COMP3670 Assignment 1
Properties of Matrices

Exercise 1.

(a) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ be a square matrix. Show that A is symmetric.

Solution: A matrix is symmetric iff it is equal to its transpose. Which means

$A_{ij} = A_{ji}$ for all i and j.

For matrix A it can be seen that row 1 is equal to column 1, row 2 is equal to column 2 and so on. ∴ A is symmetric.

(b) Compute A^2 and show its symmetric.

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1\times1+2\times2+3\times3 & 1\times2+2\times4+3\times5 & 1\times3+2\times5+3\times6 \\ 2\times1+4\times2+5\times3 & 2\times2+4\times4+5\times5 & 2\times3+4\times5+5\times6 \\ 3\times1+5\times2+6\times3 & 3\times2+5\times4+6\times5 & 3\times3+5\times5+6\times6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

Once iff $A^2 = (A^2)^T$ then the matrix is symmetric, which in this case is true as well, as $A^2_i = A^2_j$ for $i=j$.

(c) Is it true for any symmetric matrix A, A^2 is also symmetric?

Solution: As we know A is symmetric iff $A = A^T$. let's consider the square of A.

$A^2 = A \times A$. Since A is symmetric, $A = A^T$.

$A^2 = A^T \times A^T$

$A^2 = (A \times A)^T \longrightarrow A^2 = (A^2)^T$

$\therefore$ For any symmetric matrix A, A^2 is also symmetric.

(d) A is a square matrix & $f(x)$ and $g(x)$ are n-th order polynomials $\sum_{i=0}^{n} a_i x^i$. Show $f(A)$ and $g(A)$ commute, i.e, $f(A)g(A) = g(A)f(A)$ for abritrary order n.

Solution: Subbing in matrix A into the polynomials we get

$f(A) = \sum_{i=0}^{n} a_i A^i \quad g(A) = \sum_{i=0}^{n} b_i A^i$

$$f(A)g(A) = \left(\sum_{i=0}^{n} a_i A^i\right)\left(\sum_{i=0}^{n} b_i A^i\right)$$

$$= \sum_{i=0}^{n} a_i b_i A^i A^i$$

$$= \sum_{i=0}^{n} a_i b_i A^{i+i}$$

$$g(A)f(A) = \left(\sum_{i=0}^{n} b_i A^i\right)\left(\sum_{i=0}^{n} a_i A^i\right)$$

$$= \sum_{i=0}^{n} b_i a_i A^i A^i$$

$$= \sum_{i=0}^{n} b_i a_i A^{i+i}$$

$\therefore f(A)g(A) = g(A)f(A)$.

(e) let $X = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$. X is a magic square. Is $X \otimes X$ also a magic square?

Magic constant $(M) = 15$.

$$X \otimes X = \begin{bmatrix} 8 \cdot X & 1 \cdot X & 6 \cdot X \\ 3 \cdot X & 5 \cdot X & 7 \cdot X \\ 4 \cdot X & 9 \cdot X & 2 \cdot X \end{bmatrix}$$

$$= \begin{bmatrix} 64 & 8 & 48 & 8 & 1 & 6 & 48 & 6 & 36 \\ 24 & 40 & 56 & 3 & 5 & 7 & 18 & 30 & 42 \\ 32 & 72 & 16 & 4 & 9 & 2 & 24 & 54 & 12 \\ 24 & 3 & 18 & 40 & 5 & 30 & 56 & 7 & 42 \\ 9 & 15 & 21 & 15 & 25 & 35 & 21 & 35 & 49 \\ 12 & 27 & 6 & 20 & 45 & 10 & 28 & 63 & 14 \\ 32 & 4 & 24 & 72 & 9 & 54 & 16 & 2 & 12 \\ 12 & 20 & 28 & 27 & 45 & 63 & 6 & 10 & 14 \\ 16 & 36 & 8 & 36 & 81 & 18 & 8 & 18 & 4 \end{bmatrix}$$

The sum of rows, columns & diagonals for $X \otimes X$ is 225 which is 15^2.

$\therefore$ $X \otimes X$ is also a magic square where M (magic constant) is M^2 of X's M.

(f) Is $X \otimes X$ a magic square for any magic square X? Show example w/ $n=2$.

let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ where M (magic constant) $= 5$

so $A \otimes A$ magic constant must be $M^2 = 25$.

$$A \otimes A = \begin{bmatrix} 2 \cdot A & 3 \cdot A \\ 1 \cdot A & 4 \cdot A \end{bmatrix} = \begin{bmatrix} 4 & 6 & 6 & 9 \\ 2 & 8 & 3 & 12 \\ 2 & 3 & 8 & 12 \\ 1 & 4 & 4 & 16 \end{bmatrix}$$

The sum of rows, cols & diagonals $= 25$ $\therefore$ $A \otimes A$ is also a magic square.

(g) $x, y \in \mathbb{R}^2$. What conditions should x, y satisfy such that $x \otimes y \otimes x = y \otimes x \otimes y$. $x = [x_1, x_2]$ & $y = [y_1, y_2]$

$$x \otimes y \otimes x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1^2 y_1 \\ x_1 x_2 y_1 \\ y_2 x_1^2 \\ x_1 x_2 y_2 \\ x_1 x_2 y_1 \\ y_1 x_2^2 \\ x_1 x_2 y_2 \\ y_2 x_2^2 \end{bmatrix}$$

$$y \otimes x \otimes y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \otimes \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 y_1^2 \\ y_1 y_2 x_1 \\ x_2 y_1^2 \\ y_1 y_2 x_2 \\ y_1 y_2 x_1 \\ x_1 y_2^2 \\ y_1 y_2 x_2 \\ x_2 y_2^2 \end{bmatrix}$$

$x_1^2 y_1 = x_1 y_1^2$
$x_1 x_2 y_1 = y_1 y_2 x_1$
$y_2 x_1^2 = x_2 y_1^2$
$x_1 x_2 y_2 = y_1 y_2 x_2$
$x_1 x_2 y_1 = y_1 y_2 x_1$
$y_1 x_2^2 = x_1 y_2^2$
$x_1 x_2 y_2 = y_1 y_2 x_2$
$y_2 x_2^2 = x_2 y_2^2$

Solving the systems of equations we get

$x_1 = y_1$
$x_2 = y_2$ $\Rightarrow$

$x_1 = y_1$
$x_2 = y_2$ $\Rightarrow$

$x_1 = y_1$
$x_2 = y_2$

so x, y must satisfy $x = y$ such that $x \otimes y \otimes x = y \otimes x \otimes y$.

Exercise 2 Solving Linear Systems

(a) Find the set S of all solutions x of the following system $Ax = b$.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 6 & 2 & -5 \\ 2 & 1 & -1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ -2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 6 & 2 & -5 & -2 \\ 2 & 1 & -1 & 3 \\ 1 & 0 & 1 & 5 \end{array}\right] \xrightarrow{2R_3 - R_2} \left[\begin{array}{ccc|c} 6 & 2 & -5 & -2 \\ 2 & 1 & -1 & 3 \\ 0 & -1 & 3 & 7 \end{array}\right]$$

$$\xrightarrow{3R_2 - R_1} \left[\begin{array}{ccc|c} 6 & 2 & -5 & -2 \\ 0 & 1 & 2 & 11 \\ 0 & -1 & 3 & 7 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 6 & 2 & -5 & -2 \\ 0 & 1 & 2 & 11 \\ 0 & 0 & 5 & 18 \end{array}\right]$$

$$6x_1 + 2x_2 - 5x_3 = -2 \qquad x_3 = 3.6$$
$$x_2 + 2x_3 = 11 \qquad x_2 = 3.8$$
$$5x_3 = 18 \qquad x_1 = 1.4$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 + 5x_3 - 2x_2 / 6 \\ 11 - 2x_3 \\ 18/5 \end{bmatrix} = \begin{bmatrix} 1.4 \\ 3.8 \\ 3.6 \end{bmatrix}$$

(b) $\left[\begin{array}{ccccc|c} 4 & 3 & 2 & 2 & -2 & 5 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 3 & 2 & 1 & 1 & -3 & -2 \\ -1 & 0 & 1 & 1 & 1 & 16 \end{array}\right] \xrightarrow[R_3]{3R_4 +} \left[\begin{array}{ccccc|c} 4 & 3 & 2 & 2 & -2 & 5 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 3 & 2 & 1 & 1 & -3 & -2 \\ 0 & 2 & 4 & 4 & 0 & 46 \end{array}\right]$

$\xrightarrow[3R_1]{4R_3 -} \left[\begin{array}{ccccc|c} 4 & 3 & 2 & 2 & -2 & 5 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & -1 & -2 & -2 & -6 & -23 \\ 0 & 2 & 4 & 4 & 0 & 46 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccccc|c} 4 & 3 & 2 & 2 & -2 & 5 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 4 & 4 & 0 & 46 \end{array}\right] \xrightarrow{R_4-2R_2}$

$\left[\begin{array}{ccccc|c} 4 & 3 & 2 & 2 & -2 & 5 \\ 0 & 1 & 2 & 2 & 6 & 23 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -12 & 0 \end{array}\right]$

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 + 2x_4 - 2x_5 &= 5 \\ x_2 + 2x_3 + 2x_4 + 6x_5 &= 23 \\ x_3 &= x_3 \text{ free} \\ x_4 &= x_4 \text{ free} \\ x_5 &= 0 \end{aligned}$$

$x_1 = \frac{1}{4}(5 - 3x_2 - 2x_3 - 2x_4)$ $\quad x_2 = 23 - 2x_3 - 2x_4$

$$\begin{aligned} x_1 &= \frac{1}{4}(5 - 3(23 - 2x_3 - 2x_4) - 2x_3 - 2x_4) \\ &= \frac{1}{4}(5 - 69 + 6x_3 + 6x_4 - 2x_3 - 2x_4) \\ &= \frac{1}{4}(-64 + 4x_3 + 4x_4) \\ &= -16 + x_3 + x_4 \end{aligned}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -16 + x_3 + x_4 \\ 23 - 2x_3 - 2x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = \begin{bmatrix} -16 \\ 23 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Exercise 3 Inverses and Rank.

(a) Let A be invertible matrix. Show $(A^{-1})^T = (A^T)^{-1}$.

definition of inverse: $AA^{-1} = I$

transpose both sides: $(AA^{-1})^T = I^T$

transpose of product property: $A^T (A^{-1})^T = I$

we can see that A^T is the inverse of $(A^{-1})^T$ because $A^T(A^{-1})^T$ equals the identity matrix, which means:

$$A^T = ((A^{-1})^T)^{-1}$$

so if we invert both sides we get:

$$(A^T)^{-1} = (A^{-1})^T.$$

or we could have said $(A^{-1})^T$ is the inverse of A^T. Then:

$$(A^{-1})^T = (A^T)^{-1}.$$

which also proves.

(b) Find the values of $[a, b, c]^T \in \mathbb{R}^3$ for which the inverse of the matrix exists.

$$A = \begin{bmatrix} 1 & 1 & b \\ 1 & a & c \\ 1 & 1 & 1 \end{bmatrix}$$

$[A \mid I] \rightsquigarrow [I \mid A^{-1}]$

$$\cancel{\left[\begin{array}{ccc|ccc} 1 & 1 & b & 1 & 0 & 0 \\ 1 & a & c & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]}$$

If determinant of a matrix is $\neq$ zero, then inverse exists.

so if $\det(A) \neq 0$, then the inverse exists.

$$|A| = 1(a-c) - 1(1-c) + b(1-a) \neq 0.$$
$$= a - 1 + b - ab \neq 0$$

continue next page.

Now we have to find for what values of a and b, the equation is zero

$$a-1+b-ab = 0$$
$$-ab+a+b-1 = 0$$
$$-ab+a+b = 1$$
$$a(-b+1)+b = 1$$
$$b = 1-a(-b+1)$$

we can substitute b now into the equation 4.

$$a(-b+1)+1-a(-b+1) = 1$$
$$-ab+a+1-ab+a = 1$$
$$-2ab+2a+1 = 1$$

$$-2ab+2a = 0$$
$$\div 2a \qquad \div 2a$$
$$-b+1 = 0$$
$$-b = -1 \quad \therefore b = 1.$$

The above equation is also true for when $a=1$. Which means for all real values a, b if $a=1$ and/or $b=1$, the determinant is zero and the inverse does not exist. For the inverse to exist a and b cannot be 1.

$$\forall (a,b,c) \in \mathbb{R}^3 \setminus \{(1,1,c) \mid c \in \mathbb{R}\}.$$

(c) A is an arbitrary matrix in $M_{m \times n}(\mathbb{R})$, where m & n denote the number of rows and columns of A, respectively. Prove that $rk(A) = rk(A)^T$.

Assume the rank of A is r, so there are r number of linearly independent rows in A. If we transpose A into A^T, then the r linearly independent rows become r number of linearly independent columns in A^T. And since rank is determined by the numbers of non-zero rows and columns both, then A^T also has a rank of r. Therefore the rank of any matrix is equal to the rank of its transpose.

Subspaces

Exercise 4

(a) Which of the following sets are subspaces of $\mathbb{R}^n$?

(i) $A = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0\}$

Need to show: $A \neq \emptyset$, A is closed under addition, A is closed under scalar multiplication.

A contains the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ so $A \neq \emptyset$.

consider the vectors $[x_1, x_2, x_3]$ & $[x_4, x_5, x_6] \in A$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 + x_4 \\ x_2 + x_5 \\ x_3 + x_6 \end{bmatrix}$$ where $(x_1 + x_4) \geq 0$, $(x_2 + x_5) \geq 0$, $(x_3 + x_6) \geq 0$

So A is closed under addition.

Suppose $\forall a \in \mathbb{R}\ \exists -a$ and $[x_1, x_2, x_3] \in A$

$$-a \times \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -ax_1 \\ -ax_2 \\ -ax_3 \end{bmatrix} \notin A$$ because $-ax_1, -ax_2, -ax_3 \not\geq 0$

So A is not closed under scalar multiplication.

Hence A is not a subspace of $\mathbb{R}^n$.

(ii) $B = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \text{at least one } x_i \text{ is irrational}\}$

Suppose the vector $x = (\sqrt{2}, 0, \dots, 0)$. The vector x belongs to B because it has one irrational component.

Check closure under addition.

$$\begin{pmatrix} \sqrt{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \sqrt{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

since $2\sqrt{2}$ is also irrational, B is closed under addition.

Check closure under scalar multiplication

Suppose we multiply x by a scalar $c = 0$.

$cx = (0, 0, \dots, 0)$. All the components of cx are rational so cx does not belong in B.

$\therefore$ B is not a subspace of $\mathbb{R}^n$. (B also does not contain the zero vector so not a subspace).

(iii) $C = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} (-1)^{i+1} x_i \geq 0\}$

let's take the ~~first three components~~ vectors (x_1, x_2, x_3) & (x_4, x_5, x_6) and check closure under addition. Note that $x_1 < x_4$, $x_2 < x_5$, $x_3 < x_6$.

$$x_1 + x_4 = \sum_{i=1}^{n} (-1)^{i+1} (x_1 + x_4) \geq 0$$

$$= \sum_{i=1}^{n} (-1)^{i+1} x_1 + \sum_{i=1}^{n} (-1)^{i+1} x_4 \geq 0$$

$$1 \cdot x_1 + -1 \cdot x_4 \geq 0$$

$$x_1 - x_4 \ngeq 0. \quad \text{since } x_1 < x_4.$$

$\therefore$ C is ~~not closed~~ not a subspace of $\mathbb{R}^n$.

(iv) D : The set of all solutions x to the matrix equation $Ax = b$; for some matrix A and some vector b.

The zero vector is a solution to the equation $A0 = 0$ for any A. So 0 belongs to set A.

Vector addition:

Let x and y be two solutions such that $Ax = b$ & $Ay = b$.

Then $x + y$ should also be a solution.

$$A(x+y) = Ax + Ay = b + b = 2b$$

Since $A(x+y) = 2b$ is satisfied, $x+y$ is also a solution to $Ax = b$.

Scalar Multiplication:

Let x be a solution to $Ax = b$ and c is a scalar. Then cx should also be a solution to $Ax = b$.

$A(cx) = cAx = cb$. So we have cx is also a solution to $Ax = b$ by scalar multiplication applied to both sides.

Since B contains the zero vector; is closed under addition and also closed under scalar multiplication, this set of all solutions is a subspace of $\mathbb{R}^n$.

(b) Let V be an inner product space, and let W be a subspace of V. The orthogonal complement of W, denoted $W^\perp$, is defined as the set of all vectors in V that are orthogonal to every vector in W. Show that $W^\perp$ is also a subspace of V.

Solution:

Need to verify that $W^\perp$ satisfies the 3 properties for a subset to be considered a subspace for V.

1. The zero vector is in $W^\perp$.

Two vectors are orthogonal if their inner product is 0, and the inner product of 0 w/ any vector is 0.

$\therefore 0 \in W^\perp$.

2. $W^\perp$ is closed under vector addition.

Let x and y be vectors in $W^\perp$. So x & y are orthogonal to every vector in W. Suppose there is a vector z in W. we have:

$\langle x, z \rangle = 0$ and $\langle y, z \rangle = 0$ since x & y are orthogonal to z.

need to show that $x+y$ also belongs to $W^\perp$ by showing $x+y$ is orthogonal to z.

$\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle = 0 + 0 = 0$

since the inner product of $x+y$ with $z \in W$ is zero, $x+y$ is orthogonal to every vector $\in W$. $\therefore x+y \in W^\perp$.

3. $W^\perp$ is closed under scalar multiplication.

Let x be a vector in $W^\perp$ and c is a scalar constant. Let z be a vector in W. So we have:

$\langle x, z \rangle = 0$.

We need to show that cx also belongs in $W^\perp$.

$\langle cx, z \rangle = c\langle x, z \rangle = c \cdot 0 = 0$

Thus cx is orthogonal to every vector in W

$\therefore cx \in W^\perp$.

Hence, $W^\perp$ is also a subspace of V.

Exercise 5 Linear Independence

Let V and W be vector spaces. Let $T: V \to W$ be a linear transformation. The image of T is defined as:

$$\mathrm{Im}(T) = \{w \in W \mid \exists v \in V : w = T(v)\}.$$

The kernel of T is defined as:

$$\mathrm{Ker}(T) = \{v \in V \mid T(v) = 0\}.$$

We say that T is injective if $\forall u, v \in V$, $T(u) = T(v)$ implies $u = v$.

(a) Show that $T(0) = 0$.

Need to show the zero vector in the source space V is mapped to the zero vector in the target space W under transformation T. We can do so using the additivity property of a linear mapping.

$T(0+v) = T(0) + T(v)$ where $v \in V$.

$T(v) = T(0) + T(v)$

$T(v) - T(v) = T(0) + T(v) - T(v)$ subtract $T(v)$ from both sides.

$0 = T(0)$.

In conclusion the zero vector in V under T is the zero vector in W.

(b) For any integer $n \geq 1$, show that given a set of vectors $\{v_1, \ldots, v_n\}$ in V and a set of coefficients $\{c_1, \ldots, c_n\}$ in $\mathbb{R}$, that

$$T(c_1 v_1 + \ldots + c_n v_n) = c_1 T(v_1) + \ldots + c_n T(v_n)$$

Proof by induction on n:

For $n=1$:

$$T(c_1 v_1) = c_1 T(v_1) \quad T \text{ preserves scalar multiplication.}$$

$n=1$ is true. Assume formula holds for $n-1$ & prove for n.

$$T(c_1 v_1 + \ldots + c_n v_n) = T(c_1 v_1 + \ldots + c_{n-1} v_{n-1}) + T(c_n v_n)$$

$$= T(c_1 v_1) + \ldots + T(c_{n-1} v_{n-1}) + T(c_n v_n) \text{ - T preserves additivity.}$$

$$= c_1 T v_1 + \ldots + c_{n-1} T v_{n-1} + c_n T v_n \text{ - T preserves scalar multiplication.}$$

(c) Prove that $Im(T)$ is a vector subspace of W and $Ker(T)$ is a vector subspace of V.

Proof for $Im(T)$.

Suppose $a, b \in W$ and $x, y \in V$. According to $Im(T)$ $T(x) = a$ & ~~T(b)~~ $T(y) = b$. Since T is linear:

$T(x+y) = a + b$. $\therefore$ $a+b$ must also be in W and $Im(T)$.

Suppose $w \in W$ and $v \in V$. Then $T(v) = w$. For any scalar c $T(cv) = cw$ $\therefore$ cw must also be in W and $Im(T)$.

T maps the zero vector in V to the zero vector in W $\therefore$ 0_W is in W and $Im(T)$.

Hence, $Im(T)$ is closed under vector addition, scalar multiplication & contains the zero vector. $\therefore$ $Im(T)$ is a vector subspace of W.

next page

Proof Ker (T) is a vector subspace of V.

Ker (T) is the set of vectors $v \in V$ that maps onto $0_W \in W$ under T.

1. Contains zero vector:

T maps $0_V \in V$ to $0_W \in W$ because T is a linear transformation. Hence $0_V \in \text{Ker}(T)$.

2. Closure under vector addition:

Suppose $v \in V$ and $u \in V$. We know that

$\text{Ker}(T) = \{T(v) = 0_W\}$

Then $T(v) = 0_W$ & $T(u) = 0_W$. By linearity, the following must also be true:

$T(u+v) = T(u) + T(v) = 0_W + 0_W = 0_W$

$\therefore u+v \in \text{Ker}(T)$.

3. Closure under scalar multiplication.

Suppose $v \in V$ and there is a scalar c.

and $\text{Ker}(T) = \{T(v) = 0_W\}$.

Then $T(cv) = c0_W = 0_W \therefore cv \in \text{Ker}(T)$.

Since Ker (T) is closed under vector addition, scalar multiplication and contains the zero vector, we can conclude that Ker (T) is a vector subspace of V.

(d) The Rank-Nullity Theorem states that for a linear map $T: V \to W$

$$\dim(V) = \dim(\text{Ker}(T)) + \dim(\text{Im}(T)).$$

Give an example of a linear map T such that $\dim(\text{Im}(T)) = 3$ and $\dim(\text{Ker}(T)) = 2$.

Since $\dim(\text{Im}(T)) = 3$, example matrix must have a rank of 3. In turn example matrix must have a dimensional space of atleast $\mathbb{R}^3$ or higher.

Since $\dim(\text{Ker}(T)) = 2$, solving the system of equations should result in 2 free variables.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$rk(A) = 3$ (pivot columns).

$A \in \mathbb{R}^5$.

$x_1 = 0$ $x_4 = x_4$ 2 free variables.

$x_2 = 0$ $x_5 = x_5$

$x_3 = 0$

$T: \mathbb{R}^5 \to \mathbb{R}^5$

(e) Consider the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x) = Ax$, where

$$A = \begin{bmatrix} 1 & a & b \\ 1 & 1 & c \\ 1 & 1 & 1 \end{bmatrix}$$

with $[a, b, c]^T \in \mathbb{R}^3$. Find the conditions on a, b, c for which this transformation is injective.

If the columns of the matrix are linearly independent, then T is injective. And the columns will be linearly independent only if the solution to $Ax = 0$ is the trivial solution $x = 0$.

$$\left[\begin{array}{ccc|c} 1 & a & b & 0 \\ 1 & 1 & c & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 1 & a & b & 0 \\ 1 & 1 & c & 0 \\ 0 & 0 & 1-c & 0 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 1 & a & b & 0 \\ 0 & 1-a & c-b & 0 \\ 0 & 0 & 1-c & 0 \end{array}\right]$$

$$x_1 + a x_2 + b x_3 = 0$$
$$(1-a) x_2 + (c-b) x_3 = 0$$
$$(1-c) x_3 = 0$$

$$x_3 = 0$$
$$x_2 = 0$$
$$x_1 = 0$$

$$(1-a) x_2 + (c-b) \cdot 0 = 0$$
$$(1-a) x_2 = 0$$
$$x_1 + a \cdot 0 + b \cdot 0 = 0$$

we have $x = 0$.

doesn't matter what a, b, c is.

$\forall a, b, c \in \mathbb{R}$, T is injective.

Exercise 6 Inner Products.

(a) Show that if an inner product $\langle \cdot , \cdot \rangle$ is symmetric and linear in the second argument, then it is bilinear.

For an inner product to be bilinear, it must satisfy linearity in both first & second argument.

Let's say we have two inner products, one linear in first argument and the other, linear in second argument.

1. $\langle ay + bz, x \rangle = \cancel{a\langle x, y \rangle + b\langle z}$

$a\langle y, x \rangle + b\langle z, x \rangle$

2. $\langle x, ay + bz \rangle = a\langle x, y \rangle + b\langle x, z \rangle.$

If inner product is symmetric in second argument, we can rewrite inner product 2 as:

symmetry:

$\langle x, y \rangle = \langle y, x \rangle$

$a\langle y, x \rangle + b\langle z, x \rangle.$

But this is the same as inner product 1.

Hence, if the inner product is symmetric & linear in the second argument, the inner product is bilinear.

(b) Given a 2 x 2 rotation matrix R represented as

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

show that it preserves the standard inner product, i.e., for all $x, y \in \mathbb{R}^2$, we have $x^T y = (Rx)^T (Ry)$.

let x and y be $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ & $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ respectively.

$$x^T y = (Rx)^T Ry$$

$$= \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)^T \times \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}^T \begin{bmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} x_1\cos\theta - x_2\sin\theta & x_1\sin\theta + x_2\cos\theta \end{bmatrix} \begin{bmatrix} y_1\cos\theta - y_2\sin\theta \\ y_1\sin\theta + y_2\cos\theta \end{bmatrix}$$

$$= (x_1\cos\theta - x_2\sin\theta)(y_1\cos\theta - y_2\sin\theta) + (x_1\sin\theta + x_2\cos\theta)(y_1\sin\theta + y_2\cos\theta)$$

$$= x_1y_1\cos^2\theta - \cancel{x_1y_2\cos\theta\sin\theta} - \cancel{x_2y_1\sin\theta\cos\theta} + x_2y_2\sin^2\theta +$$
$$+ x_1y_1\sin^2\theta + \cancel{x_1y_2\sin\theta\cos\theta} + \cancel{x_2y_1\cos\theta\sin\theta} + x_2y_2\cos^2\theta$$

$$= x_1y_1(\cos^2\theta + \sin^2\theta) + x_2y_2(\cos^2\theta + \sin^2\theta) \quad (\cos^2\theta + \sin^2\theta = 1)$$

$$= x_1y_1 + x_2y_2 \Rightarrow \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$= x^T y.$$

(c) Now, let us consider an inner product in $\mathbb{R}^2$ defined by

find D' (in terms of R & D) $D = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ show $x^T D y = (Rx)^T D' (Ry)$.

$$x^T D y = x^T \cdot (D \cdot y) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 + y_2 \\ y_1 + 3y_2 \end{bmatrix}$$

$$= [x_1 \ x_2] \begin{bmatrix} 2y_1 + y_2 \\ y_1 + 3y_2 \end{bmatrix} = \begin{bmatrix} 2y_1 x_1 + x_2 y_2 \\ x_1 y_1 + 3y_2 x_2 \end{bmatrix}$$

$$R \cdot D = D' = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 2\cos\theta - \sin\theta & \cos\theta - 3\sin\theta \\ 2\sin\theta + \cos\theta & \sin\theta + 3\cos\theta \end{bmatrix}$$

$$(Rx)^T D' (Ry) = [x_1\cos\theta - x_2\sin\theta \quad x_1\sin\theta + x_2\cos\theta] \begin{bmatrix} 2\cos\theta - \sin\theta & \cos\theta - 3\sin\theta \\ 2\sin\theta + \cos\theta & \sin\theta + 3\cos\theta \end{bmatrix}$$

$$= (x_1\cos\theta - x_2\sin\theta)(2\cos\theta - \sin\theta) + (x_1\sin\theta + x_2\cos\theta)(2\sin\theta + \cos\theta)$$
$$+ (x_1\cos\theta - x_2\sin\theta)(\cos\theta - 3\sin\theta) + (x_1\sin\theta + x_2\cos\theta)(\sin\theta + 3\cos\theta)$$

$$= 2x_1\cos^2\theta - \cancel{x_1\cos\theta\sin\theta} - \cancel{2x_2\cos\theta\sin\theta} + x_2\sin^2\theta +$$
$$2x_1\sin^2\theta + \cancel{x_1\cos\theta\sin\theta} + \cancel{2x_2\cos\theta\sin\theta} + x_2\cos^2\theta +$$
$$x_1\cos^2\theta - \cancel{3x_1\sin\theta\cos\theta} - \cancel{x_2\sin\theta\cos\theta} + 3x_2\sin^2\theta +$$
$$x_1\sin^2\theta + \cancel{3x_1\sin\theta\cos\theta} + \cancel{x_2\sin\theta\cos\theta} + 3x_2\cos^2\theta$$

$$= 2x_1(\cos^2\theta + \sin^2\theta) + x_1(\cos^2\theta + \sin^2\theta) + x_2(\cos^2\theta + \sin^2\theta) + 3x_2(\cdots).$$

$$= 2x_1 + x_1 + x_2 + 3x_2 \Rightarrow \begin{bmatrix} 2x_1 & 1x_1 \\ 1x_2 & 3x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

(d). For $\theta = \pi/4$, compute D' explicitly.

$$R = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$D' = R^T \cdot D \cdot R$$

$$= \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3\frac{\sqrt{2}}{2} & 2\sqrt{2} \\ -\frac{\sqrt{2}}{2} & 2\sqrt{2}/2 \end{bmatrix}$$

$$= \begin{bmatrix} 3\frac{\sqrt{2}}{2} & 2\sqrt{2} \\ -\frac{\sqrt{2}}{2} & 2\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 3.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

(e) Consider $u = [1,1]^T \in \mathbb{R}^2$ and $v = [2,-1]^T \in \mathbb{R}^2$. Compute the angle between u and v under the inner product defined by D, and the angle between Ru and Rv under the inner product defined by D'.

$$\cos\theta = \frac{u^T D v}{\sqrt{u^T D u \cdot v^T D v}} = \frac{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} [2,-1]}{\sqrt{\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} [1\ 1] \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} [2,-1]}}$$

$$= \frac{\begin{bmatrix} 3 \\ 4 \end{bmatrix} [2\ -1]}{\sqrt{\begin{bmatrix} 3 \\ 4 \end{bmatrix} [1,1] \cdot \begin{bmatrix} 3 \\ -1 \end{bmatrix} [2,-1]}} = \frac{2}{\sqrt{7 \cdot 7}} = \frac{2}{7}$$

$$\theta = \cos^{-1}\left(\frac{2}{7}\right) \approx 73.4^\circ$$

continued.

$$\cos\theta = \frac{(Ru)^T D' Rv}{\sqrt{Ru^T D' Ru \cdot Rv^T D' Rv}}$$

$$\cos\theta = \frac{\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \begin{bmatrix} 3.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 3\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}}{\sqrt{\begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix} \begin{bmatrix} 3.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 3\frac{\sqrt{2}}{2} \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 3.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \begin{bmatrix} 3\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}}$$

$$\theta = \cos^{-1}\left(\frac{3}{\sqrt{1.73\ldots \cdot 4.24\ldots}}\right) \approx 65.9^\circ$$

Exercise 7 Orthogonality

(a) Let V denote a vector space together with an inner product $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{R}$. Let x, y be non-zero vectors in V. Prove or disprove that if x & y are orthogonal, then they are linearly independent.

If $x \perp y$ then $\langle x, y \rangle = \langle y, x \rangle = 0$ must be true.

Lets assume a linear combination of x and y such that $\lambda_1 x + \lambda_2 y = 0$. For x & y to be linearly independent both scalars (λ_1 & λ_2) must be zero.

continued on next page.

Taking the inner product of both sides of the linear combination w.r.t x, we get:

$$\langle \lambda_1 x + \lambda_2 y , x \rangle = \langle 0 , x \rangle$$

$$\lambda_1 \langle x, x \rangle + \lambda_2 \langle y, x \rangle = 0$$

$$\lambda_1 \langle x, x \rangle = 0 \qquad \text{since } \langle y, x \rangle = 0$$

$$\lambda_1 \|x\|^2 = 0$$

we know x is non-zero, $\therefore$ λ_1 must be equal to 0.

Similarly:

$$\langle \lambda_1 x + \lambda_2 y , y \rangle = \langle 0 , y \rangle$$

$$\lambda_1 \langle x, y \rangle + \lambda_2 \langle y, y \rangle = 0$$

$$\lambda_2 \langle y, y \rangle = 0$$

$$\lambda_2 \|y\|^2 = 0$$

y is non-zero $\therefore$ λ_2 has to be equal to 0.

$\therefore$ x and y are linearly independent.

(b) Determine if the 'vectors' defined by the functions $p(x) = 3x^2 - 1$ and $q(x) = 2x + 1$ in the inner product space of continuous functions on the interval $[0, 1]$ with the inner product defined by $\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx$ are orthogonal.

$$\langle p, q \rangle = \int_0^1 (3x^2 - 1)(2x + 1)\,dx$$

$$= \int_0^1 (6x^3 + 3x^2 - 2x - 1)\,dx$$

Take the antiderivative $\left(mx^n = \frac{m}{n+1} x^{n+1}\right)$ and apply limits.

$$\left[\frac{6}{4}x^4 + x^3 - x^2 - x\right]_0^1$$

upper limit − lower limit

$$\left(\frac{6}{4} + 1 - 1 - 1\right) - (0 + 0 - 0 - 0) = \frac{6}{4} - 1 = \frac{6-4}{4} = \frac{2}{4} = \frac{1}{2}$$

Since $\frac{1}{2} \neq 0$, $\langle p, q \rangle$ are not orthogonal.