Exercise 1 Show that every group of order ≤ 5 is abelian.

Proof. It is clear that a group of order 1 is abelian. Any group of prime order is cyclic, so we only need to check that all groups of order 4 are abelian.

Let G be a group of order 4, and $x \in G$ with $x \neq e$. So we have

$$\operatorname{ord}(x) = \begin{cases} 2 \\ 4 \end{cases}$$

Indeed since $1 \neq \operatorname{ord}(x) \mid 4$.

If $\operatorname{ord}(x) = 4$, then $\{e.x.x^2.x^3\} \leq G \Rightarrow G = \langle x \rangle$, so it is abelian.

If G has no elements of order 4, then for all $x \in G$ we have $x^2 = e \Rightarrow x = x^{-1}$, so for all $x, y \in G$ we have

$$\begin{aligned}(xy)(x^{-1}y^{-1}) &= (xy)(xy) \\ &= (xy)^2 \\ &= e\end{aligned}$$

Therefore $xy = yx$ for all $x, y \in G$. So G is abelian.

In all cases we have shown that if the order of $G \leq 5$, we have that G is abelian. □

Exercise 2 Show that there are two-isomorphic groups of order 4, namely the cyclic one, and the product of two cyclic groups of order 2.

Proof. Let G be a group of order 4, assume that it is not cyclic. In this case, from last question we know that $x^2 = e$ for all $x \in G$, so $\{e, x\} = \langle x \rangle \leq G$ let $y \in G \setminus \langle x \rangle$.

So notice that $xy \notin \{e, x, y\}$ indeed since $x, y \neq e$ and $x \neq y$. So we see by comparing order $G = \{e, x, y, xy\}$.

Defining the homomorphism

$$\begin{aligned}\varphi\colon G &\to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ x &\to (1,0) \\ y &\to (0,1)\end{aligned}$$

Since $\varphi(xy) = \varphi(x) + \varphi(y) = (1,1)$, by inspection we can see that $\ker\varphi = \{e\}$ and $\operatorname{im}\varphi = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, so:

$$G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

□

Exercise 3 Let G be a group. A **commutator** in G is an element of the form $aba^{-1}b^{-1}$ with $a, b \in G$. Let G^c be the subgroup generated by the commutators. Then G^c is called the **commutator subgroup**. Show that G^c is normal. Show that any homomorphism of G into an abelian group factors through G/G^c.

Proof. Since $(aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1}$, the set of elements containing all finite products of commutators is a group. Since any subgroup containing all commutators contains this subgroup we see that $G^c = \{x_1 x_2 \cdots x_n \mid n \in \mathbb{N} \text{ and } x_i \text{ are commutators}\}$.

Now let $g \in G$ and $aba{-}1b^{-1}$ be a commutator we see that:

$$g(aba{-}1b^{-1})g^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) = zwz^{-1}w^{-1}$$

Where $z = gag^{-1}$ and $w = gbg^{-1}$.

So we see that for any $g \in G$ and $a \in G^c$, we have:

$$\begin{aligned}gag^{-1} &= g(x_1 x_2 \cdots x_n)g^{-1} \text{ for commutators } x_i \\ &= (gx_1g^{-1})(gx_2g^{-1}) \cdots (gx_ng^{-1}) \\ &\in G^c \text{ since by above observation } gx_ig^{-1} \text{ is a commutator for all } x_i\end{aligned}$$

So $G^c \trianglelefteq G$.

Now let A be an abelian group and $\varphi\colon G \to A$ be a homomorphism. First of all we will show that φ contains G^c in it's kernel.

$$\begin{aligned}\varphi(aba^{-1}b^{-1}) &= \varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1} \\ &= e \text{ by commutating elements}\end{aligned}$$

Therefore we see that for all $x \in G^c$, let $x = x_1 \cdots x_n$ where x_i are commutators:

$$\varphi(x) = \varphi(x_1)\varphi(x_2)\cdots\varphi(x_n) = e \text{ since each } \varphi(x_i) = e \tag{1}$$

So we indeed see that $G^c \leq \ker\varphi$. So now let $\pi\colon G \to G/G^c$ be the canonical map and let $\tilde{\varphi}\colon G/G^c \to A$ be the homomorphism given by:

$$\tilde{\varphi}(xG^c) = \varphi(x)$$

Note we know that this is a homomorphism since φ is a homomorphism.

Since $G^c \leq \ker\varphi$ if $xG^c = yG^c$ we have $xy^{-1} \in G^c$ so we have $\varphi(xy^{=1}) = e \Rightarrow \varphi(x) = \varphi(y)$ so $\tilde{\varphi}(xG^c) = \tilde{\varphi}(yG^c)$, this homomorphism is indeed well-defined.

So we indeed see that there is a homomorphism $\tilde{\varphi}$ such that $\varphi = \tilde{\varphi} \circ \pi$. So φ factors through G^c. □

Exercise 4 Let $H.K$ be subgroups of a finite group G with $K \subseteq N_H$. Show that:

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

Proof. Since K is contained in the normalizer of H. Recall by an isomorphism theorem:

$$K/(H \cap K) \simeq HK/H$$

So we have:

$$\frac{|K|}{|H \cap K|} = \frac{|HK|}{|H|} \Rightarrow |HK| = \frac{|H||K|}{|H \cap K|}$$

□

Exercise 5

Goursat's Lemma. Let G, G' be groups and let H be a subgroup of $G \times G'$ such that the projections $p_1\colon H \to G$ and $p_2\colon H \to G'$ are surjective. Let N be the kernel of p_2 and N' be the kernel of p_1. One can identify N as a normal subgroup of G, and N' as a normal subgroup of G'. Show that the image of H in $G/N \times G'/N'$ is the graph of an isomorphism

$$G/N \simeq G'/N'$$

Proof. First of all notice that

$$\ker p_1 = \{(e,b) \in H\} \simeq N' = \{b \in G' \mid (e,b) \in H\} \text{ and } \ker p_2 = \{(a,e') \in H\} \simeq N = \{a \in Ga \mid (a,e') \in H\}$$

Let

$$\varphi_1\colon G \to G/N \text{ and } \varphi_2\colon G' \to G'/N'$$

Be the canonical maps.

Let $\varphi\colon H \to G/N \times G'/N'$ be given by

$$\varphi((g_1,g_2)) = (\varphi_1(g_1), \varphi_2(g_2))$$

This is a homomorphism since φ_1 and φ_2 are homomorphisms.

Lemma 1. *If $(xN, x'N'), (yN, y'N') \in \varphi(H)$ then $xN = yN \iff x'N = y'N$.*

Proof. First assume that $xN = yN$:

We have: $(xy^{-1}N, x'y'^{-1}N') = (N, x'y'^{-1}N') \in \varphi(H)$. So let $(a,b) \in H$ such that:

$$(aN, bN') = \varphi(a,b) = (N, x'y'^{-1}N')$$

So we see that $aN = N \Rightarrow a \in N \simeq \ker p_2$. This means that $(a,e') \in H$, so we see that $(e,b) = (a,e')^{-1}(a,b) \in H$, so $b \in N'$. Therefore $N' = bN = x'y'^{-1}N'$ so $x'N' = y'N'$.

The other direction is similar. □

Now we let

$$\psi\colon G/N \to G'/N' \text{ be such that } (aN, \psi(aN)) \in \varphi(H) \text{ for all } aN \in G/N$$

We will first show that this function makes sense, note that since the projection from H to G for all xN, we see that $(x,y) \in H$ for some y. So $\varphi(x,y) = (xN, yN') \in \varphi(H)$ so xN is in the projection off $\varphi(H)$ to G/N. So we see that the projection is surjective so: for all $aN \in G/N$ there exists a $bN' \in G'/N'$ such that $(aN, bN') \in \varphi(H)$. Furthermore by lemma 1 this bN' is unique. Since this bN' exists and is unique then we can let $\psi(aN) = bN'$ and this function is well-defined.

Now let $aN, cN \in G/N$ since $(aN, \psi(aN)), (cN, \psi(xN)) \in \varphi(H)$ so:

$$H \ni (aN, \psi(aN))(cN, \psi(xN)) = (acN, \psi(aN)\psi(cN)) \Rightarrow \psi(aNcN) = \psi(acN) = \psi(aN)\psi(cN)$$

So ψ is indeed a homomorphism. Finallly from lemma 1 we see that $\psi(aN) = \psi(bN)$ implies that $(aN, \psi(aN)), (bN, \psi(aN)) \in \varphi(H)$ so $aN = bN$. So this function is indeed an isomorphism. $\square$

Exercise 6 Prove that the group of inner automorphishms of a group G is normal in $\mathrm{Aut}(G)$.

Proof. For all $g \in G$ we let φ_g be the homomorphism such that

$$\varphi_g(x) = gxg^{-1}$$

Recall that an inner automorphishms is an automorphishms of the form φ_g for some $g \in G$. Now let: $I = \{\varphi_g \mid g \in G\}$. Notice that

$$\forall x \in G,\ \varphi_a \circ \varphi_b(x) = abxb^{-1}a^{-1} = (ab)x(ab)^{-1} = \varphi_{ab}(x) \Rightarrow \varphi_a \circ \varphi_b = \varphi_{ab} \in I$$

Likewise

$$\forall x \in G,\ \varphi_{a^{-1}} \circ \varphi_a(x) = a^{-1}axa^{-1}a = x \Rightarrow {\varphi_a}^{-1} = \varphi_{a^{-1}} \in I$$

Let $f \in \mathrm{Aut}(G)$, let $\varphi_g \in I$, for all $x \in G$:

$$\begin{aligned} f \circ \varphi_g \circ f^{-1}(x) &= f(gf^{-1}(x)g^{-1}) \\ &= f(g)xf(g^{-1}) \text{ since } f \text{ is a homomorphism} \\ &= f(g)xf(g)^{-1} \\ &= \varphi_{f(g)}(x) \end{aligned}$$

Since this is true for all x then we have $f \circ \varphi_g \circ f^{-1} \in I$. Since this is true for all φ_g we have $fIf^{-1} \subseteq I$, for all $f \in \mathrm{Aut}(G)$. So $I \trianglelefteq \mathrm{Aut}(G)$.

$\square$

Exercise 7 Let G be a group such that $\mathrm{Aut}(G)$ is cyclic. Prove that G is abelian.

Proof. Let N be the inner automorphishms group, since it is a subgroup of $\mathrm{Aut}(G)$ it is cylic. Now we define:

$$\begin{gathered} \varphi\colon G|rightarrowN \\ \varphi(g) \to \varphi_g \end{gathered}$$

Where φ_g is defined as in exersise 6. Let $Z(G) = \{z \in G \mid zg = gz\ \forall g \in G\}$, it is clear that $Z(G) \subseteq \ker\varphi$. Furthermore if $g \in \ker\varphi$ we have:

$$\forall x \in G\ x = \mathrm{id}(x) = \varphi_g(x) = gxg^{-1}\ \therefore gx = xg \Rightarrow g \in Z(G)$$

So we see that $\ker\varphi = Z(G)$, so we have

$$G/Z(G) \simeq N$$

Since N is cyclic so is $G/Z(G)$, so let $gZ(G)$ be a generator. Let $x.y \in G$ we have $x = g^m z$, $y = g^n z'$ for some $n, m \in \mathbb{Z}$ and $z, z' \in Z(G)$. We have:

$$\begin{aligned} xy &= g^m z g^n z' \\ &= g^m g^n z z' \\ &= g^n g^m z' z \\ &= g^n z' g^m z \\ &= yx \end{aligned}$$

Since x, y are arbitrary we see that G is indeed abelian. $\square$

Exercise 8 Let G be a group and let H, H' be subgroups. By a **double coset** of H, H' one means a subset of G of the form HxH'.

(a) Show that G is a disjoint union of double cosets.

(b) Let $\{c\}$ be a family of representatives for the double cosets. For each $a \in G$ denote by $[a]H'$ the conjugate $aH'a^{-1}$ of H'. For each c we have a decomposition into ordinary cosets

$$H = \bigcup_{x_c} x_c(H \cap [c]H')$$

where $\{x_c\}$ is a family of elements of H, depending on c. Show that the elements $\{x_c c\}$ form a family of left coset representatives for H' in G; that is,

$$G = \bigcup_{c} \bigcup_{x_c} x_c c H',$$

and the union is disjoint.

Proof. (a) First of all assume that $z \in HxH' \cap HyH'$ then let $h_1, h_2 \in H$ and $h_1', h_2' \in H'$ such that:

$$h_1 x h_1' = z = h_2 y h_2' \Rightarrow y = h_2{}^{-1} h_1 x h_1' h_2'^{-1} \Rightarrow HyH' = Hh_2{}^{-1} h_1 x h_1' h_2'^{-1} H' = HxH'$$

For any $y, x \in G$ either HxH' and HyH' are disjoint or they are equal this fact combined with the fact that for all $x \in G$ we have $x \in HxH'$ tells us that we can write G as a disjoint union of double cosets.

(b) By our assumptions we have the disjoint unions:

$$\begin{aligned} G &= \bigcup_{c} HcH' \\ &= \bigcup_{c} \bigcup_{x_c} x_c(H \cap [c]H')cH' \end{aligned}$$

But notice that for $\alpha \in x_c(H \cap [c]H')cH'$ we have:

$$\alpha = x_c c h' c^{-1} c h'' = x_c c h' h'' \in x_c c H' \text{ for some } h', h'' \in H'$$

So we see that $x_c(H \cap [c]H')cH' \subseteq x_c c H'$, the other inclusion is clear since $e \in (H \cap [c]H')$. So we have:

$$G = \bigcup_{c} \bigcup_{x_c} x_c c H' \text{ and this union is disjoint}$$

□

Exercise 9

(a) Let G be a group and H a subroup of finite index. Show that there exists a normal subgroup N of G contained in H and also of finite index.

(b) Let G be a group and let H_1, H_2 be subgroups of finite index. Prove that $H_1 \cap H_2$ has finite index.

Proof. (a) Assume that $[G\colon H] = n$ and let $\{a_1H, a_2H, \ldots, a_nH\}$ be the distinct cosets of H in G. Let $a \in G$ since we know that $aa_iH \in \{a_1H, a_2H, \ldots, a_nH\}$, so let $\sigma_a \in S_n$ be such that $aa_iH = a_{\sigma_a(i)}H$ for all i. We define

$$\begin{aligned} \varphi\colon G &\to S_n \\ a &\to \sigma_a \end{aligned}$$

Let $x \in \ker \varphi$, this means that $xa_iH = a_{e(i)}H = a_iH$ for all i. In particular we know that for one i_0 we have $a_{i_0}H = H$, so we have

$$H = a_{i_0}H = xa_{i_0}H = xH$$

This means that $x \in H$. Therefore $\ker \varphi \subseteq H$. Letting $N = \ker \varphi$, we see that this is a normal subgroup conatined in H.

Now finally note that $\text{im } \varphi \le S_n$, so we see by an isomorphism theorem:

$$[G\colon N] = |G/N| = |\text{im } \varphi| \le |S_n| = n! < \infty$$

(b) First of all, let $x, y \in G$ be such that $xH_1 = yH_1$ and $xH_2 = yH_2$, then we have $y^{-1}x \in H_1$ and $y^{-1}x \in H_2$ so $y^{-1}x \in H_1 \cap H_2$ so $x(H_1 \cap H_2) = y(H_1 \cap H_2)$

Conversely assume that $x(H_1 \cap H_2) = y(H_1 \cap H_2)$, then we have that $y^{-1}x \in H_1 \cap H_2$ so $y^{-1}x \in H_1$ and $y^{-1}x \in H_2$.

So we have shown that $x(H_1 \cap H_2) = y(H_1 \cap H_2)$ if and only if $xH_1 = yH_1$ ans $xH_2 = yH_2$.

Now finally we let, C_i be the set of distinct reperesentatives of H_i and C be the set of distinct reperesentatives of $H_1 \cap H_2$

$$\begin{array}{cccc} f\colon & C & \to & C_1 \times C_2 \\ & c(H_1 \cap H_2) & \to & (cH_1, cH_2) \end{array}$$

Since we $x(H_1 \cap H_2) = y(H_1 \cap H_2)$ if and only if $(xH_1, xH_2) = (yH_1, yH_2)$ see that this function is well-defined and injective. Therefore from set theory:

$$[G\colon H_1 \cap H_2] = |C| \leq |C_1||C_2| = [G\colon H_1][G\colon H_2] < \infty$$

□

Exercise 10 Let G be a group and let H be a subgroup of finite index. Prove that there is only a finite number of right cosets of H, and that the number of right cosets is equal to the number of left cosets.

Proof. Recall, that since H has finite index, There are only a finite number of left cosets of H in G.

$$\begin{aligned} ah = b &\iff hb^{-1} = a^{-1} \\ \therefore\ b \in aH &\iff a^{-1} \in Hb^{-1} \end{aligned}$$

From this we see that $aH = bH \iff Ha^{-1} = Hb^{-1}$.

Let $H_L = \{aH \mid a \in G\}$ the set of left cosets, and let $H_R = \{Ha \mid a \in G\}$ the set of right cosets.

We define a set map:

$$f\colon H_L \to H_R$$

Given by $f(aH) = Ha^{-1}$. Now first we will show that this function is well-defined, let $aH = bH$ this implies from above that $Ha^{-1} = Hb^{-1}$:

$$f(aH) = Ha^{-1} = Hb^{-1} = f(bH)$$

So this function is indeed well-definied. Now this is also a bijection we notice that the inverse function is given by the map:

$$g\colon H_R \to H_L \text{ by } g(Ha) = a^{-1}H$$

This function is similarly seen to be well-defined since $aH = bH \iff Ha^{-1} = Hb^{-1}$.

So we see that $[G\colon H] = |H_L| = |H_R|$. Since $[G\colon H] < \infty$, there is only a finite number of right cosets and there as a many right as left cosets. □

Exercise 11 Let G be a group, and A a normal abelian subgroup. Show that G/A operates on A by conjugation; and in this manner get a homomorphism of G/A into $\mathrm{Aut}(A)$.

Proof. We will first show that this action is well-defined: Assume that $xA = yA$ so let $a \in A$ such that $y = xa$ and let $s \in A$. So we have:

$$\begin{aligned} xA \cdot s &= xsx^{-1} \\ &= xaa^{-1}sx^{-1} \\ &= (xa)s(a^{-1}x^{-1}) \text{ since } a, s \in A \text{ and } A \text{ is abelian.} \\ &= (xa)s(xa)^{-1} \\ &= ysy^{-1} \\ &= yA \cdot s \end{aligned}$$

So this function is indeed well-defined. Now we will show that this function is a group action:

Let $xA, yA \in G/A$ and $s \in A$ we have

$$xA \cdot (yA \cdot s) = xA \cdot (ysy^{-1}) = xysy^{-1}x^{-1} = (xy)s(xy)^{-1} = (xyA) \cdot s$$

For all $s \in S$

$$eA \cdot s = ese^{-1} = s$$

So this indeed a group action.
Now let

$$\varphi\colon G/A \to \mathrm{Aut}(A)$$

Be such that for all $s \in A$:

$$\begin{aligned} \varphi(xA)(s) &= xA \cdot s \\ &= xsx^{-1} \end{aligned}$$

Note that it is clear that $\varphi(xA)$ is an automorphishms, since it is an inner-homomorphism.
Finallly it is clear that φ is a homomorphism, since the map $xA \cdot s = xsx^{-1}$ is a group action, so

$$\varphi(xyA)(s) = (xyA) \cdot s = xA \cdot (yA \cdot s) = \varphi(xA)\varphi(yA)(s) \Rightarrow \varphi(xy) = \varphi(x)\varphi(y)$$

And likewise from above we see that $\varphi(A) = \mathrm{id}$. $\square$