

**Part 1 Exercise 1** Show that every group of order  $\leq 5$  is abelian.

*Proof.* It is clear that a group of order 1 is abelian. Any group of prime order is cyclic, so we only need to check that all groups of order 4 are abelian.

Let  $G$  be a group of order 4, and  $x \in G$  with  $x \neq e$ . So we have

$$\text{ord}(x) = \begin{cases} 2 \\ 4 \end{cases}$$

Indeed since  $1 \neq \text{ord}(x) \mid 4$ .

If  $\text{ord}(x) = 4$ , then  $\{e, x, x^2, x^3\} \leq G \Rightarrow G = \langle x \rangle$ , so it is abelian.

If  $G$  has no elements of order 4, then for all  $x \in G$  we have  $x^2 = e \Rightarrow x = x^{-1}$ , so for all  $x, y \in G$  we have

$$\begin{aligned} (xy)(x^{-1}y^{-1}) &= (xy)(xy) \\ &= (xy)^2 \\ &= e \end{aligned}$$

Therefore  $xy = yx$  for all  $x, y \in G$ . So  $G$  is abelian.

In all cases we have shown that if the order of  $G \leq 5$ , we have that  $G$  is abelian.  $\square$

**Exercise 2** Show that there are two-isomorphic groups of order 4, namely the cyclic one, and the product of two cyclic groups of order 2.

*Proof.* Let  $G$  be a group of order 4, assume that it is not cyclic. In this case, from last question we know that  $x^2 = e$  for all  $x \in G$ , so  $\{e, x\} = \langle x \rangle \leq G$  let  $y \in G \setminus \langle x \rangle$ .

So notice that  $xy \notin \{e, x, y\}$  indeed since  $x, y \neq e$  and  $x \neq y$ . So we see by comparing order  $G = \{e, x, y, xy\}$ .

Defining the homomorphism

$$\begin{aligned} \varphi: G &\rightarrow \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\ x &\rightarrow (1, 0) \\ y &\rightarrow (0, 1) \end{aligned}$$

Since  $\varphi(xy) = \varphi(x) + \varphi(y) = (1, 1)$ , by inspection we can see that  $\ker \varphi = \{e\}$  and  $\text{im } \varphi = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ , so:

$$G \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

$\square$

**Exercise 3** Let  $G$  be a group. A **commutator** in  $G$  is an element of the form  $aba^{-1}b^{-1}$  with  $a, b \in G$ . Let  $G^c$  be the subgroup generated by the commutators. Then  $G^c$  is called the **commutator subgroup**. Show that  $G^c$  is normal. Show that any homomorphism of  $G$  into an abelian group factors through  $G/G^c$ .

*Proof.* Since  $(aba^{-1}b^{-1})^{-1} = bab^{-1}a^{-1}$ , the set of elements containing all finite products of commutators is a group. Since any subgroup containing all commutators contains this subgroup we see that  $G^c = \{x_1x_2 \cdots x_n \mid n \in \mathbb{N} \text{ and } x_i \text{ are commutators}\}$ .

Now let  $g \in G$  and  $aba^{-1}b^{-1}$  be a commutator we see that:

$$g(aba^{-1}b^{-1})g^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) = zwz^{-1}w^{-1}$$

Where  $z = gag^{-1}$  and  $w = gbg^{-1}$ .

So we see that for any  $g \in G$  and  $a \in G^c$ , we have:

$$\begin{aligned} gag^{-1} &= g(x_1x_2 \cdots x_n)g^{-1} \text{ for commutators } x_i \\ &= (gx_1g^{-1})(gx_2g^{-1}) \cdots (gx_ng^{-1}) \\ &\in G^c \text{ since by above observation } gx_ig^{-1} \text{ is a commutator for all } x_i \end{aligned}$$

So  $G^c \trianglelefteq G$ .

Now let  $A$  be an abelian group and  $\varphi: G \rightarrow A$  be a homomorphism. First of all we will show that  $\varphi$  contains  $G^c$  in its kernel.

$$\begin{aligned} \varphi(aba^{-1}b^{-1}) &= \varphi(a)\varphi(b)\varphi(a)^{-1}\varphi(b)^{-1} \\ &= e \text{ by commuting elements} \end{aligned}$$

Therefore we see that for all  $x \in G^c$ , let  $x = x_1 \cdots x_n$  where  $x_i$  are commutators:

$$\varphi(x) = \varphi(x_1)\varphi(x_2) \cdots \varphi(x_n) = e \text{ since each } \varphi(x_i) = e \quad (1)$$

So we indeed see that  $G^c \leq \ker \varphi$ . So now let  $\pi: G \rightarrow G/G^c$  be the canonical map and let  $\tilde{\varphi}: G/G^c \rightarrow A$  be the homomorphism given by:

$$\tilde{\varphi}(xG^c) = \varphi(x)$$

Note we know that this is a homomorphism since  $\varphi$  is a homomorphism.

Since  $G^c \leq \ker \varphi$  if  $xG^c = yG^c$  we have  $xy^{-1} \in G^c$  so we have  $\varphi(xy^{-1}) = e \Rightarrow \varphi(x) = \varphi(y)$  so  $\tilde{\varphi}(xG^c) = \tilde{\varphi}(yG^c)$ , this homomorphism is indeed well-defined.

So we indeed see that there is a homomorphism  $\tilde{\varphi}$  such that  $\varphi = \tilde{\varphi} \circ \pi$ . So  $\varphi$  factors through  $G^c$ .  $\square$

**Exercise 4** Let  $H, K$  be subgroups of a finite group  $G$  with  $K \subseteq N_H$ . Show that:

$$|HK| = \frac{|H||K|}{|H \cap K|}$$

*Proof.* Since  $K$  is contained in the normalizer of  $H$ . Recall by an isomorphism theorem:

$$K/(H \cap K) \simeq HK/H$$

So we have:

$$\frac{|K|}{|H \cap K|} = \frac{|HK|}{|H|} \Rightarrow |HK| = \frac{|H||K|}{|H \cap K|}$$

$\square$

### Exercise 5

**Goursat's Lemma.** Let  $G, G'$  be groups and let  $H$  be a subgroup of  $G \times G'$  such that the projections  $p_1: H \rightarrow G$  and  $p_2: H \rightarrow G'$  are surjective. Let  $N$  be the kernel of  $p_2$  and  $N'$  be the kernel of  $p_1$ . One can identify  $N$  as a normal subgroup of  $G$ , and  $N'$  as a normal subgroup of  $G'$ . Show that the image of  $H$  in  $G/N \times G'/N'$  is the graph of an isomorphism

$$G/N \simeq G'/N'$$

*Proof.* First of all notice that

$$\ker p_1 = \{(e, b) \in H\} \simeq N' = \{b \in G' \mid (e, b) \in H\} \text{ and } \ker p_2 = \{(a, e') \in H\} \simeq N = \{a \in G \mid (a, e') \in H\}$$

Let

$$\varphi_1: G \rightarrow G/N \text{ and } \varphi_2: G' \rightarrow G'/N'$$

Be the canonical maps.

Let  $\varphi: H \rightarrow G/N \times G'/N'$  be given by

$$\varphi((g_1, g_2)) = (\varphi_1(g_1), \varphi_2(g_2))$$

This is a homomorphism since  $\varphi_1$  and  $\varphi_2$  are homomorphisms.

**Lemma 1.** If  $(xN, x'N') \in \varphi(H)$  then  $xN = yN \iff x'N = y'N$ .

*Proof.* First assume that  $xN = yN$ :

We have:  $(xy^{-1}N, x'y'^{-1}N') = (N, x'y'^{-1}N') \in \varphi(H)$ . So let  $(a, b) \in H$  such that:

$$(aN, bN') = \varphi(a, b) = (N, x'y'^{-1}N')$$

So we see that  $aN = N \Rightarrow a \in N \simeq \ker p_2$ . This means that  $(a, e') \in H$ , so we see that  $(e, b) = (a, e')^{-1}(a, b) \in H$ , so  $b \in N'$ . Therefore  $N' = bN = x'y'^{-1}N'$  so  $x'N' = y'N'$ .

The other direction is similar.  $\square$

Now we let

$$\psi: G/N \rightarrow G'/N' \text{ be such that } (aN, \psi(aN)) \in \varphi(H) \text{ for all } aN \in G/N$$

We will first show that this function makes sense, note that since the projection from  $H$  to  $G$  for all  $xN$ , we see that  $(x, y) \in H$  for some  $y$ . So  $\varphi(x, y) = (xN, yN') \in \varphi(H)$  so  $xN$  is in the projection off  $\varphi(H)$  to  $G/N$ . So we see that the projection is surjective so: for all  $aN \in G/N$  there exists a  $bN' \in G'/N'$  such that  $(aN, bN') \in \varphi(H)$ . Furthermore by lemma 1 this  $bN'$  is unique. Since this  $bN'$  exists and is unique then we can let  $\psi(aN) = bN'$  and this function is well-defined.

Now let  $aN, cN \in G/N$  since  $(aN, \psi(aN)), (cN, \psi(cN)) \in \varphi(H)$  so:

$$H \ni (aN, \psi(aN))(cN, \psi(cN)) = (acN, \psi(aN)\psi(cN)) \Rightarrow \psi(aNcN) = \psi(acN) = \psi(aN)\psi(cN)$$

So  $\psi$  is indeed a homomorphism. Finally from lemma 1 we see that  $\psi(aN) = \psi(bN)$  implies that  $(aN, \psi(aN)), (bN, \psi(aN)) \in \varphi(H)$  so  $aN = bN$ . So this function is indeed an isomorphism.  $\square$

**Exercise 6** Prove that the group of inner automorphisms of a group  $G$  is normal in  $\text{Aut}(G)$ .

*Proof.* For all  $g \in G$  we let  $\varphi_g$  be the homomorphism such that

$$\varphi_g(x) = gxg^{-1}$$

Recall that an inner automorphism is an automorphism of the form  $\varphi_g$  for some  $g \in G$ . Now let:  $I = \{\varphi_g \mid g \in G\}$ . Notice that

$$\forall x \in G, \varphi_a \circ \varphi_b(x) = abxb^{-1}a^{-1} = (ab)x(ab)^{-1} = \varphi_{ab}(x) \Rightarrow \varphi_a \circ \varphi_b = \varphi_{ab} \in I$$

Likewise

$$\forall x \in G, \varphi_{a^{-1}} \circ \varphi_a(x) = a^{-1}axa^{-1}a = x \Rightarrow \varphi_a^{-1} = \varphi_{a^{-1}} \in I$$

Let  $f \in \text{Aut}(G)$ , let  $\varphi_g \in I$ , for all  $x \in G$ :

$$\begin{aligned} f \circ \varphi_g \circ f^{-1}(x) &= f(gf^{-1}(x)g^{-1}) \\ &= f(g)xf(g^{-1}) \text{ since } f \text{ is a homomorphism} \\ &= f(g)xf(g)^{-1} \\ &= \varphi_{f(g)}(x) \end{aligned}$$

Since this is true for all  $x$  then we have  $f \circ \varphi_g \circ f^{-1} \in I$ . Since this is true for all  $\varphi_g$  we have  $fIf^{-1} \subseteq I$ , for all  $f \in \text{Aut}(G)$ . So  $I \trianglelefteq \text{Aut}(G)$ .  $\square$

**Exercise 7** Let  $G$  be a group such that  $\text{Aut}(G)$  is cyclic. Prove that  $G$  is abelian.

*Proof.* Let  $N$  be the inner automorphisms group, since it is a subgroup of  $\text{Aut}(G)$  it is cyclic. Now we define:

$$\begin{aligned} \varphi: G &\rightarrow N \\ \varphi(g) &\rightarrow \varphi_g \end{aligned}$$

Where  $\varphi_g$  is defined as in exercise 6. Let  $Z(G) = \{z \in G \mid zg = gz \forall g \in G\}$ , it is clear that  $Z(G) \subseteq \ker \varphi$ . Furthermore if  $g \in \ker \varphi$  we have:

$$\forall x \in G \ x = \text{id}(x) = \varphi_g(x) = gxg^{-1} \therefore gx = xg \Rightarrow g \in Z(G)$$

So we see that  $\ker \varphi = Z(G)$ , so we have

$$G/Z(G) \simeq N$$

Since  $N$  is cyclic so is  $G/Z(G)$ , so let  $gZ(G)$  be a generator. Let  $x, y \in G$  we have  $x = g^m z$ ,  $y = g^n z'$  for some  $n, m \in \mathbb{Z}$  and  $z, z' \in Z(G)$ . We have:

$$\begin{aligned} xy &= g^m z g^n z' \\ &= g^m g^n z z' \\ &= g^n g^m z' z \\ &= g^n z' g^m z \\ &= yx \end{aligned}$$

Since  $x, y$  are arbitrary we see that  $G$  is indeed abelian.  $\square$

**Exercise 8** Let  $G$  be a group and let  $H, H'$  be subgroups. By a **double coset** of  $H, H'$  one means a subset of  $G$  of the form  $HxH'$ .

(a) Show that  $G$  is a disjoint union of double cosets.

(b) Let  $\{c\}$  be a family of representatives for the double cosets. For each  $a \in G$  denote by  $[a]H'$  the conjugate  $aH'a^{-1}$  of  $H'$ . For each  $c$  we have a decomposition into ordinary cosets

$$H = \bigcup_{x_c} x_c(H \cap [c]H')$$

where  $\{x_c\}$  is a family of elements of  $H$ , depending on  $c$ . Show that the elements  $\{x_c c\}$  form a family of left coset representatives for  $H'$  in  $G$ ; that is,

$$G = \bigcup_c \bigcup_{x_c} x_c c H',$$

and the union is disjoint.

*Proof.* (a) First of all assume that  $z \in HxH' \cap HyH'$  then let  $h_1, h_2 \in H$  and  $h'_1, h'_2 \in H'$  such that:

$$h_1 x h'_1 = z = h_2 y h'_2 \Rightarrow y = h_2^{-1} h_1 x h'_1 h'_2{}^{-1} \Rightarrow HyH' = H h_2^{-1} h_1 x h'_1 h'_2{}^{-1} H' = HxH'$$

For any  $y, x \in G$  either  $HxH'$  and  $HyH'$  are disjoint or they are equal this fact combined with the fact that for all  $x \in G$  we have  $x \in HxH'$  tells us that we can write  $G$  as a disjoint union of double cosets.

(b) By our assumptions we have the disjoint unions:

$$\begin{aligned} G &= \bigcup_c HcH' \\ &= \bigcup_c \bigcup_{x_c} x_c(H \cap [c]H')cH' \end{aligned}$$

But notice that for  $\alpha \in x_c(H \cap [c]H')cH'$  we have:

$$\alpha = x_c c h' c^{-1} c h'' = x_c c h' h'' \in x_c c H' \text{ for some } h', h'' \in H'$$

So we see that  $x_c(H \cap [c]H')cH' \subseteq x_c c H'$ , the other inclusion is clear since  $e \in (H \cap [c]H')$ . So we have:

$$G = \bigcup_c \bigcup_{x_c} x_c c H' \text{ and this union is disjoint}$$

□

### Exercise 9

(a) Let  $G$  be a group and  $H$  a subgroup of finite index. Show that there exists a normal subgroup  $N$  of  $G$  contained in  $H$  and also of finite index.

(b) Let  $G$  be a group and let  $H_1, H_2$  be subgroups of finite index. Prove that  $H_1 \cap H_2$  has finite index.

*Proof.* (a) Assume that  $[G : H] = n$  and let  $\{a_1 H, a_2 H, \dots, a_n H\}$  be the distinct cosets of  $H$  in  $G$ . Let  $a \in G$  since we know that  $aa_i H \in \{a_1 H, a_2 H, \dots, a_n H\}$ , so let  $\sigma_a \in S_n$  be such that  $aa_i H = a_{\sigma_a(i)} H$  for all  $i$ . We define

$$\begin{aligned} \varphi: G &\rightarrow S_n \\ a &\rightarrow \sigma_a \end{aligned}$$

Let  $x \in \ker \varphi$ , this means that  $xa_i H = a_{e(i)} H = a_i H$  for all  $i$ . In particular we know that for one  $i_0$  we have  $a_{i_0} H = H$ , so we have

$$H = a_{i_0} H = xa_{i_0} H = xH$$

This means that  $x \in H$ . Therefore  $\ker \varphi \subseteq H$ . Letting  $N = \ker \varphi$ , we see that this is a normal subgroup contained in  $H$ .

Now finally note that  $\text{im } \varphi \leq S_n$ , so we see by an isomorphism theorem:

$$[G : N] = |G/N| = |\text{im } \varphi| \leq |S_n| = n! < \infty$$

- (b) First of all, let  $x, y \in G$  be such that  $xH_1 = yH_1$  and  $xH_2 = yH_2$ , then we have  $y^{-1}x \in H_1$  and  $y^{-1}x \in H_2$  so  $y^{-1}x \in H_1 \cap H_2$  so  $x(H_1 \cap H_2) = y(H_1 \cap H_2)$

Conversely assume that  $x(H_1 \cap H_2) = y(H_1 \cap H_2)$ , then we have that  $y^{-1}x \in H_1 \cap H_2$  so  $y^{-1}x \in H_1$  and  $y^{-1}x \in H_2$ .

So we have shown that  $x(H_1 \cap H_2) = y(H_1 \cap H_2)$  if and only if  $xH_1 = yH_1$  and  $xH_2 = yH_2$ .

Now finally we let,  $C_i$  be the set of distinct representatives of  $H_i$  and  $C$  be the set of distinct representatives of  $H_1 \cap H_2$

$$\begin{aligned} f: C &\rightarrow C_1 \times C_2 \\ c(H_1 \cap H_2) &\rightarrow (cH_1, cH_2) \end{aligned}$$

Since we  $x(H_1 \cap H_2) = y(H_1 \cap H_2)$  if and only if  $(xH_1, xH_2) = (yH_1, yH_2)$  see that this function is well-defined and injective. Therefore from set theory:

$$[G: H_1 \cap H_2] = |C| \leq |C_1||C_2| = [G: H_1][G: H_2] < \infty$$

□

**Exercise 10** Let  $G$  be a group and let  $H$  be a subgroup of finite index. Prove that there is only a finite number of right cosets of  $H$ , and that the number of right cosets is equal to the number of left cosets.

*Proof.* Recall, that since  $H$  has finite index, There are only a finite number of left cosets of  $H$  in  $G$ .

$$\begin{aligned} ah = b &\iff hb^{-1} = a^{-1} \\ \therefore b \in aH &\iff a^{-1} \in Hb^{-1} \end{aligned}$$

From this we see that  $aH = bH \iff Ha^{-1} = Hb^{-1}$ .

Let  $H_L = \{aH \mid a \in G\}$  the set of left cosets, and let  $H_R = \{Ha \mid a \in G\}$  the set of right cosets.

We define a set map:

$$f: H_L \rightarrow H_R$$

Given by  $f(aH) = Ha^{-1}$ . Now first we will show that this function is well-defined, let  $aH = bH$  this implies from above that  $Ha^{-1} = Hb^{-1}$ :

$$f(aH) = Ha^{-1} = Hb^{-1} = f(bH)$$

So this function is indeed well-defined. Now this is also a bijection we notice that the inverse function is given by the map:

$$g: H_R \rightarrow H_L \text{ by } g(Ha) = a^{-1}H$$

This function is similarly seen to be well-defined since  $aH = bH \iff Ha^{-1} = Hb^{-1}$ .

So we see that  $[G: H] = |H_L| = |H_R|$ . Since  $[G: H] < \infty$ , there is only a finite number of right cosets and there as many right as left cosets. □

**Exercise 11** Let  $G$  be a group, and  $A$  a normal abelian subgroup. Show that  $G/A$  operates on  $A$  by conjugation; and in this manner get a homomorphism of  $G/A$  into  $\text{Aut}(A)$ .

*Proof.* We will first show that this action is well-defined: Assume that  $xA = yA$  so let  $a \in A$  such that  $y = xa$  and let  $s \in A$ . So we have:

$$\begin{aligned} xA \cdot s &= xsx^{-1} \\ &= xaa^{-1}sx^{-1} \\ &= (xa)s(a^{-1}x^{-1}) \text{ since } a, s \in A \text{ and } A \text{ is abelian.} \\ &= (xa)s(xa)^{-1} \\ &= ysy^{-1} \\ &= yA \cdot s \end{aligned}$$

So this function is indeed well-defined. Now we will show that this function is a group action:

Let  $xA, yA \in G/A$  and  $s \in A$  we have

$$xA \cdot (yA \cdot s) = xA \cdot (ysy^{-1}) = xysy^{-1}x^{-1} = (xy)s(xy)^{-1} = (xyA) \cdot s$$

For all  $s \in S$

$$eA \cdot s = ese^{-1} = s$$

So this indeed a group action.

Now let

$$\varphi: G/A \rightarrow \text{Aut}(A)$$

Be such that for all  $s \in A$ :

$$\begin{aligned}\varphi(xA)(s) &= xA \cdot s \\ &= xsx^{-1}\end{aligned}$$

Note that it is clear that  $\varphi(xA)$  is an automorphisms, since it is an inner-homomorphism.

Finally it is clear that  $\varphi$  is a homomorphism, since the map  $xA \cdot s = xsx^{-1}$  is a group action, so

$$\varphi(xyA)(s) = (xyA) \cdot s = xA \cdot (yA \cdot s) = \varphi(xA)\varphi(yA)(s) \Rightarrow \varphi(xy) = \varphi(x)\varphi(y)$$

And likewise from above we see that  $\varphi(A) = \text{id}$ . □

**Part 2: Semidirect product** We define  $G$  to be the **semidirect product** of  $H$  and  $N$  if  $G = NH$  and  $H \cap N = \{e\}$ .

**Exercise 12** Let  $G$  be a group and let  $H, N$  be subgroups with  $N$  normal. Let  $\gamma_x$  be conjugation by an element  $x \in G$ .

- (a) Show that  $x \rightarrow \gamma_x$  induces a homomorphism  $f: H \rightarrow \text{Aut}(N)$
- (b) If  $H \cap N = \{e\}$ , show that the map  $H \times N \rightarrow HN$  given by  $(x, y) \rightarrow xy$  is a bijection, and that this map is an isomorphism if and only if  $f$  (from part (a)) is trivial.
- (c) Conversely, let  $N, H$  be groups and let  $\psi: H \rightarrow \text{Aut}(N)$  be a given homomorphism. Let  $G$  be the set of pairs  $(x, h)$  with  $x \in N$  and  $h \in H$  and define a composition law:

$$(x_1, h_1)(x_2, h_2) = (x_1\psi(h_1)x_2, h_1h_2)$$

Show that this is a group law, and yields a semidirect product of  $N$  and  $H$ , identifying  $N$  with the set of elements  $(x, 1)$  and  $H$  with the set of elements  $(1, h)$ .

*Proof.* (a) We will first show that for all  $x \in G$  we have  $\gamma_x|_N \in \text{Aut}(N)$ , first of all recall that  $\gamma_x|_N: N \rightarrow G$  is indeed a homomorphism. Now let  $y \in \ker(\gamma_x)$  then:

$$\gamma_x(y) = xyx^{-1} = e \Rightarrow xy = x \Rightarrow y = e$$

So this is injective, finally since  $N$  is a normal subgroup of  $G$ , we see that  $\gamma_x(N) = xNx^{-1} = N$ , so this function is indeed an automorphism.

So we define our function  $f: H \rightarrow \text{Aut}(N)$ , by  $f(x) = \gamma_x$  for all  $x \in H$ . Let  $x, y \in H$ , for all  $n \in N$  we have

$$\begin{aligned}f(xy)(n) &= \gamma_{xy}(n) \\ &= xy ny^{-1} x^{-1} \\ &= x(f(y)(n))x^{-1} \\ &= (f(x) \circ f(y))(n)\end{aligned}$$

This function is indeed a homomorphism.

□

- (b) Let

$$g: H \times N \rightarrow HN \text{ be given by } (x, y) \rightarrow xy$$

Since  $HN = \{hn \mid h \in H \text{ and } n \in N\}$ , this map is clearly surjective. Now assume that  $g(x, y) = g(z, w)$ , then we have:

$$xy = zw \Rightarrow \underbrace{z^{-1}x}_{\in H} = \underbrace{wy^{-1}}_{\in N} \in H \cap N = \{e\}$$

So  $x = z$  and  $y = w$ , so  $(x, y) = (z, w)$ . So this function is injective, and so a bijection.

- ( $\Rightarrow$ ) Assume that this map is also an isomorphism, then we have for all  $x, z \in H$  and  $y, w \in N$

$$\begin{aligned} xzyw &= g(xz, yw) \\ &= g((x, y)(z, w)) \\ &= (xy)(zw) \end{aligned}$$

Therefore,  $zy = yz$  for all  $z \in H$  and  $y \in N$ , which means that:

$$f(z)(y) = zyz^{-1} = y \text{ for all } y \in N \text{ and } z \in H \Rightarrow f(z) = \text{id for all } z \in H$$

So  $f$  is trivial.

- ( $\Leftarrow$ ) Assume that  $f$  is trivial. Therefore we have for all  $x, z \in H$  and  $y, w \in N$ :

$$\begin{aligned} g((x, y)(z, w)) &= g(xz, yw) \\ &= xzyw \\ &= x(zyz^{-1})zw \\ &= x(f_z(y))zw \\ &= xyzw \\ &= g(x, y)g(z, w) \end{aligned}$$

So this  $g$  is indeed a homomorphism, and so a isomorphism.

(c) First of all we will show that this composition law is associative:

$$\begin{aligned} ((x_1, h_1)(x_2, h_2))(x_3, h_3) &= (x_1\psi(h_1)x_2, h_1h_2)(x_3, h_3) \\ &= ((x_1\psi(h_1)x_2)\psi(h_1h_2)x_3, (h_1h_2)h_3) \\ &= (x_1\psi(h_1)(x_2\psi(h_2)x_3), h_1(h_2h_3)) \\ &= (x_1, h_1)(x_2\psi(h_2)x_3, h_2h_3) \\ &= (x_1, h_1)((x_2, h_2)(x_3, h_3)) \end{aligned}$$

Now for all  $(x, h) \in N \times H$  we have:

$$(e_N, e_H)(x, h) = (e_N\psi(e_H)x, e_Hh) = (x, h) = (x\psi(h)(e_N), h e_H) = (x, h)(e_N, e_H)$$

and

$$(x, h)(\psi(h^{-1})x^{-1}, h^{-1}) = (x\psi(e_H)(x^{-1}), e_H) = (e_N, e_H)$$

So this is in deed a group law.

Now let  $N = \{(x, 1) \in G\}$  and  $H = \{(1, x) \in G\}$ , it is clear that these are subgroups of  $G$  by how we defined multiplication and inverses. We first need to show that  $N \trianglelefteq G$ : Let  $(x, 1) \in N$  and  $(n, h) \in G$ , then we have:

$$\begin{aligned} (\psi(h^{-1})n^{-1}, h^{-1})(x, 1)(n, h) &= (\psi(h^{-1})n^{-1}\psi(1)x, h^{-1})(n, h) \\ &= (\psi(h^{-1})n^{-1}x\psi(h)n, 1) \in N \end{aligned}$$

So we indeed see that  $N$  is normal.

Also notice that  $N \cap H = \{(1, 1)\}$ , by how they are defined. So we only need to show that  $G = NH$ .

Let  $(n, h) \in G$ :

$$(n, 1)(1, h) = (n\psi(1)1, 1h) = (n, h)$$

So we indeed see  $G \subseteq NH$ , the other inclusion is trivial. So this group law indeed yields a semidirect product of  $N$  and  $H$ . □

### Exercise 13

- (a) Let  $H, N$  be normal subgroups of a finite group  $G$ . Assume that the orders of  $H$  and  $G$  are relatively prime. Prove that  $xy = yx$  for all  $x \in H$  and  $y \in G$  and that  $H \times N \simeq HN$
- (b) Let  $H_1, \dots, H_r$  be normal subgroups of  $G$  such that the order of  $H_i$  is relatively prime with the order of  $H_j$  for  $i \neq j$ . Prove that

$$H_1 \times \dots \times H_r = H_1 \cdots H_r$$

*Proof.* (a) First of all recall that since  $H \cap N \leq H$  and  $H \cap N \leq N$ , we see that  $|H \cap N| \mid |H|$  and  $|H \cap N| \mid |N|$ , so  $|H \cap N| = 1$ , since  $\gcd(|H|, |N|) = 1$ . So  $|H \cap N| = \{e\}$ .

Now since  $H, N$  are normal subgroups of  $G$ , for  $x \in N$  and  $y \in H$ :

$$xyx^{-1} \in H \Rightarrow (xyx^{-1})y^{-1} \in H$$

And

$$yx^{-1}y^{-1} \in N \Rightarrow x(yx^{-1}y^{-1}) \in N$$

so  $xyx^{-1}y^{-1} \in H \cap N = \{e\} \Rightarrow xy = yx$ .

Now let  $\gamma_x$  be conjugation by an element  $x \in G$  and let  $f: H \rightarrow \text{Aut}(N)$  be the induced map. Then we have:

$$f(h)(n) = h^{-1}nh = h h^{-1}n = n \text{ for all } h \in H \text{ and } n \in N$$

So the map  $f$  is trivial, so by 12b:

$$H \times N \simeq HN$$

- (b) We will proceed by induction. The base case was shown in (a), so assume this is true for all integers less than  $r$ . We have

$$H_1 \times \dots \times H_{r-1} \simeq H_1 \cdots H_{r-1}$$

So

$$H_1 \times \dots \times H_{r-1} \times H_r \simeq H_1 \cdots H_{r-1} \times H_r$$

Since  $|H_1 \cdots H_{r-1}| = |H_1 \times \dots \times H_{r-1}| = |H_1| \cdots |H_{r-1}|$ , which is coprime to  $|H_r|$  since  $|H_r|$  is coprime to all  $|H_j|$ , with  $j < r$ . So using (a) we get the desired result. □

**Exercise 14** Let  $G$  be a finite group and  $N$  a normal subgroup such that  $N$  and  $G/N$  have relatively prime orders.

- (a) Let  $H \leq G$ , such that  $|H| = |G/N|$ . Prove that  $G = HN$
- (b) Let  $g$  be an automorphism of  $G$ . Prove that  $g(N) = N$ .

*Proof.* (a) Note since  $N$  is normal:

$$|HN| = \frac{|H||N|}{|H \cap N|}$$

By a previous question. But since the order of  $N$  and  $H$  are relatively prime, as we have seen this means  $|H \cap N| = 1$ . So we have

$$\begin{aligned} |HN| &= |H||N| \\ &= |G/N||N| \\ &= |G| \end{aligned}$$

So we see that  $|HN| = |G|$ , since  $G$  is finite and  $HN \subseteq G$ , this means that  $HN = G$ .

- (b)

**Lemma 2.** If  $H \leq G$  is such that  $|H| = |N|$ , then  $H = N$



*Proof.* Let

$$\varphi: G \rightarrow G/N$$

be the canonical homomorphism.

We note that  $\varphi(HN) = HN/N \leq G/N$ , we have:

$$|H/(H \cap N)| = |HN/N| \mid |G/N| \quad (2)$$

So let  $m \in \mathbb{N}$ :

$$\frac{|H|}{|H \cap N|} m = |G/N| \Rightarrow |N|m = |H|m = |G/N||H \cap N|$$

Now let  $p$  be a prime divisor of  $|N|$ , then  $p \mid |G/N||H \cap N|$ , since  $p \nmid |G/N|$  and  $p$  is prime we see that:  $p \mid |H \cap N|$ .

So all prime divisors of  $|N|$  divide  $|H \cap N|$ , therefore  $|N| \mid |H \cap N|$  but since  $|H \cap N| \leq |N|$  this implies that  $|H \cap N| = |N|$  so  $H \subseteq N$ . Likewise we can see that  $N \subseteq H$ . So  $H = N$ .  $\square$

Now let  $g$  be an automorphism of  $G$ . So we know that  $g(N) \leq G$  and  $|g(N)| = |N|$ . So by the lemma  $g(N) = N$ .  $\square$

### Some operations

**Exercise 15** Let  $G$  be a finite group operating on a finite set  $S$  with  $\#(S) \geq 2$ . Assume that there is only one orbit. Prove that there exists an element  $x \in G$  which has no fixed point, i.e.

$$xs \neq s \text{ for all } s \in S$$

**Exercise 16** Let  $H$  be a proper subgroup of a finite group  $G$ . Show that  $G$  is not the union of all the conjugates of  $H$ .

*Proof.* Let  $|G| = m|H|$ , where  $m > 1$ . Now let  $\{x_1 H x_1^{-1}, \dots, x_n H x_n^{-1}\}$  be the set of distinct  $\square$

**Exercise 17** Let  $X, Y$  be finite sets and  $C$  be a subset of  $X \times Y$ . For  $x \in X$  let

$$\varphi(x) = \text{number of elements } y \in Y \text{ such that } (x, y) \in C$$

Verify that

$$|C| = \sum_{x \in X} \varphi(x)$$

*Proof.*  $\square$