Flows in Networks: Network Flows

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Provide the definitions of a network and a flow.
- Give some examples of real world situations in which network flow problems might arise.

Last Time

- Last time: Discussed disaster management problem.
- Today: Talk about formalization of this and similar problems.

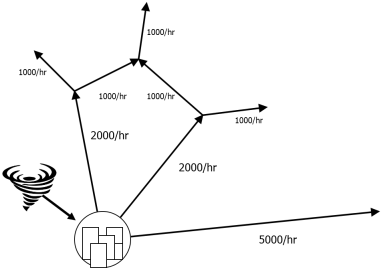
Network

Definition

A network is a directed graph G with:

- Each edge, e, is assigned a positive real capacity, C_e.
- One (or more) vertex is labelled a source.
- One (or more) vertex is labelled a sink.

Example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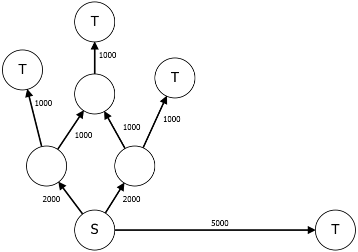
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Flows

- Next we want to be able to talk about flows (traffic) through a network.
- Rather than talking about where each car goes, we will instead concern ourselves with the total flow, f_e, through each edge e.

Flows

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- Rather than talking about where each car goes, we will instead concern ourselves with the total flow, f_e, through each edge e.
- This must satisfy two conditions:

Rate Limitation

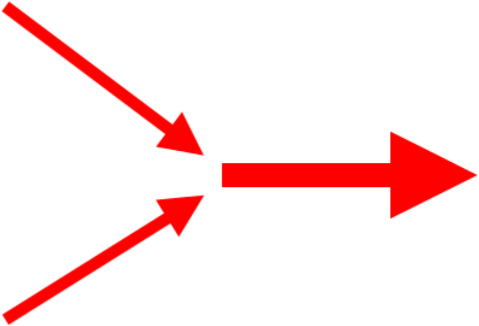
For each edge e,

$$0 \leq f_e \leq C_e.$$

Conservation of Flow

For all v not a source or sink:

$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$



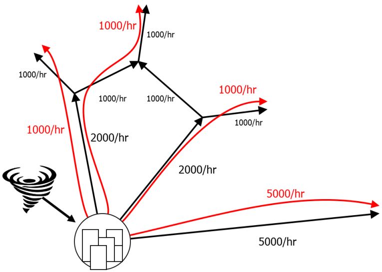
Formal Definition

Definition

A flow in a network is an assignment of a real number flow, f_e to each edge e so that

- For all e
$$0 \leq f_e \leq C_e.$$
- For all v not a source or sink
$$\sum_{e \text{ into } v} f_e = \sum_{e \text{ out of } v} f_e.$$

Example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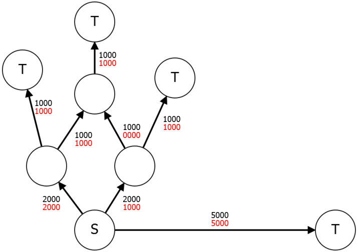
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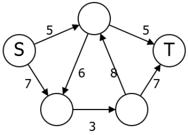
Problem

Which of the following is a valid flow for the given network?



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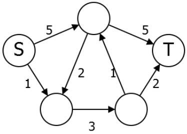

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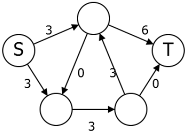
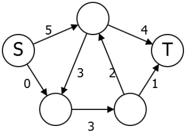
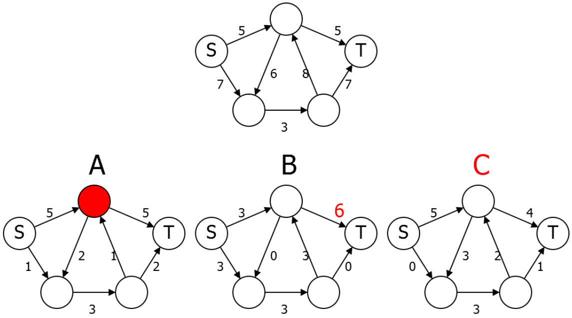


C



Solution

Only flow C is valid. A fails to conserve flow at one vertex, and B exceeds.



Examples of Network Flows

Network flows are useful to study since they can model a number of real-life phenomena.

Flows of Goods on a Transportation Network

Flows of Electricity Through the Power Grid

Flows of Water Through Pipes

Flows of Information Through a Communications Network

Flow Size

One thing to know about a flow is how much stuff is actually flowing.

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Definition

For a flow, f, the size of the flow is given by

$$|f| := \sum_{e \text{ out of a source}} f_e - \sum_{e \text{ into a source}} f_e.$$

Sinks

You can also compute this by looking at sinks.

Lemma

$$|f| = \sum_{e \text{ into a sink}} f_e - \sum_{e \text{ out of a sink}} f_e.$$

Proof

$$\begin{aligned}
0 &= \sum_{e} f_e - \sum_{e} f_e \\
&= \sum_{v} \left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \\
&= \sum_{v \text{ source or sink}} \left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right)
\end{aligned}$$

Proof

$$
\begin{aligned}
&= \sum_{v \text{ source or sink}} \left(\sum_{e \text{ into } v} f_e - \sum_{e \text{ out of } v} f_e \right) \\
&= \sum_{e \text{ into a source}} f_e + \sum_{e \text{ into a sink}} f_e \\
&\quad - \sum_{e \text{ out of a source}} f_e - \sum_{e \text{ out of a sink}} f_e \\
&= -|f| + \left(\sum_{e \text{ into a sink}} f_e - \sum_{e \text{ out of a sink}} f_e \right).
\end{aligned}
$$

Problem

How large a flow can we fit through a network?

Maxflow

Input: A network G

Output: A flow f for G with $|f|$ as large as possible.