Exercise 1 Does there exists an infinite σ-algebra which has only countably many members?

Proof. Assume that $\mathcal{M}$ is an infinite σ-algebra with countably many members in a set X. So let $\mathcal{M} = \{A_0, A_1, A_2, \ldots\}$ where $A_0 = \emptyset$.

For all $x \in X$, we define:

$$B_x \coloneqq \bigcap_{x \in A_i} A_i$$

We claim that the B_x for a partition for X

- It is clear that $\bigcup_{x \in X} B_x = X$, since for all $x \in X \Rightarrow x \in B_x$. This also tells us that $B_x \neq \emptyset$, for all x.
- Assume that $B_x \cap B_y \neq \emptyset$. So let $z \in B_x \cap B_y$. Then let

$\square$

Exercise 2 Prove the analogue of Theorem 1.8. for n functions.

Theorem 1. ***1.8*** *Let u and v be real measurable functions on a measurable space X, and Φ be a continuous mapping of the plane into a topological space Y, and define:*

$$h(x) = \Phi(u(x), v(x))$$

for $x \in X$. Then $h \colon X \to Y$ is measurable.

Exercise 3 Prove that if f is a real function on a measurable space X such that $\{x \colon f(x) \geq r\}$ is a measurable for every rational r, then f is measurable.

Proof. Recall from theorem 1.12, that if $f^{-1}((a, \infty])$ is measurable for every real a, then f is measurable.

So let $a \in \mathbb{R}$ and $\{r_n\}$ be a sequence of rational funcitons such that $r_1 < r_2 < \cdots \leq a$ and $\lim_{n \to \infty} r_n = a$.

$$A_n = \{x \colon f(x) \geq r_n\} \text{ is measurable for all } n$$

So note that, if x is such that $f(x) \geq a$ then $f(x) \geq r_n$ for all n. And if for all $n \in \mathbb{N}$ we have $r_n \leq f(x)$ then $a \leq f(x)$ since if $f(x) < a$, then there exists m such that $f(x) < r_m < a$ since $r_n \to a$.

So we see that

$$f^{-1}((a, \infty]) = \{x \colon f(x) \geq a\} = \bigcap A_n \tag{1}$$

Since each A_n is measurable then $f^{-1}((a, \infty])$ is measurable. Since $a \in \mathbb{R}$ was chosen arbitrarily, this is true for all a, so f is measurable. $\square$

Exercise 4 Let $\{a_n\}$ and $\{b_n\}$ be sequence in $[-\infty, \infty]$ and prove the following assertions:

(a)

$$\limsup_{n \to \infty} (-a_n) = -\liminf_{n \to \infty} a_n$$

(b)

$$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$$

Provide none of the sums are of the form $\infty - \infty$. Also show by an example that a strict inequality can hold.

(c) If $a_n \leq b_n$ for all n then:

$$\liminf_{n \to \infty} a_n \leq \liminf_{n \to \infty} b_n$$

Proof. (a) Note that for all k, let $b_k = \sup\{-a_k, -a{+}_{k+1}, -a_{k+2}, \ldots\}$, for all $n \geq k$:

$$-a_n \leq b_k \Rightarrow -b_k \leq a_n \Rightarrow -b_k = \inf \{a_k, a_{k+1}, \ldots\}$$

Now notice that

$$\begin{aligned} -\limsup_{n \to \infty} (-a_n) &= -\inf\{b_1, b_2, \ldots\} \\ &= \sup\{-b_1, -b_2, \ldots\} \\ &= \liminf_{n \to \infty} (a_n) \end{aligned}$$

(b) We assume that for none of the k we have $a_k + b_k$ is of the form $\infty - \infty$ and likewise we assume that $\limsup a_n + \limsup b_n$ is not of that form.

In this case all of the sums are well defined.

Let $k \in \mathbb{N}^*$, we let $A_k = \sup\{a_k, a_{k+1}, \ldots\}$ and $B_k = \sup\{b_k, b_{k+1}, \ldots\}$, and $C_k = \sup\{a_k + b_k, a_{k+1} + b_{k+1}, \ldots\}$. So for all k and $m, l \leq k \leq n$:

$$a_n + b_n \leq A_k + B_k \leq A_m + B_l$$

So

$$C_k \leq A_m + B_l \text{ for all } m, l \leq k$$

Therefore for all m, l we have:

$$\limsup_{n\to\infty}(a_n + b_n) = \inf\{C_1, C_2, \ldots\} \leq A_m + B_l$$

Indeed this is true since for all $m, l \in \mathbb{N}^*$ there is a $k \geq m.l$ so there is a $C_k \leq A_m + B_l$.

Now we will fix l, notice that since $\limsup_{n\to\infty}(a_n + b_n) \leq A_m + B_l$, for all m, So $\limsup_{n\to\infty}(a_n + b_n)$ is a lower bound for $\{A_1 + B_l, A_2 + B_l, \ldots\}$, therefore:

$$\limsup_{n\to\infty}(a_n + b_n) \leq \inf\{A_1 + B_l, A_2 + B_l, \ldots\} = \inf\{A_1, A_2, \ldots\} + B_l = \limsup_{n\to\infty}(a_n) + B_l \tag{†}$$

So (†) is true for all $l \in \mathbb{N}^*$, so similarly we see that

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(a_n) + \limsup_{n\to\infty}(b_n)$$

Now let $a_n = \cos^2(\frac{n\pi}{2})$ and $a_n = \sin^2(\frac{n\pi}{2})$. On the one hand we have:

$$a_n + b_n = \cos^2(\frac{n\pi}{2}) + \sin^2(\frac{n\pi}{2}) = 1 \text{ for all } n$$

So it is clear that $\limsup_{n\to\infty}(a_n + b_n) = \limsup_{n\to\infty}(1) = 1$.

On the other hand we know that

$$a_n = \cos^2(\frac{n\pi}{2}) = \begin{cases} 1 \text{ if } n = 0 \pmod 2 \\ 0 \text{ if } n = 1 \pmod 2 \end{cases} \quad \text{and } b_n = \sin^2(\frac{n\pi}{2}) = \begin{cases} 0 \text{ if } n = 0 \pmod 2 \\ 1 \text{ if } n = 1 \pmod 2 \end{cases}$$

So it is easy to see that $\limsup a_n = 1 = \limsup b_n$. Therefore:

$$\limsup(a_n + b_n) = 1 < 2 = \limsup(a_n) + \limsup(b_n)$$

(c) For all k let $A_k = \inf\{a_k, a_{k+1}, \ldots\}$ and $B_k = \inf\{b_k, b_{k+1}, \ldots\}$. We have:

$$A_k \leq a_n \leq b_n \text{ for all } n \geq k$$

So A_k is a lower bound of $\{b_k, b_{k+1}, \ldots\}$, so

$$A_k \leq B_k \text{ for all } k$$

Now for all n we have:

$$A_n \leq B_n \leq \sup\{B_1, B_2, \ldots\} = \liminf(b_n)$$

So $\liminf(b_n)$ is an upper bound for $\{A_1, A_2, \ldots\}$ so:

$$\liminf(a_n) = \sup\{A_1, A_2, \ldots\} \leq \liminf(b_n)$$

□

Exercise 5

(a) Suppose $f\colon X \to [-\infty, \infty]$ and $g\colon X \to [-\infty, \infty]$ are measurable. Prove that the sets:

$$\{x\colon f(x) < g(x)\}, \{x\colon f(x) = g(x)\}$$

are measurable.

(b) Prove that the set at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Proof. (a) If x is such that $f(x) < g(x)$, then there exists a r such that $f(x) < r < g(x)$, furthermore we can assume that $r \in \mathbb{Q}$. So for all $r \in \mathbb{Q}$ let us define:

$$A_r = \{x \colon f(x) < r\} \text{ and } B_r = \{x \colon r < g(x)\}$$

These sets are measurable since f, g are measurable.

Now notice that if $x \in A_r \cap B_r$, then we have $f(x) < r < g(x)$, so we are almost done! Let us just define:

$$M = \bigcup_{r \in \mathbb{Q}} (A_r \cap B_r)$$

This is a measurable set since it is a countable union of measurable sets. And for all $x \in M$, there is a $r \in \mathbb{Q}$ such that $x \in A_r \cap B_r \Rightarrow f(x) < r < g(x)$. Conversely for any $x \in X$ such that $f(x) < g(x)$, there is a $r \in \mathbb{Q}$ such that $f(x) < r < g(x)$ so $x \in M$.

So we indeed see that

$$M = \{x \colon f(x) < g(x)\}$$

is measurable.

Now notice this tells us that $S = \{x \colon g(x) < f(x)\}$ and $T = \{x \colon f(x) < g(x)\}$ are measurable. Therefore:

$$S^c = \{x \colon f(x) \leq g(x)\}$$
$$T^c = \{x \colon g(x) \leq f(x)\}$$

Are both measurable, so the set:

$$\{x \colon f(x) = g(x)\} = S^c \cap T^c = \{x \colon f(x) \leq g(x)\} \cap \{x \colon g(x) \leq f(x)\}$$

Is also measurable.

(b) Let $\{f_n\}$ be a sequence of measurable real-valued functions. Let $A = \{x \colon \lim_{n \to \infty} f_n(x) < \infty\}$

□

Exercise 6 Let X be an uncountable set, let $\mathcal{M}$ be the collection of all sets $E \subseteq X$, such that E or E^c is at most countable. We define

$$\mu(E) = \begin{cases} 0 \text{ if } E \text{ is at most countable} \\ 1 \text{ if } E^c \text{ is at most countable} \end{cases}$$

Prove that $\mathcal{M}$ is a σ-algebra in X and that μ is a measure on $\mathcal{M}$. Describe the corresponding measurable functions and their integrals.

Proof.

First we will show that $\mathcal{M}$ is a σ-algebra

We will verify the three conditions:

- Since $X^c = \emptyset$ is countable we have $X \in \mathcal{M}$
- Assume that $A \in \mathcal{M}$, then either $A = (A^c)^c$ is at most countable or A^c is at most countable so $A^c \in \mathcal{M}$.
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□