COMP3670/6670: Introduction to Machine Learning

Exercises with a **!** *denote harder ones,* **!!** *denotes very difficult, and* **!!!** *denotes optional challenge exercises.*

This tutorial will be primarily about proofs in analytic geometry. There are far too many exercises to do in the 2 hours, so you should choose some particular ones to work on. Your tutor will present some in class, and feel free to post partial solutions on Ed if you get stuck.

Question 1 **Properties of the zero vector**

Show that for any vector space V with any inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$, we have that $\mathbf{0}$ (the zero vector) is orthogonal to every vector $\mathbf{v} \in V$.

Solution. Let $\mathbf{v}$ be any vector in V

$$\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0 \cdot \mathbf{v}, \mathbf{v} \rangle = 0 \cdot \langle \mathbf{v}, \mathbf{v} \rangle = 0$$

Also, show that for any vector $\mathbf{v} \in V$, that $\{\mathbf{v}, \mathbf{0}\}$ forms a linearly dependant set.

Solution. Note that the equation

$$c_1 \mathbf{v} + c_2 \mathbf{0} = \mathbf{0}$$

has the non-trivial solution $c_1 = 0, c_2 = 1$, so these vectors form a linearly dependant set.

Question 2 **Inner products**

Prove that the standard Euclidean inner product on $\mathbb{R}^2$ given by

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + x_2 y_2$$

is an inner product.

Solution.

1. Symmetry
$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 = y_1 x_1 + y_2 x_2 = \mathbf{y} \cdot \mathbf{x}$$

2. Bilinear
$$\begin{aligned}
(\lambda \mathbf{x} + \phi \mathbf{y}) \cdot \mathbf{z} &= (\lambda \mathbf{x} + \phi \mathbf{y})_1 z_1 + (\lambda \mathbf{x} + \phi \mathbf{y})_2 z_2 \\
&= \left((\lambda \mathbf{x})_1 + (\phi \mathbf{y})_1\right) z_1 + \left((\lambda \mathbf{x})_2 + (\phi \mathbf{y})_2\right) z_2 \\
&= \left(\lambda x_1 + \phi y_1\right) z_1 + \left(\lambda x_2 + \phi y_2\right) z_2 \\
&= \left(\lambda x_1 + \phi y_1\right) z_1 + \left(\lambda x_2 + \phi y_2\right) z_2 \\
&= \lambda x_1 z_1 + \phi y_1 z_1 + \lambda x_2 z_2 + \phi y_2 z_2 \\
&= \lambda \left(x_1 z_1 + x_2 z_2\right) + \phi \left(y_1 z_1 + y_2 z_2\right) \\
&= \lambda(\mathbf{x} \cdot \mathbf{z}) + \phi(\mathbf{y} \cdot \mathbf{z})
\end{aligned}$$
We also need to show the other direction, but it follows immediately, by the property of symmetry.

3. Positive Definite
$$\mathbf{x} \cdot \mathbf{x} = x_1^2 + x_2^2 \geq 0$$
Clearly, $x_1^2 + x_2^2 = 0$ if and only if $x_1 = 0$ and $x_2 = 0$.

Question 3 **Pythagorus**

We have that any inner product induces a norm,

$$\|\mathbf{x}\| := \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

Show that for two orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$ (that is $\langle \mathbf{x}, \mathbf{y} \rangle = 0$) that the following holds

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} + \mathbf{y}\|^2$$

(This is a extension of Pythagorus' Theorem, that for a right angled triangle with hypotenuse of length c, and two other sides of length a and b, that $a^2 + b^2 = c^2$.)

Solution.

$$\|\mathbf{x} + \mathbf{y}\|^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{y} \rangle = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$$

Question 4 **! Parseval's Identity**

Let V be a vector space, together with an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$. Given a set of orthogonal vectors $\{x_1, \ldots, x_n\}$, show that

$$\left\| \sum_{i=1}^{n} x_i \right\|^2 = \sum_{i=1}^{n} \|x_i\|^2$$

Solution.

$$\begin{aligned}
\left\| \sum_{i=1}^{n} x_i \right\|^2 &= \left\langle \sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i \right\rangle \\
&= \left\langle \sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j \right\rangle \\
&= \sum_{i=1}^{n} \left\langle x_i, \sum_{j=1}^{n} x_j \right\rangle \\
&= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \langle x_i, x_j \rangle \right) \\
&= \sum_{i=1}^{n} \left(\langle x_i, x_i \rangle + \sum_{j \neq i} \langle x_i, x_j \rangle \right)
\end{aligned}$$

Note that the property of orthogonality gives us $\langle x_i, x_j \rangle = 0$ if $i \neq j$.

$$= \sum_{i=1}^{n} \langle x_i, x_i \rangle = \sum_{i=1}^{n} \|x_i\|^2$$

Question 5 **Norms**

1. Prove that the Manhatten norm (l_1 norm) on $\mathbb{R}^2$ defined by

   $$\|\mathbf{x}\|_1 := |x_1| + |x_2|$$

   is a norm. (You will need the triangle inequality on $\mathbb{R}$, $|a + b| \leq |a| + |b|$, to help you.)

   Solution. Check the three norm axioms.

(a) Absolutely homogeneous

$$\|\lambda\mathbf{x}\|_1 = |\lambda x_1| + |\lambda x_2| = |\lambda||x_1| + |\lambda||x_2| = \lambda|(|x_1| + |x_2|) = |\lambda||\mathbf{x}|$$

(b) Positive definiteness

$$\|\mathbf{x}\|_1 = |x_1| + |x_2| \geq 0$$

Clearly $|x_1| + |x_2| = 0$ if and only if $x_1 = 0$ and $x_2 = 0$.

(c) Triangle Inequality

$$\|\mathbf{x}+\mathbf{y}\|_1 = |x_1 + y_1|+|x_2 + y_2| \leq |x_1|+|y_1|+|x_2|+|y_2| = (|x_1| + |x_2|)+(|y_1| + |y_2|) = \|\mathbf{x}\|_1+\|\mathbf{y}\|_1$$

2. **!** Prove that the supremum norm (l_∞ norm) on $\mathbb{R}^2$ defined by

$$\|\mathbf{x}\|_\infty := \max\left(|x_1|, |x_2|\right)$$

is a norm. (Hint: You will need triangle inequality on $\mathbb{R}$, and the property that if $A \subseteq B$, then $\max_{x\in A} f(x) \leq \max_{x\in B} f(x)$.)

Solution. Check the three norm axioms.

(a) Absolutely homogeneous

$$\|\lambda\mathbf{x}\|_\infty = \max\left\{|\lambda x_1|, |\lambda x_2|\right\} = \max\left\{|\lambda|\,|x_1|, |\lambda|\,|x_2|\right\} = |\lambda| \max\left\{|x_1|, |x_2|\right\} = |\lambda|\|\mathbf{x}\|_\infty$$

(b) Positive definiteness

$$\|\mathbf{x}\|_\infty = \max\left(|x_1|, |x_2|\right) \geq |x_1| \geq 0$$

Clearly $\max\left(|x_1|, |x_2|\right) = 0$ if and only if the larger of $|x_1|$ and $|x_2|$ is zero, if and only if they are both zero.

(c) Triangle Inequality

$$\begin{aligned}\|\mathbf{x}+\mathbf{y}\|_\infty &= \max\left(|x_1 + y_1|, |x_2 + y_2|\right)\\ &= \max_{j\in\{1,2\}}\left(|x_j + y_j|\right)\\ &\leq \max_{i,j\in\{1,2\}}\left(|x_i + y_j|\right)\end{aligned}$$

since this allows more combinations of different components

$$\leq \max_{i,j\in\{1,2\}}\left(|x_i| + |y_j|\right)$$

by triangle inequality on $\mathbb{R}$

$$= \max_{i\in\{1,2\}}|x_i| + \max_{j\in\{1,2\}}|y_j|$$

as the previous term is maximised by choosing the largest value for $|x_i|$, and then largest value for $|y_j|$, and adding them together

$$= \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty$$

Question 6 **! Basis of a vector space**

Let V be a finite dimensional vector space, and let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for V. Suppose that for any two basis vectors $\mathbf{b}_i$ and $\mathbf{b}_j$, we can compute the inner product $\langle \mathbf{b}_i, \mathbf{b}_j \rangle$. Then, show that for any two vectors $\mathbf{u}, \mathbf{v}$ in V, we can express the inner product $\langle \mathbf{u}, \mathbf{v} \rangle$ in terms of the inner product of basis vectors $\langle \mathbf{b}_i, \mathbf{b}_j \rangle$

(Hint: Use the fact that B spans the space V.)

Solution. We can express $\mathbf{u} = \sum_i u_i \mathbf{b}_i$ and $\mathbf{v} = \sum_i v_i \mathbf{b}_i$ for some collection of constants $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$, as every vector in the space can be written as some linear combination of the basis vectors. Then,

$$\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= \left\langle \sum_i u_i \mathbf{b}_i, \sum_j v_j \mathbf{b}_j \right\rangle \\
&= \sum_i u_i \left\langle \mathbf{b}_i, \sum_j v_j \mathbf{b}_j \right\rangle \\
&= \sum_i u_i \sum_j v_j \langle \mathbf{b}_i, \mathbf{b}_j \rangle \\
&= \sum_{i,j} u_i v_j \langle \mathbf{b}_i, \mathbf{b}_j \rangle
\end{aligned}$$

Question 7 **Orthogonal matrices preserve angles and norms**

Suppose we are in the vector space $\mathbb{R}^n$, together with the standard Euclidean dot product, that is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} := \mathbf{x}^T \mathbf{y}$$

Let

$$\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be an orthogonal matrix (that is, $\mathbf{A}^{-1} = \mathbf{A}^T$.)

Show that for any vector $\mathbf{x} \in \mathbb{R}^n$ that

$$\|\mathbf{A}\mathbf{x}\|_2 = \|\mathbf{x}\|_2$$

Using the above result (or otherwise), show that if the angle between two vectors $\mathbf{x}$ and $\mathbf{y}$ is θ then the angle between $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$ is either θ, or $-\theta$ (modulo 2π).

Solution. To show the norms are the same,

$$\|\mathbf{A}\mathbf{x}\|_2 = \sqrt{(\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x})} = \sqrt{\mathbf{x}^T \left(\mathbf{A}^T \mathbf{A}\right) \mathbf{x}} = \sqrt{\mathbf{x}^T \mathbf{x}} = \|\mathbf{x}\|_2$$

To show the angles are the same, if the angle between $\mathbf{x}$ and $\mathbf{y}$ is θ, then

$$\cos\theta = \frac{\mathbf{x}^T \mathbf{y}}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2}$$

Then the cosine of the angle between $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$ is

$$\frac{(\mathbf{A}\mathbf{x})^T \mathbf{A}\mathbf{y}}{\|\mathbf{A}\mathbf{x}\|_2 \|\mathbf{A}\mathbf{y}\|_2} = \frac{\mathbf{x}^T \mathbf{A}^T \mathbf{A}\mathbf{y}}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2} = \frac{\mathbf{x}^T \mathbf{y}}{\|\mathbf{x}\|_2 \|\mathbf{y}\|_2}$$

which matches the above. If we let ϕ denote the angle between $\mathbf{A}\mathbf{x}$ and $\mathbf{A}\mathbf{y}$, then we have that

$$\cos\theta = \cos\phi$$

Since cos is an even function (i.e. it satisfies the property that $\cos x = \cos(-x)$) and cos is invertible between 0 and π, either $\theta = \phi$ (which satisfies $\cos\theta = \cos\phi$) or $\theta = -\phi$.

Given an example of an orthogonal matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, such that the angle between $\mathbf{x}$ and $\mathbf{y}$ is not the same as the angle between $\mathbf{Ax}$ and $\mathbf{Ay}$.

Solution. We choose $\mathbf{A}$ to be the matrix that flips the plane over the x-axis, namely

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

It can be quickly verified that $\mathbf{A}^{-1} = \mathbf{A}^T$. Then, choose $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It is clear that the angle from $\mathbf{x}$ to $\mathbf{y}$ is $\pi/2$. But the angle from $\mathbf{Ax} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ to $\mathbf{Ay} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is $-\pi/2$.

Question 8 **Rotation matrices preserve norms**

Given a vector $\mathbf{x} \in \mathbb{R}^2$ and the rotation matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Show that for any angle of rotation θ, we have

$$\|\mathbf{x}\|_2 = \|\mathbf{R}(\theta)\mathbf{x}\|_2$$

Solution.

$$\begin{aligned}
\|\mathbf{R}(\theta)\mathbf{x}\|_2^2 &= \left\| \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|_2^2 \\
&= \left\| \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix} \right\|_2^2 \\
&= (x_1\cos\theta - x_2\sin\theta)^2 + (x_1\sin\theta + x_2\cos\theta)^2 \\
&= x_1^2\cos^2\theta - 2x_1x_2\cos\theta\sin\theta + x_2^2\sin^2\theta + x_1^2\sin^2\theta + 2x_1x_2\sin\theta\cos\theta + x_2^2\cos^2\theta \\
&= x_1^2\left(\cos^2\theta + \sin^2\theta\right) + x_2^2\left(\cos^2\theta + \sin^2\theta\right) \\
&= x_1^2 + x_2^2 = \|\mathbf{x}\|_2^2
\end{aligned}$$

Square rooting both sides gives the result.

Question 9 **Gram-Schmidt**

Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the standard basis vectors for $\mathbb{R}^2$. Let $\mathbf{v}$ be any vector in $\mathbb{R}^2$.

Define the projection operator

$$\text{proj}_{\mathbf{u}}(\mathbf{v}) := \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle}\mathbf{u}$$

(if $\mathbf{u} = \mathbf{0}$, then we define $\text{proj}_{\mathbf{0}}(\mathbf{v}) = \mathbf{0}$.

The Gram-Schmidt algorithm takes a set of vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and proceeds as follows:

$$\begin{aligned}
&\mathbf{u}_1 = \mathbf{v}_1 \\
&\mathbf{u}_2 = \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) \\
&\mathbf{u}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_3) - \text{proj}_{\mathbf{u}_2}(\mathbf{v}_3) \\
&\dots = \dots \\
&\mathbf{u}_n = \mathbf{v}_n - \textstyle\sum_{j=1}^{n-1} \text{proj}_{\mathbf{u}_j}(\mathbf{v}_n)
\end{aligned}$$

The output $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ is a set of orthonormal vectors that spans the same set as $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ (If the dimension of the space spanned by the $\mathbf{v}_i$'s is less than n, then some of the $\mathbf{u}_i$'s will be zero.)

Suppose we are considering vectors in the vector space of $\mathbb{R}^2$.
Show that if we input $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{v}\}$ to the Gram-Schmidt algorithm, the output is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{0}\}$

Solution.

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{v}_1 = \mathbf{e}_1 \\
\mathbf{u}_2 &= \mathbf{v}_2 - \text{proj}_{\mathbf{u}_1}(\mathbf{v}_2) \\
&= \mathbf{e}_2 - \text{proj}_{\mathbf{e}_1}(\mathbf{e}_2) \\
&= \mathbf{e}_2 - \frac{\langle \mathbf{e}_2, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle}\mathbf{e}_1 \\
&= \mathbf{e}_2 \\
\text{As } \langle \mathbf{e}_1, \mathbf{e}_2 \rangle &= \mathbf{0} \\
\mathbf{u}_3 &= \mathbf{v}_3 - \frac{\langle \mathbf{v}_3, \mathbf{u}_1 \rangle}{\langle \mathbf{u}_1, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{v}_3, \mathbf{u}_2 \rangle}{\langle \mathbf{u}_2, \mathbf{u}_2 \rangle}\mathbf{u}_2 \\
&= \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{e}_1 \rangle}{\langle \mathbf{e}_1, \mathbf{e}_1 \rangle}\mathbf{e}_1 - \frac{\langle \mathbf{v}, \mathbf{e}_2 \rangle}{\langle \mathbf{e}_2, \mathbf{e}_2 \rangle}\mathbf{e}_2 \\
&= \mathbf{v} - \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 - \langle \mathbf{v}, \mathbf{e}_2 \rangle \mathbf{e}_2
\end{aligned}$$

As $\mathbf{e}_1$ and $\mathbf{e}_2$ are the standard basis vectors for $\mathbb{R}^2$, we have $\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 1$.
Note that we can decompose the vector $\mathbf{v}$ as $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$, and we have that $\langle \mathbf{v}, \mathbf{e}_1 \rangle = v_1$ and $\langle \mathbf{v}, \mathbf{e}_2 \rangle = v_2$.

$$\begin{aligned}
&= v_1\mathbf{e}_1 + v_2\mathbf{e}_2 - \langle \mathbf{v}, \mathbf{e}_1 \rangle \mathbf{e}_1 - \langle \mathbf{v}, \mathbf{e}_2 \rangle \mathbf{e}_2 \\
&= (v_1 - \langle \mathbf{v}, \mathbf{e}_1 \rangle)\mathbf{e}_1 + (v_2 - \langle \mathbf{v}, \mathbf{e}_2 \rangle)\mathbf{e}_2 \\
&= (v_1 - v_1)\mathbf{e}_1 + (v_2 - v_2)\mathbf{e}_2 \\
&= \mathbf{0}
\end{aligned}$$

Hence,

$$\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{0}\}$$

as required.

Question 10 **!!! Cauchy-Schwartz**

Prove the Cauchy-Schwartz inequality for a general inner product and corresponding induced norm:

$$\langle \mathbf{u}, \mathbf{v} \rangle \leq \|\mathbf{u}\|\|\mathbf{v}\|$$

(Hint: Let $\mathbf{z} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$, and start with the fact that $\langle \mathbf{z}, \mathbf{z} \rangle \geq 0$.)

Solution. As the hint suggests, let $\mathbf{z} = \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$.

$$\begin{aligned}
\langle \mathbf{z}, \mathbf{z} \rangle &\geq 0 \\
\left\langle \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}, \mathbf{u} - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} \right\rangle &\geq 0 \\
\langle \mathbf{u}, \mathbf{u} \rangle - \left\langle \mathbf{u}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} \right\rangle - \left\langle \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}, \mathbf{u} \right\rangle + \left\langle \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}, \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v} \right\rangle &\geq 0 \\
\|\mathbf{u}\|^2 - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\langle \mathbf{u}, \mathbf{v} \rangle - \frac{\langle \mathbf{u}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\langle \mathbf{v}, \mathbf{u} \rangle + \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle^2}\langle \mathbf{v}, \mathbf{v} \rangle &\geq 0 \\
\|\mathbf{u}\|^2 - 2\frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle} + \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle^2}\langle \mathbf{v}, \mathbf{v} \rangle &\geq 0
\end{aligned}$$

as $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$

$$\begin{aligned}
\|\mathbf{u}\|^2 - 2\frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle} + \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\langle \mathbf{v}, \mathbf{v} \rangle} &\geq 0 \\
\|\mathbf{u}\|^2 - \frac{\langle \mathbf{u}, \mathbf{v} \rangle^2}{\|\mathbf{v}\|^2} &\geq 0 \\
\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \langle \mathbf{u}, \mathbf{v} \rangle^2 &\geq 0 \\
\langle \mathbf{u}, \mathbf{v} \rangle &\leq \|\mathbf{u}\| \|\mathbf{v}\|
\end{aligned}$$

as required.