The maximum principle in forcing and the axiom of choice

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Abstract

In this paper we prove that the maximum principle in forcing is equivalent to the axiom of choice. We also look at some specific partial orders in the basic Cohen model.

Lately we have been thinking about forcing over models of set theory which do not satisfy the axiom of choice (see Miller [7, 8]). One of the first uses of the axiom of choice in forcing is:

Maximum Principle

$$p \Vdash \exists x\ \theta(x) \quad \text{iff} \quad \text{there exists a name } \tau \quad p \Vdash \theta(\tau).$$

Recall some definitions. For a partial order $\mathbb{P} = (\mathbb{P}, \trianglelefteq)$ and $p, q \in \mathbb{P}$ we say that p and q are compatible iff there exists an $r \in \mathbb{P}$ with $r \trianglelefteq p$ and $r \trianglelefteq q$. Otherwise p and q are incompatible. A subset $A \subseteq \mathbb{P}$ is an antichain iff any two distinct elements of A are incompatible. It is maximal iff every $p \in \mathbb{P}$ is compatible with some $q \in A$.

The standard definition of $p \Vdash \exists x\ \theta(x)$ is given by:

$$p \Vdash \exists x\ \theta(x) \quad \text{iff} \quad \forall q \trianglelefteq p\ \exists r \trianglelefteq q\ \exists \tau\ \ r \Vdash \theta(\tau)$$

here p, q, r range over $\mathbb{P}$ and τ is a $\mathbb{P}$-name. The usual proof of the maximum principle is to choose a maximal antichain A beneath p of such r and then choose names $(\tau_r : r \in A)$ such that $r \Vdash \theta(\tau_r)$ for each $r \in A$. Finally name τ is constructed from $(\tau_r : r \in A)$ in an argument which does not use the axiom of choice. For details the reader is referred to Kunen [6] page 226, who calls it the Maximal Principle.

Shelah [10] and Bartosyznski-Judah [1] refer to the maximum principle as the "Existential Completeness Lemma". Takeuti-Zaring [12] use "Maximum Principle" to title their Chapter 16.

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Jech [5] uses boolean valued models to do forcing proofs. He refers to the boolean algebra version of the maximum principle as: "V^B is full" , see Lemma 14.19 p.211. He notes that this is the only place in his chapter where the axiom of choice is used.

We don't know if anyone has ever wondered if the axiom of choice is necessary to prove the maximum principle. First note that the axiom of choice is needed to give the first step of the proof: Finding a maximal antichain.

Theorem 1 *The axiom of choice is equivalent to the statement that every partial order contains a maximal antichain.*

Proof
Let $(X_i \ : \ i \in I)$ be any family of nonempty pairwise disjoint sets. Let

$$\mathbb{P} = \bigcup_{i \in I} \ \omega \times X_i$$

strictly ordered by: $(n, x) \lhd (m, y)$ iff $n > m$ and $\exists i \in I \ \ x, y \in X_i$.

Note that any maximal antichain must consist of picking exactly one element out of each $\omega \times X_i$. Hence we get a choice function.
QED

The partial order used here is trivial in the forcing sense. What happens if we only consider partial orders in which every condition has at least two incompatible extensions?

In the literature on the axiom of choice there is a property called the Antichain Property (A). However, it is antichain in the sense of pairwise incomparable not pairwise incompatible. The property (A) states that every partial order contains a maximal subset A of pairwise incomparable elements (i.e. for all $p, q \in A$ if $p \unlhd q$, then $p = q$).

In ZF property (A) is equivalent to the axiom of choice (but unlike Theorem 1) property (A) is strictly weaker in set theory with atoms, i.e., it holds in some Fraenkel-Mostowski permutation model in which the axiom of choice is false. These two results are due to H.Rubin [9] and Felgner-Jech [3]. See Chapter 9 of Jech [4].

Theorem 2 *The axiom of choice is equivalent to the maximum principle.*

Proof
Let $(X_i \ : \ i \in I)$ be any family of nonempty pairwise disjoint sets. Let $\mathbb{P} = I \cup \{\mathbf{1}\}$ strictly ordered by $i \lhd \mathbf{1}$ for each $i \in I$ and the elements of I pairwise incomparable. As usual the standard names for elements of the ground model are defined by induction

$$\check{x} = \{(\mathbf{1}, \check{y}) \ : \ y \in x\}$$

and

$$\overset{\circ}{G} = \{(p, \check{p}) \ : \ p \in \mathbb{P}\}$$

is a name for the generic filter.

Then

$$\mathbf{1} \Vdash \exists x (\ \exists i \in \check{I} \cap \overset{\circ}{G} \quad x \in \check{X}_i)$$

which we may write as:

$$\mathbf{1} \Vdash \exists x \theta(x).$$

Applying the maximum principle, there exists $\mathbb{P}$-name τ such that

$$\mathbf{1} \Vdash \theta(\tau).$$

Then for each $i \in I$ we would have to have a unique $x_i \in X_i$ such that

$$i \Vdash \tau = \check{x}_i.$$

This gives us a choice function.
QED

This partial order is also trivial from the forcing point of view. A nontrivial partial order which works is

$$\mathbb{P} = (I \times 2^{<\omega}) \cup \{\mathbf{1}\}$$

which is forcing equivalent to $2^{<\omega}$. In either of these examples one can show (without using the axiom of choice) that every dense subset contains a maximal antichain. Hence we can think of them as showing that the second use of the axiom of choice in the proof of the maximum principle, the choosing of names, is also equivalent to the axiom of choice.

Note that the maximum principle holds for the suborder $I \subseteq \mathbb{P}$. So the maximum principle could fail for a partial order but hold for a dense suborder.

What can be proved without the axiom of choice in the ground model? For example, if a partial order can be well-ordered in type κ and choice holds for families of size κ, then the usual proof of the maximal principle goes thru.

We note a special case for which the maximum principle holds.

Proposition 3 *(ZF) Suppose κ is an ordinal and*

$$p \Vdash \exists \alpha < \check{\kappa} \quad \theta(\alpha)$$

then there exists a name τ such that

$$p \Vdash \theta(\tau)$$

Proof
Take τ to be a name for the least ordinal satisfying θ:

$$\tau = \{(q, \check{\beta}) \ : \ q \trianglelefteq p \text{ and } \forall \gamma \leq \beta \ \ q \Vdash \neg\, \theta(\check{\gamma})\}.$$

QED

Basic Cohen model

The Basic Cohen model $\mathcal{N}$ for the negation of the axiom of choice is described in Cohen [2] and Jech [4]. It is the analogue of Fraenkel's 1922 permutation model.

One could[2] ask: In $\mathcal{N}$ which partial orders have the maximum principle?

Definition 4 *Given infinite sets I and J let $Inj(I, J)$ be the partial order of finite injective maps from I to J, i.e., $r \in Inj(I, J)$ iff $r \subseteq I \times J$ is finite and $u, v)$ It is ordered by reverse inclusion: $r_1 \trianglelefteq r_2$ iff $r_1 \supseteq r_2$.*

Recall that in $\mathcal{N}$ the failure of the countable axiom of choice is witnessed by an infinite Dedekind finite $X \subseteq \mathcal{P}(\omega)$. We consider the following three partial orders: $Inj(\omega, \omega)$, $Inj(X, X)$, and $Inj(\omega, X)$.

We show that the maximum principle holds for one of these partial orders and fails for the other two. The easiest case is $Inj(\omega, \omega)$. The following lemma takes care of it.

[2]Since this model is the original and simplest model in which the axiom of choice fails, we think it is interesting to study its properties just for its own sake.

Lemma 5 *Suppose that the countable axiom of choice fails and $\mathbb{P}$ is a nontrivial partial order which can be well-ordered. Then $\mathbb{P}$ fails to satisfy the maximum principle.*

Proof
By nontrivial we mean that every condition has at least two incompatible extensions. Hence we can find $\langle p_n \in \mathbb{P} \ : \ n \in \omega \rangle$ such that p_n and p_m are incompatible whenever $n \neq m$. Suppose $\{X_n \ : \ n \in \omega\}$ is a family of nonempty sets without a choice function. Note that

$$\mathbf{1} \Vdash \exists x \, \forall n \in \check{\omega} \ \ (\check{p}_n \in \overset{\circ}{G} \rightarrow x \in \check{X}_n).$$

We claim that this is a witness for the failure of the maximum principle. Suppose not and let τ be $\mathbb{P}$-name for which

$$\mathbf{1} \Vdash \forall n \in \check{\omega} \ \ (\check{p}_n \in \overset{\circ}{G} \rightarrow \tau \in \check{X}_n).$$

Since $\mathbb{P}$ can be well-ordered, we may choose for each n a $q_n \trianglelefteq p_n$ and $x_n \in X_n$ such that

$$q_n \Vdash \tau = \check{x}_n.$$

But this would give a choice function for the family $\{X_n \ : \ n \in \omega\}$.
QED

Theorem 6 *In $\mathcal{N}$ the maximum principle fails for $Inj(\omega, \omega)$.*

Proof
This follows from the Lemma, since $Inj(\omega, \omega)$ is well-orderable and nontrivial, and the countable axiom of choice fails in $\mathcal{N}$.
QED

Of course, there are many partial orders for which this applies. We choose to highlight $Inj(\omega, \omega)$ because it is simple and superficially similar to the other two partial orders $\mathbb{P}_0 = Inj(X, X)$ and $\mathbb{P}_1 = Inj(\omega, X)$.

Theorem 7 *In $\mathcal{N}$ the maximum principle fails for $\mathbb{P}_0 = Inj(X, X)$.*

Proof

We start with a description of $\mathcal{N}$. Fix M a countable standard transitive model of ZFC.

Working in M let $\mathbb{P} = Fn(\omega \times \omega, 2, \omega)$ be the poset of finite partial functions, i.e., $p \in \mathbb{P}$ iff $p : D \to 2$ for some finite $D \subseteq \omega \times \omega$.

Each bijection $\tilde{\pi} : \omega \to \omega$ induces an automorphism $\pi : \mathbb{P} \to \mathbb{P}$ defined by: Given $p : D \to 2$ then $\pi(p) : E \to 2$ where $E = \{(\tilde{\pi}(i), j) \ : \ (i,j) \in D\}$ and $\pi(p)(\tilde{\pi}(i), j) = p(i,j)$ for each $(i,j) \in D$.

Let $\mathcal{G}$ be the group of automorphisms of $\mathbb{P}$ generated by $\{\pi_{i,j} \ : \ i < j < \omega\}$ where $\tilde{\pi}_{i,j} : \omega \to \omega$ is the bijection which swaps i and j.

The normal filter $\mathcal{F}$ is generated by the subgroups $\{H_n \ : \ n < \omega\}$ where $H_n = \{\pi \in \mathcal{G} \ : \ \tilde{\pi} \restriction n = id\}$. For G $\mathbb{P}$-generic over M, we let $\mathcal{N}$ with $M \subseteq \mathcal{N} \subseteq M[G]$ be the symmetric model determined by $(G, \mathcal{G}, \mathcal{F})$, so $M \subseteq \mathcal{N} \subseteq M[G]$. The model $\mathcal{N}$ is the Basic Cohen model for the negation of the axiom of choice. In $M[G]$ we define

$$x_n = \{k < \omega \ : \ \exists p \in G \ \ p(n,k) = 1\} \text{ and } X = \{x_n \ : \ n < \omega\}.$$

The set X is in $\mathcal{N}$ and $\mathcal{N}$ thinks it is Dedekind finite, so no enumeration of it is there. Recall that in $\mathcal{N}$ we define the poset $\mathbb{P}_0 = Inj(X, X)$ to be the set of all finite partial one-to-one maps from X to X. If G_0 is $\mathbb{P}_0$-generic over $\mathcal{N}$, then $\bigcup G_0$ will be the graph of a bijection from X to X.

In both posets $\mathbb{P}$ and $\mathbb{P}_0$ the trivial condition is the empty set, i.e., $\mathbf{1} = \emptyset$ and a universal name for the empty set is also the empty set. The standard names for elements of the ground model are defined by induction as $\check{x} = \{(\mathbf{1}, \check{y}) \ : \ y \in x\}$. The names for unordered and ordered pairs are

$$\{\tau_1, \tau_2\}^\circ = \{(\mathbf{1}, \tau_1), (\mathbf{1}, \tau_2)\} \text{ and } (\tau_1, \tau_2)^\circ = \{(\mathbf{1}, \{\tau_1\}^\circ), (\mathbf{1}, \{\tau_1, \tau_2\}^\circ)\}.$$

Working in $\mathcal{N}$ let

$$\Gamma = \{(r, \check{r}) \ : \ r \in \mathbb{P}_0\}$$

be the usual name for G_0, the $\mathbb{P}_0$-generic filter over $\mathcal{N}$.

Working in M let $\overset{\circ}{\Gamma}$ be a hereditarily symmetric $\mathbb{P}$-name[3] for Γ. Let $\overset{\circ}{\mathbb{P}}_0$ be a hereditarily symmetric name for $\mathbb{P}_0$. Let

$$\overset{\circ}{x}_n = \{(p, \check{k}) \ : \ p \in \mathbb{P} \text{ and } p(n,k) = 1\}.$$

[3]Yes, that's right, the name of a name.

For each n let

$$\overset{\circ}{\check{x}}_n = \{(p, (\mathbf{1}, \check{\check{k}})^\circ) \;:\; p(n,k) = 1\}.$$

This will be a $\mathbb{P}$-name for $\check{x}_n$ the standard $\mathbb{P}_0$-name for x_n. This means that if G is $\mathbb{P}$-generic over M then $\overset{\circ}{\check{x}}_n^G = \check{x}_n$, i.e, the standard name of x_n not x_n. Note that if $\tilde{\pi}$ maps column m to column m', then

$$\pi(\overset{\circ}{x}_m) = \overset{\circ}{x}_{m'} \quad \text{and} \quad \pi(\overset{\circ}{\check{x}}_m) = \overset{\circ}{\check{x}}_{m'}\ .$$

For $\sigma \in Inj(\omega, \omega)$ (the graph of a finite injection) define

$$\overset{\circ}{r}_\sigma = \{(\mathbf{1}, (\overset{\circ}{x}_i, \overset{\circ}{x}_j)^\circ) \;:\; (i,j) \in \sigma\}.$$

Note that for any $p \in \mathbb{P}$ and $\mathbb{P}$-name r if $p \Vdash r \in \overset{\circ}{\mathbb{P}}_0$, then there exists $q \leq p$ and $\sigma \in Inj(\omega, \omega)$ such that $q \Vdash r = \overset{\circ}{r}_\sigma$.

Back working in $\mathcal{N}$ note that

$$\mathbf{1} \Vdash_{\mathbb{P}_0} \exists u \;\; (\exists v \;\; u \neq v \text{ and } \exists r \in \Gamma \;\; (u,v) \in r)$$

write this as

$$\mathbf{1} \Vdash_{\mathbb{P}_0} \exists u \;\; \theta(u, \Gamma).$$

We claim that there does not exists a $\mathbb{P}_0$-name τ in $\mathcal{N}$ such that

$$\mathcal{N} \models \text{“ } \mathbf{1} \Vdash_{\mathbb{P}_0} \theta(\tau, \Gamma) \text{ ”}$$

and hence the maximal principle fails. Suppose not and let $\overset{\circ}{\tau}$ be a hereditarily symmetric $\mathbb{P}$-name for τ.

Take $p \in G$ such that

$$p \Vdash \overset{\circ}{\mathcal{N}} \models 1 \Vdash_{\overset{\circ}{\mathbb{P}}_0} \theta(\overset{\circ}{\tau}, \overset{\circ}{\Gamma}).$$

Working in M choose n so that $\text{dom}(p) \subseteq n \times \omega$ and for every $\pi \in H_n$ $\pi(\overset{\circ}{\tau}) = \overset{\circ}{\tau}$ and $\pi(\overset{\circ}{\mathbb{P}}_0) = \overset{\circ}{\mathbb{P}}_0$.

Working in $\mathcal{N}$ let $r_{id_n} = \{(x_i, x_i) \;:\; i < n\}$. We can find $r \leq r_{id_n}$ and $\check{x}_m$ such that

$$r \Vdash \tau = \check{x}_m.$$

Note that $\mathcal{N}$ will not know which subscript goes with which element of X but we know that $m \geq n$.

Working back in M find $q \leq p$ and $\sigma \in Inj(\omega, \omega)$ with $\sigma \supseteq id_n$ such that

$$q \Vdash \overset{\circ}{\mathcal{N}} \models \overset{\circ}{r}_\sigma \ \Vdash_{\overset{\circ}{\mathbb{P}}_0} \overset{\circ}{\tau} = \overset{\circ}{\check{x}}_m$$

We write this as:

$$q \Vdash \psi(\overset{\circ}{\mathcal{N}}, \overset{\circ}{r}_\sigma, \overset{\circ}{\mathbb{P}}_0, \overset{\circ}{\tau}, \overset{\circ}{\check{x}}_m)$$

Now take $N > n$ with $dom(q) \subseteq N \times \omega$, $n \leq m < N$, and $\sigma \subseteq N \times N$. Let $\pi \in \mathcal{G}$ be determined by the bijection $\tilde{\pi} : \omega \to \omega$ given by swapping the interval of columns $[n, N)$ with $[n+N, 2N)$, i.e., swap k and $N+k$ for each k with $n \leq k < N$. Note that the corresponding automorphism π of $\mathbb{P}$ has the property $\pi(\overset{\circ}{\check{x}}_m) = \overset{\circ}{\check{x}}_{m+N}$. Let

$$\sigma' = id_n \cup \{(i+N, j+N) \ : \ (i,j) \in \sigma \text{ and } i, j \geq n\}$$

and note that $\pi(\overset{\circ}{r}_\sigma) = \overset{\circ}{r}_{\sigma'}$. Since $\pi \in H_n$ it fixes $\overset{\circ}{\mathbb{P}}_0$ and $\overset{\circ}{\tau}$ so

$$\pi(q) \Vdash \psi(\overset{\circ}{\mathcal{N}}, \overset{\circ}{r}_{\sigma'}, \overset{\circ}{\mathbb{P}}_0, \overset{\circ}{\tau}, \overset{\circ}{\check{x}}_{m+N}).$$

But q and $\pi(q)$ are compatible so we may find G which is $\mathbb{P}$-generic over M containing them both. In the model corresponding model $\mathcal{N}$ we will get that

$$r_\sigma \Vdash \tau = \check{x}_m \quad \text{and} \quad r_{\sigma'} \Vdash \tau = \check{x}_{m+N}$$

but this is a contradiction because r_σ and $r_{\sigma'}$ are compatible.
QED

Example 8 *Recall that $Fn(I, J, \omega)$ is the partial order of finite maps from I to J, i.e. $r \subseteq I \times J$ is finite and $(u, v) \in r$ and $(u, w) \in r$ implies $v = w$. Some other posets in $\mathcal{N}$ for which the maximum principle fails and for which some variant of the above argument works are:*

1. $Fn(X, 2, \omega)$ $\qquad \exists u \ (\exists r \in \Gamma \ \ (u, 0) \in r)$
2. $Fn(X, \omega, \omega)$ $\qquad \exists u \ (\exists r \in \Gamma \ \ (u, 0) \in r)$

3. $Fn(X, X, \omega) \qquad \exists u\ \ (\exists r \in \Gamma\ \ (u, x_0) \in r)$

Proofs are left for the reader. Finally we show that in $\mathcal{N}$ the maximum principle holds for $\mathbb{P}_1 = Inj(\omega, X)$. Recall that this is the partial order of the finite one-to-one maps from ω into X. The key to the proof is Lemma 11, but first we note some preliminary lemmas.

Define H_n^∞ to be the subgroup of automorphisms of $\mathbb{P}$ which are determined by bijections $\tilde{\pi} : \omega \to \omega$ which are the identity on n ,i.e., $\tilde{\pi}(i) = i$ for all $i < n$. Hence H_n is $\mathcal{G} \cap H_n^\infty$. The elements of H_n^∞ do not have to be in the ground model M or even $M[G]$.

Lemma 9 *Suppose $k > n$ and $\pi \in H_n^\infty$ then there exists $\pi_1 \in H_n$ and $\pi_2 \in H_k^\infty$ such that $\pi = \pi_1 \circ \pi_2$.*

Proof
Consider any orbit of $\tilde{\pi}$ which contains at least one of the $j < k$. If it is finite, we set $\tilde{\pi}_1 = \tilde{\pi}$ on it and put $\tilde{\pi}_2$ to be the identity. If it is an infinite orbit, write it as $\{a_m\ :\ m \in \mathbb{Z}\}$ where $\tilde{\pi}(a_m) = a_{m+1}$. Since there are only finitely many a_i with $0 \leq a_i < k$, we may renumber them so that for some N any a_i with $0 \leq a_i < k$ is in the set $a_1, \ldots, a_{N-1}$. On this orbit define $\tilde{\pi}_1$ to shift the list $a_1, a_2, \ldots, a_N$ up one and send the last to the beginning, i.e., $\tilde{\pi}_1(a_i) = a_{i+1}$ for $1 \leq i < N$ and $\tilde{\pi}_1(a_N) = a_1$. Define $\tilde{\pi}_2$ to shift the $\mathbb{Z}$-chain:

$$\ldots, a_{-2}, a_{-1}, a_0, a_N, a_{N+1}, \ldots$$

i.e., $\tilde{\pi}_2(a_j) = a_{j+1}$ except when $j = 0$ and then $\tilde{\pi}_2(a_0) = a_N$.
QED

Lemma 10 *For any hereditarily symmetric $\mathbb{P}$-name τ, if every $\pi \in H_n$ fixes τ, i.e., $\pi(\tau) = \tau$, then every $\pi \in H_n^\infty$ fixes τ.*

Proof
This is proved by induction on the rank of τ. Suppose that $\pi \in H_n^\infty$ and $(p, \sigma) \in \tau$. Choose $k > n$ so that $\text{dom}(p) \subseteq k \times \omega$ and H_k fixes σ. By Lemma 9 there exists $\pi_1 \in H_n$ and $\pi_2 \in H_k^\infty$ such that $\pi = \pi_1 \circ \pi_2$. It follows that $(\pi(p), \pi(\sigma)) = (\pi_1(p), \pi_1(\sigma))$ since $\tilde{\pi}_2$ is that identity on k, so $\pi_2(p) = p$, and since by induction on rank $\pi_2(\sigma) = \sigma$. Since π_1 fixes τ we have that $(\pi(p), \pi(\sigma)) \in \tau$. It follows that $\pi(\tau) \subseteq \tau$. Applying the same argument to π^{-1} shows that $\pi^{-1}(\tau) \subseteq \tau$ and therefore $\tau \subseteq \pi(\tau)$ and so $\pi(\tau) = \tau$.
QED

Lemma 11 *Suppose G is $\mathbb{P}$-generic over M and $\mathcal{N} = \mathcal{N}_G$ is the symmetric inner model with $M \subseteq \mathcal{N} \subseteq M[G]$. Working in $M[G]$ define*

$$x_i = \{j \in \omega \ : \ \exists p \in G \ p(i,j) = 1\}$$

and let

$$G_1 = \{r \in \mathbb{P}_1 \ : \ \forall i \in \mathrm{dom}(r) \ \ r(i) = x_i\}.$$

Then G_1 is $\mathbb{P}_1$-generic over $\mathcal{N}$ and $\mathcal{N}[G_1] = M[G]$.

Conversely, if $\tilde{G}_1$ is $\mathbb{P}_1$-generic over $\mathcal{N}$, then

$$\tilde{G} = \{s \in \mathbb{P} \ : \ \forall (i,j) \in \mathrm{dom}(s) \ \ [s(i,j) = 1 \ \textit{iff} \ \exists p \in \tilde{G}_1 \ j \in p(i)]\}$$

is $\mathbb{P}$-generic over M and $\mathcal{N} = \mathcal{N}_{\tilde{G}}$.

Proof
First we see that G_1 is $\mathbb{P}_1$-generic over $\mathcal{N}$. In this proof we will use $r_\sigma \in \mathbb{P}_1$ for $\sigma \in Inj(\omega, \omega)$ to refer to the condition satisfying $r_\sigma(i) = x_{\sigma(i)}$ for each $i \in \mathrm{dom}(\sigma)$.

Working in M suppose that $\overset{\circ}{D}$ is a symmetric name and $s \in \mathbb{P}$ satisfies:

$$s \Vdash \overset{\circ}{D} \subseteq \overset{\circ}{\mathbb{P}}_1 \ \text{ is dense open.}$$

Choose n so that every π in H_n fixes $\overset{\circ}{D}$ and $\mathrm{dom}(s) \subseteq n \times \omega$. Choose $t \trianglelefteq s$, $m > n$, and a one-to-one $\sigma : m \to \omega$ such that $\sigma \supseteq id_n$ and

$$t \Vdash \overset{\circ}{r}_\sigma \in \overset{\circ}{D}$$

where

$$\overset{\circ}{r}_\sigma = \{(\check{j}, \overset{\circ}{x}_{\sigma(j)})^\circ \ : \ j < m\}.$$

Let $\pi \in H_n$ be an automorphism for which $\tilde{\pi}(\sigma(j)) = j$ for every $j < m$. It follows that

$$\pi(\overset{\circ}{r}_\sigma) = \overset{\circ}{r}_{id_m}$$

and

$$\pi(t) \Vdash \overset{\circ}{r}_{id_m} \in \overset{\circ}{D} \ .$$

Since $\pi(t) \trianglelefteq s$ and s and D were arbitrary it follows that G_1 meets every dense subset of $\mathbb{P}_1$ in $\mathcal{N}$.

Since $M[G]$ is the smallest model of ZF containing G and including M we have that $M[G] \subseteq \mathcal{N}[G_1]$. The other inclusion follows since G_1 is easily definable from G.

Next we prove the "Conversely" statement. Suppose that $D \subseteq \mathbb{P}$ is dense and in M. We must show it meets $\tilde{G}$.

Working in $\mathcal{N}$ for $s \in \mathbb{P}$ and $q \in \mathbb{P}_1$ define $s \sqsubseteq q$ as follows: For any $(i,j) \in \mathrm{dom}(s)$ we have that $i \in \mathrm{dom}(q)$ and $(s(i,j) = 1$ iff $j \in q(i))$.

We claim that

$$E = \{q \in \mathbb{P}_1 \ : \ \exists s \in D \ \ s \sqsubseteq q\}$$

is dense in $\mathbb{P}_1$. Since E is in $\mathcal{N}$ we have that E meets $\tilde{G}_1$. It follows that D meets $\tilde{G}$.

To prove E is dense work in $M[G]$. Fix $p \in \mathbb{P}_1$. Take $\pi \in \mathcal{G}$ so that $p(i) = x_{\tilde{\pi}(i)}$ for each $i \in \mathrm{dom}(p)$. Since D is dense, so is $\pi^{-1}(D)$. Take $s \in G \cap \pi^{-1}(D)$. Then $\pi(s) \in D$ and if $s_0 = s \restriction \mathrm{dom}(p)$, then $s_0 \sqsubseteq p$. By genericity it is easy to find $q \trianglelefteq p$ with $\pi(s) \sqsubseteq q$.

Finally, we show $\mathcal{N}_G = \mathcal{N}_{\tilde{G}}$. Let $\tilde{\pi} : \omega \to \omega$ be the bijection defined by $\tilde{\pi}(i) = j$ iff $\exists p \in G_1$ with $p(i) = x_j$. Then $\pi \in H_0^\infty$. Note also that $\tilde{G} = \pi(G)$.

It is a standard fact that the hereditarily symmetric $\mathbb{P}$-names in M are closed under $\mathcal{G}$. Combining Lemmas 9 and 10 gives that the same is true for any $\pi \in H_0^\infty$. To see this, suppose τ is fixed by H_n. Decompose $\pi = \pi_1 \circ \pi_2$ with $\pi_2 \in H_n^\infty$ and $\pi_1 \in \mathcal{G}$. Then $\pi(\tau) = \pi_1(\tau)$.

Note that we have that

$$\tau^G = \pi(\tau)^{\pi(G)} = \pi_1(\tau)^{\tilde{G}}$$

and hence $\mathcal{N}_G \subseteq \mathcal{N}_{\tilde{G}}$. Similarly $\mathcal{N}_{\tilde{G}} \subseteq \mathcal{N}_G$ so they are equal.
QED

Theorem 12 *In $\mathcal{N}$ the partial order $\mathbb{P}_1 = Inj(\omega, X)$ satisfies the maximum principle.*

Proof

Let $(\mathbb{P}-\text{names})^M$ be the class[4] of $\mathbb{P}$-names in M.

[4] This may be assumed to be a definable class in $M[G]$ and in $\mathcal{N}$. It is easy to see this would be true if we make the additional assumption that M is a model of $V = L$. In general one can make it true by adding a unary predicate for M to the models. See Solovay [11] p.5-6.

Working in $\mathcal{N}$ define a mapping which takes $(\mathbb{P}-\text{names})^M$ to $\mathbb{P}_1$-names as follows:

$$\hat{\tau} = \{(q, \hat{\sigma}) \ : \ \exists r\ (r, \sigma) \in \tau \text{ and } r \sqsubseteq q\}.$$

The relation $\sqsubseteq$ is defined in the proof of Lemma 11. It then follows that

$$\hat{\tau}^{\tilde{G}_1} = \tau^{\tilde{G}}$$

for any $\tilde{G}_1$ which is $\mathbb{P}_1$-generic over $\mathcal{N}$ and $\tilde{G}$ defined from it as in Lemma 11.

In $\mathcal{N}$ suppose that

$$p_0 \Vdash_{\mathbb{P}_1} \exists x\ \theta(x).$$

For any $\tilde{G}_1$ $\mathbb{P}_1$-generic over $\mathcal{N}$ with p_0 in $\tilde{G}_1$, we know that

$$\mathcal{N}[\tilde{G}_1] \models \exists x\ \theta(x)$$

by the definition of forcing. By the key Lemma 11, $\mathcal{N}[\tilde{G}_1] = M[\tilde{G}]$ and so for some τ in $(\mathbb{P}-\text{names})^M$

$$M[\tilde{G}] \models \theta(\tau^{\tilde{G}})$$

and so

$$\mathcal{N}[\tilde{G}_1] \models\ \theta(\hat{\tau}^{\tilde{G}_1}).$$

It follows that in $\mathcal{N}$

$$\forall q \trianglelefteq p_0\ \exists r \trianglelefteq q\ \exists \tau \in (\mathbb{P}-\text{names})^M\ \ r \Vdash_{\mathbb{P}_1} \theta(\hat{\tau}).$$

By using the replacement axiom in $\mathcal{N}$ and the axiom of choice in M we can find $\langle \tau_\alpha \ : \ \alpha < \kappa \rangle \in M \subseteq \mathcal{N}$ such that in $\mathcal{N}$:

$$\forall q \trianglelefteq p_0\ \exists r \trianglelefteq q\ \exists \alpha < \kappa\ \ r \Vdash_{\mathbb{P}_1} \theta(\hat{\tau}_\alpha).$$

But this existential quantifier is essentially over an ordinal, so by a proof similar to Proposition 3 we can find a name τ such that

$$p_0 \Vdash \theta(\tau)$$

and the maximum principle is proved.

Working in $\mathcal{N}$ the name τ can be found as follows. Let

$$\rho = \{(q, \hat{\tau}_\alpha) \ : \ q \trianglelefteq p_0,\ q \Vdash \theta(\hat{\tau}_\alpha),\ \text{and } \forall \beta < \alpha\ \ q \Vdash \neg\theta(\hat{\tau}_\beta)\}.$$

Then ρ is the name of a singleton $\{u\}$ where u satisfies θ. As in the usual proof of the maximum principle, to remove the enclosing braces note that $u = \cup\{u\}$, so letting

$$\tau = \cup^\circ \rho = \{(q_3, \sigma_2) \ : \ \exists (q_1, \sigma_1) \in \rho \ \ \exists q_2 \ (q_2, \sigma_2) \in \sigma_1 \ \ q_3 \trianglelefteq q_1, q_2\}$$

does the job.
QED

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