Exercise 1. Inversion Lemma [15 pts]

$$\dfrac{\dfrac{\Gamma \Longrightarrow A \supset B}{\Gamma, A \Longrightarrow A \supset B}\ \texttt{weak} \qquad \dfrac{\dfrac{}{\Gamma, A \supset B, A \Longrightarrow A}\ \texttt{init} \qquad \dfrac{}{\Gamma, A \supset B, A, B \Longrightarrow B}\ \texttt{init}}{\Gamma, A \supset B, A \Longrightarrow B}\ {\supset}\,\texttt{L}}{\Gamma, A \Longrightarrow B}\ \texttt{cut}$$

Exercise 2. New Connective (Nor) [35 pts]

Part 1. Conversion into sequent calculus [10 pts]

$$\dfrac{\Gamma, u : A, v : B \Longrightarrow p}{\Gamma \Longrightarrow A \overline{\wedge} B}\ \overline{\wedge}R \qquad \dfrac{\Gamma, A \overline{\wedge} B, A, B \Longrightarrow C}{\Gamma, A \overline{\wedge} B \Longrightarrow C}\ \overline{\wedge}L$$

Part 2. Invertibility [20 pts]

Yes, $\overline{\wedge}R$ is invertible

Lemma 1. $\Gamma \Longrightarrow A \overline{\wedge} B \Rightarrow \Gamma, u : A, v : B \Longrightarrow p$

Proof.
$$\dfrac{\dfrac{\mathcal{D}}{\Gamma, u : A, v : B \Longrightarrow A \overline{\wedge} B}\ \mathit{hypothesis} \qquad \dfrac{}{\Gamma, u : A, v : B \Longrightarrow A}\ \mathit{init} \qquad \dfrac{}{\Gamma, u : A, v : B \Longrightarrow B}\ \mathit{init}}{\Gamma, u : A, v : B \Longrightarrow p}\ \overline{\wedge}L$$

□

However, $\overline{\wedge}L$ is not invertible.
Counter-example: Γ empty, $C = \top$, $A = \bot$, $B = \top$
Clearly, we have $\Gamma \Longrightarrow A \overline{\wedge} B$
However, we cannot prove $\Gamma \Longrightarrow A$

Part 3. Focusing [5 pts]

Since the right rule is invertible, it will go in the Asynchronous phase.
Since the left rule is non-invertible, it will go in the Synchronous phase.

Exercise 3. Reducibility [50 pts]

Part 1. Definition [10 pts]

$\mathcal{R}_{A \times B} = \{M \mid \texttt{fst}\,(M) \in \mathcal{R}_A \text{ and } \texttt{snd}\,(M) \in \mathcal{R}_B\}$

Part 2. Backwards closed lemma [20 ps]

Lemma 2. ***Backwards Lemma (Part 1)***
If $M \longrightarrow M'$ *and* $M' \in \mathcal{R}_C$*, then* $M \in \mathcal{R}_C$

Proof. By induction on the structure of C.

Case. C $= A \times B$ $M \longrightarrow M'$	by assumption
$M' \in \mathcal{R}_{A\times B}$	by assumption
$\texttt{fst}\,(M') \in \mathcal{R}_A$	by definition of $\mathcal{R}_{A\times B}$
$\texttt{snd}\,(M') \in \mathcal{R}_B$	by definition of $\mathcal{R}_{A\times B}$
$\texttt{fst}\,(M) \longrightarrow \texttt{fst}\,(M')$	by `fst` rules
$\texttt{fst}\,(M) \in \mathcal{R}_A$	by i.h.
$\texttt{snd}\,(M) \longrightarrow \texttt{snd}\,(M')$	by `snd` rules
$\texttt{snd}\,(M) \in \mathcal{R}_B$	by i.h.
$M \in \mathcal{R}_{A\times B}$	by definition of $\mathcal{R}_{A\times B}$

□

Part 3. Fundamental lemma [20 pts]

Lemma 3. ***Fundamental Lemma*** *If* $\mathcal{D} : \Gamma \vdash M : A$ *and* $\mathcal{E} : \sigma \in \mathcal{R}_\Gamma$*, then* $[\sigma]M \in \mathcal{R}_A$

Proof. By structural induction on the derivation $\mathcal{D}$

Case. $\mathcal{D} = \dfrac{\Gamma \vdash M : A \quad \Gamma \vdash N : B}{\Gamma \vdash < M, N >: A \times B}$

$[\sigma]M \in \mathcal{R}_A$	by i.h.
$[\sigma]N \in \mathcal{R}_B$	by i.h.
$[\sigma]\texttt{fst}\ < M, N > \longrightarrow M$	by redux reduction
$[\sigma]\texttt{snd}\ < M, N > \longrightarrow N$	by redux reduction
$[\sigma]\texttt{fst}\ < M, N > \in \mathcal{R}_A$	by Backwards Lemma
$[\sigma]\texttt{snd}\ < M, N > \in \mathcal{R}_B$	by Backwards Lemma
$\texttt{fst}\,[\sigma] < M, N > = [\sigma]\texttt{fst}\ < M, N >$	by substitution
$\texttt{snd}\,[\sigma] < M, N > = [\sigma]\texttt{snd}\ < M, N >$	by substitution
$\texttt{fst}\,[\sigma] < M, N > \in \mathcal{R}_A$	by above
$\texttt{snd}\,[\sigma] < M, N > \in \mathcal{R}_B$	by above
$[\sigma]M \in \mathcal{R}_{A\times B}$	by definition of $\mathcal{R}_{A\times B}$

Case. $\mathcal{D} = \dfrac{\Gamma \vdash M : A \times B}{\Gamma \vdash \texttt{fst}\, M : A}$

$[\sigma]M \in \mathcal{R}_{A\times B}$	by i.h.

$\texttt{fst}\,[\sigma]M \in \mathcal{R}_A$ by definition of $\mathcal{R}_{A\times B}$
$[\sigma]\texttt{fst}\,M = \texttt{fst}\,[\sigma]M$ by substitution
$[\sigma]\texttt{fst}\,M \in \mathcal{R}_A$ by above

Case. $\mathcal{D} = \dfrac{\Gamma \vdash M : A \times B}{\Gamma \vdash \texttt{snd}\,M : B}$

$[\sigma]M \in \mathcal{R}_{A\times B}$ by i.h.
$\texttt{snd}\,[\sigma]M \in \mathcal{R}_B$ by definition of $\mathcal{R}_{A\times B}$
$[\sigma]\texttt{snd}\,M = \texttt{snd}\,[\sigma]M$ by substitution
$[\sigma]\texttt{snd}\,M \in \mathcal{R}_B$ by above

$\square$