MATH 254 Assignment 3 McGill University
Fall 2016

You should work carefully on all problems. However you only have to hand in solutions to problems 3 and 6. This assignment is due on Tuesday, October 18 in class.

Exercise 1. Show that the sets

$$A = \left\{2 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\} \quad \text{and} \quad B = \left\{\frac{1}{n} + (-1)^n : n \in \mathbb{N}\right\}$$

are bounded and determine their supremum and infimum.

Exercise 2. Let A and B be two nonempty subsets of $\mathbb{R}$. Prove that $A \cup B$ is bounded above if and only if A and B are bounded above. If it is the case, prove that $\sup(A \cup B) = \sup(\sup A, \sup B)$.

Exercise 3. [10 points] We say that a number $x_0 \in \mathbb{R}$ is an accumulation point of a set $S \subseteq \mathbb{R}$ if for every $\epsilon > 0$, $S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} \neq \emptyset$.

(i) What are the accumulation points of $\mathbb{Q}$?

(ii) Assume that S is bounded and not finite. Let A be the set of all points $x \in \mathbb{R}$ such that $(x, \infty) \cap S$ is not finite.

(a) Show that A is not empty and $\sup S$ is an upper bound of A.

(b) Show that $\sup A$ is an accumulation point of S.

Exercise 4. Prove that for every $x, y \in \mathbb{R}$, the following inequalities are true:

(i) $\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$.

(ii) $\lfloor x \rfloor + \lfloor x + y \rfloor + \lfloor y \rfloor \leq \lfloor 2x \rfloor + \lfloor 2y \rfloor$.

Exercise 5. Let $a, b \in \mathbb{R}$ be such that $a < b$. Prove that

$$\mathrm{Card}([a, b] \cap \mathbb{Z}) = \lfloor b \rfloor + \lfloor 1 - a \rfloor.$$

Exercise 6. [10 points]

(i) By modifying the proof of the Archimedean property, show that for every $z \in \mathbb{R}$, there exists $k \in \mathbb{N}$ such that $10^k > z$.

(ii) Show that the set

$$\mathbb{D} = \left\{\frac{a}{10^k} : a \in \mathbb{Z},\, k \in \mathbb{N}\right\}$$

is dense in $\mathbb{R}$, namely for every $x, y \in \mathbb{R}$ such that $x < y$, there exists $r \in \mathbb{D}$ such that $x < r < y$. *Hint: The proof of density of $\mathbb{Q}$ in $\mathbb{R}$ may inspire you.*