§4: Bounded linear functionals on L^p

Theorem 6.16

Suppose $1 \leq p < \infty$; μ is a σ-finite positive measure on X; and Φ is a bounded linear functional on $L^p(\mu)$. Then there is a unique $g \in L^q(\mu)$; where $\frac{1}{q} + \frac{1}{p} = 1$ s/t:

$$\Phi(f) = \int_X fg \, d\mu \qquad \forall f \in L^p(\mu) \quad (1)$$

Moreover we have:

$$\|\Phi\| = \|g\|_q . \qquad (2)$$

Proof: Note if g & g' satisfy (1) then:

$$\int_X (g - g') f \, d\mu = 0 \qquad \forall f \in L^p(\mu)$$

In particular $\forall f = \chi_E$ where E is a set of finite measure.
So $g - g' = 0$ a.e.

Furthermore if (1) holds, $\forall f \in L^p(\mu)$ w/ $\|f\|_p = 1$:

$$|\Phi(f)| = \left|\int_X fg \, d\mu\right|$$

$$= \|fg\|_1$$

$$\leq \|f\|_p \|g\|_q = \|g\|_q$$

$\therefore \|\Phi\| \leq \|g\|_q$.

So we just need to prove that g exists and that equality holds.

- If $\|\Phi\| = 0$, then let $g = 0$.

- O/w, $\|\Phi\| > 0$:

We first consider the case $\mu(X) < \infty$.

For any measurable set $E \subseteq X$ define:

$$\lambda(E) = \Phi(\chi_E)$$

Since Φ is linear; and $\chi_{A \cup B} = \chi_A + \chi_B$ if $A \cap B = \emptyset$, we see that λ is additive.

Now we show countable additivity; suppose $E = \bigcup_i E_i$. Let $A_k = E_1 \cup \dots \cup E_k$ and note:

$$\|\chi_E - \chi_{A_k}\|_p = \left(\int_X (\chi_E - \chi_{A_k})^p d\mu\right)^{1/p} \text{, since } p < \infty$$

$$= \left(\int_X \chi_{E \setminus A_k}^p \, d\mu\right)^{1/p}$$

$$= \mu(E \setminus A_k)^{1/p}$$

$$\mapsto 0 \text{ as } k \mapsto \infty.$$

So by continuity of Φ;

$$|\Phi(\chi_E) - \Phi(A_k)| = |\Phi(\chi_E - \chi_{A_k})| \longmapsto 0 \text{ as } k \mapsto \infty.$$

$$\therefore \lambda(E) = \Phi(\chi_E) = \lim_{k \to \infty} \Phi(A_k) = \sum_{i=1}^{\infty} \lambda(E_i).$$

Furthermore if $\mu(E) = 0$ then $\| \overset{\chi_E = 0}{\overrightarrow{\chi_E}} \|_p = 0$

so $\lambda(E) = \Phi(\chi_E) = 0.$

Thus $\lambda \ll \mu$; by Radon-Nikodym, $\exists\, g \in L^1(\mu)$ s/t $\forall$ measurable $E \subseteq X$:

$$\Phi(\chi_E) = \int_E g \, d\mu = \int_X \chi_E \, g \, d\mu$$

$$\therefore \quad \Phi(f) = \int_X f g \, d\mu.$$

for every simple measurable f; so for all

$f \in L^\infty(\mu)$.

Now we will show equality of (2):

Case 1: $p = 1$.

$$\left| \int_E g \, d\mu \right| = |\phi(\chi_E)| \leq \|\Phi\| \cdot \|\chi_E\|_1$$
$$= \|\phi\| \cdot \mu(E)$$

$\forall E \in \mathcal{M}$, so by th 1.40:

$$|g(x)| \leq \|\Phi\| \text{ a.e.}$$

so $\quad \|g\|_\infty \leq \|\Phi\|$

Case 2: $1 < p < \infty$. There is a measurable funct° α, $|\alpha| = 1$ s/t:

$$\alpha g = |g|.$$

Let $E_n = \{x : |g(x)| \le n\}$ and define

$$f = \chi_{E_n} |g|^{q-1} \alpha .$$

then on E_n:

$$|f|^p = |g|^{p(q-1)} = |g|^{q\cancel{+p-p}}$$

$f \in L^\infty(\mu)$ and:

$$\int_{E_n} |g|^q d\mu = \int_X f g \, d\mu$$

$$= \Phi(f)$$

$$\le \|\Phi\| \cdot \|f\|_p$$

$$= \|\Phi\| \left(\int_{E_n} |g|^q \right)^{1/p}$$

$$\therefore \int_X \chi_{E_n} |g|^q d\mu \le \|\Phi\|^q \quad \forall n \in \mathbb{N}^{\alpha}.$$

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$$\underrightarrow{\|g\|_q} \leq \|\Phi\|$$

Now assume that $\mu(X) = \infty$ and μ is σ-finite.
Choose $w \in L^1(\mu)$ s/t $0 < w(x) < 1$ $\forall x \in X$,
↳ like in lemma 6.9.

Then $d\tilde{\mu} = w\,d\mu$ defines a finite measure on $\mathcal{M}$ and:

$$F \mapsto w^{1/p} F$$

is a linear isometry of $L^p(\tilde{\mu})$ onto $L^p(\mu)$ since $w(x) > 0$ $\forall x \in X$.

So:

$$\Psi(F) = \Phi(w^{1/p} F)$$

is a bounded linear functional Ψ on $L^p(\tilde{\mu})$ w/

$$\|\Psi\| = \|\Phi\|$$

So by the previous part (since $\tilde{\mu}(X) < \infty$) ; $\exists$ $G \in L^q(\tilde{\mu})$ s/t

$$\Psi(F) = \int_X F G \, d\tilde{\mu}$$

Let $g = \omega^{1/q} G$; then, if $p>1$:

$$\int_X |g|^q \, d\mu = \int_X |G|^q \, d\tilde{\mu}$$

$$= \|\Psi\|^q$$

$$= \|\Phi\|^q$$

if $p=1$

$$\|g\|_\infty = \|G\|_\infty = \|\Psi\| = \|\Phi\|$$

and so

$$\Phi(f) = \Psi(\omega^{-1/p} f) = \int_X \omega^{-1/p} f G \, d\tilde{\mu}$$

$$= \int_X f \omega^{1/q} G \, \omega^{-1} d\tilde{\mu}.$$

$$= \int_X f g \, d\mu.$$

$\forall f \in L^p(\mu)$.

$\square$