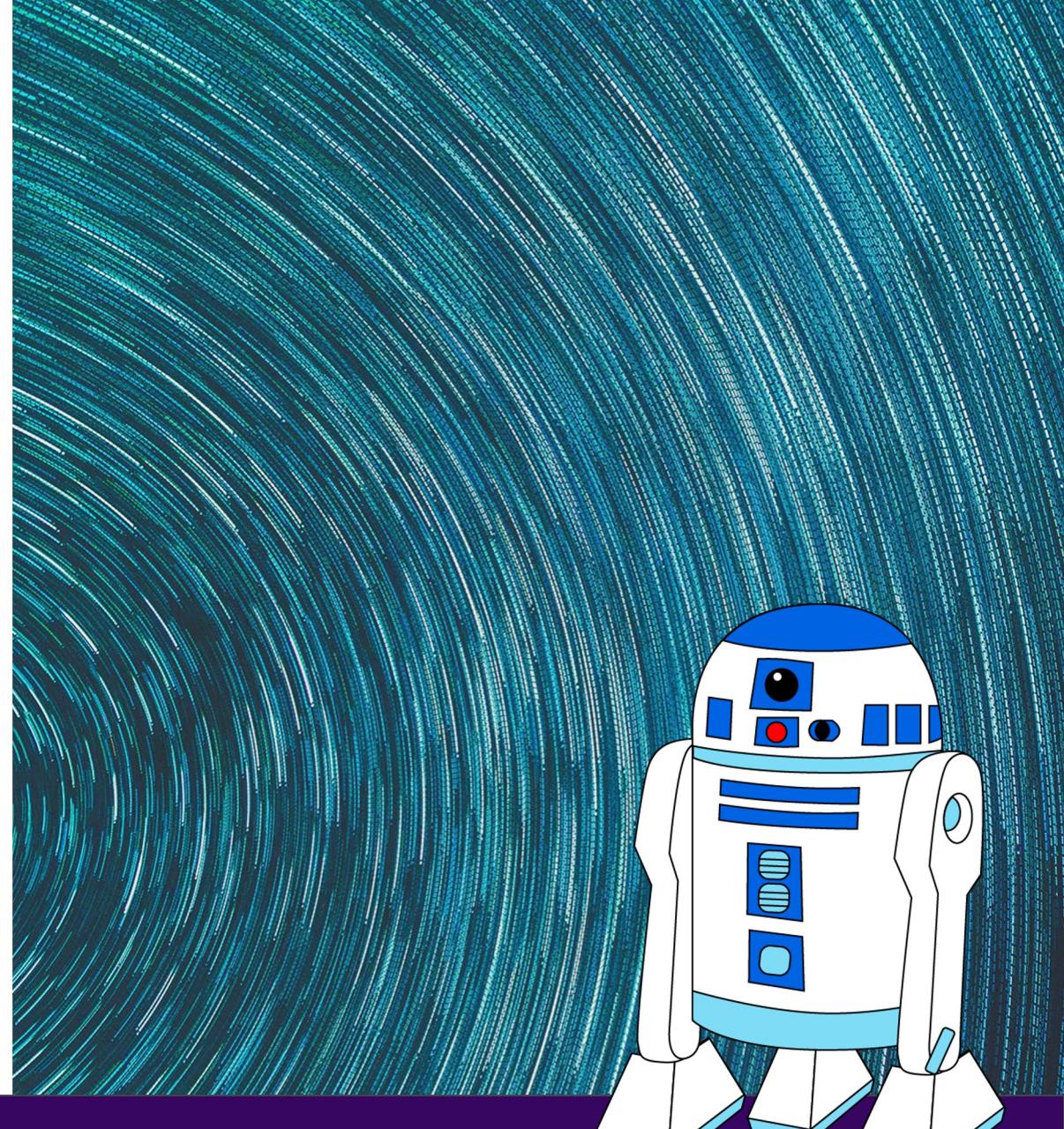


CIS 4210/5210:  
ARTIFICIAL INTELLIGENCE

# Game Formulation

Professor Chris Callison-Burch



# Games: Outline of Unit

- Part 1: Games as Search
  - Motivation
  - Game-playing AI successes
  - Game Trees
  - Evaluation Functions
- Part II: Adversarial Search
  - The Minimax Rule
  - Alpha-Beta Pruning

# May 11, 1997

may 11th game 6 : may 11 @ 3:00PM EDT | 19:00 GMT kasparov 2.5 deep blue 2.5

Home ▶ The match ▶ The players ▶ The technology ▶ Community

## Deep Blue Wins 3.5 to 2.5

### KASPAROV vs DEEP BLUE the rematch

With a dramatic victory in Game 6, Deep Blue won its six-game rematch with Champion Garry Kasparov ▶

**OVERVIEW**

▶ EVENT COVERAGE

▶ MATCH NEWS

▶ MAIN STORIES

**Commentary**  
George Plimpton on chess, Kasparov, and the limitations of computers  
▶ Read the article

**Commentary**  
Vishwanathan Anand on the legacy of Kasparov vs. Deep Blue  
▶ Read the article

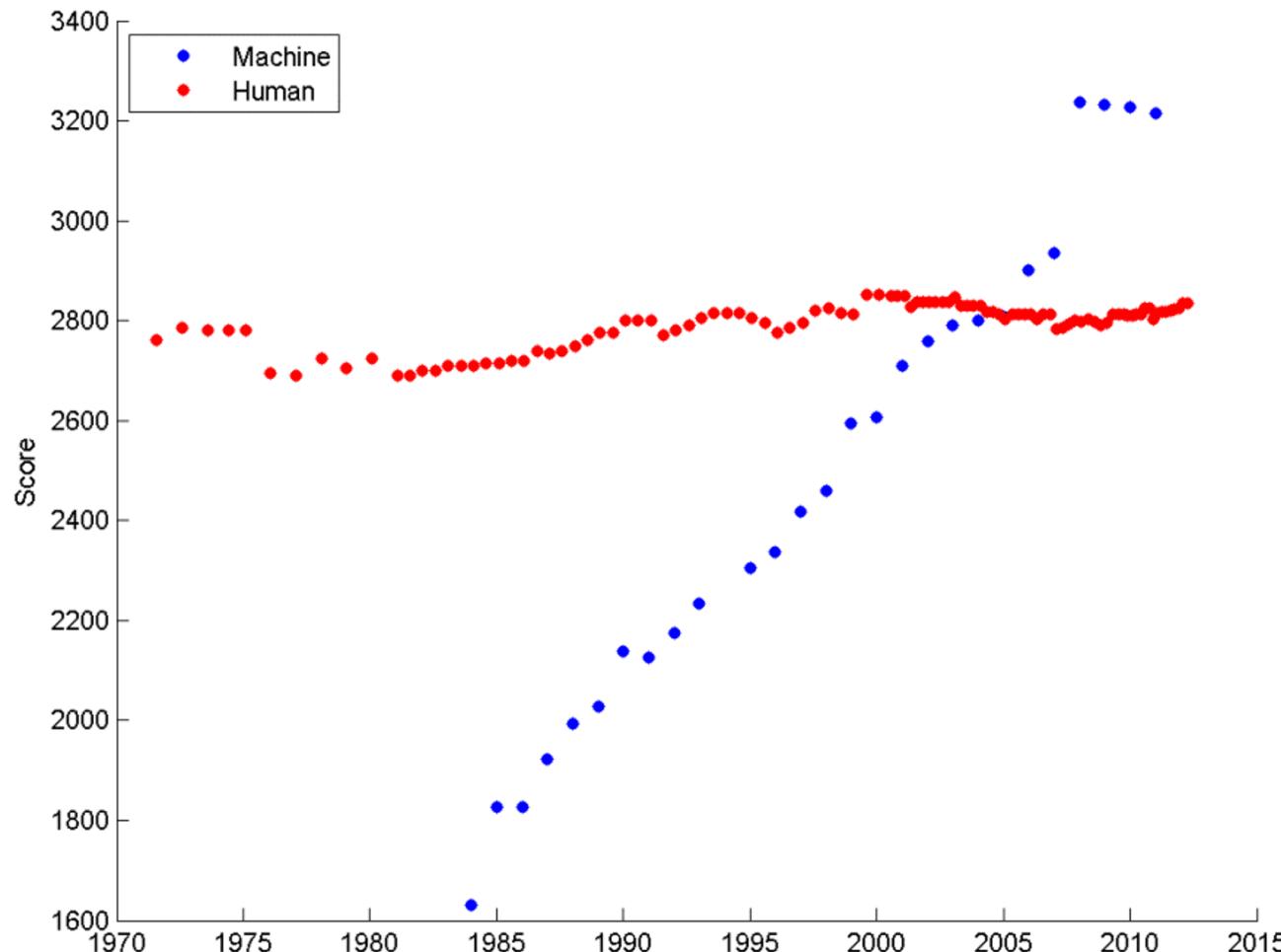
**Club Kasparov**  
Visit the virtual home of the world's greatest chess player.

**Guest essays**  
Thoughts on chess, computers, and what it all means  
▶ Read the essays...

**Community**  
During the rematch, more than 20,000 people from 120 countries joined the community to talk about the match.

**Clips from the rematch**  
Video footage from the games  
▶ Highlights from the games

# Ratings of human and computer chess champions



<https://srconstantin.wordpress.com/2017/01/28/performance-trends-in-ai/>

# AlphaGo seals 4-1 victory over Go grandmaster Lee Sedol

DeepMind's artificial intelligence astonishes fans to defeat human opponent and offers evidence computer software has mastered a major challenge

Steven Borowiec

Tuesday 15 March 2016 06.12 EDT



This article is 6 months old

Shares

613

Save for later



The world's top Go player, Lee Sedol, lost the final game of the Google DeepMind challenge match.  
Photograph: Yonhap/Reuters

Google DeepMind's AlphaGo program triumphed in its final game against South Korean Go grandmaster Lee Sedol to win the series 4-1, providing further evidence of the landmark achievement for an artificial intelligence program.

Lee started Tuesday's game strongly, taking advantage of an early mistake by AlphaGo. But in the end, Lee was unable to hold off a comeback by his opponent, which won a narrow victory.

# The Simplest Game Environment

- *Multiagent*
- *Static*: No change while an agent is deliberating.
- *Discrete*: A finite set of percepts and actions.
- *Fully observable*: An agent's sensors give it the complete state of the environment.
- *Strategic*: The next state is determined by the current state and the action executed by the agent and the actions of one other agent.

# Key properties of our games

1. Two players alternate moves
  2. Zero-sum: one player's loss is another's gain
  3. Clear set of legal moves
  4. Well-defined outcomes (e.g. win, lose, draw)
- Examples:
    - Chess, Checkers, Go,
    - Mancala, Tic-Tac-Toe, Othello ...



# More complicated games

- Most card games (e.g. Hearts, Bridge, etc.) and Scrabble
  - Stochastic, not deterministic**
  - Not fully observable: lacking in perfect information**
- Real-time strategy games
  - Continuous rather than discrete**
  - No pause between actions, don't take turns**
- Cooperative games

# Pac-Man



<https://youtu.be/-CbyAk3Sn9I>

# Formalizing the Game setup

1. Two players: **MAX** and **MIN**; **MAX** moves first.
2. **MAX** and **MIN** take turns until the game is over.
3. Winner gets award, loser gets penalty.

- **Games as search:**

*Initial state*: e.g. board configuration of chess

*Successor function*: list of (move,state) pairs specifying legal moves.

*Terminal test*: Is the game finished?

*Utility function*: Gives numerical value of terminal states.

e.g. win ( $+\infty$ ), lose ( $-\infty$ ) and draw (0)

**MAX** uses search tree to determine next move.

# How to Play a Game by Searching

- **General Scheme**
  1. Consider all legal successors to the current state ('board position')
  2. Evaluate each successor board position
  3. Pick the move which leads to the best board position.
  4. After your opponent moves, repeat.
- **Design issues**
  1. Representing the 'board'
  2. Representing legal next boards
  3. Evaluating positions
  4. Looking ahead

# Hexapawn: A very simple Game

- Hexapawn is played on a 3x3 chessboard



- Only standard pawn moves:
  1. A pawn moves forward one square onto an empty square
  2. A pawn “captures” an opponent pawn by moving diagonally forward one square, if that square contains an opposing pawn. The opposing pawn is removed from the board.

# Hexapawn: A very simple Game

- Hexapawn is played on a 3x3 chessboard



- Player P<sub>1</sub> wins the game against P<sub>2</sub> when:

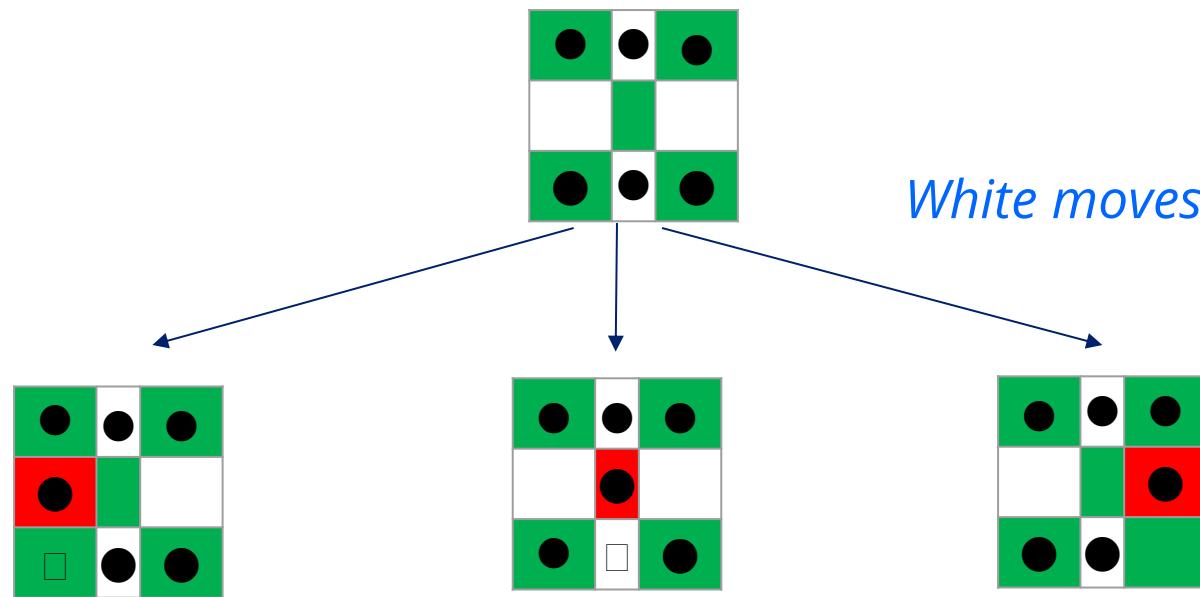
One of P<sub>1</sub>'s pawns reaches the far side of the board, or

P<sub>2</sub> cannot move because no legal move is possible.

P<sub>2</sub> has no pawns left.

*(Invented by Martin Gardner in 1962, with learning “program” using match boxes.)*

# Hexapawn: Three Possible First Moves

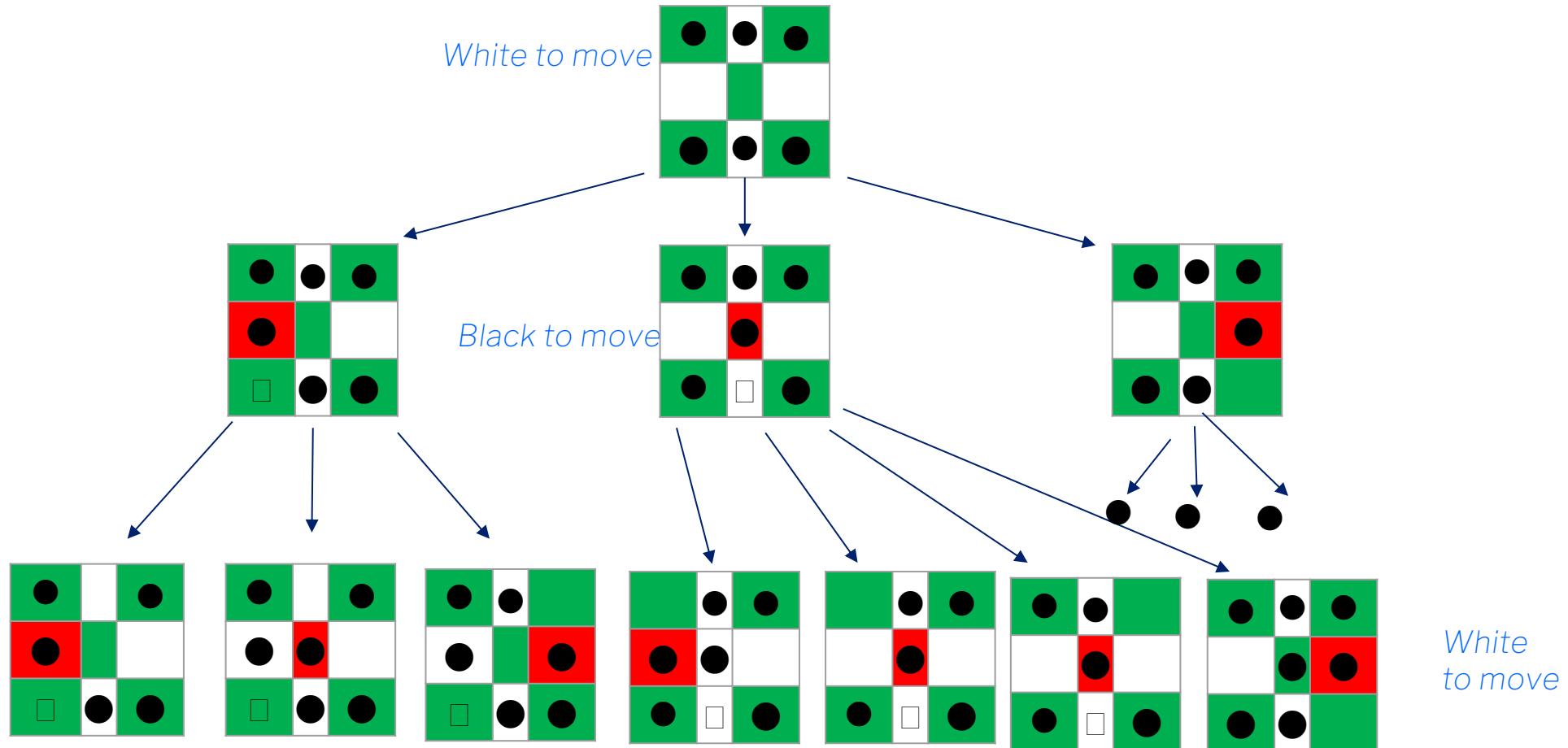


# Game Trees

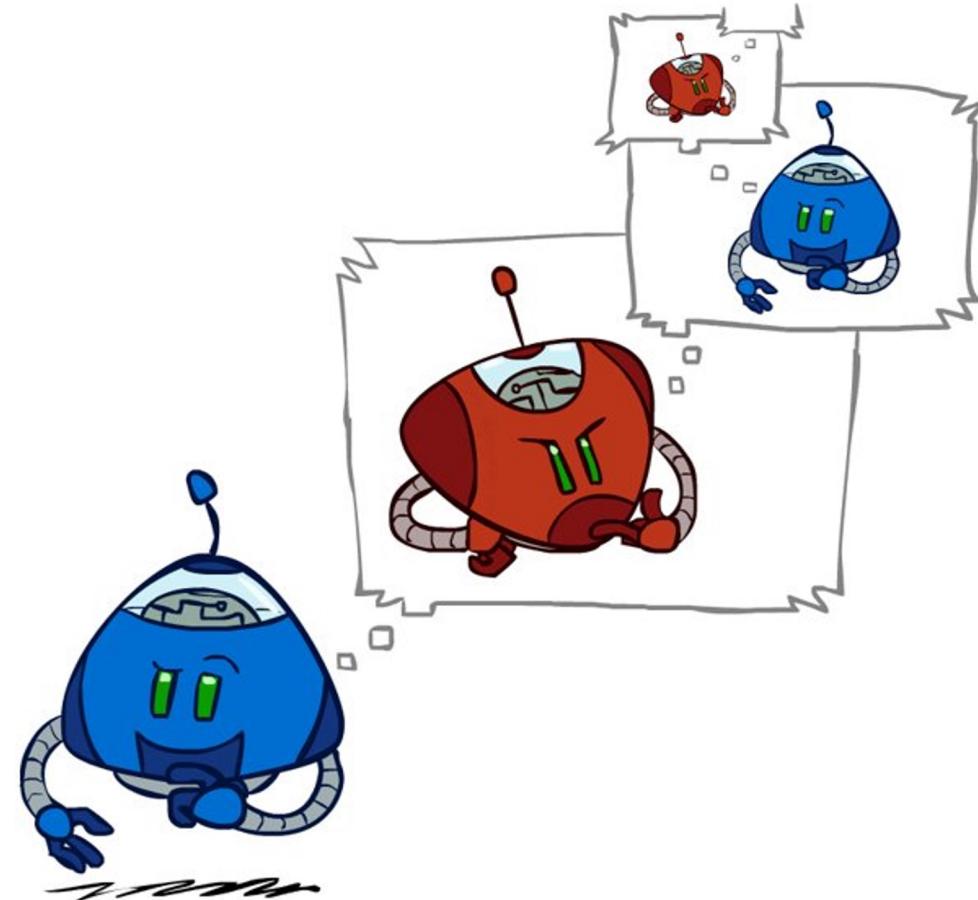
- **Represent the game problem space by a tree:**

Nodes represent 'board positions'; edges represent legal moves.  
Root node is the first position in which a decision must be made.

# Hexapawn: Simplified Game Tree for 2 Moves



# Adversarial Search



# Battle of Wits



FANDANGO  
**MOVIECLIPS** 

# Games: Outline of Unit

- Part 1: Games as Search
  - Motivation
  - Game-playing AI successes
  - Game Trees
  - Evaluation Functions**
- Part II: Adversarial Search
  - The Minimax Rule**
  - Alpha-Beta Pruning**

# MAX & MIN Nodes : An egocentric view

- Two players: MAX, MAX's opponent MIN
- *All play is computed from MAX's vantage point.*
- When MAX moves, MAX attempts to MAXimize MAX's outcome.
- When MAX's opponent moves, they attempt to MINimize MAX's outcome.
  - WE TYPICALLY ASSUME MAX MOVES FIRST:
- Label the root (level 0) MAX
- Alternate MAX/MIN labels at each successive tree level (*ply*).
- *Even levels* represent turns for MAX
- *Odd levels* represent turns for MIN

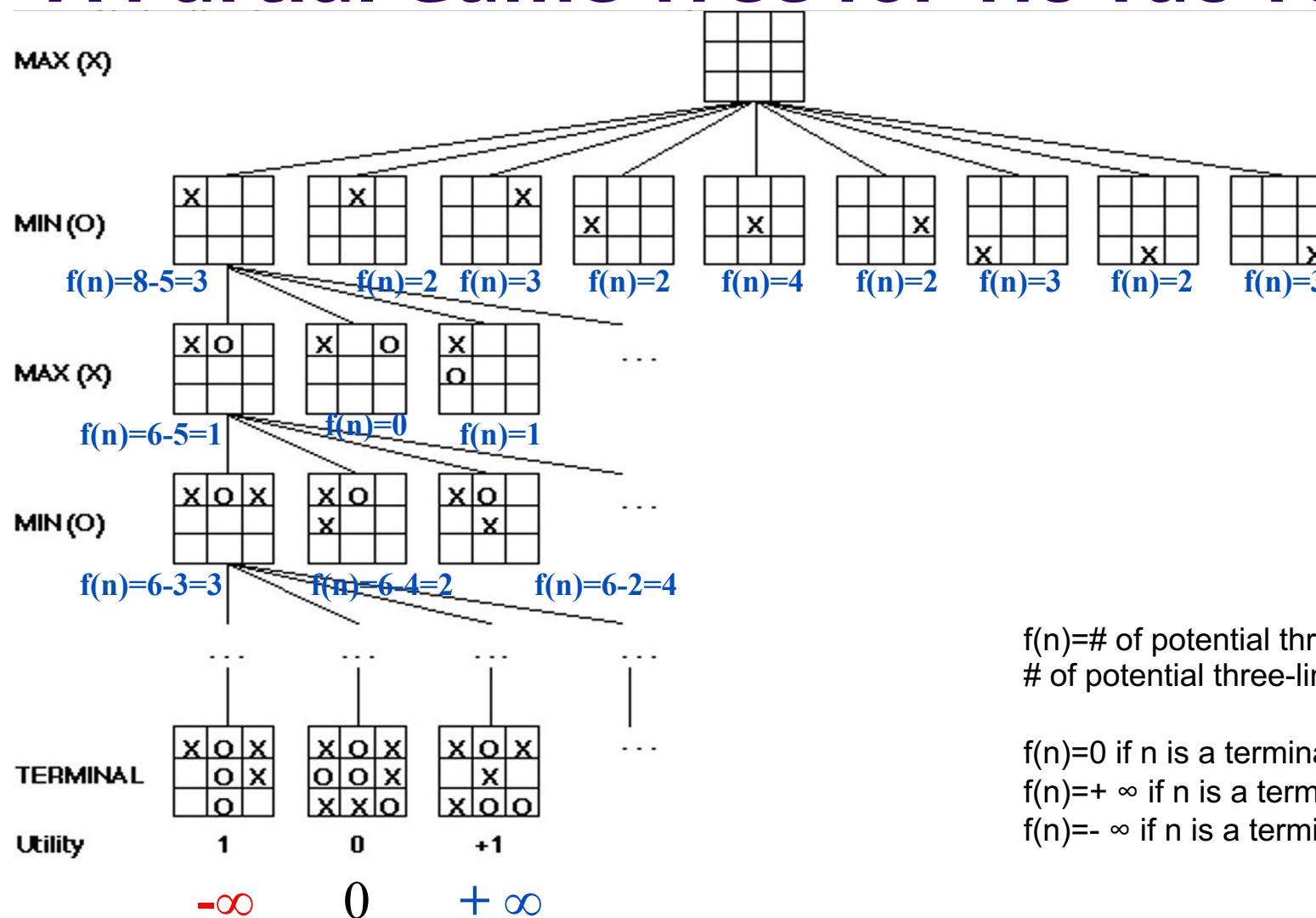
# Game Trees

- **Represent the game problem space by a tree:**  
Nodes represent ‘board positions’; edges represent legal moves.  
Root node is the first position in which a decision must be made.
- **Evaluation function  $f$  assigns real-number scores to `board positions' *without reference to path***
- **Terminal nodes represent ways the game could end, labeled with the desirability of that ending (e.g. win/lose/draw or a numerical score)**

# Evaluation functions: $f(n)$

- Evaluates how good a ‘board position’ is
- Based on *static features* of that board alone
- Zero-sum assumption lets us use one function to describe goodness for both players.
  - $f(n) > 0$  if MAX is winning in position  $n$
  - $f(n) = 0$  if position  $n$  is tied
  - $f(n) < 0$  if MIN is winning in position  $n$
- Build using expert knowledge,  
Tic-tac-toe:  $f(n) = (\# \text{ of 3 lengths open for MAX}) - (\# \text{ open for MIN})$

# A Partial Game Tree for Tic-Tac-Toe



$f(n) = \# \text{ of potential three-lines for X} - \# \text{ of potential three-line for O}$

$f(n)=0$  if n is a terminal tie  
 $f(n)=+\infty$  if n is a terminal win  
 $f(n)=-\infty$  if n is a terminal loss

# Chess Evaluation Functions

- Claude Shannon argued for a chess evaluation function in a 1950 paper
- Alan Turing defined function in 1948:  
 $f(n) = (\text{sum of A's piece values}) - (\text{sum of B's piece values})$
- More complex: weighted sum of *positional* features:  
 $\sum w_i \text{feature}_i(n)$
- Deep Blue had >8000 features

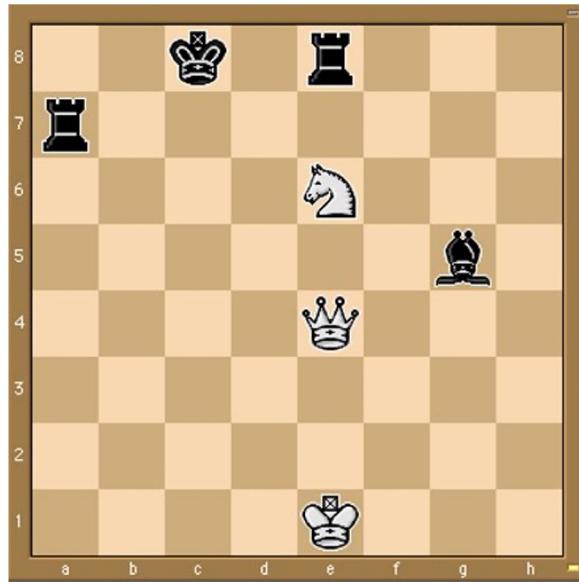
Pawn	1.0
Knight	3.0
Bishop	3.25
Rook	5.0
Queen	9.0

Pieces values for a simple Turing-style evaluation function often taught to novice chess players

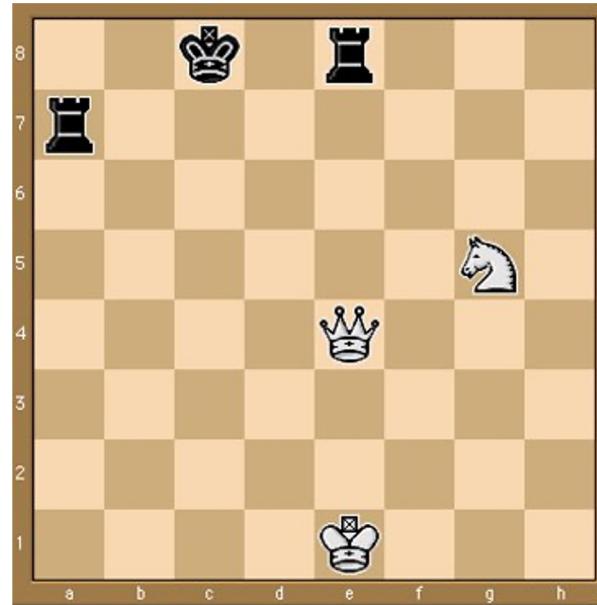
**Positive:** rooks on open files, knights in closed positions, control of the center, developed pieces

**Negative:** doubled pawns, wrong-colored bishops in closed positions, isolated pawns, pinned pieces  
*Examples of more complex features*

# Some Chess Positions and their Evaluations

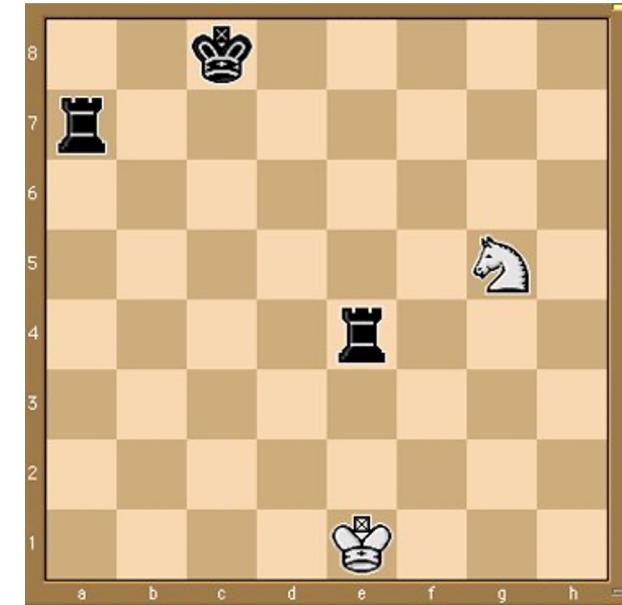


White to move  
 $f(n) = (9+3)-(5+5+3.25)$   
 $= -1.25$



... Nxe5??  
 $f(n) = (9+3)-(5+5)$   
 $= 2$

So, considering our opponent's possible responses would be wise.



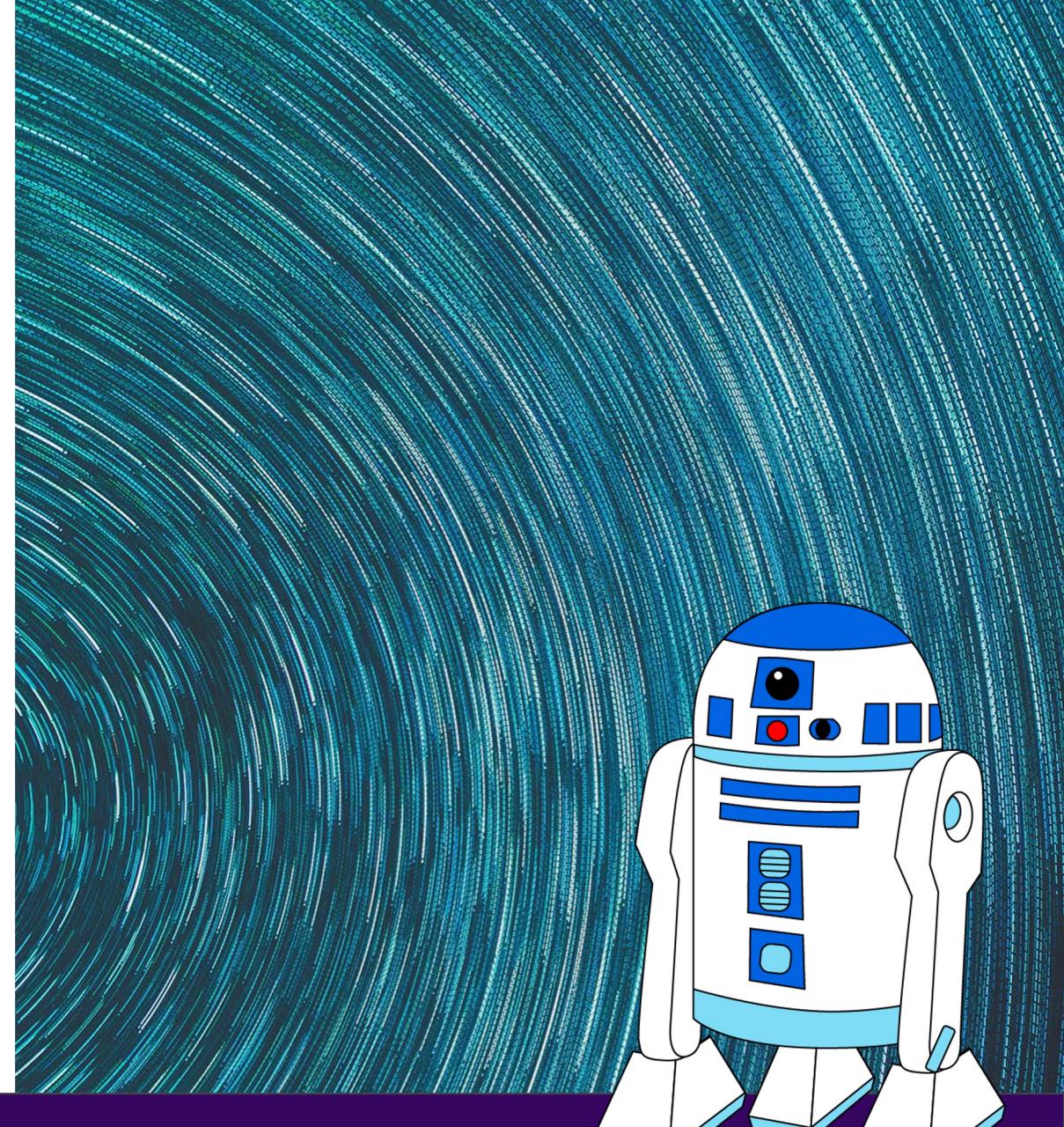
Uh-oh: Rxg4+  
 $f(n) = (3)-(5+5)$   
 $= -7$

*And black may force checkmate*

CIS 4210/5210:  
ARTIFICIAL INTELLIGENCE

# Minimax and Alpha-Beta Pruning

Professor Chris Callison-Burch



# The Minimax Rule: “Don’t play hope chess”

- Idea: Make the best move for MAX *assuming that MIN always replies with the best move for MIN*
- Easily computed by a recursive process
  - The **backed-up value** of each node in the tree is determined by the values of its children:
    - For a **MAX** node, the backed-up value is the **maximum** of the values of its children (*i.e. the best for MAX*)
    - For a **MIN** node, the backed-up value is the **minimum** of the values of its children (*i.e. the best for MIN*)

# The Minimax Procedure

Until game is over:

1. Start with the current position as a MAX node.
2. Expand the game tree a fixed number of *ply*.
3. Apply the evaluation function to the leaf positions.
4. Calculate back-up values bottom-up.
5. Pick the move assigned to MAX at the root
6. Wait for MIN to respond

# Adversarial Search (Minimax)

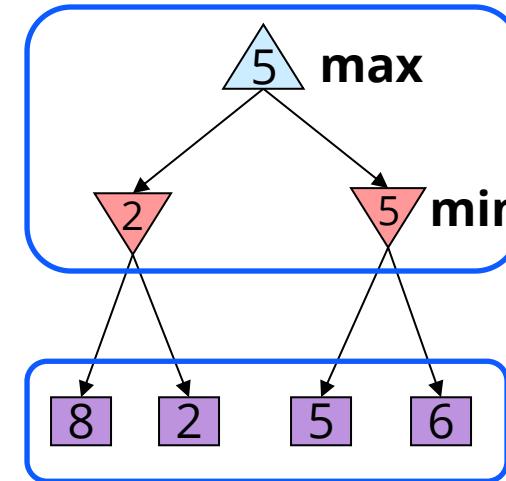
- Minimax search:

- A state-space search tree

- Players alternate turns

- Compute each node's **minimax value**: the best achievable utility against a rational (optimal) adversary

**Minimax values:  
computed recursively**



**Terminal values:  
part of the game**

# Minimax Implementation

```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

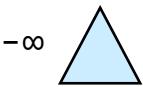


```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

# What if MIN does not play optimally?

- **Definition of optimal play for MAX assumes MIN plays optimally:**  
*Maximizes worst-case outcome* for MAX.  
(Classic game theoretic strategy)
- **But if MIN does not play optimally, MAX will do even better.**  
This theorem is not hard to prove

# Minimax Example



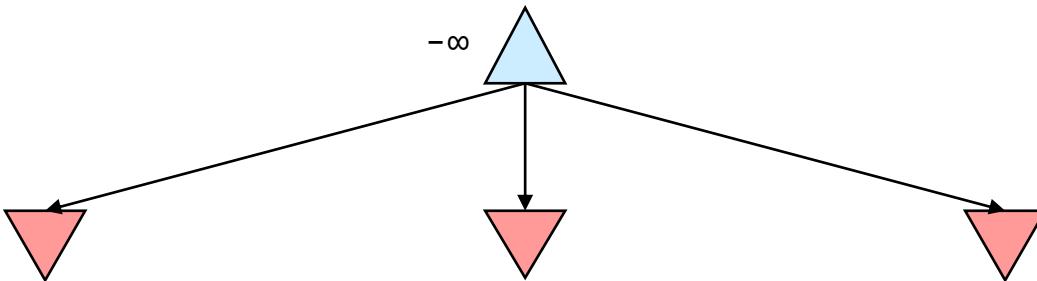
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



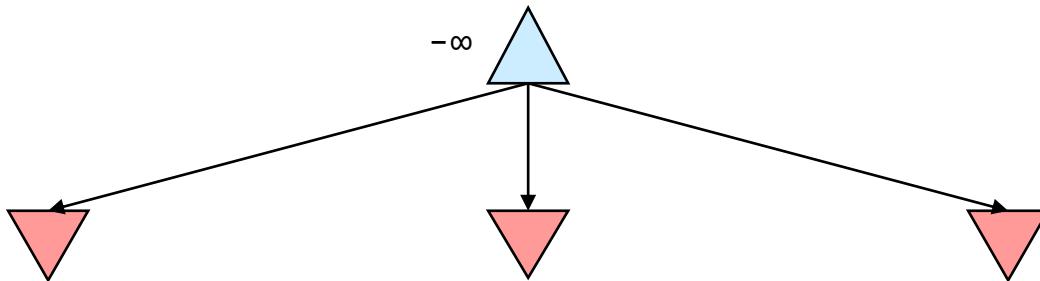
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



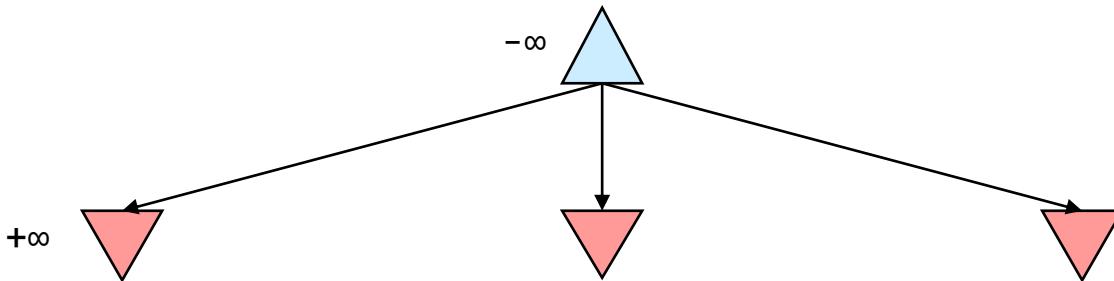
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



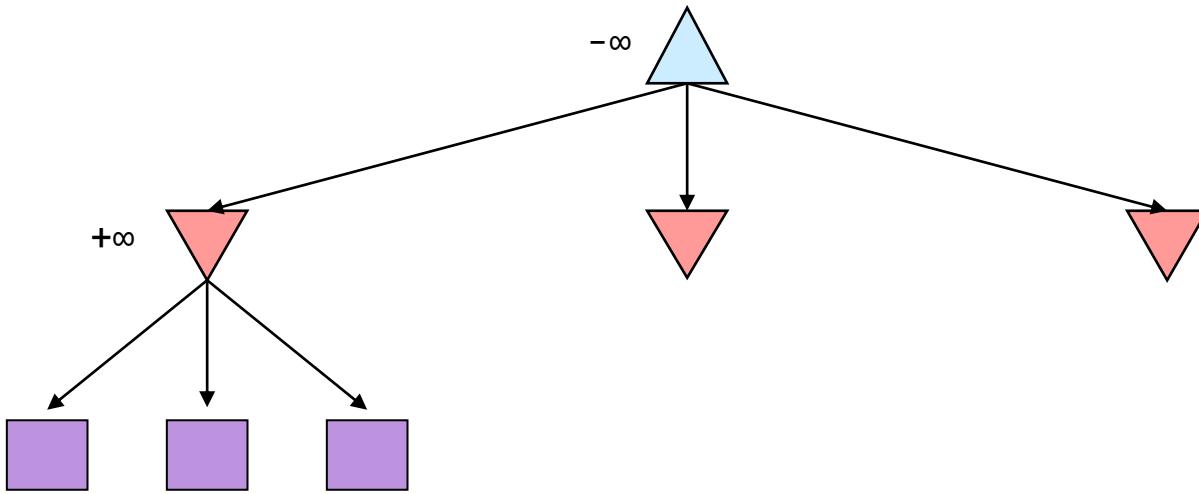
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = −∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



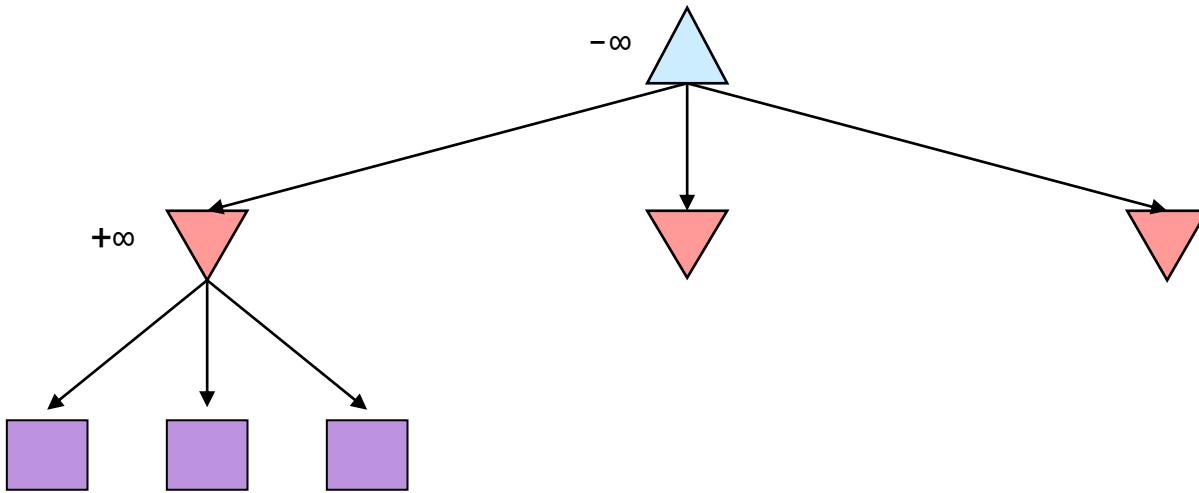
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



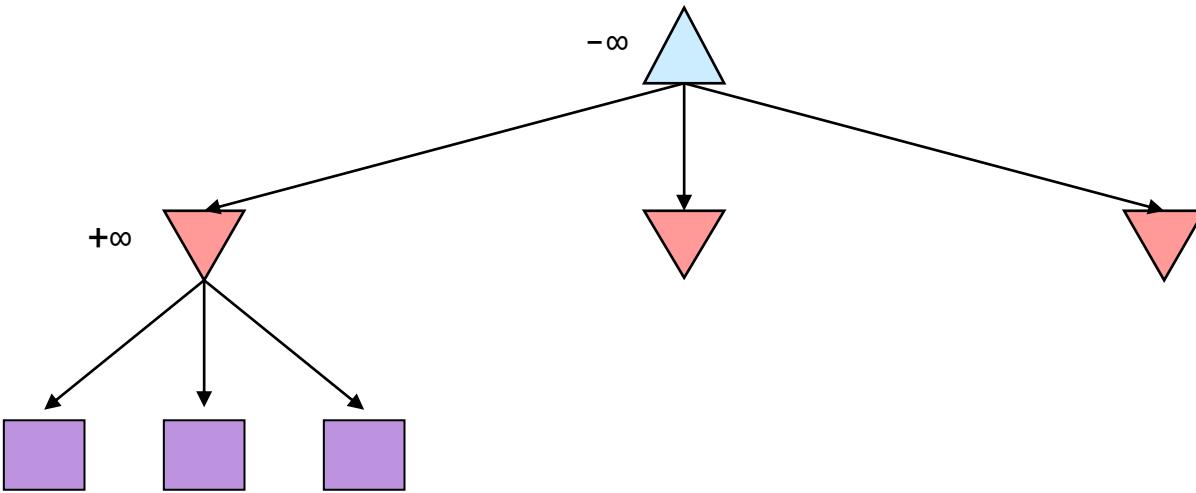
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



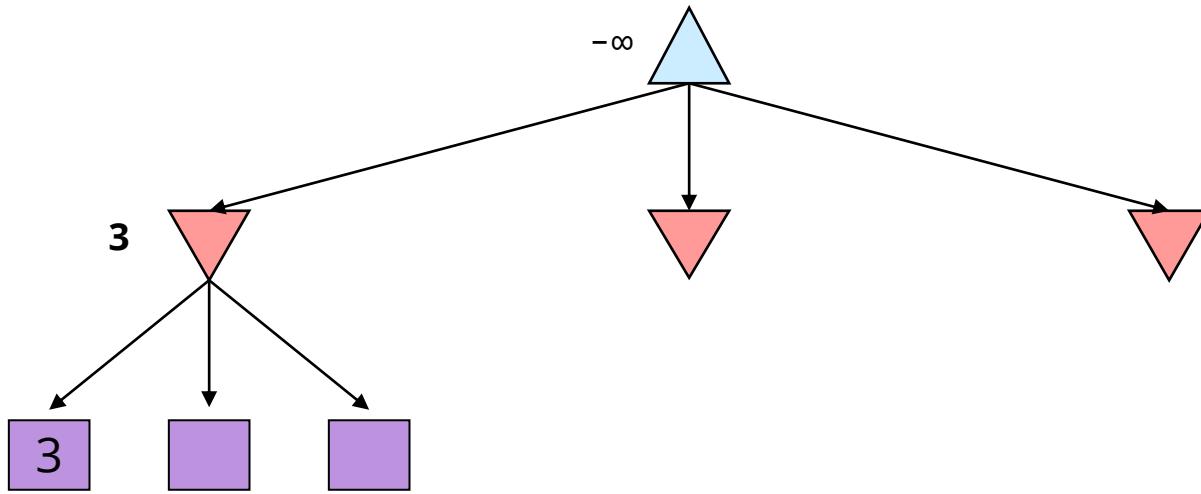
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



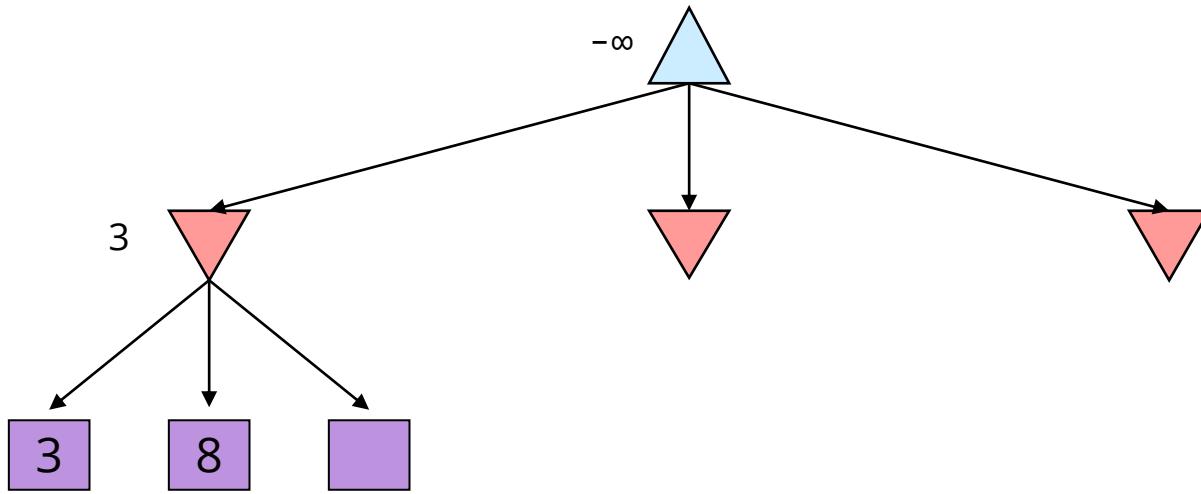
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



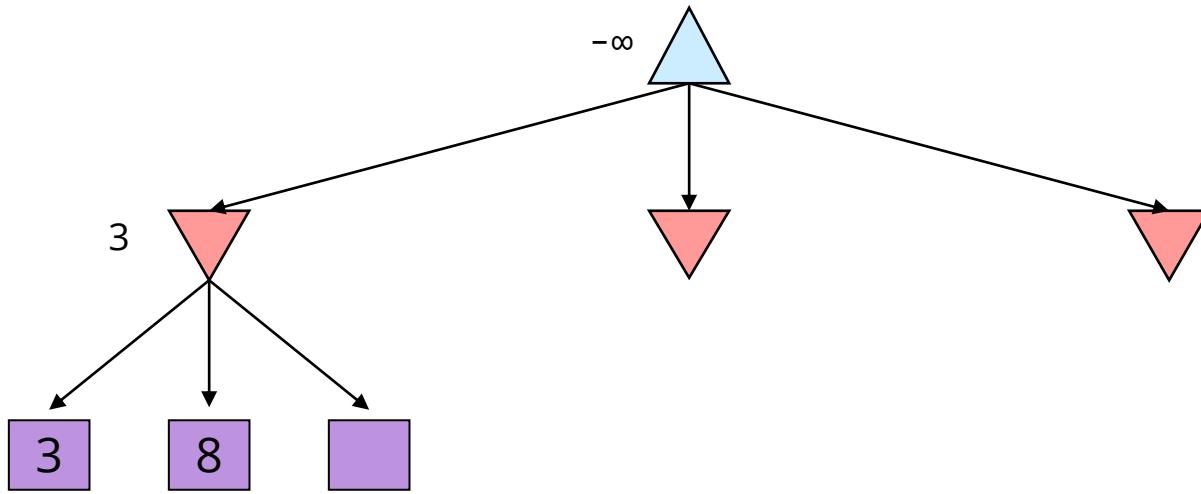
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



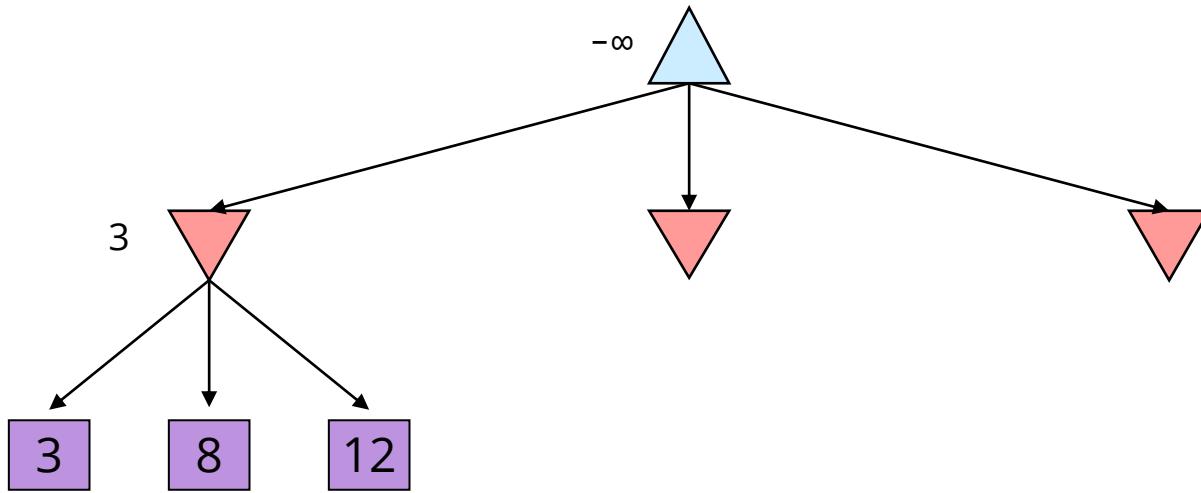
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



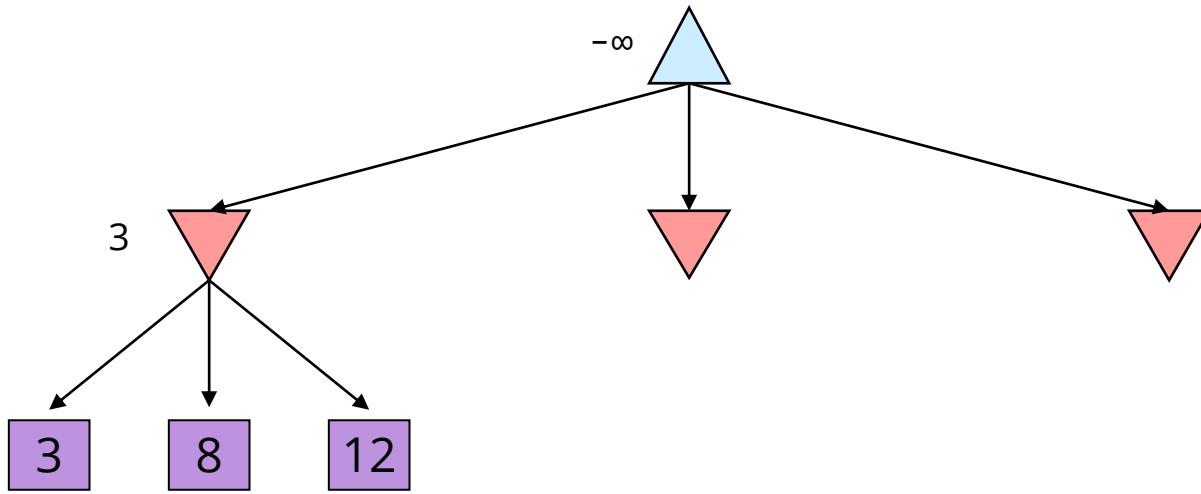
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



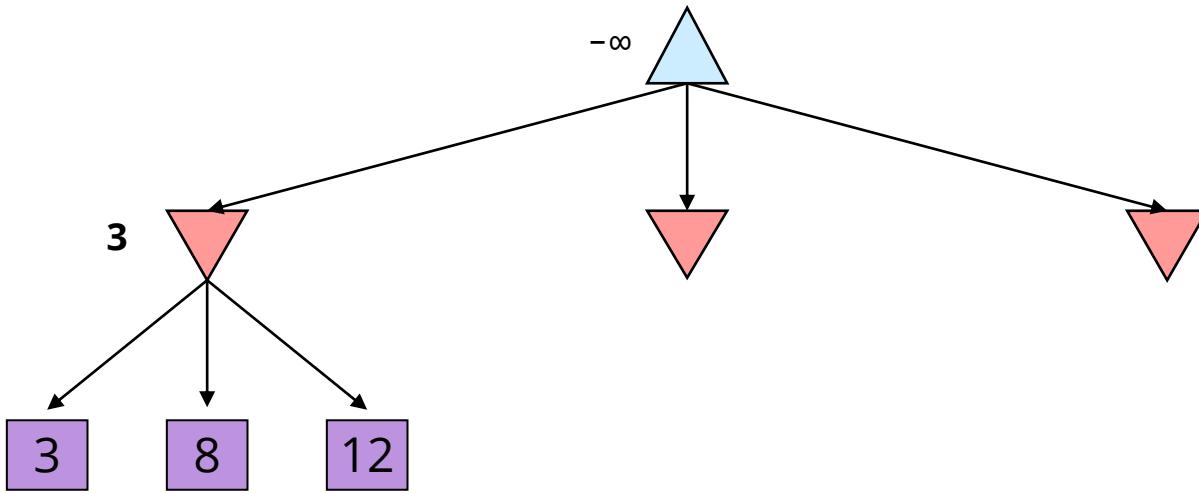
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



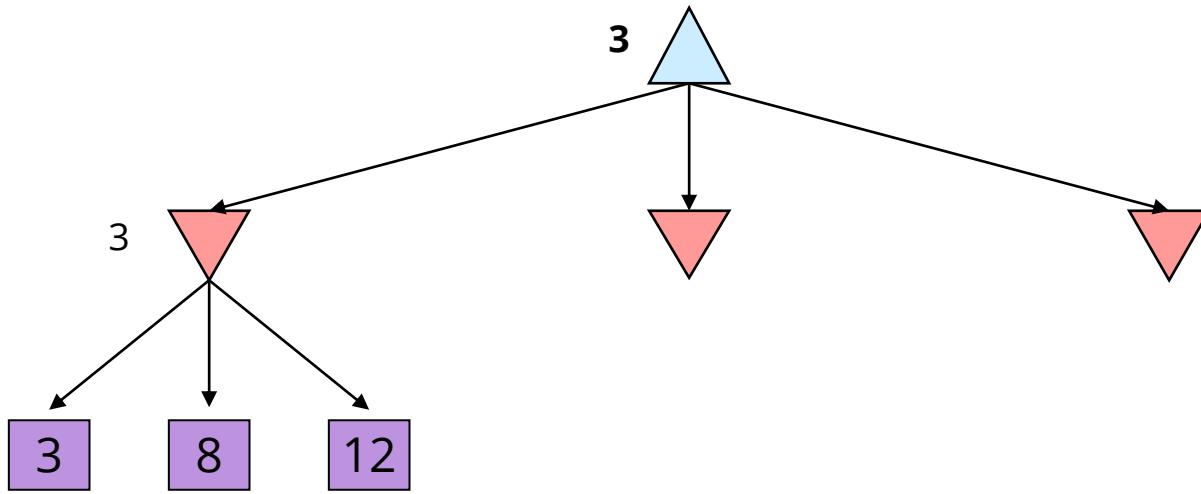
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



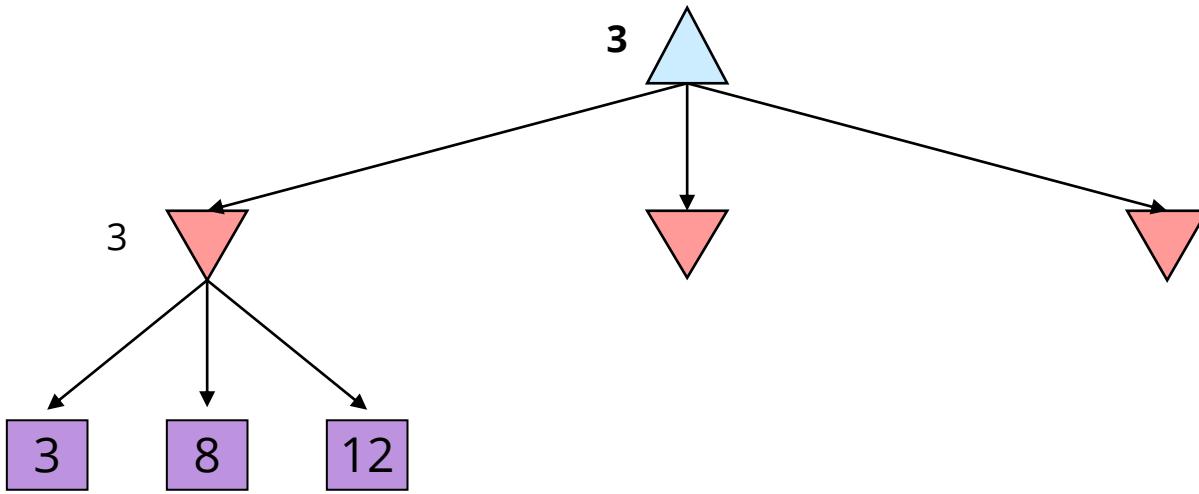
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



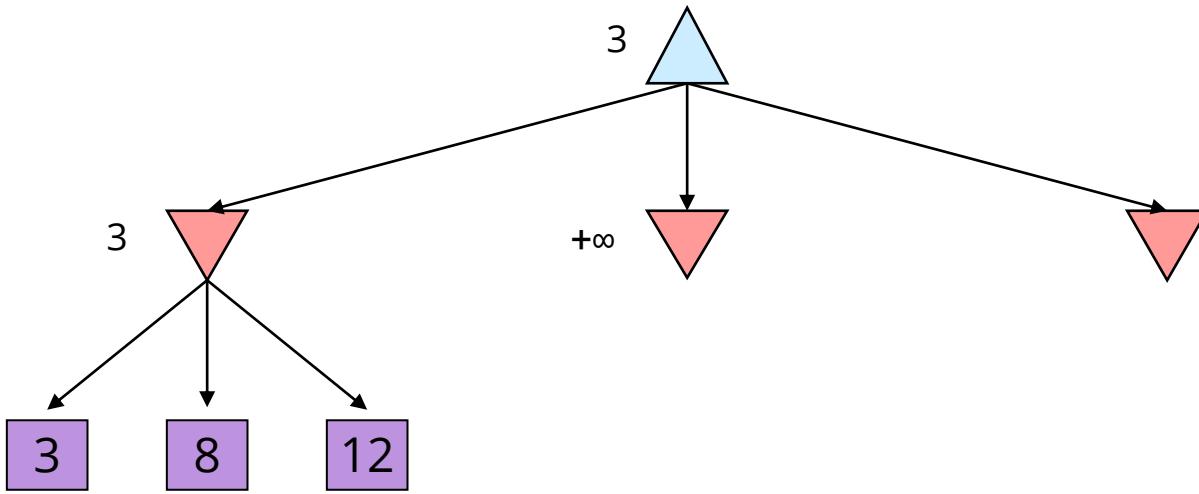
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



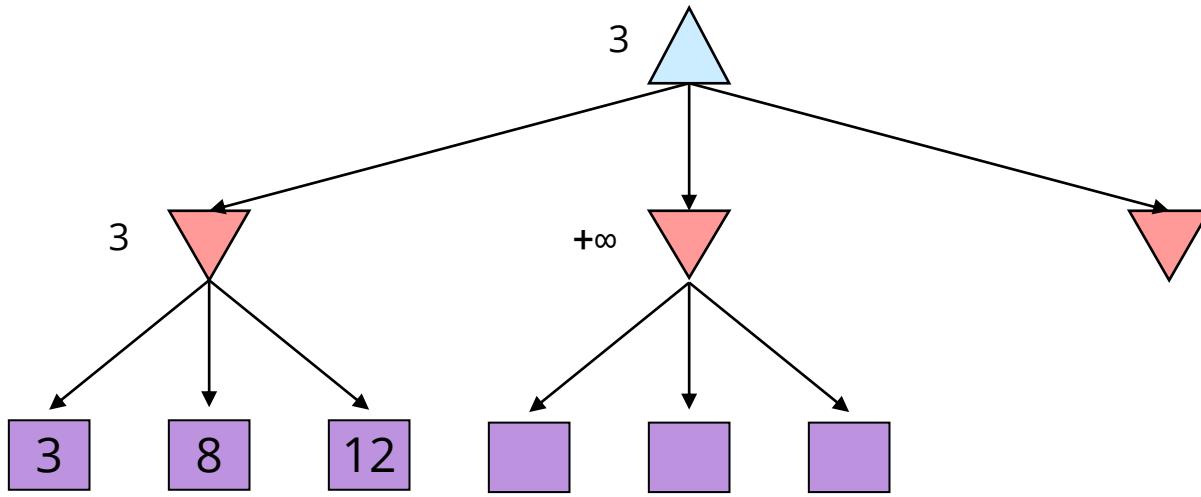
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



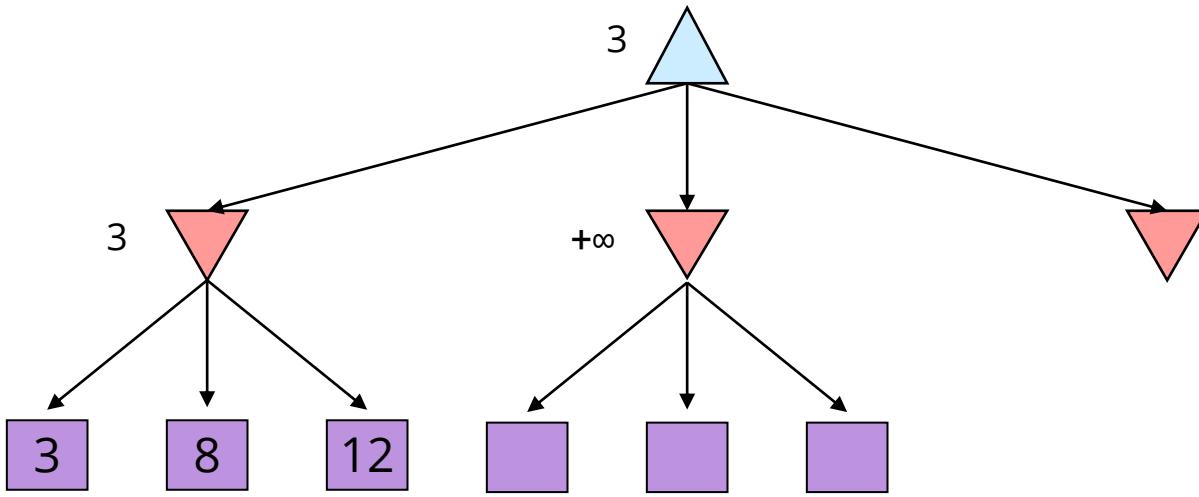
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



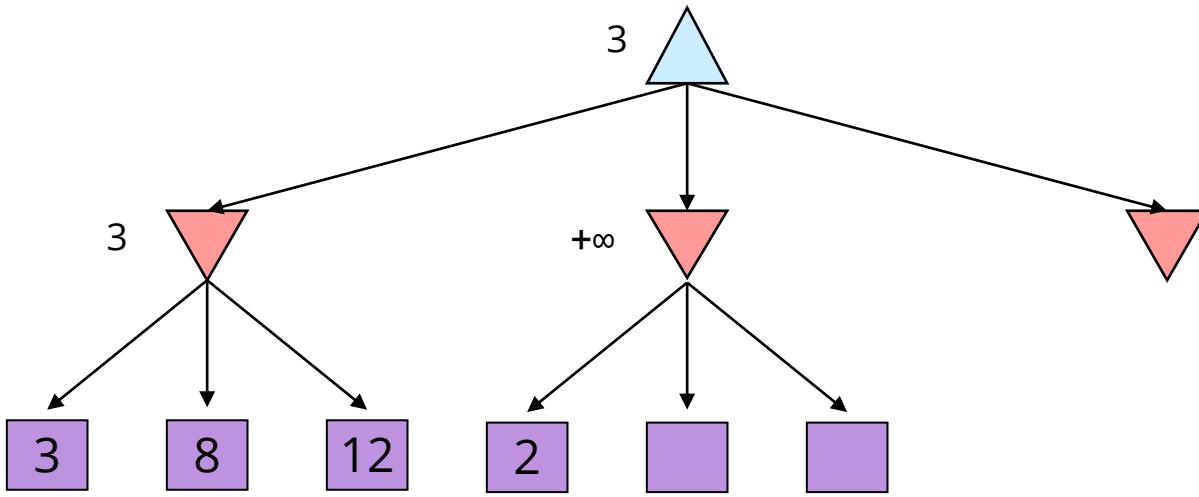
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



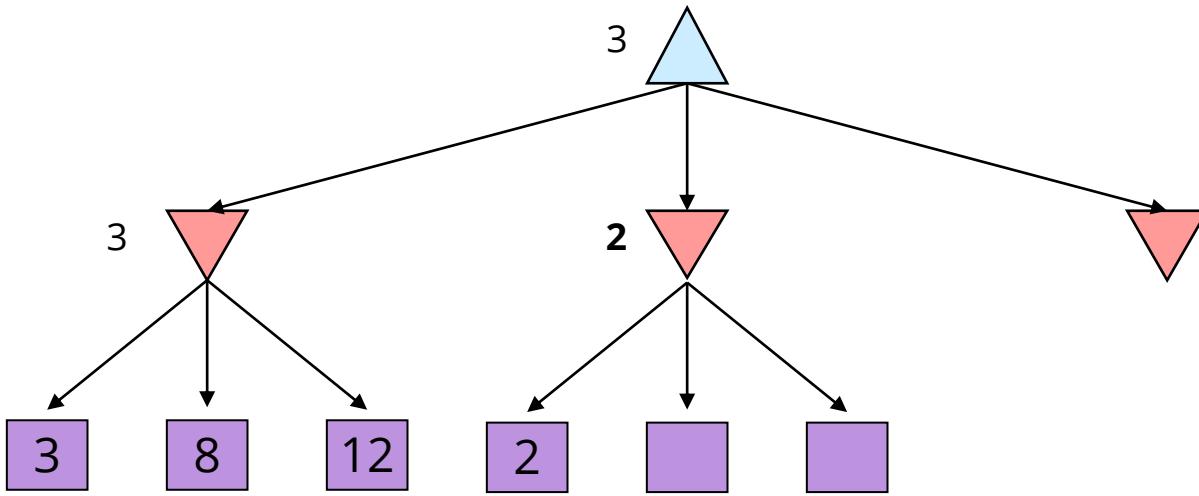
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



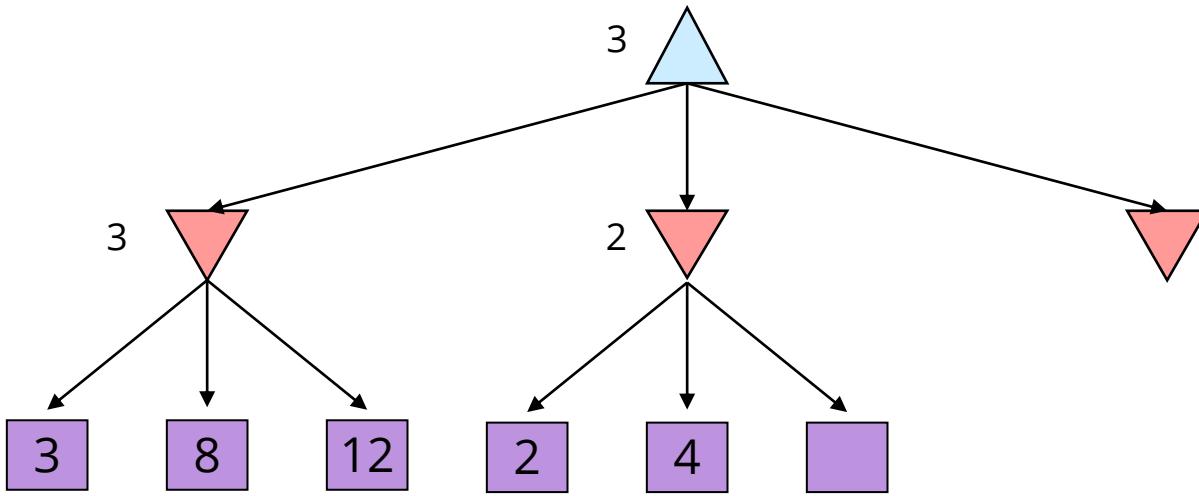
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



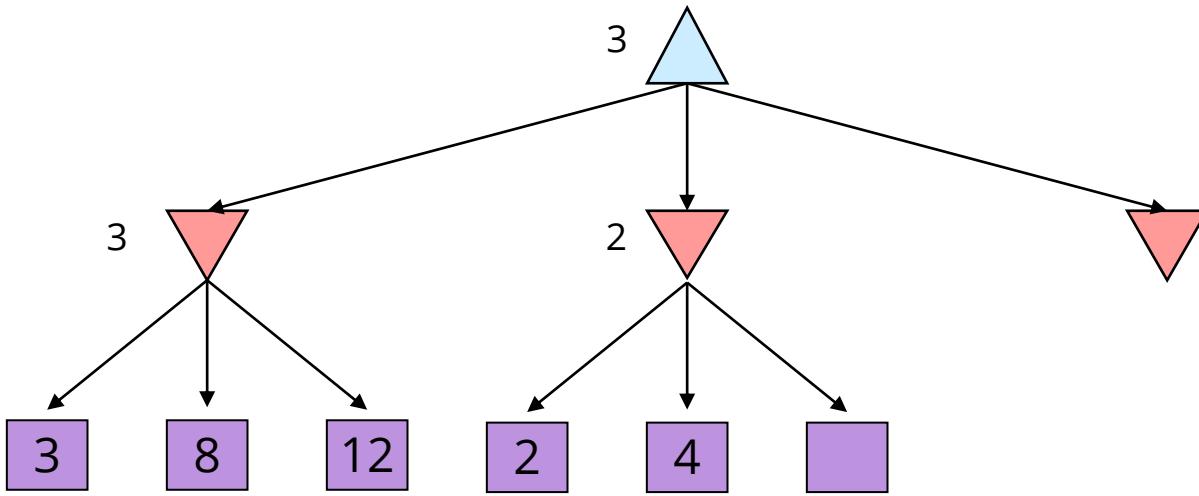
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



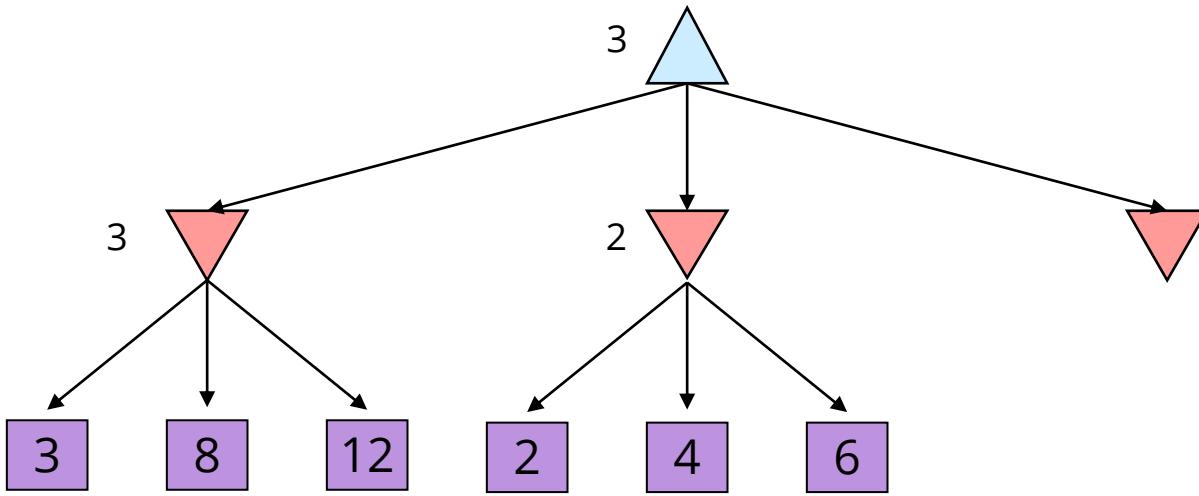
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



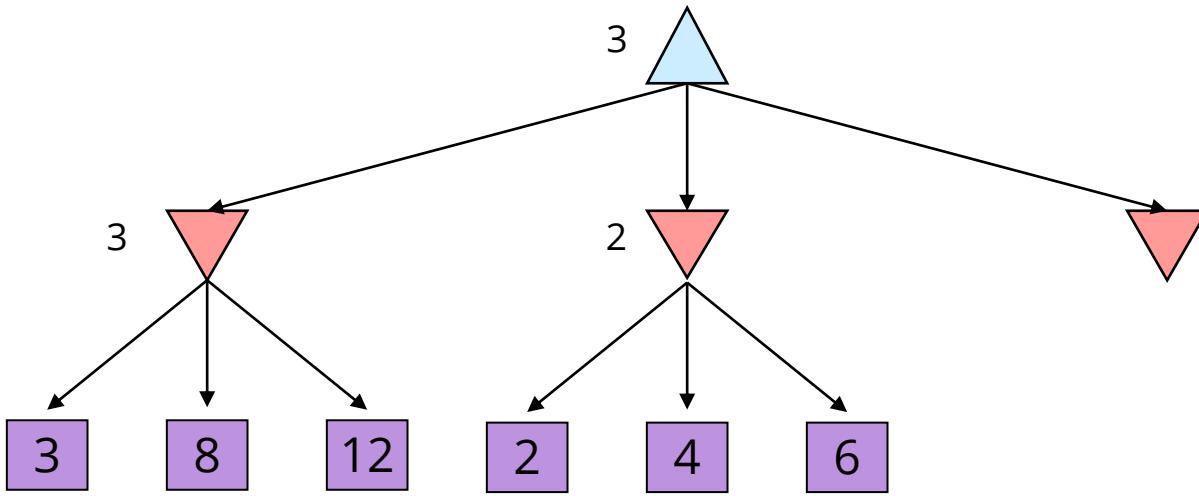
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



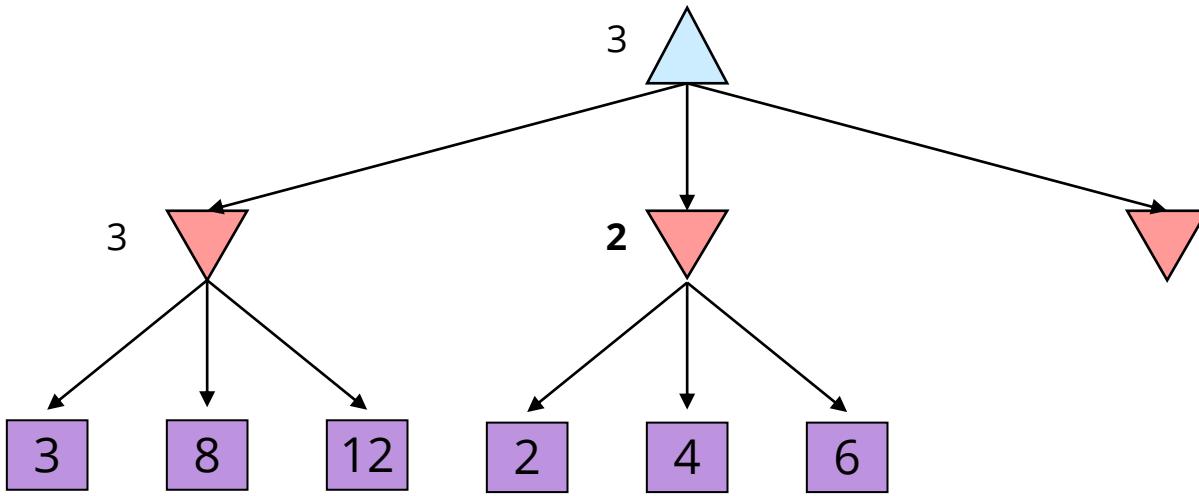
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



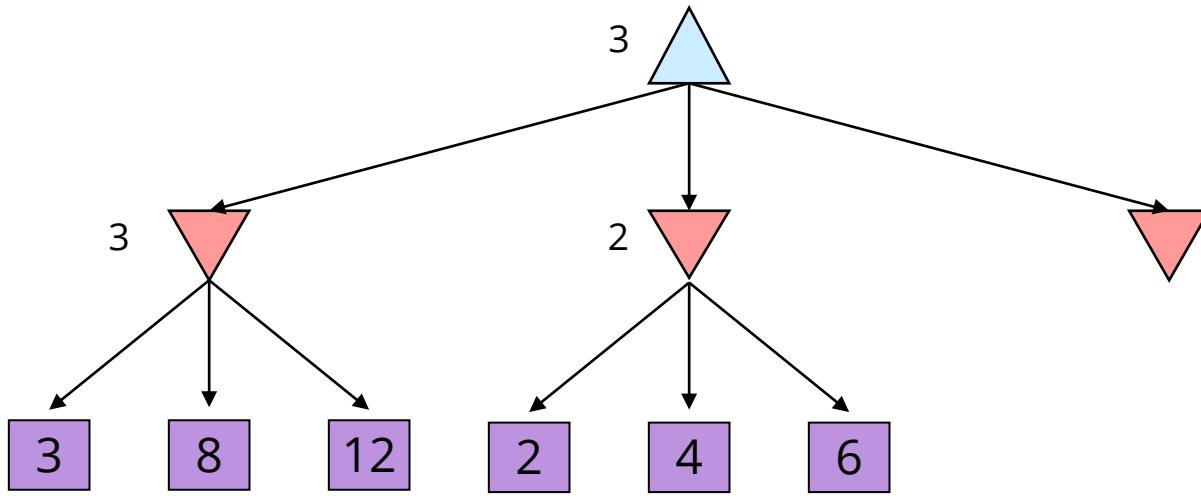
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



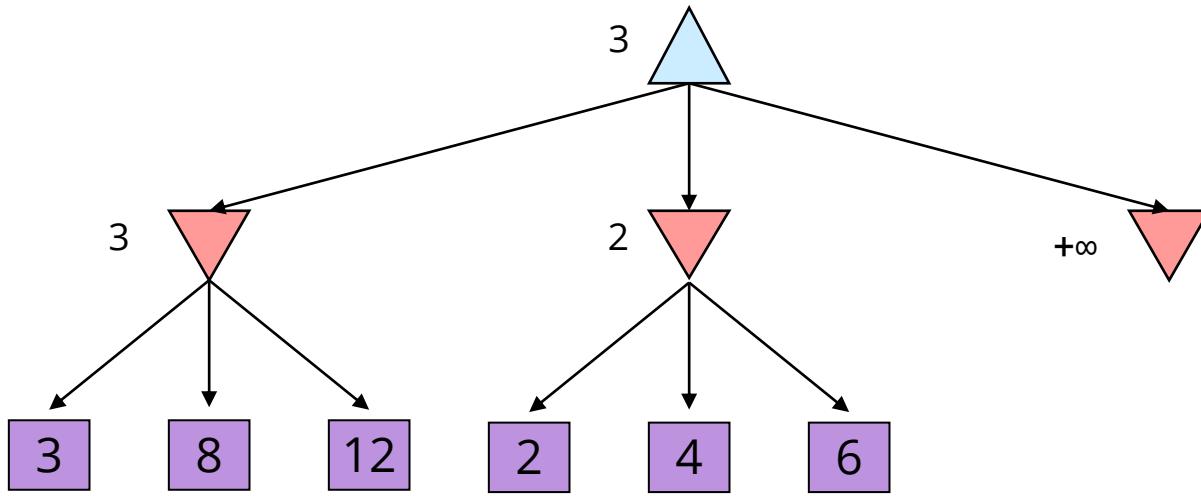
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



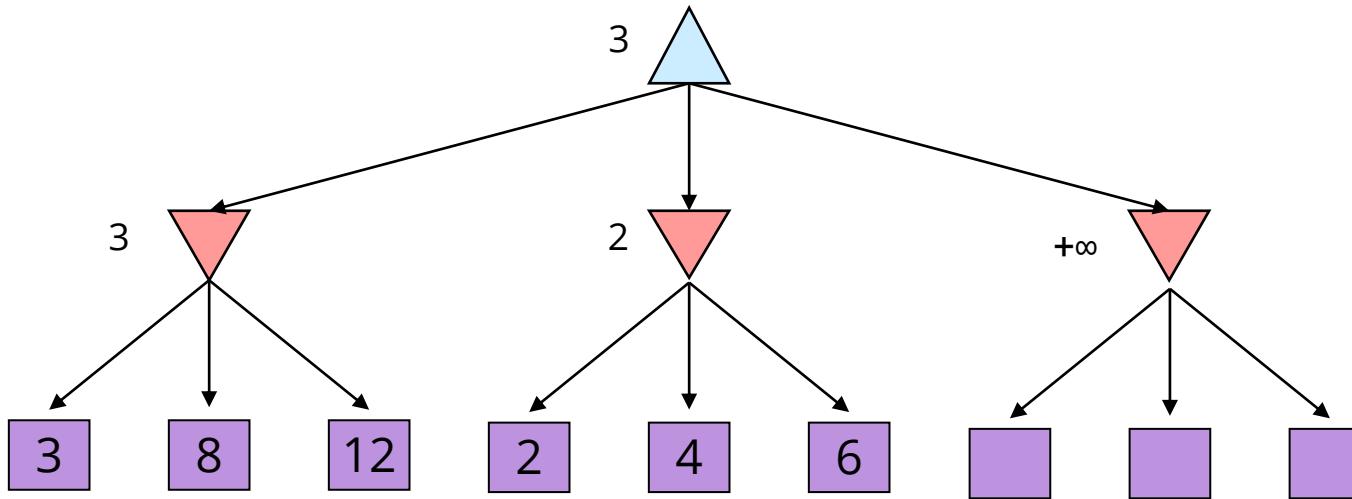
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



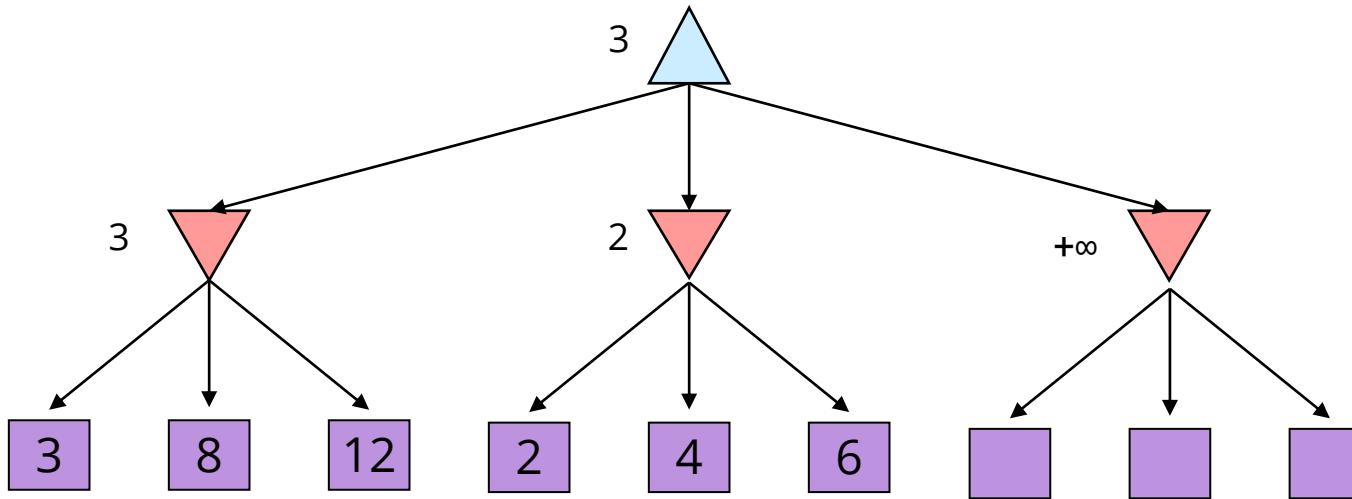
```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



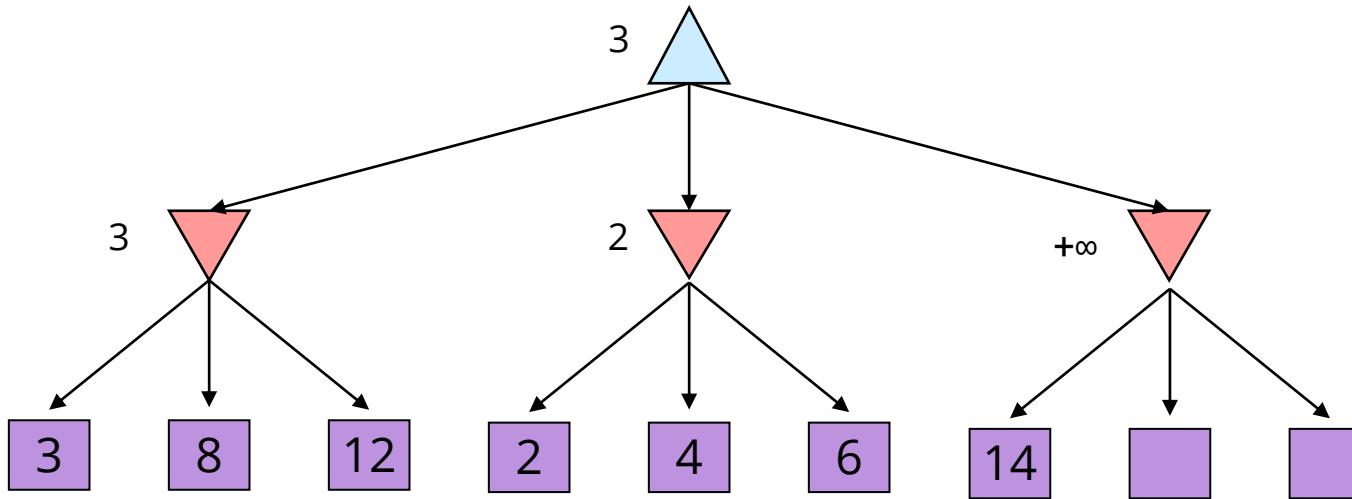
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



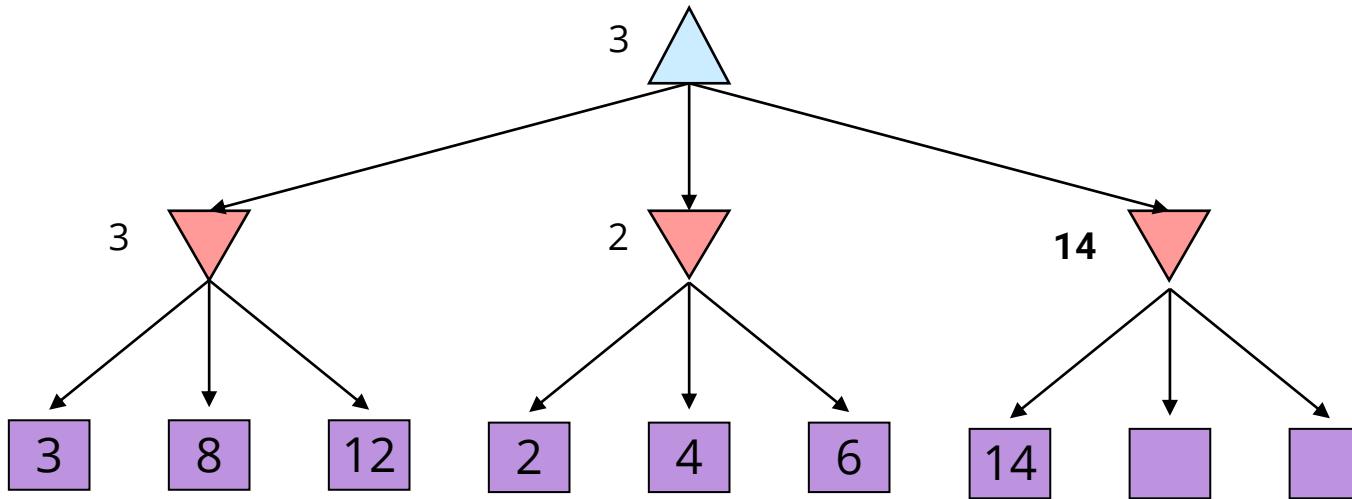
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



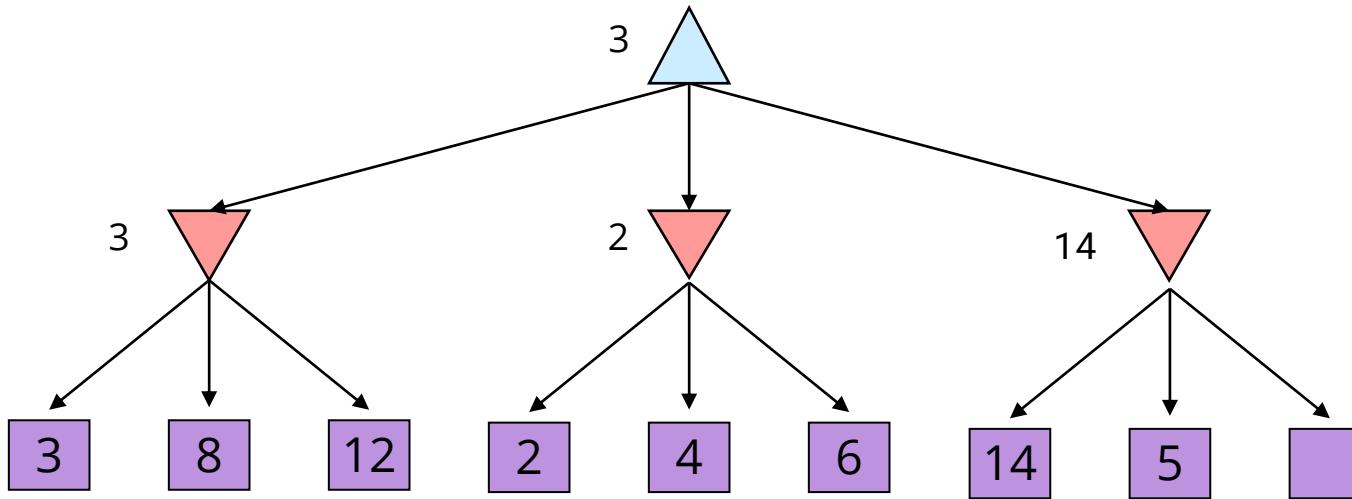
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



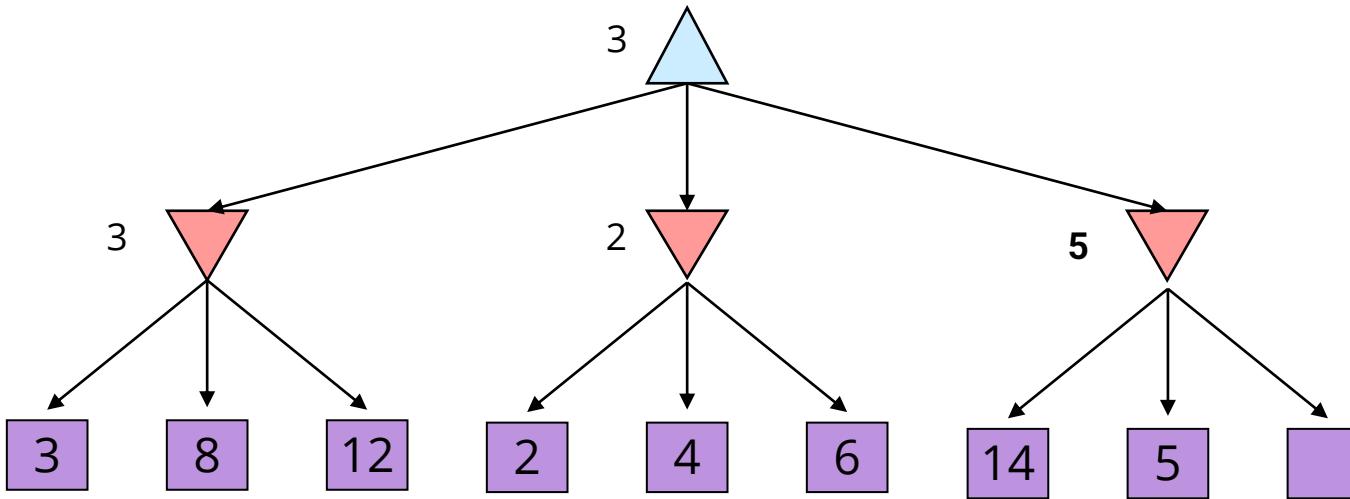
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



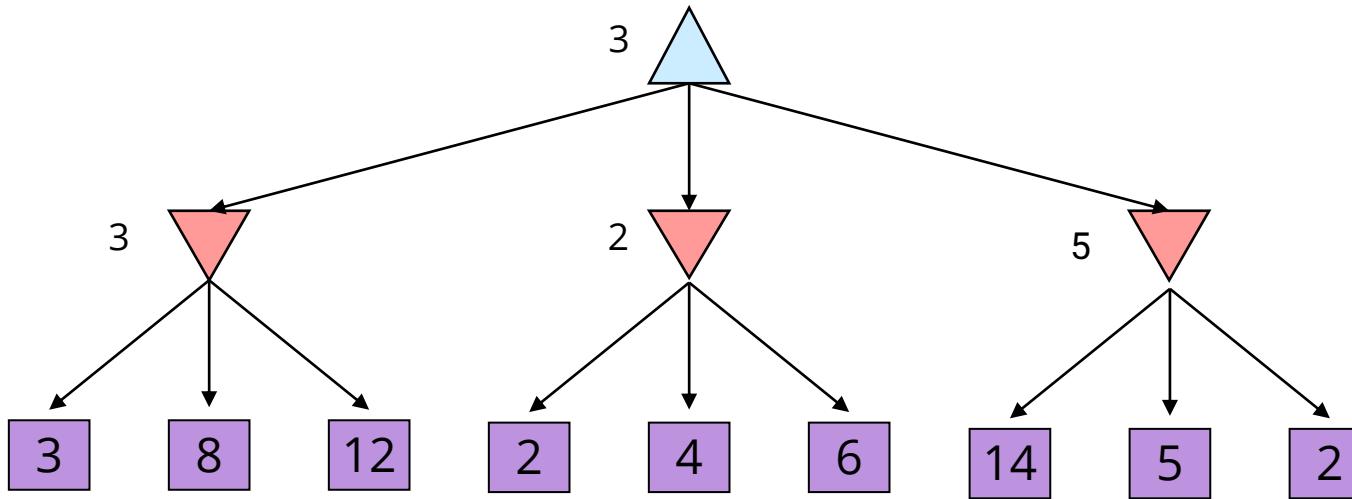
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



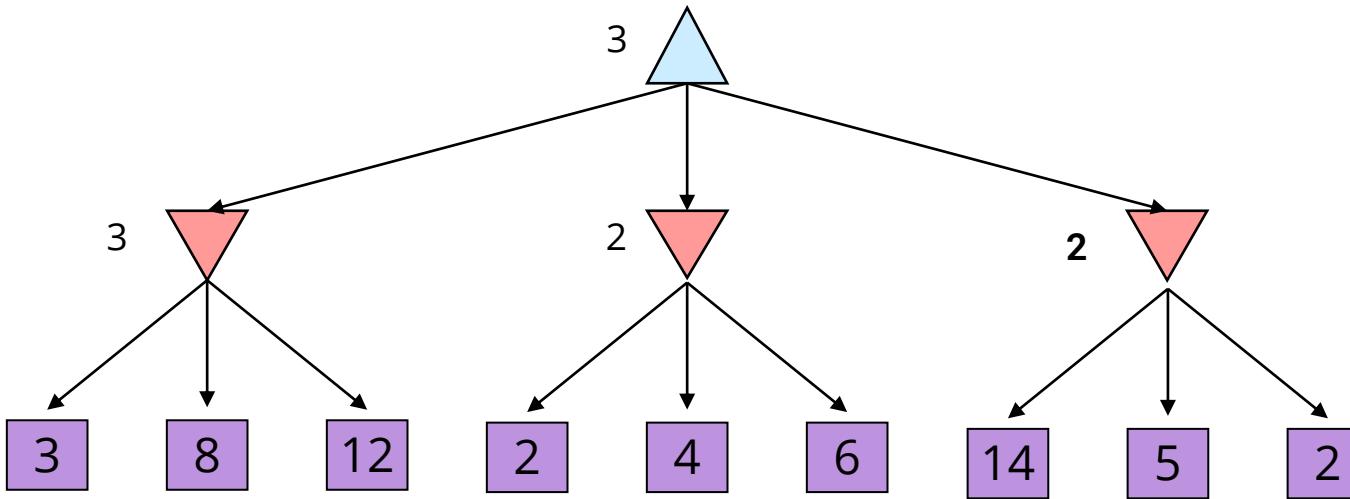
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



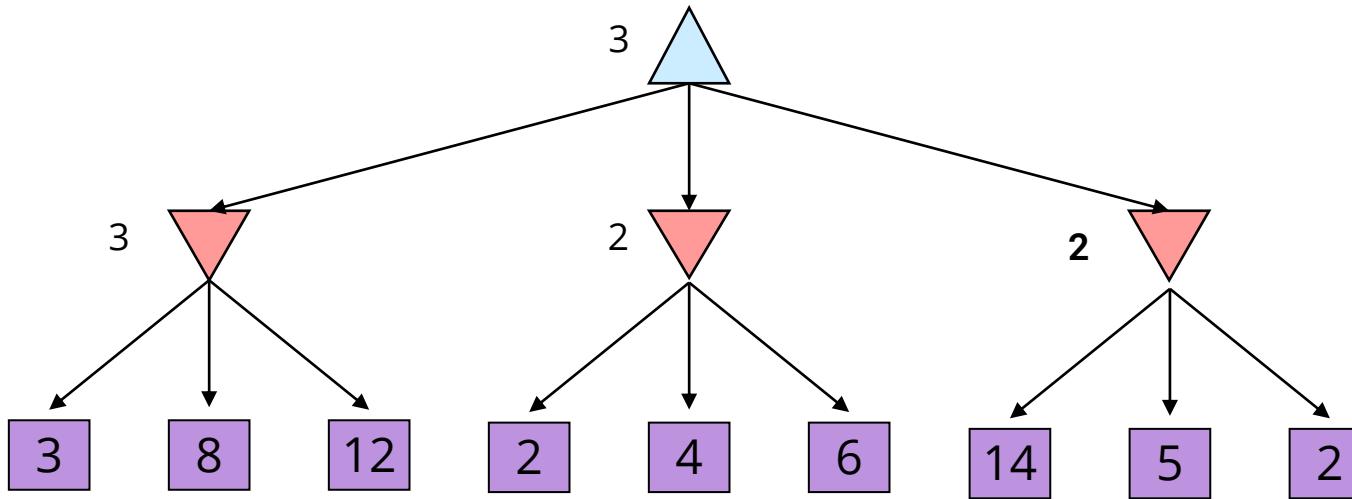
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



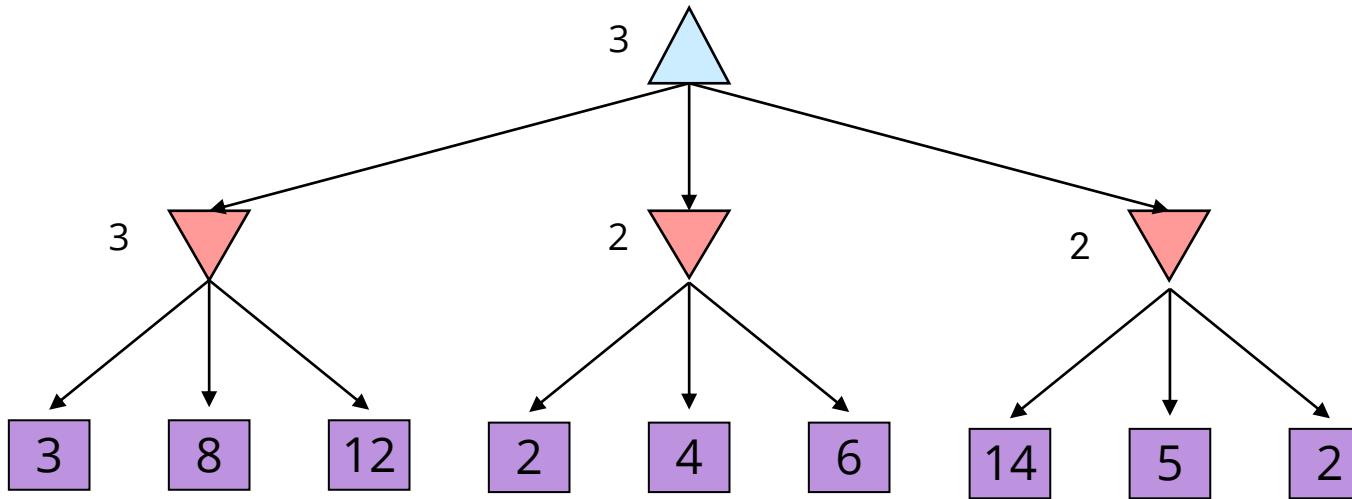
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



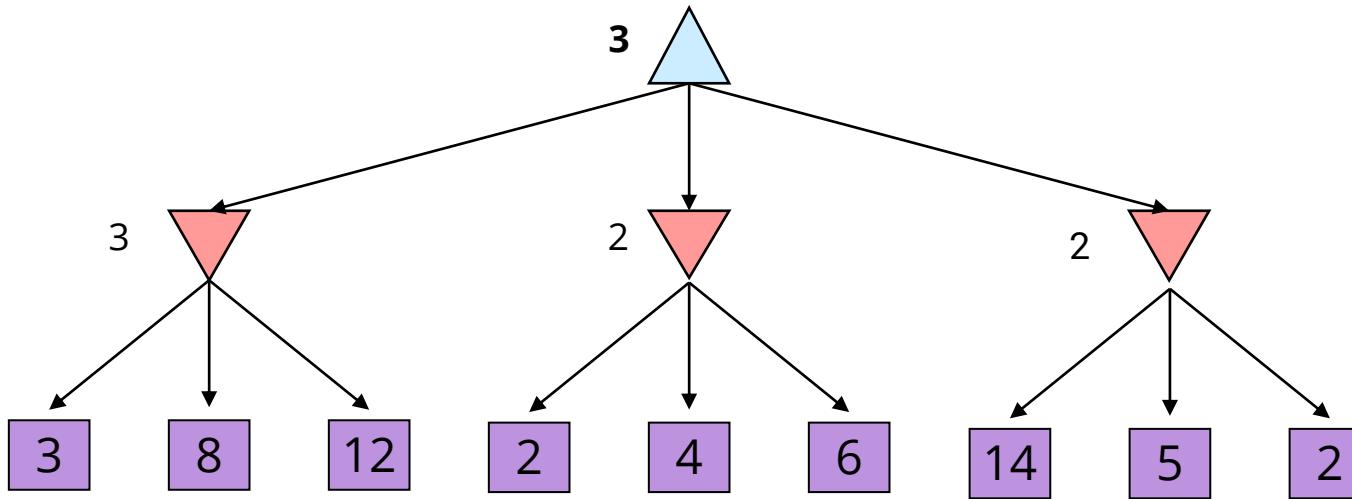
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



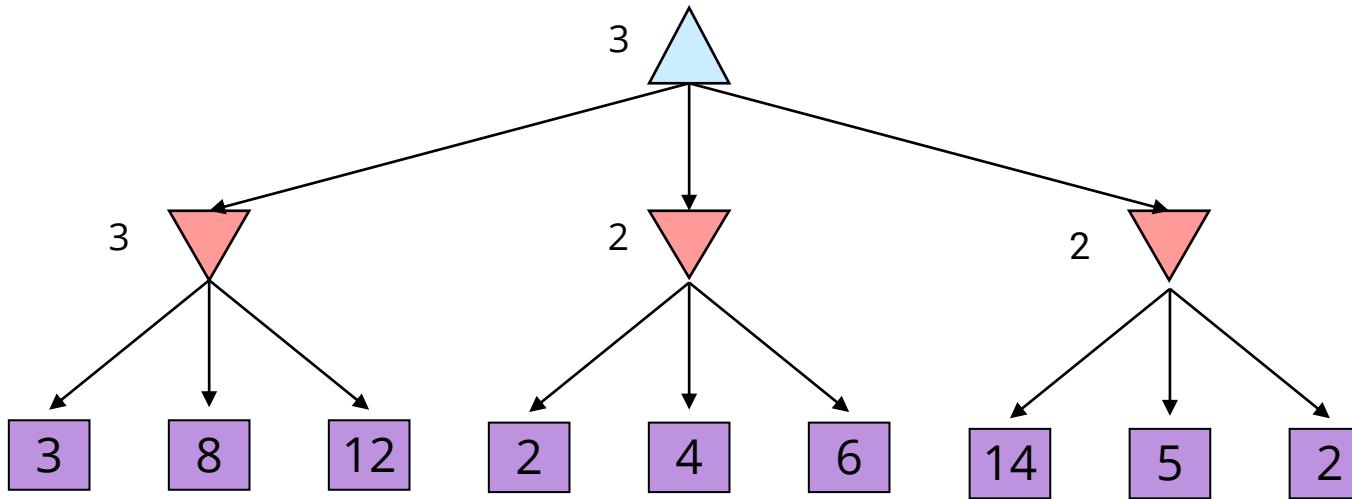
```
def max-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v = max(v, min-value(successor))
    return v
```

```
def min-value(state):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v = min(v, max-value(successor))
    return v
```

$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

# Minimax Example



```
def max-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v = max(v, min-value(successor))  
    return v
```

```
def min-value(state):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v = min(v, max-value(successor))  
    return v
```

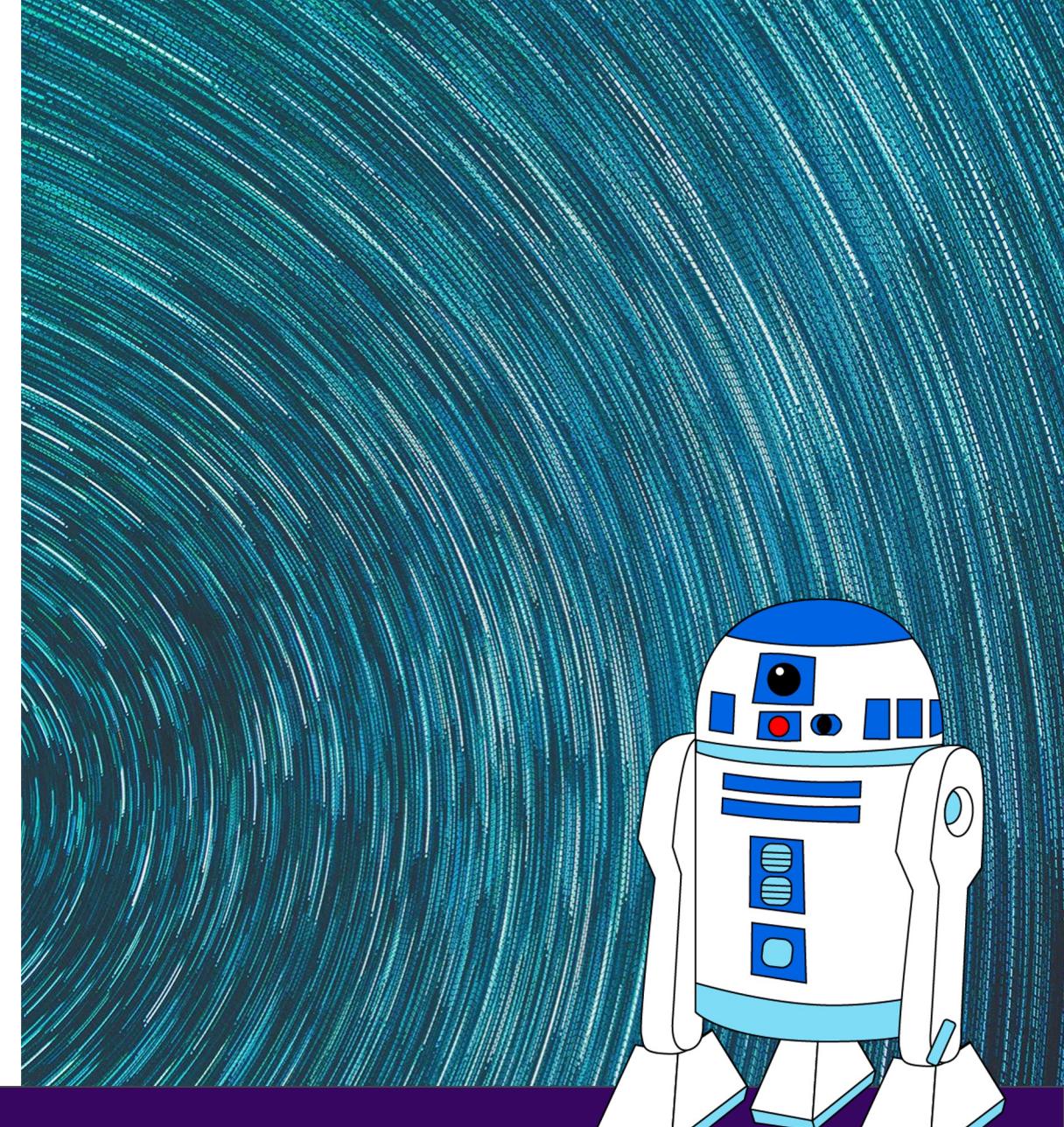
$$V(s) = \max_{s' \in \text{successors}(s)} V(s')$$

$$V(s') = \min_{s \in \text{successors}(s')} V(s)$$

CIS 4210/5210:  
ARTIFICIAL INTELLIGENCE

# Alpha-Beta Pruning

Professor Chris Callison-Burch



# Alpha-Beta Pruning

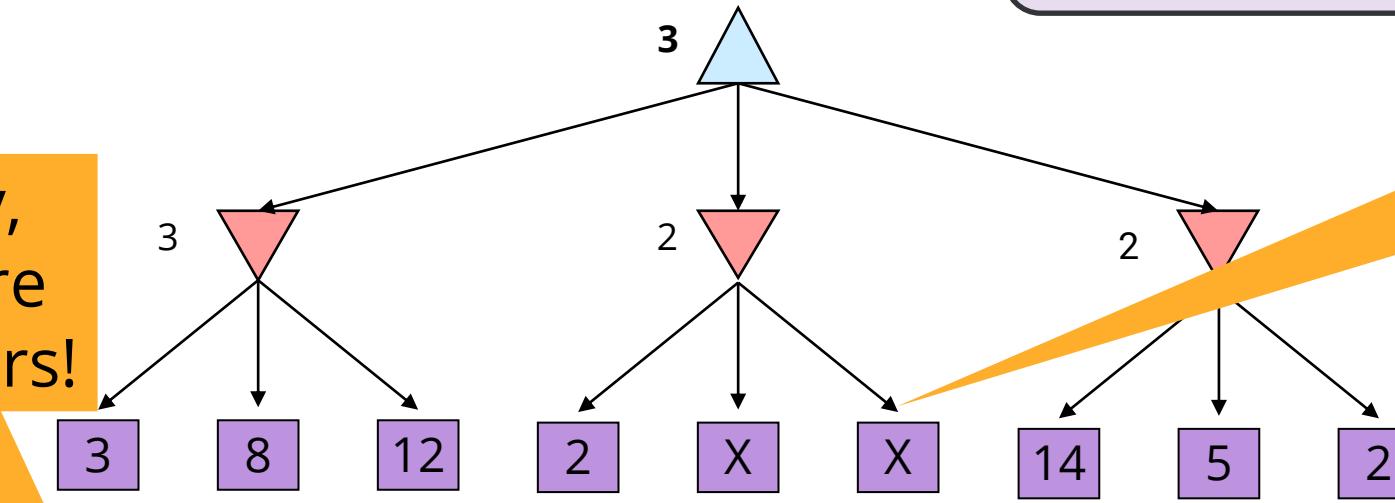
- During Minimax, keep track of two additional values:
  - $\alpha$ : MAX's current *lower* bound on MAX's outcome**
  - $\beta$ : MIN's current *upper* bound on MIN's outcome**
- MAX will never allow a move that could lead to a worse score (for MAX) than  $\alpha$
- MIN will never allow a move that could lead to a better score (for MAX) than  $\beta$
- Therefore, stop evaluating a branch whenever:
  - When evaluating a MAX node: a value  $v \geq \beta$  is backed-up
    - MIN will never select that MAX node
  - When evaluating a MIN node: a value  $v \leq \alpha$  is found
    - MAX will never select that MIN node



# Alpha-Beta Pruning

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

Returns early,  
doesn't explore  
other successors!



These branches  
never get explored

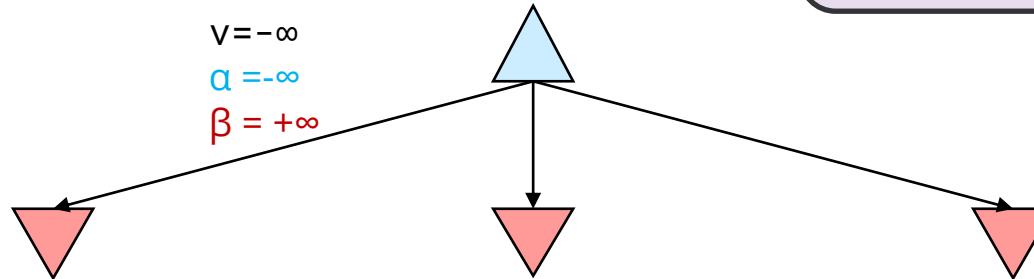
```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

Updates  $\alpha$  or  $\beta$

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root



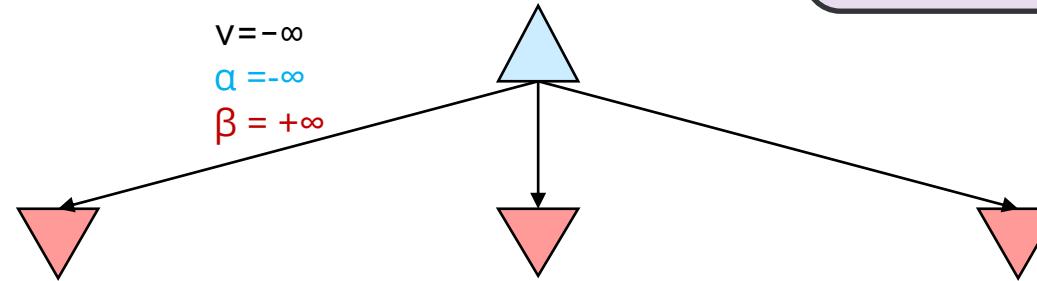
```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v =  $-\infty$   
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v =  $+\infty$   
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```



# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

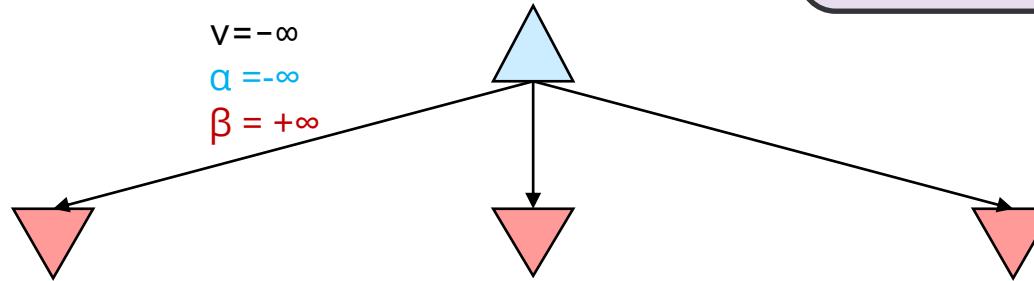


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v =  $-\infty$   
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v =  $+\infty$   
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

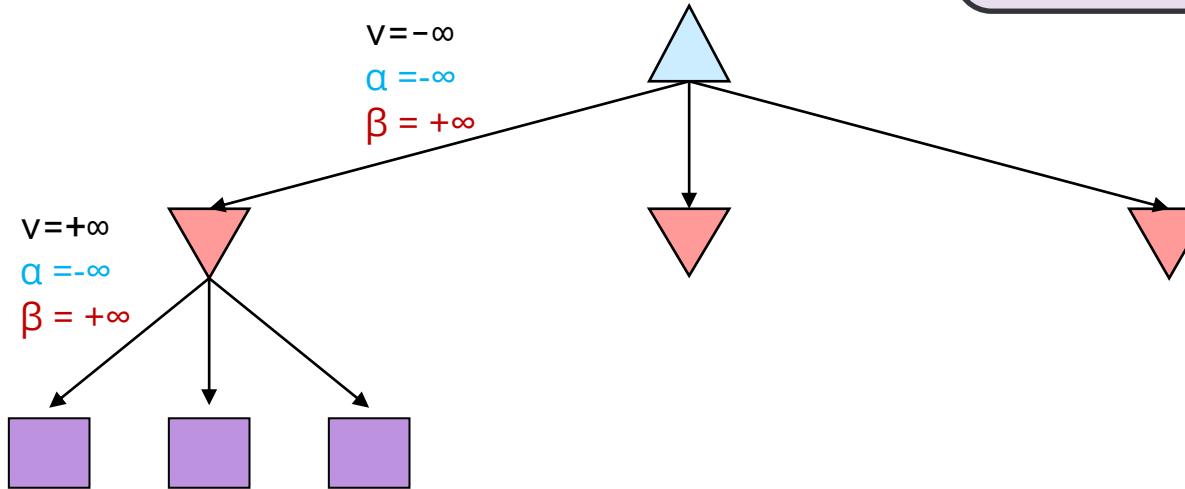


```
def max-value(state, a, b):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v =  $-\infty$   
    for each successor of state:  
        v' = min-value(successor, a, b)  
        if v' > v then v=v'  
        if v' >= b then return v  
        if v' > a then a=v'  
    return v
```

```
def min-value(state, a, b):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v =  $+\infty$   
    for each successor of state:  
        v' = max-value(successor, a, b)  
        if v' < v then v=v'  
        if v' <= a then return v  
        if v' < b then b=v'  
    return v
```

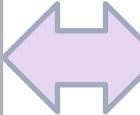
# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root



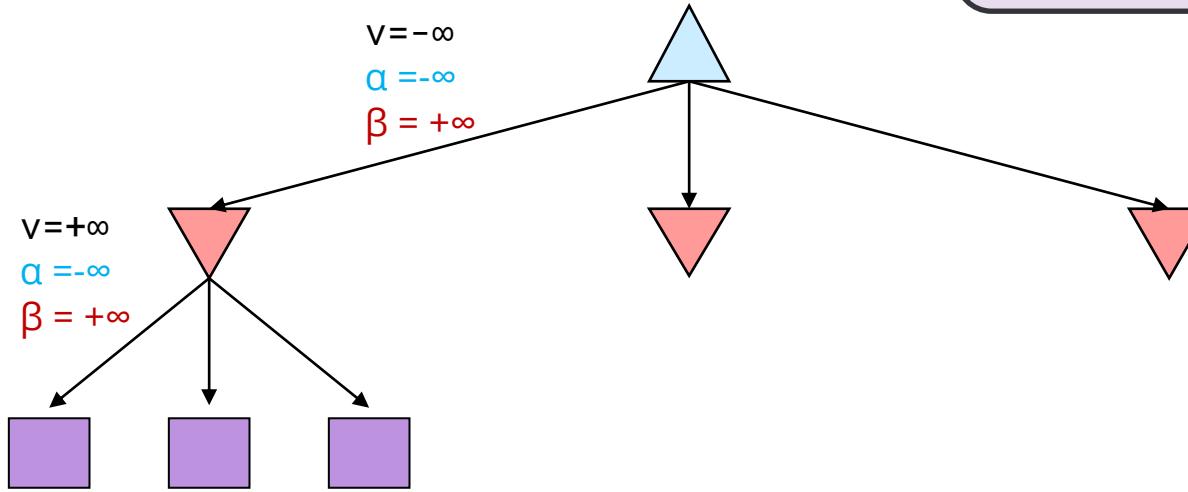
```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```



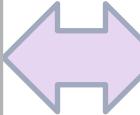
# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root



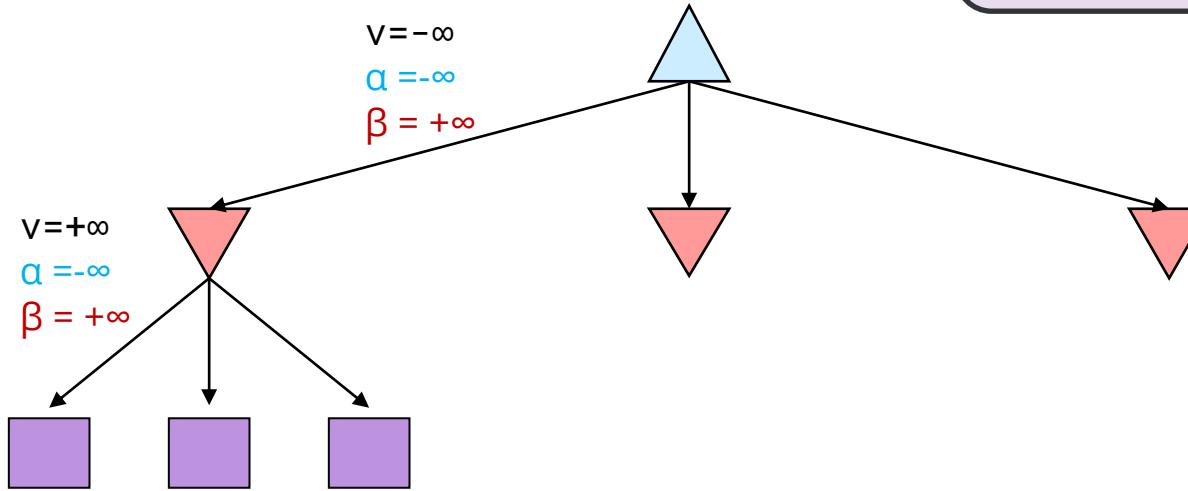
```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```



# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

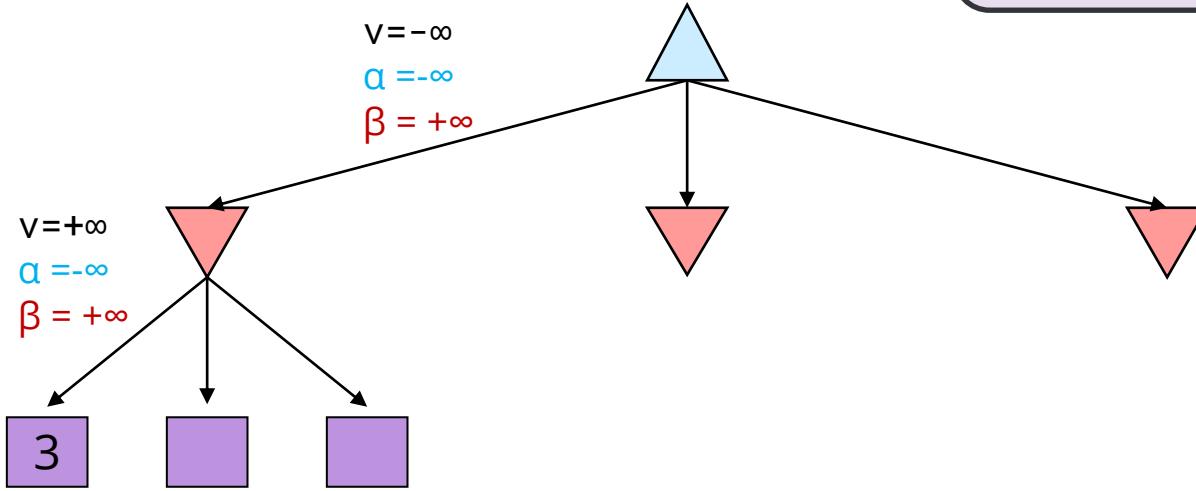


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

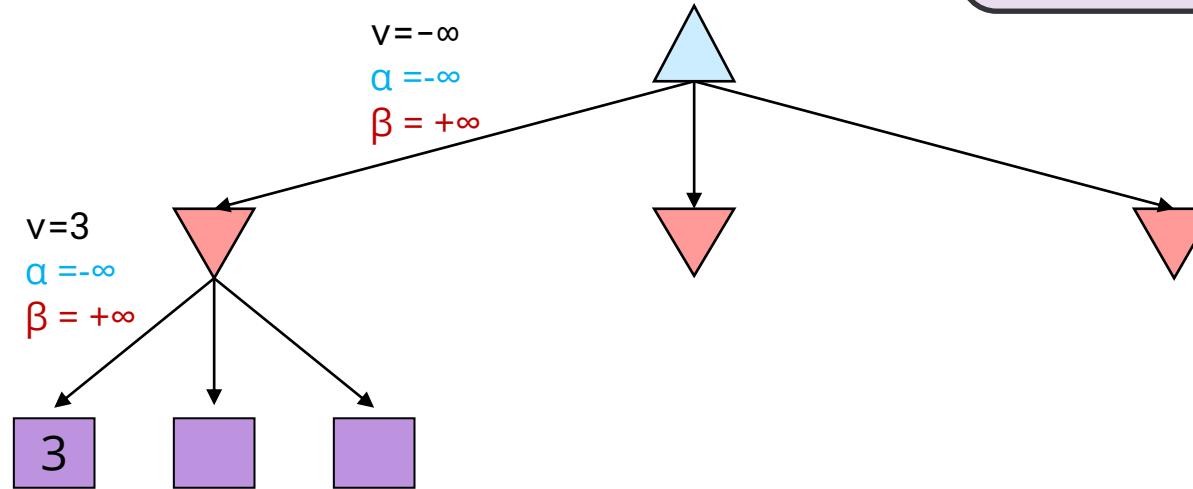


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

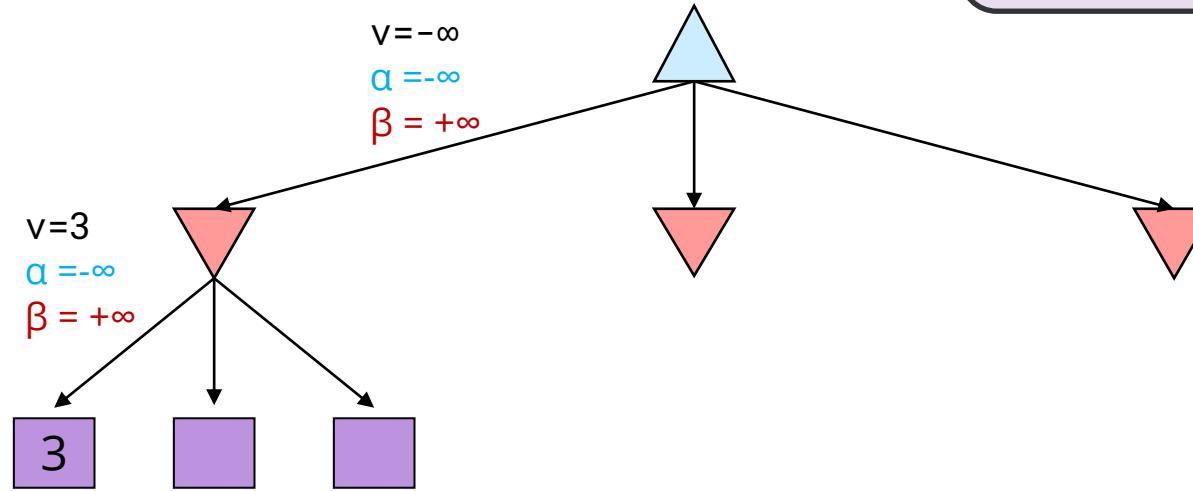


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

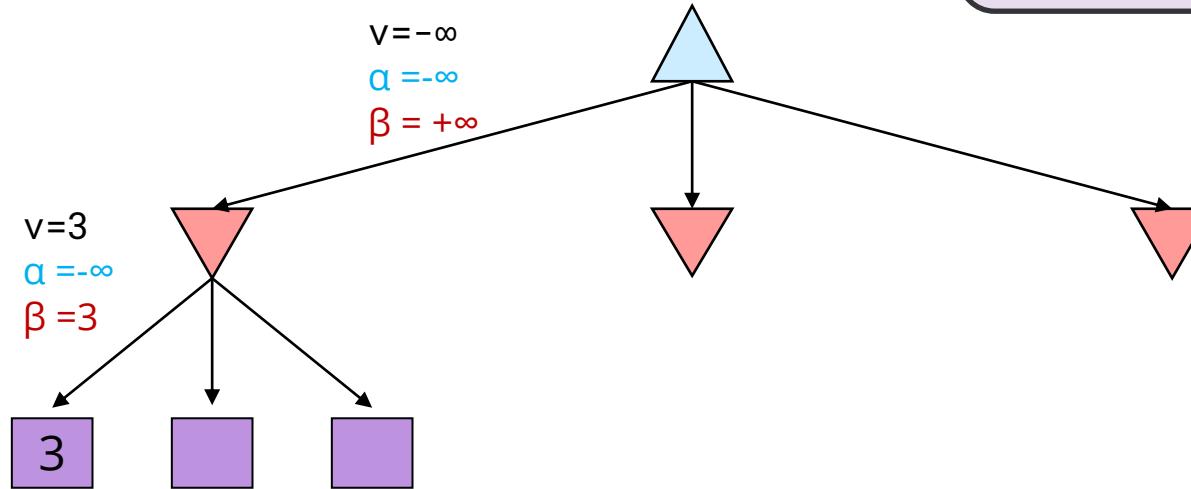


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

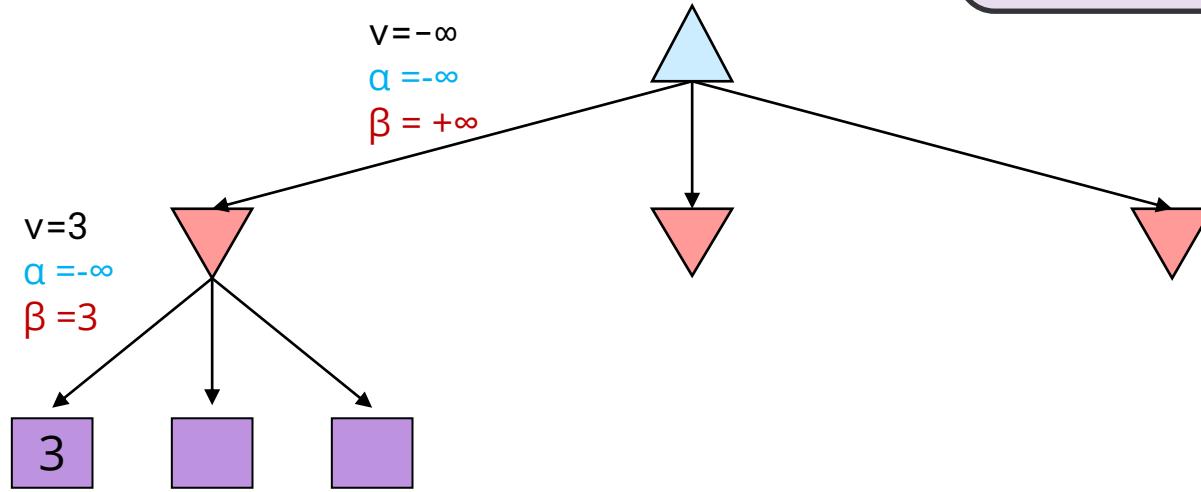


```
def max-value(state, \alpha, \beta):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -\infty  
    for each successor of state:  
        v' = min-value(successor, \alpha, \beta)  
        if v' > v then v=v'  
        if v' >= \beta then return v  
        if v' > \alpha then \alpha=v'  
    return v
```

```
def min-value(state, \alpha, \beta):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +\infty  
    for each successor of state:  
        v' = max-value(successor, \alpha, \beta)  
        if v' < v then v=v'  
        if v' <= \alpha then return v  
        if v' < \beta then \beta=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

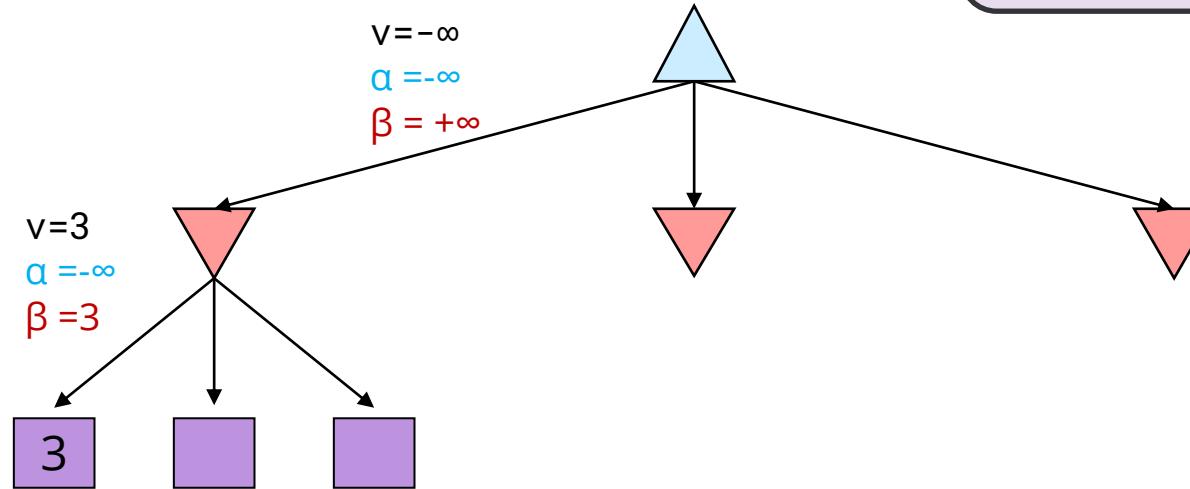


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

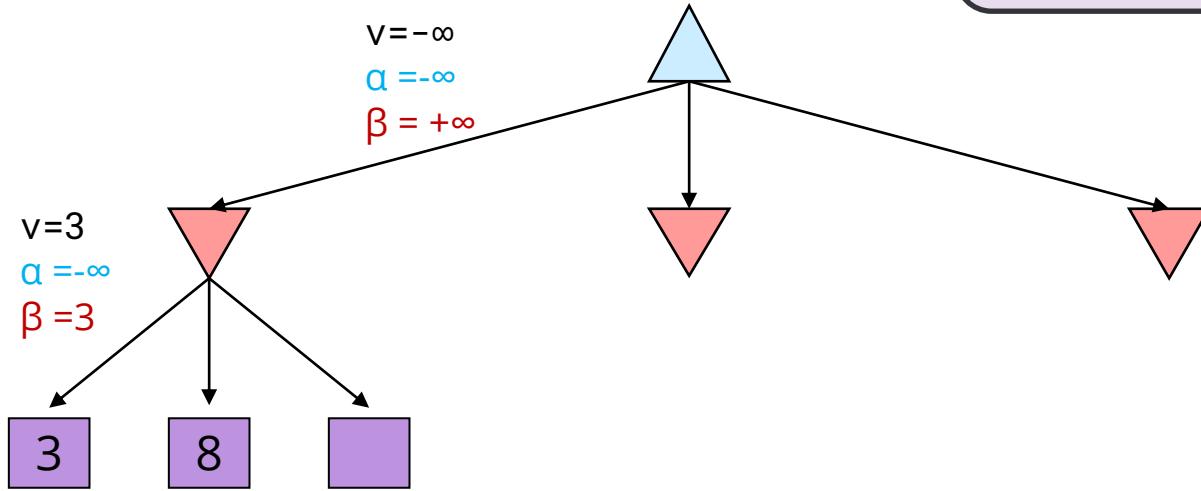


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

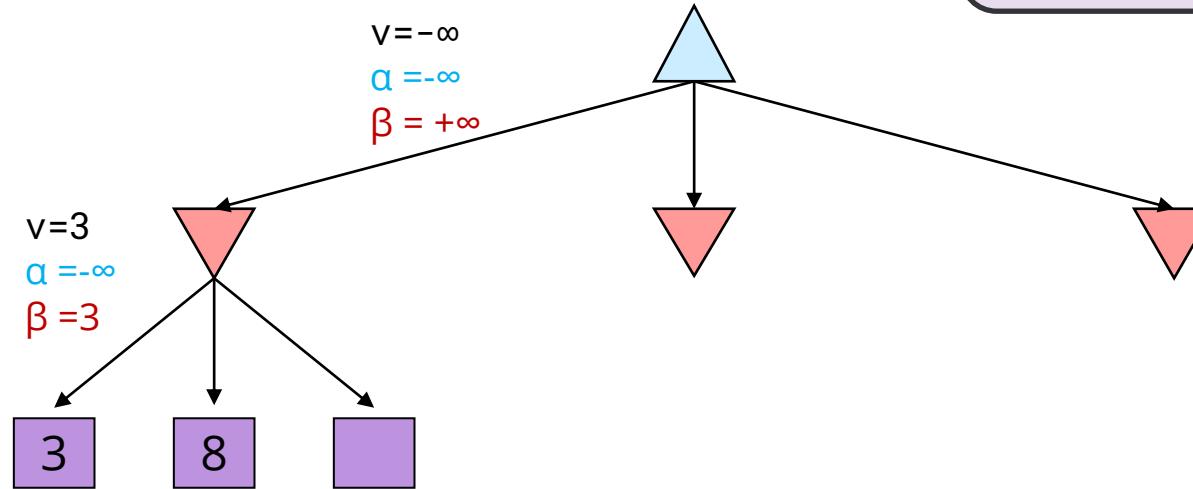


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v' = +∞  
    for each successor of state:  
        v = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

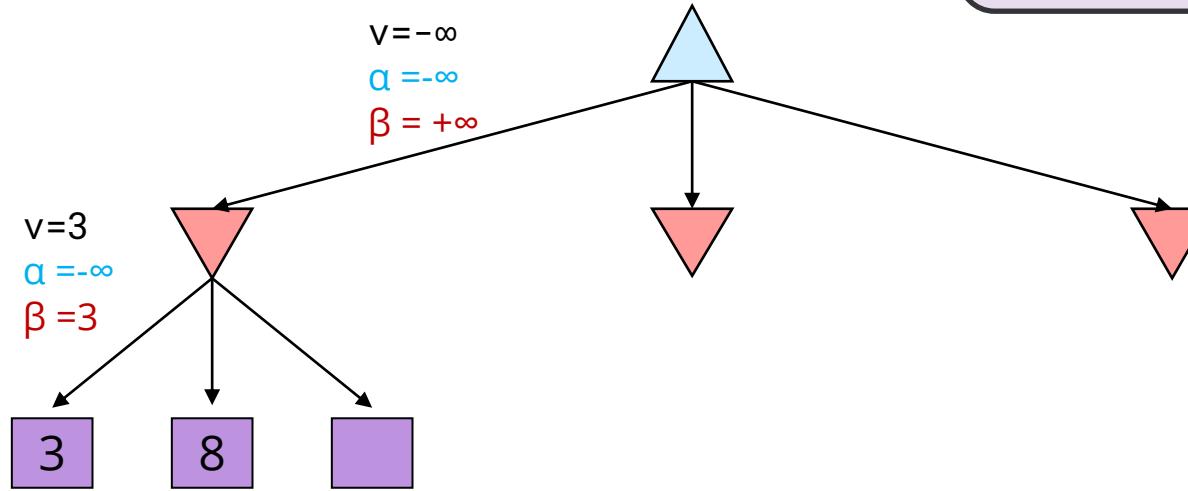


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

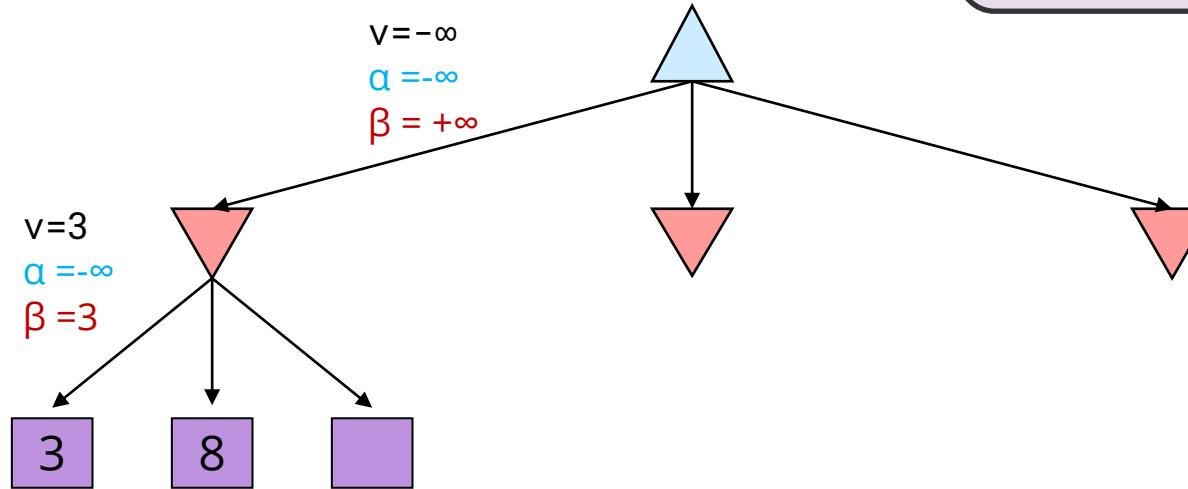


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

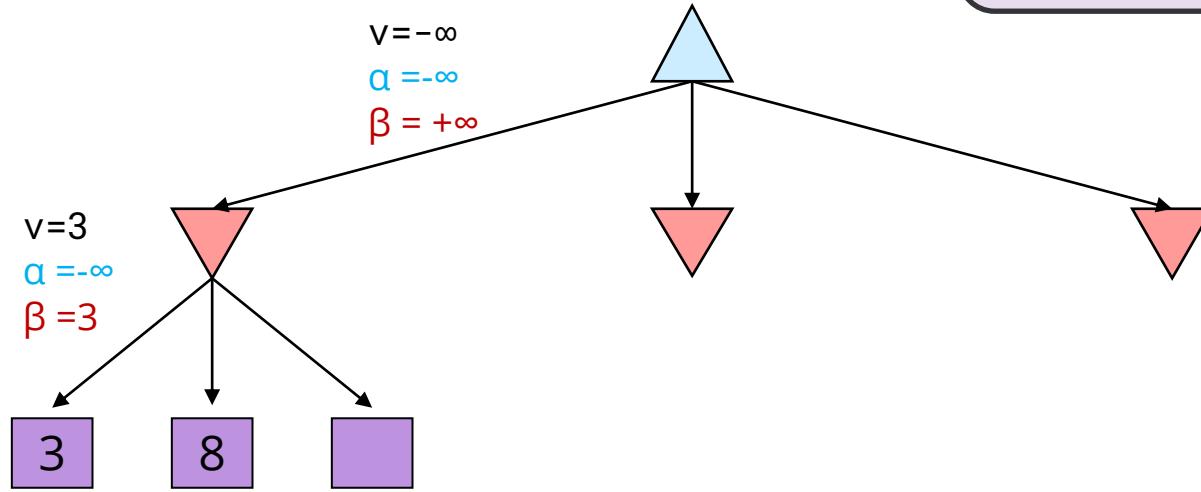


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

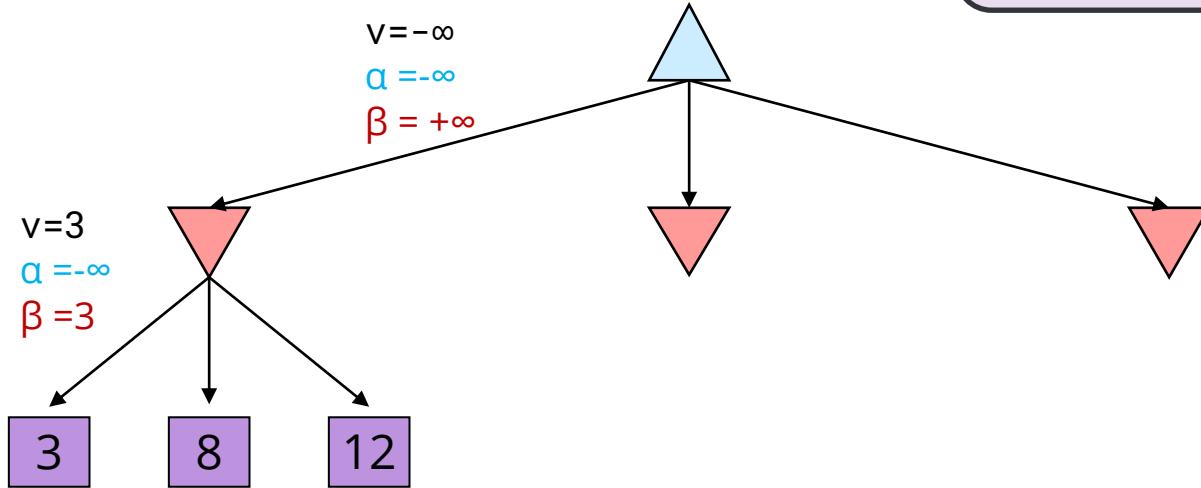


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

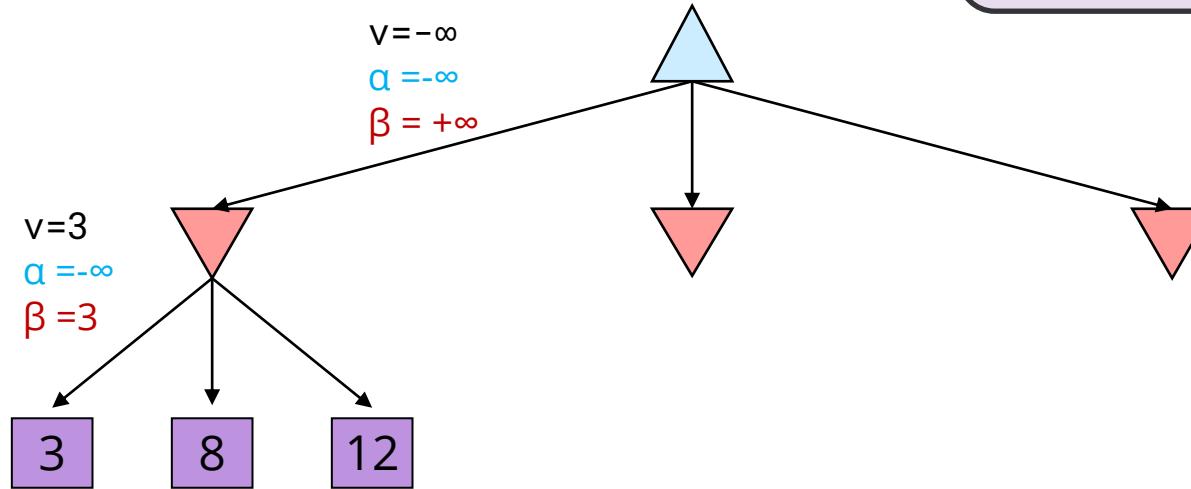


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

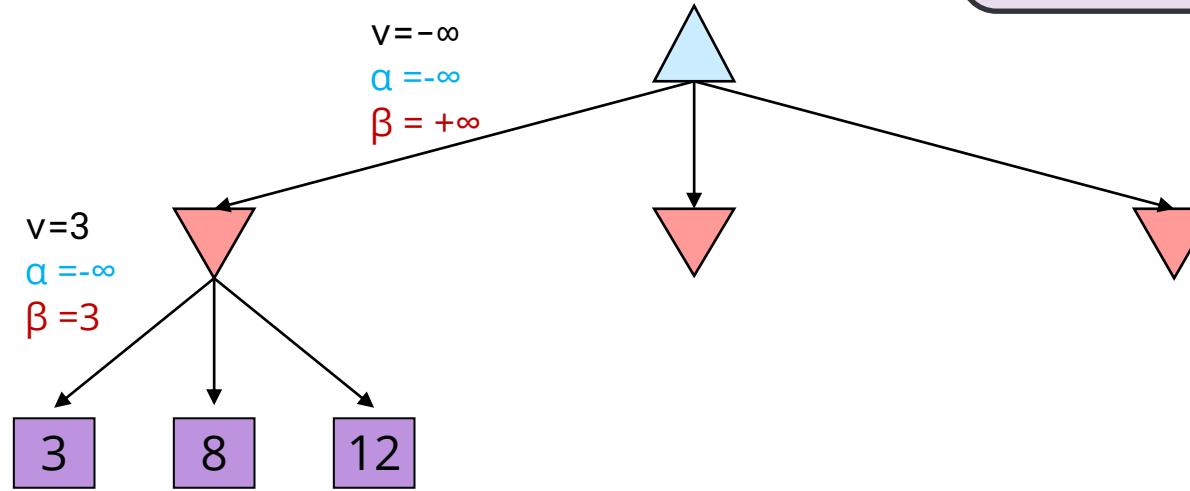


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

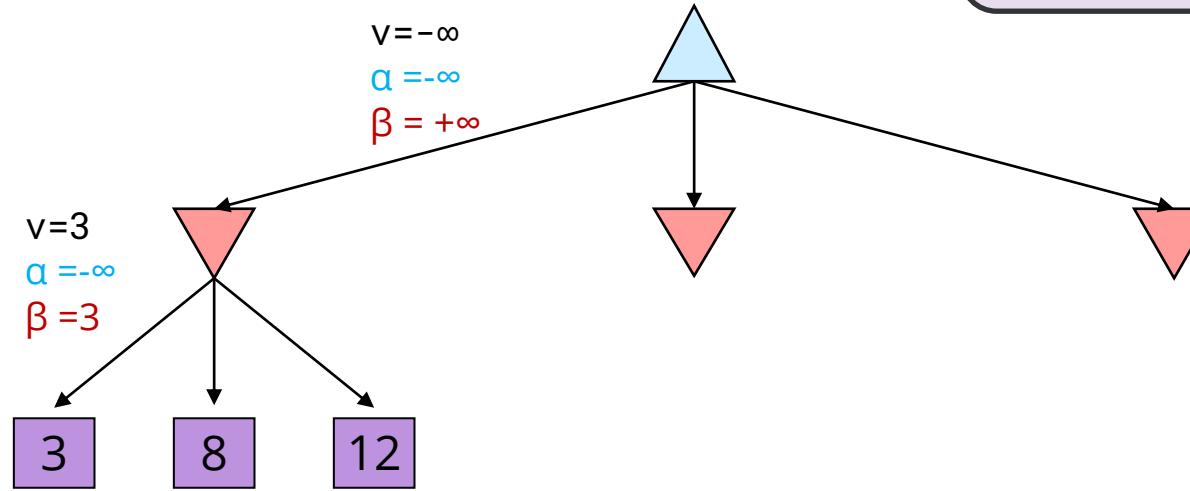


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

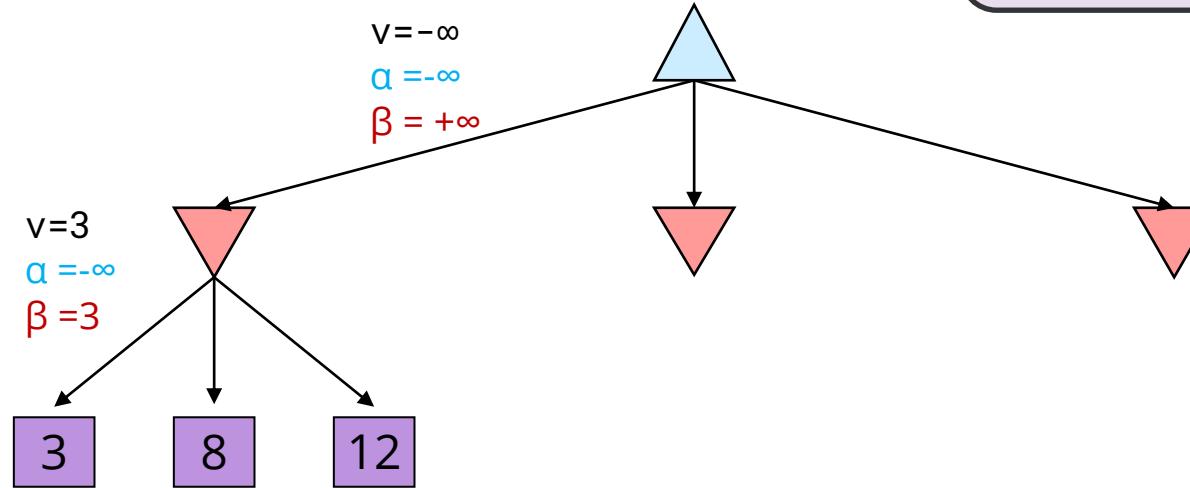


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

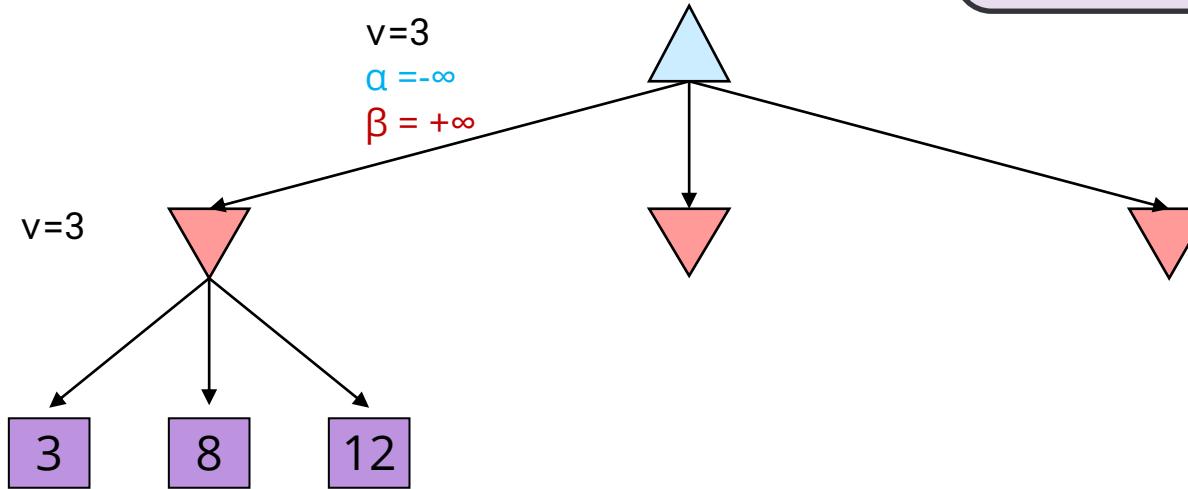


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

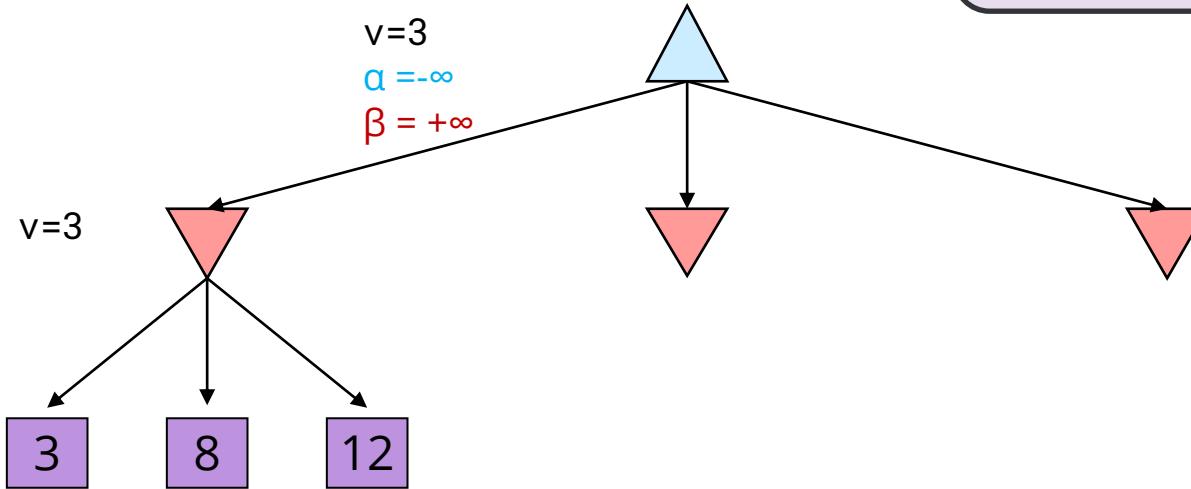


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

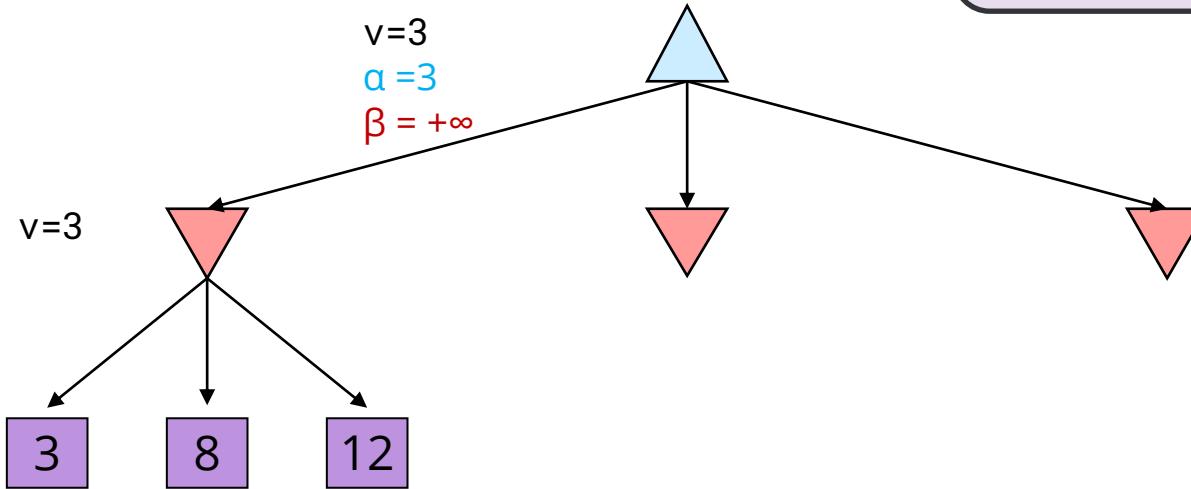


```
def max-value(state, a, b):
    if the state is a terminal state:
        return the state's utility
    initialize v = -\infty
    for each successor of state:
        v' = min-value(successor, a, b)
        if v' > v then v=v'
        if v' >= b then return v
        if v' > a then a=v'
    return v
```

```
def min-value(state, a, b):
    if the state is a terminal state:
        return the state's utility
    initialize v = +\infty
    for each successor of state:
        v' = max-value(successor, a, b)
        if v' < v then v=v'
        if v' <= a then return v
        if v' < b then b=v'
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

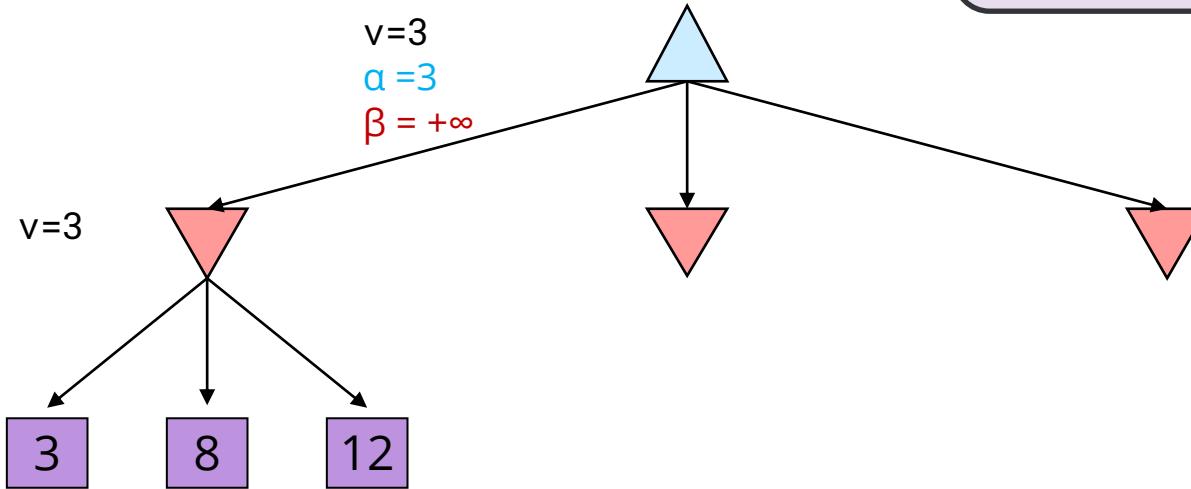


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

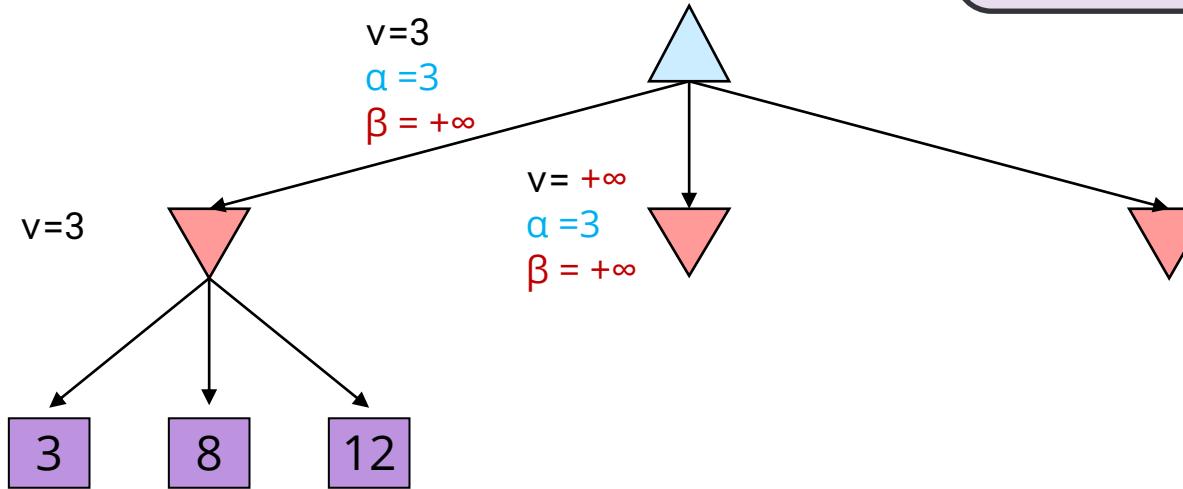


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

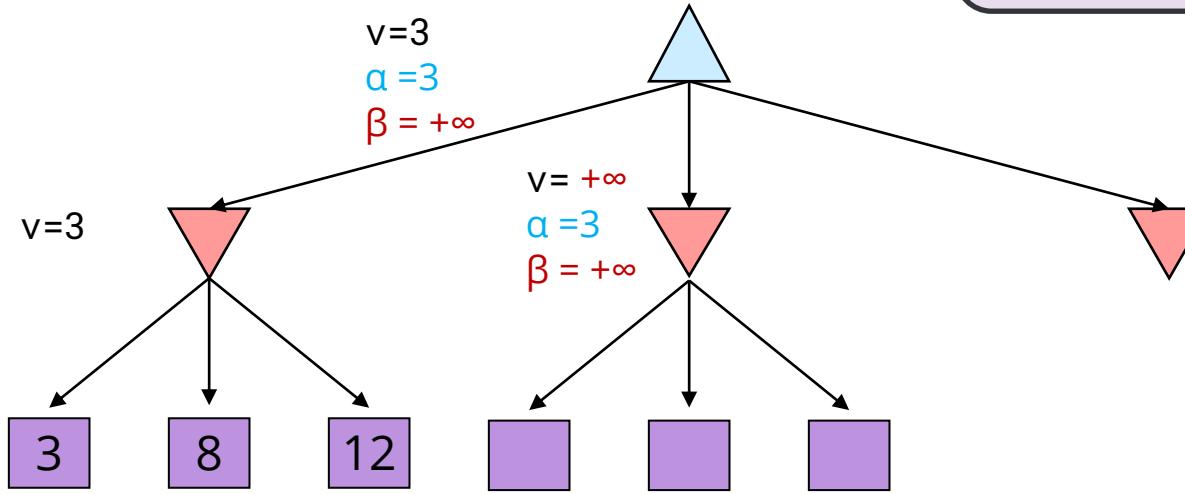


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

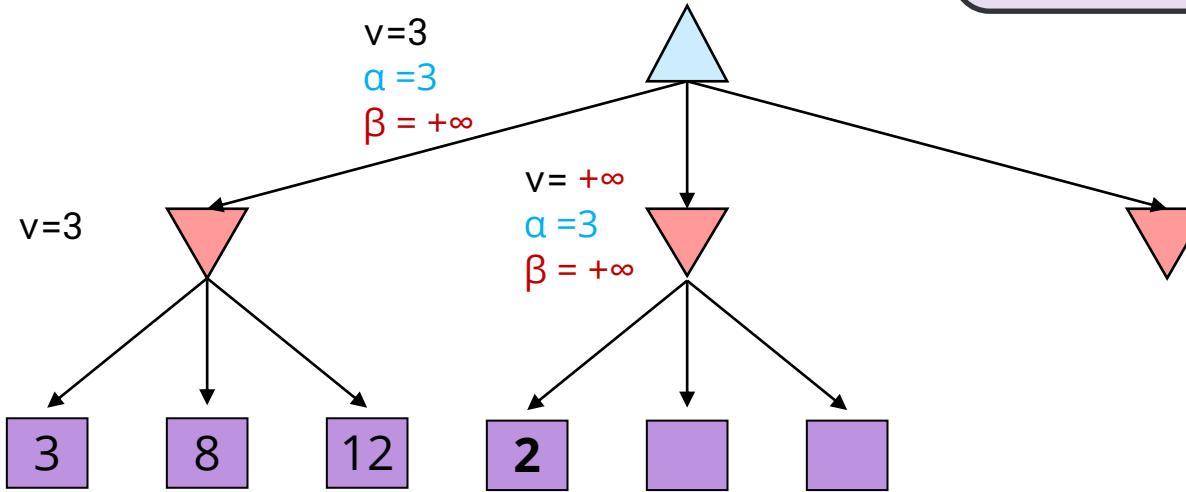


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

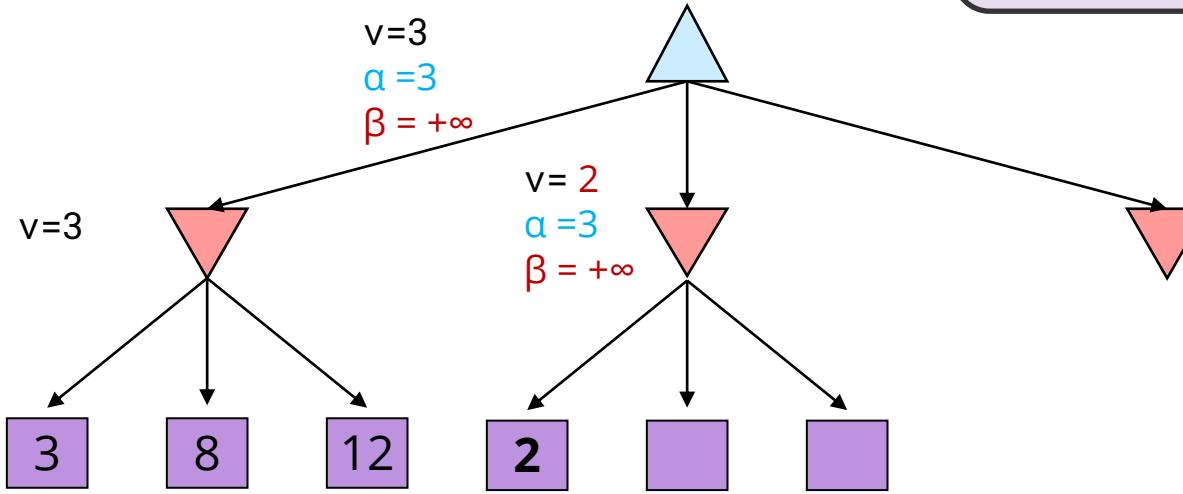


```
def max-value(state, α, β):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v' = min-value(successor, α, β)
        if v' > v then v=v'
        if v' >= β then return v
        if v' > α then α=v'
    return v
```

```
def min-value(state, α, β):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v' = max-value(successor, α, β)
        if v' < v then v=v'
        if v' <= α then return v
        if v' < β then β=v'
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

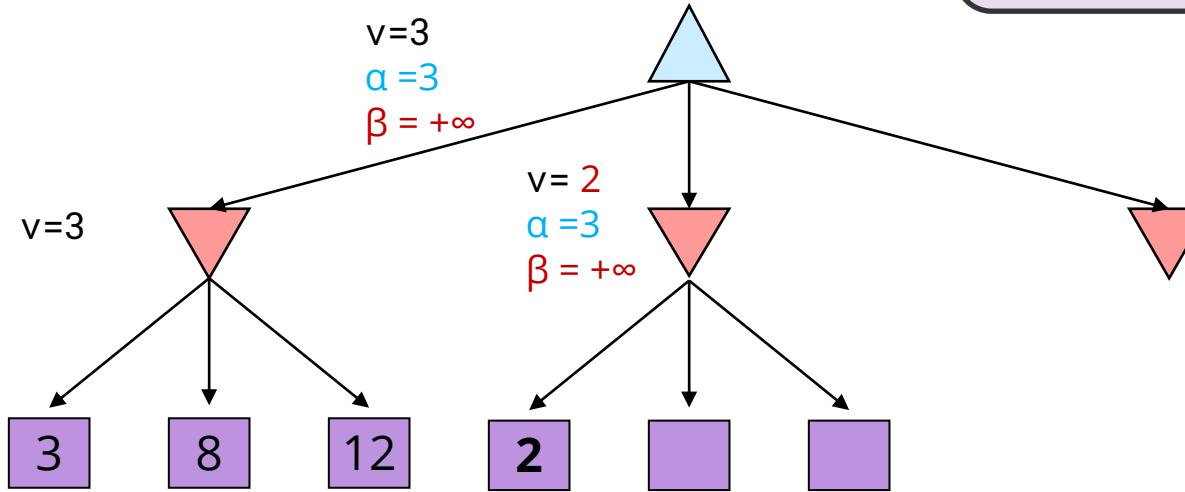


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

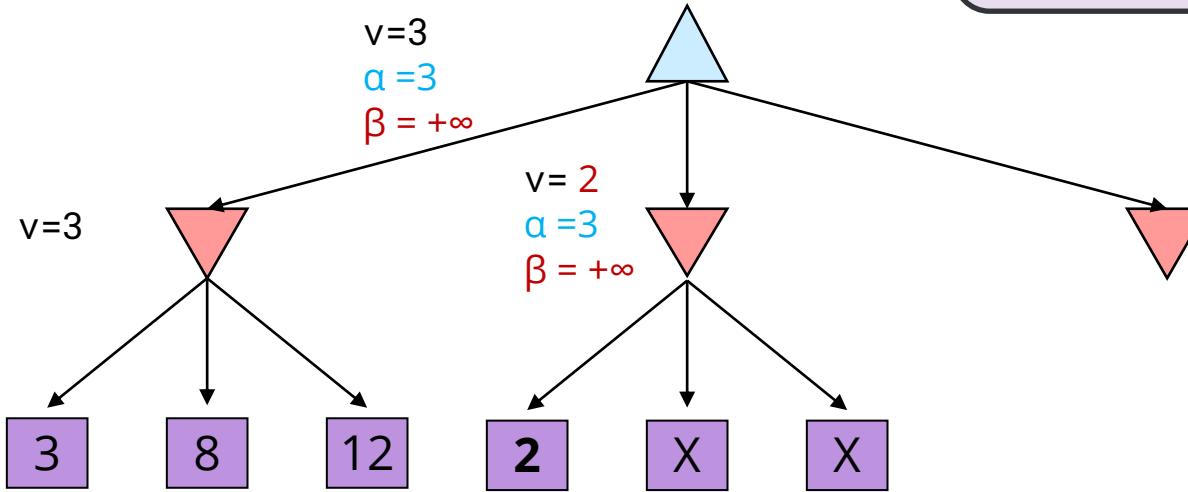


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

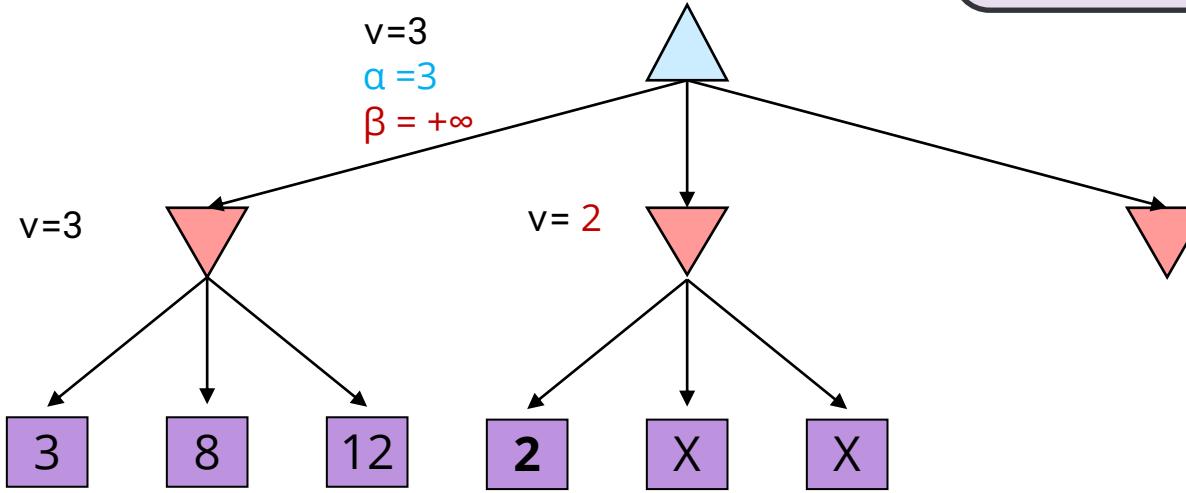


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

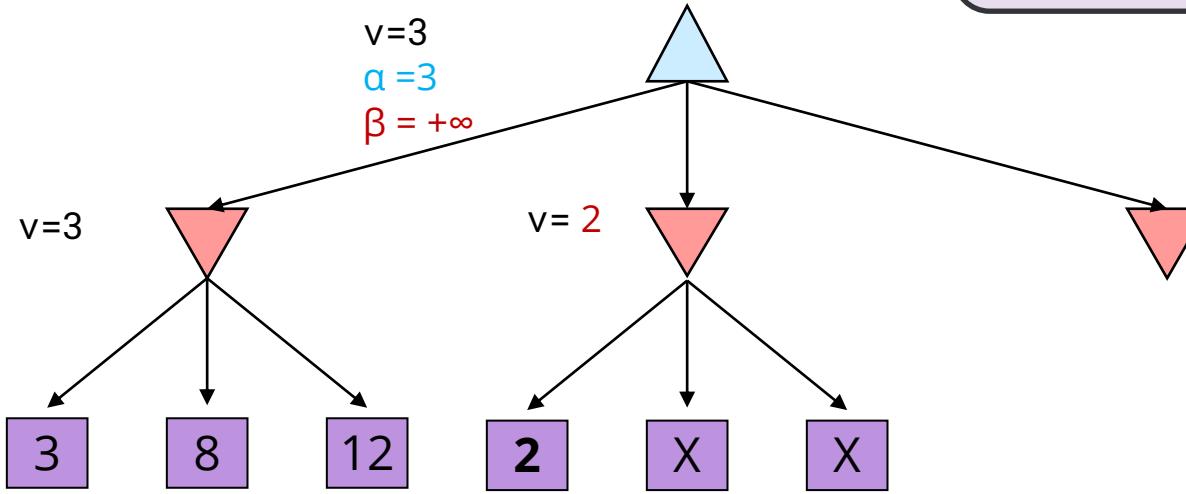


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

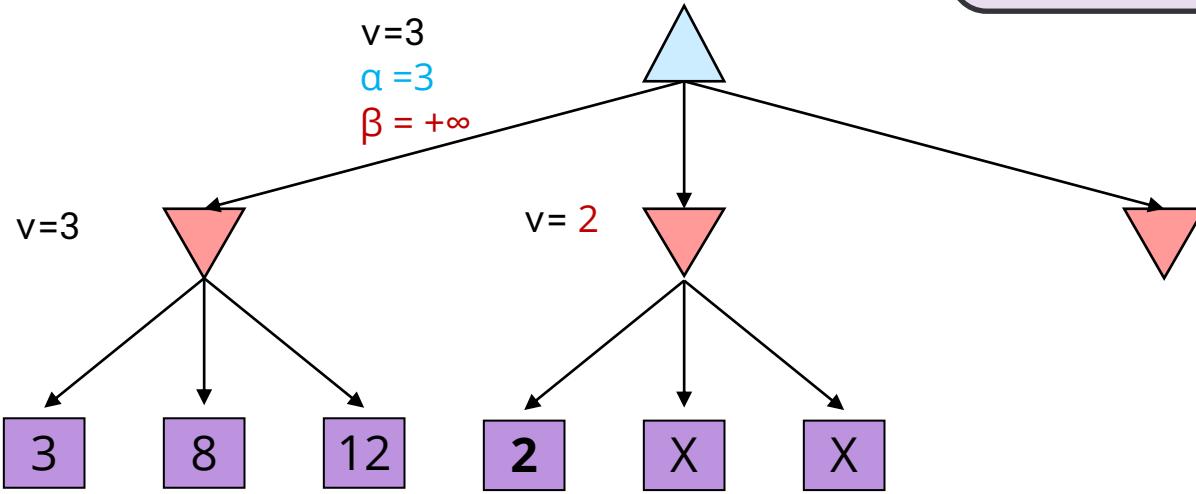


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

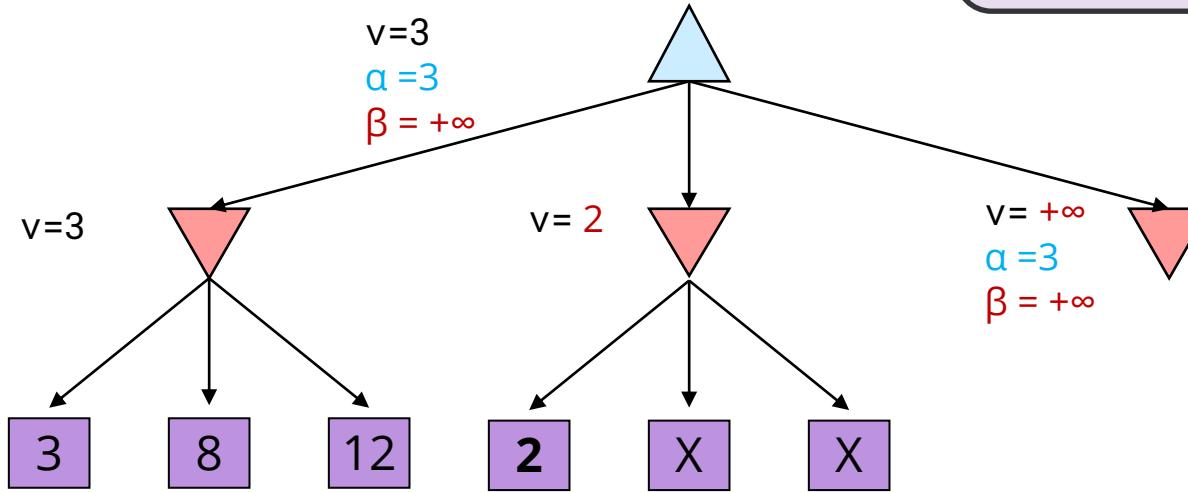


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

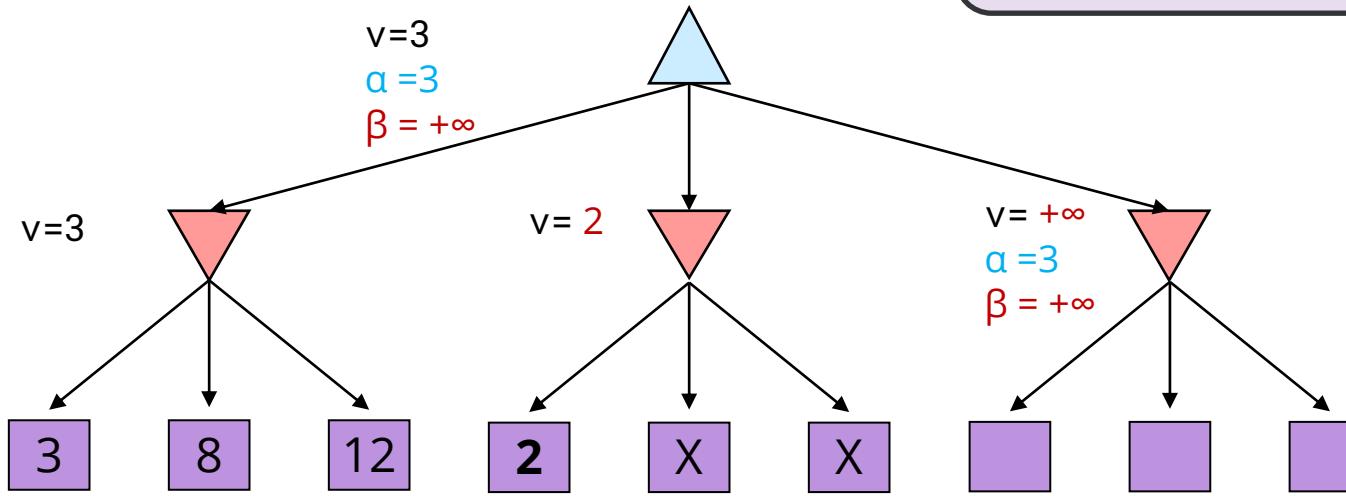


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

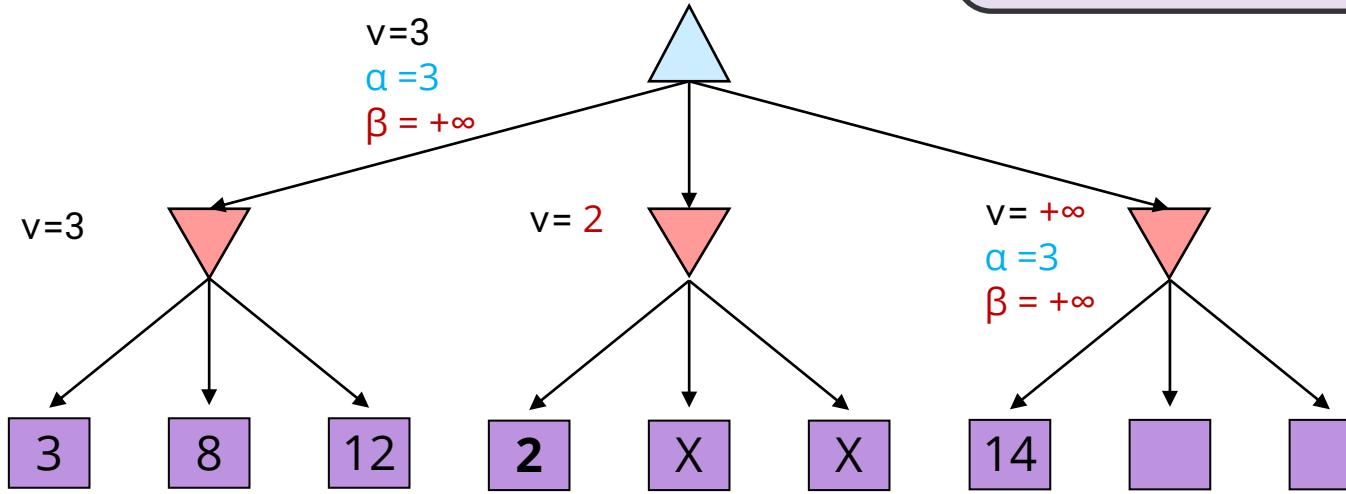


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v =  $-\infty$   
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v =  $+\infty$   
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

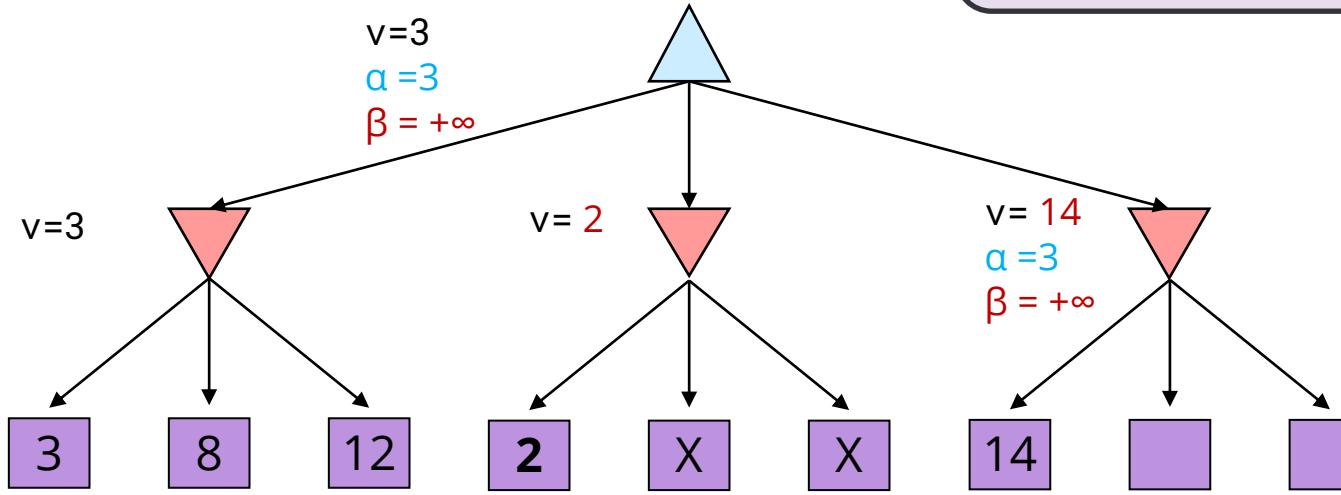


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

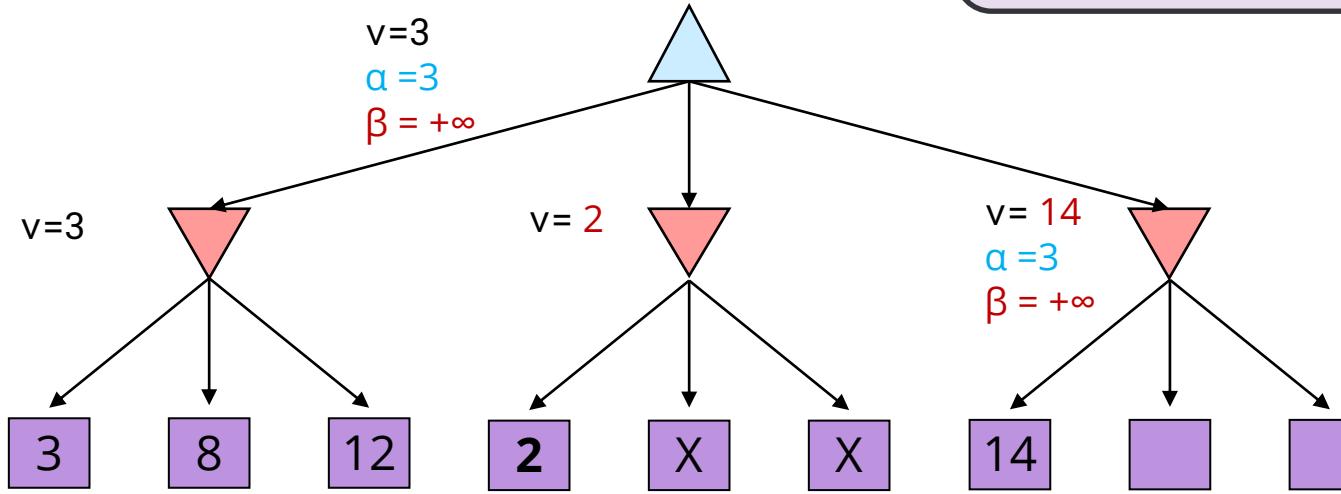


```
def max-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, a, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > a then a=v'  
    return v
```

```
def min-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, a, β)  
        if v' < v then v=v'  
        if v' <= a then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

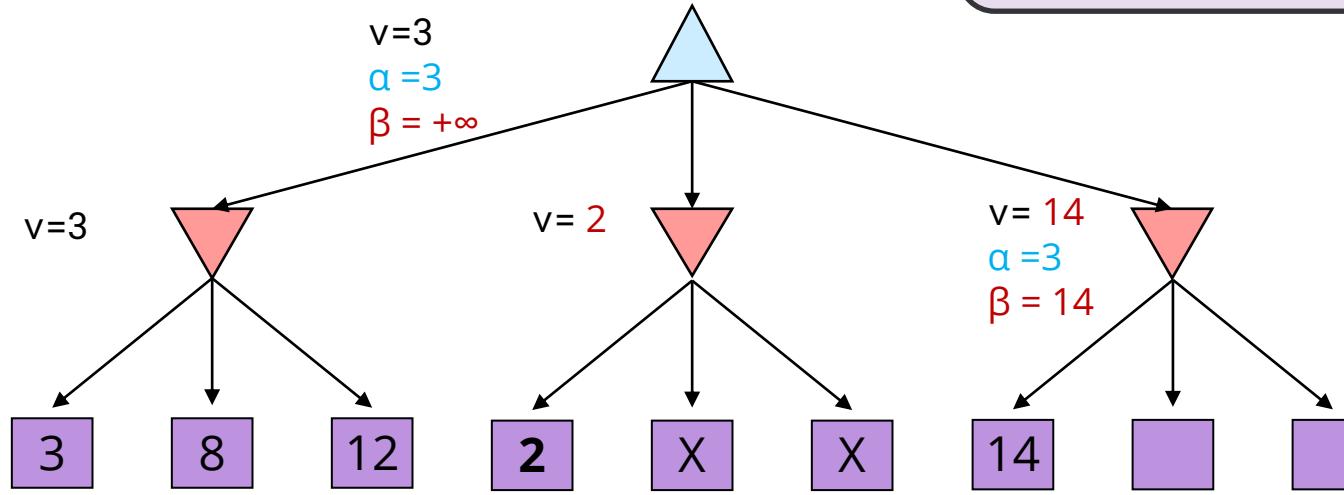


```
def max-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, a, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > a then a=v'  
    return v
```

```
def min-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, a, β)  
        if v' < v then v=v'  
        if v' <= a then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

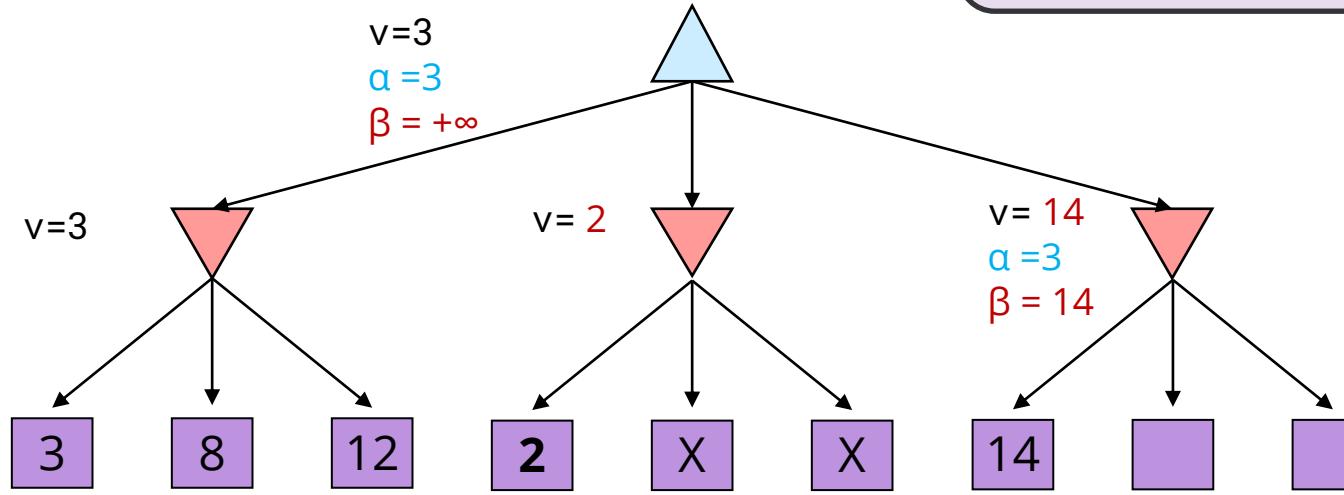


```
def max-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, a, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > a then a=v'  
    return v
```

```
def min-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, a, β)  
        if v' < v then v=v'  
        if v' <= a then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

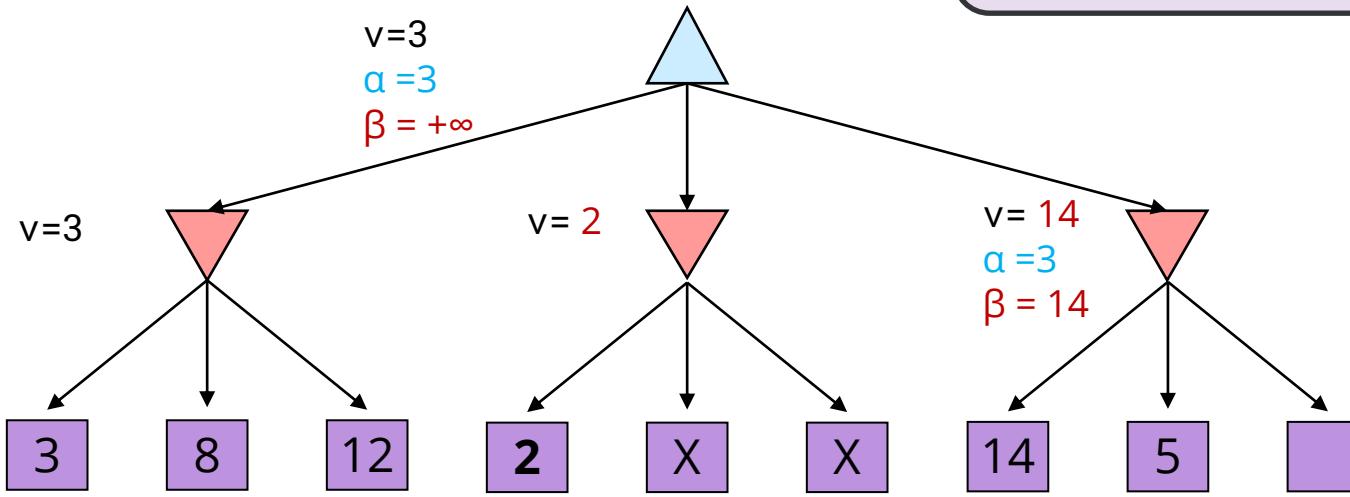


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

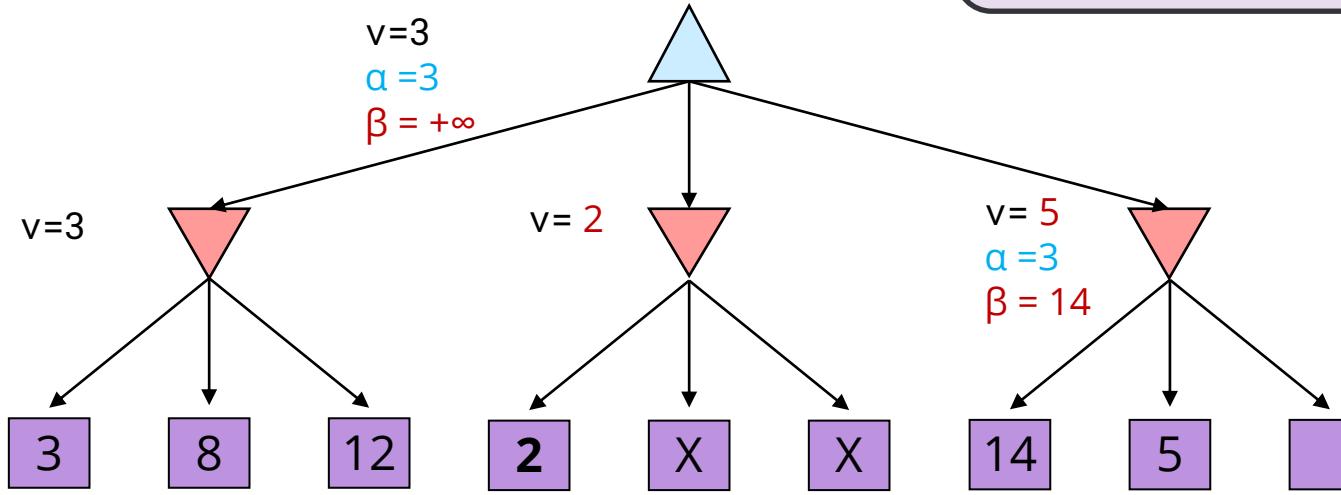


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

a: MAX's best option on path to root  
β: MIN's best option on path to root

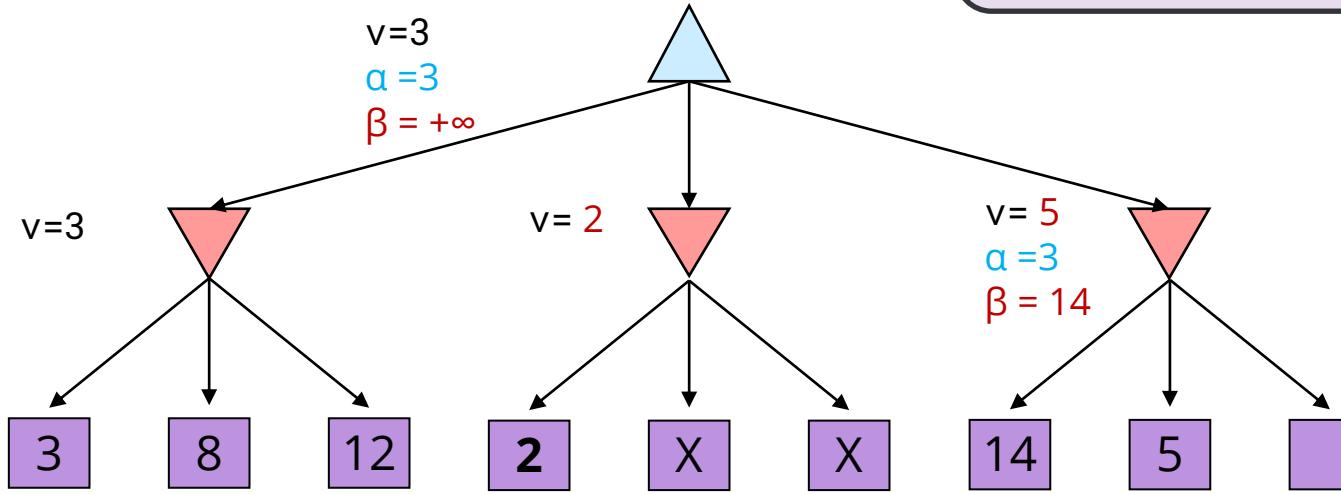


```
def max-value(state,α,β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor,α,β)  
        if v'>v then v=v'  
        if v'>=β then return v  
        if v'>α then α=v'  
    return v
```

```
def min-value(state,α,β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor,α,β)  
        if v'<v then v=v'  
        if v'<=α then return v  
        if v'<β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

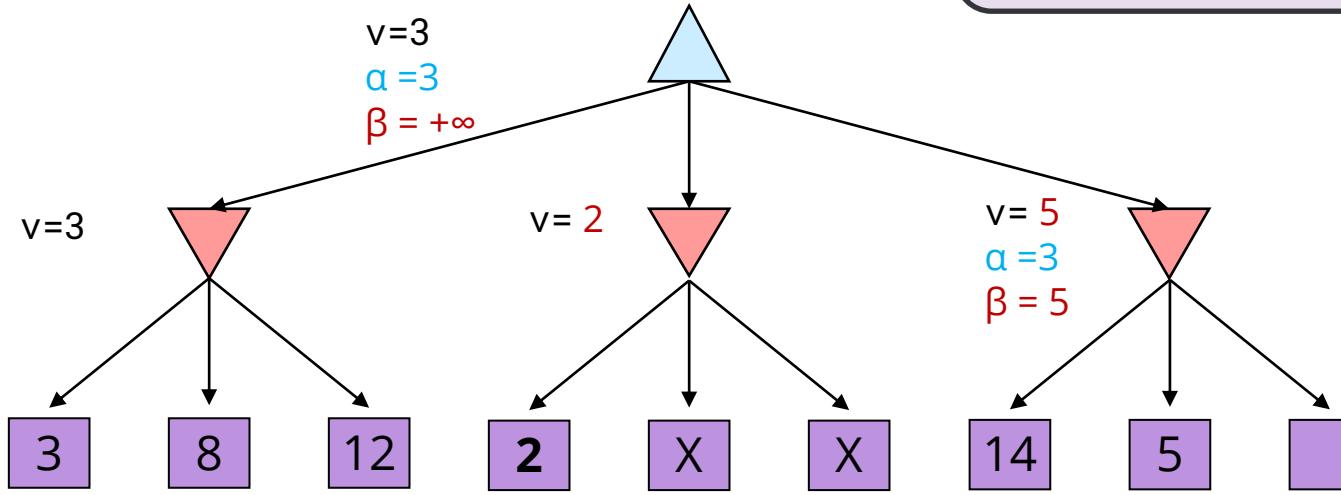


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

a: MAX's best option on path to root  
β: MIN's best option on path to root

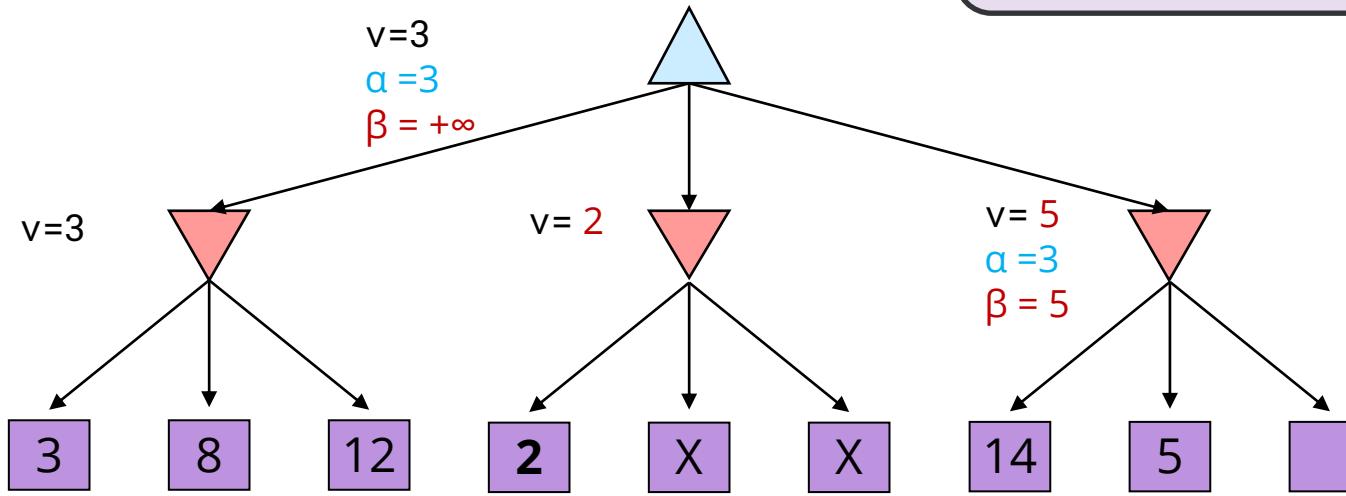


```
def max-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, a, β)  
        if v'>v then v=v'  
        if v'>=β then return v  
        if v'>a then a=v'  
    return v
```

```
def min-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, a, β)  
        if v'<v then v=v'  
        if v'<=a then return v  
        if v'<β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

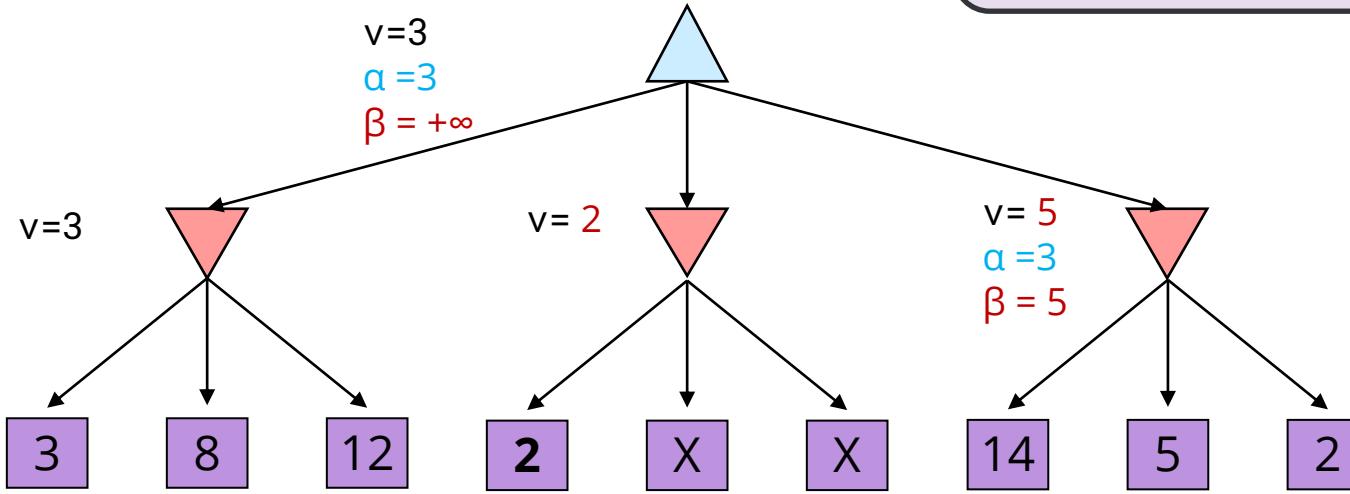


```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

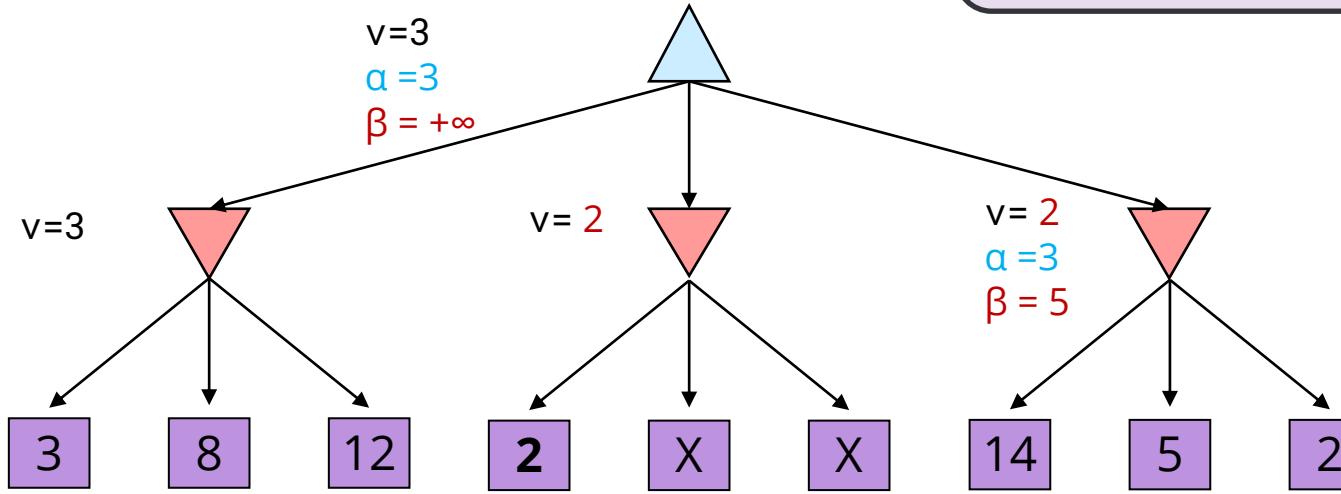


```
def max-value(state, α, β):
    if the state is a terminal state:
        return the state's utility
    initialize v = -∞
    for each successor of state:
        v' = min-value(successor, α, β)
        if v' > v then v=v'
        if v' >= β then return v
        if v' > α then α=v'
    return v
```

```
def min-value(state, α, β):
    if the state is a terminal state:
        return the state's utility
    initialize v = +∞
    for each successor of state:
        v' = max-value(successor, α, β)
        if v' < v then v=v'
        if v' <= α then return v
        if v' < β then β=v'
    return v
```

# Alpha-Beta Pruning Example

a: MAX's best option on path to root  
β: MIN's best option on path to root

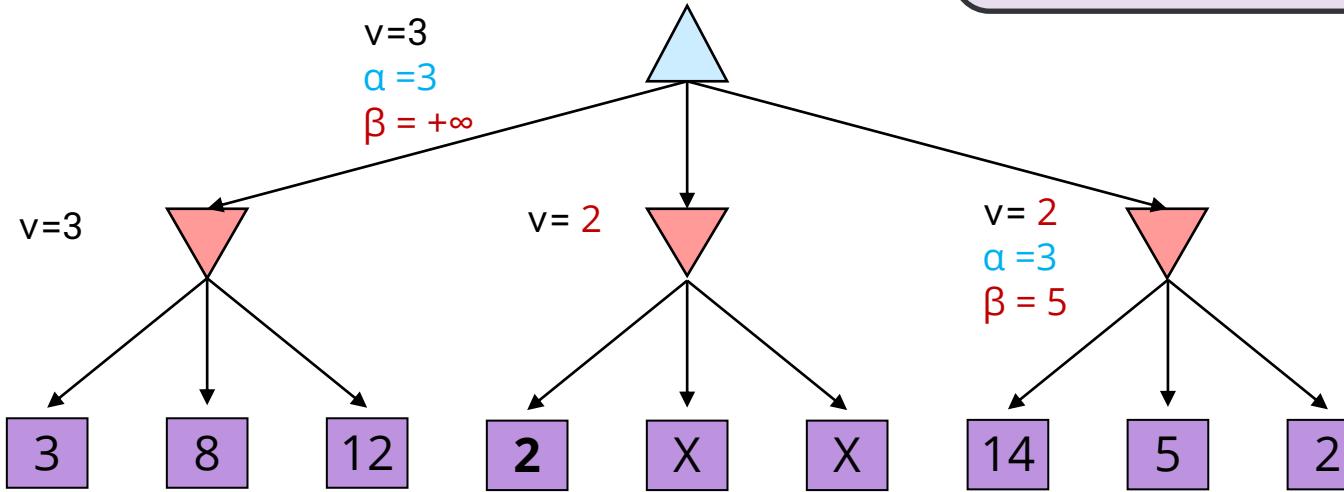


```
def max-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, a, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > a then a=v'  
    return v
```

```
def min-value(state, a, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, a, β)  
        if v' < v then v=v'  
        if v' <= a then return v  
        if v' < β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

a: MAX's best option on path to root  
β: MIN's best option on path to root

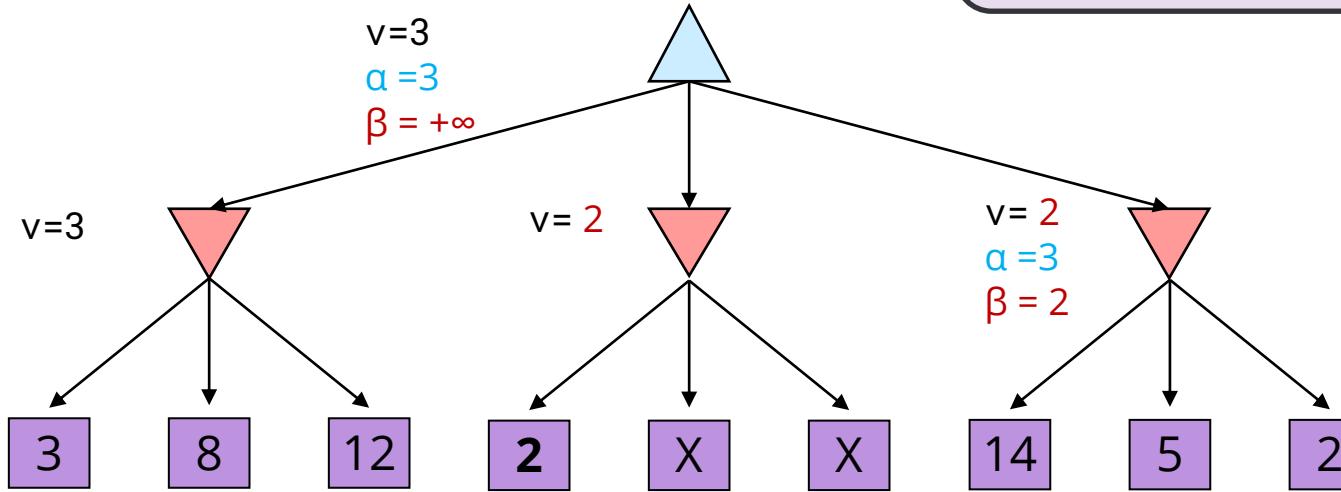


```
def max-value(state,α,β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor,α,β)  
        if v'>v then v=v'  
        if v'>=β then return v  
        if v'>α then α=v'  
    return v
```

```
def min-value(state,α,β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor,α,β)  
        if v'<v then v=v'  
        if v'<=α then return v  
        if v'<β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

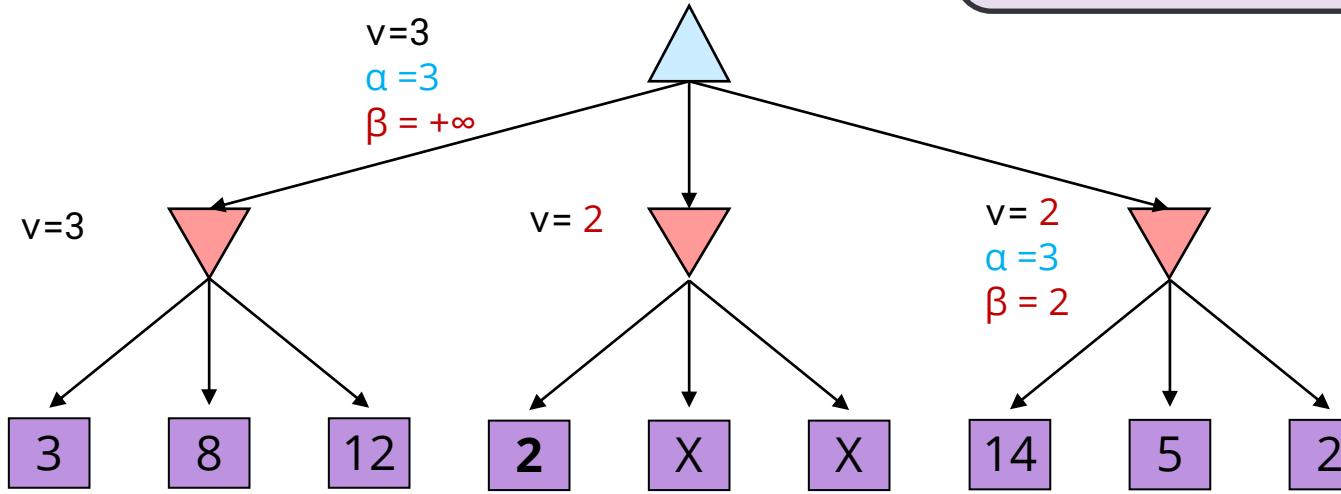


```
def max-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = -\infty$   
    for each successor of state:  
         $v' = \text{min-value}(\text{successor}, \alpha, \beta)$   
        if  $v' > v$  then  $v=v'$   
        if  $v' \geq \beta$  then return  $v$   
        if  $v' > \alpha$  then  $\alpha=v'$   
    return  $v$ 
```

```
def min-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = +\infty$   
    for each successor of state:  
         $v' = \text{max-value}(\text{successor}, \alpha, \beta)$   
        if  $v' < v$  then  $v=v'$   
        if  $v' \leq \alpha$  then return  $v$   
        if  $v' < \beta$  then  $\beta=v'$   
    return  $v$ 
```

# Alpha-Beta Pruning Example

a: MAX's best option on path to root  
β: MIN's best option on path to root

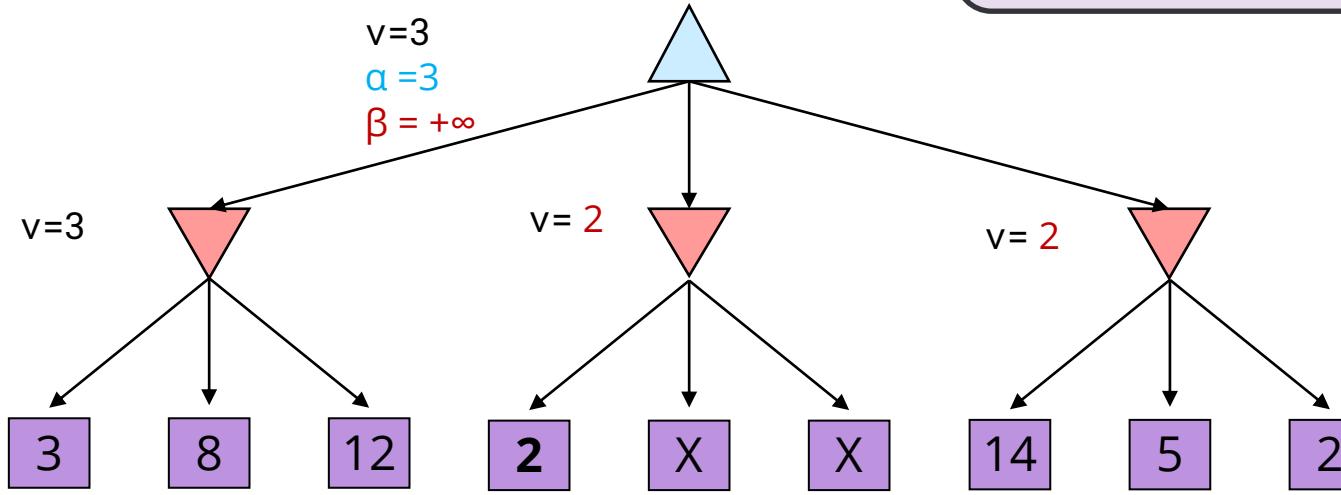


```
def max-value(state,α,β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor,α,β)  
        if v'>v then v=v'  
        if v'>=β then return v  
        if v'>α then α=v'  
    return v
```

```
def min-value(state,α,β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor,α,β)  
        if v'<v then v=v'  
        if v'<=α then return v  
        if v'<β then β=v'  
    return v
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

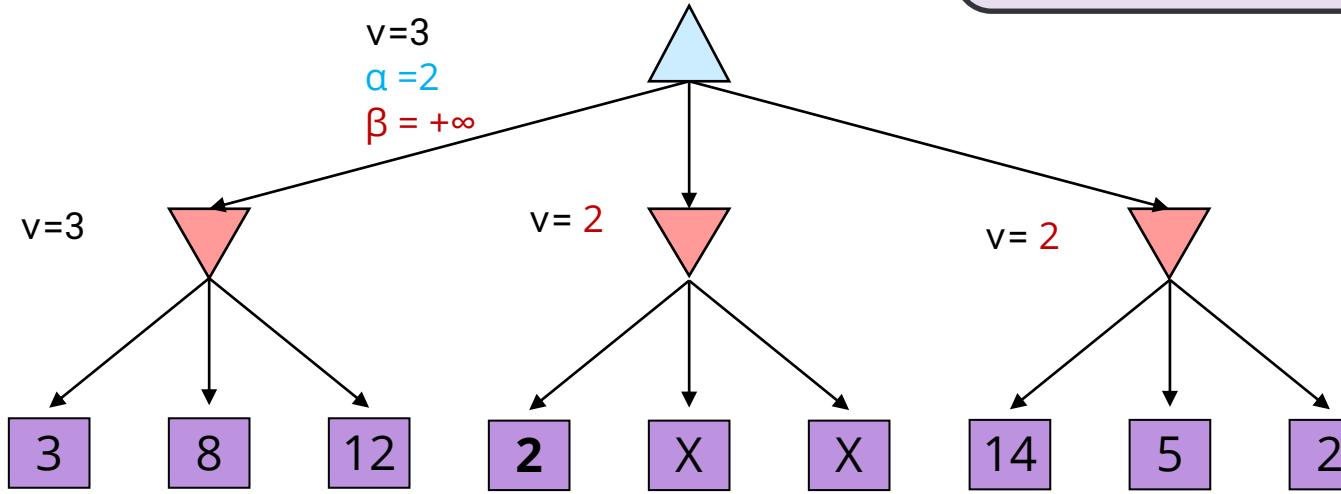


```
def max-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = -\infty$   
    for each successor of state:  
         $v' = \text{min-value}(\text{successor}, \alpha, \beta)$   
        if  $v' > v$  then  $v=v'$   
        if  $v' \geq \beta$  then return  $v$   
        if  $v' > \alpha$  then  $\alpha=v'$   
    return  $v$ 
```

```
def min-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = +\infty$   
    for each successor of state:  
         $v' = \text{max-value}(\text{successor}, \alpha, \beta)$   
        if  $v' < v$  then  $v=v'$   
        if  $v' \leq \alpha$  then return  $v$   
        if  $v' < \beta$  then  $\beta=v'$   
    return  $v$ 
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

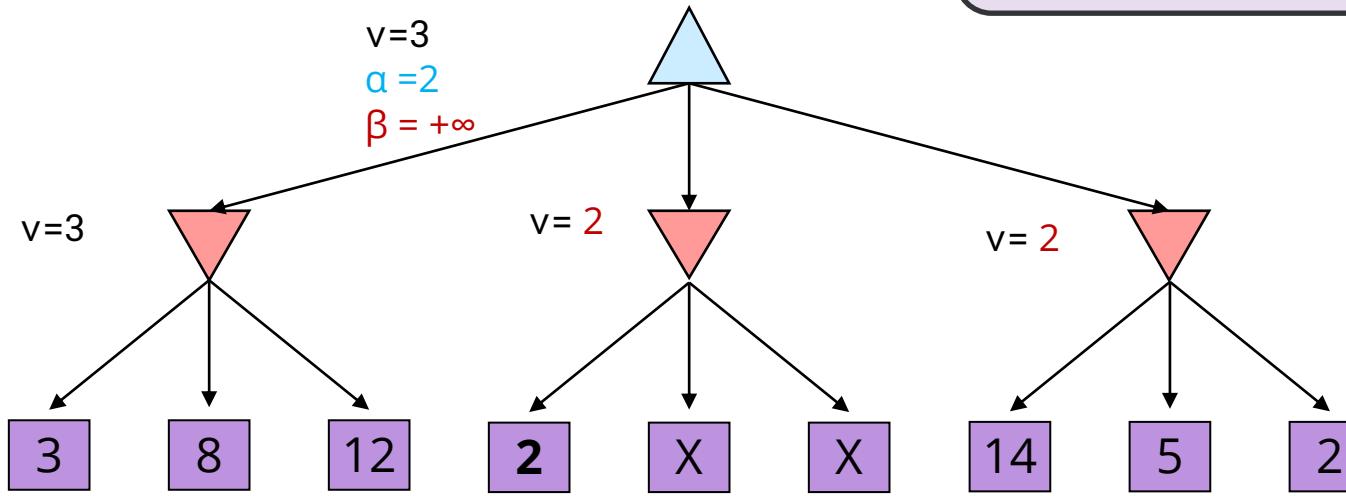


```
def max-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = -\infty$   
    for each successor of state:  
         $v' = \text{min-value}(\text{successor}, \alpha, \beta)$   
        if  $v' > v$  then  $v=v'$   
        if  $v' >= \beta$  then return  $v$   
        if  $v' > \alpha$  then  $\alpha=v'$   
    return  $v$ 
```

```
def min-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = +\infty$   
    for each successor of state:  
         $v' = \text{max-value}(\text{successor}, \alpha, \beta)$   
        if  $v' < v$  then  $v=v'$   
        if  $v' <= \alpha$  then return  $v$   
        if  $v' < \beta$  then  $\beta=v'$   
    return  $v$ 
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

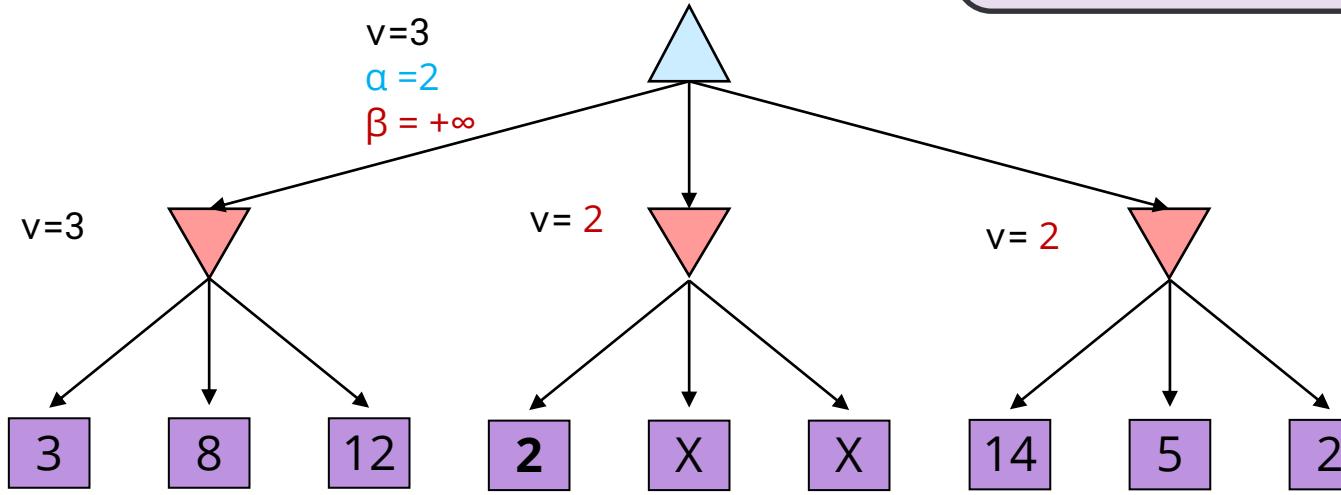


```
def max-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = -\infty$   
    for each successor of state:  
         $v' = \text{min-value}(\text{successor}, \alpha, \beta)$   
        if  $v' > v$  then  $v = v'$   
        if  $v' \geq \beta$  then return  $v$   
        if  $v' > \alpha$  then  $\alpha = v'$   
    return  $v$ 
```

```
def min-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = +\infty$   
    for each successor of state:  
         $v' = \text{max-value}(\text{successor}, \alpha, \beta)$   
        if  $v' < v$  then  $v = v'$   
        if  $v' \leq \alpha$  then return  $v$   
        if  $v' < \beta$  then  $\beta = v'$   
    return  $v$ 
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root

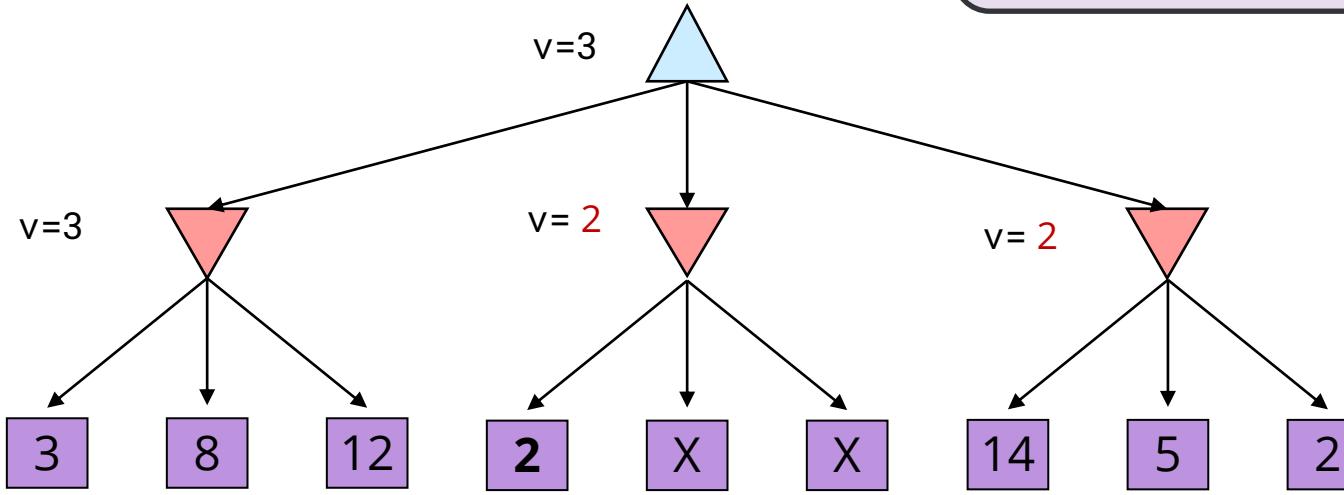


```
def max-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = -\infty$   
    for each successor of state:  
         $v' = \text{min-value}(\text{successor}, \alpha, \beta)$   
        if  $v' > v$  then  $v = v'$   
        if  $v' \geq \beta$  then return  $v$   
        if  $v' > \alpha$  then  $\alpha = v'$   
    return  $v$ 
```

```
def min-value(state,  $\alpha$ ,  $\beta$ ):  
    if the state is a terminal state:  
        return the state's utility  
    initialize  $v = +\infty$   
    for each successor of state:  
         $v' = \text{max-value}(\text{successor}, \alpha, \beta)$   
        if  $v' < v$  then  $v = v'$   
        if  $v' \leq \alpha$  then return  $v$   
        if  $v' < \beta$  then  $\beta = v'$   
    return  $v$ 
```

# Alpha-Beta Pruning Example

$\alpha$ : MAX's best option on path to root  
 $\beta$ : MIN's best option on path to root



```
def max-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = -∞  
    for each successor of state:  
        v' = min-value(successor, α, β)  
        if v' > v then v=v'  
        if v' >= β then return v  
        if v' > α then α=v'  
    return v
```

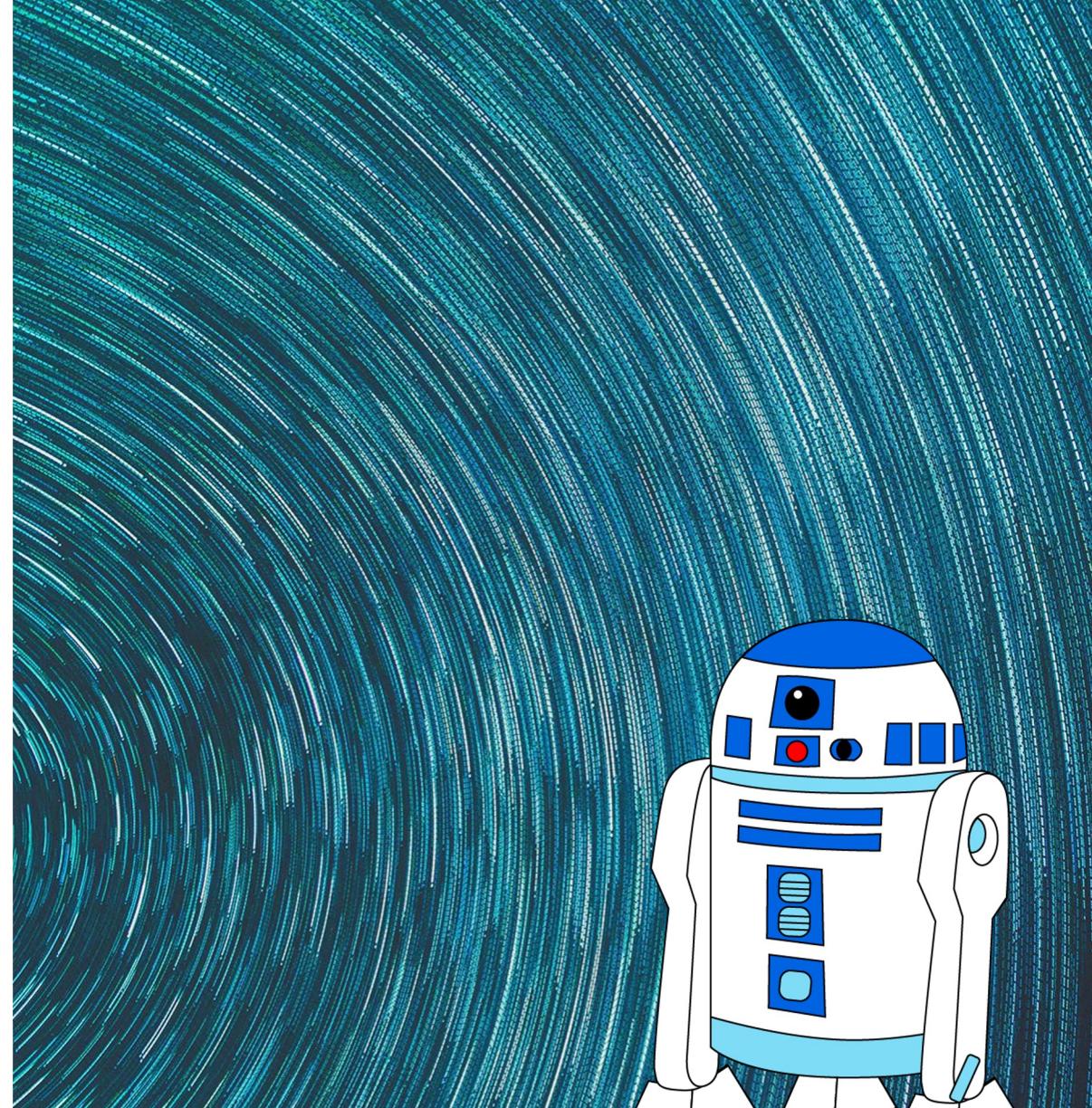
```
def min-value(state, α, β):  
    if the state is a terminal state:  
        return the state's utility  
    initialize v = +∞  
    for each successor of state:  
        v' = max-value(successor, α, β)  
        if v' < v then v=v'  
        if v' <= α then return v  
        if v' < β then β=v'  
    return v
```

CIS 4210/5210:  
ARTIFICIAL INTELLIGENCE

# Expectimax and Utilities

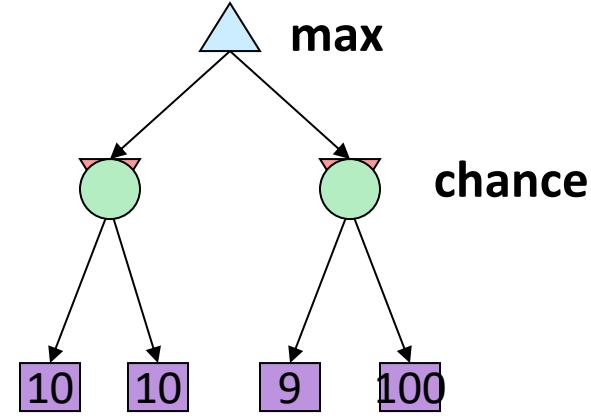
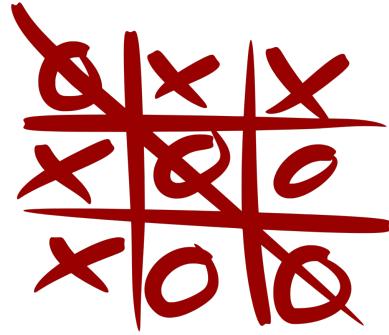
Professor Chris Callison-Burch

Many of today's slides are courtesy of Dan Klein and  
Pieter Abbeel of University of California, Berkeley



# Uncertain Outcomes

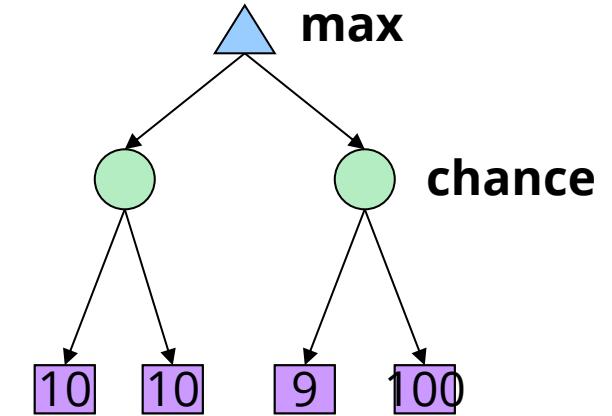




Idea: Uncertain outcomes controlled by chance, not an adversary!

# Expectimax Search

- Why wouldn't we know what the result of an action will be?
  - Explicit randomness: rolling dice
  - Unpredictable opponents: the opponent isn't optimal
  - Actions can fail: when moving a robot, wheels might slip
- Values should now reflect average-case (expectimax) outcomes, not worst-case (minimax) outcomes
- **Expectimax search:** compute the average score under optimal play
  - Max nodes as in minimax search
  - Chance nodes are like min nodes but the outcome is uncertain
  - Calculate their **expected utilities**
    - i.e. take weighted average (expectation) of children
- Later, we'll learn how to formalize the underlying uncertain-result problems as **Markov Decision Processes**

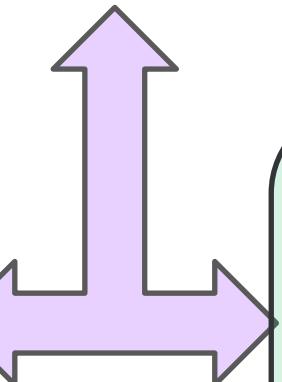


# Expectimax Pseudocode

```
def value(state):
    if the state is a terminal state: return the state's utility
    if the next agent is MAX: return max-value(state)
    if the next agent is EXP: return exp-value(state)
```

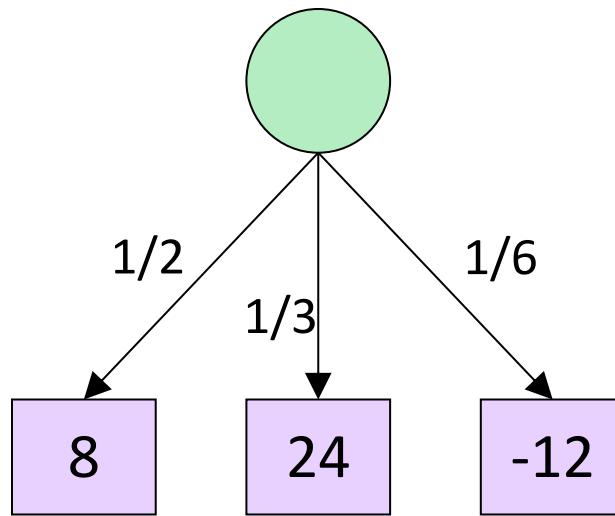
```
def max-value(state):
    initialize v = -∞
    for each successor of state:
        v = max(v, value(successor))
    return v
```

```
def exp-value(state):
    initialize v = 0
    for each successor of state:
        p = probability(successor)
        v += p * value(successor)
    return v
```

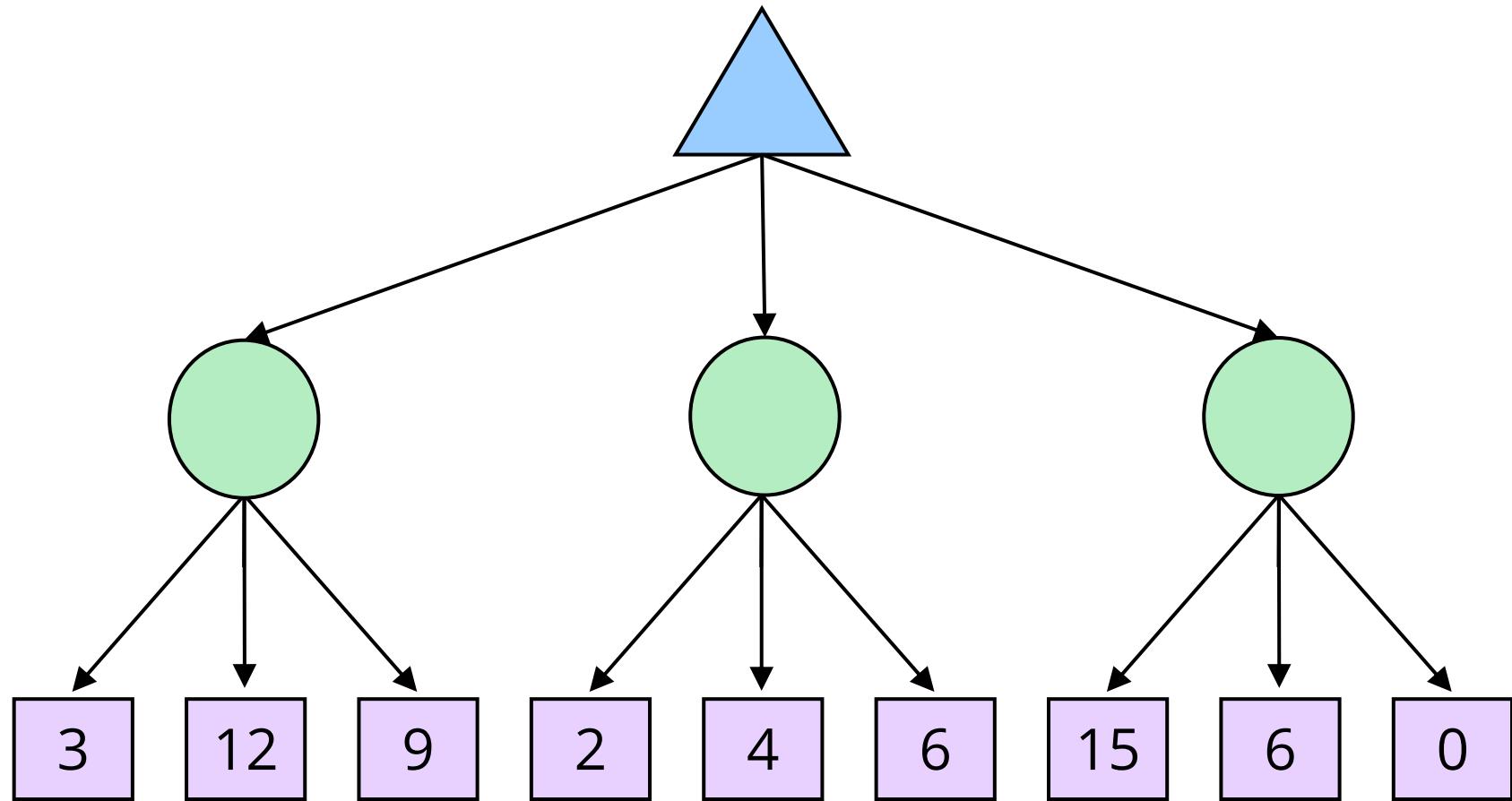


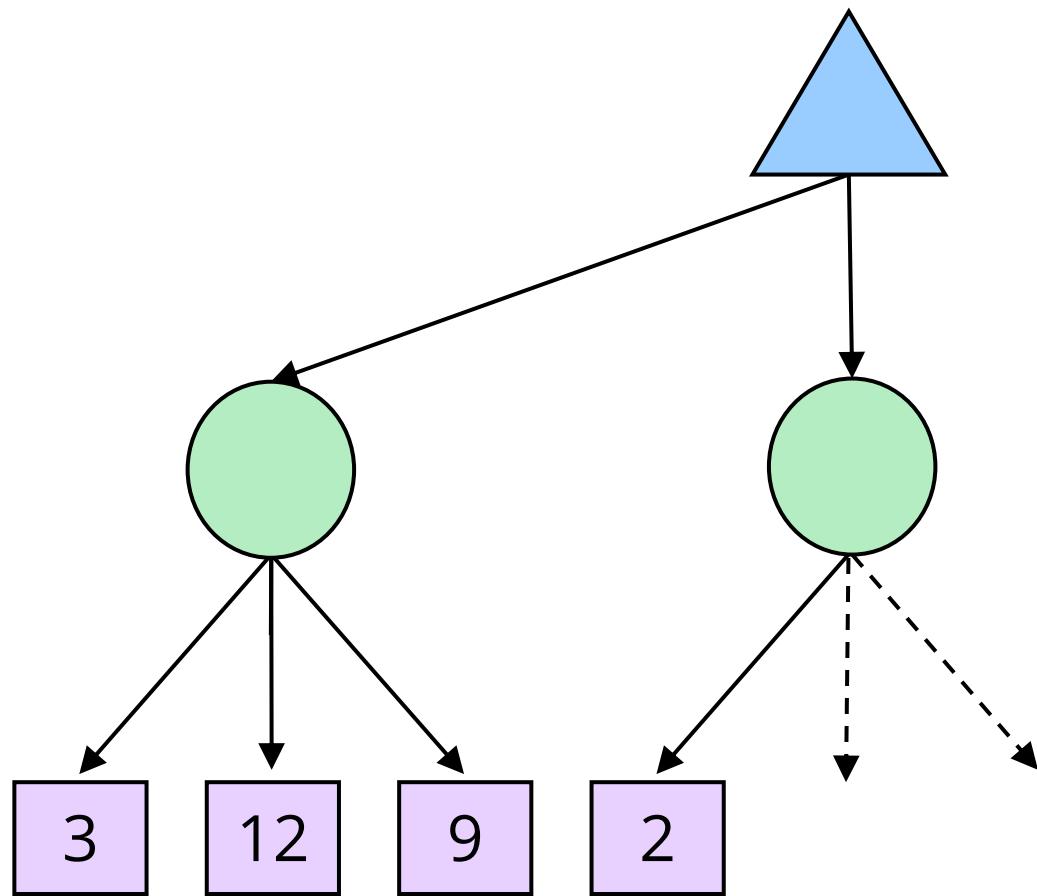
# Expectimax Pseudocode

```
def exp-value(state):  
    initialize v = 0  
    for each successor of state:  
        p = probability(successor)  
        v += p * value(successor)  
    return v
```

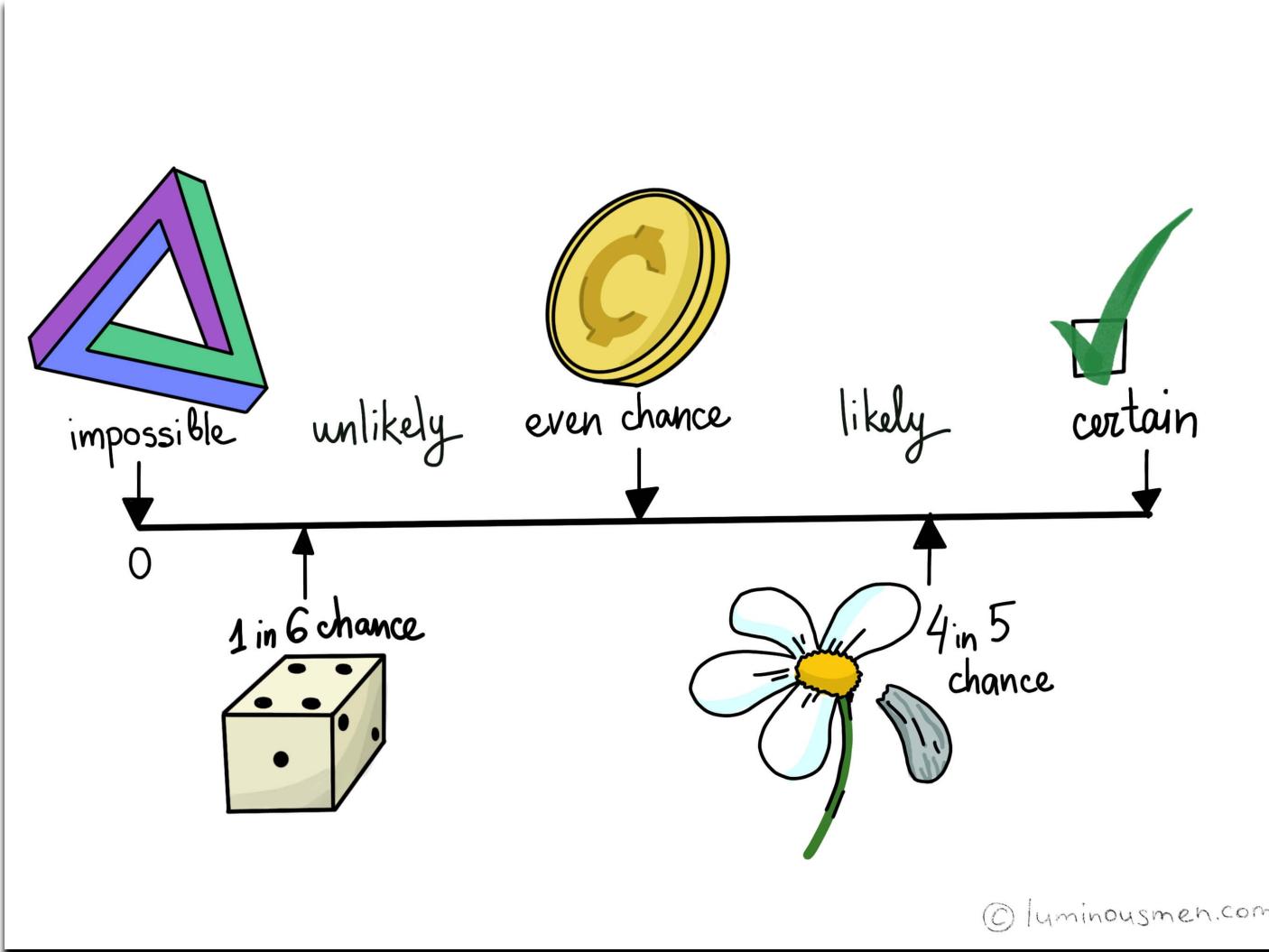


$$v = \frac{1}{2} \cdot (8) + \frac{1}{3} \cdot (24) + \frac{1}{6} \cdot (-12)$$





# Probabilities



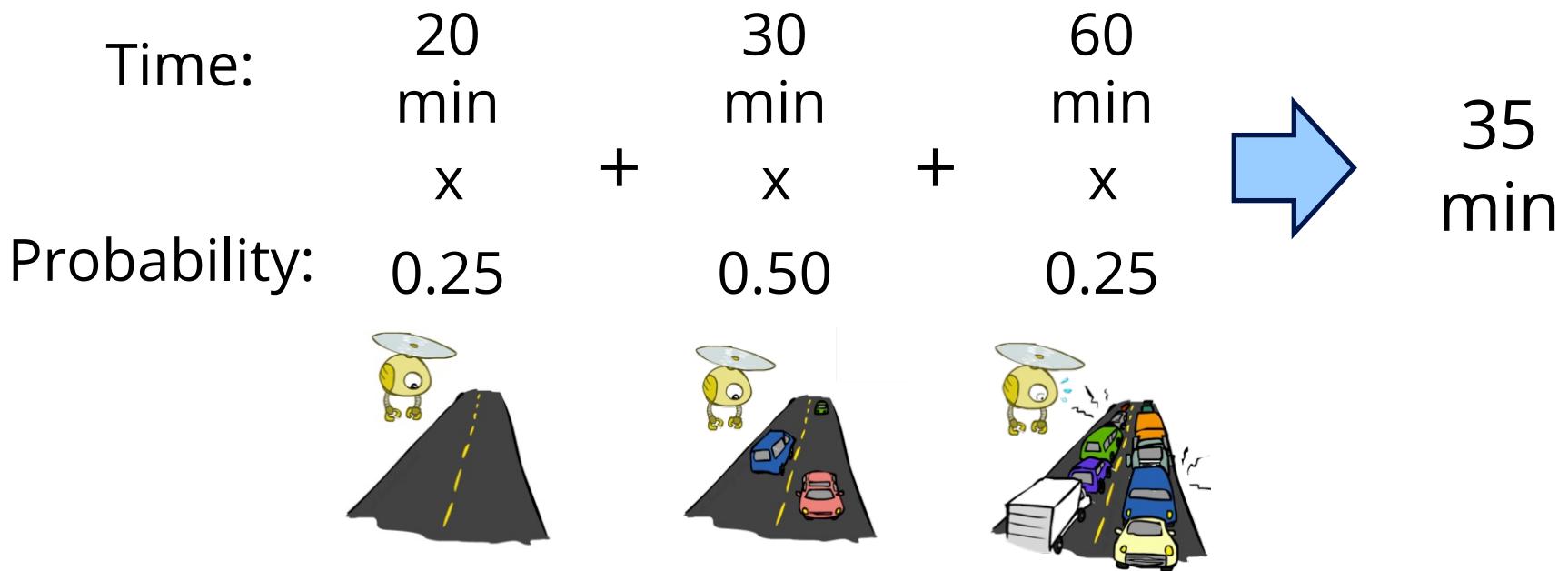
# Probabilities

- A **random variable** represents an event whose outcome is unknown
- A **probability distribution** is an assignment of weights to outcomes
- Example: Traffic on freeway
  - Random variable:  $T$  = whether there's traffic
  - Outcomes:  $T \in \{\text{none}, \text{light}, \text{heavy}\}$
  - Distribution:  $P(T = \text{none}) = 0.25, P(T = \text{light}) = 0.50, P(T = \text{heavy}) = 0.25$
- Some laws of probability (more later):
  - Probabilities are always non-negative
  - Probabilities over all possible outcomes sum to one
- As we get more evidence, probabilities may change:  
 $P(T = \text{heavy}) = 0.25, P(T = \text{heavy} | \text{Hour} = 8\text{am}) = 0.60$   
We'll talk about methods for reasoning and updating probabilities later



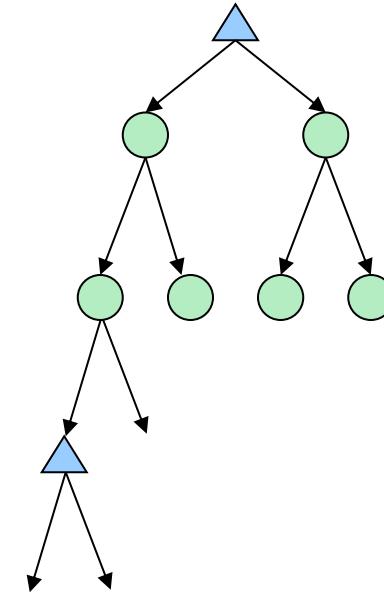
# Probabilities

- The expected value of a function of a random variable is the average, weighted by the probability distribution over outcomes
- Example: How long to get to the airport?



# What Probabilities to Use?

- In expectimax search, we have a probabilistic model of how the opponent (or environment) will behave in any state
  - Model could be a simple uniform distribution (roll a die)
  - Model could be sophisticated and require a great deal of computation
  - We have a chance node for any outcome out of our control: opponent or environment
  - The model might say that adversarial actions are likely!
- For now, assume each chance node magically comes along with probabilities that specify the distribution over its outcomes



*Having a probabilistic belief about another agent's action does not mean that the agent is flipping any coins!*

# What Probabilities to Use?

- **Objectivist / frequentist answer:**

Averages over repeated *experiments*

E.g. empirically estimating  $P(\text{rain})$  from historical observation

Assertion about how future experiments will go (in the limit)

New evidence changes the *reference class*

Makes one think of *inherently random* events, like rolling dice

- **Subjectivist / Bayesian answer:**

Degrees of belief about unobserved variables

E.g. an agent's belief that it's raining, given the temperature

E.g. agent's belief how an opponent will behave, given the state

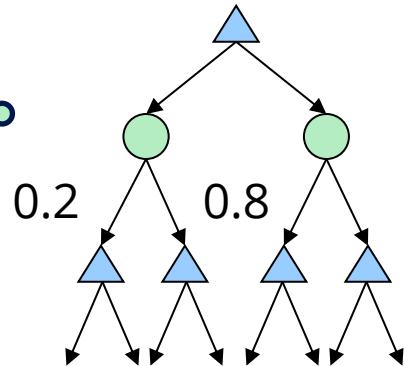
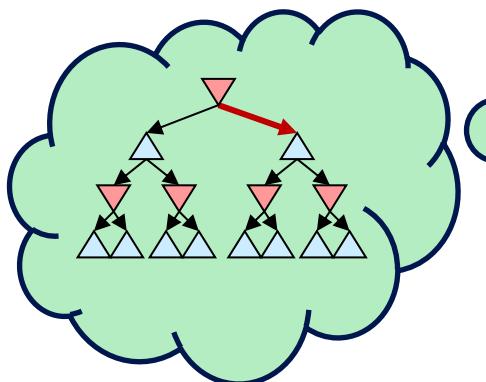
Often *learn* probabilities from past experiences (more later)

New evidence *updates beliefs* (more later)



# Quiz: Informed Probabilities

- Let's say you know that your opponent is running a depth 2 minimax, using the result 80% of the time, and moving randomly otherwise
- Question: What tree search should you use?



- **Answer: Expectimax!**

- To figure out EACH chance node's probabilities, you must run a simulation of your opponent
- This kind of thing gets very slow very quickly
- Even worse if you must simulate your opponent simulating you...
- ... except for minimax, which has the nice property that it all collapses into one game tree

- **Dice rolls increase  $b$ : 21 possible rolls with 2 dice**
  - Backgammon  $\approx 20$  legal moves
  - Depth 2  $\rightarrow 20 \times (21 \times 20)^2 = 3.5 \text{ million}$
  - Depth 3  $\rightarrow 1.2 \times 10^9 = 1.2 \text{ billion}$
- **As depth increases, probability of reaching a given search node shrinks**
  - So usefulness of search is diminished
  - So limiting depth is less damaging
  - But pruning is trickier...
- **Historic AI: TDGammon uses depth-2 search + very good evaluation function + reinforcement learning  $\rightarrow$  world-champion level play**
- **1<sup>st</sup> AI world champion in any game!**



- **E.g. Backgammon**
- **Expectiminimax**

Environment is an extra “random agent” player that moves after each min/max agent

Each node computes the appropriate combination of its children

