1. Define a sequence (x_n) recursively by $x_1 = 0$, $x_2 = 1$, $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$.

   (a) Prove by induction that $x_{2k-1} < x_{2k}$ for all $k \in \mathbb{N}$.

   By part (a) we can define intervals $I_k := [x_{2k-1}, x_{2k}]$ for all $k \in \mathbb{N}$. Prove the following:

   (b) The I_k form a nested sequence of closed and bounded intervals i.e. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$.

   (c) Show that the intervals I_k have exactly one point in common i.e. show that $\bigcap_{k \in \mathbb{N}} I_k = \{x\}$ for some $x \in \mathbb{R}$.

   (d) Show that (x_n) converges and that $\lim (x_n) = x$.

   (It can be shown that $x = \frac{2}{3}$. Proving this is *not* part of this problem.)

   Solution:

   (a) We prove this by induction.

   Base Case $k = 1$: $x_1 = 0 < 1 = x_2$ i.e. $x_1 < x_2$ which is what we had to show.

   Inductive Step : We assume that $x_{2k-1} < x_{2k}$. Then

   $$x_{2k+1} < x_{2k+2} \Leftrightarrow x_{2k+1} < \frac{1}{2}(x_{2k+1} + x_{2k}) \Leftrightarrow \frac{1}{2}x_{2k+1} < \frac{1}{2}x_{2k} \Leftrightarrow x_{2k+1} < x_{2k}$$
   $$\Leftrightarrow \frac{1}{2}(x_{2k-1} + x_{2k}) < x_{2k} \Leftrightarrow \frac{1}{2}x_{2k-1} < \frac{1}{2}x_{2k} \Leftrightarrow x_{2k-1} < x_{2k}$$

   Especially, the induction hypothesis $x_{2k-1} < x_{2k}$ implies that $x_{2k+1} < x_{2k+2}$ which is what we had to show in the inductive step.

   This proves $x_{2k-1} < x_{2k}$ for all $k \in \mathbb{N}$ and shows that the intervals $I_k := [x_{2k-1}, x_{2k}]$ are well-defined.

Before we continue with proving parts (b)–(d) we first establish some additional useful facts about the sequence (x_n):

   (i) $x_{2k+1} < x_{2k}$ for all $k \in \mathbb{N}$.

   Proof.

   $$x_{2k+1} < x_{2k} \Leftrightarrow \frac{1}{2}(x_{2k} + x_{2k-1}) < x_{2k} \Leftrightarrow \frac{1}{2}x_{2k-1} < \frac{1}{2}x_{2k} \Leftrightarrow x_{2k-1} < x_{2k}$$

   where the latter holds by (a). This completes the proof. $\qquad \square$

(ii) $x_{2k-1} < x_{2k+1}$ for all $k \in \mathbb{N}$.

Proof.

$$x_{2k-1} < x_{2k+1} \Leftrightarrow x_{2k-1} < \frac{1}{2}(x_{2k} + x_{2k-1}) \Leftrightarrow \frac{1}{2}x_{2k-1} < \frac{1}{2}x_{2k} \Leftrightarrow x_{2k-1} < x_{2k}$$

where the latter holds by (a). This completes the proof. $\qquad\square$

(iii) $x_{2k+2} < x_{2k}$ for all $k \in \mathbb{N}$.

Proof.

$$x_{2k+2} < x_{2k} \Leftrightarrow \frac{1}{2}(x_{2k+1} + x_{2k}) < x_{2k} \Leftrightarrow \frac{1}{2}x_{2k+1} < \frac{1}{2}x_{2k} \Leftrightarrow x_{2k+1} < x_{2k}$$

where the latter holds by (i). This completes the proof. $\qquad\square$

(iv) $x_{2k} - x_{2k-1} = \frac{1}{4^{k-1}}$ for all $k \in \mathbb{N}$.

Proof. We prove this by induction.
<u>Base Case $k = 1$</u>: $x_2 - x_1 = 1 = \frac{1}{4^0}$ which is what we had to show.
<u>Inductive Step</u> : We assume that $x_{2k} - x_{2k-1} = \frac{1}{4^{k-1}}$. Then

$$x_{2k+2} - x_{2k+1} = \frac{1}{2}(x_{2k+1} + x_{2k}) - x_{2k+1} = \frac{1}{2}(x_{2k} - x_{2k+1}) = \frac{1}{2}(x_{2k} - \frac{1}{2}(x_{2k} + x_{2k-1}))$$

$$= \frac{1}{2}(\frac{1}{2}x_{2k} - \frac{1}{2}x_{2k+1}) = \frac{1}{4}(x_{2k} - x_{2k-1}) \stackrel{\text{Ind. Hyp}}{=} \frac{1}{4} \cdot \frac{1}{4^{k-1}} = \frac{1}{4^k}$$

$\qquad\square$

We now continue with proving parts (b)–(d).

(b) Let $I_k = [a_k, b_k]$ i.e. $a_k = x_{2k-1}$, $b_k = x_{2k}$. We have already seen in part (a) that $a_k < b_k$ for all $k \in \mathbb{N}$. Furthermore, for all $k \in \mathbb{N}$ we have: $a_k = x_{2k-1} \overset{\text{(ii)}}{<} x_{2k+1} = a_{k+1}$ and $b_{k+1} = x_{2k+2} \overset{\text{(iii)}}{<} x_{2k} = b_k$. But this just means that $I_{k+1} \subseteq I_k$ for all $k \in \mathbb{N}$ i.e. the I_k form a nested sequence of closed and bounded intervals.

(c) By the nested interval property of $\mathbb{R}$, $\bigcap_{k \in \mathbb{N}} I_k \neq \emptyset$; more precisely: $\bigcap_{k \in \mathbb{N}} I_k = [a, b]$ where $a = \sup\{a_1, a_2, \dots\}$ and $b = \inf\{b_1, b_2, \dots\}$. Furthermore, the length of I_k equals $b_k - a_k = x_{2k} - x_{2k-1} \overset{\text{(iv)}}{=} \frac{1}{4^{k-1}}$. Since $[a, b] \subseteq I_k$ for all $k \in \mathbb{N}$ it follows that $b - a \leq \frac{1}{4^{k-1}}$ for all $k \in \mathbb{N}$, where $\lim \left(\frac{1}{4^{k-1}}\right) = 0$. This implies $b = a$ i.e. $\bigcap_{k \in \mathbb{N}} I_k = \{x\}$ where $x = a = \sup\{a_1, a_2, \dots\} = b = \inf\{b_1, b_2, \dots\}$.

(d) Let $\epsilon > 0$. Since $x = \sup\{a_1, a_2, \dots\}$, $x - \epsilon$ is not an upper bound for $\{a_1, a_2, \dots\}$. Thus there exists an $K_1 \in \mathbb{N}$ with $x - \epsilon < a_{K_1} \leq x$. Since (a_k) is increasing, this means that $x - \epsilon < a_k \leq x$ for all $k \geq K_1$.
Similarly, since $x = \inf\{b_1, b_2 \dots\}$, $x + \epsilon$ is not a lower bound for $\{b_1, b_2, \dots\}$. Thus there exists a $K_2 \in \mathbb{N}$ with $x \leq b_{K_2} < x + \epsilon$. Since (b_k) is decreasing, this means that $x \leq b_k < x + \epsilon$ for all $k \geq K_2$.
Combining these two results yields that $x - \epsilon < a_k = x_{2k-1} \leq x \leq x_{2k} = b_k < x + \epsilon$ for all $k \geq K := \max\{K_1, K_2\}$. But this means that $x_n \in V_\epsilon(x)$ for all $n \geq N := 2K - 1$ which proves that $x = \lim(x_n)$.

2. Use the definition of the limit of a sequence to show that:

(a) $\lim \left(\frac{n^2-1}{2n^2+3} \right) = \frac{1}{2}$ (b) $\lim \left(\frac{\sqrt{n}}{n+1} \right) = 0$ (c) $\lim \left(\frac{(-1)^n n}{n^2+1} \right) = 0$

Solution:

(a) Let $\epsilon > 0$. Then

$$\left| \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right| = \left| \frac{2n^2 - 2 - 2n^2 - 3}{2(2n^2+3)} \right| = \frac{5}{2(2n^2+3)} < \frac{5}{4n^2} < \epsilon \Leftrightarrow n^2 > \frac{5}{4\epsilon} \Leftrightarrow n > \sqrt{\frac{5}{4\epsilon}}$$

By the Archimedean Property, $\exists N \in \mathbb{N}$ with $N > \sqrt{\frac{5}{4\epsilon}}$. Then $\left| \frac{n^2-1}{2n^2+3} - \frac{1}{2} \right| < \epsilon$ for all $n \geq N$ which proves that $\lim \left(\frac{n^2-1}{2n^2+3} \right) = \frac{1}{2}$.

(b) Let $\epsilon > 0$. Then

$$\left| \frac{\sqrt{n}}{n+1} - 0 \right| = \frac{\sqrt{n}}{n+1} < \frac{\sqrt{n}}{n} < \frac{1}{\sqrt{n}} < \epsilon \Leftrightarrow \sqrt{n} > \frac{1}{\epsilon} \Leftrightarrow n > \frac{1}{\epsilon^2}$$

By the Archimedean Property, $\exists N \in \mathbb{N}$ with $N > \frac{1}{\epsilon^2}$. Then $\left| \frac{\sqrt{n}}{n+1} - 0 \right| < \epsilon$ for all $n \geq N$ which proves that $\lim \left(\frac{\sqrt{n}}{n+1} \right) = 0$.

(c) Let $\epsilon > 0$. Then

$$\left| \frac{(-1)^n n}{n^2+1} - 0 \right| = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}$$

By the Archimedean Property, $\exists N \in \mathbb{N}$ with $N > \frac{1}{\epsilon}$. Then $\left| \frac{(-1)^n n}{n^2+1} - 0 \right| < \epsilon$ for all $n \geq N$ which proves that $\lim \left(\frac{(-1)^n n}{n^2+1} \right) = 0$.

3. Let $P_{n,k}$, $n, k \in \mathbb{N}$, be real numbers satisfying the following:

   (a) $P_{n,k} \geq 0$ for all n, k.

   (b) $\sum_{k=1}^{n} P_{n,k} = 1$ for all n.

   (c) $\lim_{n \to \infty} P_{n,k} = 0$ for all k.

   Let (x_n) be a convergent sequence and let a sequence (y_n) be defined by

   $$y_n = \sum_{k=1}^{n} P_{n,k} x_k.$$

   Prove that (y_n) is a convergent sequence and that

   $$\lim_{n \to \infty} y_n = \lim_{n \to \infty} x_n.$$

   Solution:

   Let $\epsilon > 0$ be arbitrary. Let $x := \lim_{n \to \infty} x_n$. Then,

   $$\exists K \in \mathbb{N} \text{ such that } |x_n - x| < \epsilon/2 \quad \forall n \geq K. \tag{1}$$

   Remark: At that point, you may wonder why we have an "$\epsilon/2$" and not simply an "ϵ". Note that both are equally valid, and so would be the choices 6ϵ, 2ϵ, $\epsilon/3$ or $6\epsilon/(7\pi^2)$. The reason we choose "$\epsilon/2$" is that when considering $|y_n - x| = |\sum_{k=1}^{n} P_{n,k} x_k - x|$, we will have two main terms to bound: one for "small" k's (so that we have only finitely many $P_{n,k}$'s to control) and another one when k is large (so that we can control $|x_k - x|$). Each term will be bounded by $\epsilon/2$ to get a final elegant "ϵ". This will be made more precise later.

   Note that since $(x_n)_{n \in \mathbb{N}}$ converges, $\exists B \in \mathbb{R}$ such that $|x_n| \leq B \quad \forall n \in \mathbb{N}$. For any $k \leq K$, since $\lim_{n \to \infty} P_{n,k} = 0$,

   $$\exists N_k \in \mathbb{N} \text{ such that } |P_{n,k}| < \epsilon/(2K(B + |x|)) \quad \forall n \geq N_k. \tag{2}$$

   Remark: Here, "$\epsilon/(2K(B + |x|))$" is okay because K is, since (1), a fixed integer. The particular choice of "$\epsilon/(2K(B + |x|))$" is justified by the fact that if we continue with "$\epsilon/2$", we will end up showing something like $|y_n - x| < K(B + |x|)\epsilon + \epsilon/2$, which is logically good enough, but less elegant. Note that it is common to first try to work on a draft of the following inequalities to see what kind of bound is needed and then pick the expression that will make the proof as elegant as possible.

We define $N := 1 + \max\{N_1, N_2, \ldots, N_K, K\}$. Now, for $n \geq N$,

$$\begin{aligned}
|y_n - x| &= \left|\left(\sum_{k=1}^{n} P_{n,k} x_k\right) - x\right| && \text{by definition} \\
&= \left|\sum_{k=1}^{n} P_{n,k}(x_k - x)\right| && \text{since } \sum_{k=1}^{n} P_{n,k} = 1 \\
&\leq \sum_{k=1}^{n} |P_{n,k}(x_k - x)| && \text{by applying the triangle inequality } n \text{ times} \\
&= \sum_{k=1}^{n} P_{n,k}|x_k - x| && \text{since } P_{n,k} \geq 0 \\
&= \sum_{k=1}^{K} P_{n,k}|x_k - x| + \sum_{k=K+1}^{n} P_{n,k}|x_k - x| \\
&\leq \sum_{k=1}^{K} P_{n,k}(|x_k| + |x|) + \sum_{k=K+1}^{n} P_{n,k}\epsilon/2 && \text{by the triangle inequality and (1)} \\
&\leq \sum_{k=1}^{K} P_{n,k}(B + |x|) + (\epsilon/2) \sum_{k=K+1}^{n} P_{n,k} && \text{since } |x_k| \leq B \\
&\leq (B + |x|) \sum_{k=1}^{K} P_{n,k} + \epsilon/2 && \text{since } \sum_{k=K+1}^{n} P_{n,k} \leq 1 \\
&< (B + |x|) \sum_{k=1}^{K} \epsilon/(2K(B + |x|)) + \epsilon/2 && \text{by (2) since } n \geq N \\
&= \epsilon
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary we conclude that $(y_n)_{n\in\mathbb{N}}$ converges and that $\lim_{n\to\infty} y_n = x$.

Q.E.D.

4. Let (x_n) be a convergent sequence and let (y_n) be a sequence defined by

$$y_n = \frac{x_1 + \cdots + x_n}{n}.$$

Prove that (y_n) is a convergent sequence and that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n.$$

Solution:

This is just a special case of the previous problem since $P_{n,k} = 1/n$ satisfies (a), (b) and (c).

5. Let $b \in \mathbb{R}$ with $b > 1$. Prove that $\lim \left(\frac{n}{b^n} \right) = 0$.

Solution:

Remark: Applying Bernoulli's inequality directly to $(1+a)^n$ (i.e. using $(1+a)^n \geq 1+na$) will not solve the problem; the inequality is not sharp enough in this case:

$$\left| \frac{n}{b^n} - 0 \right| = \frac{n}{(1+a)^n} \leq \frac{n}{1+an} \to \frac{1}{a} > 0$$

We will demonstrate below that splitting $(1+a)^n$ into two factors will lead to a sufficiently sharp estimate.

Claim: For all $n \in \mathbb{N}$, $n \geq 2$ we have $(1+a)^n > \frac{n^2-1}{4}a^2 > 0$.

Proof. We distinguish 2 cases.

1. Case: n is even.

Let $n = 2k$, $k \in \mathbb{N}$. Then

$$(1+a)^n = (1+a)^{2k} = ((1+a)^k)^2 \overset{\text{Bernoulli}}{\geq} (1+ka)^2 > (ka)^2 = \left(\frac{n}{2}a\right)^2 = \frac{n^2}{4}a^2 > \frac{n^2-1}{4}a^2$$

2. Case: n is odd.

Let $n = 2k-1$, $k \in \mathbb{N}$, $k \geq 2$. Then

$$(1+a)^n = (1+a)^{2k-1} = (1+a)^k(1+a)^{k-1} \overset{\text{Bernoulli}}{\geq} (1+ka)(1+(k-1)a) > k(k-1)a^2$$
$$= \frac{n+1}{2}\left(\frac{n+1}{2} - 1\right)a^2 = \frac{(n+1)(n-1)}{4}a^2 = \frac{n^2-1}{4}a^2$$

This proves the claim. $\qquad\square$

Now we can prove convergence of the sequence. Let $\epsilon > 0$. Then

$$\left| \frac{n}{b^n} - 0 \right| = \frac{n}{(1+a)^n} \overset{\text{Claim}}{<} \frac{n}{\frac{n^2-1}{4}a^2} = \frac{4}{a^2}\frac{n}{n^2-1} < \frac{4}{a^2}\frac{n+1}{n^2-1} = \frac{4}{a^2}\frac{1}{n-1} < \epsilon$$
$$\Leftrightarrow \frac{1}{n-1} < \frac{a^2\epsilon}{4} \Leftrightarrow n-1 > \frac{4}{a^2\epsilon} \Leftrightarrow n > 1 + \frac{4}{a^2\epsilon}$$

Let $N \in \mathbb{N}$ with $N > \max\left\{2, 1+\frac{4}{a^2\epsilon}\right\}$. Then $\left|\frac{n}{b^n} - 0\right| < \epsilon$ for all $n \geq N$ which proves $\lim \left(\frac{n}{b^n}\right) = 0$.

Remark: A lot of people used theorem 3.2.11 of Bartle and Sherbert to show the limit. Others used the binomial expansion (done in the tuorials) to get the above claim. Note that in any case you have to show that a sequence is convergent before you take its limit.

6. Prove that $\lim \left(\frac{n!}{n^n} \right) = 0$.

Solution:

For all $n \in \mathbb{N}$, we have
$$\frac{n!}{n^n} = \frac{1}{n} \cdot \underbrace{\frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n}}_{\leq 1} \leq \frac{1}{n}.$$

Now we calculate $\lim \left(\frac{n!}{n^n} \right)$. Let $\epsilon > 0$. Then

$$\left| \frac{n!}{n^n} - 0 \right| = \frac{n!}{n^n} \leq \frac{1}{n} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon}.$$

Let $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon}$. Then $\left| \frac{n!}{n^n} - 0 \right| < \epsilon$ for all $n \geq N$. This proves that $\lim \left(\frac{n!}{n^n} \right) = 0$.