McGill University
Department of Mathematics and Statistics
MATH 254 Analysis 1, Fall 2015
Assignment 5

You should carefully work out **all** problems. However, you only have to hand in solutions to **problems 1, 2, 10, and 11**

This assignment is due **Monday, November 9, at the end of the class. Late assignments will not be accepted.**

1. Let (y_n) be an unbounded sequence of positive numbers satisfying $y_{n+1} > y_n$ for all $n \in \mathbb{N}$. Let (x_n) be another sequence, and suppose that the limit

$$\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

exists. Prove that

$$\lim \frac{x_n}{y_n} = \lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

Hint: You may use the Problem 3 on the Assignment 4.

Using the above result prove that for any $p \in \mathbb{N}$ the following holds:

(a)
$$\lim \frac{1^p + 2^p + \cdots n^p}{n^{p+1}} = \frac{1}{p+1}.$$

(b)
$$\lim \left(\frac{1^p + 2^p + \cdots n^p}{n^p} - \frac{n}{p+1} \right) = \frac{1}{2}.$$

(c)
$$\lim \frac{1^p + 3^p + \cdots (2n+1)^p}{n^{p+1}} = \frac{2^p}{p+1}.$$

2. Let (x_n) and (y_n) be two sequences defined recursively as follows: $x_1 = a \geq 0$, $y_1 = b \geq 0$,

$$x_{n+1} = \sqrt{x_n y_n}, \qquad y_{n+1} = \frac{x_n + y_n}{2}, \qquad n \geq 1.$$

Prove that the sequences (x_n) and (y_n) are convergent and that

$$\lim x_n = \lim y_n.$$

3. Prove that

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$$

for all $n \in \mathbb{N}$.

4. Prove that the sequence

$$x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n, \qquad n \in \mathbb{N},$$

converges.

Remark. The limit of this sequence is called the Euler-Mascheroni constant; its numerical value is 0.5772156649...It is currently unknown whether this constant is rational or irrational.

5. Prove that

$$\lim \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) = \ln 2.$$

6. Let

$$x_n = \left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{4}\right) \cdots \left(1 + \frac{1}{2^n}\right), \qquad n \in \mathbb{N}.$$

Prove that the sequence (x_n) converges.

7. Let (x_n), $x_n > 0$, be a convergent sequence. Prove that

$$\lim \sqrt[n]{x_1 x_2 \cdots x_n} = \lim x_n.$$

8. Let (x_n), $x_n > 0$, be a sequence such that the limit

$$L = \lim_{n \to \infty} \frac{x_{n+1}}{x_n}$$

exist. Prove that

$$\lim \sqrt[n]{x_n} = L.$$

Using this result, prove that

$$\lim \frac{n}{\sqrt[n]{n!}} = e.$$

9. Let (x_n) be a sequence such that for all $n, m \in \mathbb{N}$,

$$0 \le x_{n+m} \le x_n + x_m.$$

Prove that

$$\lim \frac{x_n}{n} = \inf \left\{ \frac{x_n}{n} \, : \, n \in \mathbb{N} \right\}.$$

10. Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n = \sup\{x_k \, : \, k \ge n\}$ and $S = \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.

11. Let $L \subseteq \mathbb{R}$. The set L is called open if for any $x \in L$ there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq L$. The set L is called closed if its complement $L^c = \{x \, : \, x \notin L\}$ is open.

   (a) Prove that L is closed if and only if for any converging sequence (x_n) with $x_n \in L$, the limit $x = \lim x_n$ is also an element of L.

(b) Let (x_n) be a bounded sequence. A point $x \in \mathbb{R}$ is called an accumulation point of (x_n) if there exists a subsequence (x_{n_k}) of (x_n) such that $\lim x_{n_k} = x$. We denote by L the set of all accumulation points of (x_n). By the Bolzano-Weierstrass Theorem, the set L is non-empty. Prove that L is a bounded closed set.

(c) Let (x_n) be a bounded sequence, let L be as in part (b) and let S be as in problem 1. Prove that $S = \sup L$.

12. Using Cauchy Convergence Criterion, prove that the sequence

$$x_n = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

is convergent.

13. Definition: A sequence (x_n) has bounded variation if there exists $c > 0$ such that for all $n \in \mathbb{N}$,

$$|x_2 - x_1| + |x_3 - x_2| + \cdots + |x_n - x_{n-1}| < c.$$

Show that if a sequence has bounded variation, then the sequence is converging. Find an example of a convergent sequence which does not have bounded variation.

14. Let $x_1 < x_2$ be arbitrary real numbers and

$$x_n = \frac{1}{3}x_{n-1} + \frac{2}{3}x_{n-2}, \qquad n > 2.$$

Find the formula for x_n and $\lim x_n$.

15. Let $x_1 > 0$ and

$$x_{n+1} = \frac{1}{2 + x_n}, \qquad n \geq 1.$$

Show that (x_n) is a contractive sequence and find $\lim x_n$.