McGill University

MATH 323, Winter 2017

Midterm examination

Date: 16 February, 2017
Time: 13:10–14:20

- Write your name and Student ID in the answer booklet.
- Answer all questions. The problems carry a total of 26 marks. Maximum you can score is 25.
- Explain your argument clearly. Solutions without proper justification may not receive full credit.
- Calculators are not allowed (or necessary for this test). If the final answer contains terms like e^2 or $\log_e 6$, leave it like that.
- Simplify your answer as much as possible. For example, the expression for the final answer should not be an infinite series.

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1. Jerry, Elaine, George, Kramer, and Newman compete against each other in a contest in which the five contestants get ranked one through five, and only the first, second, and third place holders get awarded. Assume all outcomes to be equally likely.

(a) Find the probability that both Jerry and Elaine get awarded.

(b) Conditional on the event that Kramer and Newman did not get any awards, find the probability that George gets first prize.

(3+3)

2. A total of $2n$ people, consisting of n couples, are seated in a straight line at random. Let X_n denote the number of couples that sit together. For example, if we have $n = 3$ couples (B_i, G_i), $1 \le i \le 3$, then $X_n = 2$ for the arrangement $B1 - G2 - B2 - B3 - G3 - G1$.

(a) Find $\mathbb{P}(X_n = n)$.

(b) Find $\mathbb{E}(X_n)$. (Hint: Use linearity of expectation.)

(3+3)

3. When a biased coin is tossed, on an average it takes 100 tosses to get a head for the first time. Toss the coin 300 times independently, and let X denote the number of heads obtained.

(a) Find $\mathbb{E}(X^2)$.

(b) Use Poisson approximation to compute $\mathbb{P}(X \geq 2)$.

(3+3)

4. The number of bacteria colonies of a certain type in samples of polluted water has a Poisson distribution with a mean of 2 per cubic centimeter (cm^3).

(a) If four 1–cm^3 samples are independently selected from this water, find the probability that at least two samples will contain one or more bacteria colonies.

(b) Find the minimum number of 1–cm^3 samples to be selected so that the probability of seeing at least one bacteria colony is greater than 0.999?

(3+3)

5. As part of a study, Jim has been asked to play a game of Go against a computer once a day for a week. He will get \$100 for participating in the study. However, he has to return \$10 for every game he loses.

Let X denote the amount Jim is left with at the end of the week. For example, if he lost all seven games, then $X = 30$. Assume that the outcomes of different games are independent, and Jim wins each game with probability p.

(a) Find $\mathbb{E}(X)$.

(b) Find $\mathrm{V}(X)$.

(1+1)