Papa Rudin Chapter 4 solutions: These are some of my solutions for the exercises in chapter 4 of “Real and Complex Analysis”, 3rd edition, by Rudin. That I wrote out in 2022, in preperation for graduate studies.

**Exercise 1 If M is a closed subspace of H, then** $M = \left(M^{\perp}\right)^{\perp}$; **does a similar statement hold if for M not necessarily closed?**

First of all, let us assume that M is closed. Let $x \in \left(M^{\perp}\right)^{\perp}$; since M is closed we know that there exists:

$$P\colon H \to M \tag{1}$$
$$Q\colon H \to M^{\perp} \tag{2}$$

Such that $x = Px + Qx$, now let $y \in M^{\perp}$ be arbitrary. Notice that:

$$0 = (x, y) = \cancelto{0}{(Px, y)} + (Qx, y), \quad \forall y \in M^{\perp}$$

$$\therefore Qx \in \left(M^{\perp}\right)^{\perp}$$

So we conclude that $Qx \in \left(M^{\perp}\right)^{\perp} \cap (M^{\perp})$, which implies that $(Qx, Qx) = 0$ so $Qx = 0$.
From this we see that

$$x = Px \in M$$
$$\therefore \left(M^{\perp}\right)^{\perp} \subseteq M$$

But notice that the other inclusion is clear, by the definition of $M^{\perp}$, for all $x \in M$ we have $(x, y) = 0$ for all $x \in M^{\perp}$ so $x \in \left(M^{\perp}\right)^{\perp}$

Now what if $M$ is not closed? Well notice that:

$$\left(M^{\perp}\right)^{\perp} = \bigcap_{x \in M^{\perp}} x^{\perp}$$

Recall that

$$x^{\perp} = \{y \in H \mid (x, y) = 0\} = \varphi_x^{-1}(0)$$

Where

$$\varphi_x\colon H \to \mathbb{C} \tag{3}$$
$$y \to (x, y) \tag{4}$$

We will show that $x^{\perp}$ is closed for all $x \in H$, it is clear for $x = 0$ so we can assume that $x \neq 0$.

Let $y_n \to y_0$ in $H$, then let $\epsilon > 0$ and $N \in \mathbb{N}^*$ such that for all $n \geq N$:

$$||y_n - y_0|| < \frac{\epsilon}{||x||}$$

So we have:

$$\begin{aligned}|\varphi_x(y_0) - \varphi_x(y_n)| &= |(x, y_0 - y_n)| \\ &\leq ||x|| \cdot ||y_0 - y_n|| \text{ by Cauchy-Schwartz} \\ &< \epsilon\end{aligned}$$

So $\varphi_x$ is continuous, so we indeed see that $x^{\perp}$ is closed for all $x \in H$, so in particular it is closed for all $x \in \left(M^{\perp}\right)^{\perp}$, so $\left(M^{\perp}\right)^{\perp}$ is closed and contains $M$. So we have:

$$M \neq \overline{M} \subseteq \left(M^{\perp}\right)^{\perp} \tag{5}$$

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**Exercise 2 Let** $\{x_n\}_{n=1}^{\infty}$ **be a linearly independent set of vectors in H. Let:**

$$u_1 = \frac{x_1}{||x_1||};\; u_n = \frac{v_n}{||v_n||} \text{ where } v_n = x_n - \sum_{i=1}^{n-1} (x_n, u_i) u_i$$

**Show that** $\{u_n\}$ **is an orthonormal set such that:** $span\{x_1, \ldots, x_N\} = span\{u_1, \ldots, u_N\}\ \forall N.$

Since

$$||u_n|| = \begin{cases} ||\frac{x_1}{|x_1|}|| = 1 \text{ if } n = 1 \\ ||\frac{v_n}{||v_n||}|| = 1 \text{ otherwise} \end{cases}$$

We will use induction to show that for all $n \in \mathbb{N}^*$, we have $(u_n, u_k) = 0$ for $k < n$:

- Note that
$$(u_2, u_1) = \frac{1}{||x_1|| \cdot ||v_2||}\left(x_2 - \frac{1}{||x_1||}\cancelto{0}{(x_2, x_1)}u_1, x_1\right) = \frac{1}{||x_1|| \cdot ||v_2||}(x_2, x_1) = 0 \tag{1}$$

- Now assume that this is true for all $2 \leq i < n$, and let $k < n$

$$\begin{aligned}
(u_k, u_n) &= \frac{1}{||v_n||}(u_k, x_n - \sum_{i=1}^{n-1}(x_n, u_i)u_i) && (2)\\
&= \frac{1}{||v_n||}\left((u_k, x_n) - \sum_{i=1}^{n-1}(u_k, (x_n, u_i)u_i)\right) && (3)\\
&= \frac{1}{||v_n||}\left((u_k, x_n) - \sum_{i=1}^{n-1}\overline{(x_n, u_i)}(u_k, u_i)\right) && (4)\\
&= \frac{1}{||v_n||}\left((u_k, x_n) - \sum_{i=1}^{n-1}(u_i, x_n)\delta_{ik}\right) && (5)\\
&= \frac{1}{||v_n||}\left((u_k, x_n) - (u_k, x_n)\right) && (6)\\
&= 0 && (7)
\end{aligned}$$

So now let $n, m \in \mathbb{N}^*$ then WLOG $n \leq m$ so $(u_n, u_m) = \delta_{n,m}$

Now we will show that $span\{x_1, \ldots, x_N\} = span\{u_1, \ldots, u_N\}$ for all $N$.

Indeed, let $N \in \mathbb{N}^*$ then notice that: $u_i \in span\{x_1, \ldots, x_N\}$ for $1 \leq i \leq N$, therefore we see that

$$span\{x_1, \ldots, x_N\} \supseteq span\{u_1, \ldots, u_N\} \tag{8}$$

Now note that

$$x_1 = ||x_1||u_1 \in span\{u_1, \ldots, u_N\} \tag{9}$$

$$x_n = ||v_n||u_n + \sum_{i=1}^{n-1}(x_n, u_i)u_i \in span\{u_1, \ldots, u_n\} \tag{10}$$

So we indeed see that these two sets have the same *span*.

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**Exercise 3**

**Exercise 4**

**Exercise 5. Let $M = \{x \in H \mid Lx = 0\} \neq H$, where L is a continuous, linear functional. Then $M^\perp$ is a space of dimension** $1$.

(Note that if $M = H$, then $M^\perp = \{0\}$, is of dimension 0).

Recall that there exists a unique $y \in H \setminus \{0\}$ such that:

$$L(x) = (x, y) \text{ for all } x \in H \tag{1}$$

Now notice that for all $x \in M$ we have $(x, y) = L(x) = 0$. $\therefore y \in M^\perp$.
Let $z \in M^\perp \setminus \{0\}$, and $x \in H$. Recall that since

$$u = L(x)z - L(z)x \in M \tag{2}$$

We have:

$$0 = (u, z) = L(x)(z, z) - L(z)(x, z) \tag{3}$$
$$\therefore L(x)(z, z) = L(z)(x, z) \tag{4}$$

So for all $x \in H$:

$$L(x) = \frac{L(x)}{||z||^2}(z, z) = \frac{L(x)}{||z||^2}(x, z) = (x, \overline{\frac{L(z)}{||z||^2}} z) \tag{5}$$

So by uniquness of $y$ we see that, $y = \overline{\frac{L(z)}{||z||^2}} z$. So $z = \alpha y$ for some $\alpha \in \mathbb{C}$.

So we see that $M^\perp = \{\alpha y \mid \alpha \in \mathbb{C}\}$, so $\dim(M^\perp) = 1$.

■ **Exercise 6**