McGill University

MATH 323, Winter 2017

Assignment 3

4 March, 2017

- The deadline for submission is 19 March 2017, 10 PM.
- Answer all questions. The problems carry a total of 16 marks. Maximum you can score is 15.
- Upload your paper as a single PDF file (scan it if using pen and paper) to the folder named "Assignment-3" located under "Assignments" tab in myCourses. Do not upload multiple pages separately. Make sure pages are not upside down.
- Write your name and Student ID on top of the first page.
- Explain your argument clearly. Solutions without proper justification may not receive full credit.
- Simplify your answer as much as possible.

---

1. In a factory, 1000 Rubik's cubes are manufactured every day. The side length of each cube is a random variable with density

$$f(x) = 2x, \quad x \in [0, 1].$$

Let μ and σ respectively denote the expectation and the standard deviation of the **volume of a cube**.

(a) Compute μ and σ.

(b) A cube is labeled defective if its volume does not fall in the interval $(\mu - \sigma, \mu + \sigma)$. Find the expected number of cubes labeled defective in a day.

(2+2)

2. Let X be a real-valued random variable with density f. Further, assume that there exists $\mu \in \mathbb{R}$ such that

$$f(\mu + x) = f(\mu - x), \quad \text{for all } x \in \mathbb{R}.$$

(a) Show that the random variables $(\mu - X)$ and $(X - \mu)$ have the same distribution, i.e., have the same cdf.

(b) Show that if $\int_{\mathbb{R}} |x|^\alpha f(x)dx < \infty$ for all $\alpha > 0$, then

$$\mathbb{E}(X) = \mu, \text{ and } \mathbb{E}\big((X-\mu)^{2n+1}\big) = 0, \text{ for any integer } n \geq 1.$$

(2+2)

3. Suppose X is a real-valued random variable.

(a) Assume that the cdf F of X is continuous and strictly increasing on $\mathbb{R}$, i.e., $F(x) < F(y)$ if $x < y$. Find the distribution of $F(X)$.

(b) Assume that X has density f. Find the pmf of $\lfloor X \rfloor$. (Recall that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, e.g., $\lfloor 3 \rfloor = \lfloor 3.7 \rfloor = 3$.)

(2+2)

4. It is a trivial fact that if X is a nonnegative random variable and $X \leq c$ for some $c \geq 0$, then $\mathbb{E}(X^n) \leq c^n$ for all $n \geq 1$. In this problem, we prove the converse:

Suppose that X is a nonnegative random variable that either has a density or is discrete. Assume further that there exists $c \geq 0$ such that

$$\mathbb{E}\big(X^n\big) \leq c^n, \text{ for every } n \geq 1.$$

(a) Show that for every $\varepsilon > 0$,

$$\mathbb{P}\big(X > c + \varepsilon\big) = 0.$$

(Hint: Use one of the many inequalities related to Markov's inequality.)

(b) Conclude from (a) that

$$\mathbb{P}\big(X \leq c\big) = 1.$$

(**Note:** The result is true even without the assumption that X is either discrete or has a density. But in this course, we are not working with expectations of general random variables.) (3+1)