Assignment #3
Honours set theory – MATH 488

Jacob Thomas Errington

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Question #1 Club sets and L-structures

Proposition 1. *Let κ be an uncountable regular cardinal and let L be a countable language. Suppose $(M_\alpha : \alpha < \kappa))$ is an increasing and continuous sequence of L-structures. Let $M = \bigcup_{\alpha<\kappa}$. Then*

$$S = \{\alpha \in \kappa : M_\alpha \prec M\}$$

is a club set.

Proof. First we show closure. Take arbitrary $A \subseteq S$ and let $\gamma = \lim A$. We want to show that $\gamma \in S$. Since κ is regular, we can deduce that $\gamma < \kappa$. We need to see that $M_\gamma \prec M$. We apply the Tarski-Vaught Test. Take an arbitrary L-formula φ and suppose that there exists an $a \in M$ such that $M \models \varphi(a)$. Since φ uses finitely many parameters from M_γ, we can find a $\beta < \gamma$ such that $\beta \in S$ and each parameter in φ is in is in M_β. Now since $M_\beta \prec M$, there is some $b \in M_\beta$ such that $M \models \varphi(b)$. But $M_\beta \subseteq M_\gamma$, so $b \in M_\gamma$. Hence the Tarski-Vaught criterion is satisfied, so $M_\gamma \prec M$. Hence $\gamma \in S$. Since γ is an arbitrary limit, this shows that S is closed under taking limits.

Next we show unboundedness. Take arbitrary $\alpha \in S$. Construct a sequence $(A_\gamma : \gamma < \kappa)$ with $A_0 = \alpha$, and whenever $\varphi(\bar{x}, x)$ is an L-formula such that $M \models \exists x : \varphi(\bar{a}, x)$, then there exists $a \in M_{i+1}$ such that $M \models \varphi(\bar{a}, a)$. At limit stages, take the union of preceding A_i. By construction, this satisfies the Tarski-Vaught criterion. Now notice that we perform choice countably many times, once for each formula, so we aren't "growing" the model too much. Next notice that we perform this countable choosing for every $\alpha < \kappa$. Hence, $\lambda = \lim A_\gamma > \alpha$ and still in S. □

Question #2 Coding formulas in HF

We establish a coding of formulas. Intuitively this is straightforward because formulas are finite objects. We assign natural number tags to each kind of object in a formula, and build up a nested tuple structure for the formula.

Constant $\varphi =$ "c" for a constant c is coded by $(0, c)$;

Free variable $\varphi =$ "x" for a variable x is coded by $(1, n)$ where n is a natural number associated with "x";

Let $v : \text{Var} \to \mathbb{N}$ denote the injection from variable names to natural numbers.

Connectives $\varphi =$ "$\psi_1 \wedge \psi_2$" is coded by $(2, [\![\psi_1]\!], [\![\psi_2]\!])$, and other connectives use other tags, e.g. $[\![\vee]\!] = 3$.

Equality $\varphi =$ "$\psi_1 = \psi_2$" is coded by $(6, [\![\psi_1]\!], [\![\psi_2]\!])$.

Membership $\varphi =$ "$\psi_1 \in \psi_2$" is coded by $(7, [\![\psi_1]\!], [\![\psi_2]\!])$.

Quantifiers

$\varphi =$ "$\exists x \in z : \psi$" is coded by $(8, v(x), v(z), [\![\psi]\!])$.

$\varphi =$ "$\exists x : \psi$" is coded by $(9, v(x), [\![\psi]\!])$.

The universal quantifier uses tags 10 and 11.

This recursively defines the operator $[\![\cdot]\!]$ mapping formulas to elements of HF, since tuples are in HF. Then, to say "φ is a formula" in the language of set theory, we write "$\exists z : [\![\varphi]\!] = z$", i.e. that φ has such a construction sequence. This is Σ_1 and hence Δ_1 because these construction sequences are unique.

For satisfaction, we just look at the semantics to construct a formula in set theory corresponding to the meaning of "$x \models \varphi(y)$". Equality, membership, logical connectives, and constants can be ported over wholesale, and all unbounded quantifiers present in φ are converted to bounded quantifiers over x. This results in a Δ_0 formula.

Question #3 The language of set theory

1. "$\mathrm{V} = \mathrm{WF}$"

   To state this in the language of set theory, we will construct V and WF using existential quantification, and then state that they are equal.

   Let $\mathrm{Seq}\,(n)$ mean "is a sequence of length n".

   (a) Constructing V. First we will see how to say "$x \in \mathrm{V}_\alpha$", with x and α being free variables.

$$\begin{aligned}
&\alpha \in \mathbf{Ord} \implies \\
&\exists s : s \in \mathrm{Seq}\,(\alpha + 1) \\
&\wedge s(0) = \emptyset \\
&\wedge \left(\forall \lambda : (\lambda \in \alpha \wedge \lambda \text{ is a limit }) \implies s(\lambda) = \bigcup_{\beta < \lambda} s(\beta) \right) \\
&\wedge (\forall \beta : \beta + 1 \leq \alpha \implies s(\beta + 1) = \mathcal{P}\left(s(\beta)\right)) \\
&\wedge x \in s(\alpha)
\end{aligned}$$

   Then, we can construct V by taking a union over ordinals.

$$\forall x : (\exists \alpha : x \in \mathrm{V}_\alpha) \iff x \in \mathrm{V}$$

   (b) Next, constructing WF. This is very similar, since $x \in \mathrm{WF}$ is defined by the formula $\exists \alpha : \alpha \in \mathbf{Ord} \wedge x \in \mathrm{V}_\alpha$. Hence we can repeat the same construction for V_α to build WF by

$$\forall x : (\exists \alpha : x \in \mathrm{V}_\alpha) \iff x \in \mathrm{WF}$$

   But these constructions work out to the same formulas! So to write $\mathrm{V} = \mathrm{WF}$ in the language of set theory is simply to write that any set whose members belong to a level of the von Neumann hierarchy is equal to itself.

$$\forall A : (\forall x : x \in A \iff (\exists \alpha : \alpha \in \mathbf{Ord} \wedge x \in \mathrm{V}_\alpha)) \implies A = A$$

   But all things are equal to themselves by reflexivity, so the left-hand side of the implication doesn't really do anything.

2. “V = L″

   We can reuse the same construction for V, but now we must construct L. The construction is essentially the same, but when looking at the successor stage in the construction, rather than use the powerset, we use $\mathrm{Def}\,(s(\beta), s(\beta))$, which we know has a (at most Δ_1) formula in the language of set theory.

   Therefore, we some defining formula $\varphi(A)$ for V and a defining formula $\psi(B)$ for L. (Specifically, if we have $\varphi(A)$, for some A then A *is* V.) So to write V = L, we can simply write

$$\forall A : \forall B : \varphi(A) \implies \psi(B) \implies A = B$$

   to say that “anything that is V is equal to anything that is L” but of course there is just one such thing of each kind, so this means “V = L″.