**MATH 254 Assignment 1** McGill University
Fall 2016

*You should work carefully on all problems. However you only have to hand in solutions to problems 1, 3, and 5. This assignment is due on Tuesday, September 20 in class.*

**Exercise 1.** [10 points] We denote $n! = 1 \times 2 \times 3 \times \cdots \times n$ for all integers $n \geq 1$. Conjecture a formula for the sum

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}$$

and prove your conjecture by using mathematical induction.

**Exercise 2.** For any integer $n \geq 1$ and real numbers $c_1, \ldots, c_n$, we denote

$$\sum_{k=1}^{n} c_k = c_1 + c_2 + \cdots + c_n\,.$$

Use mathematical induction to prove the inequality

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right)$$

for all integers $n \geq 1$ and real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$. *Hint: You may use the inequality* $2xy \leq x^2 + y^2$ *(which follows from* $(x-y)^2 \geq 0$*).*

**Exercise 3.** [10 points] Let $n \geq 2$ be an integer and $A_1, \cdots, A_n$ be $n$ sets such that $A_i \neq A_j$ for all $i, j \in \{1, \ldots, n\}$. Give a rigorous proof of the fact that at least one of the sets $A_1, \cdots, A_n$ contains none of the other sets, namely $\exists i \in \{1, \ldots, n\}\,,\ \forall j \in \{1, \ldots, n\} \setminus \{i\}\,,\ A_j \nsubseteq A_i$.

**Exercise 4.** For any set $A$, we denote $\mathcal{P}(A)$ the set of all subsets of $A$. Let $f\colon A \to B$ be a function. Show that

(i) $\forall G \in \mathcal{P}(A)$, $G \subseteq f^{-1}(f(G))$
(ii) $f$ is injective $\iff \forall G \in \mathcal{P}(A)$, $G = f^{-1}(f(G))$.
(iii) $\forall H \in \mathcal{P}(B)$, $f(f^{-1}(H)) \subseteq H$
(iv) $f$ is surjective $\iff \forall H \in \mathcal{P}(B)$, $f(f^{-1}(H)) = H$.

**Exercise 5.** [10 points] Let $f\colon A \to B$ be a function. Show that

$$f \text{ is bijective } \iff \forall G \in \mathcal{P}(A), \quad f(A \backslash G) = B \backslash f(G)$$

where $\mathcal{P}(A)$ is as in Exercise 4.

**Exercise 6.** Let $f : E \to E$ be a function.

(i) Assume that $f \circ f = f$. Show that $f$ is injective $\iff$ $f$ is surjective.
(ii) More difficult, assume that $f \circ f \circ f = f$ and show that $f$ is injective $\iff$ $f$ is surjective.