1 Differentiation

1.1 Derivatives of Measures

Theorem 1. *Suppose μ is a complex Borel measure on $\mathbb{R}^1$ and*

$$f(x) = \mu((-\infty, x)) \text{ for } x \in \mathbb{R}^1 \tag{1}$$

If $x \in \mathbb{R}^1$ and A is a complex number, TFAE

(a) f is differentiable at x and $f'(x) = A$.

(b) For all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{\mu(I)}{m(I)} - A\right| < \epsilon \tag{2}$$

for every open segment I that contains x and whose length is less than δ. Note m is the Lebesgue measure on $\mathbb{R}^1$.

Proof. $(a) \Rightarrow (b)$ We will first show this for $A = 0$.

Since $f'(x) = A$, we have, for all $\epsilon > 0$ there is a $\delta > 0$ such that for (t, x) with $|t - x| < \delta$:

$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| = \left|\frac{\mu([t, x))}{t - x} - A\right| = \left|\frac{\mu([t, x))}{m([t, x))} - A\right| < \epsilon$$

□