Linear Programming: (Optional) Duality Proofs

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Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Prove the duality and complementary slackness theorems.

Last Time

To each linear program, associate a dual program. Find non-negative combination of constraints to put bound on objective.

Duality

Theorem

A linear program and its dual always have the same (numerical) answer.

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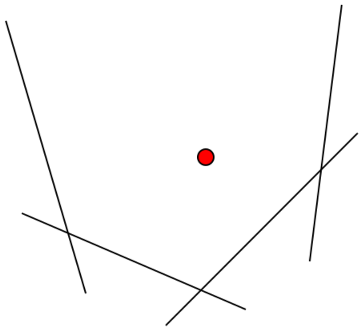
Today we prove it.

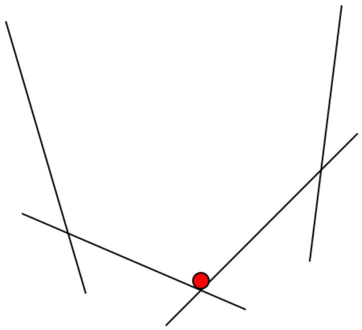
Proof of Duality (intuition)

Ball in a well.



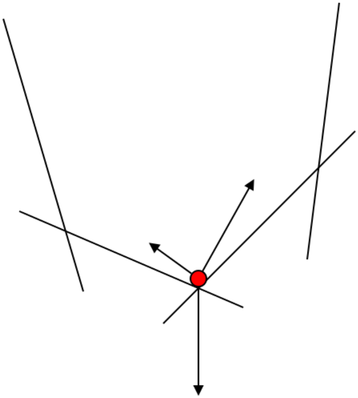
Proof of Duality (intuition)

Gravity pulls to lowest point.



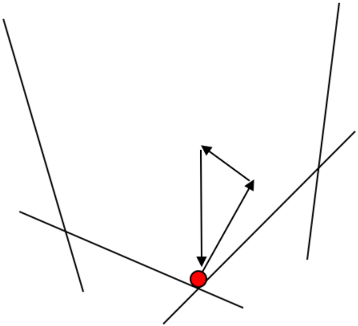
Proof of Duality (intuition)

Force of gravity cancels normal forces.



Proof of Duality (intuition)

Linear combination of normal vectors equals downward vector.



Duality Proof

- Consider solvability of system $Ax \geq b, x \cdot v \leq t$.
- Claim solution unless combination of equations yields $0 \geq 1$.
- Equivalent to original problem.

Duality Proof

Consider set $\mathcal{C}$ of combinations

$$c_1 E_1 + c_2 E_2 + \ldots + c_m E_m$$

with $c_i \geq 0$.

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If equation $0 \geq 1$ not in $\mathcal{C}$, there is a separating hyperplane.
This hyperplane correspond to a solution to the original system!

Example

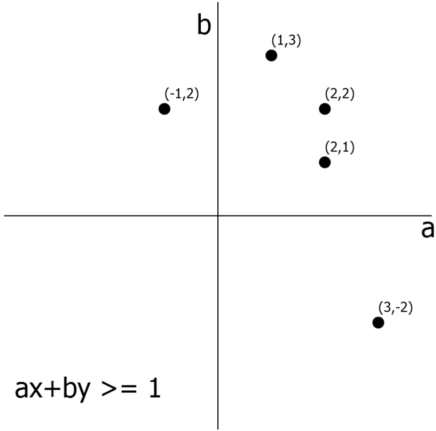
Consider system:

$$\begin{aligned} 3x - 2y &\geq 1 \\ -x + 2y &\geq 1 \\ 2x + 1y &\geq 1 \\ x + 3y &\geq 1 \\ 2x + 2y &\geq 1. \end{aligned}$$

Is there a solution?

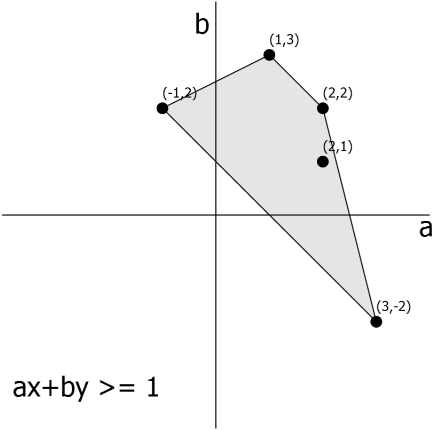
Example

Plot equations.



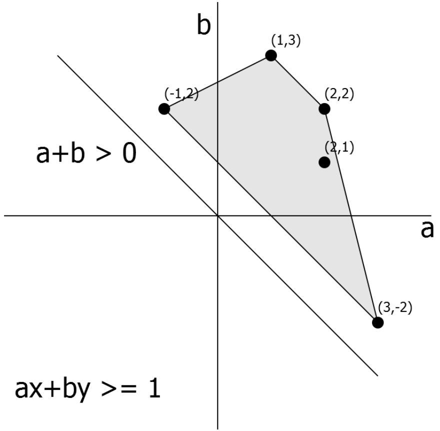
Example

Consider linear combinations.



Example

Find separator.



Example

- All equations of form $ax + by \geq 1$ with $a + b > 0$.
- If $x = y = \mathrm{Big}$, $ax + by = (a + b)\mathrm{Big} > 1$.
- Have solution $x = y = 1$!

Complementary Slackness

Theorem

Consider a primal LP:
Minimize $v \cdot x$ subject to $Ax \geq b$,
and its dual LP:
Maximize $y \cdot b$ subject to $y^T A = v$, $y \geq 0$.
Then in the solutions, $y_i > 0$ only if the i^{th} equation in x is tight.

Proof

- Solution x to primal matching solution y to the dual.
- $x \cdot v = t$.
- Combination of equations $\sum y_i E_i$ yields $x \cdot v \geq t$.
- Since final equation is tight, cannot use non-tight equations in sum.
- For each i, either E_i is tight or $y_i = 0$.