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1 Differentiation

1.1 Derivatives of Measures

Theorem 1. *Suppose μ is a complex Borel measure on $\mathbb{R}^1$ and*

$$f(x) = \mu((-\infty, x)) \text{ for } x \in \mathbb{R}^1 \tag{1}$$

If $x \in \mathbb{R}^1$ and A is a complex number, TFAE

(a) f is differentiable at x and $f'(x) = A$.

(b) For all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{\mu(I)}{m(I)} - A\right| < \epsilon \tag{2}$$

for every open segment I that contains x and whose length is less than δ. Note m is the Lebesgue measure on $\mathbb{R}^1$.

Proof. $(a) \Rightarrow (b)$ Since $f'(x) = A$, we have, for all $\epsilon > 0$ there is a $\delta > 0$ such that for (t, x) with $|t - x| < \delta$:

$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| = \left|\frac{\mu([t, x))}{t - x} - A\right| = \left|\frac{\mu([t, x))}{m([t, x))} - A\right| < \epsilon \tag{†}$$

So for $I = (a, b)$ is any open interval containg x, of length less than δ. Now let $\{t_n\}$ be such that $a < \ldots < t_n < t_{n-1} < \ldots < t_1$. ☐

Definition 1.1. Let us fix a dimension k, denote the open ball with center $x \in \mathbb{R}^k$ and radius $r > 0$ by

$$B(x, r) = \{y \in \mathbb{R}^k \colon |y - x| < r\}$$

We associate to any Borel measure μ on $\mathbb{R}^k$ the quotients:

$$(Q_r\mu)(x) = \frac{\mu(B(x, r))}{m(B(x, r))}$$

Where m is the Lebesgue measure on R^k.

We define the **symmetric derivative** to be

$$(D\mu)(x) = \lim_{r \to 0} (Q_r\mu)(x)$$

Definition 1.2. Using the same notation as above we define the **maximal function** $M\mu$, for $\mu \geq 0$, to be defined by

$$(M\mu)(x) = \sup_{0 < r < \infty} (Q_r\mu)(x)$$

Remark. The maximal function of a complex Borel measure μ is, by definition, its total variation $|\mu|$.

Proposition 1. *The functions $M\mu \colon R^k \to [0, \infty]$ are lower semicontinuous, hence measurable.*

Proof. Assume $\mu \geq 0$, and let $\lambda > 0$ and $E = \{M\mu > \lambda\}$. Fix $x \in E$. Then there is an $r > 0$ such that:

$$\mu(B(x, r)) = tm(B(x, r)) \text{ for some } t > \lambda$$

Indeed since $\sup_{0<r<\infty} \frac{\mu(B(x,r))}{m(B(x,r))} > \lambda$. So for some r, we have $\frac{\mu(B(x,r))}{m(B(x,r))} > \lambda$. Letting $t = \frac{\mu(B(x,r))}{m(B(x,r))}$ gives us the desired result.

Furthermore there is a $\delta > 0$ such that:

$$(r + \delta)^k < \frac{r^k t}{\lambda}$$

If $|y - x| < \delta$, then $B(y, r + \delta) \supseteq B(x, r)$. Therefore

$$\mu(B(y, r + \delta)) \geq \mu(B(x, r)) = tm(B(x, r)) = t[\frac{r}{(r + \delta)^k} m(B(y, r + \delta)) > \lambda m(B(y, r + \delta))]$$

Thus $B(x, \delta) \subseteq E$. So E is open. ☐