Linear Programming: Gaussian Elimination

Daniel Kane

Department of Computer Science and Engineering
University of California, San Diego

Advanced Algorithms and Complexity
Data Structures and Algorithms

Learning Objectives

- Solve a system of linear equations.
- Implement a row reduction algorithm.
- Say something about what the set of solution to a system of linear equations looks like.

Last Time

Linear programming: Dealing with systems of linear inequalities.

Linear Algebra

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For example:

$$\begin{aligned} x + y &= 5 \\ 2x + 4y &= 12. \end{aligned}$$

Method of Substitution

- Use first equation to solve for one variable in terms of the others.
- Substitute into other equations.
- Solve recursively.
- Substitute back in to first equation to get initial variable.

Example

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Substituting into second:

$$12 = 2x + 4y = 2(5 - y) + 4y = 10 + 2y.$$

So $y = 1, x = 5 - 1 = 4$.

Problem

What is the value of x in the solution to the following linear system?

$$\begin{aligned} x + 2y &= 6 \\ 3x - y &= -3. \end{aligned}$$

Solution

From the first equation, we get

$$x = 6 - 2y.$$

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Solving gives, $y = 3$, so $x = 6 - 2 \cdot 3 = 0$.

Notation

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We just store coefficients of equations in an (augmented) matrix, like so:

$$\begin{array}{c} \begin{matrix} x & y & = 1 \end{matrix} \\ \left[\begin{array}{cc|r} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right] \end{array}$$

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By subtracting. Subtracting $2(x + y = 5)$ from $(2x + 4y = 12)$ gives $2y = 2$.
Subtract twice first row from second.

Basic Row Operations

There are three basic ways to manipulate our matrix. These are called Basic row operations. Each of them gives us an equivalent system of equations.

Adding

Add/subtract a multiple of one row to another.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 2 & 4 & 12 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right]$$

Scaling

Multiply/divide a row by a non-zero constant.

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$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

Swapping

Sometimes you want to change the ordering of rows.

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Sometimes you want to change the ordering of rows. For example, swapping the first and second rows we get

$$\left[\begin{array}{cc|c} 1 & 1 & 5 \\ 0 & 1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 5 \end{array}\right]$$

Row Reduction

Row reduction uses row operations to put a matrix into a simple standard form. The idea is to simulate the substitution method.

Example

Consider the system given by the matrix:

$$\left[\begin{array}{rrrr|r} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Use first for to solve for first variable.

$$\left[\begin{array}{rrrr|r} 2 & 4 & -2 & 0 & 2 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Divide first row by 2.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ -1 & -2 & 1 & -2 & -1 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Add first row to second.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 2 & 2 & 0 & 2 & 0 \end{array}\right]$$

Example

Subtract twice first row from third.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

Example

Need to solve for next variable.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

Example

Cannot use second row.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & -2 & 2 & 2 & -2 \end{array}\right]$$

Example

Swap second and third rows.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Solve for second variable.

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 0 & 1 \\ 0 & -2 & 2 & 2 & -2 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Divide second row by -2.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Subtract twice second row from first.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Can't solve for third variable.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Solve for fourth instead.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

Example

Divide last row by -2.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Substitute into other equations.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Subtract twice third row from first.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Add third row to second.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Example

Done.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Answer

Our matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

corresponds to equations:

$$\begin{aligned} x + z &= -1 \\ y - z &= 1 \\ w &= 0. \end{aligned}$$

Solution

So for any value of z, we have solution:

$$\begin{aligned} x &= -1 - z \\ y &= 1 + z \\ w &= 0. \end{aligned}$$

RowReduce(A)

Leftmost non-zero

Swap row to top

Make entry pivot

Rescale to make pivot 1

Subtract row from others to make other entries in column 0

Repeat

RowReduce(A)

Leftmost non-zero in non-pivot row

Swap row to top of non-pivot rows

Make entry pivot

Rescale to make pivot 1

Subtract row from others to make other entries in column 0

Repeat until no more non-zero entries outside of pivot rows

Reading off Answer

- Each row has one pivot and a few other non-pivot entries.
- Gives equation writing pivot variable in terms of non-pivot variables.
- If pivot in units column, have equation $0 = 1$, so no solutions.
- Otherwise, set non-pivot variables to anything, gives answer.

Degrees of Freedom

- Your solution set will be a subspace.
- Dimension = number of non-pivot variables.
- Or n minus the number of pivot variables.
- Generally, dimension equals

$$\text{num. variables} - \text{num. equations}.$$

Runtime

- m equations in n variables.
- $\min(n, m)$ pivots.
- For each pivot, need to subtract multiple of row from each other row $O(nm)$ time.
- Total runtime: $O(nm \min(n, m))$.