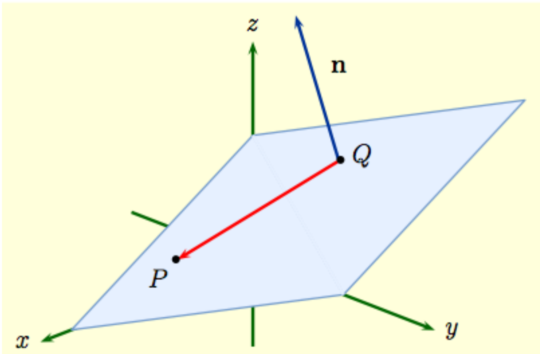
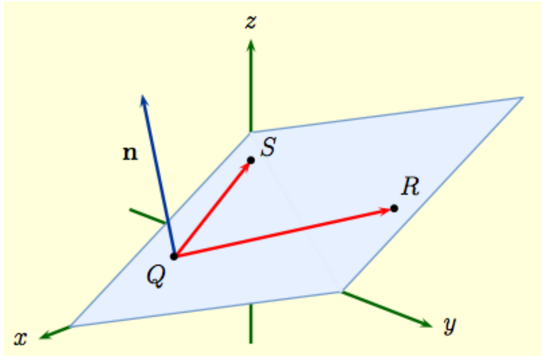
Equations of planes

Planes: To describe a line, we needed a point $\mathbf{b}$ and a vector $\mathbf{v}$ along the line. We could also start with two points $\mathbf{b}$ and $\mathbf{a}$ and take $\mathbf{v} = \mathbf{a} - \mathbf{b}$.

To describe a plane, we need a point Q and a vector $\mathbf{n}$ that is **perpendicular** to the plane. Later on, we'll see how to get $\mathbf{n}$ from other kinds of data, like the location of three points in the plane.



point-normal



three points

Let $Q(a, b, c)$ be a fixed point in the plane, $P(x, y, z)$ an arbitrary point in the plane, and $\mathbf{n} = \langle A, B, C \rangle$ the normal to the plane. If

$$\mathbf{b} = \langle a, b, c \rangle, \qquad \mathbf{r} = \langle x, y, z \rangle,$$

the vector

$$\overrightarrow{QP} = \mathbf{r} - \mathbf{b} = \langle x-a, y-b, z-c \rangle$$

lies in the plane, and is **perpendicular** to $\mathbf{n}$.

Thus $\mathbf{n} \cdot (\mathbf{r} - \mathbf{b}) = 0$. In terms of coordinates, this becomes

$$\langle A, B, C \rangle \cdot \langle x-a, y-b, z-c \rangle = 0,$$

where $\mathbf{n} = \langle A, B, C \rangle$. In other words, we get the **point-normal** equation

$$A(x-a) + B(y-b) + C(z-c) = 0.$$

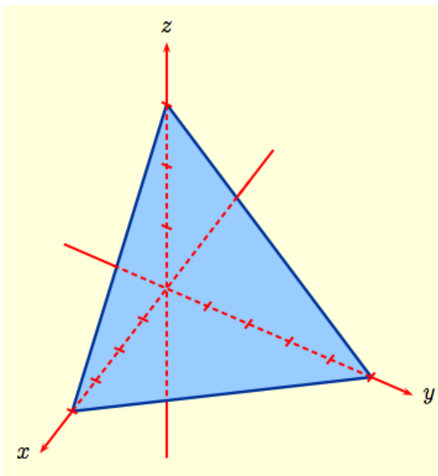
for a plane.

To emphasize the normal in describing planes, we often ignore the special fixed point $Q(a, b, c)$ and simply write

$$Ax + By + Cz = D$$

for the equation of a plane having normal $\mathbf{n} = \langle A, B, C \rangle$. Here $D = \mathbf{n} \cdot \mathbf{b} = Aa + Bb + Cc$. The next three examples show useful this way of writing planes can be.

Example 1: *Find an equation for the plane whose graph in the first octant is*



Solution: The basic idea is to look at the points of intersection of the plane and the coordinate axes: now

$$Ax + By + Cz = D$$

intersects the x-axis when $y = z = 0$, *i.e.*, when $x = D/A$. Similarly, it intersects the y-axis when $y = D/B$, and the z-axis when $z = D/C$.

Thus from the given graph

$$\frac{d}{a} = 4, \qquad \frac{d}{b} = 5, \qquad \frac{d}{c} = 3.$$

Consequently,

$$\frac{x}{4} + \frac{y}{5} + \frac{z}{3} = 1$$

is an equation for the plane.

Example 2: *Find the vector equation of the line passing through the point $P(2, -4, 3)$ and perpendicular to the plane*

$$x + 4y - 2z = 5.$$

$\mathbf{n} = \langle 1, 4, -2 \rangle$. Thus the line has

$$\mathbf{v} = \langle 1, 4, -2 \rangle$$

Chapter 15: Multiple Integrals

Solution: when the line is perpendicular to the plane, then the direction vector of the line is parallel to the normal to the plane. When the plane is

$$x + 4y - 2z \;=\; 5$$

this means the plane has normal

as direction vector for. But $P(2,\,-4,\,3)$ lies on the line, so the vector

$$\mathbf{b} \;=\; \langle\, 2,\,-4,\,3\,\rangle$$

determines a point on the line.

Consequently, in vector form the equation of the line is

$$\mathbf{r}(t) = t\mathbf{v} + \mathbf{b} = t\langle\, 1,\,4,\,-2\,\rangle + \langle\, 2,\,-4,\,3\,\rangle\,.$$

Example 3: *Find an equation for the plane passing through the point $Q(1,\,1,\,1)$ and parallel to the plane*

$$2x + 3y + z \;=\; 5\,.$$

Solution: parallel planes have the same normal. So any plane parallel to $2x + 3y + z \;=\; 5$ has normal

$$\mathbf{n} \;=\; \langle\, 2,\,3,\,1\,\rangle\,.$$

On the other hand, since $Q(1,\,1,\,1)$ lies on the parallel plane, the vector

$$\mathbf{b} \;=\; \langle\, 1,1,1\,\rangle$$

determines a point on the parallel plane.

Now let $\mathbf{r} \;=\; \langle\, x,\,y,\,z\,\rangle$ be an arbitrary point on the parallel plane. Then the vector

$$\overrightarrow{QP} \;=\; \mathbf{r} - \mathbf{b} \;=\; \langle\, x-1,\,y-1,\,z-1\,\rangle$$

lies in the plane and so will be perpendicular to $\mathbf{n}$. In this case,

$$\begin{aligned}\mathbf{n}\cdot(\mathbf{r}-\mathbf{b}) \;&=\; \langle\, 2,\,3,\,1\,\rangle\cdot\langle\, x-1,\,y-1,\,z-1\,\rangle \\ &=\; 2(x-1) + 3(y-1) + (z-1) \;=\; 0\,.\end{aligned}$$

Consequently, an equation for the plane is

$$2x + 3y + z \;=\; 6\,.$$