Flows in Networks: The Ford-Fulkerson Algorithm

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Compute maximum flows in networks.

Algorithm

Idea:

- Start with zero flow.
- Repeatedly add flow.
- Stop when you cannot add more.

Adding Flow

- Have flow f.
- Compute residual G_f.
- Any new flow $f + g$, where g a flow for G_f.
- Need to find flow for G_f.
- See if there's a source-sink path.

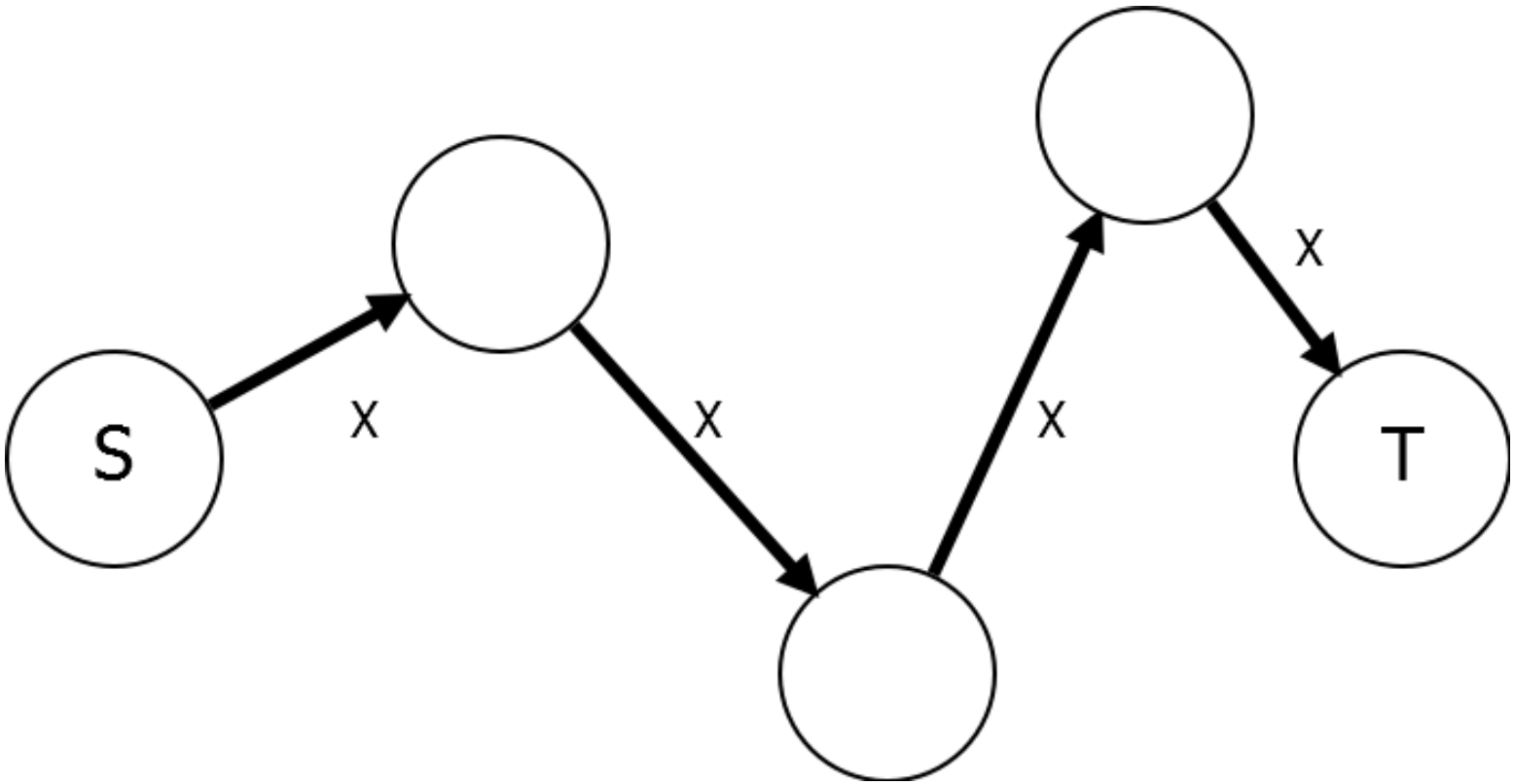
No Path

If there's no source-sink path in G_f:

- Reachable vertices define cut of size 0.
- No g of positive size.
- $|f + g| \leq |f|$.
- f is a maxflow.

Path

If there is a path, add flow along path.



Need $X \leq \min_{e \in \text{path}} C_e$.

Adding Flow

- Find flow g for G_f with $|g| > 0$.
- Replace f by $f + g$.
- $|f + g| > |f|$.

Pseudocode

Ford-Fulkerson(G)

$f \leftarrow 0$

repeat:

  Compute G_f

  Find $s - t$ path P in G_f

  if no path: return f

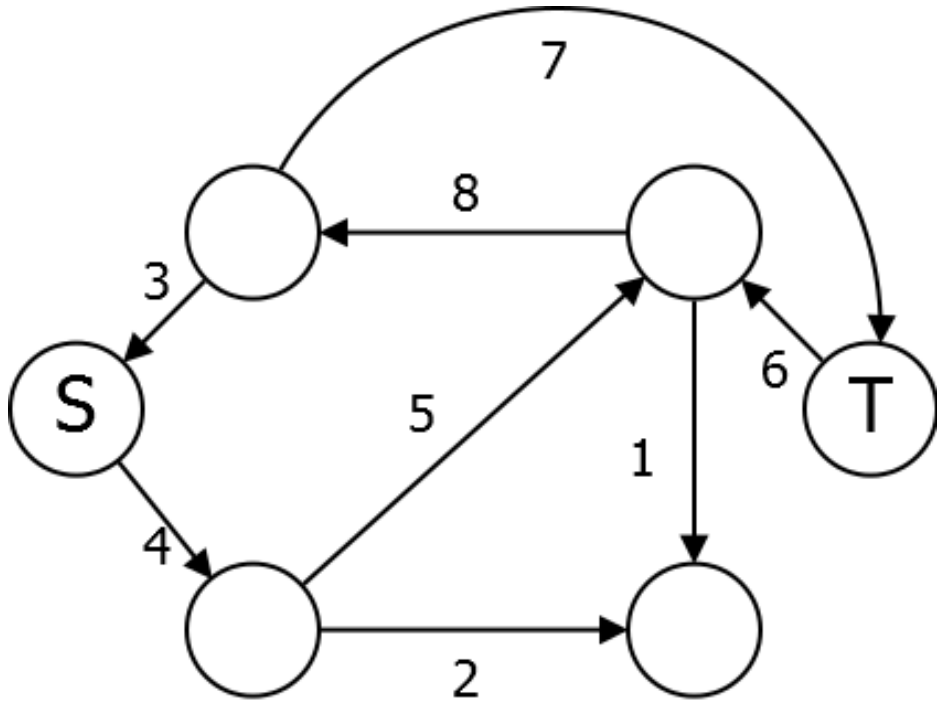
  $X \leftarrow \min_{e \in P} C_e$

  g flow with $g_e = X$ for $e \in P$

  $f \leftarrow f + g$

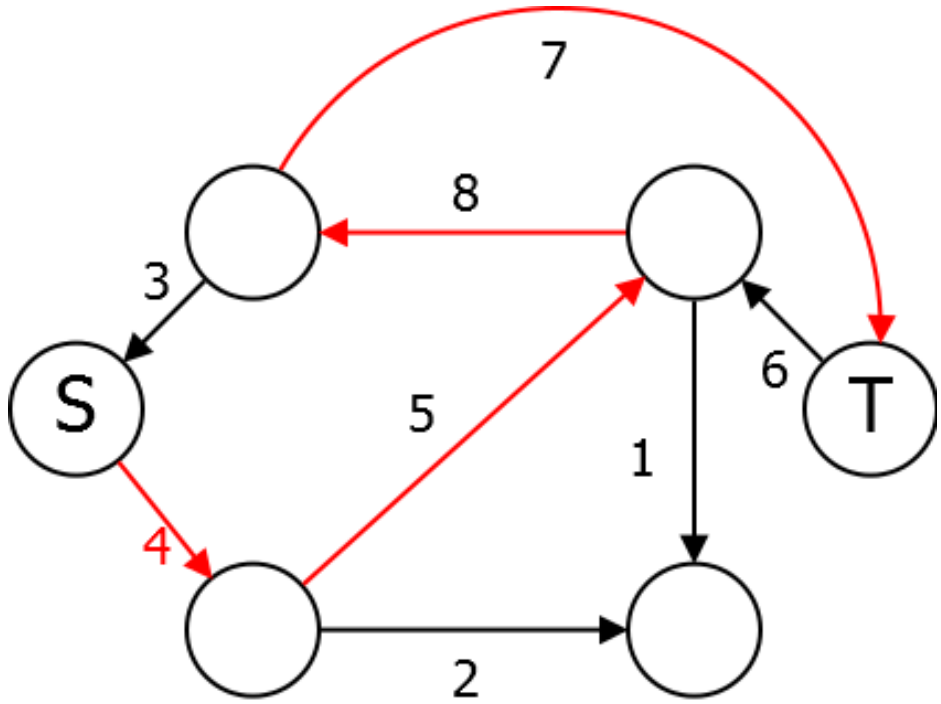
Problem

How much flow is added in one step?



Solution

Bounded by 4.



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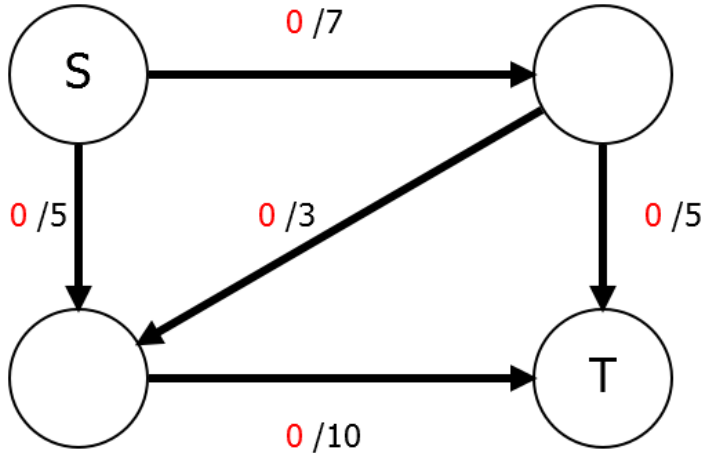

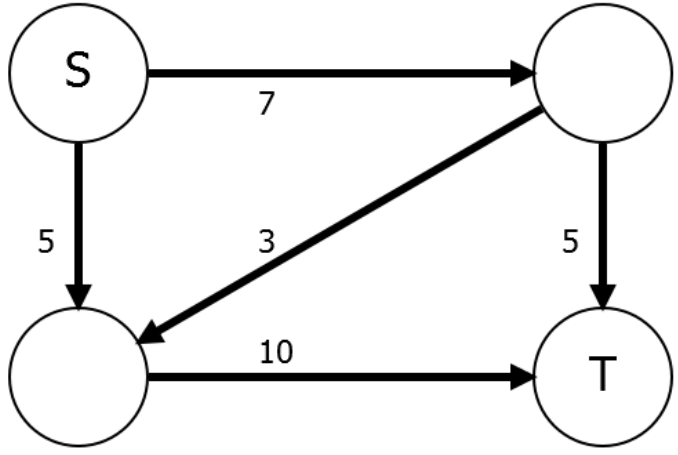

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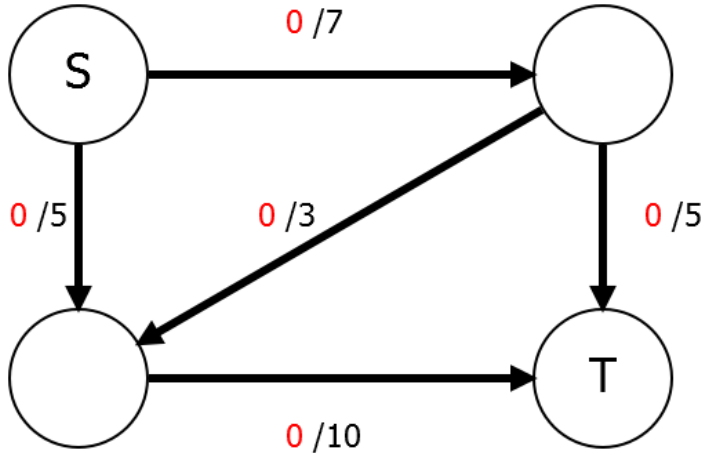

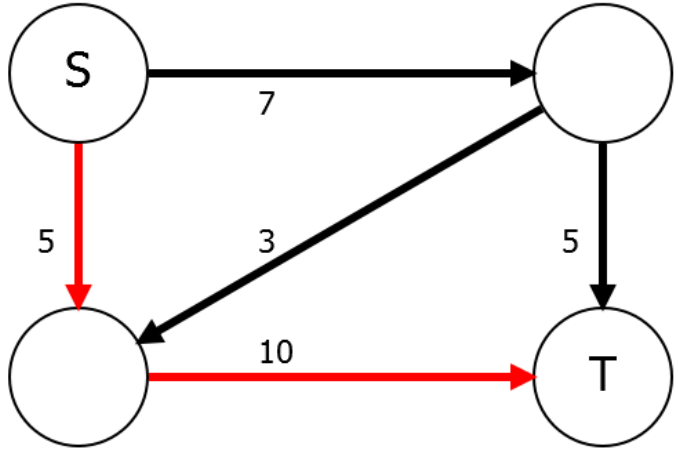

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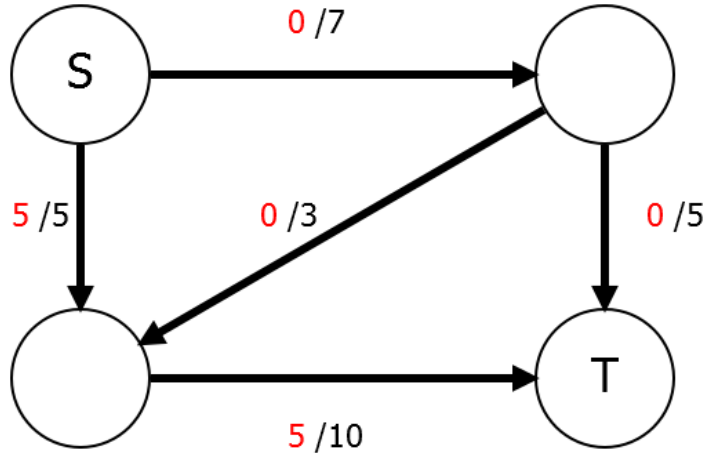

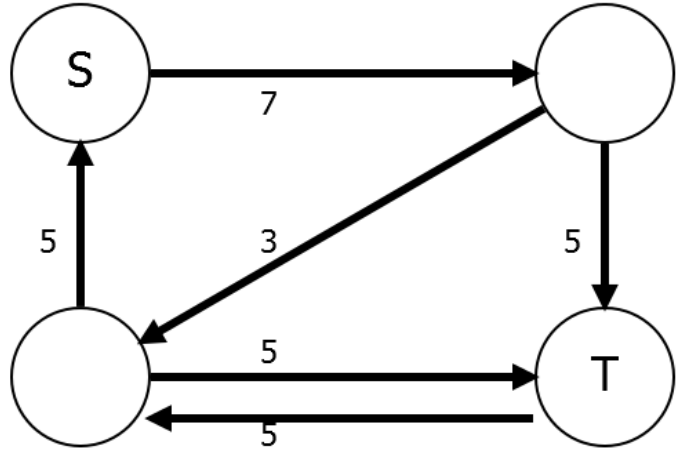

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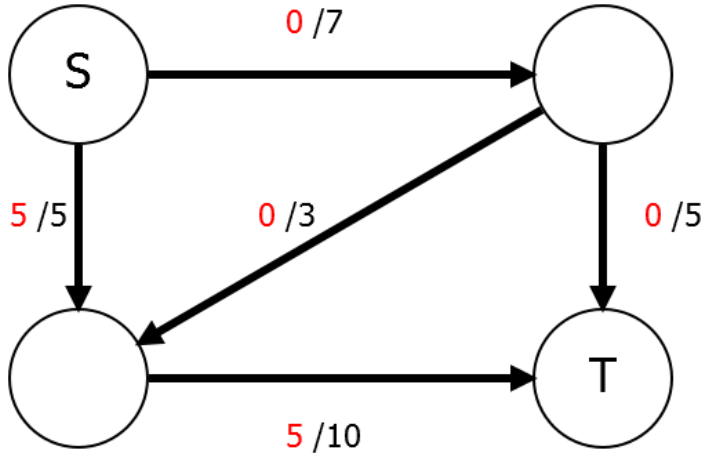

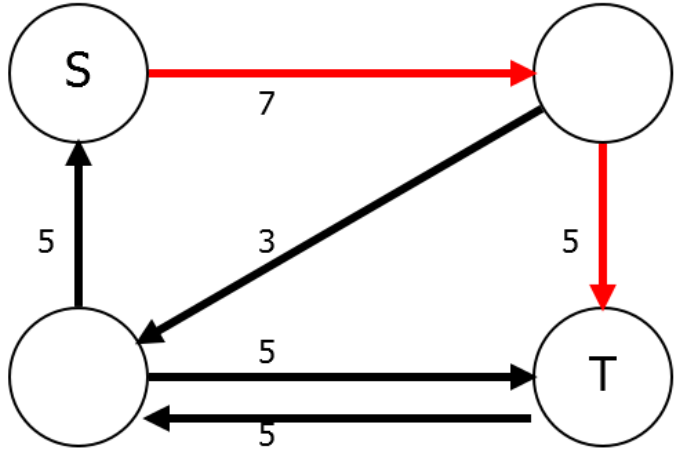

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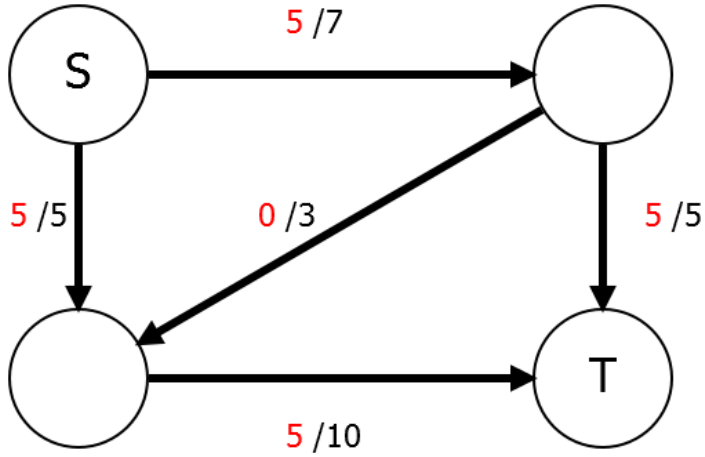

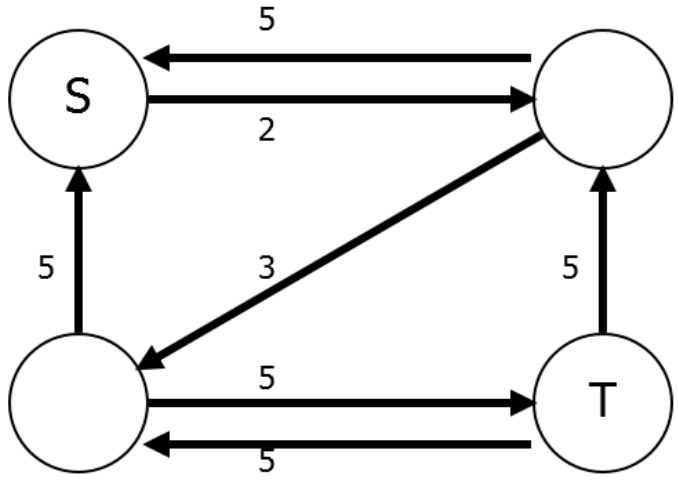

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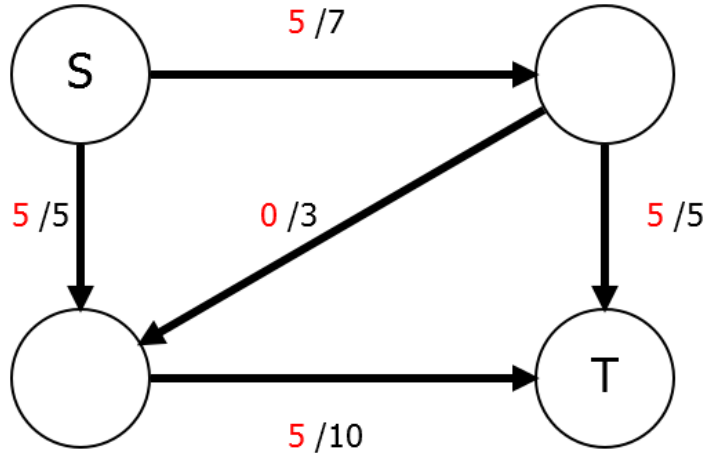

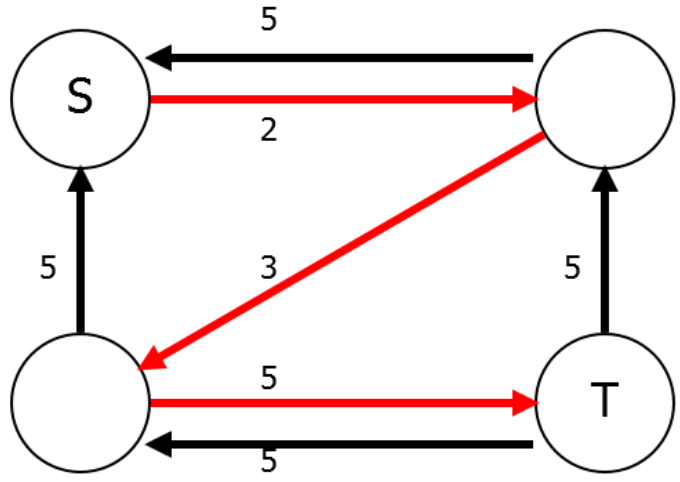

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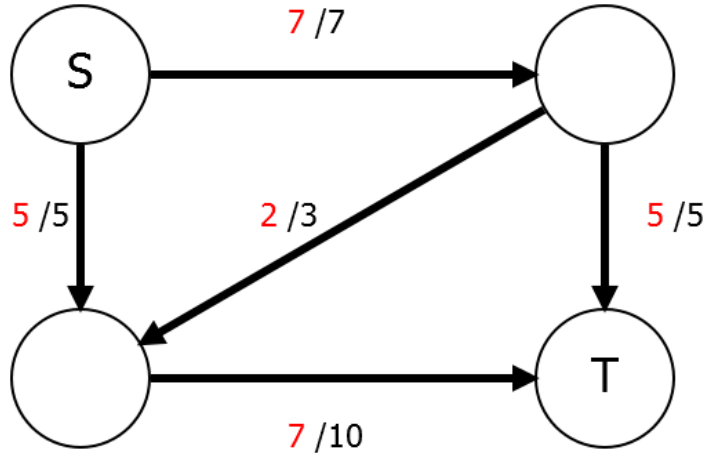

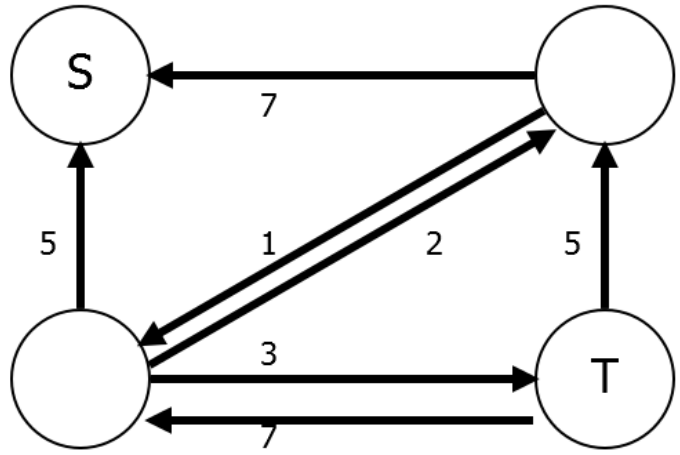

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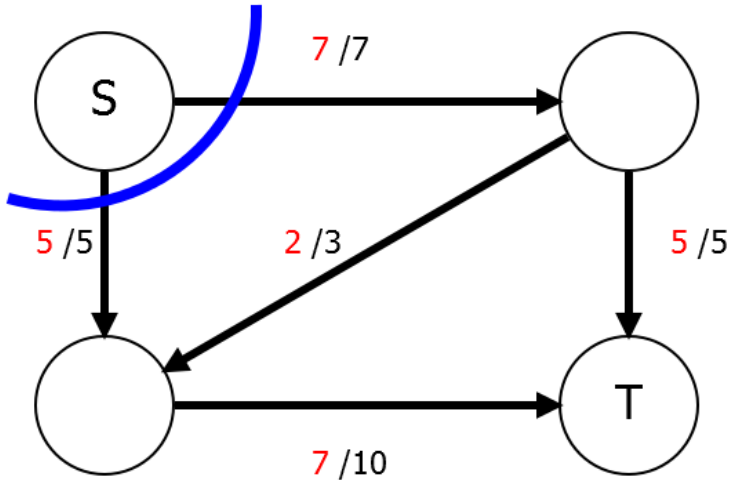

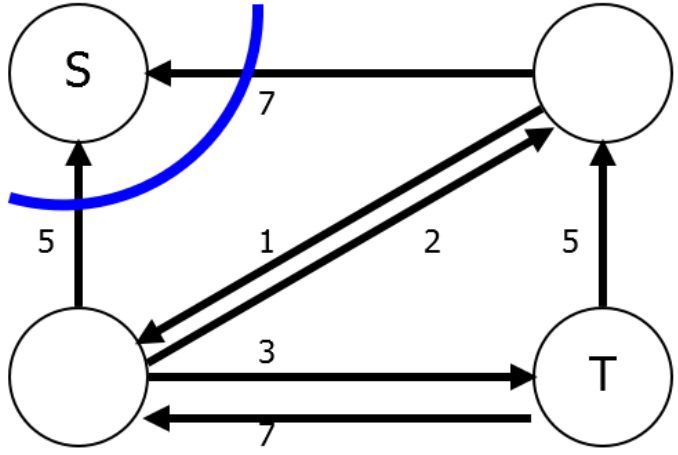

Integrality

Note that if all capacities are integers, all flows we produce are integral.

Lemma

If the network G has integral capacities, there is a maximum flow with integral flow rates.

Analysis

[Assume integral capacities]

- Can compute G_f and find P in $O(|E|)$ time.
- Each time, increase total flow by at least 1.
- Total runtime: $O(|E||f|)$.

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Note: Potentially quite large if flow is numerically large.

Non-Determinacy

Note that the algorithm says to find an $s - t$ path in G_f. There might be many valid paths to choose from. Using DFS is fast, but perhaps not the best. As we will see the way we pick our path will affect the runtime of the algorithm.