Analytical Number Theory

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1 Notation

Definition 1.1. Landau's Big-Oh, Little-Oh and $\sim$. Let a be finite or infinite:

1. We say that $f(x) \sim g(x)$ as a $x \to a$, if
$$\lim_{x \to a} \frac{f(x)}{g(x)} = 1$$
2. Big-oh: We say that $f(x) = O(g(x))$ as $x \to \infty$, if there exists $x_0 \in \mathbb{R}$, and $c > 0$, such that:
$$|f(x)| \leq cg(x) \quad \text{for all } x > x_0$$

   We say that $f(x) = O(g(x))$ as $x \to a$, if there exists $\delta > 0$, and $c > 0$, such that:
$$|f(x)| \leq cg(x) \quad \text{for all } |x - a| < \delta$$
3. Little-oh: We say that $f(x) = o(g(x))$ as $x \to a$, if
$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

Example 1.1.1. For $\sim$:

1. $\sin(x) \sim x$, as $x \to 0$.
2. Stirlings:
$n! \sim (\frac{n}{e})^n \sqrt{2\pi n}$, as $n \to \infty$.

Example 1.1.2. For Big-Oh:

1. $\sin(x) = O(1)$. As $x \to \infty$ since $\sin(x)$ is bounded.
2. $\sin(x) = x - \frac{x^3}{3!} + O(x^5)$. As $x \to 0$ since:
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1!)} \Rightarrow \sin(x) - x - \frac{x^3}{3!} = \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Example 1.1.3. For Little-Oh: $n! = o(n^n)$, since
$$\frac{n!}{n^n} \sim \frac{\sqrt{2\pi n}}{e^n} = \sqrt{2\pi} e^{\frac{1}{2}\log(n) - n}$$

2 Summation by parts

Let f be a function from $\mathbb{Z}^+$ to $\mathbb{R}$ or $\mathbb{C}$, and g a real or complex valued function of a real variable. If g' exists and is continuous on $[1, x]$ for some $x \in R$. We find that:
$$\sum_{1 \leq n \leq x} f(n)g(n) = (\sum_{1 \leq n \leq x} f(n))g(x) - \int_1^x \sum_{1 \leq n \leq t} f(t)g'(t)dt \qquad (1)$$

Proof.

$$f(n)g(n) = f(n)g(x) - f(n)\int_n^x g'(t)dt$$
$$\sum_{1\le n\le x} f(n)g(n) = \sum_{1\le n\le x} f(n)g(x) - \sum_{1\le n\le x} f(n)\int_n^x g'(t)dt$$

We can bring in all the $f(n)$ where $n \le t$, inside the same integral, and this gives us our result. $\square$

Proof. Alternate proof using telescoping: Since $\int_1^x = \int_1^2 + \int_2^3 + \cdots + \int_{\lfloor x\rfloor}^x$

$$\sum_{1\le n\le x} f(n)g(x) - \int_1^x (\sum_{1\le n\le t} f(n))g'(t)dt =$$
$$(\sum_{1\le n\le x} f(n))g(x) - f(1)(g(2)-g(1)) - (f(1)+f(2))(g(3)-g(2)) - \cdots - (\sum_{1\le n\le x} f(n))(g(x)-g(\lfloor x\rfloor))$$

By telescoping we get the desired result. $\square$

2.1 $\sum_{1\le n\le\lfloor x\rfloor}$

$\sum_{1\le n\le\lfloor x\rfloor}\frac{1}{n}$, in this situation we have $f(n) = 1$ and $g(x) = \frac{1}{x}$. So, we have: $\sum_{1\le n\le x} 1 = \lfloor x\rfloor$. We define x to be the fractional part of x, so that $\lfloor x\rfloor = x - \{x\}$.

Hence we have:

$$\begin{aligned}\sum_{1\le n\le\lfloor x\rfloor}\frac{1}{n} &= \lfloor x\rfloor\frac{1}{x} + \int_1^x \lfloor t\rfloor\frac{1}{t^2}dt \\ &= 1 - \frac{\{x\}}{x} + \log(x) - \int_1^x \frac{\{t\}}{t^2}dt \\ &= 1 - \frac{\{x\}}{x} + \log(x) - (\int_1^\infty \frac{\{t\}}{t^2}dt - \int_x^\infty \frac{\{t\}}{t^2}dt)\end{aligned}$$

Since $-\frac{\{x\}}{x} = O(\frac{1}{x})$ and $\int_x^\infty \frac{\{t\}}{t^2}dt) = O(\frac{1}{x})$, we finally get at the end:

$$\sum_{1\le n\le x}\frac{1}{n} = \log(x) + (1 - \int_1^\infty \frac{\{t\}}{t^2}dt) + O(\frac{1}{x})$$

We call $1 - \int_1^\infty \frac{\{t\}}{t^2}dt = \gamma = 0.5772\cdots$ Euler's constant.

This example tells us that:

$$\lim_{x\to\infty}(\sum_{1\le n\le x}\frac{1}{n} - \log(x)) = \gamma$$

2.2 $\log(m!)$

Consider: $\log(m!) = \sum_{1\le n\le m}\log(n)$; let $f(n) = 1$ and $g(x) = \log(x)$.

$$\begin{aligned}\log(m!) &= m\log(m) - \int_1^m \frac{\lfloor t\rfloor}{t}dt \\ &= m\log(m) - \int_1^m \frac{t-\{t\}}{t}dt \\ &= m\log(m) - (m-1) + \int_1^m \frac{\{t\}}{t}dt\end{aligned}$$

Since $0 < \int_1^m \frac{\{t\}}{t}dt < \log(m)$, we have $m\log(m) - (m-1) < \log(m!) < (m+1)\log(m) - (m-1)$

Thus:

$$\frac{m^m}{e^{m-1}} < m! < \frac{m^{m+1}}{e^{m-1}} \text{ in reality } m! \sim (\frac{m}{e})^m\sqrt{2\pi m}$$