14. Models of set theory

In this chapter we introduce the basic notions used for models of set theory: relativization, absoluteness, and reflection. Our first consistency result is in this chapter too: if ZFC is consistent, then so is ZFC+"there is no inaccessible cardinal".

Relativization

Although models of set theory are special cases of models in any first-order language, it is convenient to introduce the basic notions only for our standard language for set theory. Another aspect of our situation is that we want to consider models which are proper classes. For this reason, it is better to talk about relativization instead of the more standard notions of model theory.

The basic definition of relativization is as follows.

•. Suppose that $\mathbf{M}$ is a class. We associate with each formula φ its *relativization to* $\mathbf{M}$, denoted by $\varphi^{\mathbf{M}}$. The definition goes by recursion on formulas:

$(x = y)^{\mathbf{M}}$ is $x = y$
$(x \in y)^{\mathbf{M}}$ is $x \in y$
$(\varphi \wedge \psi)^{\mathbf{M}}$ is $\varphi^{\mathbf{M}} \wedge \psi^{\mathbf{M}}$.
$(\neg\varphi)^{\mathbf{M}}$ is $\neg\varphi^{\mathbf{M}}$.
$(\exists x\varphi)^{\mathbf{M}}$ is $\exists x[x \in \mathbf{M} \wedge \varphi^{\mathbf{M}}]$.

The more rigorous version of this definition associates with each pair ψ, φ of formulas a third formula which is called the relativization of φ to ψ.

We say that φ *holds in* $\mathbf{M}$ or *is true in* $\mathbf{M}$, iff $\varphi^{\mathbf{M}}$ holds.

The basic fact about relativization, which could be proved in an elementary logic course, is the following theorem; we supply a proof, since the theorem may not be familiar to the reader. Note that in this theorem and proof we use the rigorous version of the definition of relativization. We write $\Gamma \models \varphi$ to abbreviate the statment that every structure which is a model of each member of Γ is also a model of φ.

Theorem 14.1. *Let Γ be a set of sentences, φ a sentence, and $\psi(x, \overline{y})$ a formula, where we indicate one distinguished variable x, and a string $\overline{y}$ of other free variables. Let $\Gamma^\psi = \{\chi^\psi : \chi \in \Gamma\}$. Suppose that $\Gamma \models \varphi$. Then*

$$\Gamma^\psi \models \exists x\psi(x, \overline{y}) \rightarrow \varphi^\psi.$$

Proof. Assume the hypothesis of the theorem, let (A, E) be any structure for our set-theoretic language, and let $\overline{a}$ be a sequence of elements of A of the length of $\overline{y}$. Assume that

$$(A, E) \models \Gamma^\psi(\overline{a}) \wedge \exists x\psi(x, \overline{a}); \tag{1}$$

we want to show that $(A, E) \models \varphi^\psi(\overline{a})$. To do this, we define another structure (B, F) for our language. Let

$$B = \{b \in A : (A, E) \models \psi(b, \overline{a})\} \quad \text{and} \quad F = E \cap (B \times B).$$

Note that $B \neq \emptyset$ since $(A, E) \models \exists x\psi(x, \overline{a})$ by (1). Now we claim:

(2) For any formula $\chi(\overline{z})$ and any $\overline{c}$ in B, $(A, E) \models \chi^{\psi}(\overline{c}, \overline{a})$ iff $(B, F) \models \chi(\overline{c})$.

We prove (2) by induction on χ:

$$\begin{array}{rll}
(A,E) \models (c_0 = c_1)^{\psi} & \text{iff} & c_0 = c_1 \\
& \text{iff} & (B,F) \models c_0 = c_1; \\
(A,E) \models (c_0 \in c_1)^{\psi} & \text{iff} & (c_0, c_1) \in E \\
& \text{iff} & (c_0, c_1) \in F \\
& \text{iff} & (B,F) \models c_0 \in c_1; \\
(A,E) \models (\neg\chi)^{\psi}(\overline{c},\overline{a}) & \text{iff} & \text{not}[(A,E) \models \chi^{\psi}(\overline{c},\overline{a})] \\
& \text{iff} & \text{not}[(B,F) \models \chi(\overline{c})] \quad \text{(induction hypothesis)} \\
& \text{iff} & (B,F) \models \neg\chi(\overline{c}); \\
(A,E) \models (\chi \wedge \theta)^{\psi}(\overline{c},\overline{a}) & \text{iff} & [(A,E) \models \chi(\overline{c},\overline{a})] \text{ and } (A,E) \models \theta(\overline{c},\overline{a}) \\
& \text{iff} & [(B,F) \models \chi(\overline{c})] \text{ and } (B,F) \models \theta(\overline{c}) \\
& & \qquad\qquad \text{(induction hypothesis)} \\
& \text{iff} & (B,F) \models (\chi \wedge \theta)(\overline{c}).
\end{array}$$

We do the quantifier step in each direction separately. First suppose that $(A, E) \models (\exists x\chi)^{\psi}(x, \overline{c}, \overline{a})$. Thus

$$(A,E) \models \exists x[\psi(x,\overline{a}) \wedge \chi^{\psi}(x,\overline{c},\overline{a})]. \tag{3}$$

Choose $b \in A$ such that $(A, E) \models [\psi(b, \overline{a}) \wedge \chi^{\psi}(b, \overline{c}, \overline{a})]$. Now $(A, E) \models \psi(b, \overline{a})$ implies that $b \in B$. Hence $(A, E) \models \chi^{\psi}(b, \overline{c}, \overline{a})]$ implies by the inductive hypothesis that $(B, F) \models \chi(b, \overline{c})$. Hence $(B, F) \models \exists x\chi(x, \overline{c})$.

Conversely, suppose that $(B, F) \models \exists x\chi(x, \overline{c})$; we want to prove (3). Choose $b \in B$ such that $(B, F) \models \chi(b, \overline{c})$. Hence $(A, E) \models \psi(b, \overline{a})$, and by the inductive hypothesis, $(A, E) \models \chi^{\psi}(b, \overline{c}, \overline{a})$. So $(A, E) \models \psi(b, \overline{a}) \wedge \chi^{\psi}(b, \overline{c}, \overline{a})$. Now (3) follows.

This finishes the proof of (2).

Now by (1) and (2), $(B, F) \models \Gamma$. By an assumption of the theorem, $\Gamma \models \varphi$, so $(B, F) \models \varphi$. Now an application of (2) yields $(A, E) \models \varphi^{\psi}$.

This shows that $\Gamma^{\psi} \models \exists x\psi(x, \overline{y}) \rightarrow \varphi^{\psi}$. □

This theorem gives the basic idea of consistency proofs in set theory; we express this as follows. By "consistent" we mean "has a model".

Corollary 14.2. *Suppose that* Γ *and* Δ *are collections of sentences in our language of set theory. Suppose that* $\mathbf{M}$ *is a class, and* $\Gamma \models [\mathbf{M} \neq \emptyset$ *and "*$\mathbf{M}$ *is a model of* Δ*"*$]$. *Then* Γ *consistent implies that* Δ *is consistent.*

Although this is the form we use in practice, for the proof we give the more rigorous version:

Suppose that Γ and Δ are collections of sentences in our language of set theory. Suppose that $\psi(x,\overline{y})$ is a formula, and for every sentence $\varphi \in \Delta$, $\Gamma \models \exists x\psi(x,\overline{y}) \wedge \varphi^{\psi}$. Also assume that Γ has a model. Then Δ has a model.

Proof. Suppose to the contrary that Δ does not have a model. Then trivially $\Delta \models \neg(x=x)$. By Theorem 14.1, $\Delta^{\psi} \models \exists x\psi(x,\overline{y}) \to \neg(x=x)$. Hence by hypothesis we get $\Gamma \models \neg(x=x)$, contradiction. □

Our main examples of the use of this corollary are with $\Gamma =$ ZFC and $\Delta =$ ZFC+"there are no inaccessibles"; and with $\Gamma =$ ZF and $\Delta =$ ZFC + AC + GCH. The class $\mathbf{M}$ is more complicated to describe, and we defer that until actually ready to give the applications of the corollary.

Now we give some simple facts which will be useful in checking the axioms of ZFC in the transitive classes which we will define. So we need to refer back to the beginning of the notes, where the ZFC axioms were given.

Theorem 14.3. *The extensionality axiom holds in any nonempty transitive class.*

Proof. The relativized version of the extensionality axiom is

$$\forall x \in \mathbf{M} \forall y \in \mathbf{M}[\forall z \in \mathbf{M}(z \in x \leftrightarrow z \in y) \to x = y].$$

To prove this, assume that $x, y \in \mathbf{M}$, and suppose that for all $z \in \mathbf{M}$, $z \in x$ iff $z \in y$. Take any $z \in x$. Because $\mathbf{M}$ is transitive, we get $z \in \mathbf{M}$. Hence $z \in y$. Thus $z \in x$ implies that $z \in y$. The converse is similar. So $x = y$. □

The following theorem reduces checking the comprehension axioms to checking a closure property.

Theorem 14.4. *Suppose that $\mathbf{M}$ is a nonempty class, and for each formula φ with with free variables among $x, z, w_1, \ldots, w_n$,*

$$\forall z, w_1, \ldots, w_n \in \mathbf{M}[\{x \in z : \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\} \in \mathbf{M}].$$

Then the comprehension axioms hold in $\mathbf{M}$.

Proof. The straightforward relativization of an instance of the comprehension axioms is

$$\forall z \in \mathbf{M} \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \exists y \in \mathbf{M} \forall x \in \mathbf{M}(x \in y \leftrightarrow x \in z \wedge \varphi^{\mathbf{M}}).$$

So, we take $z, w_1, \ldots, w_n \in \mathbf{M}$. Let

$$y = \{x \in z : \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\};$$

by hypothesis, we have $y \in \mathbf{M}$. Then for any $x \in \mathbf{M}$,

$$x \in y \quad \text{iff} \quad x \in z \text{ and } \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n).$$ □

The following theorems are obvious from the forms of the pairing axiom and union axioms:

Theorem 14.5. *Suppose that* $\mathbf{M}$ *is a nonempty class and*

$$\forall x, y \in \mathbf{M} \exists z \in \mathbf{M} (x \in z \text{ and } y \in z).$$

Then the pairing axiom holds in $\mathbf{M}$. □

Theorem 14.6. *Suppose that* $\mathbf{M}$ *is a nonempty class and*

$$\forall x \in \mathbf{M} \exists z \in \mathbf{M} \left(\bigcup x \subseteq z \right).$$

Then the union axiom holds in $\mathbf{M}$. □

For the next result, recall that $z \subseteq x$ is an abbreviation for $\forall w (w \in z \to w \in x)$.

Theorem 14.7. *Suppose that* $\mathbf{M}$ *is a nonempty transitive class. Then the following are equivalent:*

(i) The power set axiom holds in $\mathbf{M}$.

(ii) For every $x \in \mathbf{M}$ *there is a* $y \in \mathbf{M}$ *such that* $\mathscr{P}(x) \cap \mathbf{M} \subseteq y$.

Proof. (i)⇒(ii): Assume (i). Thus

$$\forall x \in \mathbf{M} \exists y \in \mathbf{M} \forall z \in \mathbf{M} [\forall w \in \mathbf{M} (w \in z \to w \in x) \to z \in y]. \tag{1}$$

To prove (ii), take any $x \in \mathbf{M}$. Choose $y \in \mathbf{M}$ as in (1). Suppose that $z \in \mathscr{P}(x) \cap \mathbf{M}$. Clearly then $\forall w \in \mathbf{M}(w \in z \to w \in x)$, so by (1), $z \in y$, as desired in (ii).

(ii)⇒(i): Assume (ii). This time we want to prove (1). So, suppose that $x \in \mathbf{M}$. Choose $y \in \mathbf{M}$ as in (ii). Now suppose that $z \in \mathbf{M}$ and $\forall w \in \mathbf{M}(w \in z \to w \in x)$. Then the transitivity of $\mathbf{M}$ implies that $\forall w (w \in z \to w \in x)$, i.e., $z \subseteq x$. So by (ii), $z \in y$, as desired. □

We defer the discussion of the infinity axiom until we talk about absoluteness.

Theorem 14.8. *Suppose that* $\mathbf{M}$ *is a transitive class, and for every formula* φ *with free variables among* $x, y, A, w_1, \ldots, w_n$ *and for any* $A, w_1, \ldots, w_n \in \mathbf{M}$ *the following implication holds:*

$$\begin{aligned} &\forall x \in A \exists! y [y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)] \quad \textit{implies that} \\ &\exists Y \in \mathbf{M}[\{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y]]. \end{aligned}$$

Then the replacement axioms hold in $\mathbf{M}$.

Proof. Assume the hypothesis of the theorem. We write out the relativized version of an instance of the replacement axiom in full, remembering to replace the quantifier $\exists!$ by its definition:

$$\begin{aligned} &\forall A \in \mathbf{M} \forall w_1 \in \mathbf{M} \ldots \forall w_n \in \mathbf{M} \\ &[\forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M} \\ &\qquad [\varphi^{\mathbf{M}}(x, u, A, w_1, \ldots, w_n) \to y = u]]] \to \\ &\exists Y \in \mathbf{M} \forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[y \in Y \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)]]]. \end{aligned}$$

To prove this, assume that $A, w_1, \ldots, w_n \in \mathbf{M}$ and

$$\forall x \in \mathbf{M}[x \in A \to \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M} \\ [\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \to y = u]]].$$

Since $\mathbf{M}$ is transitive, we get

$$\forall x \in A \exists y \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \wedge \forall u \in \mathbf{M}[\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n) \to y = u]],$$

so that

$$(1) \qquad \forall x \in A \exists! y[y \in \mathbf{M} \wedge \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)].$$

Hence by the hypothesis of the theorem we get $Y \in \mathbf{M}$ such that

$$(2) \qquad \{y \in \mathbf{M} : \exists x \in A \varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)\} \subseteq Y.$$

Suppose that $x \in \mathbf{M}$ and $x \in A$. By (1) we get $y \in \mathbf{M}$ such that $\varphi^{\mathbf{M}}(x, y, A, w_1, \ldots, w_n)$. Hence by (2) we get $y \in Y$, as desired. $\square$

Theorem 14.9. *If* $\mathbf{M}$ *is a transitive class, then the foundation axiom holds in* $\mathbf{M}$.

Proof. The foundation axiom, with the defined notion $\emptyset$ eliminated, is

$$\forall x[\exists y(y \in x) \to \exists y[y \in x \wedge \forall z \in y(z \notin x)]].$$

Hence the relativized version is

$$\forall x \in \mathbf{M}[\exists y \in \mathbf{M}(y \in x) \to \exists y \in \mathbf{M}[y \in x \wedge \forall z \in \mathbf{M}[z \in y \to z \notin x]]].$$

So, take any $x \in \mathbf{M}$, and suppose that there is a $y \in \mathbf{M}$ such that $y \in x$. Choose $y \in x$ so that $y \cap x = \emptyset$. Then $y \in \mathbf{M}$ by transitivity. If $z \in \mathbf{M}$ and $z \in y$, then $z \notin x$, as desired. $\square$

Absoluteness

To treat the infinity axiom and more complicated statements, we need to go into the important notion of absoluteness. Roughly speaking, a formula is absolute provided that its meaning does not change in going from one set to a bigger one, or vice versa. The exact definition is as follows.

• Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes and $\varphi(x_1, \ldots, x_n)$ is a formula of our set-theoretical language. We say that φ is *absolute for* $\mathbf{M}, \mathbf{N}$ iff

$$\forall x_1, \ldots, x_n \in \mathbf{M}[\varphi^{\mathbf{M}}(x_1, \ldots, x_n) \text{ iff } \varphi^{\mathbf{N}}(x_1, \ldots, x_n)].$$

An important special case of this notion occurs when $\mathbf{N} = \mathbf{V}$. Then we just say that φ is *absolute for* $\mathbf{M}$.

More formally, we associate with three formulas $\mu(y, w_1, \ldots, w_m)$, $\nu(y, w_1, \ldots, w_m)$, $\varphi(x_1, \ldots, x_n)$ another formula "φ is absolute for μ, ν", namely the following formula:

$$\forall x_1, \ldots, x_n \left[\bigwedge_{1 \le i \le n} \mu(x_i) \to [\varphi^\mu(x_1, \ldots, x_n) \leftrightarrow \varphi^\nu(x_1, \ldots, x_n)] \right].$$

In full generality, very few formulas are absolute; for example, see exercise E14.5. Usually we need to assume that the sets are transitive. Then there is an important set of formulas all of which are absolute; this class is defined as follows.

- The set of Δ_0-formulas is the smallest set Γ of formulas satisfying the following conditions:
  - (a) Each atomic formula is in Γ.
  - (b) If φ and ψ are in Γ, then so are $\neg\varphi$ and $\varphi \wedge \psi$.
  - (c) If φ is in Γ, then so are $\exists x \in y\varphi$ and $\forall x \in y\varphi$.

Recall here that $\exists x \in y\varphi$ and $\forall x \in y\varphi$ are abbreviations for $\exists x(x \in y \wedge \varphi)$ and $\forall x(x \in y \to \varphi)$ respectively.

Theorem 14.10. *If* $\mathbf{M}$ *is transitive and* φ *is* Δ_0*, then* φ *is absolute for* $\mathbf{M}$.

Proof. We show that the collection of formulas absolute for $\mathbf{M}$ satisfies the conditions defining the set Δ_0. Absoluteness is clear for atomic formulas. It is also clear that if φ and ψ are absolute for $\mathbf{M}$, then so are $\neg\varphi$ and $\varphi \wedge \psi$. Now suppose that φ is absolute for $\mathbf{M}$; we show that $\exists x \in y\varphi$ is absolute for $\mathbf{M}$. Implicitly, φ can involve additional parameters $w_1, \ldots, w_n$. Assume that $y, w_1, \ldots, w_n \in \mathbf{M}$. First suppose that $\exists x \in y\varphi(x, y, w_1, \ldots, w_n)$. Choose $x \in y$ so that $\varphi(x, y, w_1, \ldots, w_n)$. Since $\mathbf{M}$ is transitive, $x \in \mathbf{M}$. Hence by the "inductive assumption", $\varphi^{\mathbf{M}}(x, y, w_1, \ldots, w_n)$ holds. This shows that $(\exists x \in y\varphi(x, y, w_1, \ldots, w_n))^{\mathbf{M}}$. Conversely suppose that $(\exists x \in y\varphi(x, y, w_1, \ldots, w_n))^{\mathbf{M}}$. Thus $\exists x \in \mathbf{M}[x \in y \wedge \varphi^{\mathbf{M}}(x, y, w_1, \ldots, w_n)$. By the inductive assumption, $\varphi(x, y, w_1, \ldots, w_n)$. So this shows that $\exists x \in y\varphi(x, y, w_1, \ldots, w_n)$. The case $\forall x \in y\varphi$ is treated similarly. $\square$

Ordinals and special kinds of ordinals are absolute since they could have been defined using Δ_0 formulas:

Theorem 14.11. *The following are absolute for any transitive class:*

(i) x is an ordinal	*(iii) x is a successor ordinal*	*(v) x is ω*
(ii) x is a limit ordinal	*(iv) x is a finite ordinal*	*(vi) x is i (each $i < 10$)*

Proof.

$$\begin{aligned}
x \text{ is an ordinal} &\leftrightarrow \forall y \in x \forall z \in y[z \in x] \wedge \forall y \in x \forall z \in y \forall w \in z[w \in y];\\
x \text{ is a limit ordinal} &\leftrightarrow \exists y \in x[y = y] \wedge x \text{ is an ordinal} \wedge \forall y \in x \exists z \in x(y \in z);\\
x \text{ is a successor ordinal} &\leftrightarrow x \text{ is an ordinal} \wedge x \neq \emptyset \wedge x \text{ is not a limit ordinal};\\
x \text{ is a finite ordinal} &\leftrightarrow \forall y[y \notin x] \vee (x \text{ is a successor ordinal}\\
&\qquad \wedge \forall y \in x(\forall z[z \notin y] \vee y \text{ is a successor ordinal}));\\
x = \omega &\leftrightarrow x \text{ is a limit ordinal} \wedge \forall y \in x(y \text{ is a finite ordinal});
\end{aligned}$$

finally, we do (vi) by induction on i. The case $i = 0$ is clear. Then

$$y = i + 1 \leftrightarrow \exists x \in y(x = i \wedge \forall z \in y[z \in x \vee z = x] \wedge \forall z \in x[z \in y] \wedge x \in y. \qquad \square$$

The following theorem, while obvious, will be very useful in what follows.

Theorem 14.12. *Suppose that S is a set of sentences in our set-theoretic language, and $\mathbf{M}$ and $\mathbf{N}$ are classes which are models of S. Suppose that*

$$S \models \forall x_1, \ldots, x_n[\varphi(x_1, \ldots, x_n) \leftrightarrow \psi(x_1, \ldots, x_n)].$$

Then φ is absolute for $\mathbf{M}, \mathbf{N}$ iff ψ is. $\square$

Of course we will usually apply this when S is a subset of ZFC.

Recall that all of the many definitions that we have made in our development of set theory are supposed to be eliminable in favor of our original language. To apply Theorem 14.12, we should note that the development of the very elementary set theory in Chapter 3 did not use the axiom of choice or the axiom of infinity. We let ZF be our axioms without the axiom of choice, and ZF − Inf the axioms ZF without the axiom of infinity.

The status of the functions that we have defined requires some explanation. Whenever we defined a function $\mathbf{F}$ of n arguments, we have implicitly assumed that there is an associated formula φ whose free variables are among the first $n + 1$ variables, so that the following is derivable from the axioms assumed at the time of defining the function:

$$\forall v_0, \ldots, v_{n-1} \exists! v_n \varphi(v_0, \ldots, v_n).$$

Recall that "$\exists! v_n$" means "there is exactly one v_n". Now if we have a class model $\mathbf{M}$ in which this sentence holds, then we can define $\mathbf{F}^{\mathbf{M}}$ by setting, for any $x_0, \ldots, x_{n-1} \in \mathbf{M}$,

$$\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \text{ the unique } y \text{ such that } \varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y).$$

In case $\mathbf{M}$ satisfies the indicated sentence, we say that $\mathbf{F}$ *is defined in* $\mathbf{M}$. Given two class models $\mathbf{M} \subseteq \mathbf{N}$ in which $\mathbf{F}$ is defined, we say that $\mathbf{F}$ is *absolute for* $\mathbf{M}, \mathbf{N}$ provide that φ is. Note that for $\mathbf{F}$ to be absolute for $\mathbf{M}, \mathbf{N}$ it must be defined in both of them.

Proposition 14.13. *Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are models in which $\mathbf{F}$ is defined. Then the following are equivalent:*
(i) $\mathbf{F}$ is absolute for $\mathbf{M}, \mathbf{N}$.
(ii) For all $x_0, \ldots, x_{n-1} \in \mathbf{M}$ we have $\mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1})$.

Proof. Let φ be as above.

Assume (i), and suppose that $x_0, \ldots, x_{n-1} \in \mathbf{M}$. Let $y = \mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1})$. Then $y \in \mathbf{M}$, and $\varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y)$, so by (i), $\varphi^{\mathbf{N}}(x_0, \ldots, x_{n-1}, y)$. Hence $\mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1}) = y$.

Assume (ii), and suppose that $x_0, \ldots, x_{n-1}, y \in \mathbf{M}$. Then

$$\begin{aligned} \varphi^{\mathbf{M}}(x_0, \ldots, x_{n-1}, y) \quad & \text{iff} \quad \mathbf{F}^{\mathbf{M}}(x_0, \ldots, x_{n-1}) = y \quad \text{(definition of } \mathbf{F}) \\ & \text{iff} \quad \mathbf{F}^{\mathbf{N}}(x_0, \ldots, x_{n-1}) = y \quad \text{(by (ii))} \\ & \text{iff} \quad \varphi^{\mathbf{N}}(x_0, \ldots, x_{n-1}, y) \quad \text{(definition of } \mathbf{F}). \qquad \square \end{aligned}$$

The following theorem gives many explicit absoluteness results, and will be used frequently along with some similar results below. Note that we do not need to be explicit about how the relations and functions were really defined in Chapter 1; in fact, we were not very explicit about that in the first place.

Theorem 14.14. *The following relations and functions were defined by formulas equivalent to Δ_0-formulas on the basis of* $\mathrm{ZF}-\mathrm{Inf}$, *and hence are absolute for all transitive class models of* $\mathrm{ZF}-\mathrm{Inf}$*:*

(i) $x \in y$
(ii) $x = y$
(iii) $x \subseteq y$
(iv) $\{x, y\}$
(v) $\{x\}$
(vi) (x, y)
(vii) $\emptyset$
(viii) $x \cup y$
(ix) $x \cap y$
(x) $x \backslash y$
(xi) $x \cup \{x\}$
(xii) x *is transitive*
(xiii) $\bigcup x$
(xiv) $\bigcap x$ *(with* $\bigcap \emptyset = \emptyset$*)*

Note here, for example, that in (iv) we really mean the 2-place function assigning to sets x, y the unordered pair $\{x, y\}$.

Proof. (i) and (ii) are already Δ_0 formulas. (iii):

$$x \subseteq y \leftrightarrow \forall z \in x(z \in y).$$

(iv):

$$z = \{x, y\} \leftrightarrow \forall w \in z(w = x \vee w = y) \wedge x \in z \wedge y \in z.$$

(v): Similarly. (vi):

$$z = (x, y) \leftrightarrow \forall w \in z[w = \{x, y\} \vee w = \{x\}] \wedge \exists w \in z[w = \{x, y\}] \wedge \exists w \in z[w = \{x\}].$$

(vii):

$$x = \emptyset \leftrightarrow \forall y \in x(y \neq y).$$

(viii):

$$z = x \cup y \leftrightarrow \forall w \in z(w \in x \vee w \in y) \wedge \forall w \in x(w \in z) \wedge \forall w \in y(w \in z).$$

(ix):

$$z = x \cap y \leftrightarrow \forall w \in z(w \in x \wedge w \in y) \wedge \forall w \in x(w \in y \rightarrow w \in z).$$

(x):

$$z = x \backslash y \leftrightarrow \forall w \in z(w \in x \wedge w \notin y) \wedge \forall w \in x(x \notin y \rightarrow w \in z).$$

(xi):

$$y = x \cup \{x\} \leftrightarrow \forall w \in y(w \in x \vee w = x) \wedge \forall w \in x(w \in y) \wedge x \in y.$$

(xii):

$$x \text{ is transitive} \leftrightarrow \forall y \in x(y \subseteq x).$$

(xiii):

$$y = \bigcup x \leftrightarrow \forall w \in y \exists z \in x(w \in z) \wedge \forall w \in x(w \subseteq y).$$

(xiv):

$$\begin{aligned}y = \bigcap x \leftrightarrow [&x \neq \emptyset \wedge \forall w \in y \forall z \in x(w \in z)\\ &\wedge \forall w \in x \forall t \in w[\forall z \in x(t \in z) \rightarrow t \in y] \vee [x = \emptyset \wedge y = \emptyset].\end{aligned}$$ □

A stronger form of Theorem 14.14. For each of the indicated relations and functions, we do not need $\mathbf{M}$ to be a model of all of ZF − Inf. In fact, we need only finitely many of the axioms of ZF − Inf: enough to prove the uniqueness condition for any functions involved, and enough to prove the equivalence of the formula with a Δ_0-formula, since Δ_0 formulas are absolute for any transitive class model. To be absolutely rigorous here, one would need an explicit definition for each relation and function symbol involved, and then an explicit proof of equivalence to a Δ_0 formula; given these, a finite set of axioms becomes clear. And since any of the relations and functions of Theorem 14.14 require only finitely many basic relations and functcions, this can always be done. For Theorem 14.14 it is easy enough to work this all out in detail. We will be interested, however, in using this fact for more complicated absoluteness results to come.

As an illustration, however, we do some details for the function $\{x, y\}$. The definition involved is naturally taken to be the following:

$$\forall x, y, z[z = \{x, y\} \leftrightarrow \forall w[w \in z \leftrightarrow w = x \vee x = y]].$$

The axioms involved are the pairing axiom and one instance of the comprehension axiom:

$$\begin{aligned}&\forall x, y \exists w[x \in w \wedge y \in w];\\ &\forall x, y, w \exists z \forall u(u \in z \leftrightarrow u \in w \wedge (u = x \vee u = y)).\end{aligned}$$

$\{x, y\}$ is then absolute for any transitive class model of these three sentences, by the proof of (iv) in Theorem 14.14, for which they are sufficient.

For further absoluteness results we will not reduce to Δ_0 formulas. We need the following extensions of the absoluteness notion.

• Suppose that $\mathbf{M} \subseteq \mathbf{N}$ are classes, and $\varphi(w_1, \ldots, w_n)$ is a formula. Then we say that φ is *absolute upwards* for $\mathbf{M}, \mathbf{N}$ iff for all $w_1, \ldots, w_n \in \mathbf{M}$, if $\varphi^{\mathbf{M}}(w_1, \ldots, w_n)$, then $\varphi^{\mathbf{N}}(w_1, \ldots, w_n)$. It is *absolute downwards* for $\mathbf{M}, \mathbf{N}$ iff for all $w_1, \ldots, w_n \in \mathbf{M}$, if $\varphi^{\mathbf{N}}(w_1, \ldots, w_n)$, then $\varphi^{\mathbf{M}}(w_1, \ldots, w_n)$.

Theorem 14.15. *Suppose that* $\varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ *is absolute for* $\mathbf{M}, \mathbf{N}$. *Then*
(i) $\exists x_1, \ldots \exists x_n \varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ *is absolute upwards for* $\mathbf{M}, \mathbf{N}$.
(ii) $\forall x_1, \ldots \forall x_n \varphi(x_1, \ldots, x_n, w_1, \ldots, w_m)$ *is absolute downwards for* $\mathbf{M}, \mathbf{N}$. □

Theorem 14.16. *Absoluteness is preserved under composition. In detail: suppose that* $\mathbf{M} \subseteq \mathbf{N}$ *are classes, and the following are absolute:*

$\varphi(x_1, \ldots, x_n)$*;*
$\mathbf{F}$*, an n-ary function ;*
For each $i = 1, \ldots, n$*, an* m*-ary function* $\mathbf{G}_i$*.*

Then the following are absolute:

(i) $\varphi(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m))$.

(ii) The m-ary function assigning to $x_1,\ldots,x_m$ the value

$$\mathbf{F}(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)).$$

Proof. We use Theorem 14.15:

$$\begin{aligned}
\varphi(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)) \leftrightarrow &\exists z_1,\ldots\exists z_n\biggl[\varphi(z_1,\ldots,z_n)\\
&\wedge\bigwedge_{i=1}^n(z_i=\mathbf{G}_i(x_1,\ldots,x_m))\biggr];\\
\varphi(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)) \leftrightarrow &\forall z_1,\ldots\forall z_n\biggl[\bigwedge_{i=1}^n(z_i=\mathbf{G}_i(x_1,\ldots,x_m))\\
&\to\varphi(z_1,\ldots,z_n)\biggr];\\
y=\mathbf{F}(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)) \leftrightarrow &\exists z_1,\ldots\exists z_n\biggl[(y=\mathbf{F}(z_1,\ldots,z_n))\\
&\wedge\bigwedge_{i=1}^n(z_i=\mathbf{G}_i(x_1,\ldots,x_m))\biggr];\\
y=\mathbf{F}(\mathbf{G}_1(x_1,\ldots,x_m),\ldots,\mathbf{G}_n(x_1,\ldots,x_m)) \leftrightarrow &\forall z_1,\ldots\forall z_n\biggl[\bigwedge_{i=1}^n(z_i=\mathbf{G}_i(x_1,\ldots,x_m))\\
&\to(y=\mathbf{F}(z_1,\ldots,z_n))\biggr]. \qquad\square
\end{aligned}$$

Theorem 14.17. *Suppose that $\mathbf{M}\subseteq\mathbf{N}$ are classes, $\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)$ is absolute for $\mathbf{M},\mathbf{N}$, and $\mathbf{F}$ and $\mathbf{G}$ are n-ary functions absolute for $\mathbf{M},\mathbf{N}$. Then the following are also absolute for $\mathbf{M},\mathbf{N}$:*

(i) $z\in\mathbf{F}(x_1,\ldots,x_m)$.

(ii) $\mathbf{F}(x_1,\ldots,x_m)\in z$.

(iii) $\exists y\in\mathbf{F}(x_1,\ldots,x_m)\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)$.

(iv) $\forall y\in\mathbf{F}(x_1,\ldots,x_m)\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)$.

(v) $\mathbf{F}(x_1,\ldots,x_m)=\mathbf{G}(x_1,\ldots,x_m)$.

(vi) $\mathbf{F}(x_1,\ldots,x_m)\in\mathbf{G}(x_1,\ldots,x_m)$.

Proof.

$$\begin{aligned}
z\in\mathbf{F}(x_1,\ldots,x_m)&\leftrightarrow\exists w[z\in w\wedge w=\mathbf{F}(x_1,\ldots,x_m)];\\
&\leftrightarrow\forall w[w=\mathbf{F}(x_1,\ldots,x_m)\to z\in w];\\
\mathbf{F}(x_1,\ldots,x_m)\in z&\leftrightarrow\exists w\in z[w=\mathbf{F}(x_1,\ldots,x_m)];
\end{aligned}$$

$$\begin{aligned}&\exists y \in \mathbf{F}(x_1,\ldots,x_m)\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)\\&\leftrightarrow \exists w \exists y \in w[w = \mathbf{F}(x_1,\ldots,x_m) \wedge \varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)];\\&\leftrightarrow \forall w[w = \mathbf{F}(x_1,\ldots,x_m) \rightarrow \exists y \in w\varphi(y,x_1,\ldots,x_m,w_1,\ldots,w_n)];\end{aligned}$$

(iv)–(vi) are proved similarly. □

We now give some more specific absoluteness results.

Theorem 14.18. *The following relations and functions are absolute for all transitive class models of* ZF − Inf*:*

(i) x is an ordered pair	*(iv)* $\mathrm{dmn}(R)$	*(vii)* $R(x)$
(ii) $A \times B$	*(v)* $\mathrm{rng}(R)$	*(viii) R is a one-one function*
(iii) R is a relation	*(vi) R is a function*	*(ix) x is an ordinal*

Note concerning (vii): This is supposed to have its natural meaning if R is a function and x is in its domain; otherwise, $R(x) = \emptyset$.

Proof.

$$\begin{aligned}
x \text{ is an ordered pair} &\leftrightarrow \left(\exists y \in \bigcup x\right)\left(\exists z \in \bigcup x\right)[x = (y,z)];\\
y = A \times B &\leftrightarrow (\forall a \in A)(\forall b \in B)[(a,b) \in y] \wedge\\
&\quad (\forall z \in y)(\exists a \in A)(\exists b \in B)[z = (a,b)];\\
R \text{ is a relation} &\leftrightarrow \forall x \in R[x \text{ is an ordered pair}];\\
x = \mathrm{dmn}(R) &\leftrightarrow (\forall y \in x)\left(\exists z \in \bigcup\bigcup R\right)[(x,z) \in R] \wedge\\
&\quad \left(\forall y \in \bigcup\bigcup R\right)\left(\forall z \in \bigcup\bigcup R\right)[(y,z) \in R \rightarrow y \in x];\\
x = \mathrm{rng}(R) &\leftrightarrow (\forall y \in x)\left(\exists z \in \bigcup\bigcup R\right)[(z,x) \in R] \wedge\\
&\quad \left(\forall y \in \bigcup\bigcup R\right)\left(\forall z \in \bigcup\bigcup R\right)[(y,z) \in R \rightarrow z \in x];\\
R \text{ is a function} &\leftrightarrow R \text{ is a relation } \wedge \left(\forall x \in \bigcup\bigcup R\right)\left(\forall y \in \bigcup\bigcup R\right)\\
&\quad \left(\forall z \in \bigcup\bigcup R\right)[(x,y) \in R \wedge (x,z) \in R \rightarrow y = z];\\
y = R(x) &\leftrightarrow [R \text{ is a function} \wedge (x,y) \in R] \vee\\
&\quad [R \text{ is not a function} \wedge (\forall z \in y)(z \neq z)] \vee\\
&\quad [x \notin \mathrm{dmn}(R) \wedge (\forall z \in y)(z \neq z)];\\
R \text{ is a one-one function} &\leftrightarrow R \text{ is a function} \wedge\\
&\quad \forall x \in \mathrm{dmn}(R) \forall y \in \mathrm{dmn}(R)[R(x) = R(y) \rightarrow x = y];\\
x \text{ is an ordinal} &\leftrightarrow x \text{ is transitive} \wedge (\forall y \in x)(y \text{ is transitive}).
\end{aligned}$$

□

Theorem 14.19. *Suppose that* $\mathbf{M}$ *is a transitive class model of* ZF − Inf *and* $\omega \in \mathbf{M}$*. Then the infinity axiom holds in* $\mathbf{M}$*.*

Proof. We have

$$\exists x \in \mathbf{M}[\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x)],$$

so by the absoluteness of the notions here we get

$$[\exists x[\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x)]]^{\mathbf{M}},$$

which means that the infinity axiom holds in $\mathbf{M}$. $\square$

Theorem 14.20. *If* $\mathbf{M}$ *is a transitive class model of* ZF*, then* $\emptyset, \omega \in \mathbf{M}$ *and* $\mathbf{M}$ *is closed under the following set-theoretic operations:*

(i) $\cup$	*(iv)* $(a, b) \mapsto \{a, b\}$	*(vii)* $\bigcup$
(ii) $\cap$	*(v)* $(a, b) \mapsto (a, b)$	*(viii)* $\bigcap$
(iii) $(a, b) \mapsto a \backslash b$	*(vi)* $x \mapsto x \cup \{x\}$	

Moreover, $\omega + 1 \subseteq \mathbf{M}$ *and* $[\mathbf{M}]^{<\omega} \subseteq \mathbf{M}$.

Proof. $\mathbf{M}$ has elements x, y such that $(x = \emptyset)^{\mathbf{M}}$ and $(y = \omega)^{\mathbf{M}}$. So $x = \emptyset$ and $y = \omega$ by absoluteness. (See Theorem 14.13(vii) and Theorem 14.19(iv).) (i)–(viii) are all very similar, so we only treat (i). Let $a, b \in \mathbf{M}$. Then because $\mathbf{M} \models$ ZF, there is a $c \in \mathbf{M}$ such that $(c = a \cup b)^{\mathbf{M}}$. By absoluteness, $c = a \cup b$.

Each $i \in \omega$ is in $\mathbf{M}$ by transitivity. Hence $\omega + 1 \subseteq \mathbf{M}$. Finally, we show by induction on n that if $x \subseteq \mathbf{M}$ and $|x| = n$ then $x \in \mathbf{M}$. This is clear for $n = 0$. Now suppose inductively that $x \subseteq \mathbf{M}$ and $|x| = n + 1$. Let $a \in x$ and set $y = x \backslash \{a\}$. So $|y| = n$. Hence $y \in \mathbf{M}$ by the inductive hypothesis. Hence $x = y \cup \{a\} \in \mathbf{M}$ by previous parts of this theorem. $\square$

Our final abstract absoluteness result concerns recursive definitions.

Theorem 14.21. *Suppose that* $\mathbf{R}$ *is a class relation which is well-founded and set-like on* $\mathbf{A}$*, and* $\mathbf{F} : \mathbf{A} \times \mathbf{V} \to \mathbf{V}$*. Let* $\mathbf{G}$ *be given by Theorem 11.8: for all* $x \in \mathbf{A}$*,*

$$\mathbf{G}(x) = \mathbf{F}(x, \mathbf{G} \restriction \mathrm{pred}(\mathbf{A}, x, \mathbf{R})).$$

Let $\mathbf{M}$ *be a transitive class model of* ZF*, and assume the following additional conditions hold:*

(i) $\mathbf{F}$*,* $\mathbf{R}$*, and* $\mathbf{A}$ *are absolute for* $\mathbf{M}$*.*

(ii) $(\mathbf{R}$ *is set-like on* $\mathbf{A})^{\mathbf{M}}$*.*

(iii) $\forall x \in \mathbf{M} \cap \mathbf{A}[\mathrm{pred}(\mathbf{A}, x, \mathbf{R}) \subseteq \mathbf{M}]$*.*

Conclusion: $\mathbf{G}$ *is absolute for* $\mathbf{M}$*.*

Proof. First we claim

(1) $(\mathbf{R}$ is well-founded on $\mathbf{A})^{\mathbf{M}}$.

In fact, by absolutenss $\mathbf{R}^{\mathbf{M}} = \mathbf{R} \cap (\mathbf{M} \times \mathbf{M})$ and $\mathbf{A}^{\mathbf{M}} = \mathbf{A} \cap \mathbf{M}$, so it follows that in $\mathbf{M}$ every nonempty subset of $\mathbf{A}^{\mathbf{M}}$ has an $\mathbf{R}^{\mathbf{M}}$-minimal element. Hence we can apply the recursion theorem within $\mathbf{M}$ to define a function $\mathbf{H} : \mathbf{A}^{\mathbf{M}} \to \mathbf{M}$ such that for all $x \in \mathbf{A}^{\mathbf{M}}$,

$$\mathbf{H}(x) = \mathbf{F}^{\mathbf{M}}(x, \mathbf{H} \restriction \mathrm{pred}^{\mathbf{M}}(\mathbf{A}^{\mathbf{M}}, x, \mathbf{R}^{\mathbf{M}})).$$

We claim that $\mathbf{H} = \mathbf{G} \restriction \mathbf{A}^{\mathbf{M}}$, which will prove the theorem. In fact, suppose that x is $\mathbf{R}$-minimal such that $x \in \mathbf{A}^{\mathbf{M}}$ and $\mathbf{G}(x) \neq \mathbf{H}(x)$. Then using absoluteness again,

$$\mathbf{H}(x) = \mathbf{F}^{\mathbf{M}}(x, \mathbf{H} \restriction \mathrm{pred}^{\mathbf{M}}(\mathbf{A}^{\mathbf{M}}, x, \mathbf{R}^{\mathbf{M}})) = \mathbf{F}(x, \mathbf{H} \restriction \mathrm{pred}(\mathbf{A}, x, \mathbf{R})) = \mathbf{G}(x),$$

contradiction. □

Theorem 14.21 is needed for many deeper applications of absoluteness. We illustrate its use by the following result.

Theorem 14.22. *The following are absolute for transitive class models of* ZF*.*
(i) $\alpha + \beta$ *(ordinal addition)*
(ii) $\alpha \cdot \beta$ *(ordinal multiplication)*
(iii) α^{β} *(ordinal exponentiation)*
(iv) $\mathrm{rank}(x)$
(v) $\mathrm{trcl}(x)$

Proof. In each case it is mainly a matter of identifying $\mathbf{A}, \mathbf{R}, \mathbf{F}$ to which to apply Theorem 14.21; checking the conditions of that theorem are straightforward.

(i): $\mathbf{A} = \mathbf{On}$, $\mathbf{R} = \{(\alpha, \beta) : \alpha, \beta \in \mathbf{On}, \text{ and } \alpha \in \beta\}$, and $\mathbf{F} : \mathbf{On} \times \mathbf{V} \to \mathbf{V}$ is defined as follows:

$$\mathbf{F}(\alpha, f) = \begin{cases} \alpha & \text{if } f = \emptyset, \\ f(\beta) \cup \{f(\beta)\} & \text{if } f \text{ is a function with domain an ordinal } \beta, \\ \bigcup_{\gamma \in \beta} f(\gamma) & \text{if } f \text{ is a function with domain a limit ordinal } \beta, \\ \emptyset & \text{otherwise.} \end{cases}$$

(ii) and (iii) are treated similarly. For (iv), take $\mathbf{R} = \{(x, y) : x \in y\}$, $\mathbf{A} = \mathbf{V}$, and define $\mathbf{F} : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ by setting, for all $x, f \in \mathbf{V}$,

$$\mathbf{F}(x, f) = \begin{cases} \bigcup_{y \in x}(f(y) \cup \{f(y)\}) & \text{if } f \text{ is a function with domain } x, \\ \emptyset & \text{otherwise.} \end{cases}$$

For (v), let $\mathbf{R} = \{(i, j) : i, j \in \omega \text{ and } i < j\}$, $\mathbf{A} = \omega$, and define $\mathbf{F} : \omega \times \mathbf{V} \to \mathbf{V}$ by setting, for all $m \in \omega$ and $f \in \mathbf{V}$,

$$\mathbf{F}(m, f) = \begin{cases} x & \text{if } m = 0, \\ f(\bigcup m) \cup \bigcup f(\bigcup m) & \text{if } m > 0 \text{ and } f \text{ is a function with domain } m, \\ \emptyset & \text{otherwise} \end{cases}$$

Then the function $\mathbf{G}$ obtained from Theorem 14.20 is absolute for transitive class models of ZF, and $\mathrm{trcl}(x) = \bigcup_{m \in \omega} \mathbf{G}(m)$. □

Theorem 14.23. *If* $\mathbf{M}$ *is a transitive model of ZF, then:*
(i) $\mathscr{P}^{\mathbf{M}}(x) = \mathscr{P}(x) \cap \mathbf{M}$ *for any* $x \in \mathbf{M}$*;*
(ii) $V_{\alpha}^{\mathbf{M}} = V_{\alpha} \cap \mathbf{M}$ *for any* $\alpha \in \mathbf{M}$*.*

Proof. (i): Assume that $x \in \mathbf{M}$. Then for any set y,

$$\begin{aligned} y \in \mathscr{P}^{\mathbf{M}}(x) \quad &\text{iff} \quad y \in \mathbf{M} \text{ and } (y \subseteq x)^{\mathbf{M}} \\ &\text{iff} \quad y \in \mathbf{M} \text{ and } y \subseteq x \\ &\text{iff} \quad y \in \mathscr{P}(x) \cap \mathbf{M}. \end{aligned}$$

(ii): Assume that $\alpha \in \mathbf{M}$. Then for any set x,

$$\begin{aligned} x \in V_\alpha^{\mathbf{M}} \quad &\text{iff} \quad x \in \mathbf{M} \text{ and } \mathrm{rank}^{\mathbf{M}}(x) < \alpha \\ &\text{iff} \quad x \in \mathbf{M} \text{ and } \mathrm{rank}(x) < \alpha \end{aligned}$$

$$x \in \mathbf{M} \text{ and } \mathrm{rank}^{\mathbf{M}}(x) < \alpha$$

$$x \in V_\alpha \cap \mathbf{M}\square$$

Proposition 14.24. *"R well-orders A" is absolute for models of ZF.*

Proof. Let $\mathbf{M}$ be a model of ZF. Suppose that $R, A \in \mathbf{M}$. Clearly

$$\begin{aligned} (R \text{ well-orders } A) \quad \text{iff} \quad &\exists x \exists f[x \text{ is an ordinal} \wedge f : x \to A \text{ is a bijection} \\ &\wedge \forall \beta, \gamma \in x[\beta < \gamma \text{ iff } (f(\beta), f(\gamma)) \in R]]. \end{aligned}$$

From this and elementary absoluteness results it is clear that $(R \text{ well-orders } A)^{\mathbf{M}}$ implies that (R well-orders A). Now suppose that (R well-orders A). Let x and f be such that x is an ordinal, $f : x \to A$ is a bijection, and $\forall \beta, \gamma \in x[\beta < \gamma \text{ iff } (f(\beta), f(\gamma)) \in R]$. Since $\mathbf{M}$ is a model of ZF, let $y, g \in \mathbf{M}$ be such that in $\mathbf{M}$ we have: y is an ordinal, $g : y \to A$ is a bijection, and $\forall \beta, \gamma \in y[\beta < \gamma \text{ iff } (g(\beta), g(\gamma)) \in R]$. By simple absoluteness results, this is really true. Then $x = y$ and $f = g$ by the uniqueness conditions in 4.17–4.19. $\square$

For this and the other absoluteness results above, remember the remark which followed Theorem 14.13: we do not have to assume that M is a model of all of ZF, but is only a model of some finite portion of it.

Consistency of no inaccessibles

Theorem 14.25. *If* ZFC *is consistent, then so is* ZF+ *"there do not exist uncountable inaccessible cardinals".*

Proof. For brevity we interpret "inaccessible" to mean "uncountable and inaccessible". Let

$$\mathbf{M} = \{x : \forall \alpha[\alpha \text{ inaccessible } \to x \in V_\alpha]\}$$

Thus $\mathbf{M}$ is a class. We claim that $\mathbf{M}$ is a model of ZFC+"there do not exist uncountable inaccessible cardinals". To prove this, we consider two possibilities.

Case 1. $\mathbf{M} = \mathbf{V}$. Then of course $\mathbf{M}$ is a model of ZFC. Suppose that α is inaccessible. Then since $\mathbf{M} = \mathbf{V}$ we have $\mathbf{V} = V_\alpha$, which is not possible, since V_α is a set. Thus $\mathbf{M}$ is a model of ZFC + "there do not exist uncountable inaccessible cardinals".

Case 2. $\mathbf{M} \neq \mathbf{V}$. Let x be a set which is not in $\mathbf{M}$. Then there is an ordinal α such that α is inaccessible and $x \notin V_\alpha$. In particular, there is an inaccessible α, and we let κ be the least such.

(1) $\mathbf{M} = V_\kappa$.

In fact, if $x \in \mathbf{M}$, then $x \in V_\alpha$ for every inaccessible α, so in particular $x \in V_\kappa$. On the other hand, if $x \in V_\kappa$, then $x \in V_\alpha$ for every $\alpha \geq \kappa$, so $x \in V_\alpha$ for every inaccessible α, and so $x \in \mathbf{M}$. So (1) holds.

Now we show that V_κ is as desired. First, we need to check all the ZFC axioms. Here we use Theorems 14.3–14.9. Now V_κ is transitive, so by Theorem 14.3, the extensionality axiom holds in V_κ.

For the comprehension axioms, we are going to apply Theorem 14.4. Suppose that φ is a formula with free variables among $x, z, w_1, \ldots, w_n$, and we are given $z, w_1, \ldots, w_n \in V_\kappa$. Let $\mathbf{A} = \{x \in z : \varphi^{\mathbf{M}}(x, z, w_1, \ldots, w_n)\}$. Then $\mathbf{A} \subseteq z$. Say $z \in V_\beta$ with $\beta < \kappa$. Then $\mathbf{A} \subseteq z \subseteq V_\beta$, so $\mathbf{A} \in \mathscr{P}(V_\beta) = V_{\beta+1} \subseteq V_\kappa$. It follows from Theorem 14.4 that the comprehension axioms hold in V_κ.

Suppose that $x, y \in V_\kappa$. Say $x, y \in V_\beta$ with $\beta < \kappa$. Then $\{x, y\} \subseteq V_\beta$, so $\{x, y\} \in V_{\beta+1} \subseteq V_\kappa$. By Theorem 14.5, the pairing axiom holds in V_κ.

Suppose that $x \in V_\kappa$. Say $x \in V_\beta$ with $\beta < \kappa$. Then $\bigcup x \subseteq V_\beta$, so $\bigcup x \in V_{\beta+1} \subseteq V_\kappa$. By Theorem 14.6, the union axiom holds in V_κ.

Suppose that $x \in V_\kappa$. Say $x \in V_\beta$ with $\beta < \kappa$. Then $x \subseteq V_\beta$. Hence $y \subseteq V_\beta$ for each $y \subseteq x$, so $y \in \mathscr{P}(V_\beta) = V_{\beta+1}$ for each $y \in \mathscr{P}(x)$. This means that $\mathscr{P}(x) \subseteq V_{\beta+1}$, so $\mathscr{P}(x) \in V_{\beta+2}$. By Theorem 14.7, the power set axiom holds in V_κ.

For the replacement axioms, we will apply Theorem 14.8. So, suppose that φ is a formula with free variables among $x, y, A, w_1, \ldots, w_n$, any $A, w_1, \ldots, w_n \in V_\kappa$, and

$$\forall x \in A \exists! y[y \in V_\kappa \wedge \varphi^{V_\kappa}(x, y, A, w_1, \ldots, w_n)].$$

For each $x \in A$, let y_x be the unique set such that $y_x \in V_\kappa$ and $\varphi^{V_\kappa}(x, y_x, A, w_1, \ldots, w_n)$, and let $\alpha_x < \kappa$ be such that $y_x \in V_{\alpha_x}$. Choose $\beta < \kappa$ such that $A \in V_\beta$. Then $A \subseteq V_\beta$, and hence $|A| \le |V_\beta| < \kappa$ by Theorem 11.5(ii). It follows that $\gamma \overset{\text{def}}{=} \bigcup\{\alpha_x : x \in A\} < \kappa$. Let

$$Y = \{z \in V_\gamma : \exists x \in A \varphi^{V_\kappa}(x, z, A, w_1, \ldots, w_n)\}.$$

Thus $Y \subseteq V_\gamma$, so $Y \in V_{\gamma+1} \subseteq V_\kappa$. Suppose that $x \in A$ and $z \in V_\kappa$ is such that $\varphi^{V_\kappa}(x, z, A, w_1, \ldots, w_n)$. Then $z = y_x$ by the above, and so $z \in Y$, as desired.

By Theorem 14.9, the foundation axiom holds in V_κ.

We have now shown that V_κ is a model of ZF $-$ Inf.

For the infinity axiom, note that by Theorem 11.4(v) we have $\omega \in V_{\omega+1} \subseteq V_\kappa$. Hence the infinity axiom holds in V_κ by Theorem 14.18.

For the axiom of choice, suppose that $\mathscr{A} \in V_\kappa$ is a family of pairwise disjoint nonempty sets, and let $\mathscr{B} \subseteq \bigcup \mathscr{A}$ have exactly one element in common with each member of $\mathscr{A}$. Say $\mathscr{A} \in V_\alpha$ with $\alpha < \kappa$. Then $\mathscr{B} \subseteq \bigcup \mathscr{A} \subseteq V_\alpha$, so $\mathscr{B} \in V_{\alpha+1} \subseteq V_\kappa$, and the axiom of choice thus holds in V_κ.

So V_κ is a model of ZFC.

Finally, suppose that $x \in V_\kappa$ and (x is an inaccessible cardinal)$^{V_\kappa}$; we want to get a contradiction. In particular, (x is an ordinal)$^{V_\kappa}$, so by absoluteness, x is an ordinal. Absoluteness clearly implies that x is infinite. We claim that x is a cardinal. For, if $f : y \to x$ is a bijection with $y < x$, then clearly $f \in V_\kappa$, and hence by absoluteness ($f : y \to x$ is a bijection and $y < x$)$^{V_\alpha}$, contradiction. Similarly, x is regular; otherwise there is an injection $f : y \to x$ with $\operatorname{rng}(f)$ unbounded in x, so clearly $f \in V_\kappa$, and absolutenss again yields a contradiction. Thus x is a regular cardinal. Hence, since κ is the smallest inaccessible, there is a $y \in x$ such that there is a one-one function g from x

into $\mathscr{P}(y)$. Again, $g \in V_\kappa$, and easy absoluteness results contradicts (x is an inaccessible cardinal)$^{V_\kappa}$. $\square$

Reflection theorems

We now want to consider to what extent sentences can reflect to proper subclasses of $\mathbf{V}$; this is a natural extension of our considerations for absoluteness.

Lemma 14.26. *Suppose that* $\mathbf{M}$ *and* $\mathbf{N}$ *are classes with* $\mathbf{M} \subseteq \mathbf{N}$. *Let* $\varphi_0, \ldots, \varphi_n$ *be a list of formulas such that if* $i \le n$ *and* ψ *is a subformula of* φ_i, *then there is a* $j \le n$ *such that* φ_j *is* ψ. *Then the following conditions are equivalent:*

(i) Each φ_i *is absolute for* $\mathbf{M}, \mathbf{N}$.

(ii) If $i \le n$ *and* φ_i *has the form* $\exists x \varphi_j(x, y_1, \ldots, y_t)$ *with* $x, y_1, \ldots, y_t$ *exactly all the free variables of* φ_j, *then*

$$\forall y_1, \ldots, y_t \in \mathbf{M}[\exists x \in \mathbf{N} \varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \to \exists x \in \mathbf{M} \varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)].$$

Note the seemingly minor respects in which (ii) differs from the definition of absoluteness: the implication goes only one direction, and on the right side of the implication we relativize φ_j to $\mathbf{N}$ rather than $\mathbf{M}$.

Proof. (i)$\Rightarrow$(ii): Assume (i) and the hypothesis of (ii). Suppose that $y_1, \ldots, y_t \in \mathbf{M}$ and $\exists x \in \mathbf{N} \varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)$. Thus by absoluteness $\exists x \in \mathbf{M} \varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t)$; choose $x \in \mathbf{M}$ such that $\varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t)$. Hence by absoluteness again, $\varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t))$. Hence $\forall x \in \mathbf{M} \varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t)$, as desired.

(ii)$\Rightarrow$(i): Assume (ii). We prove that φ_i is absolute for $\mathbf{M}, \mathbf{N}$ by induction on the length of φ_i. This is clear if φ_i is atomic, and it easily follows inductively if φ_i has the form $\neg\varphi_j$ or $\varphi_j \wedge \varphi_k$. Now suppose that φ_i is $\exists x \varphi_j(x, y_1, \ldots, y_t)$, and $y_1, \ldots, y_t \in \mathbf{M}$. then

$$\begin{aligned}
\varphi_i^{\mathbf{M}}(y_1, \ldots, y_t) &\leftrightarrow \exists x \in \mathbf{M} \varphi_j^{\mathbf{M}}(x, y_1, \ldots, y_t) \quad \text{(definition of relativization)}\\
&\leftrightarrow \exists x \in \mathbf{M} \varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \quad \text{(induction hypothesis)}\\
&\leftrightarrow \exists x \in \mathbf{N} \varphi_j^{\mathbf{N}}(x, y_1, \ldots, y_t) \quad \text{(by (ii)}\\
&\leftrightarrow \varphi_i^{\mathbf{N}}(y_1, \ldots, y_t) \quad \text{(definition of relativization)}
\end{aligned}$$

$\square$

Theorem 14.27. *Suppose that* $Z(\alpha)$ *is a set for every ordinal* α, *and the following conditions hold:*

(i) If $\alpha < \beta$, *then* $Z(\alpha) \subseteq Z(\beta)$.

(ii) If γ *is a limit ordinal, then* $Z(\gamma) = \bigcup_{\alpha<\gamma} Z(\alpha)$.

Let $\mathbf{Z} = \bigcup_{\alpha \in \mathbf{On}} Z(\alpha)$. *Then for any formulas* $\varphi_0, \ldots, \varphi_{n-1}$,

$$\forall \alpha \exists \beta > \alpha[\varphi_0, \ldots, \varphi_{n-1} \text{ are absolute for } Z(\beta), \mathbf{Z}].$$

Proof. Assume the hypothesis, and let an ordinal α be given. We are going to apply Lemma 14.26 with $\mathbf{N} = \mathbf{Z}$, and we need to find an appropriate $\beta > \alpha$ so that we can take $\mathbf{M} = Z(\beta)$ in 14.26.

We may assume that $\varphi_0, \ldots, \varphi_{n-1}$ is subformula-closed; i.e., if $i < n$, then every subformula of φ_i is in the list. Let A be the set of all $i < n$ such that φ_i begins with an existential quantifier. Suppose that $i \in A$ and φ_i is the formula $\exists x \varphi_j(x, y_1, \ldots, y_t)$, where $x, y_1, \ldots, y_t$ are exactly all the free variables of φ_j. We now define a class function $\mathbf{G}_i$ as follows. For any sets $y_1, \ldots, y_t$,

$$\mathbf{G}_i(y_1, \ldots, y_t) = \begin{cases} \text{the least } \eta \text{ such that } \exists x \in Z(\eta) \varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t) & \text{if there is such,} \\ 0 & \text{otherwise.} \end{cases}$$

Then for each ordinal ξ we define

$$\mathbf{F}_i(\xi) = \sup\{\mathbf{G}_i(y_1, \ldots, y_t) : y_1, \ldots, y_t \in Z(\xi)\};$$

note that this supremum exists by the replacement axiom.

Now we define a sequence $\gamma_0, \ldots, \gamma_p, \ldots$ of ordinals by induction on $n \in \omega$. Let $\gamma_0 = \alpha + 1$. Having defined γ_p, let

$$\gamma_{p+1} = \max(\gamma_{p+1}, \sup\{\mathbf{F}_i(\xi) : i \in A, \xi \leq \gamma_p\} + 1).$$

Finally, let $\beta = \sup_{p \in \omega} \gamma_p$. Clearly $\alpha < \beta$ and β is a limit ordinal.

(1) If $i \in A$, $y_1, \ldots, y_t \in Z(\beta)$, and $\exists x \in \mathbf{Z} \varphi_i^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, then there is an $x \in Z(\beta)$ such that $\varphi_i^{\mathbf{Z}}(x, y_1, \ldots, y_t)$.

In fact, choose p such that $y_1, \ldots, y_t \in Z(\gamma_p)$. Then $\mathbf{G}_i(y_1, \ldots, y_t) \leq \mathbf{F}(\gamma_p) < \gamma_{p+1}$. Hence an x as in (1) exists, with $x \in Z(\beta_{p+1})$.

Now the theorem follows from Lemma 14.26. $\square$

Corollary 14.28. (The reflection theorem) *For any formulas $\varphi_1, \ldots, \varphi_n$,*

$$\text{ZF} \models \forall \alpha \exists \beta > \alpha[\varphi_1, \ldots, \varphi_n \text{ are absolute for } V_\beta]. \qquad \square$$

Theorem 14.29. *Suppose that $\mathbf{Z}$ is a class and $\varphi_1, \ldots, \varphi_n$ are formulas. Then*

$$\forall X \subseteq \mathbf{Z} \exists A[X \subseteq A \subseteq \mathbf{Z} \text{ and } \varphi_1, \ldots, \varphi_n \text{ are absolute}$$
$$\text{for } A, \mathbf{Z} \text{ and } |A| \leq \max(\omega, |X|)].$$

Proof. We may assume that $\varphi_1, \ldots, \varphi_n$ is subformula closed. For each ordinal α let $Z(\alpha) = \mathbf{Z} \cap V_\alpha$. Clearly there is an ordinal α such that $X \subseteq V_\alpha$, and hence $X \subseteq Z(\alpha)$. Now we apply Theorem 14.27 to obtain an ordinal $\beta > \alpha$ such that

(1) $$\varphi_1, \ldots, \varphi_n \text{ are absolute for } Z(\beta), \mathbf{Z}.$$

Let $\prec$ be a well-order of $Z(\beta)$. Let B be the set of all $i < n$ such that φ_i begins with an existential quantifier. Suppose that $i \in B$ and φ_i is the formula $\exists x \varphi_j(x, y_1, \ldots, y_t)$, where

$x, y_1, \ldots, y_t$ are exactly all the free variables of φ_j. We now define a function H_i for each $i \in B$ as follows. For any sets $y_1, \ldots, y_t \in Z(\beta)$,

$$H_i(y_1, \ldots, y_t) = \begin{cases} \text{the } \prec\text{-least } x \in Z(\beta) \text{ such that } \varphi_i^{Z(\beta)}(x, y_1, \ldots, y_t) & \text{if there is such,} \\ \text{the } \prec\text{-least element of } Z(\beta) & \text{otherwise.} \end{cases}$$

Let $A \subseteq Z(\beta)$ be closed under each function H_i, with $X \subseteq A$. We claim that A is as desired. To prove the absoluteness, it suffices by Lemma 14.26 to take any formula φ_i with $i \in A$, with notation as above, assume that $y_1, \ldots, y_t \in A$ and $\exists x \in \mathbf{Z}\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, and find $x \in A$ such that $\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$. By (1), there is an $x \in Z(\beta)$ such that $\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$. Hence $H_i(y_1, \ldots, y_t)$ is an element of A such that $\varphi_j^{\mathbf{Z}}(x, y_1, \ldots, y_t)$, as desired.

It remains only to check the cardinality estimate. This is elementary. □

Lemma 14.30. *Suppose that $\mathbf{G}$ is a bijection from A onto $\mathbf{M}$, and for any $a, b \in A$ we have $a \in b$ iff $\mathbf{G}(a) \in \mathbf{G}(b)$. Then for any formula $\varphi(x_1, \ldots, x_n)$ and any $x_1, \ldots, x_n \in A$,*

$$\varphi^A(x_1, \ldots, x_n) \leftrightarrow \varphi^{\mathbf{M}}(\mathbf{G}(x_1), \ldots, \mathbf{G}(x_n)).$$

Proof. An easy induction on φ. □

Theorem 14.31. *Suppose that $\mathbf{Z}$ is a transitive class and $\varphi_0, \ldots, \varphi_{m-1}$ are sentences. Suppose that X is a transitive subset of $\mathbf{Z}$. Then there is a transitive set M such that $X \subseteq M$, $|M| \leq \max(\omega, |X|)$, and for every $i < m$, $\varphi_i^M \leftrightarrow \varphi_i^{\mathbf{Z}}$.*

Proof. We may assume that the extensionality axiom is one of the φ_i's. Now we apply Theorem 14.29 to get a set A as indicated there. By Proposition 13.11, there is a transitive set M and a bijection G from A onto M such that for any $a, b \in A$, $a \in b$ iff $G(a) \in G(b)$. Hence all of the desired conditions are clear, except possibly $X \subseteq M$. We show that $G[X] = X$ by proving that $G(x) = x$ for all $x \in X$. In fact, suppose that $G(x) \neq x$ for some $x \in X$, and by the foundation axiom choose y such that $G(y) \neq y$ while $G(z) = z$ for all $z \in y$. Then if $z \in y$ we have $z, y \in X \subseteq A$, and hence $z = G(z) \in G(y)$. So $y \subseteq G(y)$. If $w \in G(y)$, then $w \in M = \operatorname{rng}(G)$, so we can choose $z \in A$ such that $w = G(z)$. Then $G(z) \in G(y)$, so $z \in y$. Hence $w = G(z) = z$ and so $w \in y$. This gives $G(y) \subseteq y$, and finishes the proof. □

Corollary 14.32. *Suppose that S is a set of sentences containing ZFC. Suppose also that $\varphi_0, \ldots, \varphi_{n-1} \in S$. Then*

$$S \models \exists M \left(M \text{ is transitive, } |M| = \omega, \text{ and } \bigwedge_{i<n} \varphi_i^M \right).$$

Proof. Take $\mathbf{Z} = \mathbf{V}$ and $X = \omega$ in Theorem 14.31. □

The following corollary can be taken as a basis for working with countable transitive models of ZFC.

Theorem 14.33. *Suppose that S is a consistent set of sentences containing ZFC. Expand the basic set-theoretic language by adding an individual constant* $\mathbf{M}$*. Then the following set of sentences is consistent:*

$$S \cup \{\mathbf{M} \text{ is transitive}\} \cup \{|\mathbf{M}| = \omega\} \cup \{\varphi^{\mathbf{M}} : \varphi \in S\}.$$

Proof. Suppose that the indicated set is not consistent. Then there are $\varphi_0, \ldots, \varphi_{m-1}$ in S such that

$$S \models \mathbf{M} \text{ is transitive and } |\mathbf{M}| = \omega \to \neg \bigwedge_{i<n} \varphi_i^{\mathbf{M}};$$

it follows that

$$S \models \neg\exists\mathbf{M}\left(\mathbf{M} \text{ is transitive, } |\mathbf{M}| = \omega, \text{ and } \bigwedge_{i<n} \varphi_i^{\mathbf{M}}\right),$$

contradicting Corollary 14.32. □

Indescribability

We give an important equivalent of weak compactness. This is optional material.

An infinite cardinal κ is *first-order describable* iff there is a $U \subseteq V_\kappa$ and a sentence σ in the language for $(V_\kappa, \in, U)$ such that $(V_\kappa, \in, U) \models \sigma$, while there is no $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

Theorem 14.34. *If κ is infinite but not inaccessible, then it is first-order describable.*

Proof. ω is describable by the sentence that says that κ is the first limit ordinal; absoluteness is used. The subset U is not needed for this. Now suppose that κ is singular.

Let $\lambda = \mathrm{cf}(\kappa)$, and let f be a function whose domain is some ordinal $\gamma < \kappa$ with $\mathrm{rng}(f)$ cofinal in κ. Let $U = \{(\lambda, \beta, f(\beta)) : \beta < \lambda\}$. Let σ be the sentence expressing the following:

For every ordinal γ there is an ordinal δ with $\gamma < \delta$, U is nonempty, and there is an ordinal μ and a function g with domain μ such that U consists of all triples $(\mu, \beta, g(\beta))$ with $\beta < \mu$.

Clearly $(V_\kappa, \in, U) \models \sigma$. Suppose that $\alpha < \kappa$ and $(V_\alpha, \in, V_\alpha \cap U) \models \sigma$. Then α is a limit ordinal, and there is an ordinal $\gamma < \alpha$ and a function g with domain γ such that $V_\alpha \cap U$ consists of all triples $(\gamma, \beta, g(\beta))$ with $\beta < \gamma$. (Some absoluteness is used.) Now $V_\alpha \cap U$ is nonempty; choose $(\gamma, \beta, g(\beta))$ in it. Then $\gamma = \lambda$ since it is in U. It follows that $g = f$. Choose $\beta < \lambda$ such that $\alpha < f(\beta)$. Then $(\lambda, \beta, f(\beta)) \in U \cap V_\alpha$. Since $\alpha < f(\beta)$, it follows that α has rank less than α, contradiction.

Now suppose that $\lambda < \kappa \leq 2^\lambda$. A contradiction is reached similarly, as follows. Let f be a function whose domain is $\mathscr{P}(\lambda)$ with range κ. Let $U = \{(\lambda, B, f(B)) : B \subseteq \lambda\}$. Let σ be the sentence expressing the following:

For every ordinal γ there is an ordinal δ with $\gamma < \delta$, U is nonempty, and there is an ordinal μ and a function g with domain $\mathscr{P}(\mu)$ such that U consists of all triples $(\mu, B, g(B))$ with $B \subseteq \mu$.

Clearly $(V_\kappa, \in, U) \models \sigma$. Suppose that $\alpha < \kappa$ and $(V_\alpha, \in, V_\alpha \cap U) \models \sigma$. Then α is a limit ordinal, and there is an ordinal $\gamma < \alpha$ and a function g with domain $\mathscr{P}(\gamma)$ such that $V_\alpha \cap U$ consists of all triples $(\gamma, B, g(B))$ with $B \subseteq \gamma$. (Some absoluteness is used.) Clearly $\gamma = \lambda$; otherwise $U \cap V_\alpha$ would be empty. Note that $g = f$. Choose $B \subseteq \lambda$ such that $\alpha = f(B)$. Then $(\lambda, B, f(B)) \in U \cap V_\alpha$. Again this implies that α has rank less than α, contradiction. $\square$

We need the following little fact about the Mostowski collapse.

Theorem 14.35. *Suppose that* $\mathbf{R}$ *is a well-founded class relation on a class* $\mathbf{A}$*, and it is set-like and extensional. Also suppose that* $\mathbf{B} \subseteq \mathbf{A}$*,* $\mathbf{B}$ *is transitive,* $\forall a, b \in \mathbf{A}[a\mathbf{R}b \in \mathbf{B} \to a \in \mathbf{B}]$*, and* $\forall a, b \in \mathbf{B}[a\mathbf{R}b \leftrightarrow a \in b]$*. Let* $\mathbf{G}, \mathbf{M}$ *be the Mostowski collapse of* $(\mathbf{A}, \mathbf{R})$*. Then* $\mathbf{G} \restriction \mathbf{B}$ *is the identity.*

Proof. Suppose not, and let $\mathbf{X} = \{b \in \mathbf{B} : \mathbf{G}(b) \neq b\}$. Since we are assuming that $\mathbf{X}$ is a nonempty subclass of $\mathbf{A}$, by Proposition 13.7 choose $b \in \mathbf{X}$ such that $y \in \mathbf{A}$ and $y\mathbf{R}b$ imply that $y \notin \mathbf{X}$. Then

$$\begin{aligned}
\mathbf{G}(b) &= \{\mathbf{G}(y) : y \in \mathbf{A} \text{ and } y\mathbf{R}b\} \\
&= \{\mathbf{G}(y) : y \in \mathbf{B} \text{ and } y\mathbf{R}b\} \\
&= \{y : y \in \mathbf{B} \text{ and } y\mathbf{R}b\} \\
&= \{y : y \in \mathbf{B} \text{ and } y \in b\} \\
&= \{y : y \in b\} \\
&= b,
\end{aligned}$$

contradiction.

Lemma 14.36. *Let* κ *be weakly compact. Then for every* $U \subseteq V_\kappa$*, the structure* $(V_\kappa, \in, U)$ *has a transitive elementary extension* $(M, \in, U')$ *such that* $\kappa \in M$*.*

Proof. Let Γ be the set of all $L_{\kappa\kappa}$-sentences true in the structure $(V_\kappa, \in, U, x)_{x \in V_\kappa}$, together with the sentences

$$\begin{aligned}
&c \text{ is an ordinal,} \\
&\alpha < c \text{ (for all } \alpha < \kappa),
\end{aligned}$$

where c is a new individual constant. The language here clearly has κ many symbols. Every subset of Γ of size less than κ has a model; namely we can take $(V_\kappa, \in, U, x, \beta)_{x \in V_\kappa}$, choosing β greater than each α appearing in the sentences of Γ. Hence by weak compactness, Γ has a model $(M, E, W, k_x, y)_{x \in V_\kappa}$. This model is well-founded, since the sentence

$$\neg \exists v_0 v_1 \ldots \left[\bigwedge_{n \in \omega} (v_{n+1} \in v_n) \right]$$

holds in $(V_\kappa, \in, U, x)_{x \in V_\kappa}$, and hence in $(M, E, W, k_x, y)_{x \in V_\kappa}$.

Note that k is an injection of V_κ into M. Let F be a bijection from $M\backslash\mathrm{rng}(k)$ onto $\{(V_\kappa, u) : u \in M\backslash\mathrm{rng}(k)\}$. Then $G \stackrel{\text{def}}{=} k^{-1} \cup F^{-1}$ is one-one, mapping M onto some set N such that $V_\kappa \subseteq N$. We define, for $x, z \in N$, $xE'z$ iff $G^{-1}(x)EG^{-1}(z)$. Then G is an isomorphism from $(M, E, W, k_x, y)_{x\in V_\kappa}$ onto $\overline{N} \stackrel{\text{def}}{=} (N, E', G[W], x, G(y))_{x\in V_\kappa}$. Of course $\overline{N}$ is still well-founded. It is also extensional, since the extensionality axiom holds in $(V_\kappa, \in)$ and hence in (M, E) and (N, E'). Let H, P be the Mostowski collapse of (N, E'). Thus P is a transitive set, and

(1) H is an isomorphism from (N, E') onto $(P, \in)$.

(2) $\forall a, b \in N[aE'b \in V_\kappa \to a \in b]$.

In fact, suppose that $a, b \in N$ and $aE'b \in V_\kappa$. Let the individual constants used in the expansion of $(V_\kappa, \in, U)$ to $(V_\kappa, \in, U, x)_{a\in V_\kappa}$ be $\langle c_x : x \in V_\kappa\rangle$. Then

$$(V_\kappa, \in, U, x)_{a\in V_\kappa} \models \forall z \left[z \in k_b \to \bigvee_{w\in b} (z = k_w)\right],$$

and hence this sentence holds in $(N, E', G[W], x, G(y))_{x\in V_\kappa}$ as well, and so there is a $w \in b$ such that $a = w$, i.e., $a \in b$. So (2) holds.

(3) $\forall a, b \in V_\kappa[a \in b \to aE'b]$

In fact, suppose that $a, b \in V_\kappa$ and $a \in b$. Then the sentence $k_a \in k_b$ holds in $(V_\kappa, \in, U, x)_{x\in V_\kappa}$, so it also holds in $(N, E', G[W], x, G(y))_{x\in V_\kappa}$, so that $aE'b$.

We have now verified the hypotheses of Lemma 14.35. It follows that $H \restriction V_\kappa$ is the identity. In particular, $V_\kappa \subseteq P$. Now take any sentence σ in the language of $(V_\kappa, \in, U, x)_{x\in V_\kappa}$. Then

$$\begin{aligned}(V_\kappa, \in, U, x)_{x\in V_\kappa} \models \sigma \quad &\text{iff} \quad (M, E, W, k_x)_{x\in V_\kappa} \models \sigma\\ &\text{iff} \quad (N, E', G[W], x)_{x\in V_\kappa} \models \sigma\\ &\text{iff} \quad (P, \in, H[G[W]], x)_{x\in V_\kappa} \models \sigma.\end{aligned}$$

Thus $(P, \in, H[G[W]])$ is an elementary extension of $(V_\kappa, \in, U)$.

Now for $\alpha < \kappa$ we have

$$\begin{aligned}&(M, E, W, k_x, y)_{x\in V_\kappa} \models [y \text{ is an ordinal and } k_\alpha Ey], \quad \text{hence}\\ &(N, E', G[W], x, G(y))_{x\in V_\kappa} \models [G(y) \text{ is an ordinal and } \alpha E' G(y)], \quad \text{hence}\\ &(P, \in, H[G[W]], x, H(G(y)))_{x\in V_\kappa} \models [H(G(y)) \text{ is an ordinal and } \alpha \in H(G(y))].\end{aligned}$$

Thus $H(G(y))$ is an ordinal in P greater than each $\alpha < \kappa$, so since P is transitive, $\kappa \in P$. □

The new equivalent of weak compactness involves second-order logic. We augment first order logic by adding a new variable S ranging over subsets rather than elements. There is one new kind of atomic formula: Sv with v a first-order variable. This is interpreted as saying that v is a member of S.

Now an infinite cardinal κ is Π^1_1-*indescribable* iff for every $U \subseteq V_\kappa$ and every second-order sentence σ of the form $\forall S\varphi$, with no quantifiers on S within φ, if $(V_\kappa, \in, U) \models \sigma$, then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

Theorem 14.37. *An infinite cardinal κ is weakly compact iff it is Π^1_1-indescribable.*

Proof. First suppose that κ is Π^1_1-indescribable. By Theorem 14.34 it is inaccessible. So it suffices to show that it has the tree property. By the proof of Theorem 12.7(iii)⇒(iv) it suffices to check the tree property for a tree $T \subseteq {}^{<\kappa}\kappa$. Note that ${}^{<\kappa}\kappa \subseteq V_\kappa$. Let σ be the following sentence in the second-order language of $(V_\kappa, \in, T)$:

$$\begin{aligned}&\exists S[T \text{ is a tree under } \subset, \text{ and}\\&S \subseteq T \text{ and } S \text{ is a branch of } T \text{ of unbounded length}].\end{aligned}$$

Thus for each $\alpha < \kappa$ the sentence σ holds in $(V_\alpha, \in, T \cap V_\alpha)$. Hence it holds in $(V_\kappa, \in, T)$, as desired.

Now suppose that κ is weakly compact. Let $U \subseteq V_\kappa$, and let σ be a Π^1_1-sentence holding in $(V_\kappa, \in, U)$. By Lemma 14.36, let $(M, \in, U')$ be a transitive elementary extension of $(V_\kappa, \in, U)$ such that $\kappa \in M$. Say that σ is $\forall S\varphi$, with φ having no quantifiers on S. Now

$$\forall X \subseteq V_\kappa[(V_\kappa, \in, U) \models \varphi(X)]. \tag{1}$$

Now since $\kappa \in M$ and $(M, \in)$ is a model of ZFC, V_κ^M exists, and by absoluteness it is equal to V_κ. Hence by (1) we get

$$(M, \in, U') \models \forall X \subseteq V_\kappa \varphi^{V_\kappa}(U' \cap V_\kappa).$$

Hence

$$(M, \in, U') \models \exists\alpha\forall X \subseteq V_\alpha \varphi^{V_\alpha}(U' \cap V_\alpha),$$

so by the elementary extension property we get

$$(V_\kappa, \in, U) \models \exists\alpha\forall X \subseteq V_\alpha \varphi^{V_\alpha}(U' \cap V_\alpha).$$

We choose such an α. Since $V_\kappa \cap \mathbf{On} = \kappa$, it follows that $\alpha < \kappa$. Hence $(V_\alpha, \in, U' \cap V_\alpha) \models \sigma$, as desired. $\square$

A diagram of large cardinals

We define some more large cardinals, and then indicate relationships between them by a diagram.

All cardinals are assumed to be uncountable.

1. **regular limit cardinals**.
2. **inaccessible**.
3. **Mahlo**.

4. **weakly compact**.

5. **indescribable**. The ω-order language is an extension of first order logic in which one has variables of each type $n \in \omega$. For n positive, a variable of type n ranges over $\mathscr{P}^n(A)$ for a given structure A. In addition to first-order atomic formulas, one has formulas $P \in Q$ with P n-th order and Q $(n+1)$-order. Quantification is allowed over the higher order variables.

κ is *indescribable* iff for all $U \subseteq V_\kappa$ and every higher order sentence σ, if $(V_\kappa, \in, U) \models \sigma$ then there is an $\alpha < \kappa$ such that $(V_\alpha, \in, U \cap V_\alpha) \models \sigma$.

6. $\kappa \to (\omega)_2^{<\omega}$. Here in general

$$\kappa \to (\alpha)_m^{<\omega}$$

means that for every function $f : \bigcup_{n\in\omega}[\kappa]^n \to m$ there is a subset $H \subseteq \kappa$ of order type α such that for each $n \in \omega$, $f \restriction [H]^n$ is constant.

7. $0^\sharp$ exists. This means that there is a non-identity elementary embedding of L into L. Thus no actual cardinal is referred to. But $0^\sharp$ implies the existence of some large cardinals, and the existence of some large cardinals implies that $0^\sharp$ exists.

8. **Jónsson** κ is a Jónsson cardinal iff every model of size κ has a proper elementary substructure of size κ.

9. **Rowbottom** κ is a Rowbottom cardinal iff for every uncountable $\lambda < \kappa$, every model of type (κ, λ) has an elementary submodel of type (κ, ω).

10. **Ramsey** $\kappa \to (\kappa)_2^{<\omega}$.

11. **measurable**

12. **strong** κ is a strong cardinal iff for every set X there exists a nontrivial elementary embedding from V to $\mathbf{M}$ with κ the first ordinal moved and with $\kappa \in \mathbf{M}$.

13. **Woodin** κ is a Woodin cardinal iff

$$\forall A \subseteq V_\kappa \forall \lambda < \kappa \exists \mu \in (\lambda, \kappa) \forall \nu < \kappa \exists j [j \text{ is a nontrivial elementary embedding of } V \text{ into some set } \mathbf{M}, \text{ with } \mu \text{ the first ordinal moved, such that } j(\mu) > \nu, V_\nu \subseteq \mathbf{M}, A \cap V_\nu = j(A) \cap V_\nu]$$

14. **superstrong** κ is superstrong iff there is a nontrivial elementary embedding $j : V \to \mathbf{M}$ with κ the first ordinal moved, such that $V_{j(\kappa)} \subseteq \mathbf{M}$.

15. **strongly compact** κ is strongly compact iff for any $L_{\kappa\kappa}$-language, if Γ is a set of sentences and every subset of Γ of size less than κ has a model, then Γ itself has a model.

16. **supercompact** κ is supercompact iff for every A with $|A| \geq \kappa$ there is normal measure on $P_\kappa(A)$.

17. **extendible** For an ordinal η, we say that k is η-extendible iff there exist ζ and a nontrivial elementary embedding $j : V_{\kappa+\eta} \to V_\zeta$ with κ first ordinal moved, with $\eta < j(\kappa)$. κ is extendible iff it is η-extendible for every $\eta > 0$.

18. **Vopěnka's principle** If C is a proper class of models in a given first-order language, then there exist two distinct members $A, B \in C$ such that A can be elementarily embedded in B.

19. **huge** A cardinal κ is huge iff there is a nontrivial elementary embedding $j : V \to \mathbf{M}$ with κ the first ordinal moved, such that $\mathbf{M}^{j(\kappa)} \subseteq \mathbf{M}$.

20. $I0$. There is an ordinal δ and a proper elementary embedding j of $L(V_{\delta+1})$ into $L(V_{\delta+1})$ such that the first ordinal moved is less than δ.

In the diagram on the next page, a line indicates that (the consistency of the) existence of the cardinal above implies (the consisteny of the) existence of the one below.

EXERCISES

E14.1. For any infinite cardinal κ, let $H(\kappa)$ be the set of all x such that $|\text{trcl}(x)| < \kappa$. Prove that $V_\omega = H(\omega)$. ($H(\omega)$ is the collection of all hereditarily finite sets.) Hint: $V_\omega \subseteq H(\omega)$ is easy. For the other direction, suppose that $x \in H(\omega)$, let $t = \text{trcl}(x)$, and let $S = \{\text{rank}(y) : y \in t\}$. Show that S is an ordinal.

E14.2. Which axioms of ZFC are true in **On**?

E14.3. Show that the power set operation is absolute for V_α for α limit.

E14.4. Let M be a countable transitive model of ZFC. Show that the power set operation is not absolute for M.

E14.5. Show that V_ω is a model of ZFC $-$ Inf.

E14.6. Show that the formula $\exists x(x \in y)$ is not absolute for all nonempty sets, but it is absolute for all nonempty transitive sets.

E14.7. Show that the formula $\exists z(x \in z)$ is not absolute for every nonempty transitive set.

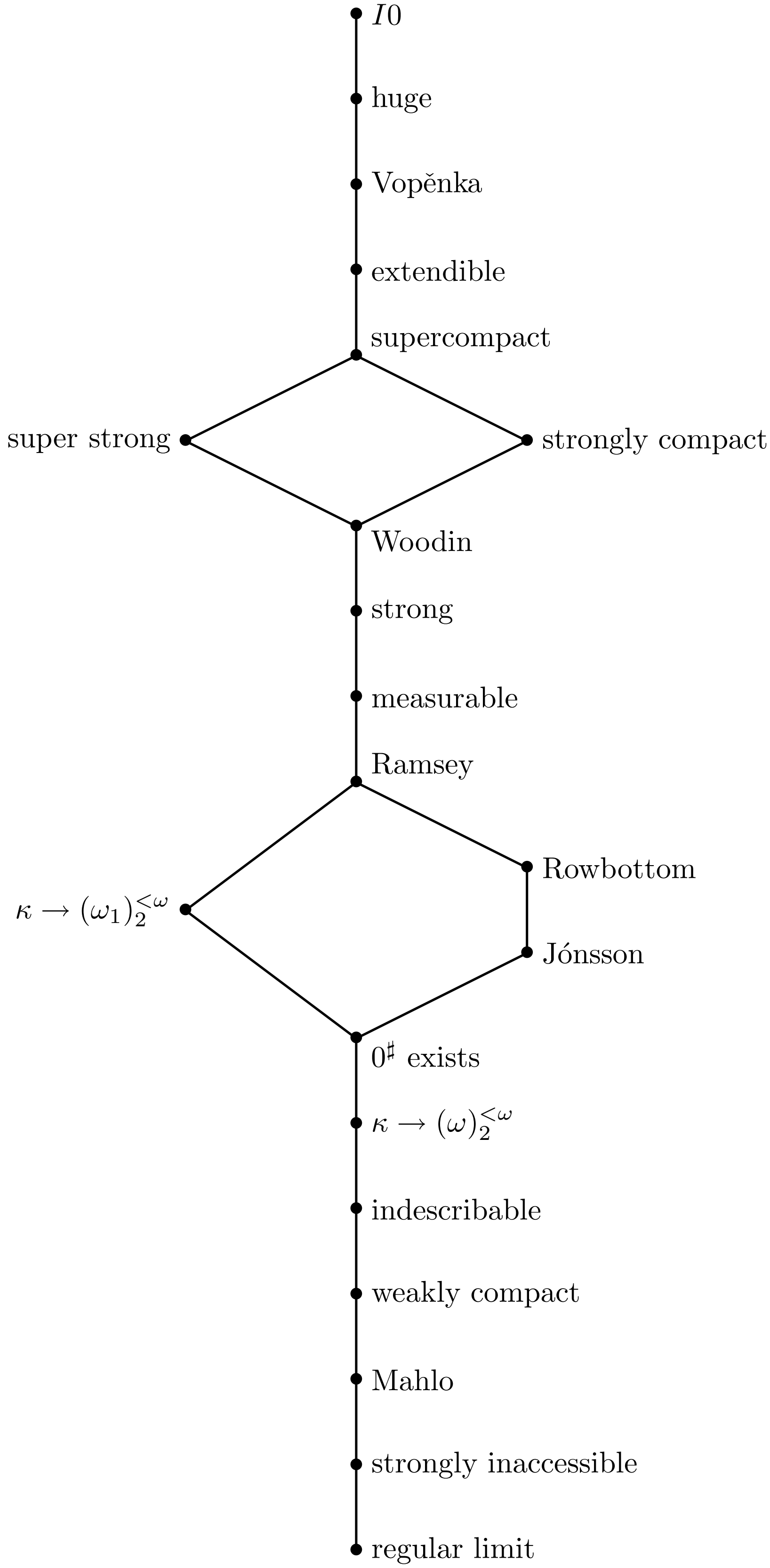
E14.8. A formula is Σ_1 iff it has the form $\exists x\varphi$ with φ a Δ_0 formula; it is Π_1 iff it has the form $\forall x\varphi$ with φ a Δ_0 formula.

(i) Show that "X is countable" is equivalent on the basis of ZF to a Σ_1 formula.

(ii) Show that "α is a cardinal" is equivalent on the basis of ZF to a Π_1 formula.

E14.9. Prove that if κ is an infinite cardinal, then $H(\kappa) \subseteq V_\kappa$.

E14.10. Prove that for κ regular, $H(\kappa) = V_\kappa$ iff $\kappa = \omega$ or κ is inaccessible.


$I0$
huge
Vopěnka
extendible
supercompact
super strong
strongly compact
Woodin
strong
measurable
Ramsey
Rowbottom
$\kappa \to (\omega_1)_2^{<\omega}$
Jónsson
$0^\sharp$ exists
$\kappa \to (\omega)_2^{<\omega}$
indescribable
weakly compact
Mahlo
strongly inaccessible
regular limit

E14.11. Assume that κ is an infinite cardinal. Prove the following:

(a) $H(\kappa)$ is transitive.

(b) $H(\kappa) \cap \mathbf{On} = \kappa$.

(c) If $x \in H(\kappa)$, then $\bigcup x \in H(\kappa)$.

(d) If $x, y \in H(\kappa)$, then $\{x, y\} \in H(\kappa)$.

(e) If $y \subseteq x \in H(\kappa)$, then $y \in H(\kappa)$.

(f) If κ is regular and x is any set, then $x \in H(\kappa)$ iff $x \subseteq H(\kappa)$ and $|x| < \kappa$.

E14.12. Show that if κ is regular and uncountable, then $H(\kappa)$ is a model of all of the ZFC axioms except possibly the power set axiom.

References

Jech, T. **Set Theory**, 769pp.

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