Some Simplex Method Examples

Example 1: (from class) Maximize: $P = 3x + 4y$ subject to:

$$\begin{cases} x + y \leq 4 \\ 2x + y \leq 5 \\ x \geq 0, y \geq 0 \end{cases}$$

Our first step is to classify the problem. Clearly, we are going to maximize our objective function, all are variables are nonnegative, and our constraints are written with our variable combinations less than or equal to a constant. So this is a *standard maximization problem* and we know how to use the simplex method to solve it.

We need to write our initial simplex tableau. Since we have two constraints, we need to introduce the two slack variables u and v. This gives us the equalities

$$x + y + u = 4$$
$$2x + y = 5$$

We rewrite our objective function as $-3x - 4y + P = 0$ and from here obtain the system of equations:

$$\begin{cases} x + y + u = 4 \\ 2x + y = 5 \\ -3x - 4y + P = 0 \end{cases}$$

This gives us our initial simplex tableau:

x	y	u	v	P	
1	1	1	0	0	4
2	1	0	1	0	5
-3	-4	0	0	1	0

To find the column, locate the most negative entry to the left of the vertical line (here this is -4). To find the pivot row, divide each entry in the constant column by the entry in the corresponding in the pivot column. In this case, we'll get $\frac{4}{1}$ as the ratio for the first row and $\frac{5}{1}$ for the ratio in the second row. The pivot row is the row corresponding to the smallest ratio, in this case 4. So our pivot element is the in the second column, first row, hence is 1.

Now we perform the following row operations to get convert the pivot column to a unit column:

$$R_2 \rightarrow R_2 - R_1$$
$$R_3 \rightarrow R_3 + 4R_1$$

Our simplex tableau has transformed into:

x	y	u	v	P	
1	1	1	0	0	4
1	0	-1	1	0	1
1	0	4	0	1	16

Notice that all of the entries to the left of the vertical line in the last row are non-negative. We must stop here!

To read off the solution from the table, first find the unit columns in the table. The variables above the unit columns are assigned the value in the constant column in the row where the 1 is in the unit column. Every variable above a non-unit column is set to 0. So here $y = 4$, $v = 1$, $P = 16$, $x = 0$, and $u = 0$.

Thus, our maximum occurs when $x = 0, y = 4$ and the maximum value is 16.

Example 2 (from class) Minimize: $C = -2x + y$ subject to:

$$\begin{cases} x + 2y \leq 6 \\ 3x + 2y \leq 12 \\ x \geq 0, y \geq 0 \end{cases}$$

Classify the problem. Clearly, this is a minimization problem...but it's not the standard problem since our constraints involve linear expressions with the variables less than or equal to a constant. We use the trick that minimizing this function C is the same as maximizing the function $P = -C$. This problem is then equivalent to:

Maximize $P = 2x - y$ subject to:

$$\begin{cases} x + 2y \leq 6 \\ 3x + 2y \leq 12 \\ x \geq 0, y \geq 0 \end{cases}$$

Introducing the slack variable u and v, our initial simplex tableau is:

x	y	u	v	P	
1	2	1	0	0	6
3	2	0	1	0	12
-2	1	0	0	1	0

Our pivot element is the 3 in row 2 column 1. We do the following sequence of row operations to convert this column into a unit column:

$$R_2 \to \frac{1}{3}R_1$$
$$R_1 \to R_1 - R_2$$
$$R_3 \to R_3 + 2R_2$$

We have the tableau:

x	y	u	v	P	
0	4 /3	1	-1 /3	0	2
1	2 /3	0	1 /3	0	4
0	5 /3	0	1 /3	1	8

Since the last row has all positive entries to the left of the vertical line, we are done. Our maximum of P occurs at $x = 4, y = 0$ and the maximum value of P is 8. Since $P = -C$ our minimum for C is -8.

Thus $C = -8$ is the minimum for C and C is minimized at $(4, 0)$.

Example 3 (from class) Minimize $C = 2x + 5y$ subject to

$$\begin{cases} x + 2y \geq 4 \\ 3x + 2y \geq 3 \\ x \geq 0, y \geq 0 \end{cases}$$

This problem is a standard minimization problem. We call this problem the *primal problem*. To solve it, we first write the tableau

x	y	
1	2	4
3	2	3
2	5	

Now take this tableau and interchange its columns and rows, labeling the first two columns u, v:

u	v	
1	3	2
2	2	5
4	3	

From here we get the tableau of the *dual problem* by negating the last row of this table and then inserting unit columns and x, y, and P between the v the constant column:

u	v	x	y	P	
1	3	1	0	0	2
2	2	0	1	0	5
-4	-3	0	0	1	0

Now we use the simplex algorithm to get a solution to the dual problem. The pivot element is the 1 in the first column, first row. We do the following sequence of row operations to reduce this column to a unit column:

$$R_2 \to R_2 - 2R_1$$
$$R_3 \to R_3 + 4R_1$$

and arrive at the final tableau:

u	v	x	y	P	
1	3	1	0	0	2
0	-4	-2	1	0	1
0	9	4	0	1	8

The solution for the primal problem appears underneath the slack variables (in this case x and y) in the last row of of the final tableau. The maximum of the dual problem is the same as the minimum for the primal problem so the minimum for C is 8 and this value occurs at $x = 4, y = 0$. Note that the dual problem has a maximum at $u = 2$ and $v = 0$.

Example 4: Suppose when I solve a simplex problem I find that my slack variable u takes on a positive value. Did I do something wrong? Explain.

Answer. No. Basically, when one of the slack variables takes on a positive value it means that in maximizing our objective function we stayed "below" one of our constraints. For example, if u is the slack variable corresponding to a constraint on labor hours used and the value of u is 12 in our optimal solution, it means we have 12 remaining labor hours available.

Example 5: What do I conclude if the last row to the left of the vertical line in my simplex tableau contains all zeros?

Answer. There are infinitely many solutions to the optimization problem.