MATH323 - Tutorial 3 (January 25th)

More Counting & Probability Problems

1. (Exercise 2.66) There are 20 labourers who are assigned to four different construction jobs. The first job (considred to be very undesirable) required 6 laborers, the second, third, fourth required 4, 5, 5 workers respectively. Of the 20 labourers, 4 labourers are members of a particular ethnic group. What is the probability that
   a) an ethnic member is assigned to each group?
   b) no ethnic group member is assigned to a type 4 job?

Solution:

$$|S| = \binom{20}{6\ 4\ 5\ 5} = \frac{20!}{6!\,4!\,5!\,5!}$$

a) Let E be the event that an ethnic member is assigned to each group.
The four members can be assigned in $\binom{4}{1\ 1\ 1\ 1} = 4!$ many ways. (i.e. 1 per group)
The remaining 16 labourers can be assigned in $\binom{16}{5\ 3\ 4\ 4}$ many ways. (Note: the # of workers per job went down by 1 because they were already filled by the members of the ethnic group).

$$|E| = 4! \times \frac{16!}{5!\,3!\,4!\,4!} \Rightarrow \mathbb{P}(E) = \left(\frac{4!\,16!}{5!\,3!\,4!\,4!}\right) \Big/ \left(\frac{20!}{6!\,4!\,5!\,5!}\right)$$

$$= \frac{16!\,5!\,6!}{20!\,3!}$$

b) Let E be the event that no ethnic member is assigned to a type 4 group. Of the 16 non-ethnic members, there are $\binom{16}{5}$ ways they can be assigned to job 4. With the remaining 15 members, there are $\binom{15}{6\ 4\ 5}$ ways of assigning them to the remaining 3 jobs.

Thus, $|E| = \binom{16}{5}\binom{15}{6\ 4\ 5} \Rightarrow \mathbb{P}(E) = \binom{16}{5}\binom{15}{6\ 4\ 5} \Big/ \binom{20}{6\ 4\ 5\ 5}$

2. (Exercise 2.54) 3 undergraduates and 5 graduates are available to fill certain government posts. 4 students are to be randomly selected from this group, find the probability that exactly 2 undergraduates will be among the 4 chosen.

Solution: $|S| = \binom{8}{4} = 2 \times 7 \times 5 = 70$

Let E be the event that exactly 2 undergraduates will be among the 4 chosen. Of the 3 undergrads, there are $\binom{3}{2}$ ways of selecting 2. Of the 5 graduates, there are $\binom{5}{2}$ ways of selecting the remaining 2 positions.

$$|E| = \binom{3}{2} \times \binom{5}{2}$$
$$= \frac{3!}{2!2!} \times \frac{5!}{3!2!} = 5 \times 3 = 15$$

$$\Rightarrow \mathbb{P}(E) = \frac{15}{70}$$

3. (Exercise 2.46) Ten teams are playing in a basketball tournament, where the teams are randomly assigned to games 1, 2, 3, 4, 5. Suppose half are tier I and the other half are tier II. What is the probability that at least one of the matches will feature a tier I vs. tier I match or a tier II v.s. tier II match.

Solution:
$$|S| = \binom{10}{2\ 2\ 2\ 2\ 2} \Big/ 5! = \frac{10 \times 9 \times 8 \times 7 \times 6}{2^5}$$

Let E be the event of having at least one match with the same tier type teams playing against each other.
Then $\bar{E}$ is the event that all matches are tier I v.s. tier II teams.

$$|\bar{E}| = \binom{5}{1\ 1\ 1\ 1\ 1} = 5!$$

$$\Rightarrow \mathbb{P}(E) = 1 - \mathbb{P}(\bar{E}) = 1 - \frac{5!(2!)^5}{10 \times 9 \times 8 \times 7 \times 6}$$ {see last page for details}

4. (Exercise 2.56) A student prepares for an exam by studying a list of ten problems. She can solve 6 problems. The instructor selects 5 problems of the ten to put on the exam. What is the probability that she can solve all 5 problems?

Solution: $|S| = \binom{10}{5} = \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} = 2\times 2\times 7\times 9$

Let E be the event she can solve all 5 problems. Then,

$$|E| = \binom{6}{5} = 6$$

$$\Rightarrow \mathbb{P}(E) = \frac{|E|}{|S|} = \frac{6}{2\times 2\times 7\times 9} = \frac{1}{2\times 7\times 3} = \frac{1}{42}$$

5. (Exercise 2.102) Diseases I and II are prevalent among people in a certain population. It is assumed 10% will contract disease I eventually, and that 15% will contract disease II eventually, and 3% will contract both diseases.

a) What is the probability that a randomly chosen person will contract at least one disease?

b) What is the conditional probability that a randomly chosen person will contract both diseases, given that he or she has contractly at least one disease?

Solution: Let A be the event of contracting disease I and B be the event of contracting disease II. $\mathbb{P}(A) = 0.10$, $\mathbb{P}(B) = 0.15$, $\mathbb{P}(A\cap B) = 0.03$

a) At least one disease $\Rightarrow A\cup B$

$$\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A\cap B) = 0.10 + 0.15 - 0.03 = 0.22$$

b) $\mathbb{P}(A\cap B \mid A\cup B) = \dfrac{\mathbb{P}((A\cap B)\cap(A\cup B))}{\mathbb{P}(A\cup B)} = \dfrac{\mathbb{P}(A\cap B)}{\mathbb{P}(A\cup B)} = \dfrac{0.03}{0.22} = \dfrac{3}{22}$

6. (Exercise 2.104) If A and B are two events, prove that
$$\mathbb{P}(A \cap B) \geq 1 - \mathbb{P}(\bar{A}) - \mathbb{P}(\bar{B})$$

Solution: Observe the following

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \leq 1$$
$$\Rightarrow 1 \geq \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$
$$\Rightarrow \mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$$
$$= 1 - \mathbb{P}(\bar{A}) + 1 - \mathbb{P}(\bar{B}) - 1$$
$$= 1 - \mathbb{P}(\bar{A}) - \mathbb{P}(\bar{B})$$
$$\Rightarrow \mathbb{P}(A \cap B) \geq 1 - \mathbb{P}(\bar{A}) - \mathbb{P}(\bar{B})$$

7. (Exercise 2.137) Five identical bowls are labeled $1, 2, \ldots, 5$. Bowl i contains i white and $5-i$ black balls, with $i = 1, 2, \ldots, 5$. A bowl is randomly selected and two balls are randomly selected (without replacement) from that bowl.

a) What is the probability that both balls selected are white?

b) Given that both balls selected are white, what is the probability that bowl 3 was selected?

Solution:

Let A_i be the event both balls are white from bowl i

Let B_i be the event bowl i is selected, $i = 1, \ldots, 5$.

$\mathbb{P}(B_i) = 0.2$, and we assume $A = \bigcup_i A_i$

a) $$\mathbb{P}(A) = \sum_{i=1}^{5} \mathbb{P}(A_i \mid B_i)\mathbb{P}(B_i) = 0.2\left[0 + \left(\tfrac{2}{5}\right)\left(\tfrac{1}{4}\right) + \left(\tfrac{3}{5}\right)\left(\tfrac{2}{4}\right) + \left(\tfrac{4}{5}\right)\left(\tfrac{3}{4}\right) + 1\right]$$
$$= \left(\tfrac{1}{5}\right)(2) = \tfrac{2}{5} = \tfrac{20}{50}$$

b) $$\mathbb{P}(B_3 \mid A) = \frac{\mathbb{P}(B_3 \cap (A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5))}{\mathbb{P}(A)}$$
$$= \frac{\mathbb{P}(B_3 \cap A_3)}{\mathbb{P}(A)} = \frac{(1/5)(3/10)}{20/50} = \frac{3}{20}.$$

Problem 3 - Explanation

10 teams of which 5 are tier I and 5 are tier II assigned to play five games to which we assume there is no natural ordering in the five games played.

Let S be the sample space of all the five games played.
Let $|S|$ be the number of ways of arranging the 10 teams into the 5 games.

Thus, $|S| = \binom{10}{2\ 2\ 2\ 2\ 2} \Big/ 5!$ ← there is no natural ordering in the five games played

↑ we need to put 10 teams into 5 games/groups of 2

i.e. if tier I teams are denoted $A_1, \ldots, A_5$ and tier II as $B_1, \ldots, B_5$ then we do not want to count the following two setups as 2 separate occurrences

	Game 1	Game 2	Game 3	Game 4	Game 5
→	A_1B_1	A_2B_2	A_3B_3	A_4B_4	A_5B_5
→	A_2B_2	A_1B_1	A_3B_3	A_4B_4	A_5B_5

and so we divide $\binom{10}{2\ 2\ 2\ 2\ 2}$ by $5!$.

Let $\bar{E}$ be the event all games are tier I v.s. tier II. We can count $\bar{E}$ by fixing all the tier I teams in a set order and varying all tier II team orderings.
i.e.

tier I teams : $A_1 \quad A_2 \quad A_3 \quad A_4 \quad A_5$
tier II teams : ${}^5P_5 = 5!$ many ways of arranging B_1 through B_5

Thus, $|\bar{E}| = 5!$ which accounts for all possible matches between tier I and tier II teams.

Therefore, $\mathbb{P}(E) = 1 - \mathbb{P}(\bar{E}) = 1 - \dfrac{(5!)(5!)}{\binom{10}{2\ 2\ 2\ 2\ 2}}$ ■