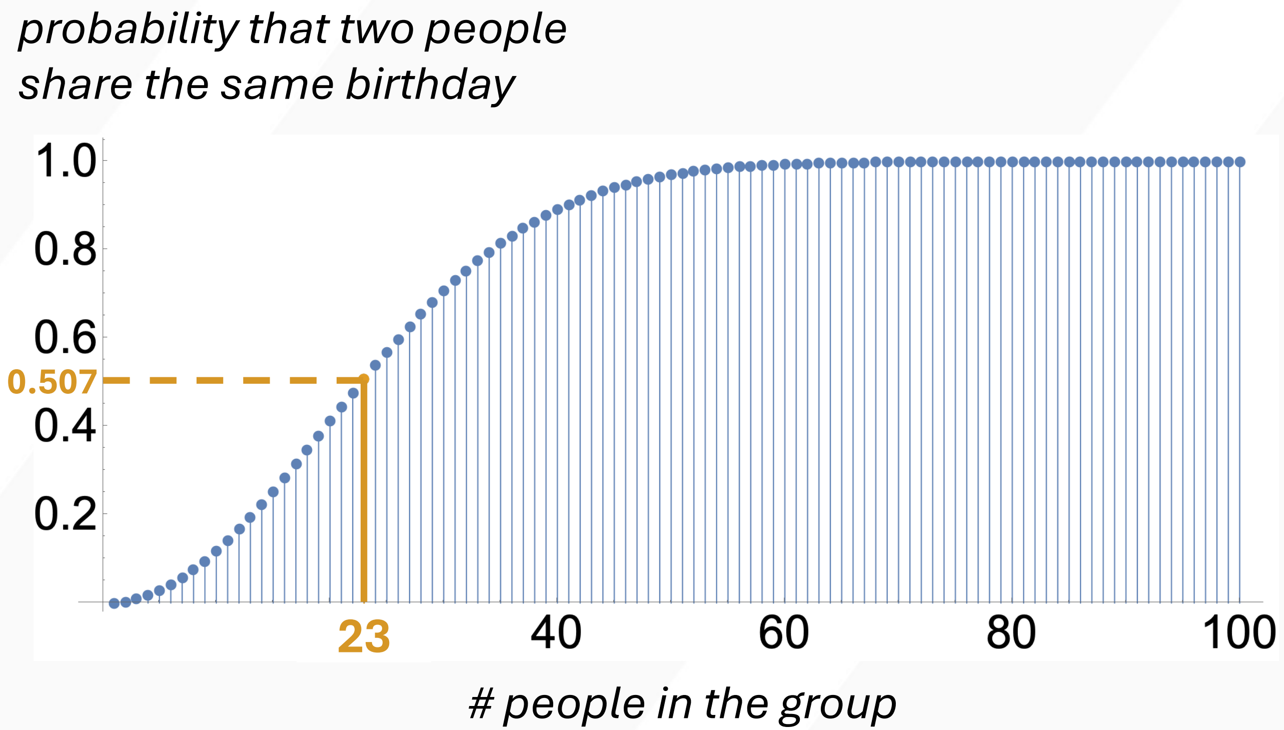
Motivating Example - Birthday Problem

What is the minimum number of people needed in a group for there to be at least a 50% probability that two people share the same birthday?

Answer: 23.

We will come back to this example in a later lecture.



Lecture 1

Probability Space

Introduction to Probability § 1.1

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Probability Space $(\Omega, \mathcal{F}, P)$

"Omega"

- sample space: Ω - the set of all possible outcomes

"omega"

An **outcome** is a possible result of an experiment, typically denoted by ω.

- collection of events: $\mathcal{F}$ - a collection of subsets of Ω

An **event** is a collection of outcomes, typically denoted by upper case letters A, B, etc..

- probability measure: P - a function from $\mathcal{F}$ into $[0,1]$

P assigns each event $A \in \mathcal{F}$ a probability $P(A)$, which satisfies the

Kolmogorov's Axioms (1933):

i. for an event $A \in \mathcal{F}$, $0 \leq P(A) \leq 1$,

ii. $P(\Omega) = 1$ and $P(\emptyset) = 0$,

iii. if $A_1, A_2, \ldots$ are pairwise *disjoint* events (i.e., $A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots.$$

The words "collection" and "set" are often used interchangeably.

Example - Flipping a Fair Coin

Probability space $(\Omega, \mathcal{F}, P)$:

- sample space: $\Omega = \{H, T\}$
- collection of events: $\mathcal{F} = \{\emptyset, \{H\}, \{T\}, \{H, T\}\}$
- probability measure: $P(\emptyset) = 0, P(\{H\}) = \frac{1}{2}, P(\{T\}) = \frac{1}{2}, P(\{H, T\}) = 1$

Kolmogorov's axioms are satisfied since

i. $0 \leq P(\emptyset), P(\{H\}), P(\{T\}), P(\{H, T\}) \leq 1$,

ii. $P(\Omega) = 1$ and $P(\emptyset) = 0$,

iii. $P(\{H\} \cup \emptyset) = P(\{H\}) + P(\emptyset)$,

$P(\{H\} \cup \{T\}) = P(\{H\}) + P(\{T\}), \ldots$

* $P(\{H\} \cup \{H, T\}) \neq P(\{H\}) + P(\{H, T\})$ ..the reason is that $\{H\}$ and $\{H, T\}$ are not disjoint

Example - Rolling a Standard Six-Sided Die

Probability space $(\Omega, \mathcal{F}, P)$:

- sample space: $\Omega = \{1,2,3,4,5,6\}$
- collection of events: $\mathcal{F} = \{A: A \subseteq \Omega\} = \{\emptyset, \{1\}, \{2\}, \dots, \{6\}, \dots, \{3,5\}, \dots, \{1,3,5\}, \dots, \{2,4,6\}, \dots, \{1,2,3,4,5,6\}\}$
- probability measure: P is given by

$$P(\{1\}) = P(\{2\}) = P(\{3\}) = P(\{4\}) = P(\{5\}) = P(\{6\}) = \frac{1}{6}$$

Once we determine the probabilities for the six events $\{1\}, \{2\}, \dots, \{6\}$, we can derive the probabilities for all other events in $\mathcal{F}$ (the size of $\mathcal{F}$ is $2^6 = 64$) using Kolmogorov's axioms.

A possible event in Ω is

$$A = \{\text{the outcome is divisible by } 3\} = \{3,6\} \in \mathcal{F}.$$

By Kolmogorov's axiom *iii*,

$$P(A) = P(\{3\}) + P(\{6\}) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

Example - Flipping a Fair Coin Three Times

Probability space $(\Omega, \mathcal{F}, P)$:

- sample space: $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

- collection of events: $\mathcal{F} = \{A: A \subseteq \Omega\}$

- probability measure: P is given by

$$P(\{HHH\}) = P(\{HHT\}) = P(\{HTH\}) = P(\{HTT\})$$
$$= P(\{THH\}) = P(\{THT\}) = P(\{TTH\}) = P(\{TTT\}) = \frac{1}{8}$$

A possible event in Ω is

$$A = \{\text{exactly two heads}\} = \{HHT, HTH, THH\} \in \mathcal{F}.$$

By Kolmogorov's axiom *iii*,

$$P(A) = P(\{HHT\}) + P(\{HTH\}) + P(\{THH\}) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$$