COMP 360 - Fall 2015 - Assignment 4

Due: 6:00 pm Nov 24th.

General rules: In solving these questions you may collaborate with other students but each student has to write his/her own solution. There are in total 110 points, but your grade will be considered out of 100. You should drop your solutions in the assignment drop-off box located in the Trottier Building.

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1. (10 Points) Show that the following problem is in PSPACE:
   - Input: A CNF ϕ.
   - Output: The number of the truth assignments that satisfy ϕ.

2. (10 Points) Given a set P of n points on the plane, consider the problem of finding the smallest circle containing all the points in P. Show that the following is a 2-factor approximation algorithm for this problem. Pick a point x in P, and set r to be the distance of the farthest point in P from x. Output the circle centered at x with radius r.

3. (10 Points) Consider the following optimization version of the Subset-Sum problem: Given positive integers $\{w_1, \ldots, w_n\}$ and a positive integer m. We want to find a set $S \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in S} w_i \leq m$ and is maximized. Show that the following is a $\frac{1}{2}$-factor approximation algorithm:
   - Set $S := \emptyset$.
   - Sort the numbers such that $w_1 \geq w_2 \geq \ldots \geq w_n$.
   - For $i = 1, \ldots, n$:
   - if it is possible add i to S without violating $\sum_{i \in S} w_i \leq m$, then add i to S.

4. Consider the MAX-SAT problem: Given a CNF formula ϕ on variables $x_1, \ldots, x_n$, find a truth assignment to the variables that maximizes the total number of satisfied clauses.
   (a) (10 Points) Show that the following is a $\frac{1}{2}$-factor approximation algorithm for MAX-SAT (meaning that: the output of the algorithm is always at least half of the optimum): Let σ_{true} be the truth assignment that assigns True to every variable, and σ_{false} be the truth assignment that assigns False to every variable. Compute the number of clauses satisfied by σ_{true} and σ_{false}, and output the better assignment.
   (b) (5 Points) Give a tight example: An input instance where this algorithm performs as bad as the $\frac{1}{2}$ factor.

5. (10 Points) Problem 10 of Chapter 11: Suppose you are given an $n \times n$ grid graph G. Associated with each node v is an integer weight $w(v) \geq 0$. You may assume that all the weights are distinct. Your goal is to choose an independent set S of nodes of the grid, so that the sum of the weights of the nodes in S is as large as possible. (The sum of the weights of the nodes in S will be called its total weight.) Consider the following greedy algorithm for this problem.

- Start with $S := \emptyset$.
- While some node remains in G:
  - Pick a node v of maximum weight.
  - Add v to S.
  - Delete v and its neighbors from G
- Endwhile.

Show that this algorithm returns an independent set of total weight at least $\frac{1}{4}$ times the maximum total weight of any independent set in the grid graph G.

6. Consider a directed bipartite graph $G = (V, E)$. We want to eliminate all the directed cycles of length 4 by removing a smallest possible set of <u>vertices</u>.

   (a) (5 points) Let $\mathcal{C}_4$ denote the set of all cycles of length 4 in the graph. Show that the following integer program models the problem:

   $$\begin{array}{lll} \min & \sum_{v\in V} x_v & \\ \text{s.t.} & \sum_{u\in C} x_u \geq 1 & \forall C \in \mathcal{C}_4 \\ & x_u \in \{0,1\} & u \in V \end{array}$$

   (b) (5 points) Why does the optimal solution to the following relaxation provides a lower bound for the optimal answer to the above integer linear program? In other words why it is not necessary to have the constraints $x_u \leq 1$ in the relaxation?

   $$\begin{array}{lll} \min & \sum_{v\in V} x_v & \\ \text{s.t.} & \sum_{u\in C} x_u \geq 1 & \forall C \in \mathcal{C}_4 \\ & x_u \geq 0 & \forall u \in V \end{array}$$

   (c) (10 points) Give a simple 4-factor approximation algorithm for the problem based on rounding the solution to the above linear program.

   (d) (15 points) Let L and R denote the set of the vertices in the two parts of the bipartite graph. (Every edge has one endpoint in L and one endpoint in R). Let x^* denote an optimal solution to the linear program in Part (b). We round x^* in the following way:

   For every $u \in V$,

   - if $u \in R$ and $x^*_u \geq 1/2$, set $\widehat{x}_u = 1$.
   - if $u \in L$ and $x^*_u > 0$, set $\widehat{x}_u = 1$.
   - Otherwise set $\widehat{x}_u = 0$.

   Show that $\widehat{x}$ is a feasible solution to the integer linear program.

   (e) (10 points) Consider the dual of the relaxation:

   $$\begin{array}{lll} \max & \sum_{C\in\mathcal{C}_4} y_C & \\ \text{s.t.} & \sum_{C\in\mathcal{C}_4, u\in C} y_C \leq 1 & \forall u \in V \\ & y_C \geq 0 & \forall C \in \mathcal{C}_4 \end{array}$$

   and let y^* be an optimal solution to the dual. Use the complementary slackness to prove the following statement: For every $C \in \mathcal{C}_4$ either we have $|\{u : \widehat{x}_u = 1\}| \leq 3$ or $y^*_C = 0$.

   (f) (10 points) Use the complementary slackness and the previous parts to show that our rounding algorithm is a 3-factor approximation algorithm.