The Annihilator Method

The annihilator method is an easier way to solve higher order nonhomogeneous differential equations with constant coefficients. An annihilator is a linear differential operator that makes a function go to zero. In other words, differentiate it a certain amount of times and its derivative is eventually zero.

First we discuss how we find annihilators for different types of functions.

Function	Annihilator
Polynomials of degree m	D^{m+1}
$e^{\alpha x}$ *(polynomial of degree m)	$(D-\alpha)^m$
$x^{m-1}e^{\alpha x}\cos\beta x$ or $x^{m-1}e^{\alpha x}\sin\beta x$	$[(D-\alpha)^2+\beta^2]^m$

Examples:

→ $3x^2-6x+1$ Annihilator: D^3

→ x^2-e^x Annihilator: $D^3(D-1)$

Note: We had to find the annihilator of each function. Then the annihilator of the whole function will be given by the product of the annihilators.

→ $xe^{3x}\cos 5x$ Annihilator: $[(D-3)^2+25]^2$

Now we can solve higher order nonhomogeneous differential equations using the annihilator method.

Example:

→ $y''+6y'+y=e^{3x}-\sin x$

1. Apply the annihilator to both sides. This gives a characteristic equation in terms of the linear operator D.

$$A[y''+6y'+8y]=A[e^{3x}-\sin x]$$

$$(D-3)(D+1)(D^2+6D+8)=0$$

2. Solve the characteristic equation in terms of r.

$$(D-3)(D^2+1)(D^2+6D+8)=0$$

$$(r-3)(r^2+1)(r^2+6r+8)=0$$

$$(r-3)(r^2+1)(r+4)(r+2)=0$$

$$r=3, r=\pm i, r=-4, r=-2$$

3. Write the general solution to the differential equation.

$$y=c_1e^{3x}+c_2\cos x+c_3\sin x+c_4e^{-4x}+c_5e^{-2x}$$

4. Determine which part of the general solution is the homogeneous solution by solving the corresponding homogeneous equation.

$$y''+6y'+y=0 \rightarrow y_H=c_1e^{-4x}+c_2e^{-2x}$$

5. Solve for the coefficients of the particular solution.

$$y_P = c_3 e^{3x} + c_4 \cos x + c_5 \sin x$$

--Find the first and second derivatives of y_P.

$$y_P' = 3c_3 e^{3x} - c_4 \sin x + c_5 \cos x \qquad y_P'' = 9c_3 e^{3x} - c_4 \cos x - c_5 \sin x$$

--Plug derivatves into the differential equation and solve for the coefficients.

$$y'' + 6y' + y = e^{3x} - \sin x$$

$$\downarrow$$

$$9c_3 e^{3x} - c_4 \cos x - c_5 \sin x + 6\left(3c_3 e^{3x} - c_4 \sin x + c_5 \cos x\right) + 8\left(c_3 e^{3x} + c_4 \cos x + c_5 \sin x\right) = e^{3x} - \sin x$$

$$9c_3 e^{3x} - c_4 \cos x - c_5 \sin x + 18c_3 e^{3x} - 6c_4 \sin x + 6c_5 \cos x + 8c_3 e^{3x} + 8c_4 \cos x + 8c_5 \sin x = e^{3x} - \sin x$$

$$35c_3 e^{3x} + 7c_4 \cos x - 6c_4 \sin x + 7c_5 \sin x + 6c_5 \cos x = e^{3x} - \sin x$$

--Equate the coefficients together and solve.

$$\begin{cases} 35c_3 = 1 \\ 7c_4 + 6c_5 = 0 \\ -6c_4 + 7c_5 = -1 \end{cases} \rightarrow \begin{cases} c_3 = 1/35 \\ c_4 = -7/85 \\ c_5 = 6/65 \end{cases}$$

The general solution to the differential equation is given by

$$y = c_1 e^{-4x} + c_2 e^{-2x} + \frac{1}{35} e^{3x} - \frac{7}{85} \cos x + \frac{6}{65} \sin x \,.$$