MATH 254 Tutorial 2 (Sets and Functions):

Problem 1 (Characteristic Function): Let S be a set. For a subset $A \subseteq S$, we define its characteristic function $\chi_A : S \to \{0, 1\}$ by:

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Prove the following facts about characteristic functions (all sets in this problem are subsets of a fixed set S).

a) $\chi_A = \chi_B$ if and only if $A = B$.

b) If A has finitely many elements, then $\Sigma_{x \in S} \chi_A(x) = n(A)$, where $n(A)$ is the number of elements of A.

c) $\chi_{S-A} = 1 - \chi_A$, where 1 is the constant function on S.

d) $\chi_{A_1 \cap ... \cap A_n} = \chi_{A_1} ... \chi_{A_n}$

e) $\chi_{A_1 \cup ... \cup A_n} = \Sigma_i \chi_{A_i} - \Sigma_{i<j} \chi_{A_i} \chi_{A_j} + \Sigma_{i<j<k} \chi_{A_i} \chi_{A_j} \chi_{A_k} - ... + (-1)^{n-1} \chi_{A_1} ... \chi_{A_n}$. (Hint: Use mathematical induction on the number of sets n.)

f) $n(A_1 \cup ... \cup A_n) = \Sigma_i n(A_i) - \Sigma_{i<j} n(A_i \cap A_j) + \Sigma_{i<j<k} n(A_i \cap A_j \cap A_k) - ... + (-1)^{n-1} n(A_1 \cap ... \cap A_n)$, where all A_i have finitely many elements. (Hint: Use parts b and e or repeat an inductive proof similar to part e.)

g) Using part a, prove that $(A \Delta B) \Delta C = A \Delta (B \Delta C)$, where Δ is the symmetric difference defined by $A \Delta B = (A - B) \cup (B - A)$. (Hint: Use $\chi_{A \Delta B} = \chi_A + \chi_B - 2\chi_A \chi_B$ twice to compute $\chi_{(A \Delta B) \Delta C}$.)

h) part g proves that Δ is associative, so there is no ambiguity in $A_1 \Delta ... \Delta A_n$ without any parentheses. Prove that $A_1 \Delta ... \Delta A_n$ is precisely the subset of S whose elements belong to oddly many of $A_1, ..., A_n$. (Hint: Use mathematical induction on the number of sets n.)

Problem 2: Let $f : A \to B$ be a function. Prove the following facts about image and pre-image:

a) $f(E \cup F) = f(E) \cup f(F)$, where $E, F \subseteq A$.

b) $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$, where $E, F \subseteq B$.

c) $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$, where $E, F \subseteq B$.

d) $f(E \cap F) \subseteq f(E) \cap f(F)$, where $E, F \subseteq A$.

e) Give an example for strict inclusion in part d.

f) Prove that we have equality in part d for any two subsets $E, F \subseteq A$ if and only if f is injective.

Problem 3: Let $f : A \to B$ be a function. Prove the following facts about image and pre-image:

a) $E \subseteq f^{-1}(f(E))$, where $E \subseteq A$.

b) Give an example for strict inclusion in part a.

c) Prove that we have equality in part a for any subset $E \subseteq A$ if and only if f is injective.

d) $f(f^{-1}(E)) \subseteq E$, where $E \subseteq B$.

e) Give an example for strict inclusion in part d.

f) Prove that we have equality in part d for any subset $E \subseteq B$ if and only if f is surjective.

Problem 4: Give examples of functions such that $f \circ g \neq g \circ f$ and $(f \circ g) \circ h \neq f \circ (g \circ h)$.

Problem 5: Let $f : A \to B$ and $g : B \to C$ be functions. Prove the following facts about composition:

a) If f and g are injective, then $g \circ f$ is injective.

b) If $g \circ f$ is injective, then f is injective.

c) If f and g are surjective, then $g \circ f$ is surjective.

d) If $g \circ f$ is surjective, then g is surjective.

e) Give an example where f is not surjective and g is not injective but $g \circ f$ is bijective.

f) Prove that composing from left or right with an bijection doesn't change injectivity or surjectivity.

g) If any two of f, g and $g \circ f$ are bijective, then the third one is also bijective.

Problem 6: Prove that a function $f : A \to B$ is injective if and only if it has a left inverse $g : B \to A$ such that $g \circ f : A \to A$ is the identity map. Then, prove problem 5 part b again using this problem. Moreover, when the set A has at least 2 elements, prove that this left inverse g is unique if and only if f is bijective.