The Austalian National University
School of Computing

Semester 2, 2024
Week 7 - Tutorial

COMP3670/6670: Introduction to Machine Learning

Question 1 — Computing Eigenvalues and Eigenvectors

Given the following matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

1. Compute the determinant of $\mathbf{A}$.

   Solution.

$$\det \mathbf{A} = 2 \cdot 1 \cdot 2 + 0 + 0 - 1 \cdot 1 \cdot 1 - 0 - 0 = 4 - 1 = 3$$

2. What is the characteristic equation of this matrix?

   Solution. We form the matrix $\mathbf{A} - \lambda\mathbf{I}$, and compute it's determinate to obtain the characteristic equation.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{bmatrix} 2-\lambda & 0 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 0 & 2-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(\mathbf{A} - \lambda\mathbf{I}) &= (2-\lambda)(1-\lambda)(2-\lambda) + 0 + 0 - 1 \cdot (1-\lambda) \cdot 1 - 0 - 0 \\ &= (2-\lambda)^2(1-\lambda) - (1-\lambda) \\ &= (1-\lambda)\left((2-\lambda)^2 - 1\right) \end{aligned}$$

   We could have also achieved the same result via Gaussian elimination to a upper triangular matrix, and then applied the previous proven property.

3. Find the eigenvalues, and their algebraic multiplicity.

   Solution. Solving $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, we obtain,

$$1 - \lambda = 0 \text{ or } (2-\lambda)^2 - 1 = 0$$

   In the first case, we have $\lambda = 1$. In the second case, we have $(2-\lambda)^2 = 1$, so $2 - \lambda = \pm 1$. This means that we have $2 - \lambda = 1 \Rightarrow \lambda = 1$, or that $2 - \lambda = -1 \Rightarrow \lambda = 3$.
   Hence, we have that $\lambda = 1$ is an eigenvalue with algebraic multiplicity of 2, and $\lambda = 3$ is an eigenvalue with algebraic multiplicity of 1.

4. For each eigenvalue, compute the corresponding eigenspaces.

   Solution. For $\lambda = 1$ we need to find all $\mathbf{x}$ that solves the equation $\mathbf{A}\mathbf{x} = \mathbf{x}$. We form the homogeneous system $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$, and solve

$$\left[\begin{array}{ccc|c} 2-1 & 0 & 1 & 0 \\ 1 & 1-1 & 1 & 0 \\ 1 & 0 & 2-1 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

   Hence we have that $x_1 = -x_3$ and that both x_2 and x_3 are free variables. The solution set is

$$E_1 = \left\{ \begin{bmatrix} -\beta \\ \alpha \\ \beta \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\} = \text{span}[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}]$$

For $\lambda = 3$ we need to find all $\mathbf{x}$ that solves the equation $\mathbf{Ax} = 3\mathbf{x}$. We form the homogeneous system $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$, and solve

$$\left[\begin{array}{ccc|c} 2-3 & 0 & 1 & 0 \\ 1 & 1-3 & 1 & 0 \\ 1 & 0 & 2-3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence we obtain $x_1 = x_3$ and $x_2 = x_3$, providing the solution set

$$E_3 = \left\{ \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\} = \text{span}[\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}]$$

5. Show that $\mathbf{A}$ is diagonalizable.

   Solution. We need to show that $\mathbf{A}$ is non-defective, that is, that the set of all eigenvectors (the eigenspectrum) spans $\mathbb{R}^3$. We can demonstate that by finding three linearly independent eigenvectors. From Question 3.4 we have a description of the eigenspaces in terms of the spans of basis vectors. By combining the eigenspaces, if the three basis vectors span $\mathbb{R}^3$, then $\mathbf{A}$ is diagonalizable. Using Gaussian elimination, we can show this to be true.

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. Find an invertible matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and a diagonal matrix $\mathbf{D} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

   Solution. We can read off the eigenvectors to form the columns of $\mathbf{P}$, and the corresponding eigenvalues (in the same order) to form the diagonal elements of $\mathbf{D}$.

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   The inverse of $\mathbf{P}$ can be found via the standard row reduction algorithm.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \end{array}\right]$$

   Hence,

$$\mathbf{P}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

   It can be then verified by standard matrix multiplication that $\mathbf{A} = \mathbf{PDP}^{-1}$.

7. Give a closed form for $\mathbf{A}^n$ for any $n \geq 0$.

Solution. Since $\mathbf{A} = \mathbf{PDP}^{-1}$, we can compute $\mathbf{A}^n$.

$$\begin{aligned}
\mathbf{A}^n &= \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^n \left(\frac{1}{2} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \right) \\
&= \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3^n & 0 & 3^n \\ -1 & 2 & -1 \\ -1 & 0 & 1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 3^n + 1 & 0 & 3^n - 1 \\ 3^n - 1 & 2 & 3^n - 1 \\ 3^n - 1 & 0 & 3^n + 1 \end{bmatrix}
\end{aligned}$$

Question 2 **Singular Value Decomposition**

Compute the singular value decomposition of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Solution. We need to compute the eigenbasis of $\mathbf{A}^T\mathbf{A}$ to obtain $\mathbf{V}$, and the eigenbasis for $\mathbf{AA}^T$ to obtain $\mathbf{U}$. The matrix $\mathbf{\Sigma}$ is given by the square roots of the non-zero eigenvalues of either $\mathbf{A}^T\mathbf{A}$ or $\mathbf{AA}^T$, padded with zeros to be the same shape as $\mathbf{A}$.
So,

$$\mathbf{AA}^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

We solve for the eigenvalues by constructing the characteristic equation,

$$\det(\mathbf{AA}^T - \lambda \mathbf{I}) = (1-\lambda)^2(2-\lambda) - 2(1-\lambda)$$

and solving for when $\det(\mathbf{AA}^T - \lambda \mathbf{I}) = 0$, we obtain the solutions

$$\lambda = 3, 1, 0$$

For $\lambda = 3$, we compute the eigenspace by solving $(\mathbf{AA}^T - 3\mathbf{I})\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 1-3 & 0 & 1 & 0 \\ 0 & 1-3 & 1 & 0 \\ 1 & 1 & 2-3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We obtain the solution set

$$E_3 = \text{span}[\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}]$$

We need to choose a unit eigenvector from this set, so we choose

$$\mathbf{v}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

For $\lambda = 1$, we compute the eigenspace by solving $(\mathbf{A}\mathbf{A}^T - \mathbf{I})\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 1-1 & 0 & 1 & 0 \\ 0 & 1-1 & 1 & 0 \\ 1 & 1 & 2-1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We obtain the solution set

$$E_1 = \text{span}[\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}]$$

and choose a unit eigenvector from this span,

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda = 0$, we compute the eigenspace by solving $\mathbf{A}\mathbf{A}^T\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We obtain the solution set

$$E_0 = \text{span}[\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}]$$

and choose a unit eigenvector from this span,

$$\mathbf{v}_0 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

We have now obtained the eigenspectra of $\mathbf{A}$, which provides an orthonormal basis of $\mathbb{R}^3$, as $\mathbf{A}\mathbf{A}^T$ is symmetric, by the spectral theorem. We stack the three eigenvectors together into a matrix, and call the result $\mathbf{U}$.

$$\mathbf{U} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix}$$

We create a diagonal matrix out of the eigenvalues (in the same order as the eigenvectors) and call the result $\mathbf{D}$.

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We thus have an eigenvalue decomposition of $\mathbf{A}\mathbf{A}^T$, by an orthogonal matrix $\mathbf{U}$ and diagonal matrix $\mathbf{D}$. (You can verify that $\mathbf{U}^{-1} = \mathbf{U}^T$).

$$\mathbf{A}\mathbf{A}^T = \mathbf{U}\mathbf{D}\mathbf{U}^T$$

Now, repeat the process for $\mathbf{A}^T\mathbf{A}$.

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

We solve for the eigenvalues by constructing the characteristic equation,

$$\det(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}) = (2 - \lambda)^2 - 1$$

and solving for when $\det(\mathbf{A}^T\mathbf{A} - \lambda\mathbf{I}) = 0$, we obtain the solutions

$$\lambda = 3, 1$$

For $\lambda = 3$, we compute the eigenspace by solving $(\mathbf{A}^T\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 2-3 & 1 & 0 \\ 1 & 2-3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We obtain the solution set

$$E_3' = \text{span}[\begin{bmatrix} 1 \\ 1 \end{bmatrix}]$$

and choose a unit vector from this span,

$$\mathbf{v}_3' = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 1$, we compute the eigenspace by solving $(\mathbf{A}^T\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$.

$$\left[\begin{array}{cc|c} 2-1 & 1 & 0 \\ 1 & 2-1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

We obtain the solution set

$$E_1' = \text{span}[\begin{bmatrix} -1 \\ 1 \end{bmatrix}]$$

and choose a unit vector from this span,

$$\mathbf{v}_1' = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We have now obtained the eigenspectra of $\mathbf{A}$, which provides an orthonormal basis of $\mathbb{R}^2$, as $\mathbf{A}^T\mathbf{A}$ is symmetric, by the spectral theorem. We stack the two eigenvectors together, and call the result $\mathbf{V}$.

$$\mathbf{V} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We stack the corresponding eigenvalues in the same order, and call the result $\mathbf{D}'$.

$$\mathbf{D}' = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

We thus have an eigenvalue decomposition of $\mathbf{A}^T\mathbf{A}$, by an orthogonal matrix $\mathbf{V}$ and diagonal matrix $\mathbf{D}'$. (You can verify that $\mathbf{P}^{-1} = \mathbf{P}^T$).

$$\mathbf{A}^T\mathbf{A} = \mathbf{V}\mathbf{D}'\mathbf{V}^T$$

Now, to obtain the singular value decomposition, we let

$$\mathbf{\Sigma} = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(the non-zero singular values (square roots of eigenvalues) of $\mathbf{A}^T\mathbf{A}$ (or of $\mathbf{A}\mathbf{A}^T$), padded with zeros to match the shape of $\mathbf{A}$. Hence, we have the SVD

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$

which can be verified to mutiply back out to

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$