Linear Programming: (Optional) The Ellipsoid Algorithm

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Advanced Algorithms and Complexity
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Learning Objectives

- Understand the basic ideas behind the ellipsoid algorithm.
- Explain the practical differences between the ellipsoid and simplex algorithms.

Last Time

Simplex algorithm.

- Solves LPs.
- Works well most of the time.
- Exponential in some cases.

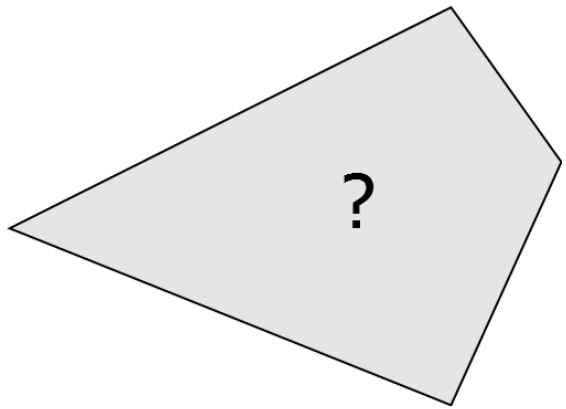
Today

Ellipsoid algorithm.

- Solves LPs.
- Polynomial time in all cases.
- Often not as good in practice.

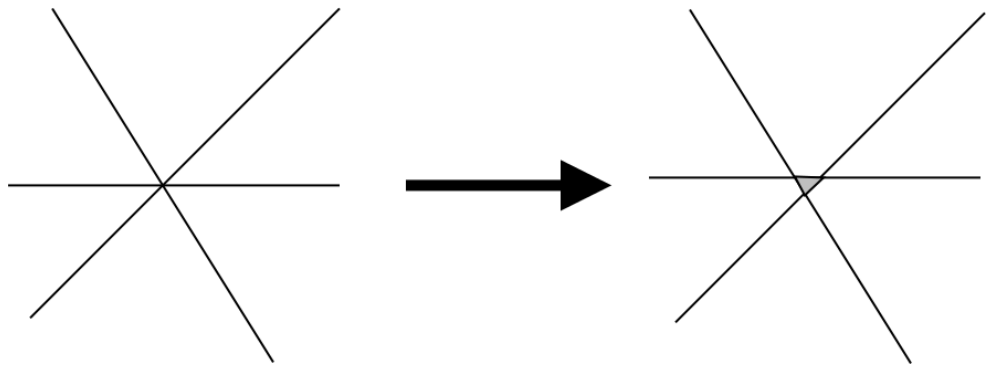
Formulation

Ellipsoid solves the satisfiability version of a Linear Program.



Step I

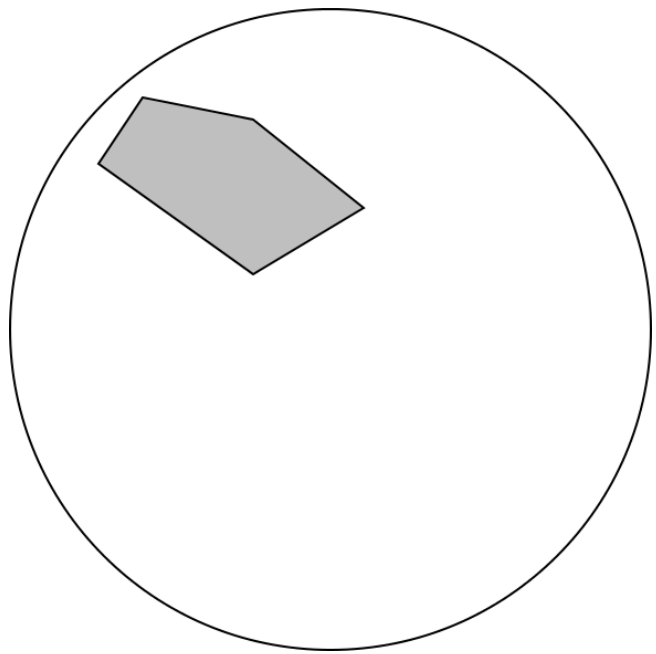
Relax all equations a tiny bit.



Solution set (if exists) has positive volume.

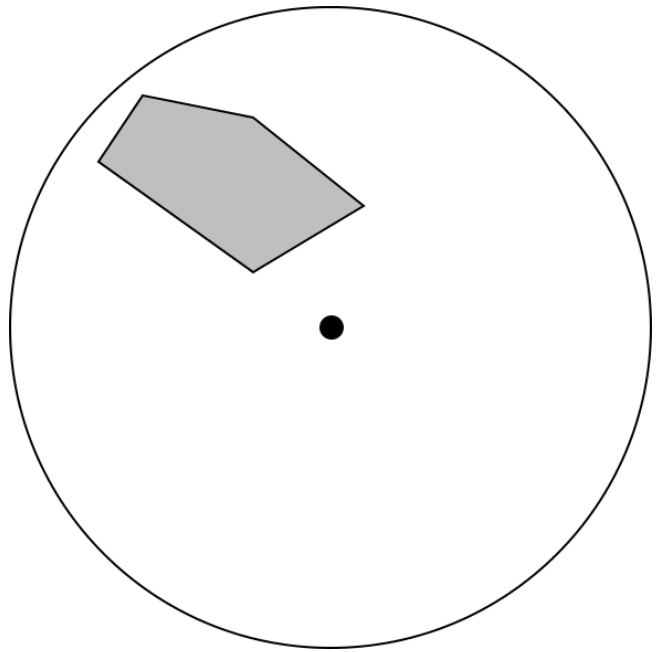
Step II

Bound solution set in large ball.



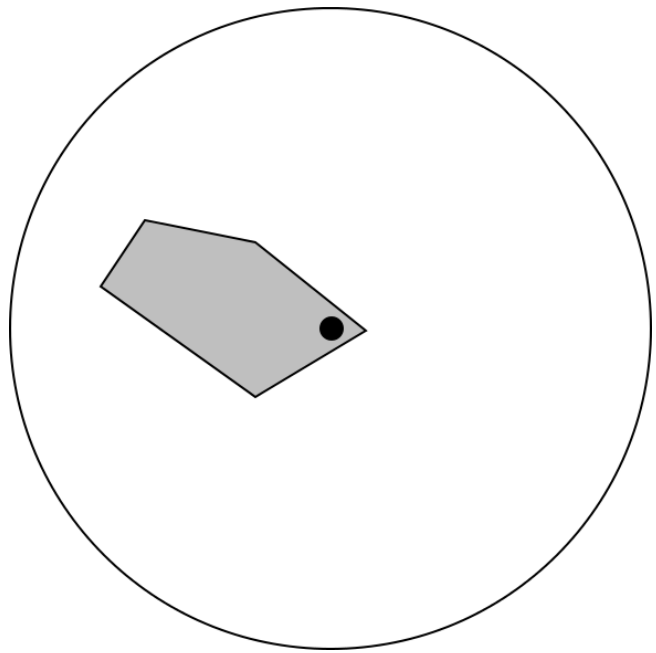
Step III

Have ellipsoid that contains all solutions. See if center of ellipsoid is a solution.



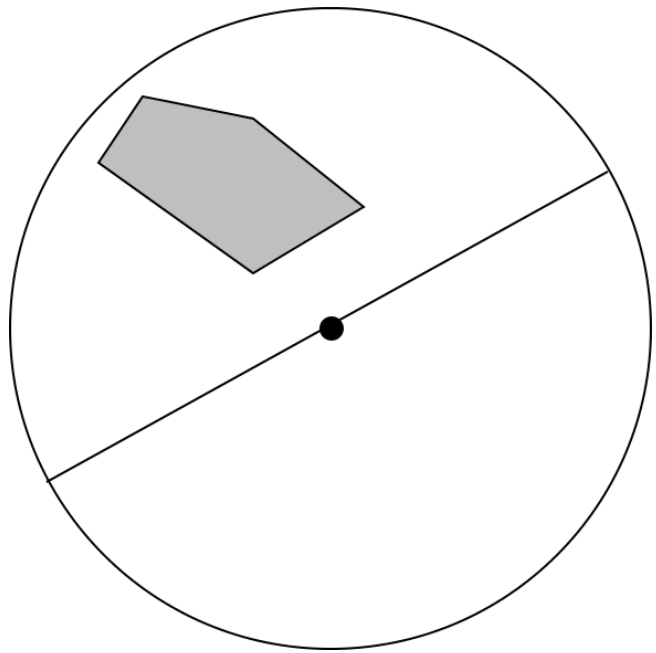
Step IIIa

If it is a solution, system is satisfiable.



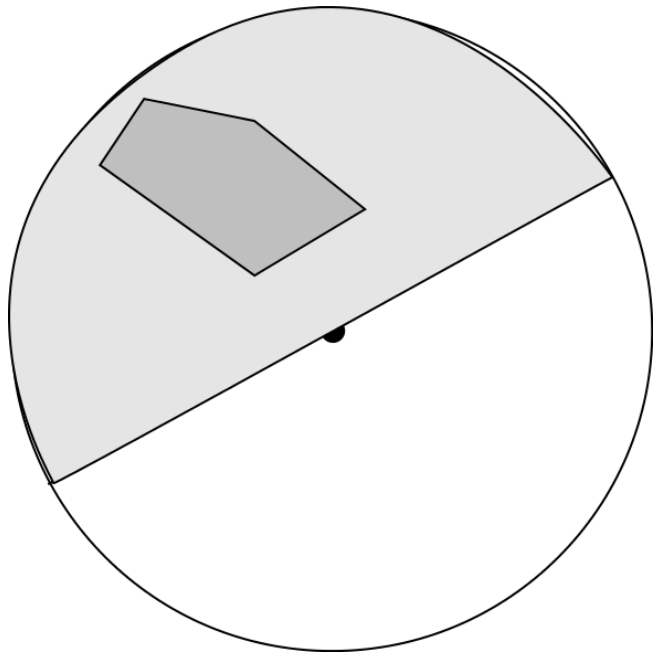
Step IIIb

If not solution, find separating hyperplane.



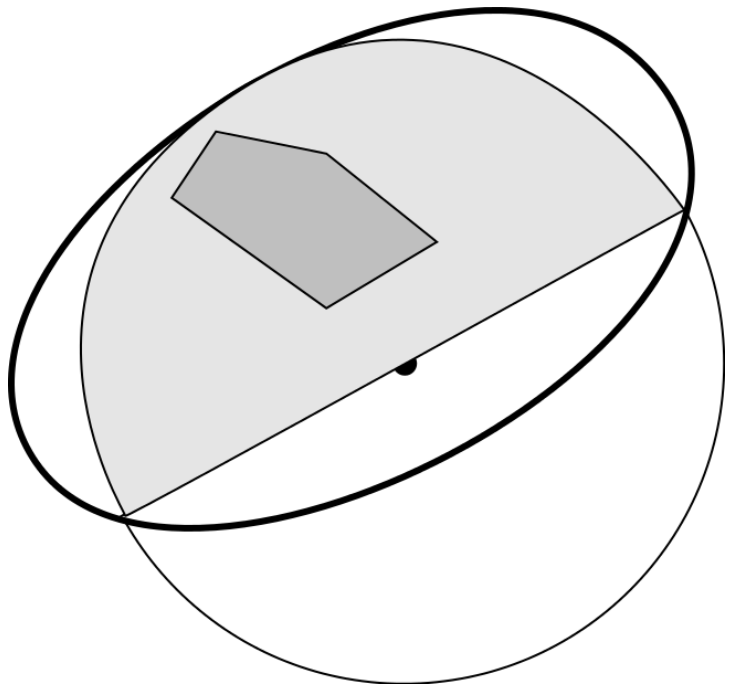
Step IIIb

Solutions now contained in half-ellipsoid.



Step IIIb

Half-ellipsoid contained in new ellipsoid of smaller volume.



Step IV

Repeat until ellipsoid is too small to contain solution set. In this case, there must be no solutions.

Runtime

Ellipsoid runs in polynomial time

$$O((m + n^2)n^5 \log(nU))$$

where

n = Dimension

m = Number of Equations

U = Numerical size of coefficients

Runtime

Runtime is:

- Polynomial!

Runtime

Runtime is:

- Polynomial!
- But a bad polynomial.
- That also depends (logarithmically) on the coefficient size.

Oracles

Also, note that the Ellipsoid algorithm doesn't really need the equations. It just needs a separation oracle. That is an algorithm that given an $x \notin \mathcal{C}$ gives a hyperplane that separates x from $\mathcal{C}$. This lets you employ Ellipsoid in contexts when you cannot necessarily enumerate the constraints.

Summary

The ellipsoid algorithm

- Is another way to solve LPs.
- Has better worst-case performance than simplex.
- However, is usually slower.
- Can run with only a separation oracle.