Solutions to selected problems

1. Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, and let $\phi : \mathbb{R}^n \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}^n$ be smooth functions satisfying

$$M\gamma''(t) = -(D\phi(\gamma(t))^T, \qquad t \in \mathbb{R}.$$

Show that the quantity

$$E(t) = \tfrac{1}{2}(\gamma'(t))^T M\gamma'(t) + \phi(\gamma(t)),$$

is independent of $t \in \mathbb{R}$, i.e., E is a *conserved* along $\gamma(t)$.

Solution: For any differentiable functions $A : \mathbb{R} \to \mathbb{R}^{n\times m}$ and $B : \mathbb{R} \to \mathbb{R}^{m\times \ell}$, we have $(AB)' = A'B + AB'$, because

$$(AB)'_{ij} = \sum_{k=1}^{m}(A_{ik}B_{kj})' = \sum_{k=1}^{m}(A'_{ik}B_{kj} + A_{ik}B'_{kj}) = (A/B)_{ij} + (AB')_{ij},$$

for $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,\ell\}$. Now we apply this to differentiate E:

$$E' = \tfrac{1}{2}(\gamma'')^T M\gamma' + \tfrac{1}{2}(\gamma')^T M\gamma'' + D\phi(\gamma)\gamma',$$

where all functions of t are understood to be evaluated at $t \in \mathbb{R}$. Since M is symmetric, we have $(\gamma')^T M\gamma'' = (\gamma'')^T M\gamma'$ (exercise!), which yields

$$E' = (\gamma'')^T M\gamma' + D\phi(\gamma)\gamma' = \big((\gamma'')^T M + D\phi(\gamma)\big)\,\gamma'.$$

Now, the equation $M\gamma''(t) = -(D\phi(\gamma(t))^T$ becomes $(\gamma'')^T M = -D\phi(\gamma)$ upon transposing, which means that $E'(t) = 0$ for all $t \in \mathbb{R}$.

2. Let $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a smooth, matrix valued function of a single variable. Assuming that $A(t)$ is an invertible diagonal matrix for all $t \in \mathbb{R}$, show that

$$(\det A)' = \det(A)\,\mathrm{tr}(A^{-1}A').$$

Solution: Let us write

$$A = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & a_{nn} \end{pmatrix},$$

so that we have $\det A = a_{11}a_{22}\cdots a_{nn}$. Since A is invertible, $\det A \neq 0$, and hence $a_{kk} \neq 0$ for any k. Then we compute

$$\begin{aligned}(\det A)' &= (a_{11}a_{22}\cdots a_{nn})' = a'_{11}a_{22}\cdots a_{nn} + a_{11}a'_{22}\cdots a_{nn} + \ldots + a_{11}a_{22}\cdots a'_{nn} \\ &= a_{11}a_{22}\cdots a_{nn}\left(\frac{a'_{11}}{a_{11}} + \frac{a'_{22}}{a_{22}} + \ldots + \frac{a'_{nn}}{a_{nn}}\right)\end{aligned}$$

We can recognize that the expression between the brackets is the trace of some matrix. What is that matrix? If we recall

$$A' = \begin{pmatrix} a'_{11} & 0 & \dots & 0 \\ 0 & a'_{22} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & a'_{nn} \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} 1/a_{11} & 0 & \dots & 0 \\ 0 & 1/a_{22} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1/a_{nn} \end{pmatrix},$$

we immediately see that

$$A^{-1}A' = \begin{pmatrix} a'_{11}/a_{11} & 0 & \dots & 0 \\ 0 & a'_{22}/a_{22} & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & a'_{nn}/a_{nn} \end{pmatrix}.$$

Therefore we conclude

$$\begin{aligned} (\det A)' &= a_{11}a_{22}\cdots a_{nn}\left(\frac{a'_{11}}{a_{11}} + \frac{a'_{22}}{a_{22}} + \ldots + \frac{a'_{nn}}{a_{nn}}\right) \\ &= \det(A)\left(\frac{a'_{11}}{a_{11}} + \frac{a'_{22}}{a_{22}} + \ldots + \frac{a'_{nn}}{a_{nn}}\right) \\ &= \det(A)\,\mathrm{tr}(A^{-1}A'). \end{aligned}$$

3. Show that the product rule $(fg)' = f'g + fg'$ for single variable functions is a special case of the multivariable chain rule.

   Solution: Let the function $\Psi : \mathbb{R} \to \mathbb{R}^2$ given by $\Psi(t) = (f(t), g(t))$, and let $u : \mathbb{R}^2 \to \mathbb{R}$ be defined by $u(x, y) = xy$. Then we see that $f(t)g(t) = u(\Psi(t)) = (u \circ \Psi)(t)$, and thus

$$[f(t)g(t)]' = Du(\Psi(t))\Psi'(t).$$

   We have

$$Du(x, y) = \begin{pmatrix} \partial_x u & \partial_y u \end{pmatrix} = \begin{pmatrix} y & x \end{pmatrix}, \qquad \text{and} \qquad \Psi'(t) = \begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix},$$

   leading to

$$[f(t)g(t)]' = \begin{pmatrix} g(t) & f(t) \end{pmatrix}\begin{pmatrix} f'(t) \\ g'(t) \end{pmatrix} = g(t)f'(t) + f(t)g'(t),$$

   which is of course the product rule for $(fg)'$.

4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function satisfying $f(tx) = tf(x)$ for all $t > 0$ and $x \in \mathbb{R}^n$. Show that if f is differentiable at 0, then f is linear, i.e., $f(x) = b^T x$ for some $b \in \mathbb{R}^n$.

Solution: By Caratheodory's criterion, there exists a function $F : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, continuous at 0, such that

$$f(x) = f(0) + F(x)x, \qquad x \in \mathbb{R}^n.$$

Since $f(0) = tf(0)$ for all $t > 0$, for instance taking $t = 2$, we infer that $f(0) = 0$. Thus we have $f(x) = F(x)x$ for all $x \in \mathbb{R}^n$. Now the property $f(tx) = tf(x)$ becomes $tF(tx)x = tf(x)$, that is,

$$f(x) = F(tx)x, \qquad t > 0, \quad x \in \mathbb{R}^n.$$

As F is continuous at 0, $F(tx)x$ tends to $F(0)x$ as $t \to 0$. With $b = F(0)^T$, this yields

$$f(x) = F(0)x = b^T x, \quad x \in \mathbb{R}^n.$$

5. Let $\phi : (0,\infty) \to \mathbb{R}$ be a smooth function, and let $u(x) = \phi(|x|)$ for $x \in \mathbb{R}^n \setminus \{0\}$. Show that

$$\Delta u(x) = \partial_1^2 u(x) + \ldots + \partial_n^2 u(x) = \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|),$$

for $x \in \mathbb{R}^n \setminus \{0\}$.

Solution: Dor $x \in \mathbb{R}^n \setminus \{0\}$, we have

$$\partial_k u(x) = \phi'(|x|)\partial_k|x|, \qquad \text{and} \qquad \partial_k^2 u(x) = \phi''(|x|)(\partial_k|x|)^2 + \phi'(|x|)\partial_k^2|x|.$$

To compute $\partial_k|x|$, write $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$, and use the chain rule, as

$$\partial_k|x| = \partial_k\sqrt{x_1^2 + \ldots + x_n^2} = \frac{2x_k}{2\sqrt{x_1^2 + \ldots + x_n^2}} = \frac{x_k}{|x|}.$$

Furthermore, we infer

$$\partial_k^2|x| = \partial_k\frac{x_k}{|x|} = \frac{|x| - x_k\partial_k|x|}{|x|^2} = \frac{|x|^2 - x_k^2}{|x|^3},$$

and substituting these into the expression for $\partial_k^2 u$, we get

$$\partial_k^2 u(x) = \frac{x_k^2}{|x|^2}\phi''(|x|) + \frac{|x|^2 - x_k^2}{|x|^3}\phi'(|x|).$$

Finally, summing over k, we conclude that

$$\begin{aligned}\Delta u(x) &= \sum_{k=1}^n \partial_k^2 u(x) = \frac{x_1^2 + \ldots + x_n^2}{|x|^2}\phi''(|x|) + \frac{n|x|^2 - x_1^2 - \ldots - x_n^2}{|x|^3}\phi'(|x|)\\ &= \phi''(|x|) + \frac{n-1}{|x|}\phi'(|x|).\end{aligned}$$

6. Show that the equation $z^3 + ze^{xy} - xy = 0$ defines z as a function of $(x, y) \in \mathbb{R}^2$. Is $z = z(x, y)$ differentiable?

Solution: First of all, we observe that x and y appear only in combination xy, so that upon substitution $t = xy$, the equation becomes $z^3 + ze^t - t = 0$. We introduce the notation $f(z, t) = z^3 + ze^t - t$, and formulate our plan as follows.

(i) For any $t \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $f(t, z) = 0$.

(ii) The solution z of $f(t, z) = 0$ is unique, i.e., $f(t, z) = f(t, w) = 0$ implies $z = w$.

(iii) The dependence $z = z(t)$ is differentiable.

In what follows, we will carry out this program.

(i) Note that the implicit function theorem would be able to give only a solution $z = z(t)$ for t in some small intergal at a time. It also does not guarantee that $z = z(t)$ is the only possible solution even when we consider t in that small interval. Since we need to show that $f(z, t) = 0$ *defines* a function of $t \in \mathbb{R}$, we need to use a different method. Thus fix an arbitrary $t \in \mathbb{R}$, and let $g(z) = f(z, t)$. We want to show that $g(a) \leq 0$ and $g(b) \geq 0$, for $a = \min\{0, -1, t\}$ and $b = \max\{0, t\}$. The intermediate value theorem would then guarantee that there exists $z \in [a, b]$ such that $g(z) = 0$. We have

$$g(b) = \max\{0, t^3\} + \max\{0, t\}e^t - t = \begin{cases} t(t^2 + e^t - 1) & \text{for } t \geq 0 \\ -t & \text{for } t < 0, \end{cases}$$

and so $g(b) \geq 0$ either way. On the other hand, we have

$$g(a) = \min\{0, -1, t^3\} + \min\{0, -1, t\}e^t - t = \begin{cases} -t & \text{for } t \geq 0 \\ -1 - e^t - t & \text{for } -1 \leq t < 0 \\ t(t^2 + e^t - 1) & \text{for } t < -1, \end{cases}$$

As $-1 - t \leq 0$ in the second case, and $t^2 - 1 > 0$ in the third case, we conclude that $g(a) \leq 0$ in all cases. By the intermediate value theorem, there exists $z \in [a, b]$ such that $g(z) = 0$, that is, for any $t \in \mathbb{R}$, there exists $z \in \mathbb{R}$ such that $f(z, t) = 0$.

(ii) Now we need to show that for any fixed $t \in \mathbb{R}$, the equation $f(z, t) = 0$ has a unique solution $z \in \mathbb{R}$. We shall use here a monotonicity argument. Thus fix an arbitrary $t \in \mathbb{R}$, and let $g(z) = f(z, t)$ as before. We have

$$g'(z) = 3z^2 + e^t > 0, \qquad z \in \mathbb{R}.$$

Suppose that $g(z) = g(w) = 0$ for some $z < w$. Then by the mean value theorem (or by Rolle's theorem), there exists $\xi \in (z, w)$ such that $g'(\xi) = 0$. But we have just proven that $g' \neq 0$ everywhere. This means that $g(z) = g(w) = 0$ cannot hold for different values $z \neq w$, and hence $g(z) = 0$ has a unique solution. To conclude, the relation $f(z, t) = 0$ defines a function $z = z(t)$ of $t \in \mathbb{R}$.

(iii) Finally, for each $t_* \in \mathbb{R}$, we apply the implicit function theorem to $f(z, t) = 0$ around the point $(z_*, t_*) = (z(t_*), t_*)$, to infer that there is a continuously differentiable function $z = h(t)$ for t varying in a small interval I centred at t_*, such that $f(h(t), t) = 0$ for all $t \in I$. However, as $f(z, t) = 0$ is uniquely solvable for z, this function $h(t)$ must agree with our previously defined function $z(t)$ for $t \in I$. Therefore $z(t)$ is continuously differentiable at $t = t_*$. Since $t_* \in \mathbb{R}$ is arbitrary, we conclude that $z(t)$ is continuously differentiable everywhere. Note that the application of the implicit function theorem is justified by the fact that $f(z, t)$ is continuously differentiable function of $(z, t) \in \mathbb{R}^2$, and that $\partial_z f(z, t) = 3z^2 + e^t \neq 0$ everywhere.

7. Show that the equation $x + y + z + \cos(xyz) = 0$ can be solved for $z = z(x, y)$ in an open set containing the origin. Find the plane tangent to $z = z(x, y)$ at the origin.

   Solution: Define $\phi : \mathbb{R}^3 \to \mathbb{R}$ by

$$\phi(x, y, z) = x + y + z + \cos(xyz).$$

It is easy to check that ϕ is continuously differentiable in $\mathbb{R}^3$. For $x = y = 0$, we have $\phi(0, 0, z) = z + \cos(0 \cdot z) = z + 1$, and so we need to take $p = (0, 0, -1)$ in order for $\phi(p) = 0$ to hold. For the implicit function theorem to be applicable to $\phi = 0$ at p and to be able to give us a function $z = z(x, y)$, what remains is to verify that the z-derivative of ϕ is nonzero at p. We have

$$\partial_z \phi(x, y, z) = 1 - xy \sin(xyz),$$

and hence

$$\partial_z \phi(0, 0, -1) = 1 \neq 0.$$

Therefore, with the purpose of writing z as a function of (x, y), the implicit function theorem can be applied to $\phi(x, y, z) = 0$ at p, which yields the existence of $\delta > 0$, and a continuously differentiable function $h : (-\delta, \delta)^2 \to \mathbb{R}$, such that $\phi(x, y, h(x, y)) = 0$ for all $(x, y) \in (-\delta, \delta)^2$. Note that $h(0, 0) = -1$.

To find the equation of the tangent plane, we need to compute the partial derivatives of h at the origin. From the implicit function theorem, we infer

$$\partial_x h(0, 0) = -\frac{\partial_x \phi(p)}{\partial_z \phi(p)} = -1, \qquad \partial_y h(0, 0) = -\frac{\partial_y \phi(p)}{\partial_z \phi(p)} = -1.$$

Recall that

$$h(x, y) = h(0, 0) + \partial_x h(0, 0)x + \partial_y h(0, 0)y + o(\max\{|x|, |y|\}),$$

and recognize that the linear part of this is the equation of the tangent plane of the surface at p. By substituting the relevant values to the equation, we conclude that the equation of the tangent plane is

$$z = -1 - x - y, \qquad \text{or} \qquad x + y + z + 1 = 0.$$

8. The point $p = (1, -1, 1)$ lies on the surfaces

$$x^3(y^3 + z^3) = 0, \qquad (x - y)^3 - z^2 = 7.$$

Show that, in an open set containing p, the curve of intersection of the surfaces can be parameterized by x, that is, the curve can be described by a system of equations of the form $\{y = f(x),\, z = g(x)\}$.

Solution: Define $\phi : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$\phi(x, y, z) = \begin{pmatrix} x^3(y^3 + z^3) \\ (x - y)^3 - z^2 - 7 \end{pmatrix}.$$

It is easy to check that $\phi(p) = 0$, and also that ϕ is continuously differentiable in $\mathbb{R}^3$. In order to apply the implicit function theorem, we now need to verify that a certain derivative of ϕ at p is invertible. Thus let us call $\beta = (y, z)$, and compute

$$D_\beta \phi(x, y, z) = \begin{pmatrix} 3x^3y^2 & 3x^3z^2 \\ -3(x - y)^2 & -2z^2 \end{pmatrix},$$

and hence

$$\det D_\beta \phi(p) = \det \begin{pmatrix} 3 & 3 \\ -12 & -2 \end{pmatrix} = 3 \cdot (-2) - 3 \cdot (-12) = 30 \neq 0.$$

Therefore, with the purpose of writing (y, z) as a function of x, the implicit function theorem can be applied to $\phi(x, y, z) = 0$ at p, which yields the existence of $\delta > 0$, and a continuously differentiable function $\gamma : (1 - \delta, 1 + \delta) \to \mathbb{R}^2$, such that $\phi(x, \gamma(x)) = 0$ for all $x \in (1 - \delta, 1 + \delta)$. Note that the notation $\phi(x, \gamma(x))$ simply means $\phi(x, \gamma_1(x), \gamma_2(x))$, where γ_1 and γ_2 are the components of the function γ.

Aside: The velocity vector γ' can be computed from

$$\gamma' = -(D_\beta \phi)^{-1} \partial_x \phi,$$

where

$$[D_\beta \phi(x, y, z)]^{-1} = \begin{pmatrix} -2z^2 & -3x^3z^2 \\ 3(x - y)^2 & 3x^3y^2 \end{pmatrix},$$

and

$$\partial_x \phi(x, y, z) = \begin{pmatrix} 3x^2(y^3 + z^3) \\ 3(x - y)^2 \end{pmatrix}.$$

Note that in all function evaluations we must have $x \in (1 - \delta, 1 + \delta)$ and $(y, z) = \gamma(x)$.