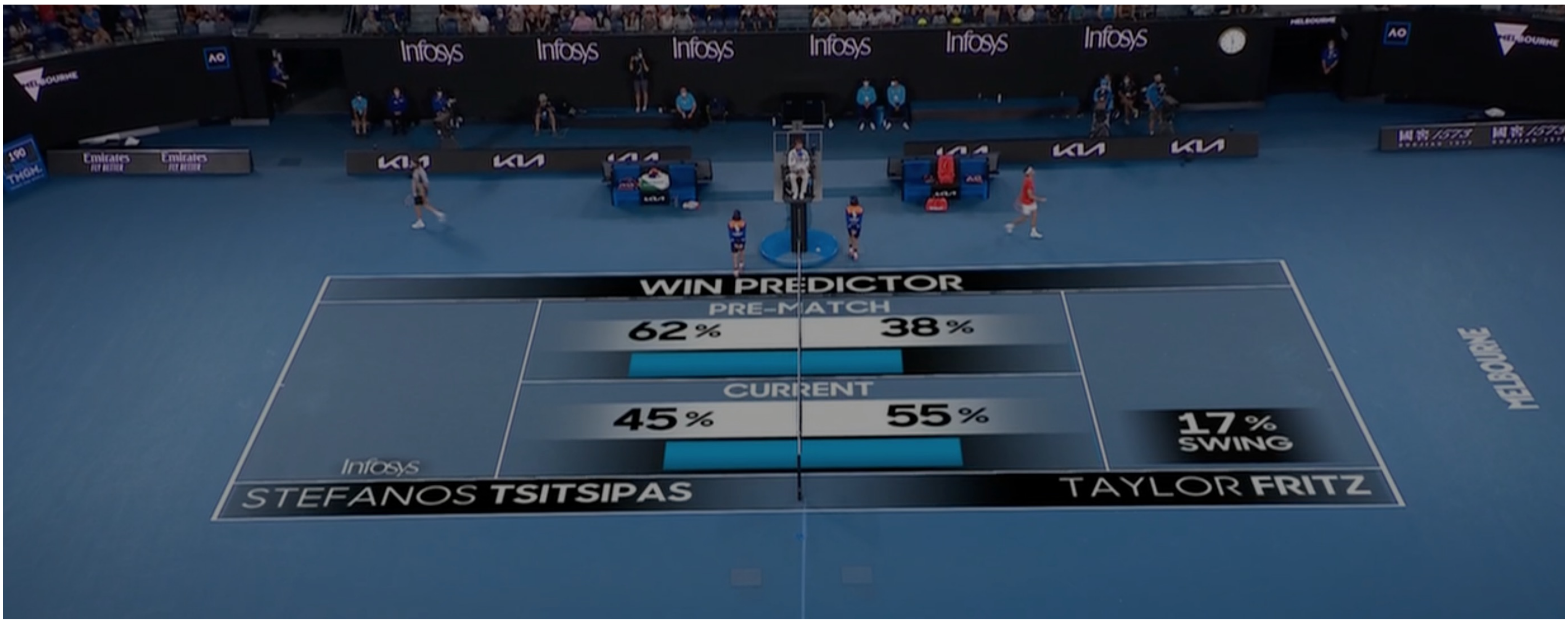
Projective Transformations

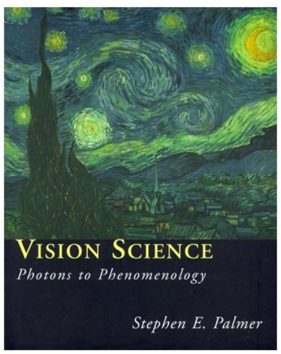
aka Collineations
aka Homographies

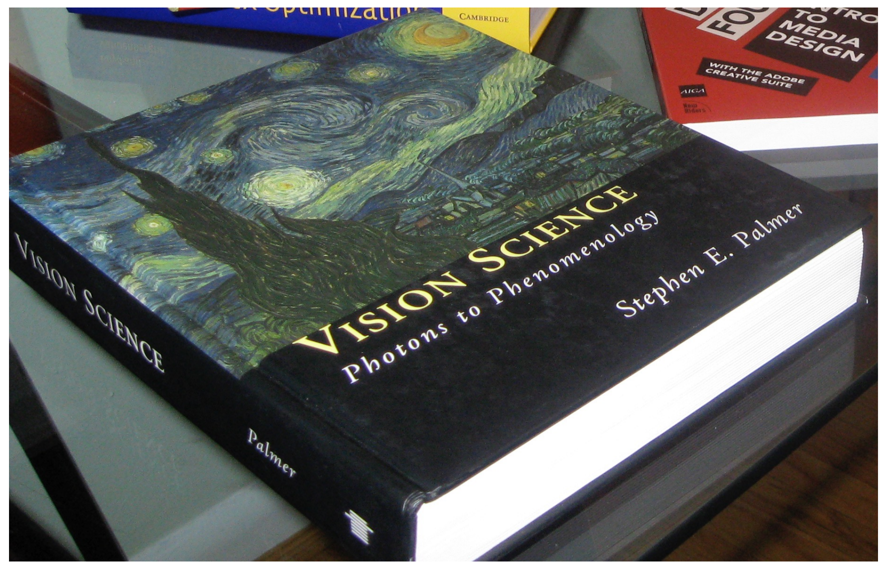
Kostas Daniilidis

Virtual Billboards

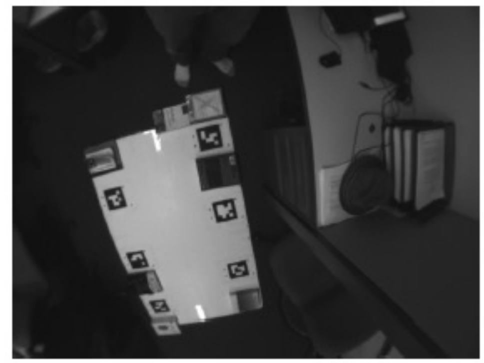


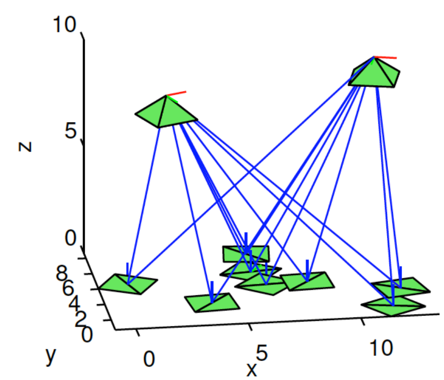
A perspective projection of a plane (like a camera image) is always a projective transformation





Using the projective transformation the pose of a robot with respect to a planar pattern:





Projective Transformation

Definition

A **projective transformation** is any invertible matrix transformation $\mathbb{P}^2 \to \mathbb{P}^2$.

A projective transformation A maps p to $p' \sim Ap$.

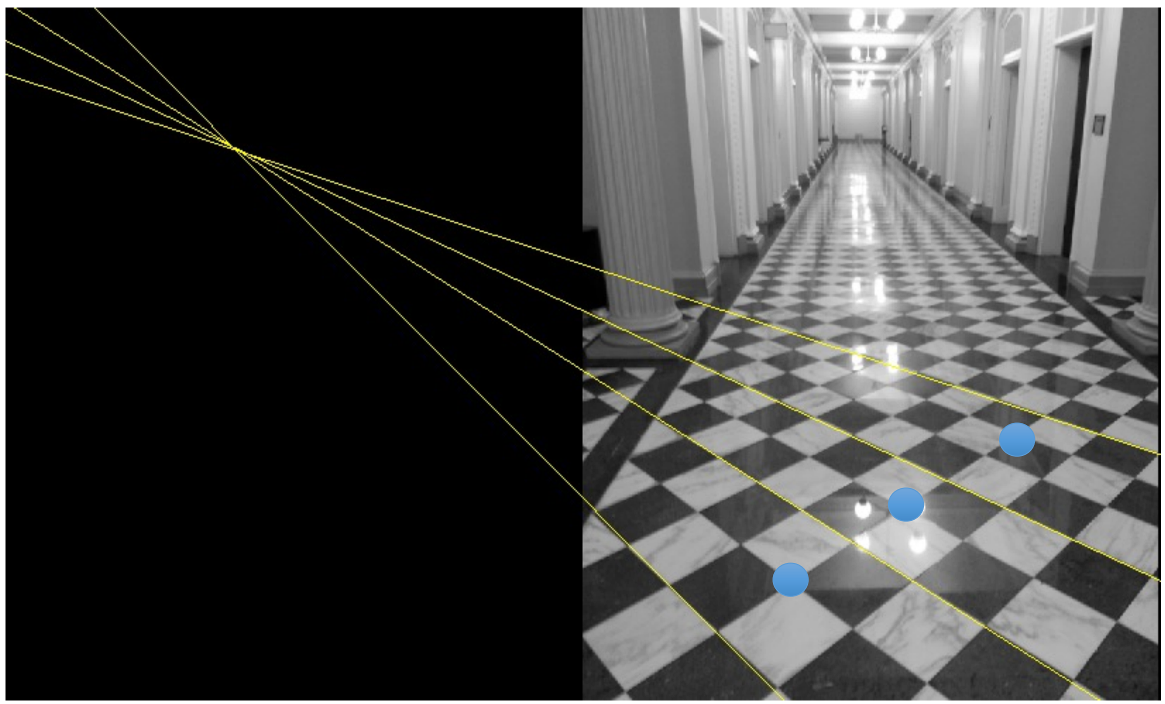
Invertibility means that $\det(A) \neq 0$ and that there exists $\lambda \neq 0$ such that $\lambda p' = Ap$.

Observe that we will write either $p' \sim Ap$ or $\lambda p' = Ap$.

A projective transformation is also known as **collineation or homography.**

A projective transformation preserves incidence:

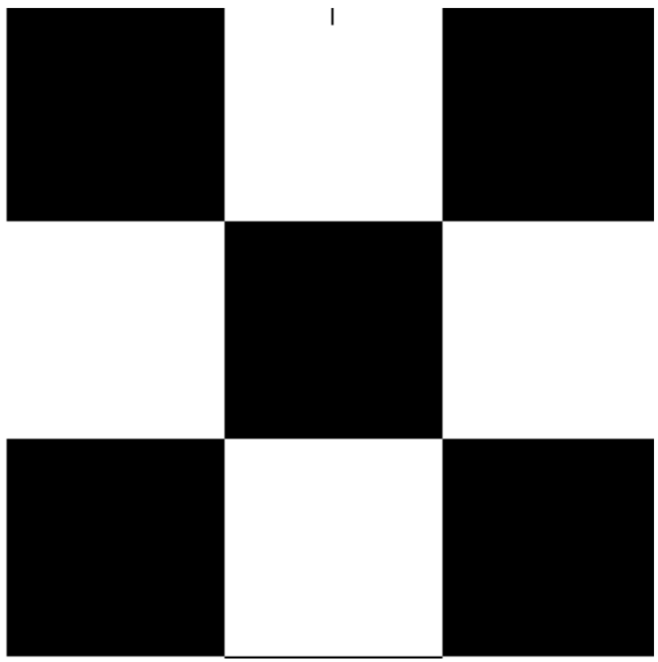
- Three collinear points are mapped to three collinear points.
- and three concurrent lines are mapped to three concurrent lines.



Compute Projective Transformations Using 4 Points

Kostas Daniilidis

How can we compute the projective transformation between

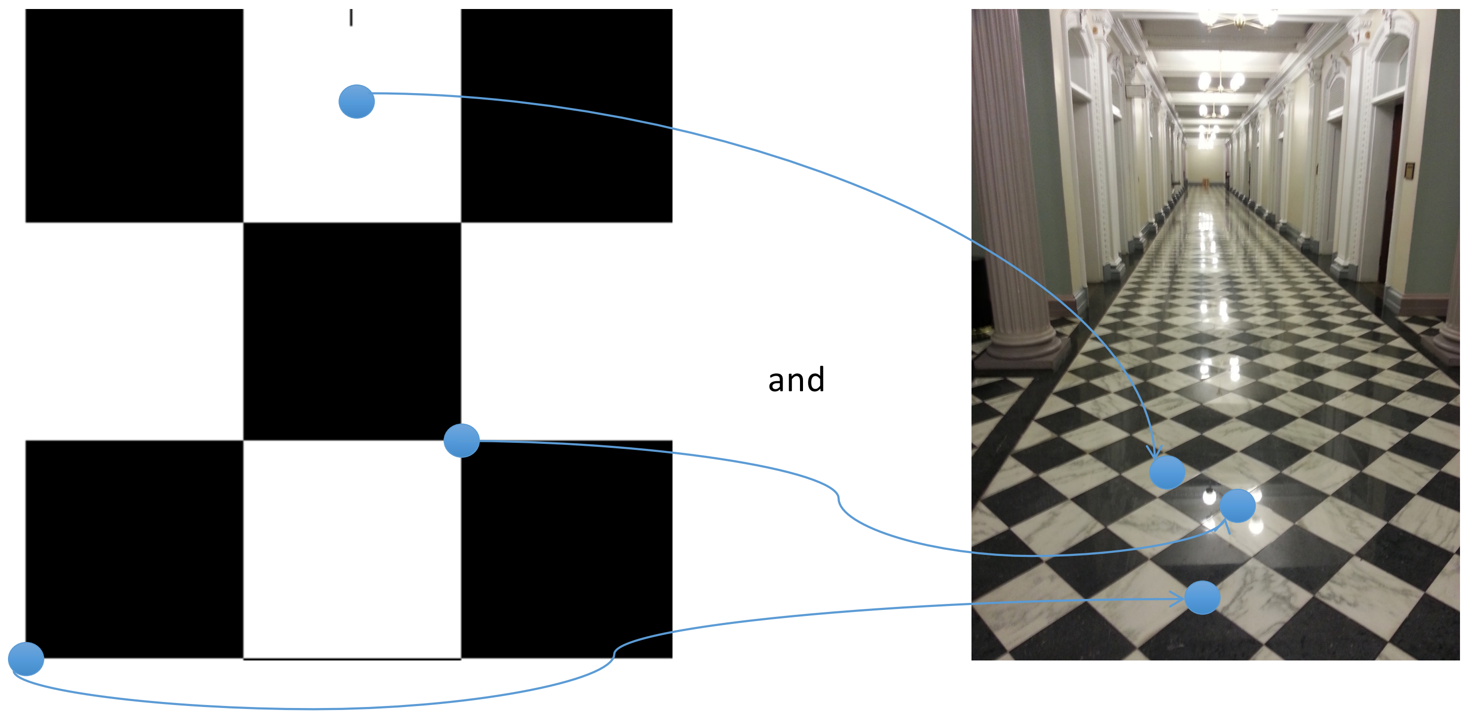


and

Floor tiles measured in [m]

Points in pixel coordinates

The result of such a transformation would map any point in one plane to the corresponding point in the other



Floor tiles measured in [m]

Points in pixel coordinates

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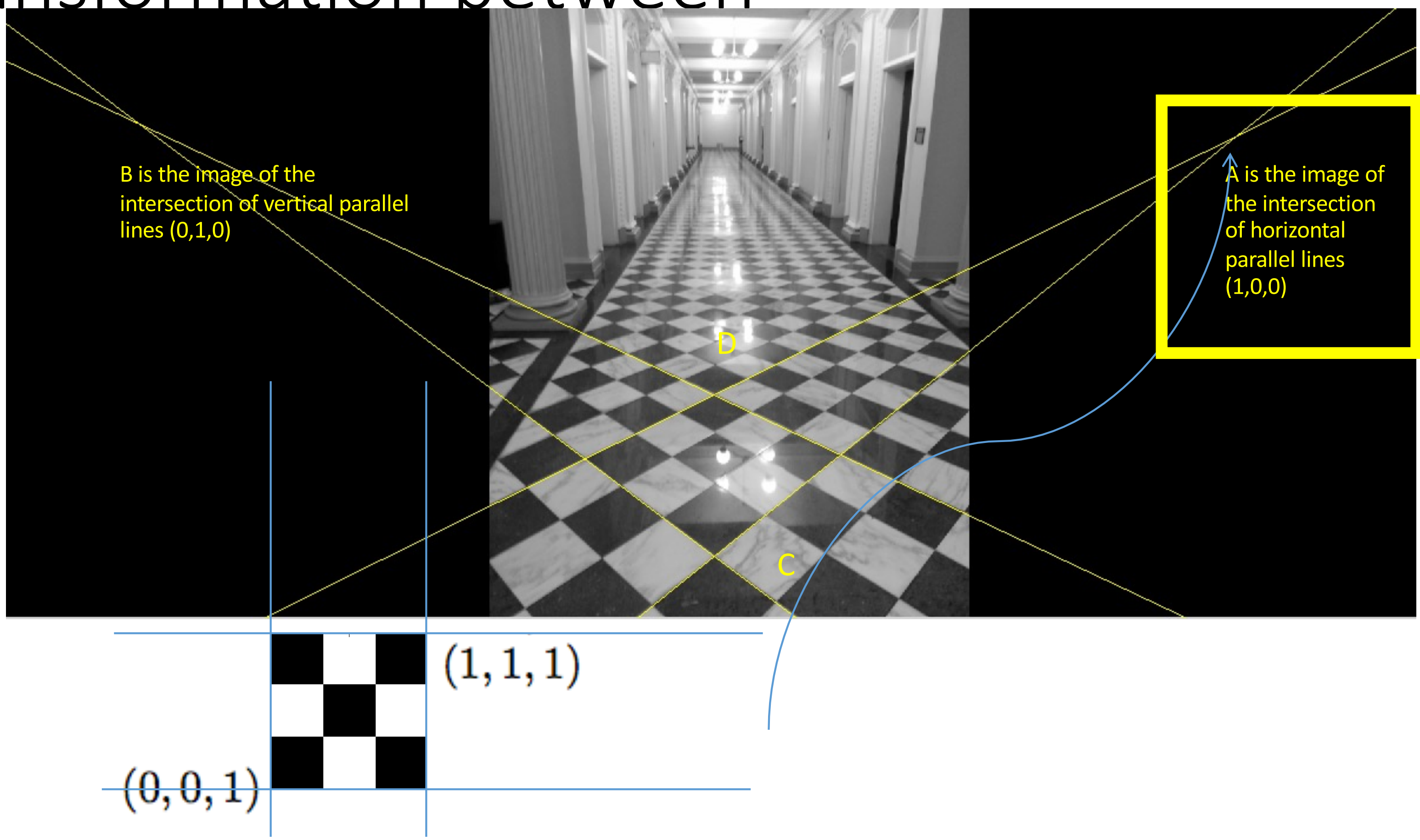

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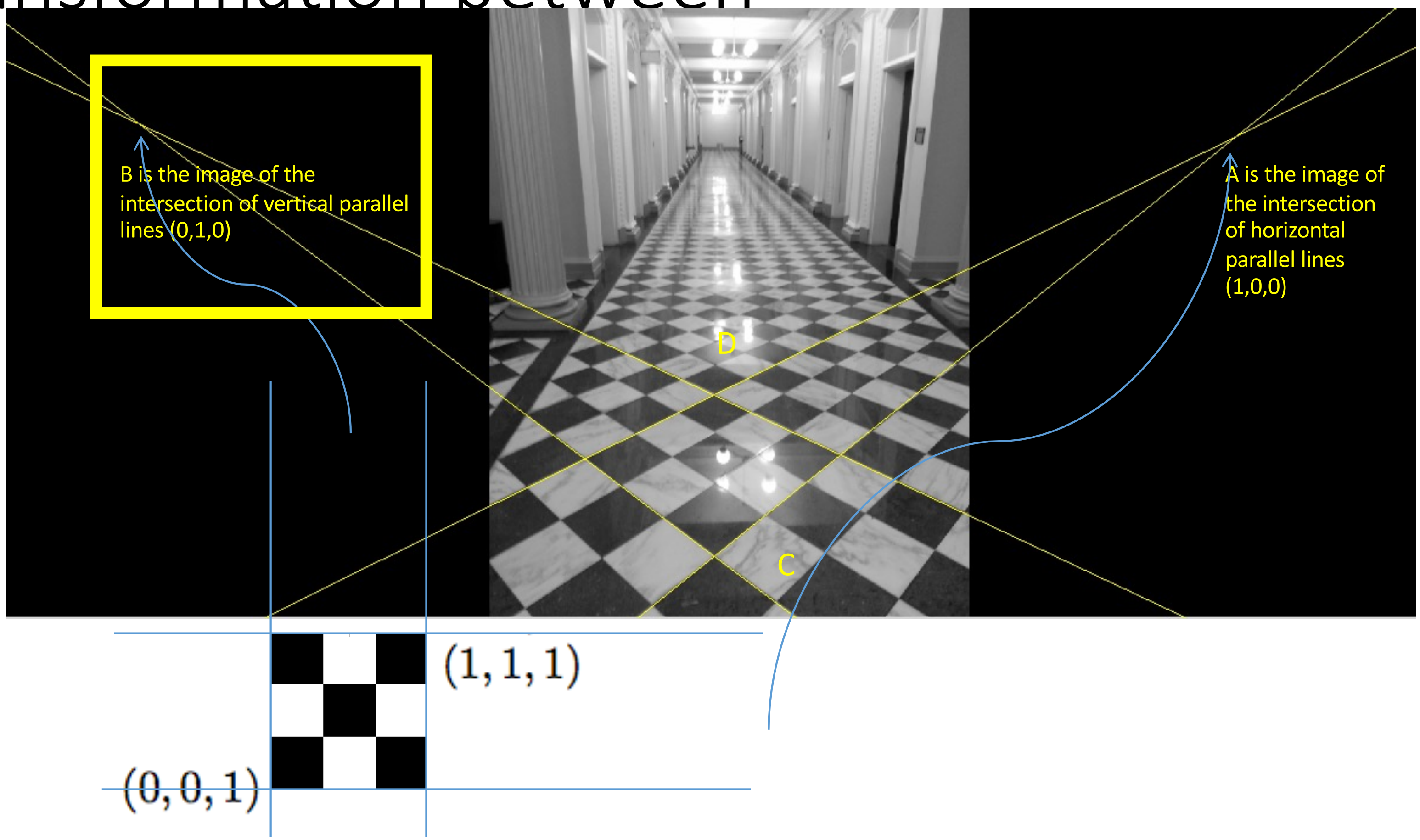

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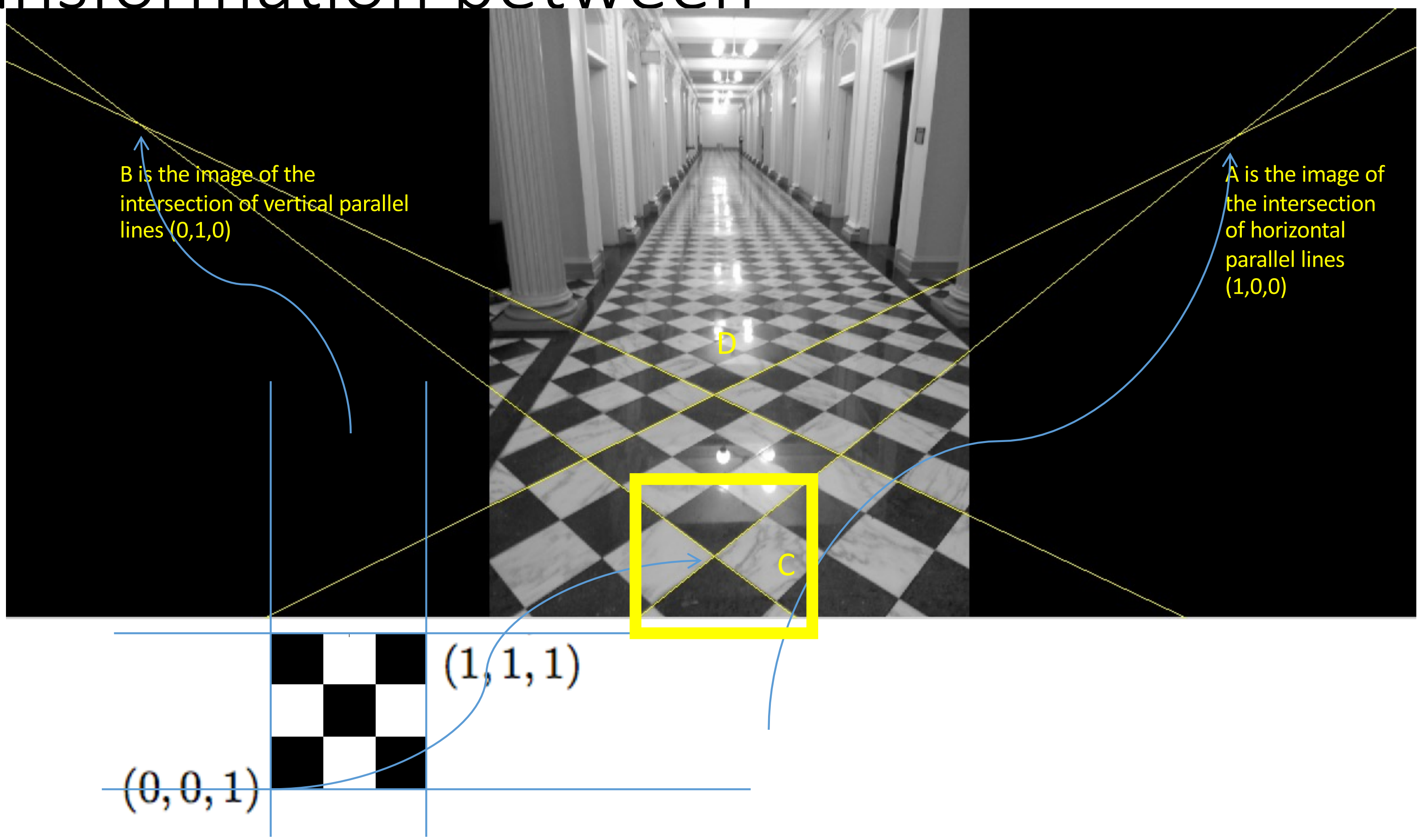


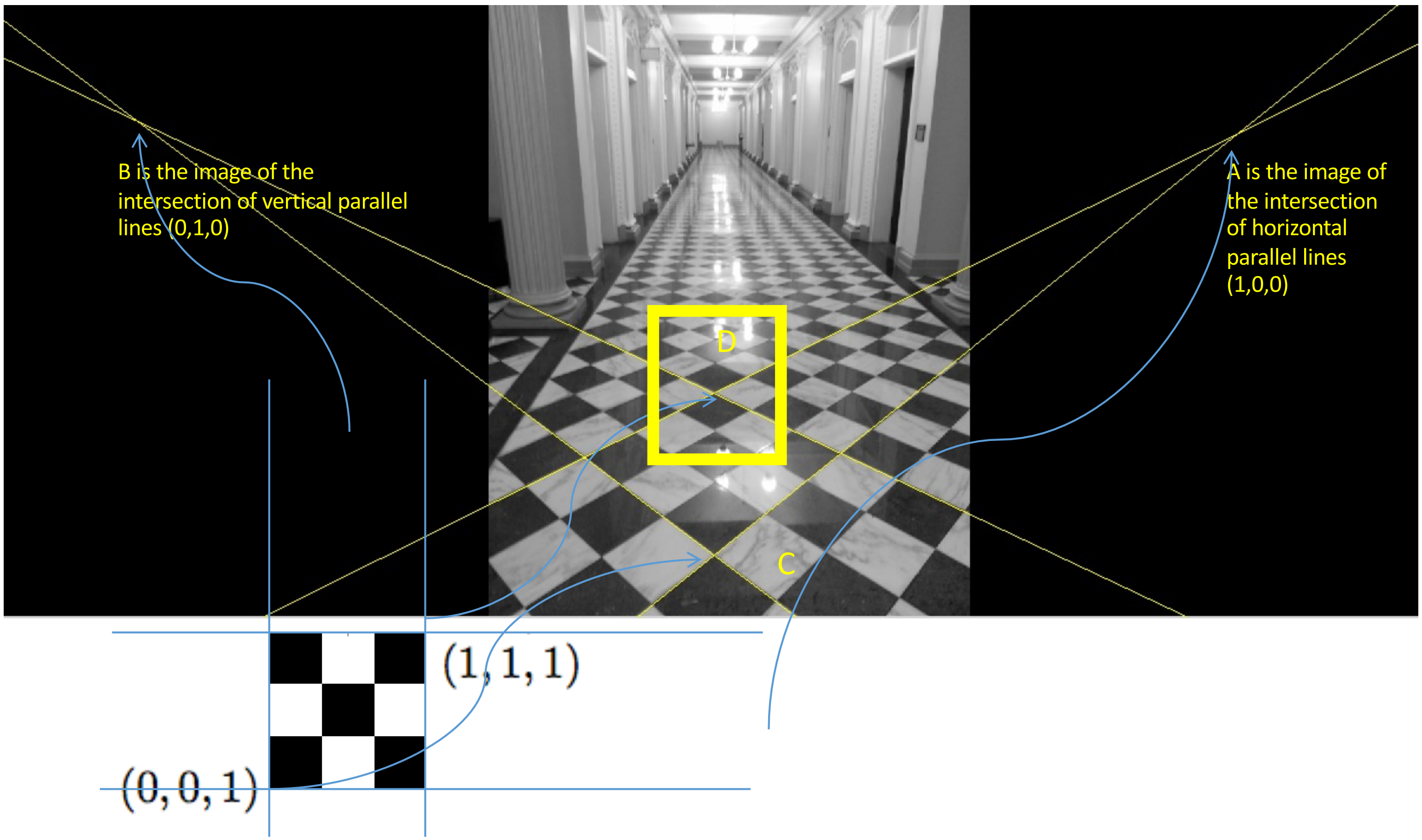
Assume that a mapping A maps the three points
$(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ to the non-collinear points A,B,C

with coordinate vectors a, b and $c \in \mathbb{P}^2$. Then the following is a possible projective transformation:

$$\begin{pmatrix} a & b & c \end{pmatrix} = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with 3 degrees of freedoms α, β and γ. This means 3 points do not suffice to compute a projective transformation.

Let us introduce a 4th point D



Let us assume that the same A maps $(1,1,1)$ to the point d. Then, the following should hold:

$$\lambda d = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

hence

$$\lambda d = \alpha a + \beta b + \gamma c.$$

There always exist such $\lambda, \alpha, \beta, \gamma$ because four elements of $\mathbb{R}^3 \setminus \{(0,0,0)\}$ are always linearly dependent.

Because a, b, c are not collinear, there exist unique $\alpha/\lambda, \beta/\lambda, \gamma/\lambda$ for writing this linear combination.

Since A is the same as A/λ we solve for α, β, γ such that $d = \alpha a + \beta b + \gamma c$, which can be written as a linear system

$$\begin{pmatrix} a & b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = d.$$

Since a, b, c are not collinear we can always find a unique triple α, β, γ. The resulting projective transformation is $A = \begin{pmatrix} \alpha a & \beta b & \gamma c \end{pmatrix}$.

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Four points not three of them collinear suffice to recover unambiguously a projective transformation.

Knowledge of this projective transformation makes Virtual Billboards possible!

Computing proj. transformations with more points

$$\mathbf{x}' \sim H\mathbf{x}$$

$$\lambda \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\lambda x' = h_{11}x + h_{12}y + h_{13}$$

$$\lambda y' = h_{21}x + h_{22}y + h_{23}$$

$$\lambda = h_{31}x + h_{32}y + h_{33}$$

Computing proj. transformations with more points

$$x' = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}}$$

$$y' = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$\Rightarrow$$

$$-h_{11}x - h_{12}y - h_{13} + h_{31}xx' + h_{32}yx' + h_{33}x' = 0$$

$$-h_{21}x - h_{22}y - h_{23} + h_{31}xy' + h_{32}yy' + h_{33}y' = 0$$

$$\Downarrow$$

$$a_x = \begin{pmatrix} -x & -y & -1 & 0 & 0 & 0 & xx' & yx' & x' \end{pmatrix}$$

$$a_y = \begin{pmatrix} 0 & 0 & 0 & -x & -y & -1 & xy' & yy' & y' \end{pmatrix}$$

$$h = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{21} & h_{22} & h_{23} & h_{31} & h_{32} & h_{33} \end{pmatrix}^T$$

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} h = 0$$

Computing proj. transformations with more points

Our matrix H has 8 degrees of freedom, and so, as each point gives 2 sets of equations, we will need 4 points to solve for h uniquely. So, given four points (such as the corners provided for this assignment), we can generate vectors a_x and a_y for each, and concatenate them together:

$$A = \begin{pmatrix} a_{x,1} \\ a_{y_1} \\ \vdots \\ a_{x,n} \\ a_{y,n} \end{pmatrix}$$

$$Ah = 0$$

As A is a 8x9 matrix, there is a unique null space. Normally, we can use MATLAB's **null** function, however, due to noise in our measurements, there may not be an h such that Ah is exactly 0. Instead, we have, for some small $\vec{\epsilon}$:

$$Ah = \vec{\epsilon} \tag{18}$$

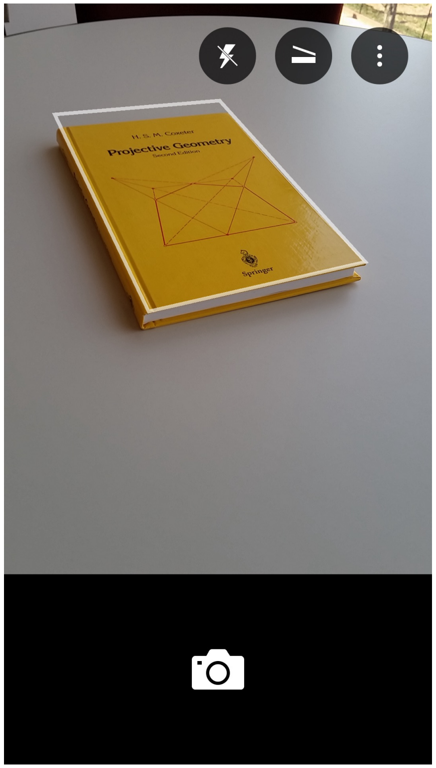
To resolve this issue, we can find the vector h that minimizes the norm of this $\vec{\epsilon}$. To do this, we must use the SVD, which we will cover in week 3. For this project, all you need to know is that you need to run the command:

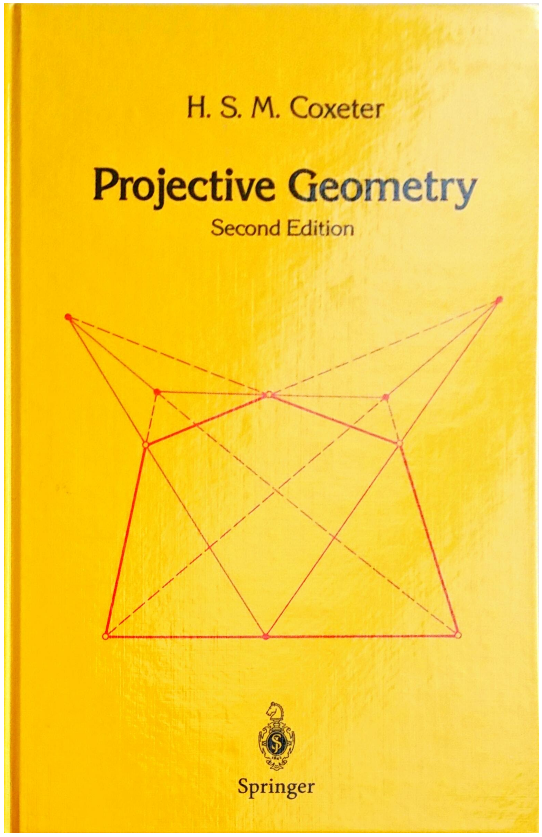
```
[U, S, V] = svd(A);
```

The vector h will then be the last column of V, and you can then construct the 3x3 homography matrix by reshaping the 9x1 h vector.

Microsoft Office Lens App

Office Lens


H. S. M. Coxeter
Projective Geometry
Second Edition
Springer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What happens when the original set of points is not a square?

Find projective transformation mapping $(a, b, c, d) \rightarrow (a', b', c', d')$:

To determine this mapping we go through the four canonical points.

We find the mapping from $(1,0,0), etc$ to (a,b,c,d) and we call it T:

$$a \sim T(1,0,0)^T, etc$$

We find the mapping from $(1,0,0), etc$ to (a',b',c',d') and we call it T':

$$a' \sim T'(1,0,0)^T, etc$$

Then, back-substituing $(1,0,0)^T \sim T^{-1}a, etc$ we obtain that

$$a' = T'T^{-1}a, etc$$

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Projective transformation of lines

$\boxed{\text{If } A \text{ maps a point to } Ap\text{, then where does a line } l \text{ map to?}}$

Line equation in original plane

$$l^T p = 0$$

Line equation in image plane $p' \sim Ap$

$$l^T A^{-1} p' = 0$$

implies that $l' = A^{-T} l$.

Projective transformation of lines

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Line equation in original plane

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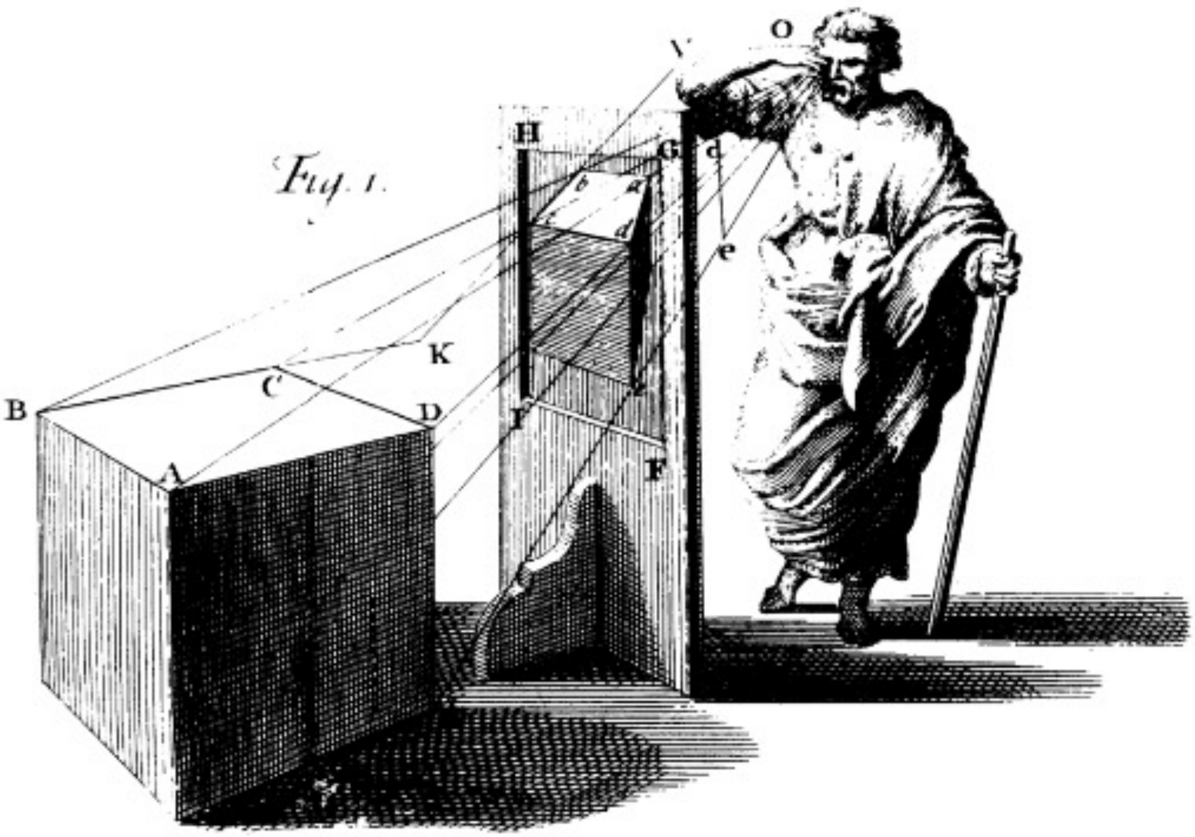


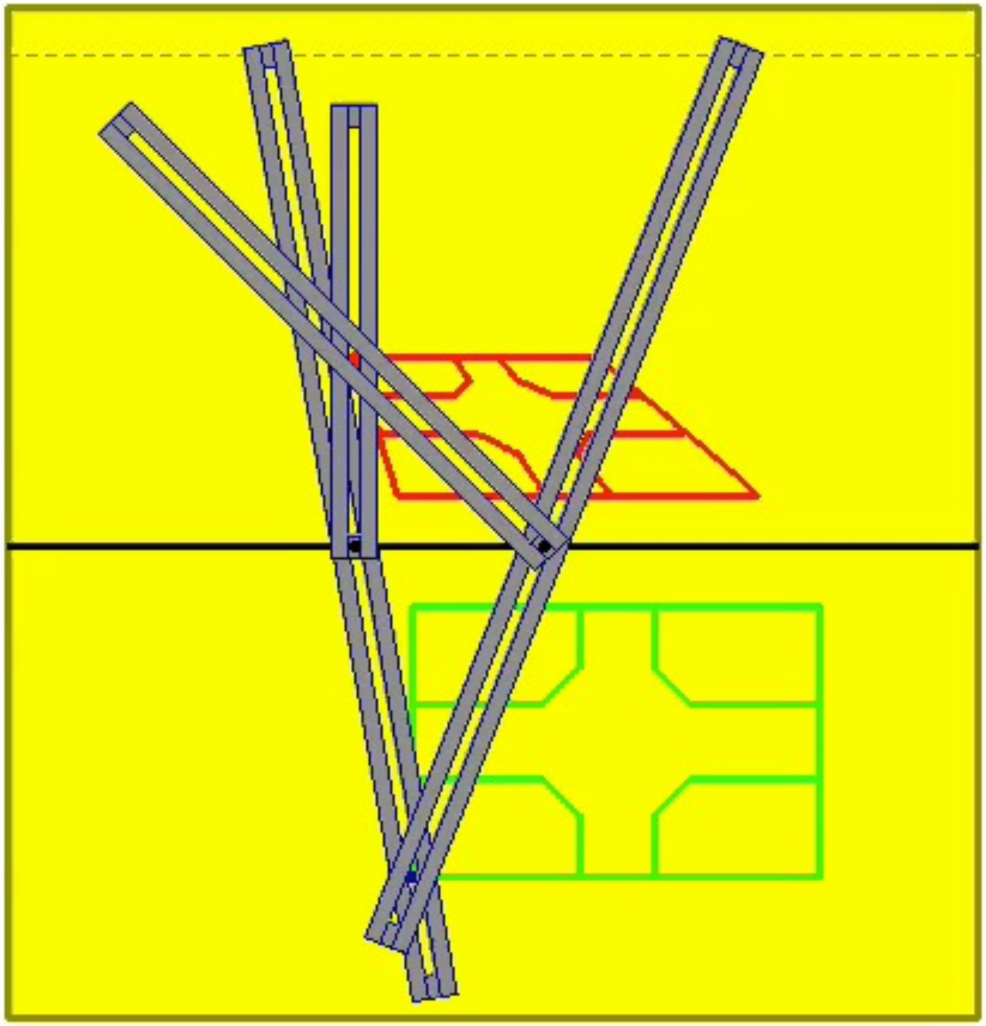
Fig. 1.
V
O
H
G
c
b
a
d
e
K
C
B
D
A
I
F

Bi-perspectograph 1752

LAMBERT'S TWO-DIMENSIONAL PERSPECTOGRAPH (1).

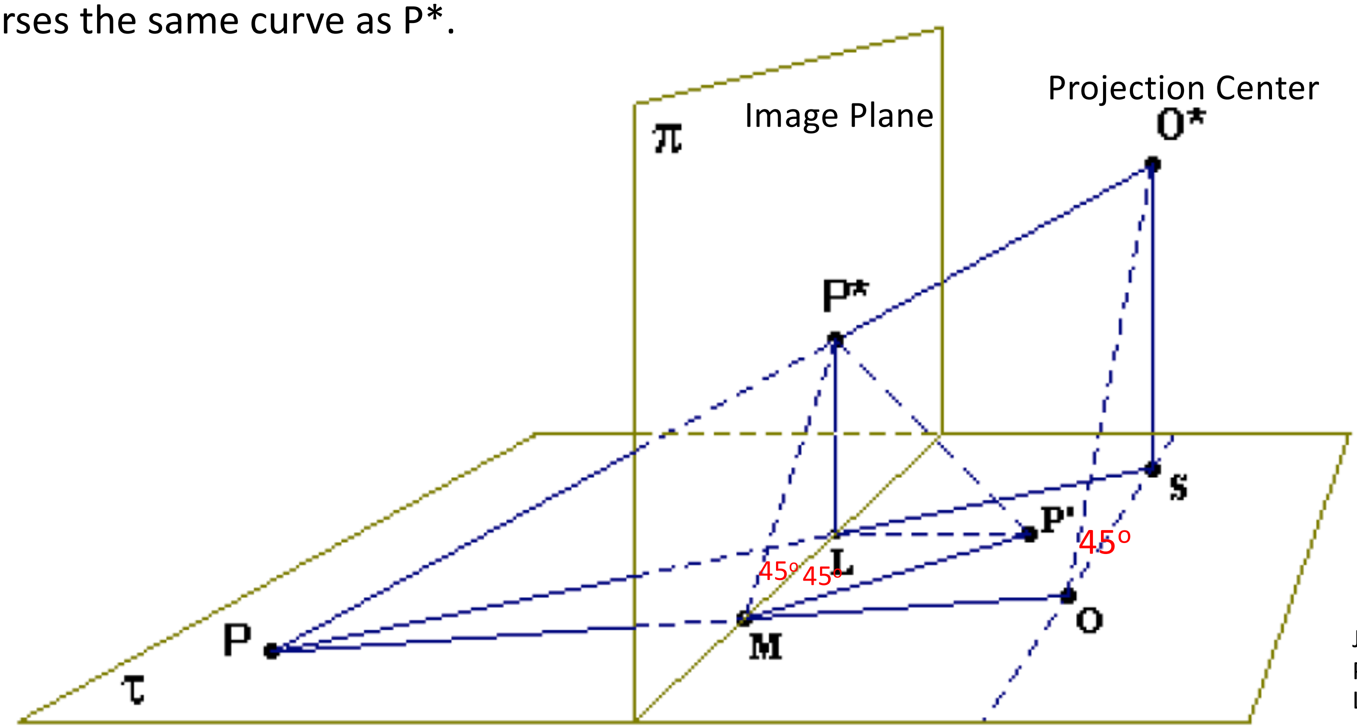
(From: J. H. Lambert, "*Anlage zur Perspektive*", manuscript, August 1752; "*Essai sur la Perspective*", edizione Peiffer – Laurent, 1981)

Mechanical realization



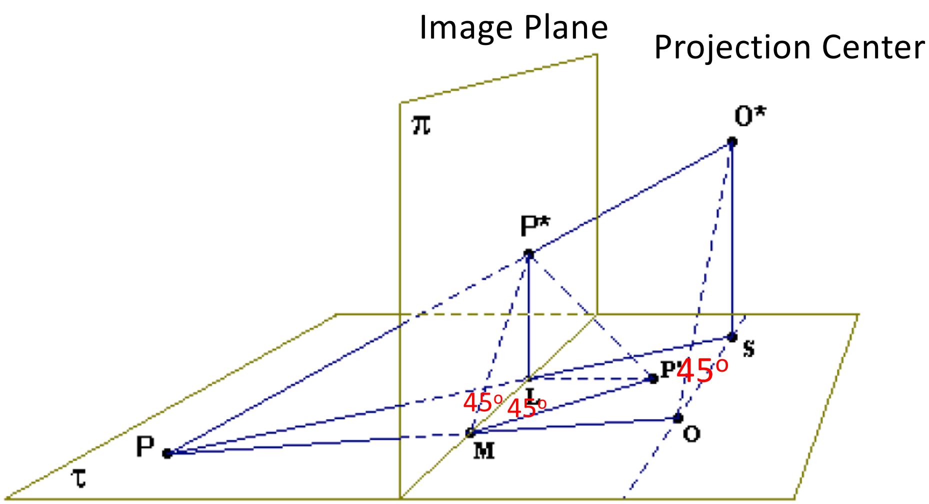
Geometric explanation of bi-perspectograph:

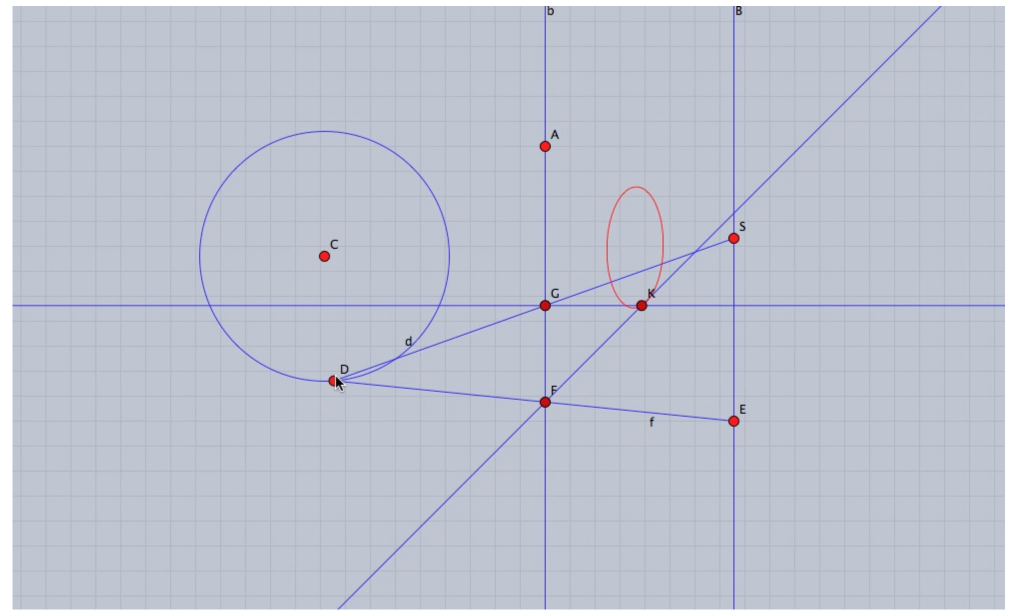
How can we draw a congruent copy of the image plane on the ground plane?
Select O such that OS=SO*. This means angle SOO*=45^O and hence angle LMP* is 45^O as well.
Draw line at L perpendicular to LM. Draw line at M with fixed angle 45^O. Call their intersection P'. Then triangle P*LM is congruent to P'LM.
Hence, P' traverses the same curve as P*.



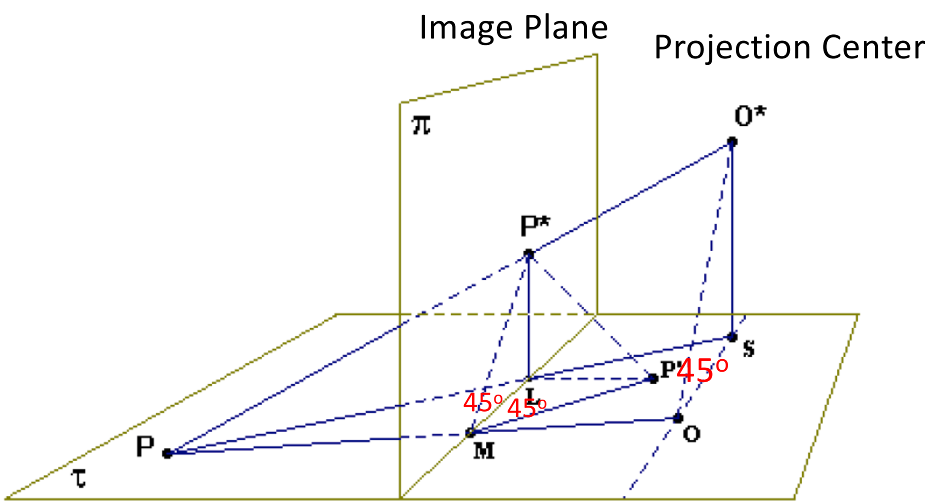
J. H. Lambert, Anlage zur Perspektive, edizione Peiffer - Laurent, 1981

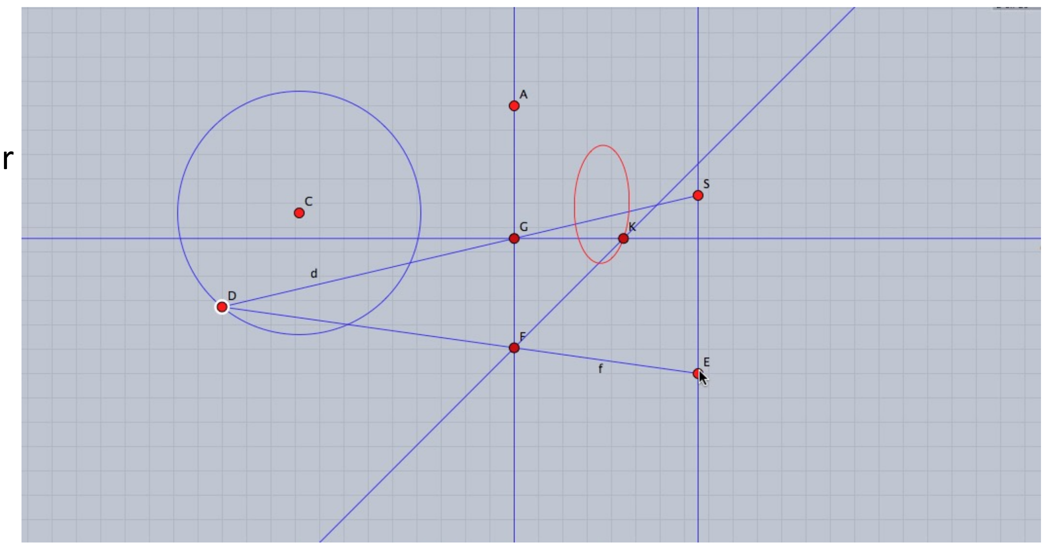
A circle is projected into an ellipse





Effect of height of camera on ellipse shape: the lower the camera, the more squeezed is the ellipse.





Effect of distance of projection center from image plane:
Only the size not the shape of the ellipse changes!