THE EDUCATION LINK

PHYSICS-I

Chapter # 04

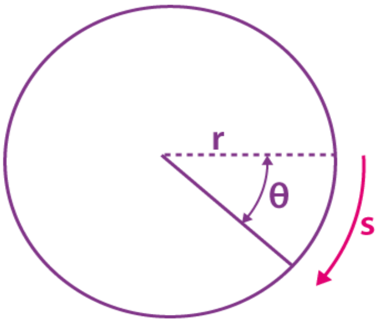
ROTATIONAL AND CIRCULAR MOTION

ROTATIONAL MOTION:

"Rotational motion happens when the body itself is spinning on its axis."

Example: The earth spinning on its axis and the turning shaft of an electric motor.

ANGULAR DISPLACEMENT:



"The angle subtended at the center of the circular path by an arc traced by the moving object in a certain time interval is called angular distance. It is denoted by 'θ' (theta)"

OR

"It is defined as the angle with its vertex at the center of a circle whose sides cut off an arc on the circle equal to its radius."

Explanation:

$$S = r\,\theta$$

$$\boxed{\theta = S / r}$$

When $\theta = 360^o$, then $S = 2\,\pi\,r$ (circumference of the circle)

- $360^o = 2\,\pi\,r / r$
- $360^o = 2\,\pi$
- $1^o = 2\,\pi / 360$
- $1^o = \pi / 180$ rad
- $180^o = \pi$ rad
- 1 rad $= 180^o / \pi$
- **1 rad = 57.3º**

RADIAN:

"The ratio of the length of the arc and the radius of the circle is equals to angle in radians."

REVOLUTION:

"The revolution is one complete rotation of a body."

ANGULAR VELOCITY:

"The rate of change of angular displacement is called angular velocity."

SI Unit: rad/sec

Formula:

$$\omega = \frac{\Theta}{t}$$

Where ω is the angular velocity, θ is the angular displacement, and t is the change in time t. By convention, positive angular velocity indicates counterclockwise rotation, while negative is clockwise.

AVERAGE ANGULAR VELOCITY

The average angular velocity of a rotating rigid body is the ratio of the angular displacement to the time interval.

$$\omega_{avg} = \frac{\Delta\Theta}{\Delta t} = \frac{\Theta_2 - \Theta_1}{t_1 - t_2}$$

INSTANTANEOUS ANGULAR VELOCITY

The instantaneous angular velocity is defined as the limit of the average angular velocity as the time interval approaches zero.

$$\omega_{ins} = \lim_{\Delta t \to 0} \frac{\Delta\Theta}{\Delta t} = \frac{d\Theta}{dt}$$

ANGULAR ACCELERATION:

"The rate of change of angular velocity of a body is called its angular acceleration."

Mathematically,

$$\alpha_{av} = \Delta\omega / \Delta t$$

But $$\alpha_{av} = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t}$$

RELATION BETWEEN LINEAR AND ANGULAR QUANTITIES:

Consider an object revolving in circle, which involve angular as well as linear motions i.e. stadium race ground, moving in circular laps but counted in linear terms. The equations of both motions as under:

- $\theta = \frac{S}{r}$ → (i)
- $v = \frac{s}{t}$ → (ii)
- $\omega = \frac{\theta}{t}$ → (iii)

Therefore, combining and substituting s/r for θ in eq (iii), we obtain

- $\omega = \frac{S/r}{t}$
- $\omega\,(r) = \frac{S/r}{t}\,(r)$
- $\omega\,(r) = \frac{s}{t}$ *whereas, s/t = v*
- $\boxed{\mathbf{V = r\,\omega}}$ → (iv)

V = linear velocity of point on circle also known as tangential velocity
r = radius
ω = angular speed.

Similarly,
If V = r ω is divide both sides by t then,

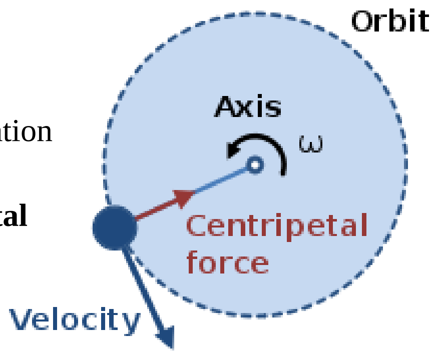
- $V/t = r\,(\omega/t)$
- $\boxed{\mathbf{a = r\,\alpha}}$ → (V)

a = Linear or tangential acceleration
α = angular acceleration
r = radius

CENTRIPETAL FORCE:

“The force which compels a body to retain its motion along a circular path and always directed towards the center of circular path is known as the centripetal force.”

Explanation:



- An object moving in a circle has **acceleration** because its **velocity** changes direction constantly.
- According to **Newton’s second law of motion**, acceleration is caused by a **force** acting on the object.
- The force that causes circular motion is called **centripetal force**, which means **center-seeking**.
- If the centripetal force is removed, the object will move in a **straight-line** tangent to the circle.

For Example:
A stone is tied at one end of the string, while the other end is held in hand. Now this stone is whirled in a circle. In order to keep the stone alone the circular path, the string must be pulled inward with a force which produces acceleration in the stone directed towards the hand. If the string breaks, the inward pull stops suddenly and the stone flies off along the tangent to the circular path.

Formula:

$$F= mv^2/r$$

CENTRIPETAL ACCELERATION:

Definition:

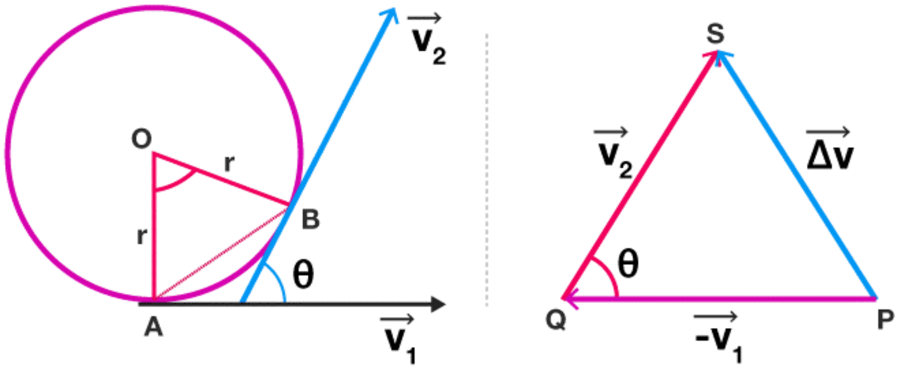
“The acceleration of the object moving in a circular path with constant speed, towards the center of the circular path is called centripetal acceleration.”

Explanation:
If a body moves in a circle with constant speed, the magnitude of the velocity remains constant but its direction changes continuously. Due to this change in velocity, the body must have an acceleration which is directed towards the center of the circle.

EXPRESSIONS FOR CENTRIPETAL ACCELERATION

ASSUMPTION:

To derive an expression for the centripetal acceleration and force, let us consider a body of mass ‘m’ moving along a circular path of radius ‘r’ with a constant linear speed ‘v’.
Let the body moves from point ‘A’ to ‘B’ on the circular path in time interval ‘Δt’. The linear velocity V_1 at point ‘A’ changes to V_2 at ‘B’ during this interval. This change in velocity is due to the change in direction of motion. In figure, observe that the triangle formed by the velocity vectors and the one formed by radius r and ΔS are similar. Both ΔPQR and ΔABC are similar. The two equal sides of velocity vector are given as $V_1=V_2=V$.



DERIVATION:

- $\Delta V / V = \Delta S / r$
- $\Delta V = \Delta S \times V / r$
- Dividing both sides with **Δt**
- $\frac{\Delta V}{\Delta t} = \frac{V \times \Delta S}{r \times \Delta t}$
- Since, $\frac{\Delta S}{\Delta t} = v$ and $\frac{\Delta V}{\Delta t} = a_c$
- $\mathbf{a_c = \frac{V2}{r}}$

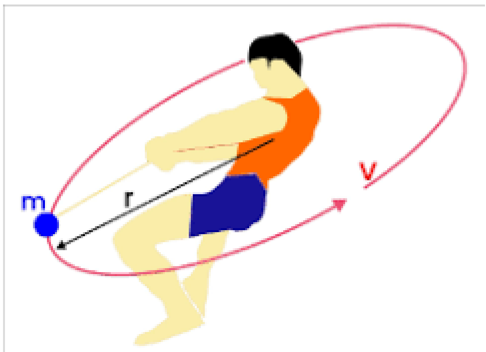
It is useful to show centripetal acceleration in terms of angular velocity.
Since, $V = r\,\omega$

- $a_c = \frac{(r\,\omega)^2}{r}$
- $\mathbf{a_c = r\,\omega^2}$

CENTRIPETAL ACCELERATION CAUSED BY TENSION FORCE:

An acceleration must be produced by force. Any force or combination of forces can cause a centripetal or radial acceleration. For examples are the tension in the rope constraint the motion of the ball, the force of Earth's gravity on the Moon to keep them in orbit, friction between roller skates, a rink floor, and a banked roadway's force on a car. Any net force causing uniform circular motion is called a centripetal force. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration.

CONSIDERATION:
If a ball is tied to the end of a string and whirling in a circle, the ball accelerates towards the center of the circle as shown in figure. The centripetal force which causes the inward acceleration is from the tension in the string caused by the person's hand pulling the string plus the weight of the ball also having its implications. The centripetal force on the object equals the tension of the string plus the weight of the ball, both acting toward the center of the vertical circle.



MATHEMATICALLY:

$F_C = F_t + F_w$

$F_t = \frac{m\,V2}{r} - mg$

The tension is exerted inward toward the center of the vertical circle, while the weight is directed away from the center of the vertical circle.

If the string breaks there is no longer a resultant force acting on the ball, so it will continue its motion in a straight line at constant speed.

FORCES ACTING ON BANKED ROAD

DEFINITION:

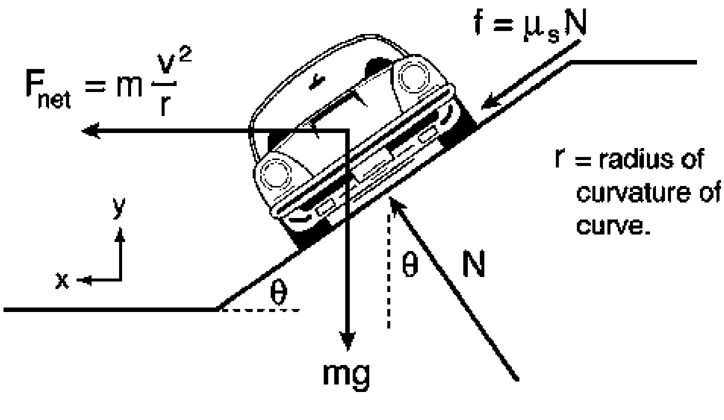
"A banked curve is a curve that has its surface at angle with respect to the ground on which the curve is positioned."

REASON:

- The primary reason for banking curves is to reduce reliance on friction. On a non-banked curve, a car experiences a force of static friction pointing towards the center of the circular pathway. This frictional force generates centripetal acceleration, facilitating the car's movement along the curve.
- In contrast, a banked curve introduces a normal force acting at an angle, creating a component that contributes to the required centripetal acceleration. Consequently, for certain velocities, no friction is needed to navigate the curve, enhancing the ability to turn even under slippery conditions like ice or water.

CONSIDERATION:

To derive an expression for the angle (θ) of an ideally banked curve, we consider an ideal scenario where the net external force equals the horizontal centripetal force, and friction is absent. This simplifies the forces acting on the car to its weight and the normal force of the road.



DERIVATION:

We will derive an expression for **Θ** for an ideally banked curve. For ideal banking, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force **N** in the horizontal and vertical directions must equal the centripetal force and the weight of the car.

A free -body diagram for a car on a frictionless banked curve. If the angle **ϴ** is ideal for the speed and radius r, then the net external force equals the necessary centripetal force. The only two external forces acting on the car are its weight and the normal force of the road **N.**

These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude $\frac{\boldsymbol{m\,V2}}{\boldsymbol{r}}$. As it is a crucial force and is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, so this must equal the centripetal force, that is,

N sin **ϴ** = $\frac{m\,V2}{r}$ → (i)

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From figure, we note that the vertical component of the normal force is N cos **ϴ**, and the only other vertical force is the car's weight. These must be equal in magnitude.

- N cos **ϴ** = mg → (ii)
- Divide (ii) by (i)
- mg x $\frac{N\,sin\,\theta}{N\,cos\,\theta} = \frac{m\,V2}{r}$
- mg $\tan\theta = \frac{m\,V2}{r}$
- $\tan\theta = \frac{V2}{rg}$

$$\theta = \tan^{-1}\frac{V2}{rg}$$

<u>ORBITAL VELOCITY:</u>

"Orbital velocity is the speed required to achieve orbit around a heavenly body, such as a planet or a star."

EXPLANATION:

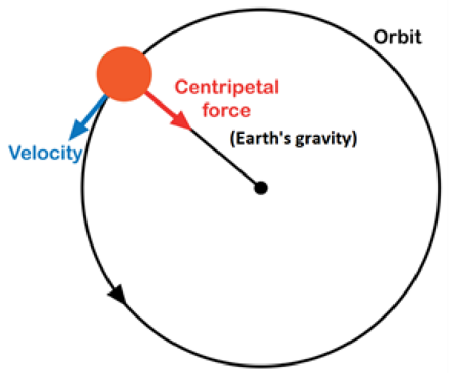
This requires traveling at a sustained speed that:

- Aligns with the heavenly body's rotational velocity.
- Is fast enough to counteract the force of gravity pulling the orbiting object toward the body's surface.
- An airplane can travel in the sky, but it does not travel at a velocity fast enough to sustain orbit around the earth. This means that once the airplane's engines are turned

off, the plane will slow down and be pulled back down to earth, via the force of gravity.

- By contrast, a satellite (such as the one that powers your phone's GPS or the one that transmits a Direct TV signal) does not need to expend fuel to maintain its orbit around the earth. This is because such satellites travel at a velocity that overrides the force of gravity.

ORBITAL SPEED



CONSIDERATION

Suppose a satellite with mass $M_{\text{satellite}}$ orbiting a central body with mass M_{central}. The central body could be a planet, the moon, the sun, or any other heavenly body which may be capable of causing reasonable acceleration over a less massive body nearly.

DERIVATION:

When the satellite is moving in a circular motion, then the net centripetal force acting upon this orbiting satellite is.

- $F_c = \frac{M\ satellite\ V2}{R}$

This net centripetal force is the resultant of gravitational force which attracts the satellite towards the central body.

- $F_G = \frac{\text{G M satellite M central}}{R^2}$
- $F_G = F_c$
- $\frac{M\ satellite\ V^2}{R} = \frac{\text{G M satellite M central}}{R^2}$
- $V^2 = \frac{\text{G M central}}{R}$

$$\boxed{V = \sqrt{\frac{\text{G M central}}{R}}}$$

$G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

M central = 5.98×10^{24} m

R = distance from the center of earth

TIME PERIOD:

Since a planet or a satellite is traveling in circular motion when in orbit, its orbital time periods T to travel the circumference of the orbit 2 Π R, the linear speed is,

$$V = \frac{\mathbf{2\ \Pi\ R}}{T}$$

Substituting ‘V’

$$V^2 = \frac{\text{G M central}}{R}$$

$$\left(\frac{\mathbf{2\ \Pi\ R}}{T}\right)^2 = \frac{\text{G M central}}{R}$$

$$\mathbf{T^2 = \frac{4\ \Pi\ 2\ R3}{G\ M\ central}}\quad \textbf{(KEPLERS' 3}^{\textbf{rd}}\textbf{ LAW)}$$

Where:

T = Time Period

R = Orbital Radius

G = Universal Gravitational Constant

M = Mass of object

MOMENT OF INERTIA

“Moment of inertia is the property of the body by virtue of it resists angular acceleration, which is the sum of the products of the mass of each particle in the body with the square of its distance from the axis of rotation.”

MATHEMATICALLY:

The moment of inertia can be expressed in terms of its individual masses as the sum of the product of each individual mass and the squared perpendicular distance to the axis of rotation.

$I = \Sigma\ mr^2$
I = Moment of Inertia
m = Mass
r = distance to axis of rotation

Unit & Dimension:

The moment of inertia is measured in kilogram square meters (kg m^2), its dimensional formula is $[M^1L^2T^0]$.

Factors Depending Upon:

The moment of inertia depends upon the following factors.

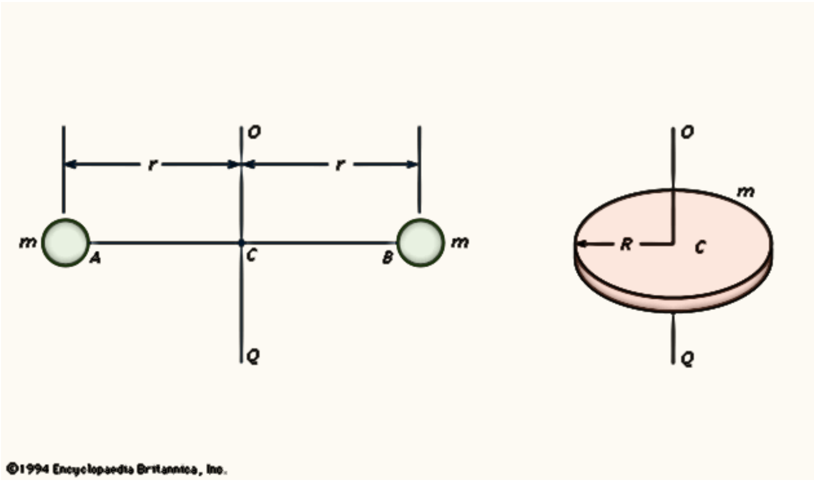
- ✓ Shape and size of the body
- ✓ The density of body
- ✓ Axis of rotation (distribution of mass relative to the axis)

ROTATIONAL INERTIA OF TWO PARTICLE SYSTEM:

Consider a rigid body containing two particles of mass m connected by a rod of length L with negligible mass.

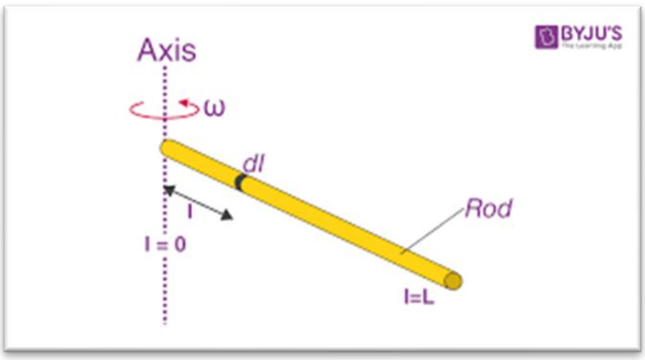
a) Two particles each at perpendicular distance 1⁄2 L from the axis of rotation. For two particles each at perpendicular distance1⁄2 L from the axis of rotation, we have:

- $I = \Sigma mr^2 = (m)(1/2\ L)^2 + (m)(½\ L)^2$
- $\mathbf{I = mL^2}$



b) Rotational inertia I of the body about an axis through left end of rod and parallel to the first axis. Here it is simple to find I use either method. The perpendicular distance r is zero for the particle on left and L for the partial on the right. We have:

- $I = m(0)^2 + m\ L^2$
- $\mathbf{I = m\ L^2}$



SECOND METHOD:
As I_c, about an axis through the center of mass and because the axis here is parallel to that center of axis, we can apply the parallel-axis theorem.

- $I = I_c + Mh^2$
- $I = ½ mL^2 + (2m) (1/2 L)^2$
- $\mathbf{I = m L^2}$

- **Moment of Inertia of Various bodies:**

Consider a solid cylinder with mass M, and radius R. The moment of inertia of a solid cylinder rotating about its central axis is given by the formula:

$\mathbf{I = \frac{1}{2} MR^2}$

- **Moment of Inertia of a Hollow Cylinder:**

Consider a hollow cylinder with mass M, inner radius a, and outer radius b. The moment of inertia of a hollow cylinder rotating about its central axis is given by the formula:

$\mathbf{I = \frac{1}{2} m (a^2 + b^2)}$

- **Moment of Inertia of a Sphere:**

Consider a solid sphere with mass M and radius R. The moment romans hd and radios R. The of inertia of a solid sphere rotating about its center is given by the formula:

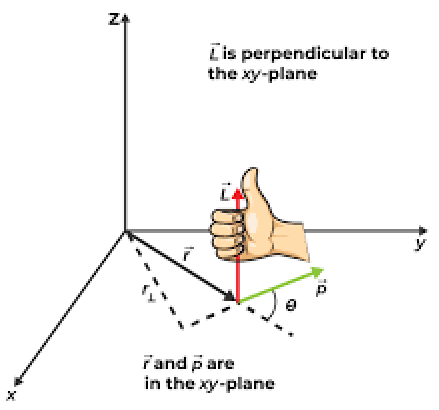
$\mathbf{I = \frac{2}{5} MR^2}$

ANGULAR MOMENTUM:

A body in translation motion has a linear velocity and linear momentum. Similarly if a body has rotational motion about some axis, it has angular velocity and angular momentum.

"The angular momentum of a body about any axis is the product of its linear Momentum and the perpendicular distance of the body from the axis".

CONSIDERATION:
Suppose a point particle of mass m in Cartesian coordinates at position r with respect to origin O, is having momentum p. The point particle is accelerating around a fixed point e.g. Earth revolving around the Sun.



EXPLANATION:
The angular momentum L, of a point particle is the vector product of its position/moment arm and linear momentum.

- L = r x p

The magnitude of angular momentum is

- L= r p Sin θ

Where θ is angle between position and momentum vectors.
Its direction is perpendicular to both position and linear momentum and determined by right-hand rule.

- L= mvr Sin θ
- L= mvr

LAW OF CONSERVATION OF ANGULAR MOMENTUM

STATEMENT:
"When no external torque acts upon a body or system, the total angular momentum of the system about and axis remains constant".

OR

"It states that the total angular momentum of a system always remains constant if net external torque acting on the system is zero."

PROOF:

Given Conditions:

- A centripetal force (F) acts on the particle, directing it towards the center to maintain a circular path.
- Centripetal force always points towards the center, producing no torque about an origin at the center of the circle.
- **Magnitude Constant:**
  - The magnitude of angular momentum ($mr\omega^2$) is constant.
- **Direction Constant:**
  - Angular momentum (L) remains constant in direction as motion is confined to the plane of rotation.

- L = r x p
- L = (r || + r ⊥) x p
- L = (r || x p) + (r ⊥ x p)

Since,

That the angular momentum has a component L ⊥ = r || x p which is perpendicular to the axis of rotation and is not conserved. This is not a problem because, relative to the origin, the particle experiences a torque Ꚍ ⊥ = r || x F where F is the centripetal force acting along - r ⊥.

- r || = constant
- $\frac{dL\perp}{dt}$ = r || x $\frac{dp}{dt}$ = r || x F = Ꚍ ⊥

- $\frac{dL}{dt} = \tau$
- $\boldsymbol{\frac{dL}{dt} = 0}$

Where, $\tau = 0$
L = constant
i.e. the angular momentum of a particle is conserved if the net external force is zero.

TORQUE

"The rotating effect of a body is called torque."

OR

"The cross product of moment arm and force is called torque."

MATHEMATICALLY:

Torque = moment arm × Force

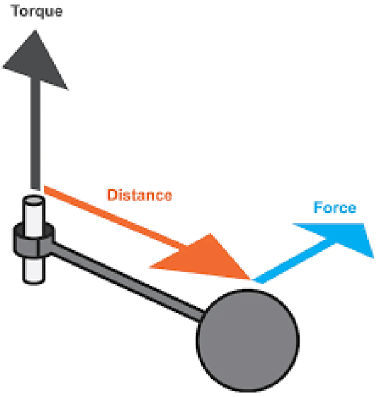
Let a body of mass 'm' is at point 'P' in xy-plane and makes position vector r. If force F is acting on it due to which the body is rotated then the cross product of them is equal to torque that is

- $\tau = r \times F$
- $\tau = F\, r \sin\theta$
- **Unit: N.m**
- Positive torque for anticlockwise rotation and negative torque for clockwise rotation.

RELATION BETWEEN TORQUE, MOMENT OF INERTIA AND ANGULAR ACCELERATION

CONSIDERATION:

Consider a particle of mass *m* rotating in a circle of radius *r* at the end of a string. A single force *F* acts on the mass.



DERIVATION:

- $\tau = r F \rightarrow$ (i)
- $a_{tan} = r\,\alpha \rightarrow$ (ii)

Since, F = ma

- $F = m\,a$

Multiply both sides with r and put eq ii

- $r F = m r^2 \alpha$

$$\boxed{\tau = I\,\alpha}$$

Now let us consider a rotating rigid object, such as wheel rotating about an axis through its center, which could be an axle. We can think of the wheel as containing of many particles located at various distances from the axis of rotation. We can apply eq to each particle of the object, and then sum over all the particles. The sum of the various torques is just the total torque,

- $\Sigma \tau = (\Sigma mr^2)\ \alpha$
- $I = \Sigma mr^2 = mr_1^2 + mr_2^2 + \ldots + mr_n^2$

$$\boxed{\Sigma \tau = \mathbf{I}\ \alpha}$$