MATH323 - Tutorial 7 (Feb. 22nd /17)

Def: The k^{th} moment of a random variable Y taken about the origin is defined to be $\mathbb{E}(Y^k) = \mu_k'$.

The moment-generating function $m(t) := \mathbb{E}(e^{tY})$ for random variable Y exists if there exists a positive constant b such that $m(t)$ is finite for $|t| \leq b$.

Theorem: If $m(t)$ exists, then for any positive integer k,

$$\frac{d^k}{dt^k} m(t) \Big|_{t=0} = m^{(k)}(0) = \mathbb{E}(Y^k) = \mu_k' .$$

1. Derive the moment generating function for the Binomial distribution and use it to calculate the mean and variance.

Solution:

Let $Y \sim \text{Binomial}(n,p)$, then $\mathbb{P}(Y=y) = \binom{n}{y} p^y (1-p)^{n-y}$ where $y \in \{0,1,\dots,n\}$ and $0 \leq p \leq 1$.

$$m(t) = \mathbb{E}(e^{tY}) = \sum_{y=0}^{n} \binom{n}{y} p^y (1-p)^{n-y} e^{ty}$$

$$= \sum_{y=0}^{n} \binom{n}{y} (e^t p)^y (1-p)^{n-y}$$

which by the binomial expansion

$$= (e^t p + (1-p))^n$$

$$m(t) = (1-p+pe^t)^n$$

$$\frac{d}{dt} m(t) = n(1-p+pe^t)^{n-1} pe^t$$

$$\frac{d^2}{dt^2} m(t) = np\left\{ e^t (1-p+pe^t)^{n-1} + e^t (n-1)(1-p+pe^t)^{n-2} pe^t \right\}$$

$$\frac{d}{dt} m(t)\Big|_{t=0} = n(1-p+pe^0)^{n-1} pe^0$$
$$= n(1-p+p)^{n-1} p$$
$$= np = \mathbb{E}(Y)$$

$$\frac{d^2}{dt^2} m(t)\Big|_{t=0} = np\left\{ e^0 (1-p+pe^0)^{n-1} + e^0 (n-1)(1-p+pe^0)^{n-2} pe^0 \right\}$$
$$= np\{1+(n-1)p\} = np(1+np-p)$$
$$= np + n^2p^2 - np^2 = \mathbb{E}(Y^2)$$

$$V(Y) = \mathbb{E}(Y^2) - (\mathbb{E}(Y))^2$$
$$= np + n^2p^2 - np^2 - (np)^2$$
$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2} = np - np^2 = np(1-p) \blacksquare$$

2. (Exercise 3.156, 8): Suppose that Y is a random variable with moment-generating function $m(t)$.

a) what is $m(0)$?

b) If $W = 3Y$, show the moment-generating function of W is $m(3t)$.

c) If $X = Y-2$, show the moment-generating function of X is $e^{-2t} m(t)$.

d) If $Z = aY+b$, show the moment-generating function of Z is $e^{tb} m(at)$.

Solution:

a) $m(t) = \mathbb{E}(e^{tY})$

$\Rightarrow m(0) = \mathbb{E}(e^{0 \cdot Y}) = \mathbb{E}(1) = 1$

b) Let $m_W(t)$ be the mgf of W.

$$m_W(t) = \mathbb{E}(e^{tW}) = \mathbb{E}(e^{t(3Y)}) = \mathbb{E}(e^{(3t)Y})$$
$$= m(3t)$$

c) Let $m_X(t)$ be the mgf of X.

$$m_X(t) = \mathbb{E}(e^{tX}) = \mathbb{E}(e^{t(Y-2)}) = \mathbb{E}(e^{tY-2t})$$
$$= \mathbb{E}(e^{-2t} e^{tY})$$
$$= e^{-2t}\mathbb{E}(e^{tY})$$
$$= e^{-2t} m(t).$$

d) Let $m_Z(t)$ be the mgf of Z.

$$m_Z(t) = \mathbb{E}(e^{tZ}) = \mathbb{E}(e^{t(aY+b)})$$
$$= \mathbb{E}(e^{(at)Y} e^{tb})$$
$$= e^{tb}\, \mathbb{E}(e^{(at)Y})$$
$$= e^{tb}\, m(at) \quad \blacksquare$$

Tchebysheff's Theorem

Let Y be a random variable with mean μ and finite variance σ^2. Then, for any constant $k > 0$,

$$\mathbb{P}(|Y-\mu| < k\sigma) \geq 1 - \frac{1}{k^2} \quad \text{or} \quad \mathbb{P}(|Y-\mu| \geq k\sigma) \leq \frac{1}{k^2}.$$

3. (Exercise 3.170) The U.S. mint produces dimes with an average diameter of 0.5 inch and standard deviation 0.01. Using Tchebysheff's theorem, find a lower bound for the number of coins in a lot of 400 coins that are expected to have a diameter between 0.48 and 0.52.

Solution:

Let Y be the diameter of a coin. Then,

$$\mathbb{P}(0.48 < Y < 0.52)$$
$$= \mathbb{P}(0.48 - 0.50 < Y - 0.50 < 0.52 - 0.50)$$
$$= \mathbb{P}(-0.02 < Y - 0.50 < 0.02)$$
$$= \mathbb{P}(|Y - 0.50| < 0.02)$$
$$= \mathbb{P}(|Y - \underset{\mu}{0.50}| < 2\,(\underset{\sigma}{0.01}))$$

Then by Tchebysheff's theorem,

$$\mathbb{P}(|Y - 0.50| < 2\,(0.01)) \geq 1 - \frac{1}{2^2}$$
$$= 1 - \frac{1}{4}$$
$$= 3/4$$

Thus, a lower bound on the probability is $3/4$ which implies a lower bound for the number of coins out of 400 is $400 \times 3/4 = 300$. $\blacksquare$

A look ahead

	Discrete Random Variable	Continuous Random Variable
Support :	Finite or infinitely countable	Uncountable
Probability :	probability mass function $\mathbb{P}(X=x)=p(x)$	cumulative distribution function $\mathbb{P}(X\leq x) = F(x)$ density function $f(x) = \frac{dF(x)}{dx} = F'(x)$
Total Probability :	$\sum_{supp(X)} p(x) = 1$	$\int_{supp(X)} f(x)dx = F(\infty) = 1$
Expected Value :	$\sum_{supp(X)} xp(x) = \mathbb{E}(X)$	$\int_{supp(X)} x f(x)dx = \mathbb{E}(X)$
Second moment :	$\sum_{supp(X)} x^2 p(x) = \mathbb{E}(X^2)$	$\int_{supp(X)} x^2 f(x)\,dx = \mathbb{E}(X^2)$
Expectation of a function of X :	$\sum_{supp(X)} g(x)p(x) = \mathbb{E}(g(X))$	$\int_{supp(X)} g(x)f(x)\,dx = \mathbb{E}(g(X))$
Moment Generating function :	$m(t) = \sum_{supp(X)} e^{tx} p(x)$	$m(t) = \int_{supp(X)} e^{tx} f(x)dx$
Tchebysheff's Theorem :	Holds	Holds