# Properties of Matrices

1.

(a) Let $A = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{pmatrix}$. i) Show that A is symmetric.

A square matrix is symmetric if $A = A^T$ or row $i$ = col $j$ $\forall\, i=j$. We can see that row 1 = col 1 and so on.

ii) Determine if it is positive definite.

Calculate eigenvalues. $P_A(\lambda) = \det(A - \lambda I) = 0$

$$P_A(\lambda) = \det\left(\begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right) = \begin{vmatrix} 2-\lambda & 3 & 4 \\ 3 & 5-\lambda & 6 \\ 4 & 6 & 8-\lambda \end{vmatrix}$$

$$\det(A^{3\times 3}) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$P_A(\lambda) = 2-\lambda((5-\lambda)(8-\lambda)-36) - 3(3(8-\lambda)-24) + 4(18-4(5-\lambda))$$

$$= 2-\lambda(40-13\lambda+\lambda^2-36) - 3(24-3\lambda-24) + 4(18-20+4\lambda)$$

$$= (2-\lambda)(4-13\lambda+\lambda^2) + 9\lambda + 16\lambda - 8$$

$$= -26\lambda + 2\lambda^2 - 4\lambda + 13\lambda^2 - \lambda^3 + 9\lambda + 16\lambda - 8$$

$$= -5\lambda + 15\lambda^2 - \lambda^3 = 0 \Rightarrow \lambda(\lambda^2 - 15\lambda + 5) = 0$$

$$= \lambda_1 = 0, \left(\lambda_2 = \frac{15 - \sqrt{15^2 - 4\cdot 1\cdot 5}}{2}, \lambda_3 = \frac{15 + \sqrt{15^2 - 4\cdot 1\cdot 5}}{2}\right)$$

$= \lambda_1 = 0$, $\lambda_2 = 0{,}35$, $\lambda_3 = 14{,}65$. Since one of the eigenvalues is 0, A is positive semi-definite.

b) For matrix A, i) compute $A^3$.

$$\begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 29 & 45 & 58 \\ 45 & 70 & 90 \\ 58 & 90 & 116 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 425 & 660 & 850 \\ 660 & 1025 & 1320 \\ 850 & 1320 & 1700 \end{bmatrix}$$

ii) Show $A^3$ is also symmetric. $A^{3^T} = A^3$ $\therefore$ symmetric.

c) A is a $\square$ matrix & $f(x)$ is an n-th order polynomial $\sum_{i=0}^{n} a_i X^i$ where $a_i$ are random real numbers. Show $f(A)A = A f(A)$.

$$f(A)A = \sum_{i=0}^{n} a_i A^i \cdot A = A \cdot \sum_{i=0}^{n} a_i A^i = A f(A).$$

also distributing A over the sum results in $\sum_{i=0}^{n} a_i A^{i+1}$ for both.

d) There are matrices $A^{n \times k}$ and $B^{r \times s}$. Matrix of order $nr \times ks$ in block form:

$\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1k}B \\ a_{21}B & a_{22}B & \cdots & a_{2k}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nk}B \end{bmatrix}$ is called the Kronecker product $A \otimes B$ of the matrices A and B. Find the conditions on a, b, c, d, e, f, g, h such that $X \otimes Y = Y \otimes X$ where X and $Y \in \mathbb{R}^{2\times 2}$ defined as:

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $Y = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$. $A \otimes B$ is essentially $\begin{bmatrix} a_{ij}B & a_{ij}B \\ a_{ij}B & a_{ij}B \end{bmatrix}$ for $A \& B \in \mathbb{R}^{2\times 2}$.

$$X \otimes Y = \begin{bmatrix} a \cdot Y & b \cdot Y \\ c \cdot Y & d \cdot Y \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} ae & af \\ ag & ah \end{bmatrix} & \begin{bmatrix} be & bf \\ bg & bh \end{bmatrix} \\ \begin{bmatrix} ce & cf \\ cg & ch \end{bmatrix} & \begin{bmatrix} de & df \\ dg & dh \end{bmatrix} \end{bmatrix}$$

$$Y \otimes X = \begin{bmatrix} e \cdot X & f \cdot X \\ g \cdot X & h \cdot X \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} ea & eb \\ ec & ed \end{bmatrix} & \begin{bmatrix} fa & fb \\ fc & fd \end{bmatrix} \\ \begin{bmatrix} ga & gb \\ gc & gd \end{bmatrix} & \begin{bmatrix} ha & hb \\ hc & hd \end{bmatrix} \end{bmatrix}$$

we then have:

| | | |
|---|---|---|
| $ae = ea$ | $be = fa$ | $\therefore$ |
| $af = eb$ | $bf = fb$ | $a = e$ |
| $ag = ec$ | $bg = fc$ | $b = f$ |
| $ah = ed$ | $bh = fd$ | $c = g$ |
| $ce = ga$ | $de = ha$ | $d = h$ |
| $cf = gb$ | $df = hb$ | |
| $cg = gc$ | $dg = hgc$ | |
| $ch = gd$ | $dh = hd$ | |

# Solving Linear Systems

Find the set S of all solutions x of $Ax = b$.

(a) $A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & -2 & 6 \\ 3 & 6 & -3 & 9 \\ 4 & 8 & -4 & 11 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}$ $\quad [A|b] = \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 2 & 4 & -2 & 6 & 2 \\ 3 & 6 & -3 & 9 & 3 \\ 4 & 8 & -4 & 11 & 5 \end{array}\right]$

$$\xrightarrow{R_2 : R_2 - 2R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 6 & -3 & 9 & 3 \\ 4 & 8 & -4 & 11 & 5 \end{array}\right] \xrightarrow{R_3 : R_3 - 3R_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 8 & -4 & 11 & 5 \end{array}\right] \xrightarrow{R_4 : R_4 - 4R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{array}\right] ; \quad \begin{array}{r} x_1 + 2x_2 - x_3 + 3x_4 = 1 \\ -x_4 = 1 \end{array} \quad \begin{array}{|l} x_1 = 1 - 2x_2 + x_3 - 3x_4 \\ x_4 = -1 \end{array}$$

Since $x_4 = -1$, we get for $x_1$: $x_1 = 1 - 2x_2 + x_3 - 3 \cdot -1$

Therefore we have

$$S = \begin{cases} x_1 = 4 - 2x_2 + x_3 \\ x_2 = x_2 \\ x_3 = x_3 \\ x_4 = -1 \end{cases}$$

b) $A = \begin{bmatrix} 1 & -1 & 2 & 3 & -1 \\ 2 & -2 & 4 & 6 & -2 \\ 3 & -3 & 6 & 9 & -3 \\ 4 & -4 & 8 & 11 & -4 \\ 5 & -5 & 10 & 14 & -5 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$

RREF steps:

$R_2 = R_2 - 2R_1$
$R_3 = R_3 - 3R_1$
$R_4 = R_4 - 4R_1$
$R_5 = R_5 - 5R_1$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -1 & 2 & 3 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{r} x_1 - x_2 + 2x_3 + 3x_4 - x_5 = 1 \\ -x_4 = 0 \end{array}$$

$$x_1 = 1 + x_2 - 2x_3 - 3 \cdot (0) + x_5$$
$$= 1 + x_2 - 2x_3 + x_5$$

$$S = \begin{cases} x_1 = 1 + x_2 - 2x_3 + x_5 \\ x_2 = x_2 \\ x_3 = x_3 \\ x_4 = 0 \\ x_5 = x_5 \end{cases} \quad \text{where } x_2, x_3, x_5 \in \mathbb{R}$$

# Inverses and Rank

Let $A \in \mathbb{M}_{n\times n}(\mathbb{R})$ be a matrix such that $A^2 = A$

(a) Show that if $A$ is invertible, then $A = I$.

$A$ is invertible so $A^{-1}$ exists.

left multiplication: $A^{-1}(A \cdot A) = A^{-1} \cdot A \Rightarrow IA = I \Rightarrow A = I$

right multiplication: $(A \cdot A) \cdot A^{-1} = A \cdot A^{-1} \Rightarrow AI = I \Rightarrow A = I$

(b) Show that $rk(A) = tr(A)$. The trace of $A$ is defined as the sum of the diagonal elements of $A$, i.e. $tr(A) = \sum_{i=1}^{n} a_{ii}$

Theorem 4.17 (from book) states $tr(A) = \sum_{i=1}^{n} \lambda_i$ (sum of eigenvalues)

Let $\lambda$ be an eigenvalue of $A$. If we have a corresponding eigenvector $x \in \mathbb{R}^n \setminus \{0\}$, then: $Ax = \lambda x$. given $A^2 = A \Rightarrow$ $A^2 x = \lambda^2 x$. But since $A^2 = A$, we also get $Ax = \lambda^2 x$ which is also $\lambda x = \lambda^2 x$ $= \lambda^2 x - \lambda x = 0$ thus $\lambda(\lambda - 1)$ since $x \neq \vec{0}$

$\therefore$ we have $\lambda = 0$ or $\lambda = 1$.

Given theorem 4.17, the trace of $A$ is a sum of 0's and 1's or just 1's. And the rank of $A$ is the number of non-zero eigenvalues, i.e. the number of ones.

Given that $A$ is also equal to $I$, it is easy to see $rk(A) = tr(A)$

$$I^{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where rank $= 3$ (3 non-zero rows)

where trace $= 3$ $(1+1+1 = 3)$.

# Subspaces

(a) Which of the following sets are subspaces of $\mathbb{R}^n$?

(i) $E = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 + \cdots + x_n^2 \leq 1\}$

Need to show $E \neq \emptyset$. E is closed under addition and E is closed under scalar multiplication.

E contains the zero vector $(0, 0, \ldots, 0)$ and $0^2 + 0^2 \ldots 0^2 \leq 1$.

So first condition is satisfied.

Consider $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n) \in E$.

$x + y = ((x_1+y_1)^2 + (x_2+y_2)^2 + \cdots + (x_n+y_n)^2)$

$= x_1^2 + 2x_1y_1 + y_1^2 + x_2^2 + 2x_2y_2 + y_2^2 + \cdots + x_n^2 + 2x_ny_n + y_n^2$

$= (x_1^2 + x_2^2 + \cdots + x_n^2) + (y_1^2 + y_2^2 + \cdots + y_n^2) + 2(x_1y_1 + \cdots + x_ny_n)$

$\because$ The last portion of the equation could yield a sum $\geq 2$.

$\therefore$ E is not closed under addition

$\therefore$ the set E is not a subspace of $\mathbb{R}^n$.

(ii) $F = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 x_2 \cdots x_n = 0\}$

Need to show $F \neq \emptyset$. F is closed under addition &

F is closed under scalar multiplication.

F contains the zero vector $(0, 0, \dots, 0)$ so the first condition is satisfied as $(0 \cdot 0 \cdot 0 \cdot \ldots \cdot 0) = 0$.

Show that F is closed under addition:

let $x = (x_1, x_2, \dots, x_n)$ & $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$

consider $x+y = ((x_1+y_1)(x_2+y_2) \cdots (x_n y_n))$

$$= x_1 x_2 + x_1 y_2 + y_1 x_2 + y_1 y_2 + \cdots = 0$$

However this is not always true since $x, y \in \mathbb{R}^n$, so they could take values $> 0$ or $< 0$. Thus $x+y \neq 0 \ \forall x, y \in \mathbb{R}^n$.

Show F is closed under scalar multiplication:

Suppose there is a scalar constant $c = 0$.

$c \cdot (x_1 x_2 \cdots x_n) = 0 \ \forall x_1, x_2 \cdots x_n \in \mathbb{R}^n \therefore$ F is closed under scalar multiplication.

But since F is not closed under addition,

F is not a subspace of $\mathbb{R}^n$.

(b) V is an inner product space & U and W are subspaces of V. Define $U+W = \{u+w : u \in U, w \in W\}$. Show that $(U+W)^\perp = U^\perp \cap W^\perp$. $\perp$ = orthogonal complement.

Using the definition of orthogonal complements, $(U+W)^\perp$ is a $\subseteq$ of V and contains all vectors $v \in V$ that are $\perp$ to every vector $u+w$, $u \in U$ and $w \in W$ such that $\langle v, u+w \rangle = 0$.

To prove $(U+W)^\perp = U^\perp \cap W^\perp$ show:

* $U^\perp \cap W^\perp \subseteq (U+W)^\perp$ and $(U+W)^\perp \subseteq U^\perp \cap W^\perp$.

let $y \in (U+W)^\perp$ and show $y \in U^\perp \cap W^\perp$.

we know $v \in U+W$ and by definition $v = u+w$

Then $\langle y, v \rangle = \langle y, u+w \rangle = \langle y, u \rangle + \langle y, w \rangle$

$= 0 + 0 = 0$. so $y \perp v$ for $v \in U+W \therefore y \in (U+W)^\perp$

and $y \in U^\perp \cap W^\perp$.

Therefore $(U+W)^\perp = U^\perp \cap W^\perp$.

## Linear Transformations and Injectivity

let $V$ and $W$ be vector spaces, and let $T: V \rightarrow W$ be a linear transformation

The image of $T$ is defined as:

$$Im(T) = \{w \in W \mid \exists v \in V \text{ such that } w = T(v)\}.$$

The kernel of $T$ is defined as:

$$Ker(T) = \{v \in V \mid T(v) = 0\}.$$

We say that $T$ is injective if $\forall\, u,v \in V$, $T(u) = T(v) \Rightarrow u = v$.

(a) Prove that $T$ is injective iff $Ker(T) = \{0\}$.

if $v \in Ker(T)$ and $T(v) = 0$ then $v$ must be $0$. So $0$ is the only vector in $Ker(T)$ $\therefore Ker(T) = \{0\}$.

Furthermore if $\exists\, u,v \in V \mid T(u) = T(v)$ then $u$ must also be $0$ as $v$ is $0$. so $u \in ker(T)$.

So for all $u,v \in V$, $T(u) = T(v)$ making $T$ injective only when $Ker(T) = \{0\}$.

(b) Consider the transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by $T(x) = Ax$, where

$$A = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ c & 0 & 1 \end{bmatrix}$$

with $a,b,c \in \mathbb{R}$.

Find the condition on $a,b$ and $c$ for which this transformation is injective.

$T(x) = Ax$. For $T$ to be injective, $Ker(T) = \{0\}$. Since $Ker(T) = \{x \in \mathbb{R}^3 \mid T(x) = 0\}$ then $Ax = 0$.

RREF A:

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ c & 0 & 1 \end{bmatrix} \xrightarrow{R_3 : R_3 - cR_1} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & -ca & 1 \end{bmatrix} \xrightarrow{R_3 : R_3 + caR_2} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1+cab \end{bmatrix}$$

For $T$ to be injective, $1+cab \neq 0$ meaning $cab \neq -1$

## Linear Transformations and Inner Products

let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation defined by the matrix $A = [[1,2],[2,3]]$.

(a) Show that $T$ is not an orthogonal transformation.

Show $AA^T = I = A^TA$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

since $A^TA \neq I$, $T$ is not orthogonal transformation

(b) Consider the inner product defined by the matrix $D = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Show that this inner product is preserved under $T$, i.e., $\forall x,y \in \mathbb{R}^2$, $(Tx)^T D (Ty) = x^T D y$.

if $T$ preserves the inner product then $\langle Tx, Ty \rangle = \langle x, y \rangle$

$T(x)$ is also $Ax$ $\therefore \langle Tx, Ty \rangle = \langle Ax, Ay \rangle = (Ax)^T Ay$ $= x^T A^T A y$. But $A^TA \neq I$ and $A^TA \neq D$ in $x^T D y$.

so $(Tx)^T D y T$ can't be equal to $x^T D y$.

(c) Let $u = [1,1]^T$ and $v = [1,-1]^T$.

i) Compute the angle b/w $u$ and $v$ under the inner product $D$.

$$u^T D v = [1,1]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = [3,3]\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$$

$$\omega = \cos^{-1}\left(\frac{\langle u,v\rangle}{\|u\|\,\|v\|}\right) = \cos^{-1}\left(\frac{u^T D v}{\sqrt{u^T D u \; v^T D v}}\right) = \cos^{-1}\left(\frac{0}{\|u\|\,\|v\|}\right)$$

$$\omega = \cos^{-1}(0) = 90^\circ \text{ or } \frac{\pi}{2}$$

ii) Show that this angle is preserved under T.

$$\cos\omega = \frac{\langle Au, Av\rangle}{\|Au\|\,\|Av\|} = \frac{u^T A^T D A v}{\sqrt{u^T A^T D A u \; v^T A^T D A v}}$$

For angle to be preserved under the transformation, $A^T D A$ must $= D$.

$$A^T D A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 23 \\ 23 & 38 \end{bmatrix} \neq D$$

$\therefore$ the transformation T does not preserve the angle under the inner product D.