Flows in Networks: Maxflow-Mincut

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Advanced Algorithms and Complexity
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Learning Objectives

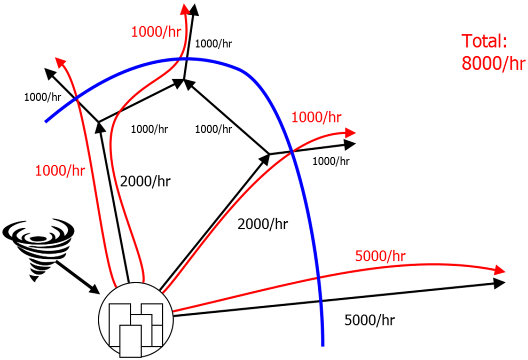
- Understand the relationship between flows and cuts.
- Produce a cut with size matching that of a maximum flow.
- Identify when a flow is maximum.

Problem

In order to find maxflows, we need a way of verifying that they are optimal.
In particular, we need techniques for bounding the size of the maxflow.

Idea

Recall our original example:



Idea

Find a bottleneck for the flow. All flow needs to cross the bottleneck.

Cuts

Definition

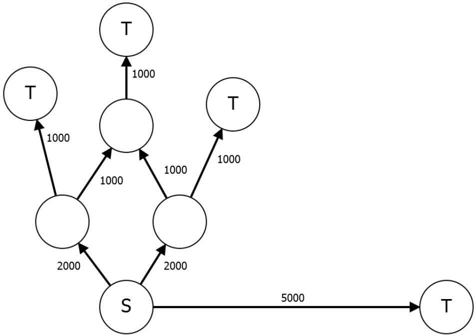
Given a network G, a cut $\mathcal{C}$, is a set of vertices of G so that $\mathcal{C}$ contains all sources of G and no sinks of G.

The size of a cut is given by

$$|\mathcal{C}| := \sum_{e \text{ out of } \mathcal{C}} C_e$$

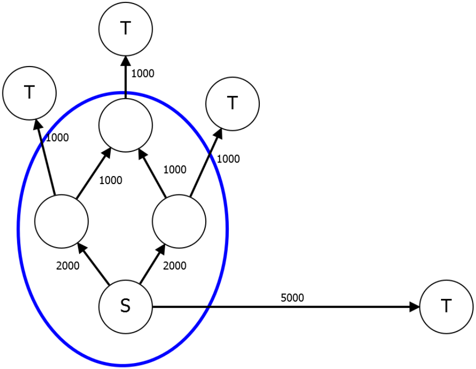
Example

Network G.



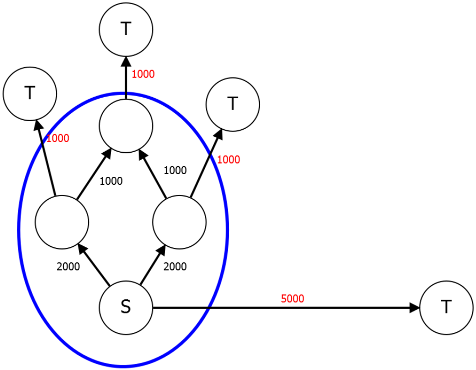
Example

Cut $\mathcal{C}$.



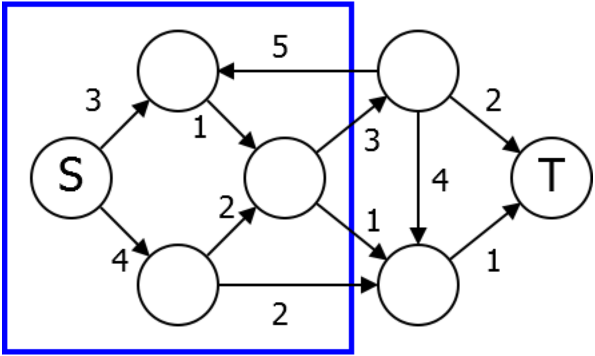
Example

Edges cut. Total size 8000.



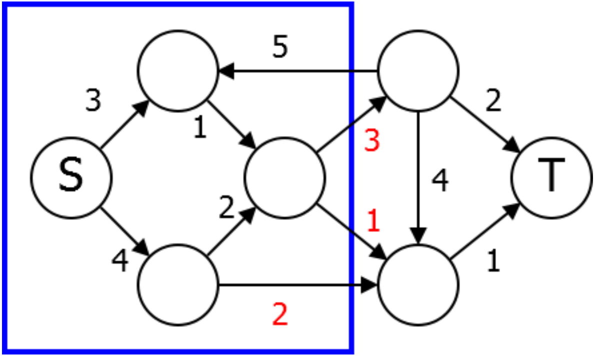
Problem

What is the size of the cut below?



Solution

$1 + 2 + 3 = 6.$



Bound

Lemma

Let G be a network. For any flow f and any cut $\mathcal{C}$,

$$|f| \leq |\mathcal{C}|.$$

Proof

$$
\begin{aligned}
|f| &= \sum_{v \text{ source}} \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right) \\
&= \sum_{v \in \mathcal{C}} \left(\sum_{e \text{ out of } v} f_e - \sum_{e \text{ into } v} f_e \right) \\
&= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\
&\leq \sum_{e \text{ out of } \mathcal{C}} C_e = |\mathcal{C}|.
\end{aligned}
$$

Bounds

In other words, for any cut $\mathcal{C}$, we get an upper bound on the maxflow. In particular,

$$\text{Maxflow} \leq |\mathcal{C}|.$$

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Question: Is this bound good enough?
Surprisingly, it is...

Maxflow-Mincut

Theorem

For any network G,

$$\max_{\text{flows f}} |f| = \min_{\text{cuts } \mathcal{C}} |\mathcal{C}|.$$

Maxflow-Mincut

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In other words, there is always a cut small enough to give the correct upper bound.

A Special Case

What happens when Maxflow $= 0$?

- There is no path from source to sink.
- Let $\mathcal{C}$ be the set of vertices reachable from sources.
- There are no edges out of $\mathcal{C}$.
- So $|\mathcal{C}| = 0$.

The General Case

- Let f be a maxflow for G.
- Note that G_f has maxflow 0.
- There is a cut $\mathcal{C}$ with size 0 in G_f.
- Claim: $|\mathcal{C}| = |f|$.

Proof

$$
\begin{aligned}
|f| &= \sum_{e \text{ out of } \mathcal{C}} f_e - \sum_{e \text{ into } \mathcal{C}} f_e \\
&= \sum_{e \text{ out of } \mathcal{C}} C_e - \sum_{e \text{ into } \mathcal{C}} 0 \\
&= |\mathcal{C}|.
\end{aligned}
$$

Conclusion

- We have found an f and $\mathcal{C}$ with $|f| = |\mathcal{C}|$.
- By Lemma, cannot have larger $|f|$ or smaller $|\mathcal{C}|$.
- So $\max |f| = \min |\mathcal{C}|$.

Summary

- Can always check if flow is maximal by finding matching cut.
- f a maxflow only if there is no source-sink path in G_f.
- We will use this in an algorithm next time.