Exercise 2: Prove that the unit ball (open or closed) is convex in every normed linear space.

Proof. Let B be the unit ball in a normed linear space X, let $x, y \in B$ and $\lambda \in [0,1]$. Then we have:

$$\begin{aligned} ||\lambda x + (1-\lambda)y|| &\leq \lambda ||x|| + (1-\lambda)||y|| && (1) \\ &\leq \lambda + (1-\lambda) = 1 && (2) \end{aligned}$$

$\therefore \lambda x + (1-\lambda)y \in B$.

$\square$

Exercise 8: Let X be a normed linear space, and let X^* be its dual space with the norm

$$||f|| = \sup\{|f(x)| \colon ||x|| \leq 1\}$$

(a) **Prove that X^* is a Banach space.**

(b) **Prove that the mapping $f \to f(x)$ is, for each $x \in X$, a bounded linear functional on X^*, of norm $||x||$**

(c) **Prove that $\{||x_n||\}$ is bounded if $\{x_n\}$ is a sequence in X such that $\{f(x_n)\}$ is bounded for every $f \in X^*$.**

Proof. (a) Note is clear that X^* is a vector space, now let $f, g \in X^*$ and $\alpha \in \mathbb{F}$ (where $\mathbb{F}$ is the field for which X^* is a vs over), we have:

$$\begin{aligned} |f(x) + g(x)| &\leq f(x) + g(x) \leq ||f|| + ||g||, \text{ for all } ||x|| \leq 1 \\ \therefore ||f+g|| &= \sup\{|f(x) + g(x)| \mid ||x|| \leq 1\} \leq ||f|| + ||g|| \end{aligned}$$

$$\begin{aligned} ||\alpha f|| &= \sup\{|\alpha f(x)| \mid ||x|| \leq 1\} \\ &= |\alpha| \sup\{|f(x)| \mid ||x|| \leq 1\} \\ &= |\alpha| \cdot ||f|| \end{aligned}$$

$$0 \leq |f(x)| \text{ for all } x \Rightarrow 0 \leq ||f||$$

$$\begin{aligned} \text{If } ||f|| = 0 &\Rightarrow \sup\{|f(x)| \mid ||x|| \leq 1\} = 0 \\ &\Rightarrow |f(x)| = 0 \text{ for all } ||x|| \leq 1 \end{aligned}$$

But in this case, notice that for all $x \in X \neq \{0\}$, we have $\frac{x}{||x||}$ has norm 1 and:

$$|f(x)| = ||x|||f(\frac{x}{||x||})| = 0 \iff f(x) = 0.$$

So $f = 0$.

Finally we will show that X^* is complete. Let $\{f_n\}$ be a cauchy sequence in X^*. So let $\epsilon > 0$ and $N \geq 1$ such that for all $n, m \geq N$:

$$||f_n - f_m|| < \epsilon \iff |f_n(x) - f_m(x)| < \epsilon \text{ for all } ||x|| \leq 1 \tag{3}$$

So notice this means that for all $x \in X \setminus \{0\}$, we have:

$$|f_n(x) - f_m(x)| = ||x|| \cdot |f_n(\frac{x}{||x||}) - f_m(\frac{x}{||x||})| < ||x||\epsilon \tag{4}$$

So $\{f_n(x)\}$ is a cauchy sequence for all $x \in X$, so we define $f \colon X \to \mathbb{F}$ by:

$$f(x) = \lim_{n \to \infty} f_n(x) \tag{5}$$

Now it is clear that f is a linear map, since $\{f_n\}$ is a Cauchy sequence, it is bounded. Indeed let N be such that $|||f_n|| - ||f_N||| < 1 \Rightarrow ||f_n|| \leq 1 + ||f_N||$ for all $n \geq N$. So let M be such that $||f_n|| < M$, for all n,

Therefore for $||x|| \leq 1$, let N be such that $|f(x) - f_N(x)| < 1$:

$$|f(x)| \leq |f(x) - f_N(x)| + |f_N(x)| < 1 + M \tag{6}$$

So $|f(x)| < 1 + M$ for all $||x|| \leq 1$, so we have $||f|| < 1 + M < \infty$. So $f \in X^*$.

(b) For $x \in X$, let $\Lambda_x \colon X^* \to \mathbb{F}$, be $\Lambda_x(f) = f(x)$. Notice that $\Lambda_x(f+g) = (f+g)(x) = f(x) + g(x) = \Lambda_x(f) + \Lambda_x(g)$ and $\Lambda_x(\alpha f) = \alpha f(x) = \alpha \Lambda_x(f)$. Furthermore, for $||f|| \leq 1$ we have:

$$|\Lambda_x(f)| = |f(x)| \leq ||f|| \cdot ||x|| \leq ||x|| \tag{7}$$

So this function is indeed bounded. Now let $f \colon \{\alpha x \colon \alpha \in \mathbb{F}\} \to \mathbb{F}$

$$f(\alpha x) = \|\alpha x\| \tag{8}$$

We can extend this to a bounded linear functional on X, such that $||f|| = ||f||_{\{\alpha x \colon \alpha \in \mathbb{F}\}} = 1$.

So we have $|\Lambda_x(f)| = |f(x)| = ||x|| \Rightarrow ||\Lambda_x|| = ||x||$.

(c) For all $n \in \mathbb{N}^*$,

$$F_n \colon X^* \to \mathbb{F} \text{ be given by } F_n(f) = f(x_n) \tag{9}$$

Let f be such that $||f|| \leq 1$, we have: $|F_n(f)| = |f(x_n)| \leq ||x_n|| \iff ||F_n|| \leq ||x_n|| < \infty$. On the otherhand, in a method similar to in (b), where we define $g(\alpha x_n) = ||\alpha x_n||$, we see that $||F_n|| = ||x_n||$

Finally, note that since $\{f(x_n)\}$ is bounded for all f:

$$\sup_{n \in \mathbb{N}} |F_n(f)| = \sup_{n \in \mathbb{N}} |f(x_n)| < \infty \quad \forall f \in X^* \tag{10}$$

So by Banach-Steinhaus, there exists a $M < \infty$ such that for all $n \in \mathbb{N}$, we have that:

$$||x_n|| = ||F_n|| \leq M; \text{ for all } n \tag{11}$$

So $\{||x_n||\}$ is bounded.

$\square$

Exercise 9. Prove the following four statements.

(a) **If** $y = \{y_i\} \in \ell^1$ **and** $\Lambda x = \sum x_i y_i$, **for every** $x = \{x_i\} \in c_0$, **then** Λ **is a bounded linear functional on** c_0, **and** $||\Lambda|| = ||y||_1$. **Moreover, every** $\Lambda \in (c_0)^*$ **is obtained in this way. So,** $(c_0)^* = \ell^1$

(b) **In the same sense,** $(\ell^1)^* = \ell^\infty$

(c) **Every** $y \in \ell^1$ **induces a bounded linear functional on** ℓ^∞**, as in** (a)**. However,** $(\ell^\infty)^*$ **contains nontrivial functionals that vanish on all of** c_0

(d) c_0 **and** ℓ^1 **are seperable but** ℓ^∞ **is not.**

Proof. (a) Let $x = \{x_i\} \in c_0$, notice that:

$$|x_i y_i| \leq ||x||_\infty |y_i| \text{ for all } i \tag{12}$$

Since $y \in \ell^1$, we see that $\sum ||x||_\infty |y_i|$, so by the comparision theorem, $\Lambda x = \sum x_i y_i$ converges absolutely. This is true for all $x \in c_0$ so we have for $x, z = \{z_i\} \in c_0$ and $\alpha \in \mathbb{C}$ we have:

$$\Lambda(x+z) = \sum (x_i + z_i) y_i = \sum x_i y_i + \sum z_i y_i = \Lambda x + \Lambda z \tag{13}$$

$$\Lambda(\alpha x) = \sum \alpha x_i y_i = \alpha \Lambda x \tag{14}$$

$$|\Lambda(x)| = |\sum x_i y_i| \leq \sum |x_i y_i| \leq ||x||_\infty \sum |y_i| = ||x||_\infty ||y||_1 \tag{15}$$

The final equation tells us that for $||x||_\infty \leq 1$, we have $|\Lambda(x)| \leq ||y||_1$ so $\Lambda(x)$ is indeed a bounded linear functional.

Now for all $n \in \mathbb{N}^*$, let $y^n = \{y_i{}^n\}$ is such that

$$y_i{}^n = \begin{cases} 0 \text{ if } i > n \text{ or } y_i = 0 \\ \\ \frac{\bar{y_i}}{|y_i|} \text{ otherwise} \end{cases}$$

Note is clear that $y^n \in c_0$ and that $||y^n||_\infty = 1$, now let $\epsilon > 0$ and N be such that for all $n \geq N$: $|\sum_{i=1}^\infty |y_i| - \sum_{i=1}^n |y_i|| = \sum_{i=n+1}^\infty |y_i| < \epsilon$.

$$||\Lambda(y^n)| - ||y||_1| = |\sum_{i=1}^n |y_i| - \sum_{i=1}^\infty |y_i|| = \sum_{i=n+1}^\infty |y_i| < \epsilon \tag{16}$$

So $|\Lambda(y^n)| < ||y||_1 + \epsilon$ We can find such an element for all $\epsilon > 0$, so we see that $||\Lambda|| = ||y||_1$.

Now let, $\Lambda \in (c_0)^*$, we define $y = \{y_i\} = \{\Lambda(e_i)\}$, where e_i is the sequence such that the i^{th} element is 1 and all other elements are 0.

Now let $x^n = \{{x_i}^n\}$, where $x_i^n = \begin{cases} 0 \text{ if } i > n \text{ or } y_i = 0 \\ \\ \frac{\bar{y_i}}{|y_i|} \text{ otherwise} \end{cases}$

Then we have:

$$|\sum_{i=1}^{n} |y_i|| = |\sum_{i=1}^{n} y_i {x_i}^n| = |\sum_{i=1}^{n} \Lambda(e_i) x_i^n| = |\Lambda(\sum_{i=1}^{n} e_i {x_i}^n)| = |\Lambda(x^n)| \leq ||\Lambda|| \tag{17}$$

Since this is true for all n, and Λ is bounded. Then we see that $\sum_{i=1}^{\infty} |y_i| < \infty$, so $y \in \ell^1$.

Now let $x = \{x_i\} \in c_0$, then:

$$\Lambda(\sum_{i=1}^{n} e_i x_i) = \sum_{i=1}^{n} x_i \Lambda(e_i) = \sum_{i=1}^{n} x_i y_i$$

This is true for all n, so by taking limits using the fact that Λ is bounded we get that:

$$\Lambda(x) = \Lambda(\sum_{i=1}^{\infty} e_i x_i) = \sum_{i=1}^{\infty} \Lambda(e_i) x_i = \sum_{i=1}^{\infty} y_i x_i \text{ for all } x \tag{18}$$

(b) Let $\Lambda \in (\ell^1)^*$, let $y = \{y_i\}$ where $y_i = \Lambda(e_i)$, where we define e_i as above. Since $||e_i||_1 = 1$ for all i:

$$|\Lambda(e_i)| \leq ||\Lambda||, \ \forall i; \ \therefore ||y||_\infty = \sup |\Lambda(e_i)| \leq ||\Lambda|| < \infty \tag{19}$$

For all $x = \{x_i\} \in \ell^1$ we have:

$$\Lambda(x) = \sum x_i \Lambda(e_i) = \sum x_i y_i \tag{20}$$

Now if $||x||_1 \leq 1$, we see that:

$$|\Lambda(x)| = |\sum x_i y_i| \leq ||y||_\infty ||x||_1 \leq ||y||_\infty \Rightarrow ||\Lambda|| \leq ||y||_\infty \tag{21}$$

Likewise since we can show that there is a i, such that $||e_i||_1 = 1$ and $||\Lambda(e_i) - ||y||_\infty|| = ||y_i - ||y||_\infty|| < \epsilon$ So $||\Lambda|| = ||y||_\infty$.

Likewise if we have a $\{y_i\} \in \ell^\infty$, then for all $x = \{x_i\} \in \ell^1$, we see that since $|y_i x_i| \leq ||y||_\infty |x_i|$ for all i. Then $\sum y_i x_i$ converges absolutely and so if we define $\Lambda(x) = \sum y_i x_i$ for all $x \in \ell^1$ we can see similarly to before that this is a bounded linear functional with $||\Lambda|| = ||y_i||_\infty$.

(c) Let $y = \{y_i\} \in \ell^1$, the same argument as in (a) can be used to show that $\Lambda x = \sum y_i x_i$, for all $x = \{x_i\} \in \ell^\infty$. Is a bounded linear functional on ℓ^∞ with $||\Lambda|| = ||y||_1$.

But let $c = \{\{x_n\} \in \ell^\infty \colon \{x_n\} \text{ converges in } \mathbb{C}\}$, now notice that $c \subseteq \ell^\infty$, since all convergent sequences are bounded. Furthermore, if $\{x_n\}, \{z_n\} \in c$ then from properties of the limit we have $\{x_n + z_n\} \in c$ and $\{\alpha x_n\} \in c$ for all $\alpha \in \mathbb{C}$.

So we see that c is a subspace of ℓ^∞. So let us define, $\gamma \colon c \to \mathbb{C}$, by:

$$\gamma(x) = \lim_{n\to\infty} x_n, \text{ where } x = \{x_n\} \tag{22}$$

This is clearly a linear functional on c, furtheremore for $x \in c$, with $||x||_\infty \leq 1$ then we have:

$$|\gamma(x)| = |\lim_{n\to\infty} x_n| = \lim_{n\to\infty} |x_n| \leq 1. \tag{23}$$

Recall we can pull out the limit since $|\cdot|$ is a continuous function and since $|x_n| \leq 1$, for all n we have that $\lim |x_n| \leq 1$.

Since the sequence $\{1, 1, 1 \ldots\} \in c$, we have $|\gamma(\{1, 1, \ldots\})| = \lim |1| = 1$. So $||\gamma|| = 1$.

Now by the Hahn-Banach Theorem, γ can be extended to a bounded linear functional Γ on ℓ^∞ such that $||\Gamma|| = ||\gamma||$.

Now notice that for all $x \in c_0$, we have $\Gamma(x) = \gamma(x) = \lim_{n\to\infty} x_n = 0$. So assume that there exists $y = \{y_n\} \in \ell^1$ such that:

$$\Gamma(x) = \sum x_i y_i \text{ for all } x_i \tag{24}$$

Then let $e^i \in c_0$, be the sequence such that $e^i{}_n = \delta^i{}_n = \begin{cases} 1 \text{ if } i = n \\ 0\text{otherwise} \end{cases}$. Now notice that:

$$y_i = \Gamma(e^i) = 0 \text{ for all } i \tag{25}$$

So $y = 0$ and $\Gamma = 0$, but this is impossible since $||\Gamma|| = 1$ (recall that $\Gamma(\{1, 1, 1, \dots\}) = 1$).
Therefore $\Gamma \in (\ell^\infty)^*$ but is not given by a $y \in \ell^1$.

(d) For all $k \geq 0$, let $T_k = \{(x_1, x_2, \dots, x_k, 0, 0, 0, \dots) \mid x_i \in \mathbb{C}\}$ and let $T = \bigcup_{k\geq 1} T_k$, we will first show that T is dense in c_0 and ℓ^1, then we will find a countable set that whose closure contains T.

- First of all it is clear that $T \subseteq c_0$.
  Let $x = \{x_1, x_2, \dots\} \in c_0$, let $\epsilon > 0$ and $N \geq 1$ be such that $|x_n| < \epsilon$, for all $n > N$. Now let:
  $$x^N = \{x_1, x_2, \dots, x_N, 0, 0, \dots\} \in T \tag{26}$$
  We have:
  $$||x - x^N||_\infty = ||(0, \dots, 0, x_{N+1}, x_{N+2}, \dots)||_\infty = \sup_{n>N} |x_n| \leq \epsilon \tag{27}$$
  So we can find a sequence $\{x^N\} \in T$ such that $x^N \to c_0$. So, $c_0 \subseteq \overline{T}$ therefore we indeed see that T is dense in c_0.
- Once again it is clear that $T \subseteq \ell^1$.
  Now let $y = \{y_1, y_2, \dots\} \in \ell^1$, and let $\epsilon > 0$, since $\{\sum_{i=1}^n |y_i|\}$ is a convergent sequence, it is a cauchy sequence. So let $N \geq 1$ be such that
  $$|\sum_{i=1}^{N} |y_i| - \sum_{i=1}^{m} |y_i|| = \sum_{i=N+1}^{m} |y_i| < \epsilon \text{ for all } m > N \tag{28}$$
  Since this is true for all $m > N$, this means that if $y^N = \{y_1, y_2, \dots, y_N, 0, 0, \dots\} \in T$ we have:
  $$||y - y^N||_1 = \sum_{i=1}^{\infty} |y_i - y_i{}^N| = \sum_{i=N+1}^{\infty} |y_i| = \sup_{m>N} \left(\sum_{i=N+1}^{m} |y_i| \right) < \epsilon \tag{29}$$
  So once again we indeed see that T is dense in ℓ^1.

Finally if for all k, let

$$S_k = \{(q_1 + ip_1, \dots, q_k + ip_k, 0, 0, \dots) \mid p_n, q_n \in \mathbb{Q}\} \text{ and } S = \bigcup_{k\geq 1} S_k \tag{30}$$

Then we see that S is the countable union of countable sets, so it's countable, furthermore since $\mathbb{Q}[i]$ is dense in $\mathbb{C}$, we see that $T \subseteq \overline{S}$, so we see that S is dense in c_0 and in ℓ^1. So they are indeed seperable.

Now let $V = \{(x_1, x_2, \dots) \in \ell^\infty \mid x_i \in \{0, 1\} \text{ for all } i \geq 1\}$. Note that for any $x, y \in V$ such that $x \neq y$, we have $x_i \neq y_i$ for some i, so WLOG we have $x_i = 1$ and $y_i = 0$, and so we have:

$$||x - y||_\infty = \sup_n |x_n - y_n| = 1 \tag{31}$$

So for all $x \in V$, we define $B(x, \frac{1}{2}) = \{z \in \ell^\infty \mid ||x - z||_\infty < \frac{1}{2}\}$. So $B(x, \frac{1}{2}) \cap B(y, \frac{1}{2}) = \emptyset$, for all $x \neq y$. Now let S be a dense subset of ℓ^∞, then notice that since $B(x, \frac{1}{2})$ is open we have that:

$$S \cap B(x, \frac{1}{2}) \neq \emptyset, \text{ for all } x \in V \text{ say } v_x \in S \cap B(x, \frac{1}{2}) \tag{32}$$

Now notice that $v_x \neq v_y$ for all $x, y \in V$ such that $x \neq y$.

So we notice that $\{v_x \mid x \in V\} \subseteq S$, but since we have an injection from $[0,1]$ to V, by $\sum_{i=1}^{\infty} 2^{x_i} \to \{x_i\}$, we see that V is uncountable. Therefore, $\{v_x \mid x \in V\}$ is uncountable so S is also. Since S is an arbitrary dense set in ℓ^∞, we see that ℓ^∞ is not seperable.

□

Exercise 11. For $0 < \alpha \leq 1$, let Lip α **denote the space of all complex functions f on $[a, b]$ for which**

$$M_f = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s-t|^\alpha} < \infty$$

(a) **Prove that** Lip α **is a Banach space, if** $||f|| = |f(a)| + M_f$.

(b) **Prove that** Lip α **is a Banach space, if** $||f|| = M_f + \sup_x |f(x)|$.

Proof. Notice that for all $f, g \in \text{Lip } \alpha$, we have for all $s \neq t$:

$$\frac{|(f(s) + g(s)) - (f(t) + g(t))|}{|s-t|^\alpha} \leq \frac{|f(s) - f(t)|}{|s-t|^\alpha} + \frac{|g(s) - g(t)|}{|s-t|^\alpha} \leq M_f + M_g$$

Therefore, $M_{f+g} \leq M_f + M_g$.

And for all $\omega \in \mathbb{C}$ we have:

$$\sup_{s \neq t} \frac{|\omega f(s) - \omega f(t)|}{|s-t|^\alpha} = |\omega| \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s-t|^\alpha} = |\omega| M_f$$

Furthermore, let $f \in \text{Lip } \alpha$ and $\epsilon > 0$, for all $s, t \in [a, b]$ with $s \neq t$, such that $|s - t| < \sqrt[\alpha]{\frac{\epsilon}{M_f}}$, we have:

$$\frac{|f(s) - f(t)|}{|s-t|^\alpha} < M_f \Rightarrow |f(s) - f(t)| < M_f |s-t|^\alpha < \epsilon$$

So all $f \in \text{Lip } \alpha$ are uniformely continuous.

(a) For all $f, g \in \text{Lip } \alpha$ and $\omega \in \mathbb{C}$:

$$||f + g|| = |f(a) + g(a)| + M_{f+g} \leq |f(a)| + |g(a)| + M_f + M_g = ||f|| + ||g||$$
$$||\omega f|| = |\omega| \cdot |f(a)| + |\omega| M_f = |\omega| \cdot ||f||$$

Si this is indeed a norm on Lip α. Now Let $\{f_n\}$ be a Cauchy sequence in Lip α.

So we let $\epsilon > 0$ and $N \geq 1$ such that for $n \geq m \geq N$:

$$\sup_{s \neq t} \frac{|(f_n(s) - f_m(s)) - (f_n(t) - f_m(t))|}{|s-t|^\alpha} = M_{f_n - f_m} \leq |f_n(a) + f_m(a)| + M_{f_n - f_m} = ||f_n - f_m|| < \epsilon$$

This means that for all $s \neq t$ we have:

$$|(f_n(s) - f_m(s)) - (f_n(t) - f_m(t))| < \epsilon |s-t|^\alpha$$

(b) For all $f, g \in \text{Lip } \alpha$ and $\omega \in \mathbb{C}$:

$$||f + g|| = \sup_x |f(x) + g(x)| + M_{f+g} \leq \sup_x |f(x)| + \sup_x |g(x)| + M_f + M_g = ||f|| + ||g||$$
$$||\omega f|| = |\omega| \cdot \sup_x |f(x)| + |\omega| M_f = |\omega| \cdot ||f||$$

Si this is indeed a norm on Lip α. Now Let $\{f_n\}$ be a Cauchy sequence in Lip α. Then note for every $\epsilon > 0$, let $N \geq 1$ such that for all $m, n \geq N$ we have:

$$\sup_x |f_n(x) - f_m(x)| \leq \sup_x |f_n(x) + f_m(x)| + M_{f_n + f_m} = ||f_n - f_m|| < \epsilon \tag{33}$$

Therefore for all $x \in [a, b]$ we have $|f_n(x) - f_m(x)| < \epsilon$, so $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$. So we define $f \colon [a, b] \to \mathbb{C}$ such that

$$f(x) = \lim_{n \to \infty} f(x) \tag{34}$$

Not done

□

Exercise 13.

Proof. (a) For all $n \in \mathbb{N}^*$, let:

$$F_n\colon X \to \mathbb{C}, \text{ such that } F_n(x) = f_n(x) \tag{35}$$

Now note that since $\lim_{n\to\infty} f_n(x)$ exists for all $x \in X$, we have that $\{f_n(x)\}$ is a bounded sequence for all $x \in X$. Therefore:

$$\sup_{n\in\mathbb{N}^*} |F_n(x)| = \sup_{n\in\mathbb{N}^*} |f_n(x)| < \infty \text{ for all } x \tag{36}$$

So by Banach-Steinhaus, there exists $M < \infty$ such that:

$$\sup_{||x||\leq 1} |f_n(x)| = ||F_n|| \leq M \text{ for all } n \in \mathbb{N}^* \tag{37}$$

But if we let $V = \{x \in X\colon ||x|| < 1\} = B(0,1)$, be the open unit ball. Then:

$$|f_n(x)| \leq ||F_n|| \leq M \text{ for all } x \in V, \ \& \ n = 1, 2, 3, \dots \tag{38}$$

(b) Let $\epsilon > 0$, and for $N = 1, 2, 3, \dots$ let:

$$A_N = \{x \in X\colon |f_m(x) - f_n(x)| \leq \epsilon, \text{ if } m \geq N \text{ and } n \geq N\} \tag{39}$$

First we claim that A_N is closed for all N. This is indeed true let, $\{x_k\}$ be a sequence in A_N such that $x_k \to x \in X$. Then since f_n is continuous for all n we see that:

$$|f_m(x) - f_n(x)| = \lim_{k\to\infty} |f_m(x_k) - f_n(x_k)| \leq \epsilon, \text{ since } |f_m(x_k) - f_n(x_k)| \leq \epsilon \text{ for all } k \tag{40}$$

So $x \in A_N$, so A_N is indeed closed.

Now let $x \in X$ be arbitrary, since $\{f_n(x)\}$ converges, there is some N such that $x \in A_N$, by the definition of a Cauchy sequence. So we see that $X = \bigcup A_N$, since X is complete by Baire's category theorem it is not of first category, so there is a N such that $\overline{A_N} = A_N$ has a nonempty interior.

Let V be a non-empty open set in A_N. Then we have that:

$$\begin{aligned} &|f_m(x) - f_n(x)| \leq \epsilon \text{ for all } x \in V, \text{ for all } m, n \geq N \\ \therefore\ &|f(x) - f_n(x)| = |\lim_{m\to\infty} f_m(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \leq \epsilon \text{ for all } x \in V, \text{ for all } n \geq N \end{aligned}$$

□