Math 488, Assignment 6

1. Suppose M is a ctm and $\mathbb{P} \in M$ is a forcing notion. Show that if $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M, then G is an ultrafilter.

2. Let M be a ctm and $\mathbb{P} \in M$ be a forcing notion. Show that $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over M if and only if G is a filter and for every maximal antichain A in $\mathbb{P}$ such that $A \in M$ we have $G \cap A \neq \emptyset$.

3. (Maximal Principle) Let $\mathbb{P}$ be a forcing notion. Suppose $p \in \mathbb{P}$ and $\varphi(x)$ is a formula in the forcing language. Show that if $p \Vdash \exists x \varphi(x)$, then there is a name $\dot{x}$ such that $p \Vdash \varphi(\dot{x})$.

4. Let κ be an uncountable cardinal. Write $\mathrm{Coll}(\omega, \kappa)$ for the set of all finite partial functions from ω to κ, ordered by reverse inclusion. Show that κ is countable in $V[G]$ and all cardinals from V which are bigger than κ remain cardinals in $V[G]$.

5. A forcing $\mathbb{P}$ is *σ-closed* if for any sequence p_n in $\mathbb{P}$ such that $p_{n+1} \leq p_n$ there exists $p \in \mathbb{P}$ with $p \leq p_n$ for all n. Show that if $\mathbb{P}$ is σ-closed then $\mathbb{P}$ does not add reals, i.e. $V \cap 2^\omega = V[G] \cap 2^\omega$ for any $\mathbb{P}$-generic G.

6. (0-1 law) A forcing $\mathbb{P}$ is *weakly homogeneous* if for any $p, q \in \mathbb{P}$ there is an automorphism φ of $\mathbb{P}$ such that $\varphi(p)$ and q are compatible. Show that if $\mathbb{P}$ is weaklly homogeneous and σ is a sentence in the forcing language, then either $\mathbb{1} \Vdash \sigma$ or $\mathbb{1} \Vdash \neg\sigma$.

7. Given two forcing notions $\mathbb{P}$ and $\mathbb{Q}$ in a ctm M satisfying ZFC*, consider $\mathbb{P} \times \mathbb{Q} \in M$ ordered coordinatewise. Let $G \subseteq \mathbb{P}$ and $H \subseteq \mathbb{Q}$. Show that the following are equivalent:
   (i) G is $\mathbb{P}$-generic over M and H is $\mathbb{Q}$-generic over $M[G]$,
   (ii) $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over M.