Math 488, Assignment 5

Given a set X, a function $f : P(X) \to P(X)$ is *monotone* if $A \subseteq B$ implies that $f(A) \subseteq f(B)$.

1. (Knaster–Tarski fixed point theorem) Suppose X is a set and $f : P(X) \to P(X)$ is a motonone function. Show that f has a fixed point.

2. (Banach's lemma) Suppose $f : X \to Y$ and $g : Y \to X$ are functions. Show that there exists a set $C \subseteq X$ such that $g(Y \setminus f(C)) = X \setminus C$. Deduce the Cantor–Schröder–Bernstein theorem.

Given a group G acting on a set X and two subsets $A, B \subseteq X$ write $A \preceq B$ if A is equidecomposable with a subset of B.

3. Prove that if $A \preceq B$ and $B \preceq A$, then A and B are equidecomposable.

An action of a group G on a set X is *countably paradoxical* if there exist a countable collection of pairwise disjoint subsets $\{A_0, A_1, \ldots, B_0, B_1, \ldots\}$ of X such that $X = \bigcup_n g_n A_n = \bigcup_n h_n B_n$ for some $g_n, h_n \in G$.

4. Show that the circle is countably paradoxical with respect to the action of $\mathrm{SO}_2(\mathbb{R})$. Conclude that the unit ball in $\mathbb{R}^2$ is countably paradoxical with respect to the action of the isometry group of $\mathbb{R}^2$.

A κ*-additive measure* on a famlily of sets A is a function $\mu : A \to [0, \infty)$ such that $\mu(\emptyset) = 0$ and for every $\lambda < \kappa$ and family $(a_i : i < \lambda)$ of pairwise disjoint sets we have $\mu(\bigcup_{i<\lambda} a_i) = \sum_{i<\lambda} \mu(a_i)$. μ is called *nontrivial* if it is not constant zero and $\mu(A) = 0$ whenever $A = \{x\}$.

A σ*-additive measure* is the same as an ω_1-additive measure. A *finitely additive measure* is an ω-additive measure.

5. Show that for every set X there exists a nontrivial finitely additive measure (taking only two values) defined on all subsets of X.

6. Show that there does not exist a nonzero finitely additive measure defined on all subsets of $\mathbb{R}^3$ which is invariant under translations and rotations.

7. Show that there does not exist a nonzero countably additive measure defined on all subsets of $\mathbb{R}^2$ which is invariant under translations and rotations

An *Ulam matrix* is a family of subsets of ω_1 indexed by $\omega_1 \times \omega$, i.e. $(A_{\alpha,n} : \alpha \in \omega_1, n \in \omega)$ such that
(1) $A_{\alpha,n} \cap A_{\beta,n} = \emptyset$ for every $\alpha \neq \beta \in \omega_1$ and $n \in \omega$,
(2) $\omega_1 \setminus \bigcup_{n\in\omega} A_{\alpha,n}$ is countable for every $\alpha < \omega_1$

8. Show that an Ulam matrix exists.

9. Show that there does not exist a nontrivial σ-additive measure on the family of all subsets of ω_1.

A cardinal κ is *real-valued measurable* if there exists a nontrivial κ-additive measure on the powerset of κ.

10. Show that if κ is real-valued measurable, then κ is weakly inaccessible.

11. Show that if κ is the least cardinal such that there exists a nontrivial σ-additive measure on the powerset of κ, then κ is real-valued measurable.

    Conclude that the following statement is not provable in ZFC: there exists an infinite set X and a nontrivial σ-additive measure on all subsets of X.