Linear Programming: Convex Polytopes

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Advanced Algorithms and Complexity
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Learning Objectives

- Understand what a convex polytope is and why it is relevant to linear programming.
- Get a feel for what a convex polytope looks like.
- Prove some basic facts about convex polytopes.

Linear Programs

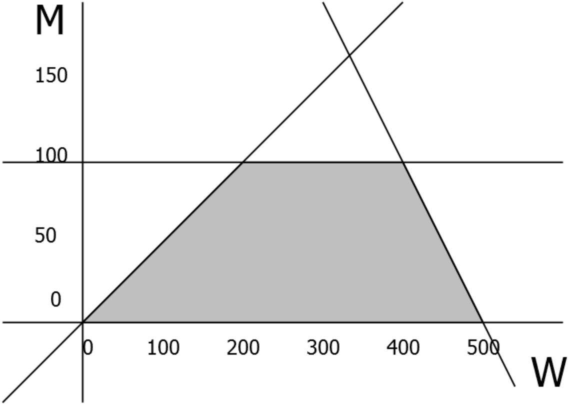
Optimize linear function given linear inequality constraints.

Linear Programs

Optimize linear function given linear inequality constraints.
Want to understand region of points defined by inequalities.

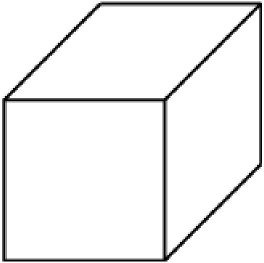
Example

From factory example



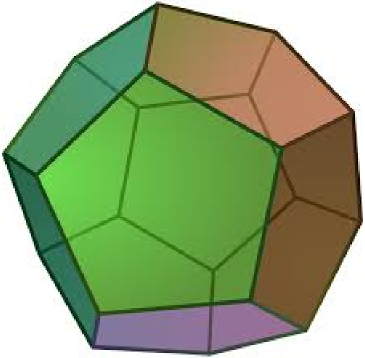
Example II

Equations: $0 \leq x, y, z \leq 1$ give cube.



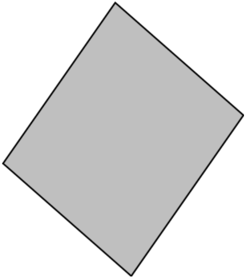
In General

Get what's called a convex polytope



Hyperplanes

A single linear equation defines a hyperplane.



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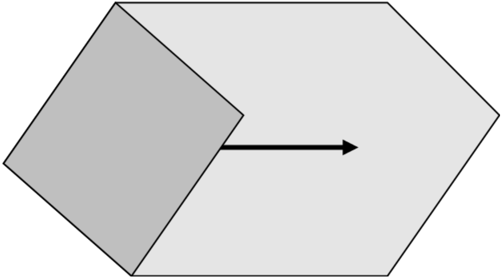

An inequality, defines a halfspace.

Polytopes

So a system of linear inequalities, defines a region bounded by a bunch of hyperplanes.

Definition

A polytope is a region in $\mathbb{R}^n$ bounded by finitely many flat surfaces.

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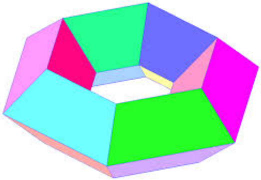
Definition

A polytope is a region in $\mathbb{R}^n$ bounded by finitely many flat surfaces. These surfaces may intersect in lower dimensional facets (like edges), with zero-dimensional facets called vertices.

More Conditions

But not every polytope is possible.

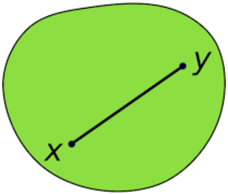


Everything must be on one side of each face.

Convexity

Definition

A region $\mathcal{C} \subset \mathbb{R}^n$ is convex, if for any $x, y \in \mathcal{C}$, the line segment connecting x and y is contained in $\mathcal{C}$.



Convexity

Lemma

An intersection of halfspaces is convex.

Proof

- Defined by $Ax \geq b$.
- Need for $x, y \in \mathcal{C}$ and $t \in [0, 1]$, $tx + (1-t)y \in \mathcal{C}$.

$$\begin{aligned} A(tx + (1-t)y) &= tAx + (1-t)Ay \\ &\geq tb + (1-t)b \\ &= b. \end{aligned}$$

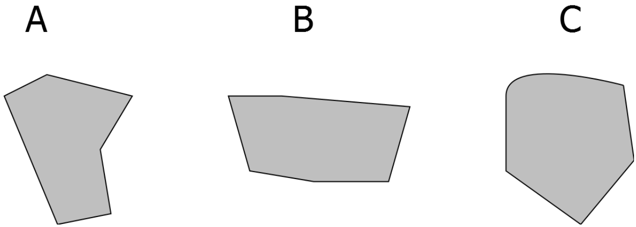
Convex Polytope

Theorem

The region defined by a system of linear inequalities is always a convex polytope.

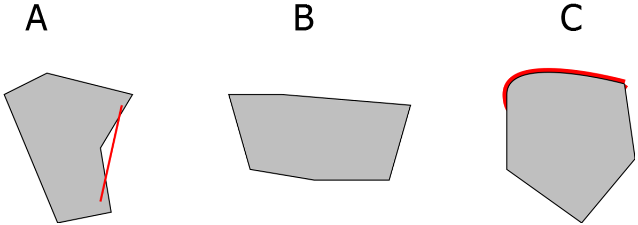
Problem

Which of these figures is a convex polytope?



Solution

Only B.



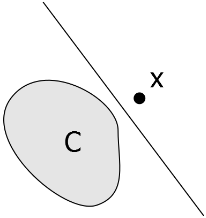
Lemmas

We will conclude with a couple important lemmas about convex polytopes.

Separation

Lemma

Let $\mathcal{C}$ be a convex region and $x \notin \mathcal{C}$ a point. Then there is a hyperplane H separating x from $\mathcal{C}$.



Separation

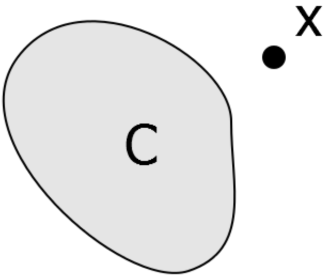
Lemma

Let $\mathcal{C}$ be a convex region and $x \notin \mathcal{C}$ a point. Then there is a hyperplane H separating x from $\mathcal{C}$.

Note that if $\mathcal{C}$ is given by a system of linear inequalities, we can just find one of the defining inequalities that x violates.

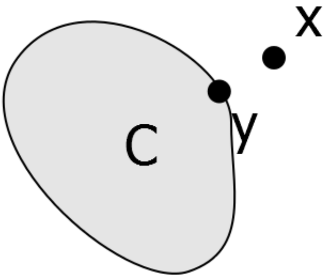
Proof (Optional)

Start with x.



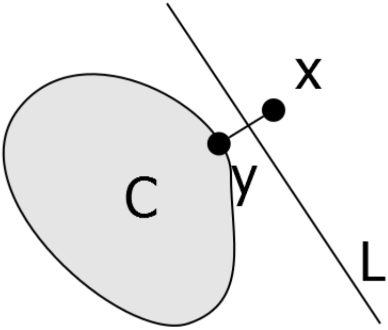
Proof (Optional)

Let y be closest point in $\mathcal{C}$



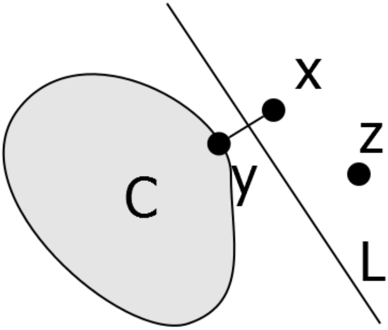
Proof (Optional)

Let L be the perpendicular bisector of xy.



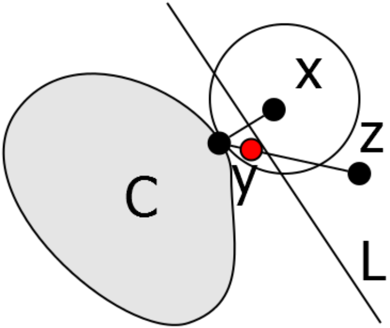
Proof (Optional)

If $z \in \mathcal{C}$ on wrong side of L,



Proof (Optional)

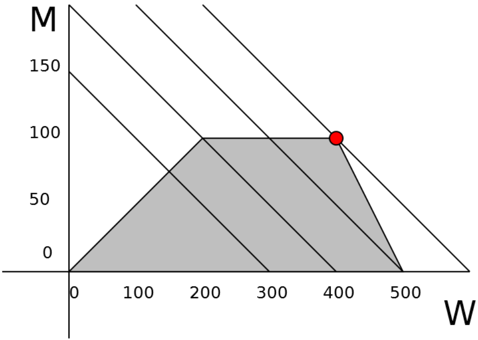
yz contains point closer to x. Contradiction.



Extreme Points

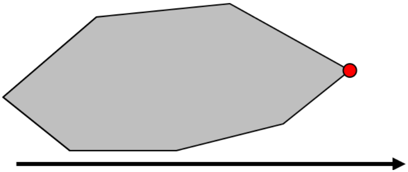
Lemma

A linear function on a polytope takes its minimum/maximum values on vertices.



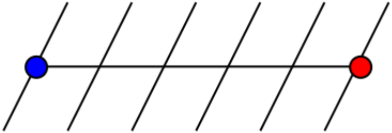
Intuition

The corners are the only extreme points.
Optima must be there.



Idea

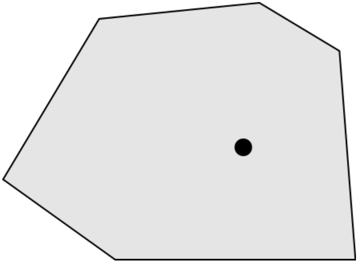
Linear function on segment takes extreme values on ends.



Use to push towards corners.

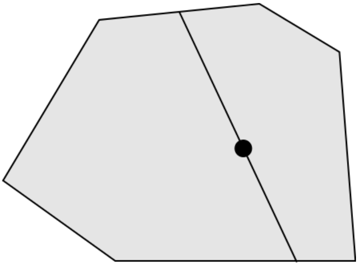
Proof

Start at any point.



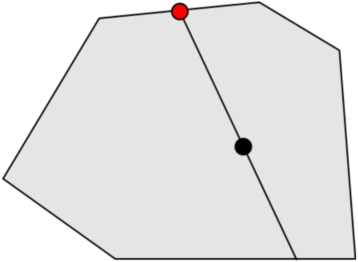
Proof

Pick line through point.



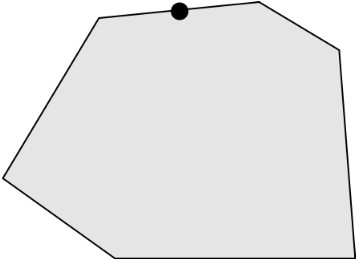
Proof

Extreme values at endpoint.



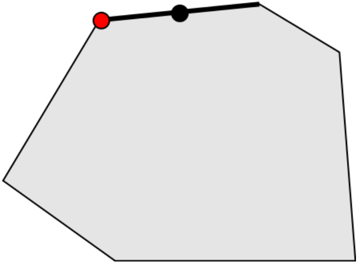
Proof

Point is on a facet.



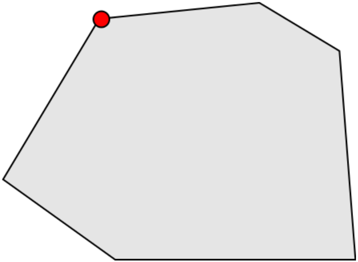
Proof

Repeat to push to lower dimensional facet.



Proof

Eventually on a vertex.



Summary

- Region determined by LP always convex polytope.
- Optimum always at vertex.
- Can separate from outside points by hyperplanes.