§ 3: Consequences of the Radon-Nikodym Theorem

Theorem 6.12

Let μ be a complex measure on a σ-algebra $\mathcal{M}$ in X. Then there is a measurable function h such that: $|h(x)| = 1$ for all $x \in X$ s/t:

$$d\mu = h\, d|\mu|$$

This is sometimes referred as the polar decomposition (representation) of μ.

Proof: Note that $\mu \ll |\mu|$, then by Radon-N. there is some $h \in L^1(|\mu|)$ s/t:

$$d\mu = h\, d|\mu|$$

Now let $A_r = \{x : |h(x)| < r\}$ for $r > 0$. Let $\{E_j\}$ be a partition of A_r. Then:

$$\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} h \, d|\mu| \right|$$

$$\leq \sum_j r |\mu|(E_j)$$

$$= r|\mu|(A_r)$$

$\therefore$

$|\mu|(A_r) \leq r|\mu|(A_r)$ so if $r<1$ then $|\mu|(A_r)=0$.

So $|h| \geq 1$ a.e. (Since $A_1 = \{x : |h(x)| < 1\}$ is s/t $\mu(A_1)=0$)

On the other hand, if $|\mu|(E) > 0$ then:

$$\left| \frac{1}{|\mu|(E)} \int_E h \, d|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1.$$

By theorem 1.40; $|h| \leq 1$ a.e.

$\therefore$ $B = \{x \in X : |h(x)| \neq 1\}$ is a set of measure 0; so we can redefine h s/t $h(x)=1$ on B.

(Indeed if $h^*(x) = \begin{cases} h(x) & \text{if } x \notin B \\ 1 & \text{if } x \in B \end{cases} \Rightarrow h^* \sim h$ so $h^* = h$ in $L^1(|\mu|)$

Theorem 6.13 Suppose that μ is a positive measure on $\mathfrak{M}$; $g \in L^1(\mu)$ and:

$$\lambda(E) = \int_E g \, d\mu$$

Then

$$|\lambda|(E) = \int_E |g| \, d\mu$$

Proof: By theorem 6.12; $\exists$ h s/t $|h| = 1$ and $d\lambda = h \, d|\lambda|$; furthermore; $d\lambda = g \, d\mu$ we have:

$$h \, d|\lambda| = g \, d\mu$$

$$\Rightarrow \quad d|\lambda| = h^{-1} g \, d\mu = \bar{h} g \, d\mu$$

Since $|\lambda| \geq 0$ and $\mu \geq 0 \Rightarrow \bar{h} g \geq 0$ a.e. $[\mu]$; so that $\bar{h} g = |g|$ a.e. $[\mu]$

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The Hahn decomposition theorem.

Let μ be a real measure on a σ-algebra $\mathcal{M}$ in a set X. Then $\exists$ sets $A, B \in \mathcal{M}$ s/t $A \cup B = X$, $A \cap B = \emptyset$ and:

$$\begin{cases} \mu^+(E) = \mu(A \cap E) \\ \mu^-(E) = -\mu(B \cap E) \end{cases}$$

We say that A carries all the positive mass of μ since $\mu(E) \geq 0$ if $E \subseteq A$ and B carries all the negative mass of μ since $\mu(E) \leq 0$ if $E \subseteq B$.

The pair (A, B) is called the Hahn-decomposition of X induced by μ.

Proof: $d\mu = h\, d|\mu|$, where $|h| = 1$. Since μ is real, it follows that h is real, hence $h = \pm 1$.

Let:

$$A = \{x : h(x) = 1\}, \quad B = \{x : h(x) = -1\}.$$

Since $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and since:

$$\frac{1}{2}(1+h) = \begin{cases} h & \text{on } A \\ 0 & \text{on } B \end{cases}$$

So for any $E \in \mathcal{M}$:

$$\mu^+(E) = \frac{1}{2}\int_E (1+h)\, d|\mu|$$

$$= \int_{E\cap A} h\, d|\mu|$$

$$= \mu(E \cap A)$$

Since $\mu(E) = \mu(E\cap A) + \mu(E \cap B)$ and $\mu = \mu^+ - \mu^-$
we see that

$$\mu^-(E) = \mu^+(E) - \mu(E) = \cancel{\mu(E\cap A)} - \cancel{\mu(E\cap A)} - \mu(E\cap B)$$

Corollary If $\mu = \lambda_1 - \lambda_2$ where λ_1 & λ_2 are pos measures, then $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.

Proof: Since $\mu \leq \lambda_1$:

$$\mu^+(E) = \mu(E \cap A) \leq \lambda_1(E \cap A) \leq \lambda_1(E)$$

Likewise for μ^-.

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