MATH 254 Tutorial 6 (Dense Subsets and Sequences):

Problem 1 (Baire Category Theorem): A subset $U \subseteq \mathbb{R}$ is called **open** if for any $a \in U$ there is a real number $r > 0$ such that $(a-r, a+r) \subseteq U$. A subset $U \subseteq \mathbb{R}$ is called **dense** if for any real numbers $x < y$ there is an $a \in U$ such that $x < a < y$. In this problem, we want to prove that any countable intersection of open dense subsets of real numbers is dense and see a few numbers of its applications.

a) Let $U \subseteq \mathbb{R}$ be dense. Prove that U is nonempty. Moreover, if we have $U \subseteq V \subseteq \mathbb{R}$, then V is also dense.

b) Let $x < y$ be two real numbers and $U \subseteq \mathbb{R}$ be open and dense. Prove that there are real numbers $x' < y'$ such that $[x', y'] \subseteq U \cap (x, y)$.

c) Let $x < y$ be two real numbers and $U_n \subseteq \mathbb{R}$ be open and dense for each $n \in \mathbb{N}$. Using part b and mathematical induction, prove that there are numbers $x_n < y_n$ for each $n \in \mathbb{N}$ such that $[x_1, y_1] \subseteq U_1 \cap (x, y)$ and $[x_{n+1}, y_{n+1}] \subseteq U_{n+1} \cap (x_n, y_n)$ for all $n \in \mathbb{N}$.

d) Let $U_n \subseteq \mathbb{R}$ be open and dense for each $n \in \mathbb{N}$. Using part c, prove that $\cap_{n=1}^{\infty} U_n$ is dense in real numbers (In particular, the intersection is nonempty).

e) Let $C \subseteq \mathbb{R}$ be countable. Using part d, prove that $\mathbb{R} - C$ is dense in real numbers (In particular, it is nonempty).

f) Using part e, give a new proof of uncountablity of real numbers.

g) Using part e, give a new proof of density of irrational numbers in real numbers.

h) Prove that any union of open subsets of real numbers is open and any finite intersection of open subsets of real numbers is open. Give an example for part d such that the intersection is not open. Moreover, prove that the empty set, $\mathbb{R}$ and open intervals are among open subsets of real numbers.

Problem 2: Let (x_n) be a sequence of natural numbers (integers) converging to $x \in \mathbb{R}$.

a) Prove that the limit x should be a natural number (an integer).

b) Prove that the sequence should be eventually constant i.e. there is an $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$.

Problem 3: Let (x_n) be a sequence of real numbers. Prove that (x_n) converges to 0 if and only if $(|x_n|)$ converges to 0. Give an example to show that the convergence of $(|x_n|)$ need not imply the convergence of (x_n).

Problem 4: Let (x_n) be a sequence of real numbers converging to x.

a) Prove that if $x > 0$, then the sequence should be eventually positive i.e. there is an $N \in \mathbb{N}$ such that $x_n > 0$ for all $n \geq N$. Does this statement remain true if we replace both $>$ with $\geq$, $<$ or $\leq$?

b) Prove that if the sequence is eventually non-negative, then the limit x should be non-negative. Does this statement remain true if we replace both non-negative with positive, non-positive or negative?

Problem 5: Let (x_n) and (y_n) be two sequences of real numbers.

a) Prove that if (x_n) and $(x_n + y_n)$ are convergent, then (y_n) is convergent.

b) Give an example of divergent sequences (x_n) and (y_n) such that both $(x_n + y_n)$ and $(x_n.y_n)$ are convergent.

c) Prove that if (x_n) converges to 0 and (y_n) is bounded, then $(x_n.y_n)$ converges to 0. Give an example of convergent sequence (x_n) and divergent sequence (y_n) such that $(x_n.y_n)$ is convergent.

d) Prove that if (x_n) and $(x_n.y_n)$ are convergent and $lim(x_n) \neq 0$, then (y_n) is convergent.

e) Assume that (x_n) and $(x_n.y_n)$ are convergent but (y_n) is divergent. Should we have $lim(x_n) = 0$? What about $lim(x_n.y_n) = 0$?

Problem 6: Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_n - y_n| < \epsilon$ for all $n \geq N$. Does it follow that (y_n) is convergent?

Problem 7: Let (x_n) and (y_n) be two sequences of real numbers converging to x and y, respectively. Prove that $(max\{x_n, y_n\})$ and $(min\{x_n, y_n\})$ converge to $max\{x, y\}$ and $min\{x, y\}$, respectively.