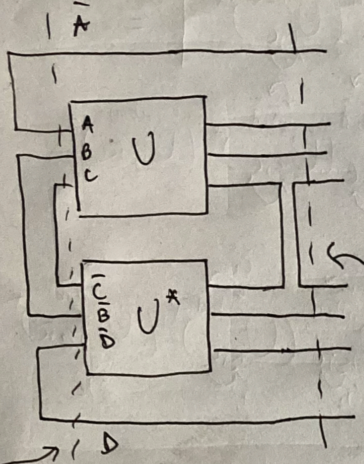

$|A| = |\bar{A}| = |\bar{D}| = |D| = 2$

$|B| = |\bar{B}| = d$

$|C| = |\bar{C}| = 2$

$$\Big[\,{}^{\mathcal{H}}_{\mathcal{H}} = \frac{1}{\sqrt{|\mathcal{H}|}} \sum_{i=0}^{|\mathcal{H}|-1} |ii\rangle$$

$$|\psi_{in}\rangle = \frac{1}{\sqrt{8d}} \sum_{k,i,j=0,1} \sum_{l=0}^{d-1} |\underset{\bar{A}}{i}\,\underset{A}{i}\,\underset{B}{j}\,\underset{C}{k}\,\underset{\bar{C}}{k}\,\underset{\bar{B}}{j}\,\underset{\bar{D}}{l}\,\underset{D}{l}\rangle$$

$$U_{ABC} = \sum_{abc,a'b'c'} U^{abc}_{a'b'c'} |\underset{A}{a}\,\underset{B}{b}\,\underset{C}{c}\rangle\langle \underset{A}{a'}\,\underset{B}{b'}\,\underset{C}{c'}|$$

$$U^*_{\bar{C}\bar{B}\bar{D}} = \sum_{\substack{\bar{c},\bar{b},\bar{d}\\ \bar{c}',\bar{b}',\bar{d}'}} \left(U^{\bar{c}\bar{b}\bar{d}}_{\bar{c}'\bar{b}'\bar{d}'}\right)^* |\underset{\bar{C}}{\bar{c}}\,\underset{\bar{B}}{\bar{b}}\,\underset{\bar{D}}{\bar{d}}\rangle\langle \underset{\bar{C}}{\bar{c}'}\,\underset{\bar{B}}{\bar{b}'}\,\underset{\bar{D}}{\bar{d}'}|$$

$$U_{ABC} \otimes U^*_{\bar{C}\bar{B}\bar{D}} |\psi_{in}\rangle = \frac{1}{\sqrt{8d}} \sum^{0,1}_{\substack{i,a,b,c,\\ \bar{c},\bar{d},l}} \sum^{d-1}_{b,\bar{b}} \Bigg[\sum_{j=0}^{d-1} \sum_{k=0,1} U^{abc}_{ijk} \left(U^{\bar{c}\bar{b}\bar{d}}_{kjl}\right)^* |\underset{\bar{A}}{i}\,\underset{A}{a}\,\underset{B}{b}\,\underset{C}{c}\,\underset{\bar{C}}{\bar{c}}\,\underset{\bar{B}}{\bar{b}}\,\underset{\bar{D}}{\bar{d}}\,\underset{D}{l}\rangle \Bigg]$$

$$\Big]\Big[\,{}^{C}_{\bar{C}} = \frac{1}{2} \sum_{m,n=0,1} |\underset{C}{m}\,\underset{\bar{C}}{m}\rangle\langle \underset{C}{n}\,\underset{\bar{C}}{n}| \equiv \Pi_{C\bar{C}}$$

$$|\hat{\psi}_f\rangle = \Pi_{C\bar{C}} \cdot U_{ABC} \otimes U^*_{\bar{C}\bar{B}\bar{D}} |\psi_{in}\rangle =$$

$$= \frac{1}{\sqrt{32d}} \sum_{\substack{iam\\ \bar{d}l=0,1}} \sum_{b,\bar{b}=0}^{d-1} \Bigg[\sum_{j=0}^{d-1} \sum_{k,m=0,1} U^{abm}_{ijk} \left(U^{m\bar{b}\bar{d}}_{kjl}\right)^* \Bigg] |i\,a\,b\,m\,m\,\bar{b}\,\bar{d}\,l\rangle$$

$$\langle \hat\psi_f | \hat\psi_f \rangle = \frac{1}{32d} \sum_{\substack{iam\\ \bar d e}}^{0,1} \sum_{b,\bar b=0}^{d-1} \left| \sum_{j=0}^{d-1} \sum_{k,m}^{0,1} U_{ijk}^{abm} \left(U_{kje}^{m\bar b\bar d}\right)^* \right|^2 =$$

$$= \frac{1}{32d} \sum_{\substack{iam\\ \bar d e\, km\\ k\bar m}}^{0,1} \sum_{\substack{b,\bar b, j,\\ \bar j = 0}}^{d-1} U_{ijk}^{abm} U_{\bar k \bar j e}^{\bar m b \bar d} \left(U_{kje}^{m\bar b\bar d}\right)^* \left(U_{\bar i \bar j \bar k}^{ab\bar m}\right)^*$$

$$[\![\langle \hat\psi_f | \hat\psi_f \rangle]\!] = \frac{1}{32d} \sum_{\text{see above}} \left[U_{ijk}^{abm} U_{\bar k \bar j e}^{\bar m b \bar d} \left(U_{kje}^{m\bar b\bar d}\right)^* \left(U_{\bar i \bar j \bar k}^{ab\bar m}\right)^* \right]$$

with i_1, j_1 on the first factor, i_2, j_2 on the second, i_1', j_1' on the third and i_2', j_2' on the fourth.

Weingarten integrals

$$\int_{U_d} U_{i_1 j_1} U_{i_2 j_2} U^*_{i_1' j_1'} U^*_{i_2' j_2'}\, dU = \sum_{\tau,\sigma\in S_2} \delta_{i_1 i'_{\sigma(1)}} \delta_{i_2 i'_{\sigma(2)}} \delta_{j_1 j'_{\tau(1)}} \delta_{j_2 j'_{\tau(2)}} Wg(\sigma\tau^{-1}, N)$$

$$S_2 = \{ \mathbb{1}, \text{SWAP} \} = \{ (1)(2), (21) \} = \{ 1^2, 2 \}$$

$$Wg(\cancel{\mathbb{1}}\, 1^2, N) = \frac{1}{N^2-1} \quad ; \quad Wg(2, N) = -\frac{1}{N(N^2-1)}$$

where $N = 4d$

σ	τ					
$\mathbb{1}$	$\mathbb{1}$	~~$\delta_{i_1 j_1}\delta_{i_2 j_2}\delta_{i_1' j_1'}\delta_{i_2' j_2'}$~~	$(i_1 i_1')(i_2 i_2')$	$(j_1 j_1')(j_2 j_2')$	$(a\bar m)(b\bar b)(m\bar d)(a\bar m)$	$(ik)(kl)(\bar k i)$
$\mathbb{1}$	Σ	~~$\delta\delta\delta\delta$~~	"	$(j_1 j_2')(j_2 j_1')$	"	$(k\bar u)(jj)$
Σ	$\mathbb{1}$	~~$\delta\delta\delta\delta$~~	$(i_1 i_2')(i_2 i_1')$		$(m\bar m)$	
Σ	Σ	~~$\delta\delta\delta\delta$~~	"		"	

$$[\![\langle \psi_f | \psi_f \rangle]\!] = \frac{1}{32d} \left[\frac{1}{16d^2-1} \left(d^3 \cdot 2^3 + d^3 \cdot 2^7 \right) - \frac{1}{4d(16d^2-1)} \left(d^2 \cdot 2^5 + d^4 \cdot 2^8 \right) \right]$$

$$= \frac{1}{4d} \frac{d^2}{16d^2-1} \left[d + 16d - \frac{1}{4d} \left(1 + d^2 \right) \right] =$$

$$= \frac{4d}{16d^2-1} \left[2d - \frac{1+d^2}{4d} \right] = \frac{7d^2-1}{16d^2-1}$$

LOOK HERE

$$= \frac{1}{4} \frac{16d^2 + d^2 - (1+d^2)}{16d^2 - 1} = \frac{1}{4} \quad \forall d.$$

Probability of entanglement swap

Let $|\psi_f\rangle = \frac{|\hat{\psi}_f\rangle}{\langle\hat{\psi}_f|\hat{\psi}_f\rangle^{1/2}}$

we'd like to calculate

$$\mathbb{E}\left[\frac{\langle\hat{\psi}_f|\Pi_{\bar{A}D}|\hat{\psi}_f\rangle}{\langle\hat{\psi}_f|\hat{\psi}_f\rangle}\right]$$

However this is probably to hard. So let's calculate

$$\frac{\mathbb{E}[\langle\hat{\psi}_f|\Pi_{\bar{A}D}|\hat{\psi}_f\rangle]}{\mathbb{E}[\langle\hat{\psi}_f|\hat{\psi}_f\rangle] = \frac{1}{4}}$$

$$\Pi_{\bar{A}D}|\hat{\psi}_f\rangle = \frac{1}{\sqrt{128d}} \sum_{\ell\bar{\ell}am\bar{d}}^{0,1} \sum_{b\bar{b}=0}^{d-1} \left[\sum_{j=0}^{d-1} \sum_{k,m,i}^{0,1} U^{abm}_{ijk} (U^{m\bar{b}\bar{d}}_{kji})^*\right] |\ell a b m \bar{m} \bar{b} \bar{d} \bar{\ell}\rangle$$

$$|\langle\Pi_{\bar{A}D}|\hat{\psi}_f\rangle|^2 = \frac{1}{128d} \sum_{\ell am\bar{d}}^{0,1} \sum_{b\bar{b}=0}^{d-1} \left|\sum_{j=0}^{d-1} \sum_{kmi}^{0,1} U^{abm}_{ijk} (U^{m\bar{b}\bar{d}}_{kji})^*\right|^2$$

$$= \frac{1}{128d} \sum_{\substack{\ell am\bar{d} \\ km\bar{k}\bar{m} \\ i\bar{i}}}^{0,1} \sum_{\substack{b\bar{b}=0 \\ j\bar{j}}}^{d-1} U^{abm}_{ijk} U^{\bar{m}\bar{b}\bar{d}}_{\bar{k}\bar{j}\bar{i}} (U^{m\bar{b}\bar{d}}_{kji})^* (U^{ab\bar{m}}_{\bar{i}\bar{j}\bar{k}})^*$$

$$\mathbb{E}[|\Pi_{\bar{A}\bar{D}}|\hat{\psi}_f\rangle|^2] = \frac{1}{128d} \sum_{\text{as above}} \frac{1}{16d^2-1} \Big[UUU^*U^* + UUU^*U^* + $$
$$ - \frac{1}{4d}\left(UUU^*U^* + UUU^*U^*\right)\Big] =$$

$$= \frac{1}{128d} \sum_{\text{as above}} \frac{1}{16d^2-1} \Big[(am)(b\bar{b})(m\bar{d})(\bar{m}a)(ik)(\bar{i}\bar{k}) + (m\bar{m})(k\bar{k})(j\bar{j})(i\bar{i}) +$$
$$(m\bar{m})(ik)(\bar{i}\bar{k})\Big]$$
$$- \frac{1}{4d}\Big((am)(b\bar{b})(m\bar{d})(\bar{m}a)(k\bar{k})(j\bar{j})(i\bar{i}) + (m\bar{m})(ik)(\bar{i}\bar{k})$$

$$= \frac{1}{128d} \frac{1}{16d^2-1}\left[d^3 \cdot 2^5 + d^3 \cdot 2^7 - \frac{1}{4d}\left(d^2 \cdot 2^5 + d^4 \cdot 2^7\right)\right]$$

$$= \frac{d}{4} \frac{1}{16d^2-1}\left[5d - \frac{1}{4d}\left(1+4d^2\right)\right] =$$

$$\cancel{= \frac{1}{16} \frac{1}{16d^2-1}\left[20d^2 - \left(1+4d^2\right)\right] = \frac{1}{16}}$$

~~again independent of d !?~~

$$\frac{d}{16d^2-1}\left(5d - \frac{1+4d^2}{4d}\right) = \frac{1}{4}$$

CORRECT (?)