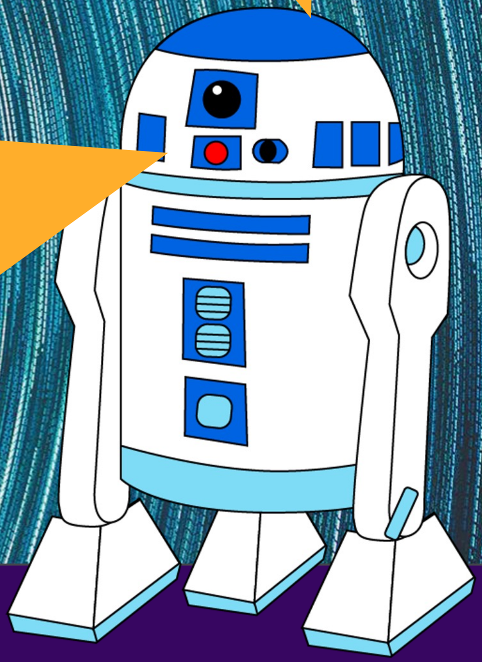
CIS 4210/5210:
ARTIFICIAL INTELLIGENCE

# Constraint Satisfaction Problems

Professor Chris Callison-Burch

HW4 is due Wednesday by 11:59pm.

There will be no lecture on Thursday because I’ll be traveling.



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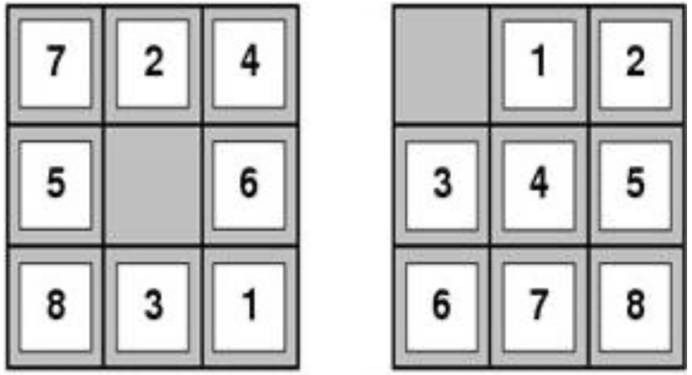
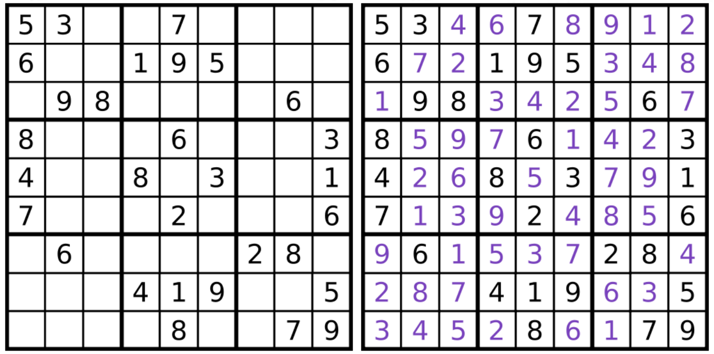
# What is Search For?

- **Assumptions about the world: a single agent, deterministic actions, fully observed state, discrete state space**

- **Planning: sequences of actions**
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics give problem-specific guidance

- **Identification: assignments to variables**
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems





# Big idea

- **Represent the *constraints* that solutions must satisfy in a uniform *declarative* language**
- **Find solutions by *GENERAL PURPOSE* search algorithms with no changes from problem to problem**
  - No hand-built transition functions
  - No hand-built heuristics

- **Just specify the problem in a formal declarative language, and a general-purpose algorithm does everything else!**

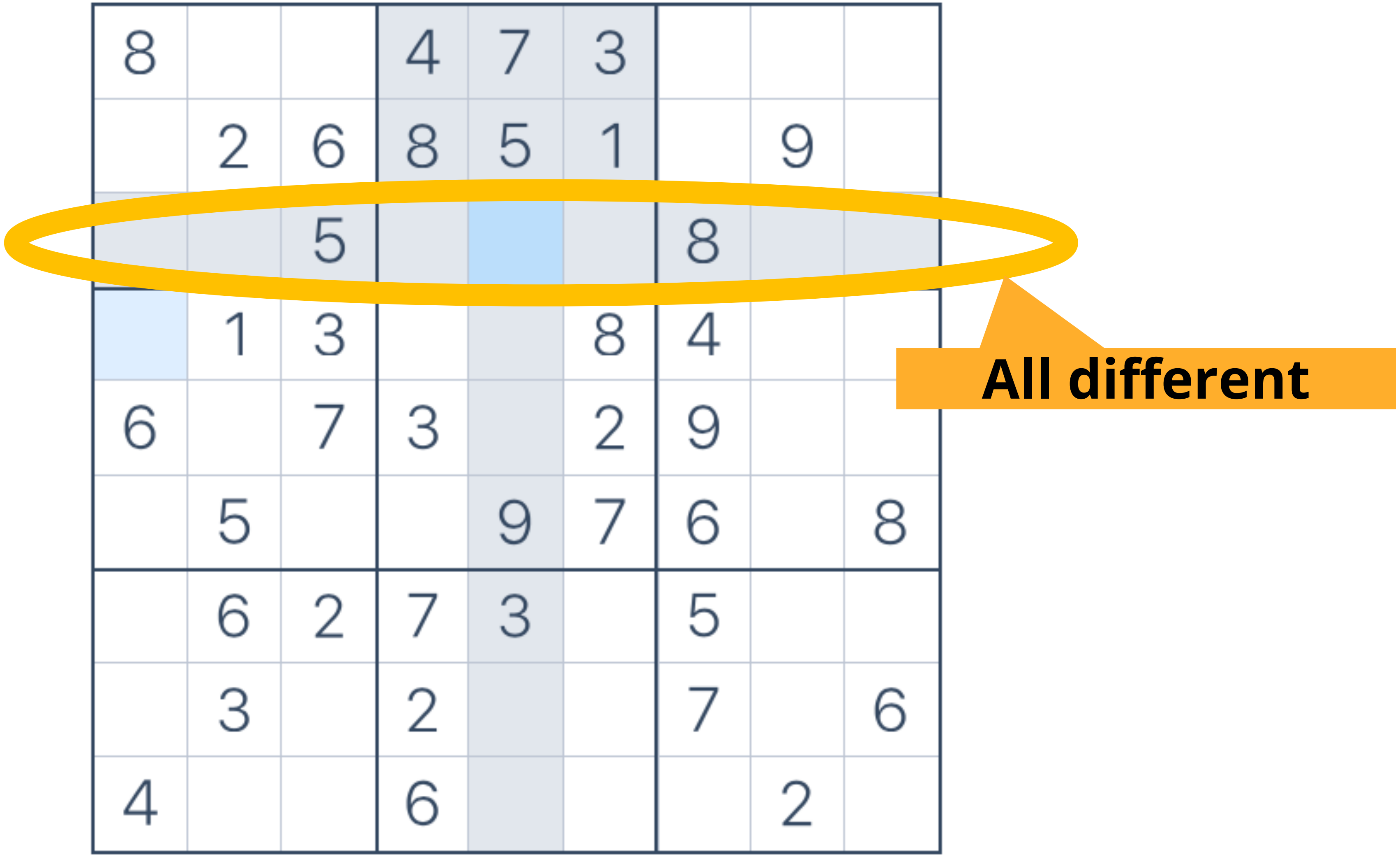
# Constraint Satisfaction Problems

## A CSP consists of:

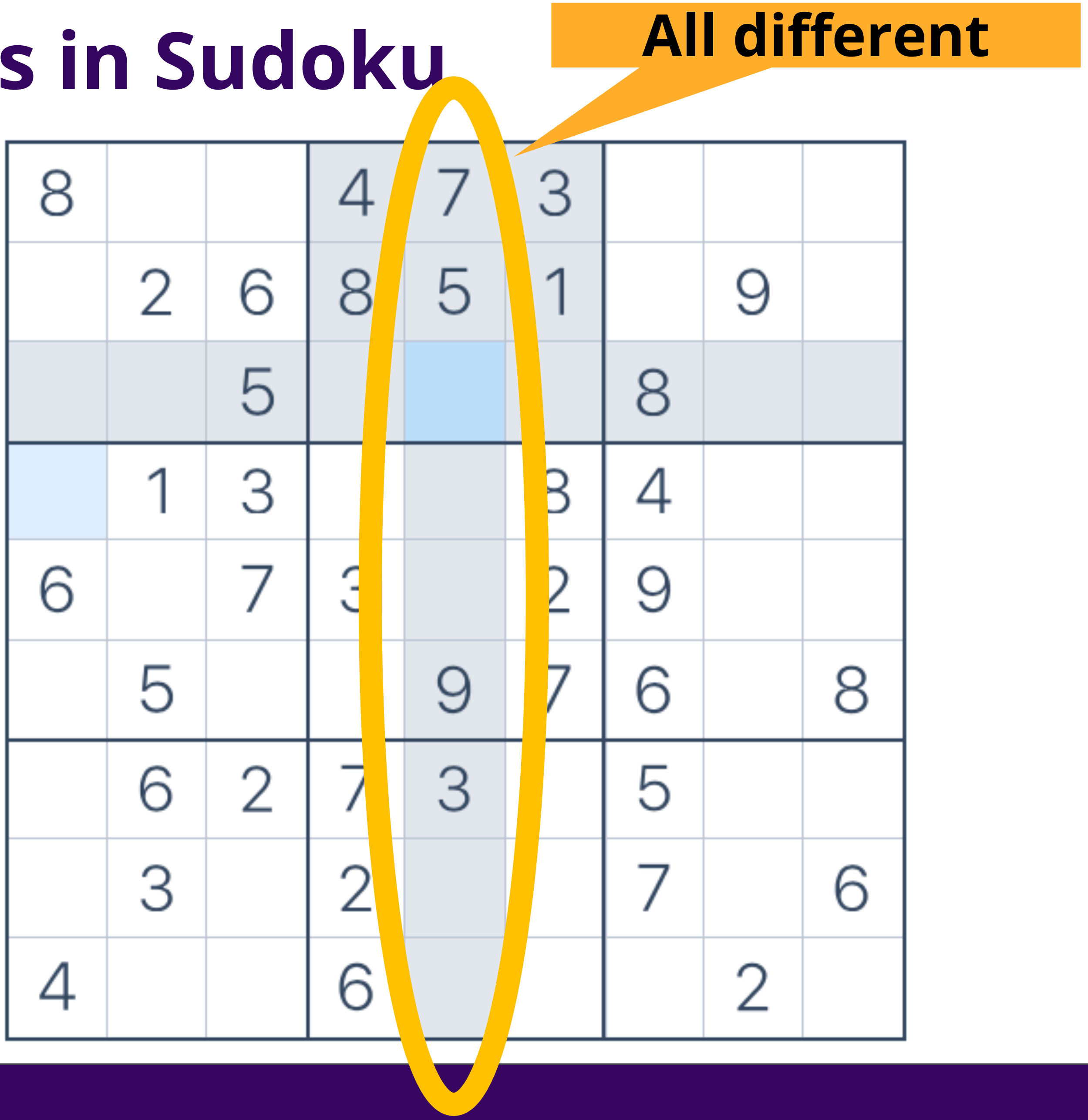
- *Finite set of variables* $X_1, X_2, \ldots, X_n$
- *Nonempty* ***domain*** *of possible values* for each variable $D_1, D_2, \ldots D_n$ *where* $D_i = \{v_1, \ldots, v_k\}$
- *Finite set of constraints* $C_1, C_2, \ldots, C_m$
  - Each *constraint* $C_i$ limits the values that variables can take, e.g., $X_1 \neq X_2$ A *state* is defined as an *assignment* of values to some or all variables.

- **A *consistent* assignment does not violate the constraints.**
- **Example problem: Sudoku**

# Constraints in Sudoku



# Constraints in Sudoku



# Constraints in Sudoku

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 8 | | | 4 | 7 | 3 | | | |
| | 2 | 6 | 8 | 5 | 1 | | 9 | |
| | | 5 | | | | 8 | | |
| | 1 | 3 | | | 8 | 4 | | |
| 6 | | 7 | 3 | | 2 | 9 | | |
| | 5 | | | 9 | 7 | 6 | | 8 |
| | 6 | 2 | 7 | 3 | | 5 | | |
| | 3 | | 2 | | | 7 | | 6 |
| 4 | | | 6 | | | | 2 | |

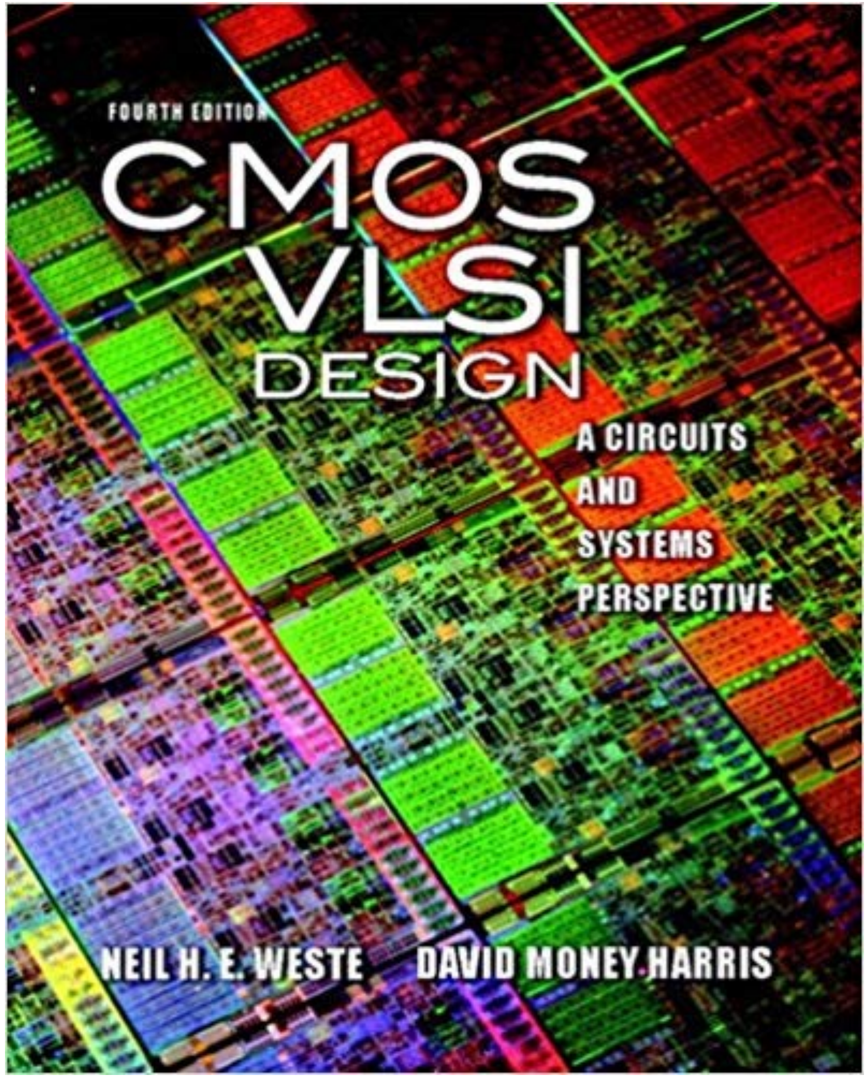
All different

# Constraint satisfaction problems

- An assignment is *complete* when every variable is assigned a value.
- A *solution* to a CSP is a *complete, consistent* assignment.
- Solutions to CSPs can be found by a completely *general purpose* algorithm, given only the formal specification of the CSP.
- Beyond our scope: CSPs that require a solution that maximizes an *objective function*.

# Applications

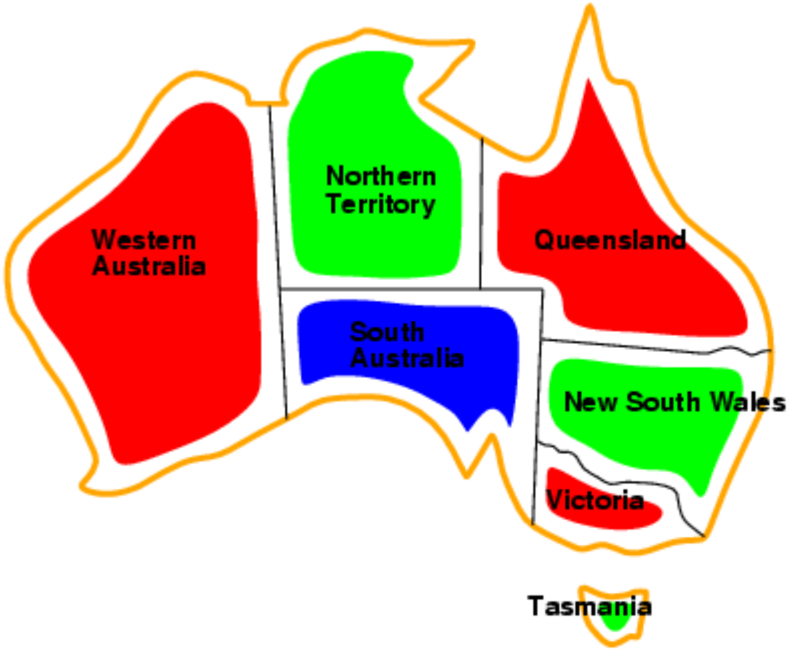
- **Map coloring**
- **Scheduling problems**
  - Job shop scheduling
  - Scheduling the Webb Space Telescope
- **Floor planning for VLSI**
- **Sudoku**
- **…**



# Example: Map-coloring


- **Variables:** ***WA, NT, Q, NSW, V, SA, T***
- **Domains:** $D_i$ **= {red,green,blue}**
- **Constraints: adjacent regions must have different colors**
  - e.g., WA ≠ NT
    - So (WA,NT) must be in {(red,green),(red,blue),(green,red), …}

# Example: Map-coloring




**Solutions: complete and consistent assignments**

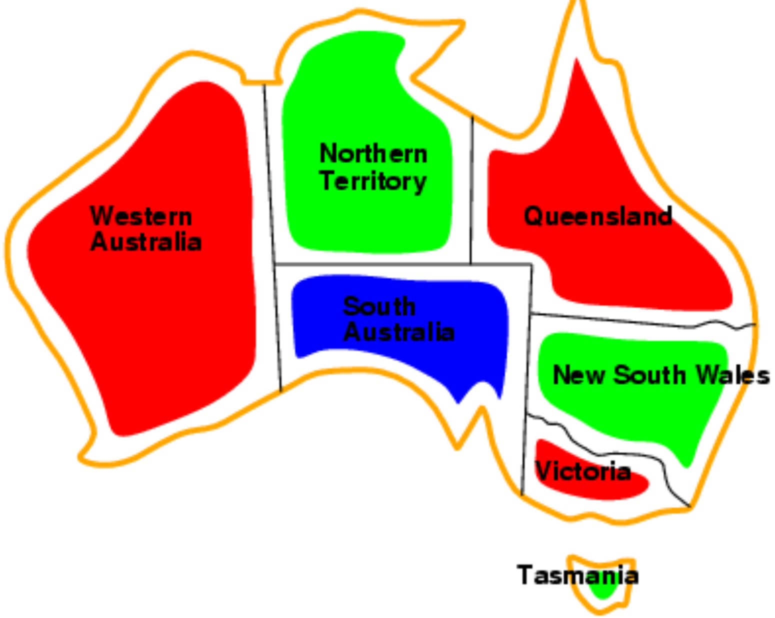
- e.g., WA = red, NT = green,Q = red, NSW = green,
  V = red, SA = blue, T = green

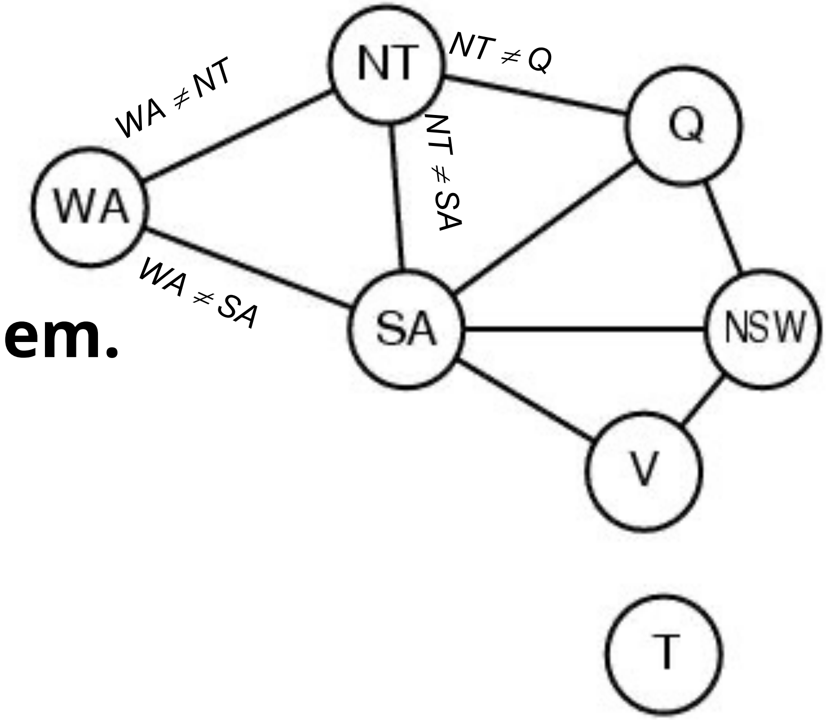
# Benefits of CSP

- **Clean specification of many problems, generic goal, successor function & heuristics**
  - Just represent problem as a CSP & solve with general package
- **CSP "knows" which variables violate a constraint**
  - And hence where to focus the search
- ***CSPs:* Automatically prune off all branches that violate constraints**
  - (State space search could do this only by *hand-building constraints into the successor function)*

# CSP Representations

- *Constraint graph:*
  - *nodes* are variables
  - *arcs* are (binary) constraints
- *Standard representation pattern:*
  - variables with values
- *Constraint graph* simplifies search.
  - e.g. Tasmania is an independent subproblem.
- *This problem: A binary CSP:*
  - each constraint relates two variables



# Varieties of CSPs

- ***Discrete variables***
  - finite domains:
    - $n$ variables, domain size $d$ → $O(d^n)$ complete assignments
    - e.g., Boolean CSPs, includes Boolean satisfiability (NP-complete)
  - infinite domains:
    - integers, strings, etc.
    - e.g., job scheduling, variables are start/end days for each job
    - need a constraint language, e.g., $StartJob_1 + 5 \leq StartJob_3$
- ***Continuous variables***
  - e.g., start/end times for Space Telescope observations
  - linear constraints solvable in polynomial time by linear programming

# Varieties of constraints

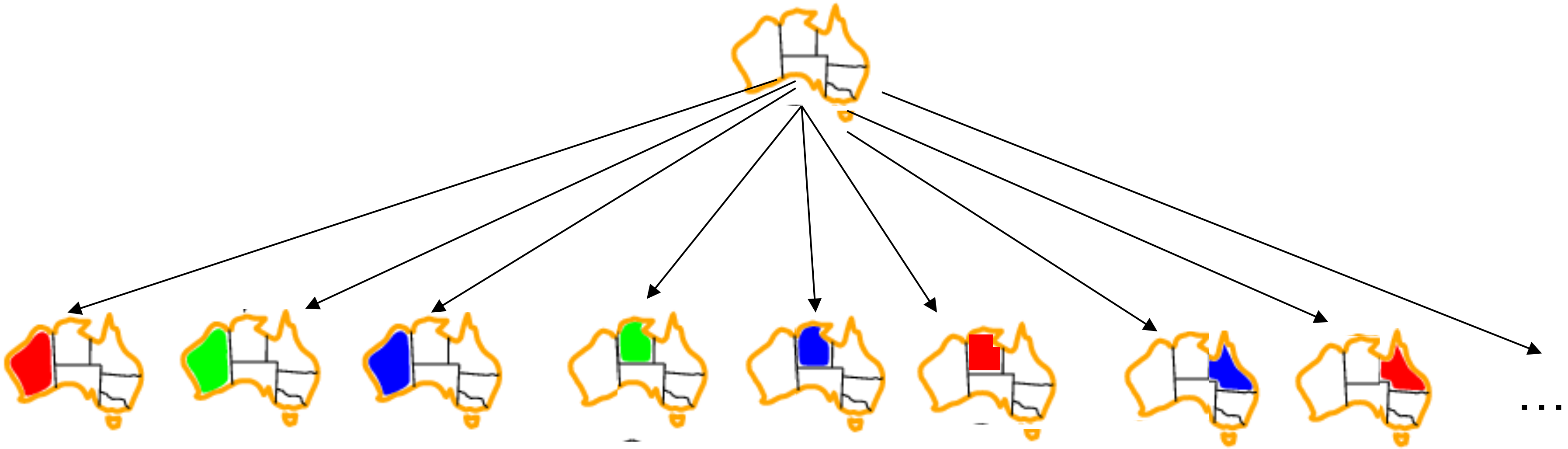
- ***Unary*** **constraints involve a single variable,**
  - e.g., SA ≠ green
- ***Binary*** **constraints involve pairs of variables,**
  - e.g., SA ≠ WA
- ***Higher-order*** **constraints involve 3 or more variables**
  - e.g., crypt-arithmetic column constraints
- ***Preference*** **(soft constraints) e.g. *red* is better than *green* can be represented by a cost for each variable assignment**
  - Constrained optimization problems.

# Idea 1: CSP as a search problem

- **A CSP can easily be expressed as a search problem**
  - *Initial State:* the empty assignment {}.
  - *Successor function:* Assign value to any unassigned variable *provided that there is not a constraint conflict*.
  - *Goal test:* the current assignment is complete.
  - *Path cost:* a constant cost for every step.

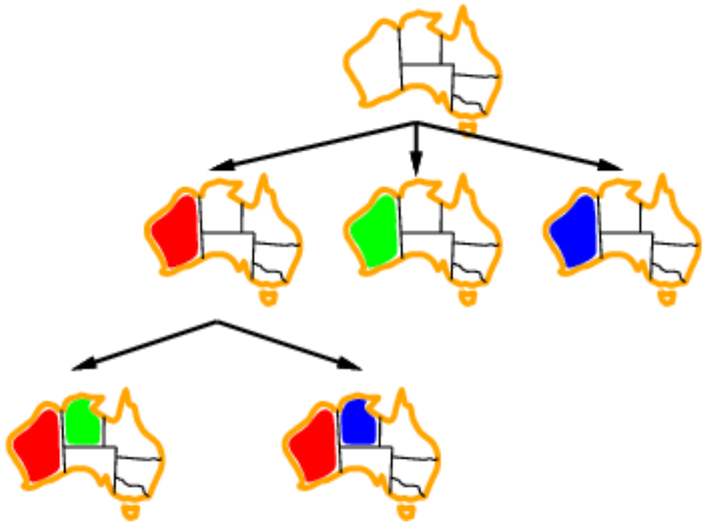
- **Solution is always found at depth *n*, for *n* variables**
  - Hence Depth First Search can be used

# Search and branching factor




- n variables of domain size d
- Branching factor at the root is n*d
- Branching factor at next level is (n-1)*d
- Tree has n!*$d^n$ leaves

# Search and branching factor

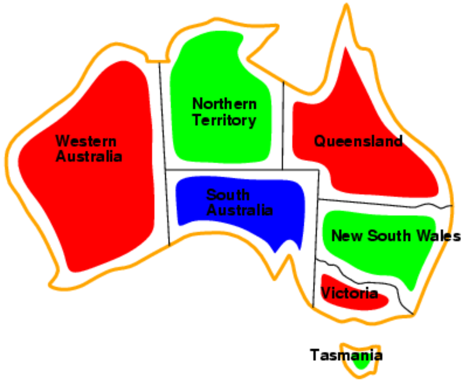


- **The variable assignments are *commutative***
  - Eg *[ step 1: **WA = red**; step 2: **NT = green** ]* equivalent to *[ step 1: **NT = green**; step 2: **WA = red** ]*
  - Therefore, a *tree search,* not a *graph search*
- **Only need to consider assignments to a single variable at each node**
  - $b = d$ and there are $d^n$ leaves *(n variables, domain size d )*

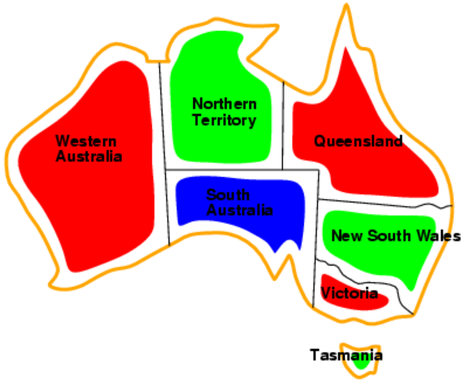
# Search and *Backtracking*

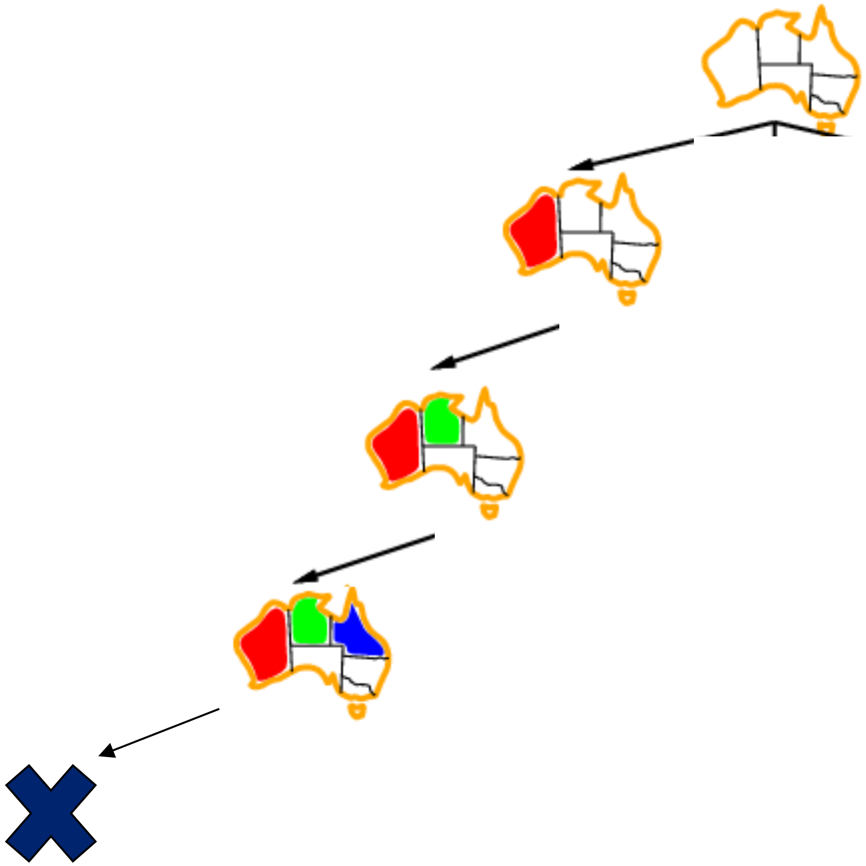
- **Depth-first search for CSPs with single-variable assignments is called *backtracking* search**
- **The term backtracking search is used for a depth-first search that chooses values for one variable at a time and backtracks when a variable has no legal values left to assign.**
- **Backtracking search is the basic *uninformed* algorithm for CSPs**

# Backtracking example



# Backtracking example



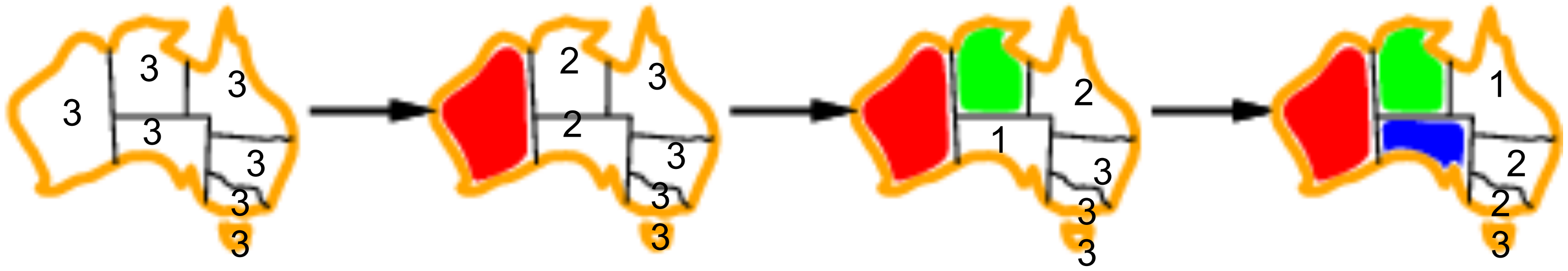



# Idea 2: Improving backtracking efficiency

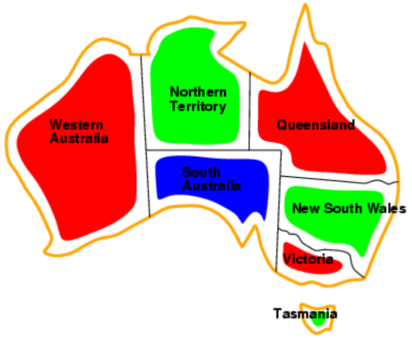
- ***General-purpose*** **methods &** ***general-purpose*** **heuristics can give huge gains in speed,** ***on average***
- **Heuristics:**
  - Q: Which variable should be assigned next?
    1. **Most constrain*ed* variable**
    2. **(if ties:) Most constrain*ing* variable**
  - Q: In what order should that variable's values be tried?
    3. **Least constraining *value***
  - Q: Can we detect inevitable failure early?
    4. **Forward checking**

# Heuristic 1: Most constrain*ed* variable

- Choose a variable with the ***fewest legal values***



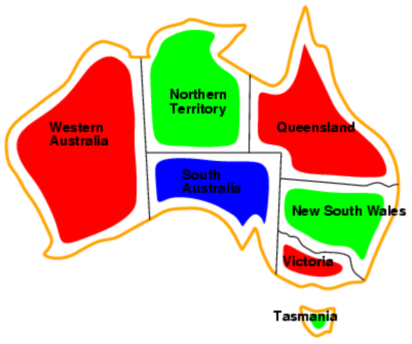

- a.k.a. ***minimum remaining values (MRV)*** heuristic



# Heuristic 2: Most constrain*ing* variable

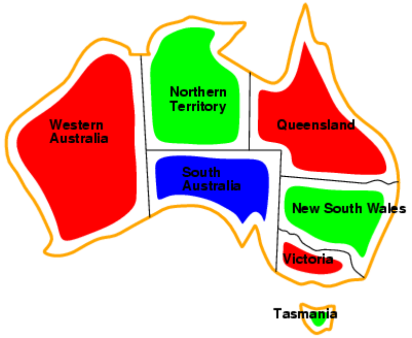
- **Tie-breaker among most constrained variables**
- **Choose the variable with the *most constraints on remaining variables***

These two heuristics together lead to immediate solution of our example problem



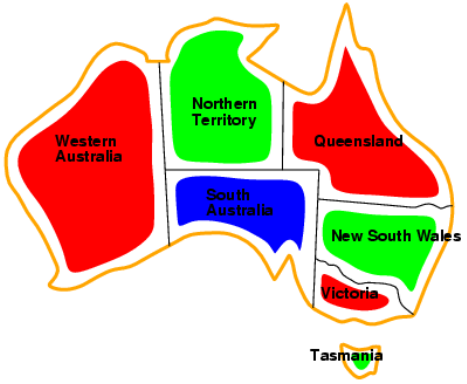
# Heuristic 3: Least constraining *value*

- **Given a variable, *choose the least constraining value:***
  - the one that rules out the fewest values in the remaining variables


Note: demonstrated here independent of the other heuristics

# Heuristic 4: Forward checking

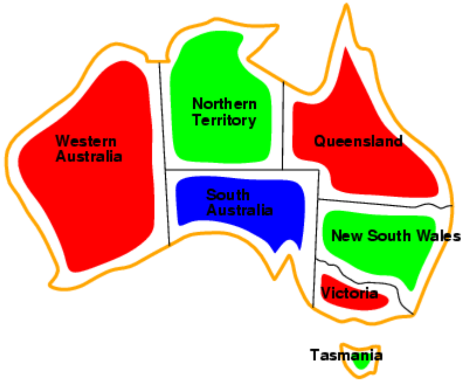



- **Idea**:
  - Keep track of *remaining* legal values for *unassigned* variables
  - Terminate search when any unassigned variable has no remaining legal values


New data structure

*(A first step towards Arc Consistency & AC-3)*

# Forward checking

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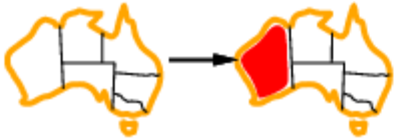
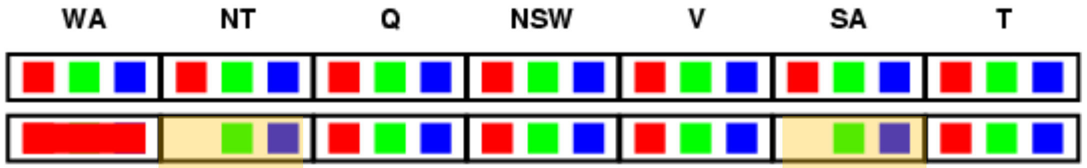
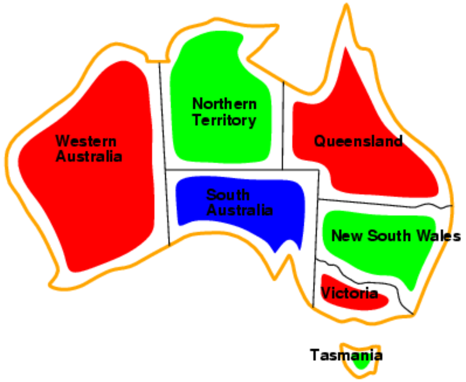




# Forward checking

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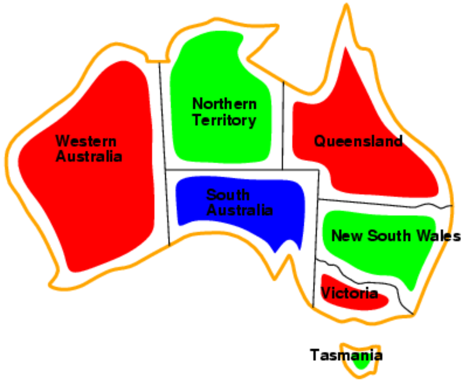
# Forward checking

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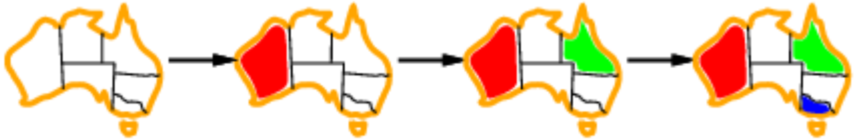
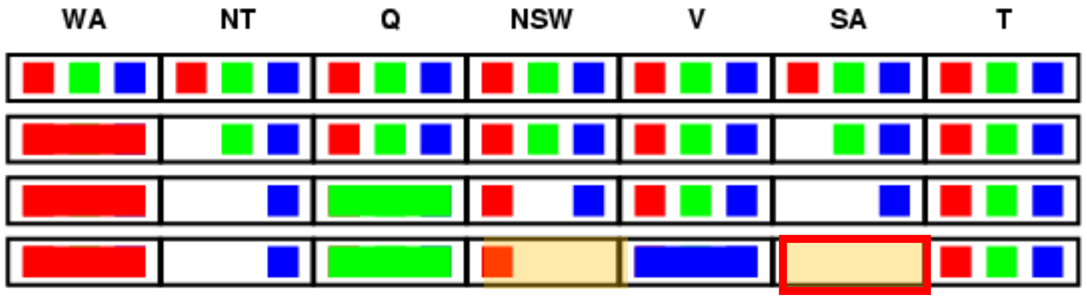
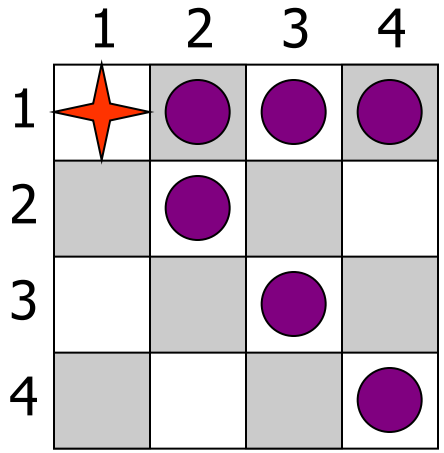
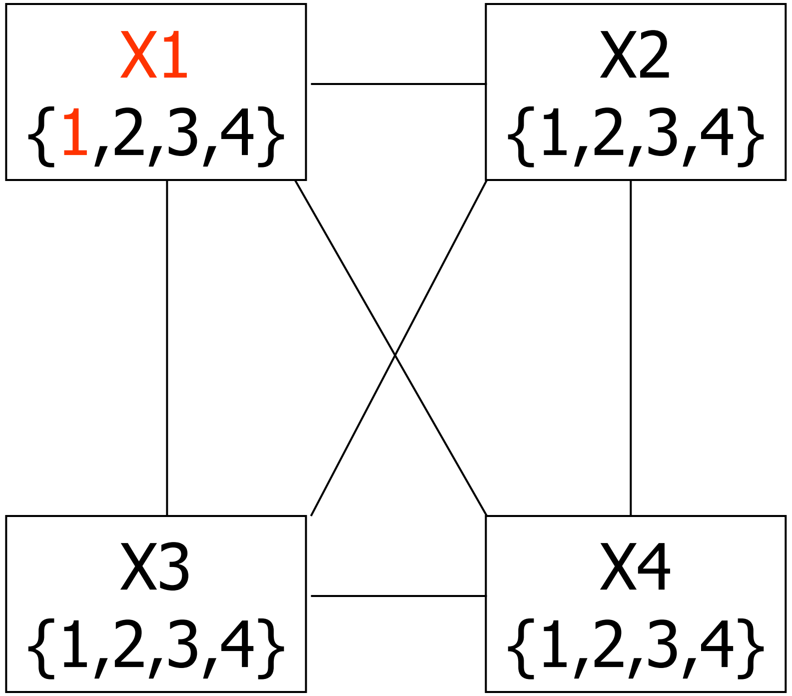





**Terminate! No possible value for SA**

# Example: 4-Queens Problem




Assign value to unassigned variable

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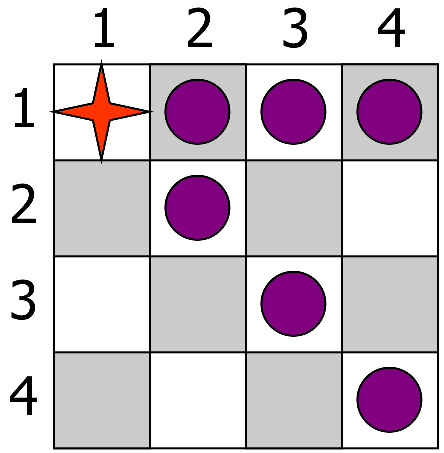


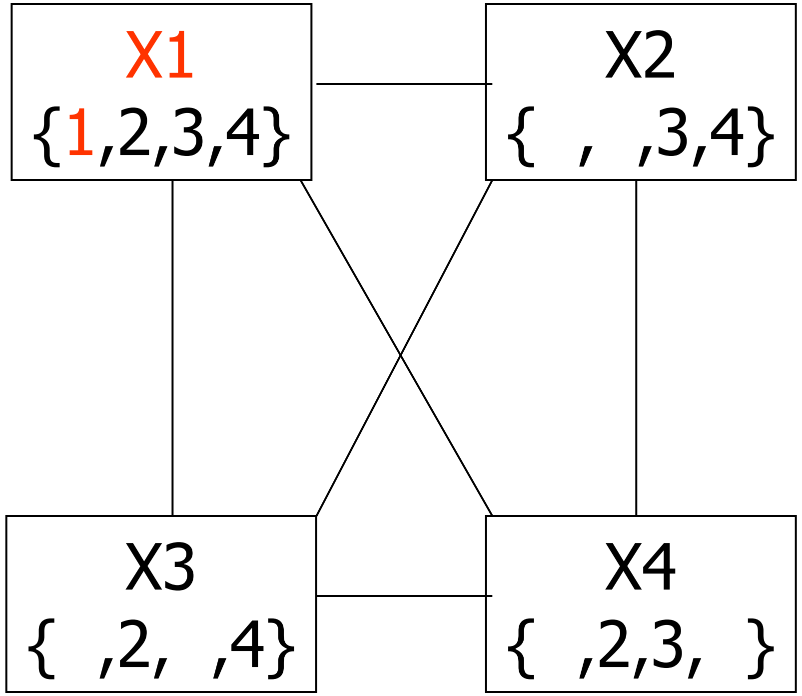


Forward check!

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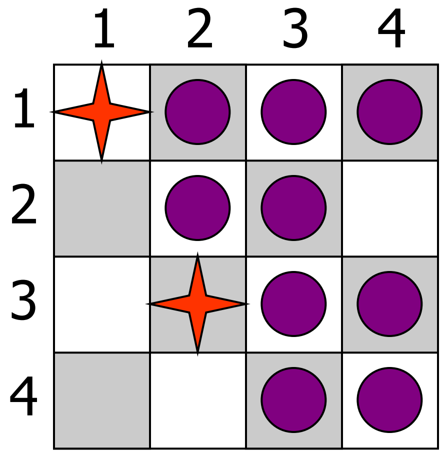
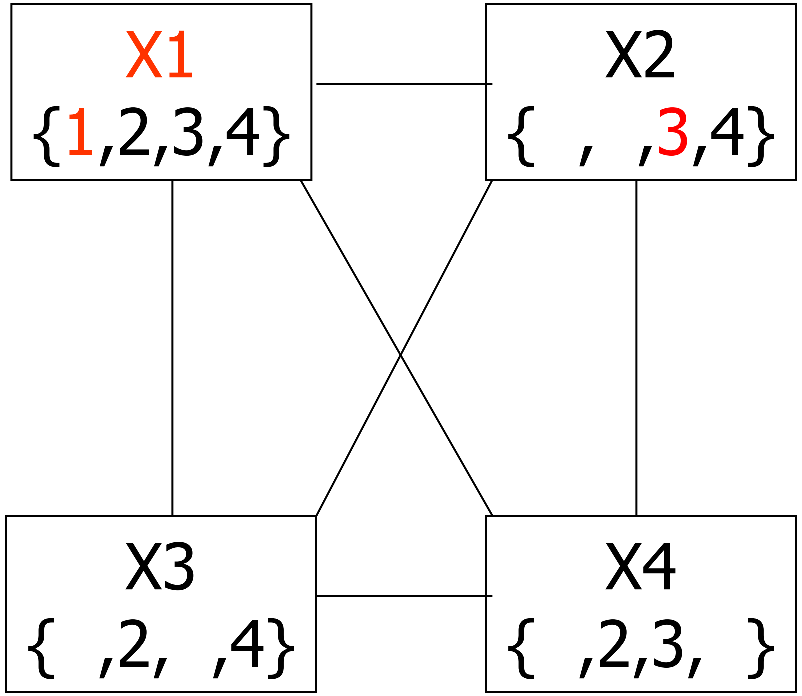
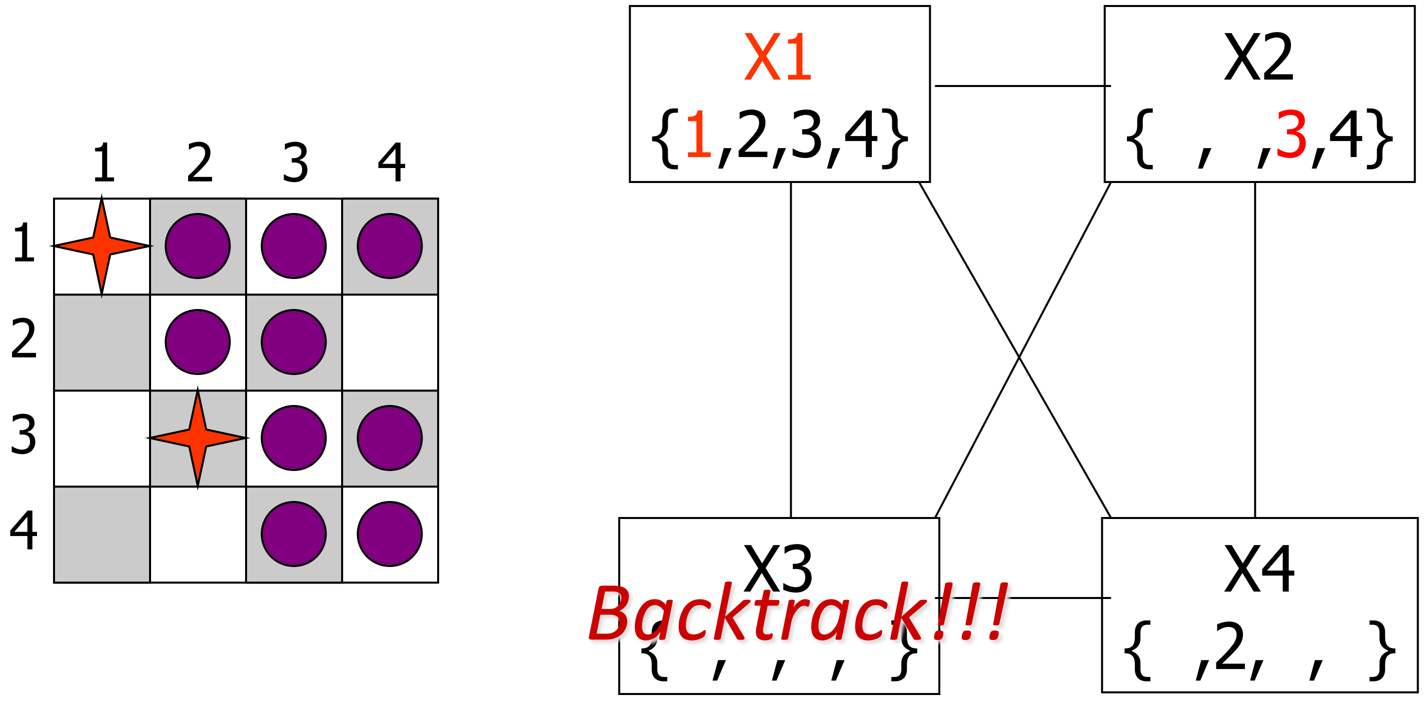



Assign value to unassigned variable

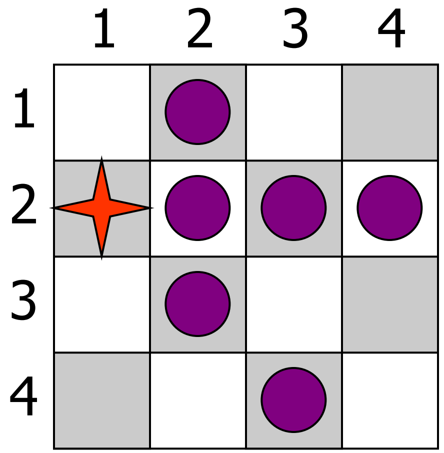
# Example: 4-Queens Problem

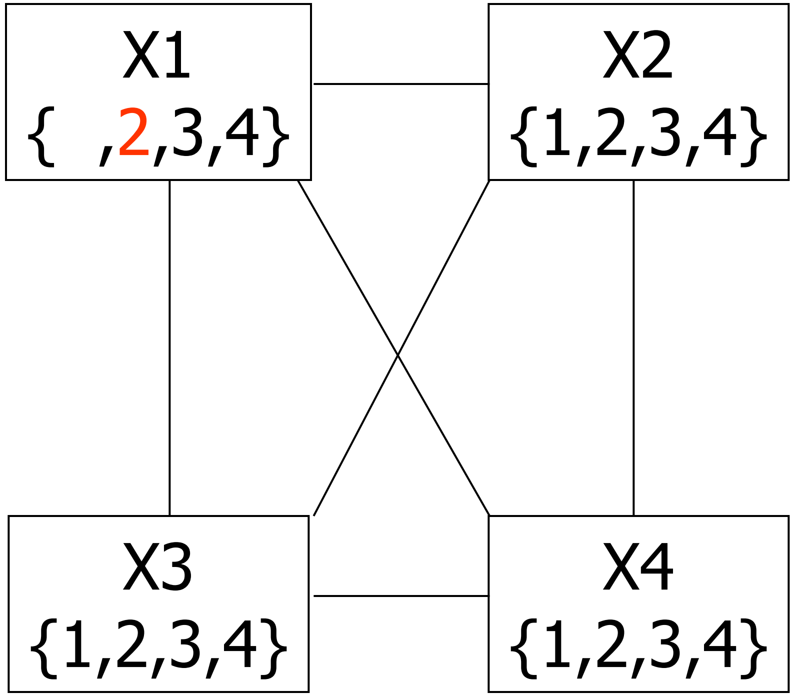



Forward check!

# Example: 4-Queens Problem

*Picking up a little later after two steps of backtracking....*

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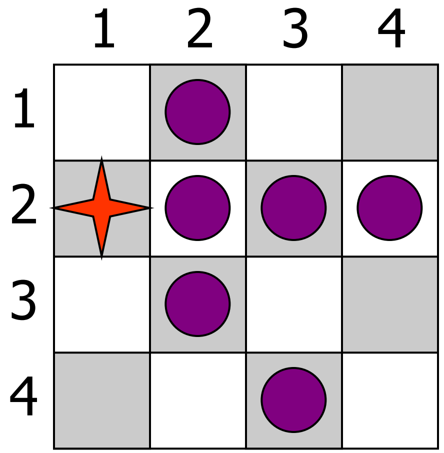


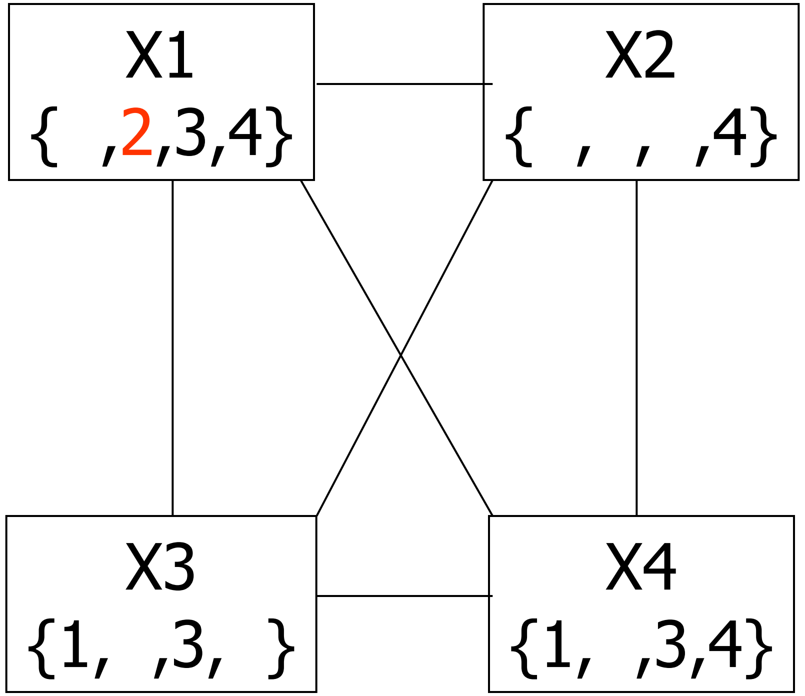


Forward check!

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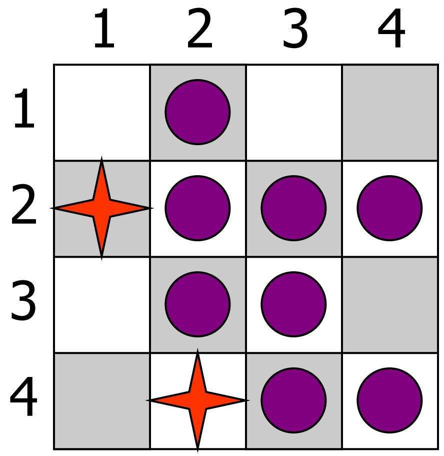
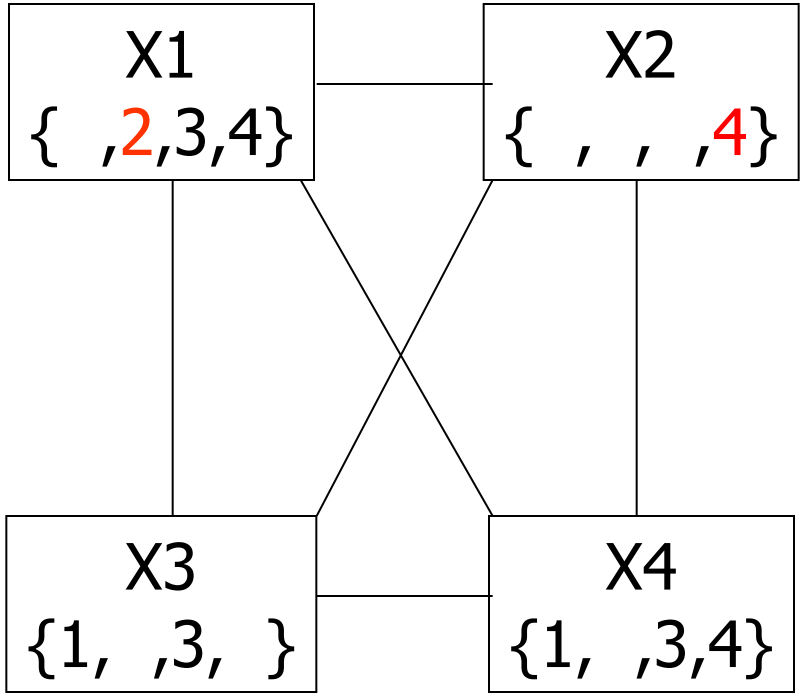
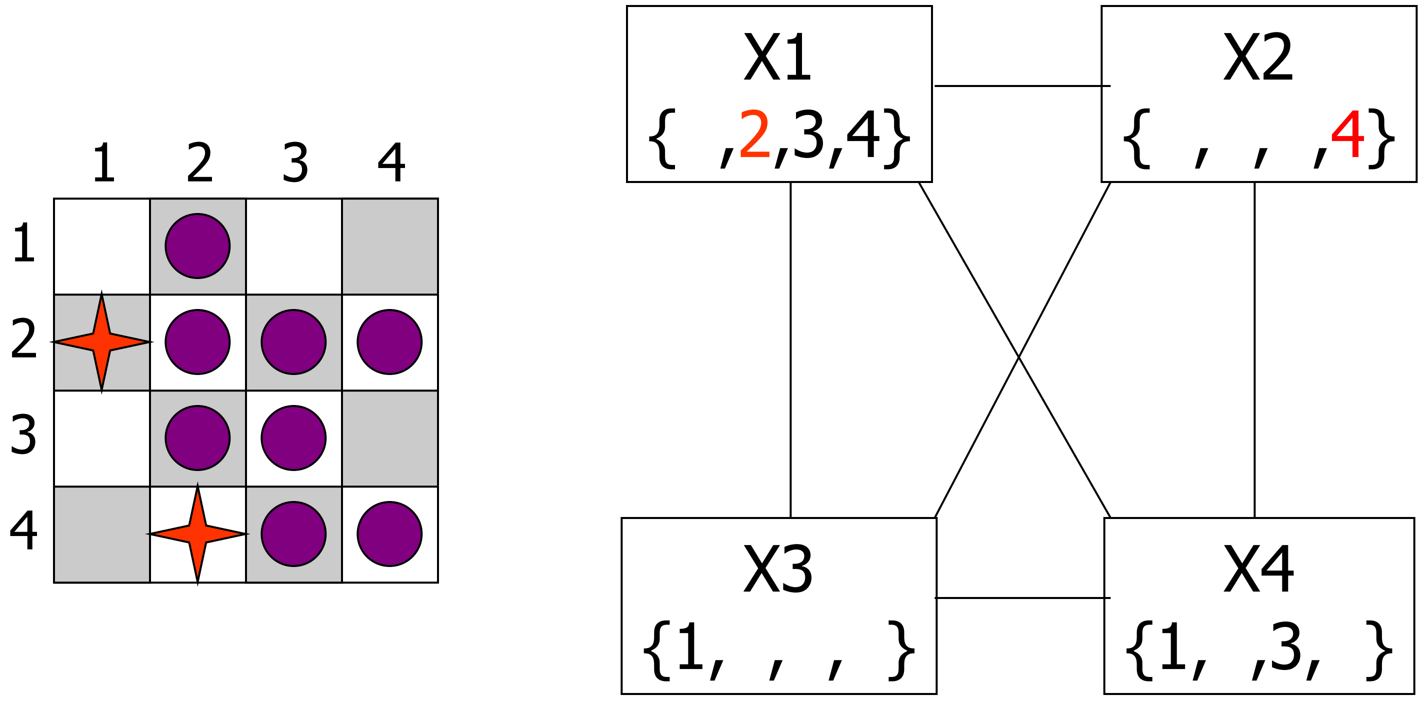


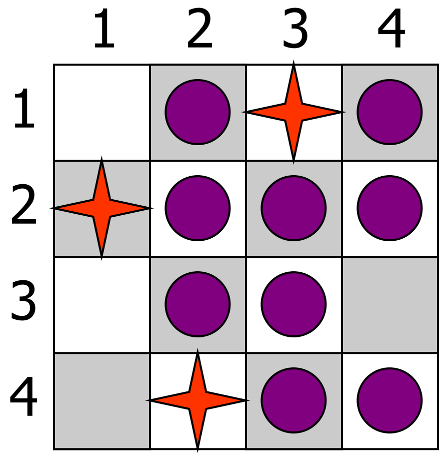

Assign value to unassigned variable

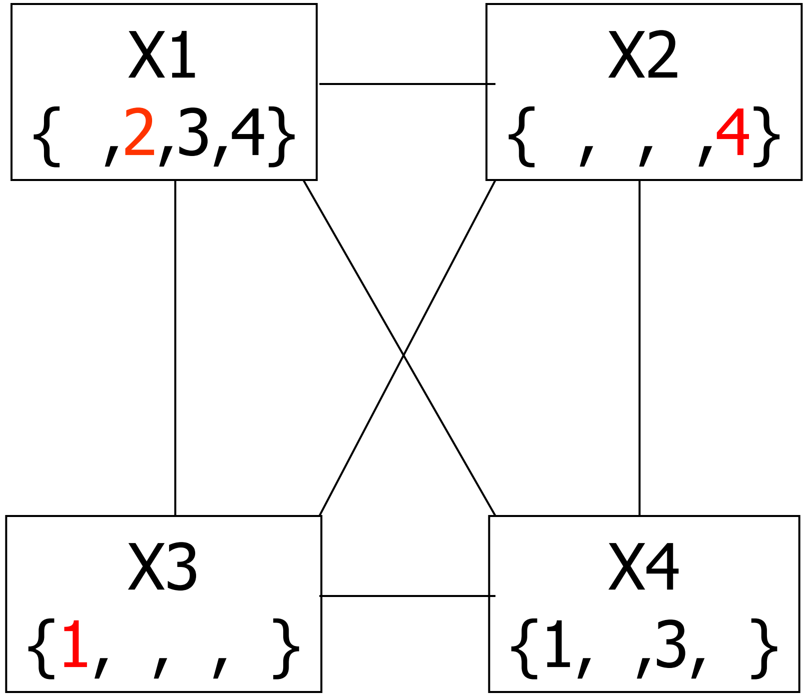
# Example: 4-Queens Problem




Forward check!

# Example: 4-Queens Problem




Assign value to unassigned variable

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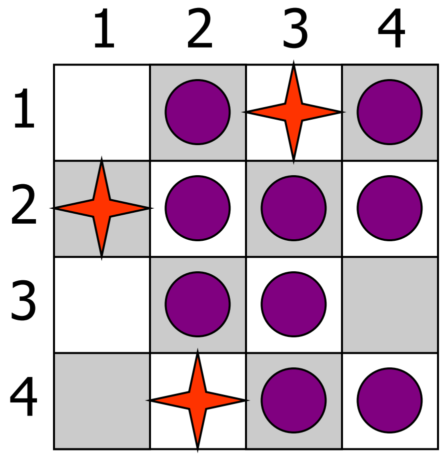
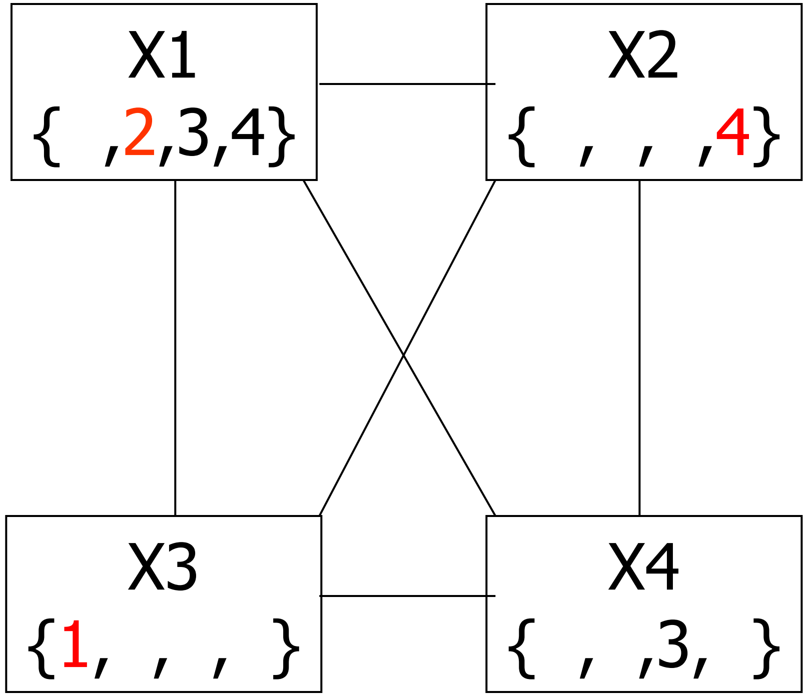
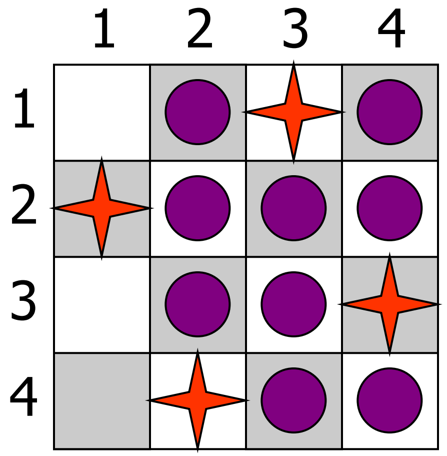
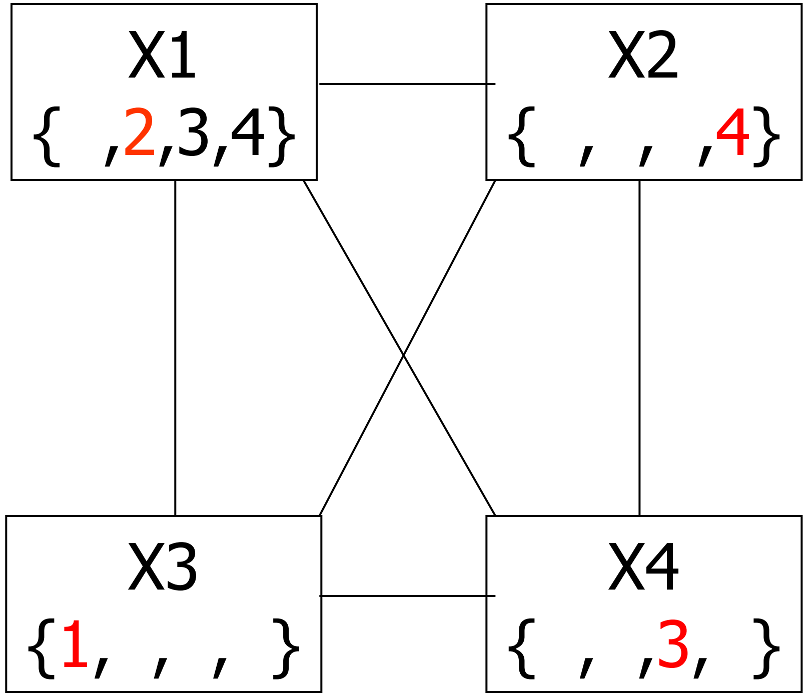



Forward check!

# Example: 4-Queens Problem




Assign value to unassigned variable

# Towards Constraint propagation


- **Forward checking** propagates information from *assigned* to *unassigned* variables, but doesn't provide early detection for all failures:


- NT and SA cannot both be blue!
- **Constraint propagation** goes beyond forward checking & repeatedly enforces constraints locally

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# Arc Consistency, Constraint Propagation & AC-3

Professor Chris Callison-Burch

# Idea 3 (*big* idea): *Inference* in CSPs

- **CSP solvers combine search *and inference***
  - Search
  - *Constraint propagation (inference)*
    - Eliminates possible values for a variable if the value would violate local consistency
  - *Can do inference first, or intertwine it with search*
    - You'll investigate this in the Sudoku homework

**Search = assign a value to a variable**

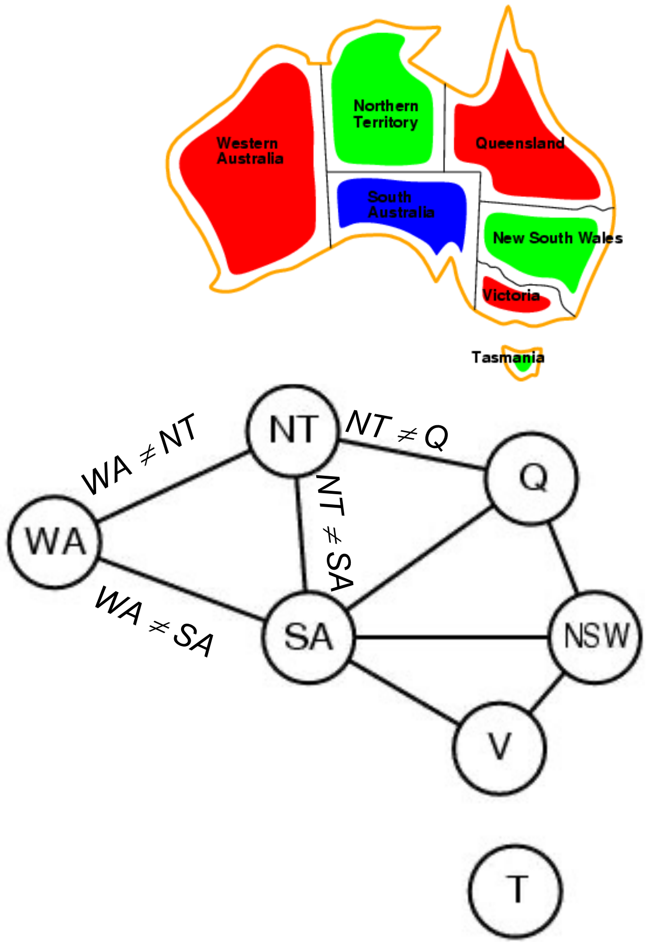
**Inference = use constraints to reduce number of legal values for a variable**

# Local Consistency

- **Node consistency: satisfies unary constraints**
  - This is trivial!
- **Arc consistency: satisfies binary constraints**
  - $X_i$ is arc-consistent with respect to $X_j$
  - If for every value $v$ in $D_i$
  - There is some value $w$ in $D_j$ that satisfies the binary constraint on the arc between $X_i$ *and* $X_j$

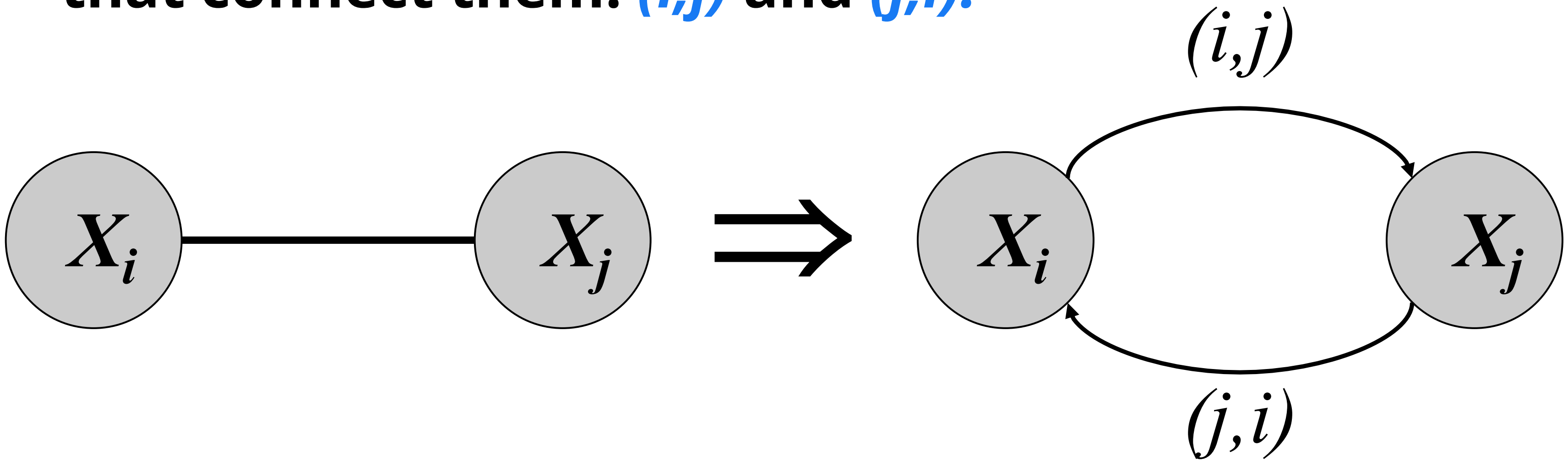
# CSP Representations

- *Constraint graph:*
  - *nodes* are variables
  - *edges are constraints*



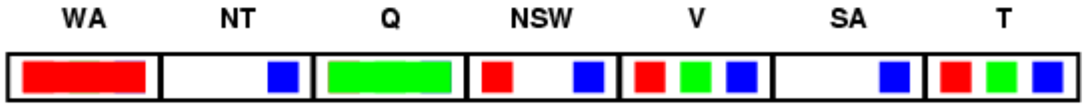
# Edges to Arcs: From Constraint Graph to Directed Graph

- Given a pair of nodes $X_i$ and $X_j$ connected by a constraint ***edge***, we represent this not by a single undirected edge, but a ***pair of directed arcs***.
  - For a connected pair of nodes $X_i$ and $X_j$, there are ***two*** arcs that connect them: ***(i,j)*** and ***(j,i).***



# Arc consistency

- Simplest form of propagation makes each arc consistent
- $X \rightarrow Y$ is consistent iff
  for every value $x$ of $X$ there is some allowed $y$



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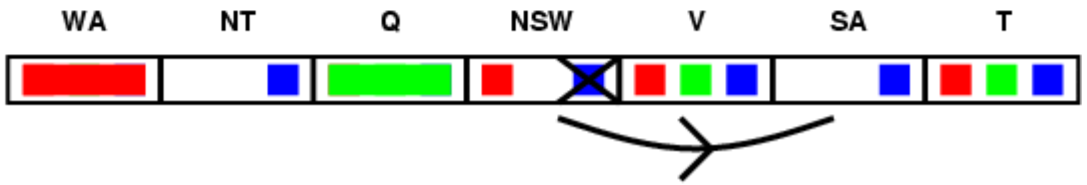
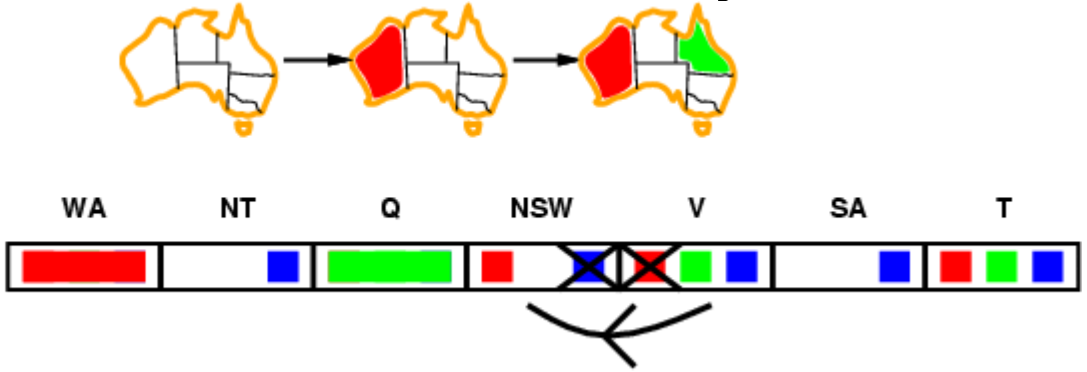


# Arc consistency

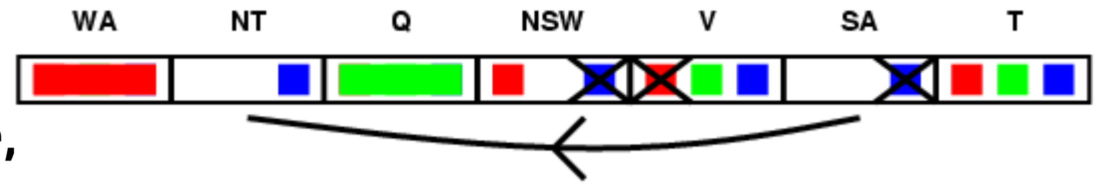
- **Simplest form of propagation makes each arc consistent**
- ***X* → *Y* is consistent iff**
  **for every value *x* of *X* there is some allowed *y***




- **If *X* loses a value, recheck neighbors of *X***

# Arc consistency

- Simplest form of propagation makes each arc consistent
- *X* →*Y* is consistent iff
  for every value *x* of *X* there is some allowed *y*




- If *X* loses a value,
- Detects failure earlier than forward checking
- Can be run as a preprocessor or after each assignment

# Arc consistency

**An arc *(i,j) is* arc consistent if and only if every value *v* on $X_i$ is consistent with some label on $Y_j$.**

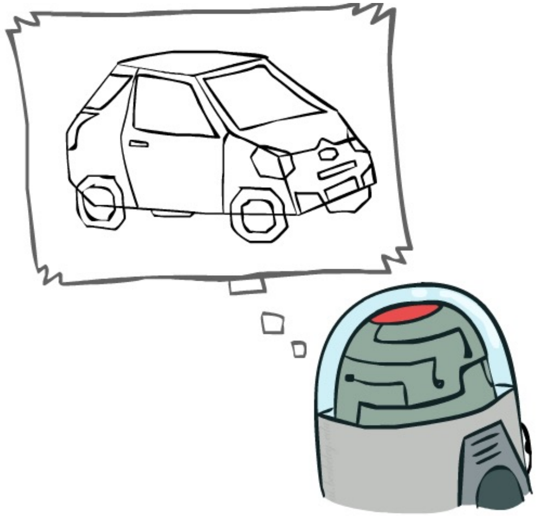
**To make an arc *(i,j)* arc consistent,**
**for each value *v* on $X_i$ ,**
**if there is no label on $Y_j$ consistent with *v***
**then remove *v* from $X_i$**

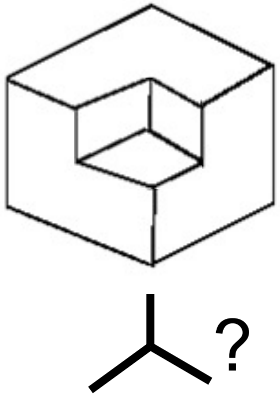
- **Given *d* values, checking arc (i,j) takes $O(d^2)$ time worst case**

**d is the size of the domain – the number of values**

# Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects
- An early example of an AI computation posed as a CSP

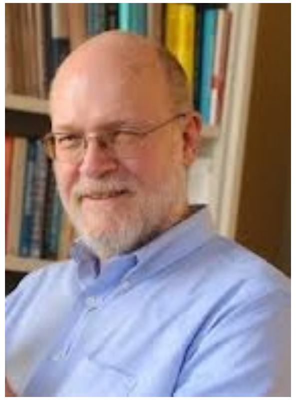





- **Approach:**
  - Each intersection is a variable
  - Adjacent intersections impose constraints on each other
  - Solutions are physically realizable 3D interpretations

Slide credit: Dan Klein and Pieter Abbeel
http://ai.berkeley.edu

# Replacing Search: Constraint Propagation Invented…



**Dave Waltz's insight:**

- By ***iterating*** over the graph, the arc-consistency *constraints* can be *propagated* along arcs of the graph.
- *Search*: Use constraints to *add* labels to find *one solution*
- *Constraint Propagation*: Use constraints to *eliminate labels* to simultaneously find *all solutions*

# The Waltz/Mackworth Constraint Propagation Algorithm

1. **Assign *every* node in the constraint graph a set of *all* possible values**
2. **Repeat until there is no change in the set of values associated with any node:**
    3. For each node $i$:
        4. For each neighboring node $j$ in the picture:
            5. Remove any value from $i$ which is not arc consistent with $j$.

# Inefficiencies: Towards AC-3

1. **At each iteration, we only need to examine those $X_i$ *where at least one neighbor of $X_i$ has lost a value* in the previous iteration.**
2. **If $X_i$ loses a value only because of arc inconsistencies with $Y_j$, we *don't need to check* $Y_j$ on the next iteration.**
3. **Removing a value on $X_i$ can only make $Y_j$ arc-inconsistent with respect to $X_i$ itself. Thus, we only need to check that *(j,i)* is still arc-consistent.**

**These insights lead a much better algorithm…**

# AC-3

function AC-3(*csp*) return the CSP, possibly with reduced domains

inputs: *csp*, a binary csp with variables $\{X_1, X_2, \ldots, X_n\}$

local variables: *queue*, a queue of arcs initially the arcs in *csp*

while queue is not empty do

  $(X_i, X_j) \leftarrow$ *queue*.pop()

  if REMOVE-INCONSISTENT-VALUES($X_i, X_j$) then

    for each $X_k$ in NEIGHBORS[$X_i$] – {$X_j$} do
      add ($X_k, X_i$) to queue

Keep track of what arcs we need to process

function REMOVE-INCONSISTENT-VALUES($X_i, X_j$) return *true* iff we remove a value

*removed* $\leftarrow$ *false*

for each *x* in DOMAIN[$X_i$] do

  if no value *y* in DOMAIN[$X_j$] allows (x,y) to satisfy
    the constraints between $X_i$ and $X_j$

  then delete x from DOMAIN[$X_i$]; *removed* $\leftarrow$ *true*

return *removed*

Add back arcs to neighbors whenever a node had values removed

# AC-3: Worst Case Complexity Analysis

- **All nodes can be connected to *every* other node,**
  - so each of ***n*** nodes must be compared against ***n-1*** other nodes,
  - so total # of arcs is ***n*(n-1), i.e. O(n²)***
- **If there are *d* values, checking arc (i,j) takes $O(d^2)$ time**
- **Each arc (i,j) can only be inserted into the queue *d* times**
- **Worst case complexity: $O(n^2 d^3)$**

**(For *planar* constraint graphs, the number of arcs can only be *linear in N and* the time complexity is only $O(nd^3)$)**

# When to Iterate, When to Stop?

The crucial principle:

*If a value is removed from a node $X_i$,*
*then the values on all of $X_i$'s neighbors must be reexamined.*

Why? *Removing* a value from a node may result in one of the neighbors becoming arc *inconsistent*, so we need to check…

(but each neighbor $X_j$ can only become inconsistent with respect to the removed values on $X_i$)

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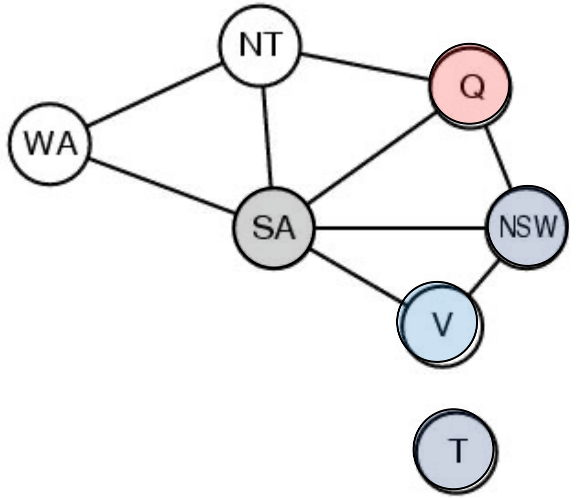
# Other techniques for speeding up finding solutions for CSPs

Professor Chris Callison-Burch

# Chronological backtracking

- DFS does Chronological backtracking
  - If a branch of a search fails, backtrack to the most recent variable assignment and try something different
  - But this variable may not be related to the failure
- Example: Map coloring of Australia
  - Variable order
    - Q, NSW, V, T, SA, WA, NT.
  - Current assignment:
    - Q=red, NWS=green, V=blue, T= red
  - SA cannot be assigned anything
  - But reassigning T does not help!

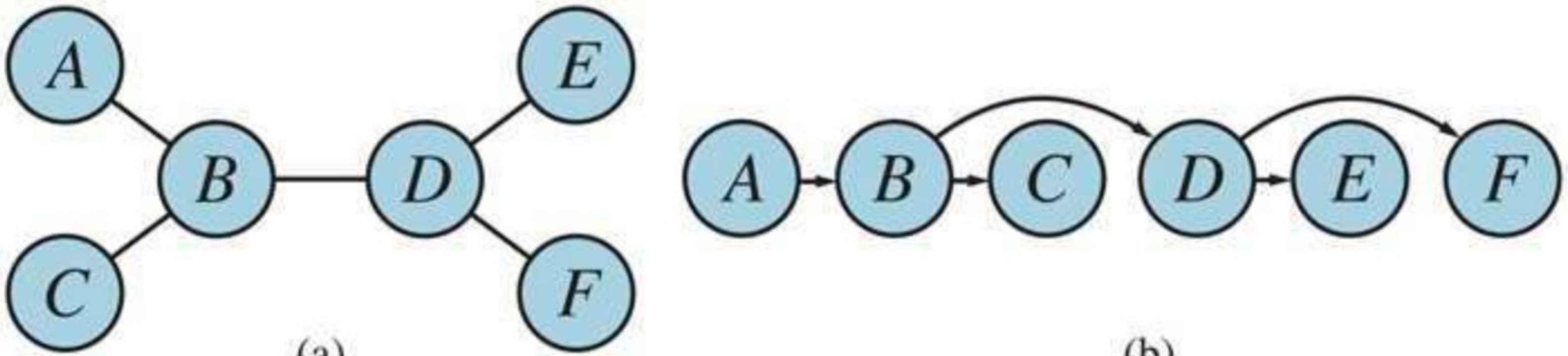


# Backjumping: Improved backtracking

- Find "the conflict set"
  - Those variable assignments that are in conflict
  - Conflict set for SA: {Q=red, NSW=green, V=blue}
- Jump back to reassign one of those conflicting variables
- Forward checking can build the conflict set
  - See textbook for details

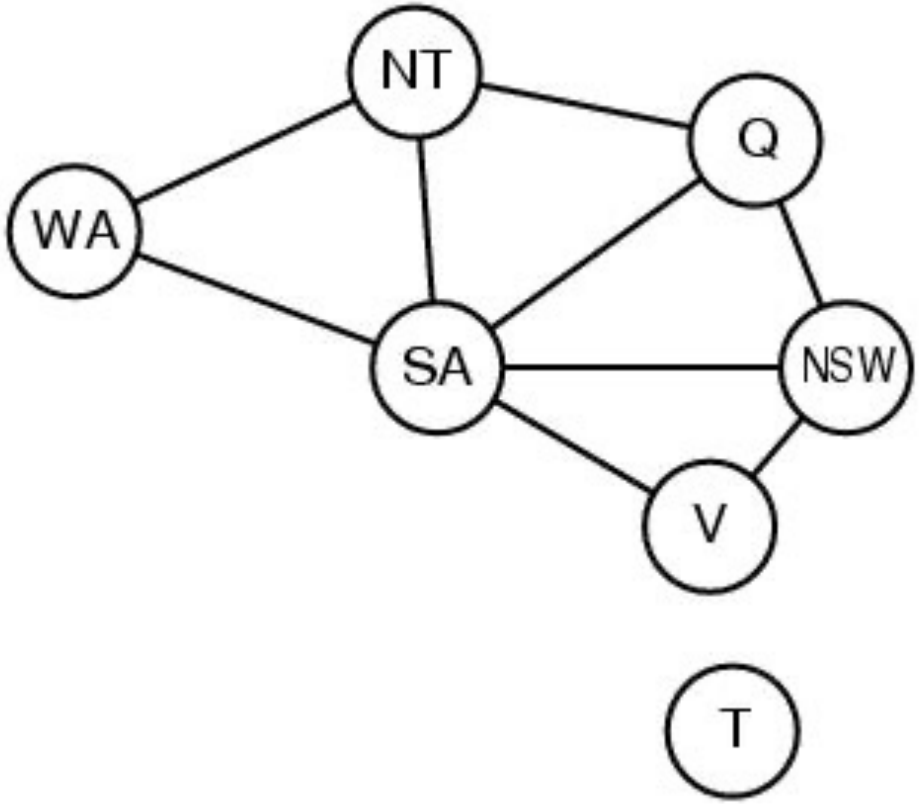
# Simple CSPs can be solved quickly

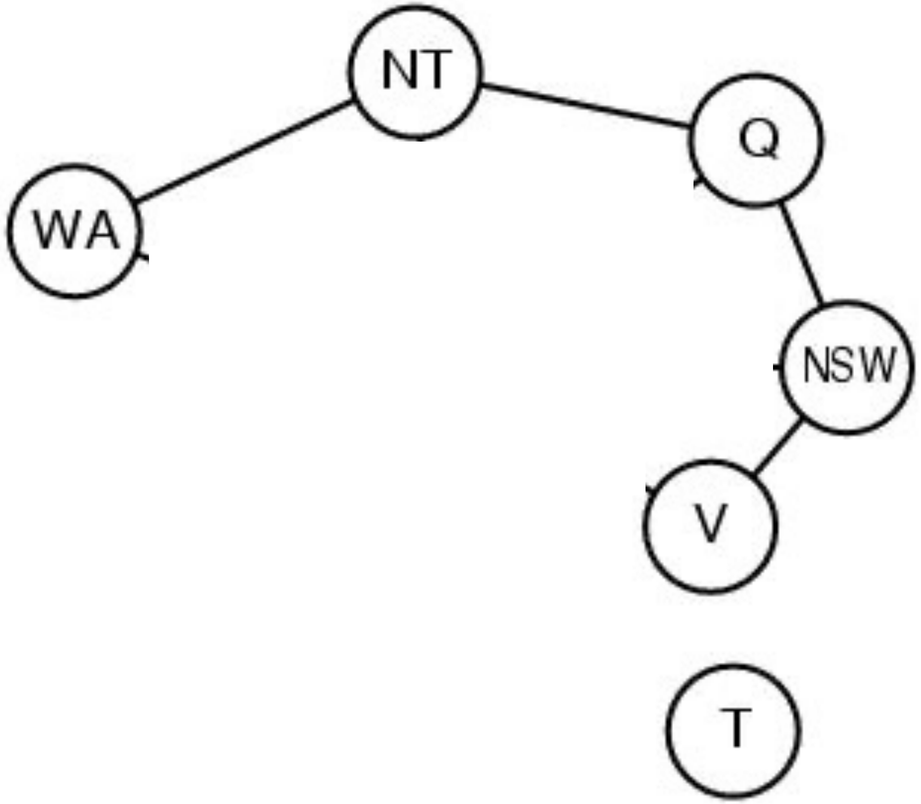
1. Completely independent subproblems
   e.g. Australia & Tasmania
   - Easiest
2. Constraint graph is a tree
   - Any two variables are connected by only a single path
   - Permits solution in time linear in number of variables
   - Do a topological sort and just march down the list




(a) (b)

# Cutset conditioning



# Local search for CSPs

- Local search like hill-climbing search for nearby solutions that improve an objective function.
- To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators reassign variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - choose value that violates the fewest constraints
  - i.e., hill-climb with $h(n)$ = total number of violated constraints

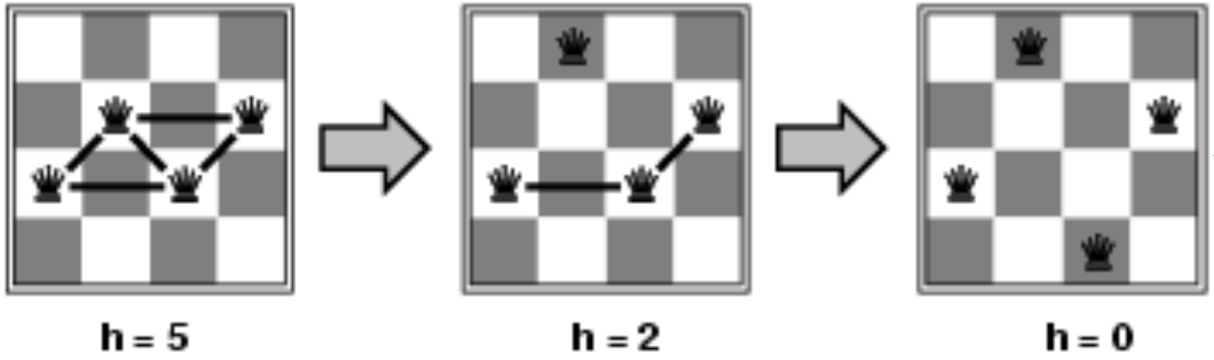
# Min-Conflicts algorithm

**function** MIN-CONFLICTS(*csp, max_steps*) **returns** a solution or failure
    **inputs**: *csp*, a constraint satisfaction problem
        *max_steps*, the number of steps allowed before giving up

*current* ← an initial complete assignment for *csp*
**for** i = 1 to max_steps **do**
    **if** *current* is a solution for *csp* **then return** *current*
    *var* ← a randomly chosen conflicted variable from cp. VARIABLES
    *value* ← the value *v* for var that minimizes CONFLICTS(csp, var, *v, current*)
    set *var* = *value in current*
**return** *failure*

# Min-Conflicts Example: n-queens

- States: 4 queens in 4 columns ($4^4$ = 256 states)
- Actions: move queen in column
- Goal test: no attacks
- Evaluation: *h(n)* = number of attacks




**Given random initial state, local min-conflicts can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 1,000,000)**

**Min-conflicts reduced time to scheduled Hubble Space Telescope from 3 weeks to 10 minutes**

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# Next week: Logical Agents