McGill University
Department of Mathematics and Statistics
MATH 254 Analysis 1, Fall 2015
Assignment 6

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 1 and 2.**

This assignment is due **Monday, November 16,** at the end of the class. **Late assignments will not be accepted.**

For questions 7–9 you may use, without proof, that $|\sin x| \le |x|$ for all $x \in \mathbb{R}$ and that $\sin x < x$ for all $x > 0$. You may also use all trigonometric identities covered in standard Calculus courses, including the sum-to-product formulas.

1. Let (x_n) be a sequence such that $x_n > 0$ for $n \in \mathbb{N}$. Set

$$y_n = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}, \qquad n \in \mathbb{N}.$$

   (a) Suppose that (x_n) is a convergent sequence. Prove that

$$\lim y_n = \lim x_n.$$

   Hint: You may use Problem 3 on the Assignment 4.

   (b) Suppose that $\lim x_n = +\infty$. Prove that $\lim y_n = +\infty$.

2. Let $a > 0$ and let (x_n) be a sequence defined recursively as $x_1 = \sqrt{a}$, $x_{n+1} = \sqrt{a + x_n}$, $n \ge 1$. Prove that (x_n) is convergent and find $\lim x_n$.

3. Let $x_1 \in \mathbb{R} \setminus \{0\}$ and let

$$x_{n+1} = x_n + \frac{1}{x_n} \qquad \forall n \in \mathbb{N}$$

   (a) Prove that $\lim (x_n) = +\infty$ if $x_1 > 0$.

   (b) Prove that $\lim (x_n) = -\infty$ if $x_1 < 0$.

4. Find

$$\lim \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{n^2}\right).$$

5. Find

$$\lim \left(\frac{2^3 - 1}{2^3 + 1}\right)\left(\frac{3^3 - 1}{3^3 + 1}\right)\cdots\left(\frac{n^3 - 1}{n^3 + 1}\right).$$

6. Let $a > 0$. Prove that

$$\lim n\left(a^{\frac{1}{n}} - 1\right) = \ln a.$$

7. Let (x_n) be a convergent sequence with $\lim x_n = x$. Prove that the sequence $(\sin x_n)$ converges and that

$$\lim \sin x_n = \sin x.$$

   Hint: Prove this result first in the special case that $\lim x_n = 0$.

8. Let (x_n) be defined recursively as $x_1 = 1$, $x_{n+1} = \sin x_n$, $n \geq 1$. Prove that (x_n) is convergent and find $\lim x_n$.

9. Let
$$x_n = \sin\left(\pi\sqrt{n^2+1}\right), \qquad n \in \mathbb{N}$$
Prove that (x_n) is convergent and find $\lim x_n$.