Hartshorne solutions, Chapter 1!

Notation Recall:

- $A = k[x_1, \ldots, x_n]$
- If $T \subseteq A$ then $Z(T) = \{P \in \mathbb{A}^n \mid f(P) \text{ for all } f \in T\}$
- If $Y \subseteq \mathbb{A}^n$ then $I(Y) = \{f \in A \mid f(P) = 0 \text{ for all } P \in Y\}$
- If $Y \subseteq \mathbb{A}^n$ is an affine algebraic set, then $A(Y) = A/I(Y)$ is the affine coordinate ring.

Part 1: Affine Varieties **Exercise 1**

(a) Let Y be the plane curve $y = x^2$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

Proof. We have

$$A(Y) = k[x,y]/(y - x^2)$$

Let

$$\varphi\colon k[x,y] \to k[t]$$

Be the homomorphism given by:

$$\begin{cases} \varphi(x) = t \\ \varphi(y) = t^2 \varphi(a) = a \text{ for all } a \in k \end{cases}$$

It is clear that this homomorphism is surjective. Now let $f \in \ker\varphi$, so we have $f(x) = \sum_{i=1}^n f_i(x) y^i$, where $f_i \in k[x]$. Notice that

$$f(x, x^2) = \sum_{i=1}^n f_i(x) x^{2i} = \sum_{i=1}^n f_i(t) t^{2i} = \varphi(f) = 0$$

(Note we can identity $k[x]$ with $k[t]$ trivially)

So we see that $f(x,y)$ when looked at as a polynomial in $k[x][y]$ has a root at $y = x^2$. So we have $f \in (y - x^2)$. So $\ker\varphi \subseteq (y - x^2)$, the other inclusion is obvious. So we see that $\ker\varphi = (y - x^2)$. By an isomorphim theorem:

$$k[x,y]/(y - x^2) \simeq k[t]$$

□

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

Proof. Let

$$\psi\colon k[x,y] \to k[t, t^{-1}]$$

Be a homomorphism given by

$$\begin{cases} \psi(x) = t \\ \psi(y) = t^{-1} \\ \psi(a) = a \text{ for } a \in k \end{cases}$$

This function is surjective since any $f \in k[t, t^{-1}]$ is of the form $f(t) = a_0 + \sum_{i=1}^n a_i t^{-n} + \sum_{i=1}^m b_i t^m$. So we take $\psi(a_0 + \sum_{i=1}^n a_i y^n + \sum_{i=1}^m b_i x^m)$.

First of all it is clear that $(xy - 1) \subseteq \ker\psi$, so let $f \in \ker\psi$. □

(c) Let f be any irreducible quadratic polynomial in $k[x,y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?