McGill University
Department of Mathematics and Statistics

MATH 254 Analysis 1, Fall 2015
Assignment 3

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 1,2.**

This assignment is due **Monday, October 26, at the end of the class**. **Late assignments will not be accepted.**

1. Let x be a real number. Show that, for any $\varepsilon > 0$, there exist two rationals q and q' such that $q < x < q'$ and $|q - q'| < \varepsilon$.

2. Let A and B be two nonempty subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded above if and only if both A and B are bounded above. If it is the case, prove that $\sup(A \cup B) = \sup(\sup A, \sup B)$.

3. Let S be a nonempty and bounded subset of $\mathbb{R}$.
   (a) Prove that $S \subseteq [\inf S, \sup S]$.
   (b) Prove that if J is a closed interval containing S, then $[\inf S, \sup S] \subseteq J$.

4. For any $n \in \mathbb{N}$ let $I_n = (0, \frac{1}{n})$ and $J_n = [0, \frac{1}{n}]$. Show that $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$ and $\bigcap_{n\in\mathbb{N}} J_n = \{0\}$.

5. If f is a function $f : D \to \mathbb{R}$ one says that f is bounded above (resp. bounded below, bounded) if the image of D under f i.e. $f(D) = \{f(x) : x \in D\}$ is bounded above (resp. bounded below, bounded). If f is bounded above (resp. bounded below), then one denotes by $\sup f$ the supremum of $f(D)$ (resp. by $\inf f$ the infimum of $f(D)$).

   Assume that two functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are bounded above.
   (a) Prove that $f(x) \leq g(x)$ for all $x \in D$ implies $\sup f \leq \sup g$.
   (b) Show that the converse it not true by providing a concrete counterexample.
   (c) Prove that $f(x) \leq g(y)$ for all $x, y \in D$ if and only if $\sup f \leq \inf g$.

6. Define a sequence $(x_n)_{n\in\mathbb{N}}$ by $x_1 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$ for any $n \in \mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ is decreasing and bounded below by $\sqrt{2}$. Prove that $(x_n)_{n\in\mathbb{N}}$ is a sequence of rational numbers converging to $\sqrt{2}$.