# A set that is neither CE nor co-CE

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**Proposition 1.** If $Q \leq_m P$ and if $P$ is CE then so is $Q$. If $P$ is co-CE so is $Q$.

**Proof** . Suppose that $f$ is the required mapping reduction: then $w \in Q$ if and only if $f(w) \in P$. Now assume that $P$ is CE. If I want to know whether $w \in Q$ I will ask the algorithm for $P$ if $f(w) \in P$, if $(w) \in P$ I will eventually find this out but if it is not, the algorithm may loop forever. If $P$ is co-CE I have an algorithm $B$ with the property that for any word **not** in $P$ the algorithm will eventually tell me that it is not in $P$; for words that are in $P$ the algorithm may loop forever. Now I want to know if $w \in Q$, I ask $B$ if $f(w) \in P$, if it is **not** then I will find out eventually but if it is in $P$ $B$ may loop forever. Thus $Q$ is also co-CE. ∎

**Corollary 2.** If $Q \leq_m P$ and if $Q$ is not CE then neither is $P$.

**Proof** . This is just the contrapositive of the proposition. ∎

Now for the main point of this note. Consider the set:

$$EQ_{TM} = \{\langle M_1,\ M_2\rangle \mid L(M_1) = L(M_2)\}.$$

I claim that this set is neither CE nor co-CE. Recall that the set $A_{TM}$ is not decidable but it is CE. Recall also that a set is decidable if it is both CE and co-CE, so $A_{TM}$ is not co-CE. Its complement $\overline{A_{TM}}$ is co-CE but not CE. First, I will show that $\overline{A_{TM}} \leq_m EQ_{TM}$, which will establish that $EQ_{TM}$ is not CE. Then I will show that $A_{TM} \leq_m EQ_{TM}$, which will show that $EQ_{TM}$ is not co-CE.

To show the first claim we proceed as follows. Suppose I want to know whether $w \notin L(M)$ for some Turing machine $M$. Define two Turing machines $M_1$ and $M_2$ as follows. $M_1$ rejects everything, thus $L(M_1) = \emptyset$. $M_2$ checks if the input word is $w$, if it is not $M_2$ will reject it. If the input word is $w$ then $M_2$ simulates $M$ on $w$: if $M$ accepts $w$ then $M_2$ will as well. Thus $\langle M,\ w\rangle \in \overline{A_{TM}}$ if and only if $L(M_1) = L(M_2)$. Thus if $EQ_{TM}$ is CE then so is $\overline{A_{TM}}$; but we know that the latter is not.

To show the second claim we proceed as follows. This time I want to know if $w \in L(M)$. I define $M_1$ to accept everything and $M_2$ as follows. $M_2$ checks its input $x$: if $x \neq w$ then $M_2$ accepts it, if $x = w$ then $M_2$ simulates $M$ on $w$ and accepts $w$ if $M$ does. In this case $\langle M, w\rangle \in A_{TM}$ iff $L(M_1) = L(M_2)$. Thus if $EQ_{TM}$ is co-CE so is $A_{TM}$ but we know that $A_{TM}$ is not co-CE.