\lim \left(\frac{n}{b^n}\right) = 0$.

6. Prove that $\lim \left(\frac{n!}{n^n}\right) = 0$.