In this chapter we will study the core of Galois theory, the group of automorphisms of a finite (and sometimes infinite) Galois extension at length.

1 Galois extensions

Definition 1.1. Let K be a field and let G be a group of automorphisms of K. We let

$$K^G = \{x \in K \mid x^\sigma = x \text{ for all } \sigma \in G\}$$

We call this the **fixed field** of G.

Definition 1.2. An algebraic extension K of a field k is called **Galois** if it is normal and seperable.

The group of automorphisms of K over k is called the **Galois group** of K over k, and is denoted $G(K/k)$, $G_{K/k}$, $\mathrm{Gal}(K/k)$ or simply G.

This is the main theorem of the Galois theory or finite Galois extensions.

Theorem 1. *Let K be a finite Galois extension of k, with Galois group G. There is a bijection between the set of subfields E of K, containing K and the set of subgroups H of G, given by $E = K^H$. The field E is Galois over k if and only if H is normal in G, and if that is the case, then the map $\sigma \to \sigma|_E$ induces an isomorphism of G/H onto the Galois group of E over k.*