COMP 525 Winter 2017
Assignment 3
Due Date: 21st February 2017

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Please turn in questions 1 through 8. There are **8** questions for credit and one question to stretch your abilities. Unlike the usual spiritual growth questions this is not beyond the realm of possibility. If you are ambitious, try it.

Question 1[10 points] Exercise 4.12 on page 224.

Question 2[10 points] Exercise 4.22 on page 226.

Question 3[10 points] Exercise 5.2 on page 301.

Question 4[10 points] Exercise 5.5 on page 302.

Question 5[15 points] Exercise 5.6 (a), (b), (d), (g) and (k); page 303.

Question 6[10 points] Exercise 6.2 (a) and (b) on page 433.

Question 7[10 points] Exercise 6.3 (a) and (b) on page 433.

In the next question we will compare the expressive power of LTL and a version of the μ-calculus adapted to paths.

Question 8[25 points] Consider the following syntax for LTL:

$$\phi ::== p \in \mathcal{P} | \phi_1 \wedge \phi_2 | \neg\phi | \bigcirc \phi | \Diamond\phi | \Box\phi | \phi_1 U \phi_2$$

where $\mathcal{P}$ is a set of atomic propositions. The semantics is – as usual – defined in terms of infinite execution paths and the temporal operators have their usual meanings. We can enrich the language by introducing fixed point operators and dropping all the temporal operators except $\bigcirc$ to get the logic μ-LTL below

$$\phi ::== \mathcal{P} | X | \phi_1 \wedge \phi_2 | \neg\phi | \bigcirc \phi | \mu X.\phi(X) | \nu X.\phi(X)$$

where X stands for a variable that ranges over formulas and μ and ν are least and greatest fixed point operators respectively. We have an important syntactic restriction on formulas called *syntactic monotonicity*: We are not allowed to put a fixed-point operator in front of a formula unless the X occurs in the scope of an *even* number of negations. The semantics of μ and ν are given in terms of fixed points as follows.

Given a formula with no free variables in it we define satisfaction as we have done in class and then we define the *denotation* of a formula φ to be $[\![\varphi]\!] = \{\sigma \in (2^{\mathcal{P}})^\omega | \sigma \models \varphi\}$. Given a formula with a free variable X in it we interpret it not as a set of sequences but as a function from sequences to sequences:

$$[\![\varphi(X)]\!] = S \mapsto [\![\varphi[S/X]]\!],$$

where we treat an arbitrary subset S of sequences as if it were a formula when, for example, working out if $\sigma \models X \wedge \psi$. In this example we would say $\sigma \models S \wedge \psi$ if $\sigma \in S$ and $\sigma \models \psi$. Now it is easy to see that the formulas obeying the syntactic monotonicity restriction define monotone functions on the complete lattice consisting of all sets of sequences. Thus they have greatest and least fixed points. These are the denotations of the formulas $\nu X.\varphi(X)$ and $\mu X.\varphi(X)$ respectively.

With the fixed point operators present we do not need any of the LTL operators except $\bigcirc$. For example, we can write $\Box\phi \equiv \nu X.\phi \wedge \bigcirc X$. We can then use negation to get $\Diamond$.

1. What is the problem with writing the definition of $\Diamond$ directly in terms of fixed points? [3 points]
2. Show how to write a formula in μ-LTL that allows one to say that a path has the property that p (which is some fixed proposition) holds in the first state and every alternate state after that; we call this formula $odd(p)$. Note, I am **not** saying that p does **not** hold in any of the other states. Thus, your formula must be satisfied by paths like: $(pq)^\omega$ as well as paths like $pqpqppppqpqpp\ldots$ or p^ω. Here we are overloading the symbol p to stand for the state where only p (the proposition) is true. [10 points]
3. What would happen if you used the other fixed-point operator in your formula? [2 points]
4. Explain why the formula $p \wedge \Box(p \Rightarrow \bigcirc\neg p) \wedge \Box(\neg p \Rightarrow \bigcirc p)$ does not express $odd(p)$. [10 points]

Question 9[0 points] Prove that LTL cannot express the formula $odd(p)$. This shows that with the fixed-point operators we have a more powerful logic. This is for your spiritual growth but I think it is not impossibly hard. If you are ambitious give it a shot.