Math 340 - Mar. 17

Enumeration (Counting)

Let S be a finite set. A collection of subsets of S, $S_1, S_2, \ldots, S_k$, is a partition of S if $S_1 \cup S_2 \cup \ldots \cup S_k = S$ and $S_i \cap S_j = \emptyset \ \forall i \neq j$

$[n] = \{1, 2, \ldots, n\}$

Ex. $\{1,3\}, \{4\}, \{2,5\}$ is a partition of $[5]$.

Sum Rule:

If $S_1, S_2, \ldots, S_k$ is a partition of S then $|S_1| + |S_2| + \ldots + |S_k| = |S|$

Divison Rule:

If $S_1, S_2, \ldots, S_k$ is a partition of S and $|S_i| = L$ for every i then $k = \frac{|S|}{L}$.

partition pieces

Product Rule:

Let S be a set of sequences $(s_1, s_2, \ldots, s_k)$. Suppose that for any choice of $(s_1, s_2, \ldots, s_{i-1})$ of elements of a sequence in S, there are exactly n_i ways to choose the element s_i such that the sequence $(s_1, s_2, \ldots, s_{i-1}, s_i)$ can be extended to a sequence in S then

$$|S| = n_1 n_2 n_3 \ldots n_k$$

Ex. Poker hands.

Four of a kind → four cards of one rank and one card of a different rank

Need to choose:

	# choices
- rank of four cards	13
- rank of remaining card	12
- suit of remaining card	4

Number of possibilities: $13 \cdot 12 \cdot 4 = 624$ hands

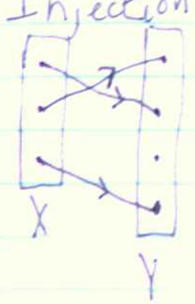
A <u>function</u> $f: X \to Y$ assigns to every $x \in X$ unique $f(x) \in Y$.

f is an <u>injection</u> (or one-to-one) if for every element $y \in Y$ there exists at most one $x \in X$ s.t. $f(x) = y$.

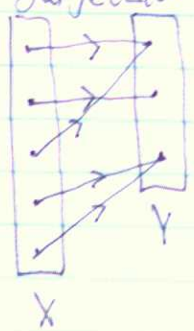
f is a <u>surjection</u> if for every $y \in Y$ there exists at least one $x \in X$ such that $f(x) = Y$.

f is a <u>bijection</u> if f is an injection and a surjection.
ie. $\forall y \in Y$ there exists a unique (exactly one) $x \in X$ s.t. $f(x) = y$.

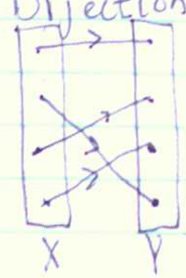
Injection



Surjection



Bijection



- perfect matching

If $f: X \to Y$ is an injection, then $|X| \leq |Y|$
If $f: X \to Y$ is a surjection, then $|X| \geq |Y|$
If $f: X \to Y$ is a bijection, then $|X| = |Y|$

A bijective proof of $|X| = N$ provides a bijection $f: X \to Y$ such that we know that $|Y| = N$.

<u>Number of functions</u>
$f: [n] \to [k]$
$\{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$

Any type of function: k^n number of functions
$(f(1), f(2), \ldots, f(n))$ by the product rule

Number injective functions $f: [k] \to [n]$:

$$\underbrace{n}_{\text{choices for } f(1)} \underbrace{(n-1)}_{\text{choices } f(2)} \cdots \underbrace{(n-k+1)}_{\text{choices for } f(k)} = \frac{n!}{(n-k)!}$$

Number bijective functions $f: [n] \to [n] = n!$
(any injection from $[n]$ to $[n]$ is a bijection)

<u>Number of subsets of $[n]$ of size k?</u> $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$

Proof:
$S =$ injections $[k] \to [n] \leftrightarrow$ ordered sequences of k elements of $[n]$
For every subset of size k of $[n]$ there are $k!$ ordered sequences of elements of this subset.

So S can be partitioned into parts corresponding to different subsets with each part of size $k!$. So by the division rule the number of parts is

$$\frac{n!}{(n-k)!} \Big/ k! = \binom{n}{k}$$

↑ # injections ↑ size of parts

□

Binomial Formula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Reminder of proof:

$$\underbrace{(x+y)(x+y)\dots(x+y)}_{n \text{ terms}}$$

The coefficient of $x^k y^{n-k}$ is equal to the number of ways of choosing k terms in the product out of n from which we select x : $\binom{n}{k}$

□

$x = y = 1$

$$2^n = \sum_{k=0}^{n} \binom{n}{k} = \underbrace{\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}}_{\substack{\downarrow \\ \#\text{ of all subsets of } [n] \\ \text{by the sum rule}}}$$

Bijective proof: that the # of subsets of $[n] = 2^n$.

$F: \{\text{all subsets of } [n]\} \to \{\text{functions } f: [n] \to \{0,1\}\}$

$$F: S \longrightarrow f(x) = \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

Bijective because every set will give a different $f(x)$, and given $f(x)$ we can find S to match it.

We know $|\{\text{functions } f: [n] \to \{0,1\}\}| = 2^n$

$\Rightarrow |\{\text{all subsets of } [n]\}| = 2^n$

□

$x = -1$ $y = 1$ for $n \geq 1$

$$0^n = 0 = \binom{n}{0}(-1)^0 + \binom{n}{1}(-1)^1 + \dots + \binom{n}{n}(-1)^n$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots$$

$$\Rightarrow \underbrace{\binom{n}{0} + \binom{n}{2} + \dots}_{\substack{\text{even \# elements} \\ Y\text{-collection}}} = \underbrace{\binom{n}{1} + \binom{n}{3} + \dots}_{\substack{\#\text{ subsets of } n \text{ with odd \# elements;} \\ \text{let } X \text{ be the collection of these subsets}}}$$

Then $|X| = |Y|$.

Bijective Proof: $F: X \to Y$

$$F(S) = S \,\Delta\, \{1\} = \begin{cases} S \setminus \{1\}, & \text{if } 1 \in S \\ S \cup \{1\}, & \text{if } 1 \notin S \end{cases}$$

F is obviously injective.

Using the same rule, one can define $F^*: Y \to X$ and $F^*(F(S)) = S$ and $F(F^*(S)) = S$

Thus F^* is the inverse of $F \Rightarrow F$ is a bijection.

□

Theorem:

The number of solutions of the equation

$$x_1 + x_2 + \dots + x_k = n$$

where $x_1, x_2, \dots, x_k$ are non-negative integers is $\binom{n+k-1}{k-1}$

Proof:

We will construct a bijection

$\{\text{solutions}\} \to \{k-1 \text{ element subsets } [n-k+\square]\}$

$\| $ sequences of n dots and $k-1$ bars (separators)

map can be inverted

separator

$(k-1)^{st}$ separator

$$(x_1, x_2, \dots, x_k) \to \underbrace{\cdot\,\cdot}_{\substack{x_1 \\ \text{dots}}} \,|\, \underbrace{\cdot\,\cdot\,\cdot}_{\substack{x_2 \\ \text{dots}}} \,|\, \dots \,|\, \underbrace{\cdot\,\cdot\,\cdot}_{x_k \text{ dots}}$$

total # dots $= n$

Since it can be inverted $\Rightarrow$ bijection

□

Theorem:

There are n^{n-2} trees with the vertex set $[n]$.
(There are n^{n-2} labelled trees on n vertices

Examples:

$n=1$

$1^{-1}=1$

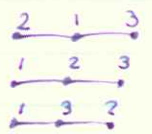
$n=2$

$2^0=1$

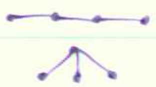
$n=3$

3 choices for label of central vertex

$3^1=3$



$n=4$



ways of labelling a star $=4$

4 choices of labelling the middle vertex

$4^2=16$

6 choices for middle vertices × 2 choices to join end vertices $=12$

There are $4!=24$ ways to label ordered paths
Every path is counted twice so $24/2=12$.

Proof:

We will show that there are n^n trees with vertex $[n]$ and a red vertex and a blue vertex chosen in the tree (possibly the same).

There are $n \cdot n = n^2$ ways of choosing these vertices in every tree so there will be $n^n/n^2 = n^{n-2}$ trees with no special vertices.

We will construct a bijection:

$F: \{f:[n]\to[n]\} \to \{\text{the set of such trees}\}$

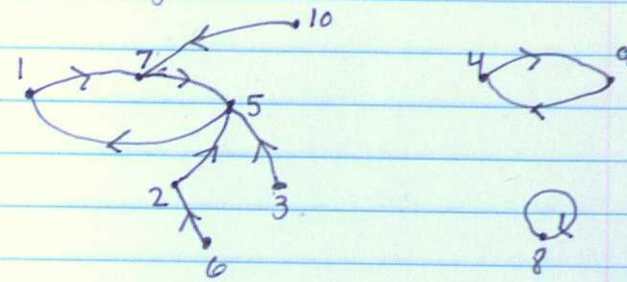
set of all functions
n^n functions

$n=10$

$f:[10]\to[10]$

x	$f(x)$
1	7
2	5
3	5
4	9
5	1
6	2
7	5
8	8
9	4
10	7

graph G with directions from f by joining x to $f(x)$ by a directed edge



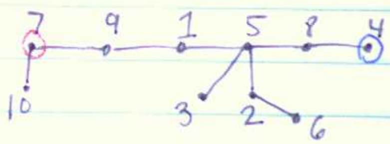
3 cycles.
Let C be the set of vertices in cycles of G: $C=\{1,4,5,7,8,9\}$

Construct a tree:
List C in order

x	1	4	5	7	8	9
$f(x)$	7	9	1	5	8	4

← reordering of C

P:



Let P be a path of values of C in order.

The remaining vertices are joined to this path by edges as in G.

Continued next class.