Borcherd Algebraic Geometry 1

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0 Prologue

Some quick notes summarising the "Algebraic geometry 1" video lectures from R.E. Borcherds, found here

1 Introduction

1.1 Examples

1.1.1 Pythagorean triangles

Problem: How do we classify all Pythagorean triangles.

We will look at two ways of solving this:

1. **Algebraic way:** We want to solve

$$x^2 + y^2 = z^2 \text{ with } x, y, z \text{ coprime integers} \tag{1}$$

If we look at the equation mod 4 we notice that $x^2, y^2, z^2 \equiv 0, 1 \mod 4$, since the squares mod 4 all take these forms.

So z is odd and WLOG we assume that x is even and y is odd. We rearange the equation:

$$y^2 = z^2 - x^2 = (z - x)(z + x) \tag{2}$$

Assume that $z - x = dm_1$ and $z + x = dm_2$, therefore we have that $2z = d(m_1 + m_2)$ and $2x = d(m_2 - m_1)$, then since $d \mid 2z$ and $d \mid 2x$, and $\gcd(x, z) = 1$ we have two cases, either d divides both x and z, which would imply that $d = 1$.

Or d divides 2 which means that $d = 1$, or $d = 2$. But note that since x, z are of opposite parity $z + x$ is odd so $d \neq 2$.

So in all cases, $d = 1$. So $(z - x)$ and $(z + x)$ are coprime.

But since their product is a square this implies that $z - x$ and $z + x$ are squares. so:

$$z - x = r^2, \text{ and } z + x = s^2, \text{ where } s, r \text{ are odd and coprime} \tag{3}$$

So we conclude that $z = \frac{r^2+s^2}{2}$, $x = \frac{s^2-r^2}{2}$, $y = rs$ for any r, s odd and coprime.

2. **Geometric solution** Let $X = \frac{x}{z}$, $Y = \frac{y}{z}$ and we want to solve

$$X^2 + Y^2 = 1, \ X, Y \text{rational} \tag{4}$$

So we are looking for rational points on the unit circle.

Note if we draw the line from $(-1, 0)$ to (X, Y) on the unit circle with $X, T \in \mathbb{Q}$.It will intersect the y-axis at the point $(0, t)$ where $t = \frac{Y}{X+1} \in \mathbb{Q}$.

Conversely, if we are given t we can find (X, Y), since we know that

$$Y = t(X + 1) \text{ and } t^2(X + 1)^2 + X^2 = 1 \Rightarrow (X + 1)((t^2 + 1)X + t^2 - 1) = 0$$

And finding roots we see that $X = \frac{1-t^2}{1+t^2}$ and $Y = \frac{2t}{1+t^2}$, for $t \in \mathbb{Q}$.

So there is a correspondence between points on the circle except for the point at $(-1, 0)$ and points on the $y-$axis. This is what is called a Birational Equivalence.

Definition 1.1. Birational Equivalence An equivalence excepts on subsets of co-dimension at least 1.

Treating this problem as a geometrical problem gives us additional insights. Indeed, for example the circle forms a group of rotations with operation:

$$(x_1, y_1) \times (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \tag{5}$$

This is the cosine and sign of the sum of two angles, indeed if $(x_1, y_1) = (\cos\theta_1, \sin\theta_1)$ and $(x_2, y_2) = (\cos\theta_2, \sin\theta_2)$ then:

$$(\cos\theta_1, \sin\theta_1) \times (\cos\theta_2, \sin\theta_2) = (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2, \dots) = (\cos(\theta_1 + \theta_2), \sin(\theta_1 + \theta_2)) \tag{6}$$

This is the simplest example of what is called an Algebraic group.

Definition 1.2. Algebraic Groups We can think of this as functor from (commutative) Rings to Groups.

$$G \colon R \to (\{(x, y) \in R^2 \mid x^2 + y^2 = 1\}, \times) \tag{7}$$

Where the operation is defined as above, and the identity is $(1, 0)$ and $(x, y)^{-1} = (x, -y)$.

Example 1.2.1. $G(\mathbb{C}) = \{(x, y) \in \mathbb{C} \mid x^2 + y^2 = 1\}$
But note that $1 = x^2 + y^2 = \underbrace{(x + iy)}_{z}\underbrace{(x - iy)}_{\overline{z}}$. So we see that

$$G(\mathbb{C}) = \{(x, y) \in \mathbb{C} \mid x^2 + y^2 = 1\} \simeq \{z \in \mathbb{C} \mid z \text{ is invertible}\} = \mathbb{C}^* \tag{8}$$

Summary There are many ways to view a circle:

1. Subset of $\mathbb{R}^2$
2. Polynomial $x^2 + y^2 - 1 \to$ Algebraic set
3. Ideal $(x^2 + y^2 - 1)$ in ring $\mathbb{R}[x, y]$.
4. Ring $\mathbb{R}[x, y]/(x^2 + y^2 - 1)$ = coordinate ring of S^1. Can be seen as the set of polynomials on the circle.
5. (Smooth) manifold
6. Group (Algebraic Group)
7. Functor from Rings to Groups or Sets (Grothendieck)

2 Two cubic curves

In this section we will discuss some cubic cubes.

1. $y^2 = x^3 + x^2$

   There is almost a 1-1 correspondence between (x, y) rational on this curve and $t \in \mathbb{Q}$, via $t = \frac{y}{x}$, the slope of the line through (x, y) and the origin. Indeed since $y = tx$, we have(if $x \neq 0$):

$$t^2x^2 = x^3 + x^2 \Rightarrow t^2 = 1 + x \Rightarrow x = t^2 - 1 \text{ and } y = t^3 - t \tag{1}$$

We don't quite get a 1-1 correspondence because $t = 1$ and $t = -1$ both correspond to $(x, y) = (0, 0)$.

So we can think of this cubic curve as a copy of $\mathbb{Q}$, but two of these points are mapped to the same point.

Definition 2.1. Resolution of Singluarity A singularity is a "bad" point of our curve, and a resolution is getting a "nice" map from a curve without singularities to our curve.

The resolution in the above case is done by a process called "blowing-up".

Remark. Hironaka, showed that blowing-up resolves singularites in zero characteristic. (The problem in non-zero characteristic is still unsolved).

Remark. Finding rational points on curves can be difficult. For example:

$$x^n + y^n = 1 \Rightarrow X^n + Y^n = Z^n \text{ where } x = X/Z \text{ and } y = Y/Z \tag{2}$$

This is Fermat's Last Theorem, which was very hard to solve.

$x^3 + y^3 = 9$

Note on this curve we can define an algebraic operation "+", if we add in a point at infinity. In that case, the point at infinity is the identity "0", and a, b, c on the curve lie on a line if and only if $a + b + c = 0$ in the group. To check that the group operation is associative we use the fact that: $a_1 + a_2 + \cdots = b_1 + b_2 + \cdots \iff$ there is a rational function with poles at a_i and zeroes at the b_i.

Definition 2.2. Groups of this kind are called **elliptic curves**, there are the 1-dimensional case of what is called **Abelian varieties**. Abelian varieties are algebraic groups that are "projective", roughly they have no missing points.