1 Differentiation

1.1 Derivatives of Measures

Theorem 1. *Suppose μ is a complex Borel measure on $\mathbb{R}^1$ and*

$$f(x) = \mu((-\infty, x)) \text{ for } x \in \mathbb{R}^1 \tag{1}$$

If $x \in \mathbb{R}^1$ and A is a complex number, TFAE

(a) f is differentiable at x and $f'(x) = A$.

(b) For all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{\mu(I)}{m(I)} - A\right| < \epsilon \tag{2}$$

for every open segment I that contains x and whose length is less than δ. Note m is the Lebesgue measure on $\mathbb{R}^1$.

Proof. $(a) \Rightarrow (b)$ Since $f'(x) = A$, we have, for all $\epsilon > 0$ there is a $\delta > 0$ such that for (t, x) with $|t - x| < \delta$:

$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| = \left|\frac{\mu([t, x))}{t - x} - A\right| = \left|\frac{\mu([t, x))}{m([t, x))} - A\right| < \epsilon \tag{†}$$

So for $I = (a, b)$ is any open interval containg x, of length less than δ. Now let $\{t_n\}$ be such that $a < \ldots < t_n < t_{n-1} < \ldots < t_1$. □

Definition 1.1. Let us fix a dimension k, denote the open ball with center $x \in \mathbb{R}^k$ and radius $r > 0$ by

$$B(x, r) = \{y \in \mathbb{R}^k \colon |y - x| < r\}$$

We associate to any Borel measure μ on $\mathbb{R}^k$ the quotients:

$$(Q_r\mu)(x) = \frac{\mu(B(x, r))}{m(B(x, r))}$$

Where m is the Lebesgue measure on R^k.

We define the **symmetric derivative** to be

$$(D\mu)(x) = \lim_{r \to 0} (Q_r\mu)(x)$$

Definition 1.2. Using the same notation as above we define the **maximal function** $M\mu$, for $\mu \geq 0$, to be defined by

$$(M\mu)(x) = \sup_{0 < r < \infty} (Q_r\mu)(x)$$

Remark. The maximal function of a complex Borel measure μ is, by definition, its total variation $|\mu|$.

Proposition 1. *The functions $M\mu \colon R^k \to [0, \infty]$ are lower semicontinuous, hence measurable.*

Proof. Assume $\mu \geq 0$, and let $\lambda > 0$ and $E = \{M\mu > \lambda\}$. Fix $x \in E$. Then there is an $r > 0$ such that:

$$\mu(B(x, r)) = tm(B(x, r)) \text{ for some } t > \lambda$$

Indeed since $\sup_{0<r<\infty} \frac{\mu(B(x,r))}{m(B(x,r))} > \lambda$. So for some r, we have $\frac{\mu(B(x,r))}{m(B(x,r))} > \lambda$. Letting $t = \frac{\mu(B(x,r))}{m(B(x,r))}$ gives us the desired result.

Furthermore there is a $\delta > 0$ such that:

$$(r + \delta)^k < \frac{r^k t}{\lambda}$$

If $|y - x| < \delta$, then $B(y, r + \delta) \supseteq B(x, r)$. Therefore

$$\mu(B(y, r + \delta)) \geq \mu(B(x, r)) = tm(B(x, r)) = t[\frac{r}{(r + \delta)}^k m(B(y, r + \delta)) > \lambda m(B(y, r + \delta))]$$

Thus $B(x, \delta) \subseteq E$. So E is open. □

Lemma 2. *If W is the union of a finite collection of balls $B(x_i, r_i)$, with $i \leq i \leq N$. Then there is a set $S \subseteq \{1, \ldots, N\}$ so that:*

(a) *the balls $B(x_i, r_i)$ with $i \in S$ are disjoint,*

(b)
$$W \subseteq \bigcup_{i \in S} B(x_i, 3r_i),$$

(c)
$$m(W) \leq 3^k \sum_{i \in S} m(B(x_i, r)i).$$

Proof. Order the balls $B_i = B(x_i, r_i)$ such that $r_1 \geq r_2 \geq \cdots \geq r_N$. Put $i_1 = 1$, discard all the B_j that intersect with B_{i_1}. Let B_{i_2} the first of our remaining balls, and discard all B_j with $j > i_2$ that intersect B_{i_2}, and let B_{i_3} be the first of the remaining ones, etc...

This process stops after a finite number of steps, since we only have a finite collection of balls, and we let $S = \{i_1, i_2, \ldots\}$. (a) holds by definition and (c) follows from (b) since $m(B(x_i, 3r_i)) = 3^k m(B(x_i, r_i))$.

So we just need to show (b). But notice for every discarded B_j, $B_j \cap B_i \neq \emptyset$ for some $i \in S$, where $r_i > r_j$. Assume that $X \in B_j \cap B_i$. We see that for all $x \in B_j$ we have:

$$\begin{aligned}
|x - x_i| &\leq |x - X| + |X - x_i| \\
&\leq |x - x_j| + |x_j - X| + |X - x_i| \\
&< r_j + r_j + r_i \text{ since } x, X \in B_j \text{ and } X \in B_i \\
&< 3r_i \text{ since } r_j \leq r_i
\end{aligned}$$

So we see that $B_j \subseteq B(x_i, 3r_i)$. This gives us (b). □

The maximal theorem

Theorem 3. *If μ is a complex Borel measure on $\mathbb{R}^k$ and λ is a positive number, then*

$$m\{M\mu > \lambda\} \leq 3^k \lambda^{-1} ||\mu|| \tag{i}$$

Here $||\mu|| = |\mu|(\mathbb{R}^k)$ and $m\{M\mu > \lambda\}$ is an abbreviation of $m(\{x \in \mathbb{R}^k : (M\mu)(x) > \lambda\})$

Proof. Fix μ and λ. Let K be a compact subset of the open set $\{M\mu > \lambda\}$. Each $x \in K$ is the center of an open ball B for which

$$|\mu|(B) > \lambda m(B)$$

Some finite collection of these B's covers K and Lemma 2 tells us there is a disjoint subcollection $\{B_1, \ldots, B_n\}$ such that:

$$m(K) \leq 3^k \sum_1^n m(B_i) \leq 3^k \lambda^{-1} \sum_1^n |\mu|(B_i) \leq 3^l \lambda^{-1} ||\mu||$$

The disjointess of the B_i's was used in the last inequality. So (i) follows by taking the supremum over all compact $K \subseteq \{M\mu > \lambda\}$. □

Weak L^1 If $f \in L^1(\mathbb{R}^k)$ and $\lambda > 0$, then

$$m\{|f| > \lambda\} \leq \lambda^{-1} ||f||_1$$

because, if we let $E = \{|f| > \lambda\}$, we have:

$$\lambda m(E) \leq \int_R |f| dm \leq \int_{\mathbb{R}^k} |f| dm = ||f||_1$$

Definition 1.3. Any measurable function f for which:

$$\lambda m\{|f| > \lambda\}$$

is a bounnded funtion of λ on $(0, \infty)$ is said to belong to **weak** L^1

So from above we see that the weak L^1 contains L^1. But it is also larger since for example if we let $f = \frac{1}{x}$ on $(0, 1)$, then for any $\lambda > 0$, we have

$$\frac{1}{x} > \lambda \iff x < \frac{1}{\lambda}$$

So we have $\lambda \cdot m\{|f| > \lambda\} \leq \lambda \cdot m(0, \frac{1}{\lambda}) = 1 < \infty$. So $\frac{1}{x}$ is weak L^1.

Definition 1.4. We associate to each $f \in L^1(\mathbb{R}^k)$ its **maximal function** $Mf\colon \mathbb{R}^k \to [0, \infty]$ by setting

$$(Mf)(x) = \sup_{0<r<\infty} \frac{1}{m(B_r)} \int_{B(x,r)} |f| \; dm$$

If we identify f with the measure μ given by $d\mu = f \; dm$, we see that this defintion agrees with the previously defined $M\mu$. So theorem 3 states that the "maximal operator" M sends L^1 to weak L^1, witha bound (namely 3^k) that depends only on the space $\mathbb{R}^k$, i.e: For every $f \in L^1(\mathbb{R}^k)$ and every $\lambda > 0$

$$m\{Mf > \lambda\} \leq 3^k \lambda^{-1} ||f||_1$$

Lebesgue points

Definition 1.5. If $f \in L^1(\mathbb{R}^k)$, any $x \in \mathbb{R}^k$ for which it is true that

$$\lim_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)| \; dm(y) = 0$$

is called a **Lebesgue point** of f.

For example this equation holds if f is continuous at the point x. More generally this equation holds, if the averages of $|f - f(x)|$ are not too small on the balls centered at x, i.e. The Lebesgue points of f are thepoints where f doesn't oscillate too much.

Theorem 4. *If $f \in L^1(R^k)$, then almost every $x \in \mathbb{R}^k$ is a Lebesgue point of f.*

Index