MATH 248 PROBLEM SET 4

DUE TUESDAY NOVEMBER 22

1. Prove the following.
   (a) Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ be open sets. Then $A \cup B$ and $A \cap B$ are open.
   (b) Let $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ be closed sets. Then $A \cup B$ and $A \cap B$ are closed.
   (c) Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\phi : \Omega \to \mathbb{R}$ be a continuous function. Then the set $U = \{x \in \Omega : \phi(x) > 0\}$ is open.
   (d) Let $C \subset \mathbb{R}^n$ be a closed set, and let $\phi : C \to \mathbb{R}$ be a continuous function. Then the set $K = \{x \in C : \phi(x) \geq 0\}$ is closed. Thus $K_0 = \{x \in C : \phi(x) = 0\}$ is closed.
2. What is the maximum value of the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) = x_1^2 x_2^2 \cdots x_n^2$ on the sphere $S^{n-1} = \{x \in \mathbb{R}^n : x^T x = 1\}$?
3. Let $a \in \mathbb{R}$ and $b \in \mathbb{R}^n$. Find, with full justifications, the maximum and minimum values of the function
$$f(x) = \frac{a + b^T x}{1 + x^T x},$$
over $\mathbb{R}^n$.
4. If a triangle has sides of lengths x, y, z, so its perimeter is $p = x + y + z$, then its area is given by Heron's formula $A = \sqrt{s(s-x)(s-y)(s-z)}$, where $s = \frac{1}{2}p$. Show that, among all triangles with a given perimeter, the one with the largest area is equilateral.
5. In this exercise, we will study the critical points of the function
$$u(x, y) = -\frac{1}{R} - \frac{\mu}{\rho} - \frac{r^2}{(1 + \mu^2)a^3},$$
where $r = \sqrt{x^2 + y^2}$, $\rho = \sqrt{(x-a)^2 + y^2}$, $R = \sqrt{(x + \mu a)^2 + y^2}$, and $a > 0$ and $\mu \geq 0$ are real parameters. This function is defined in $\Omega = \mathbb{R}^2 \setminus \{(-\mu a, 0), (a, 0)\}$, and appears as the effective potential field for a test mass (satellite) moving in the gravitational field generated by a small body (Earth) at $(a, 0)$ and a large body (Sun) at $(-\mu a, 0)$, with respect to the coordinate system that is following the rotation of both bodies. In particular, in this coordinate system, the two bodies are fixed, but the rotation of the coordinate system manifests itself through the centrifugal term $\frac{r^2}{(1+\mu^2)a^3}$. With respect to a non-rotating coordinate system, the two bodies would be circling around their common centre of mass, which is the origin in our rotating coordinate system xy. The parameter μ is assumed to be *small*, and it represents the mass ratio between the bodies.
   (a) Show that u has exactly one critical point of the form $p_1 = (x_1, 0)$ with $x_1 < -\mu a$. Show that $x_1 = x_1(\mu)$ as a function μ is differentiable at $\mu = 0$, with $x_1(0) = -a$

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and $x_1'(0) = -\frac{17}{12}a$, meaning that

$$x_1 = -a - \frac{17}{12}a\mu + o(\mu),$$

for μ small. Classify this critical point.

(b) For $\mu > 0$, show that u has exactly one critical point of the form $p_2 = (x_2, 0)$ with $-\mu a < x_2 < a$, and exactly one critical point of the form $p_3 = (x_3, 0)$ with $x_3 > a$. Show that

$$(x_{2,3} - a)^3 = \mp a^3\mu + o(\mu) \qquad \text{as} \quad \mu \to 0.$$

Classify these critical points.

(c) Show that there are exactly two critical points (say, p_4 and p_5) off the x-axis. Compute the coordinates of these points exactly, and classify them. The classification might depend on the value of μ.

(d) Sketch the graph of $u(x, 0)$ as x varies. Sketch some level curves of u in two dimensions. You can use a graphing software. Give explanation relating the sketches with what you found in (a)–(c).