Math 488, Assignment 4

1. Let X be a set and $A \subseteq X$. Show that the closure of A in βX is equal to $\hat{A}$.

2. Given a set X and a compact Hausdorff space C, let $f : X \to C$ be a function. Suppose p is an ultrafilter on X. Prove that there exists a unique point $z \in C$ such that for every neighborhood U of z we have
$$\{x \in X : f(x) \in U\} \in p.$$
The point $z \in C$ as above is denoted by $\lim_p f$ and called the *limit of f over the ultrafilter p.*

3. Given a set X and a compact Hausdorff space C, let $f : X \to C$ be a function. Show that f has a a unique continuous extension $\beta f : \beta X \to C$.

4. Let $(S, \cdot)$ be a semigroup. Show that $(\beta S, \cdot)$ is also a left-topological semigroup (i.e. show associatitivity of multiplication and and that left multiplication is continuous) and prove that for $p, q \in \beta S$ we have
$$p \cdot q = R_q(p).$$

5. Show that if S is a semigroup with cancellations, then for $p, q \in \beta S$ if $p \cdot q \in S$, then $p \in S$ and $q \in S$.

6. Deduce the following version of the Hales–Jewett theorem from the one proved in class: for any natural numbers n and m there is $N \in \mathbb{N}$ such that whenever $d > N$ and n^d is partitioned into m-many pieces, then one of the pieces contains a combinatorial line.

7. The van der Waerden theorem says that for any given natural numbers n and m, there is some number N such that whenever $d > N$ is partitioned into m-many pieces, then one of the pieces contains n integers in arithmetic progression. Deduce the van der Waerden theorem from the Hales–Jewett theorem.