Matrix Norm

Matrix norm is the generalisation of vector norms.
We call a function $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ a matrix norm, if $\forall A, B \in \mathbb{R}^{m\times n}$, $c \in \mathbb{R}$, the following conditions are satisfied:

I. $\|A\| \geq 0$ with equality attained iff A is a zero matrix.

II. $\|cA\| \leq |c| \, \|A\|$

III. $\|A+B\| \leq \|A\| + \|B\|$

IV. $\|AB\| \leq \|A\| \|B\|$

1. Suppose $\|\cdot\|$ is a matrix norm on $\mathbb{R}^{n\times n}$. Prove whether the function $\|\cdot\|_C : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by $\|A\|_C = \|C^{-1}AC\|$ is a matrix norm for arbitrary $A \in \mathbb{R}^{n\times n}$ and invertible $C \in \mathbb{R}^{n\times n}$

• 1. if $A=0$; $\|A\|_C = \|C^{-1} 0 C\| = 0 \geq 0$ $\therefore$ I is satisfied

2. $\|cA\|_C = \cancel{|c| \|A\|_C \leq |c| \|C^{-1}AC\|}$ $\|C^{-1}(cA)C\| \leq |c| \, \|C^{-1}AC\|$
$\leq |c| \, \|A\|_C$ $\therefore$ II is satisfied

3. $\|A+B\|_C = \|C^{-1}(A+B)C\| = \|C^{-1}(AC+BC)\| = \|C^{-1}AC + C^{-1}BC\|$
$\leq \|C^{-1}AC\| + \|C^{-1}BC\|$ $\forall A, B, C \in \mathbb{R}^{n\times n}$ $\therefore$ III is satisfied.

4. $\|AB\|_C = \|C^{-1}AC\| \|C^{-1}BC\| \leq \|A\|_C \|B\|_C$ $\forall A, B, C \in \mathbb{R}^{n\times n}$
$\therefore$ IV is satisfied.

So $\|\cdot\|_C : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by $\|A\|_C$ is indeed a matrix norm.

2. A function $\|\cdot\| : \mathbb{R}^{n \times n} \to \mathbb{R}$ is defined as $\|A\| = \max_{i \in \{1 \dots n\}} \sum_{j=1}^{n} |A_{ij}|$. The expression computes the maximum value of the col-sum of abs values ($|\cdot|$) of the entries for all rows. Prove whether this fn is a matrix norm.

1. If $A = 0$; then $\sum_{j=1}^{n} |0_{ij}| \geq 0$

2. $\|cA\| = \sum_{j=1}^{n} |c||A_{ij}| = |c| \cdot \sum_{j=1}^{n} |A_{ij}| \leq |c| \, \|A\|$

3. $\|A+B\| = \sum_{j=1}^{n} |(A+B)_{ij}| \leq \sum_{j=1}^{n} |A_{ij} + B_{ij}| \leq \sum_{j=1}^{n} |A_{ij}| + \sum_{j=1}^{n} |B_{ij}|$

$\leq \|A\| + \|B\|$

4. $\|AB\| = \sum_{j=1}^{n} |(AB)_{ij}| = \sum_{k=1}^{n} \left| \sum_{j=1}^{n} A_{ij} B_{jk} \right| \leq \sum_{j=1}^{n} |A_{ij}| \cdot \sum_{k=1}^{n} |B_{jk}|$

$\leq \|A\| \cdot \|B\|$. $\therefore$ The function is a matrix norm

Positive Definite Matrices

Given that $A \in \mathbb{R}^{n \times n}$ is positive definite, it is known that any diagonal entry of A, A_{ii} is positive. Prove whether matrix B with $B_{ij} = A_{ij}(A_{ii}A_{jj})^k$ is also positive definite, where $k \in \mathbb{R}$.

we know that: $x^T A x = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i A_{ij} x_j = \left\langle \sum_i x_i a_i , \sum_j x_j a_j \right\rangle > 0$

similarly

$x^T B x = \sum_i^n \sum_j^n x_i B_{ij} x_j = \sum_i^n \sum_j^n x_i A_{ij} (A_{ii} A_{jj})^k x_j$. Splitting up the terms, we have A_{ij} and $(A_{ii}A_{jj})^k$. We know $A_{ij} > 0$ thus positive. We also know diagonal entries are positive so A_{ii} & $A_{jj} > 0$. $k \in \mathbb{R}$ so k ensures $(A_{ii}A_{jj})^k$ is positive. $\therefore$ $x^T B x > 0$ and thus also positive definite. (for non-zero x).

Matrix Calculus

consider the following function of $X \in \mathbb{R}^{n \times m}$:

$$f(x) = \exp\left\{-\frac{1}{2} \operatorname{tr}\left[(X-M)^T U^{-1} (X-M) V\right]\right\}$$

where $M \in \mathbb{R}^{n \times m}$, $U \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{m \times m}$, and tr is the trace.

U, V are positive definite. Find the gradient of f, $\nabla_x f$.

$$\frac{df(x)}{dx} = e^{-\frac{1}{2} \operatorname{tr}[(X-M)^T U^{-1} (X-M) V]} \cdot \frac{d}{dx} -\frac{1}{2} \operatorname{tr}\left[(X-M)^T U^{-1} (X-M) V\right]$$

$$\frac{d}{dx} -\frac{1}{2} \operatorname{tr}\left[(X-M)^T U^{-1} (X-M) V\right] = \frac{d}{dx} -\frac{1}{2} \operatorname{tr}\left((X^T - M^T) U^{-1} (X-M) V\right)$$

$$= \frac{d}{dx} -\frac{1}{2} \operatorname{tr}\left(X^T U^{-1} - M^T U^{-1} \cdot (X-M) V\right)$$

$$= \frac{d}{dx} -\frac{1}{2} \operatorname{tr}\left((X^T U^{-1} X - X^T U^{-1} M - M^T U^{-1} X + M^T U^{-1} M) V\right)$$

$$= \frac{d}{dx} -\frac{1}{2} \operatorname{tr}\left(X^T U^{-1} X V - X^T U^{-1} M V - M^T U^{-1} X V + M^T U^{-1} M V\right)$$

$$= \cancel{\frac{d}{dx}} -\frac{1}{2} \frac{\partial}{\partial x} \underbrace{\operatorname{tr}(X^T U^{-1} X V)}_{a} - \frac{\partial}{\partial x} \underbrace{\operatorname{tr}(X^T U^{-1} M V)}_{b} - \frac{\partial}{\partial x} \underbrace{\operatorname{tr}(M^T U^{-1} X V)}_{c} + 0 \text{ (no X)}$$

$$a = V \operatorname{tr}\left(\frac{\partial}{\partial x} X^T U^{-1} X\right) = V X^T (U^{-1} + U^{-1^T}) = V X^T 2 U^{-1} \quad \left(\begin{array}{l} U \text{ is positive definite} \\ \operatorname{tr}(AX) = X \ \& \ \frac{d}{dx} \operatorname{tr}(X^T A X) \end{array}\right)$$

$$b = (U^{-1} M V)^T = U^{-1^T} M^T V^T \quad \left(\frac{\partial}{\partial x} \operatorname{tr}(X^T A)\right)$$

$$c = V M^T U^{-1} \quad \left(\frac{\partial}{\partial x} \operatorname{tr}(AXB)\right)$$

$$\frac{df(x)}{dx} = f(x) \cdot -\frac{1}{2} \left(V X^T 2 U^{-1} - U^{-1^T} M^T V^T - V M^T U^{-1}\right)$$