Assignment 3

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Problem 5.4

(i) Not valid. When proving $P(n) \to P(n+1)$, you may not assume that what you are trying to prove is true.

(ii) Valid. When proving $P(n) \to P(n+1)$, you may assume $P(n+1)$ is false in order to establish a proof by contradiction.

Problem 5.3

(c) $P(2)$ is T and $P(n) \to (P(n^2) \wedge P(n-2))$ is T for $n \geq 2$.

$$P(2) \to P(4) \to P(16) \to P(256) \to \ ...$$

$$P(2) \leftarrow P(4) \leftarrow P(6) \leftarrow P(8) \leftarrow \ ...$$

$P(n)$ is T for $n \geq 2$.

Problem 5.11

(b) $P(n) : 1 + \frac{1}{2}ln(n) \leq H_n \leq 1 + ln(n)$.

Claim: $P(n)$ is T for all $n > 0$.

Proof:

Base case: $P(1)$ is T.

Induction: assume $P(n)$, prove $P(n+1) : 1 + \frac{1}{2}ln(n+1) \leq H_{n+1} \leq 1 + ln(n+1)$.

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}$$
$$H_{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} + \frac{1}{n+1}$$
$$H_{n+1} = H_n + \frac{1}{n+1}$$

We know that for $0 \leq x \leq \frac{1}{2}, -2x \leq ln(1-x) \leq -x$.

If we plug in $\frac{1}{n+1}$ for x, we get:

$$x = \frac{1}{n+1}$$
$$\frac{-2}{n+1} \leq ln(1 - \frac{1}{n+1}) \leq \frac{-1}{n+1}$$
$$ln(1 - \frac{1}{n+1}) = ln(\frac{n+1}{n+1} - \frac{1}{n+1}) = ln(\frac{n}{n+1}) = ln(n) - ln(n+1)$$
$$\frac{-2}{n+1} \leq ln(n) - ln(n+1) \leq \frac{-1}{n+1}$$
$$\frac{2}{n+1} \geq ln(n+1) - ln(n) \geq \frac{1}{n+1}$$
$$\frac{1}{n+1} \leq ln(n+1) - ln(n) \leq \frac{2}{n+1}$$

Take the right part of that inequality:

$$ln(n+1) - ln(n) \leq \frac{2}{n+1}$$
$$\frac{1}{2}ln(n+1) - \frac{1}{2}ln(n) \leq \frac{1}{n+1}$$
$$\frac{1}{2}ln(n+1) \leq \frac{1}{2}ln(n) + \frac{1}{n+1}$$
$$1 + \frac{1}{2}ln(n+1) \leq 1 + \frac{1}{2}ln(n) + \frac{1}{n+1} \leq H_n + \frac{1}{n+1} = H_{n+1}$$
$$1 + \frac{1}{2}ln(n+1) = H_{n+1}$$

We just proved the first part of $P(n+1)$. Now take the left part of the inequality:

$$\frac{1}{n+1} \leq ln(n+1) - ln(n)$$
$$ln(n+1) - ln(n) \geq \frac{1}{n+1}$$
$$ln(n+1) \geq \frac{1}{n+1} + ln(n)$$
$$1 + ln(n+1) \geq ln(n) + 1 + \frac{1}{n+1} \geq H_n + \frac{1}{n+1} = H_{n+1}$$
$$H_{n+1} \leq 1 + ln(n+1)$$
$$\therefore P(n+1) \text{ is T.}$$

Therefore, $P(n)$ is true for all $n > 0$.

Problem 5.43

(a) Allow (x, y) to represent the x- and y-coordinate of the robot. Also treat the starting position of the robot as the origin $(0, 0)$ of the grid. There are four directions the robot can move, and each move modifies its position as dictated by this table:

old position	move	new position	change in (x, y)	change in x + change in y
(x, y)	down-left	$(x-1, y-1)$	$(-1, -1)$	-2
(x, y)	down-right	$(x+1, y-1)$	$(1, -1)$	0
(x, y)	up-left	$(x-1, y+1)$	$(-1, 1)$	0
(x, y)	up-right	$(x+1, y+1)$	$(1, 1)$	2

The sum of the x- and y-coordinate of the robot $x+y$ is 0 at the start. Adding any even number to an even number will result in an even number. The robot's $x+y$ is even, and any move it makes will add an even number (-2, 0, or 2) to its $x+y$. Therefore, its $x+y$ will always be even.

The shaded square is at $(1, 0)$. Its $x+y$ is 1, an odd number. The robot's $x+y$ will always be even, therefore no sequence of moves takes the robot to the shaded square.

(b) The robot now cannot move up one square and right one square. It has a new ability to move up two squares and right one square (in one move). It has all the other moves it had previously. I will demonstrate that this effectively grants it the ability to move in any orthogonal direction.

For simplicity, assume that:

- DL = robot moves down one and left one
- DR = robot moves down one and right one
- UL = robot moves up one and left one
- UUR = robot moves up **two** and right one
- U = robot moves up one
- D = robot moves down one
- L = robot moves left one
- R = robot moves right one
- UR = robot moves up one and right one

The last five moves can't be done by the robot in one step, but they can be achieved through series of steps listed in the third column of this table:

old position	new move	series of moves used to achieve the new move	change in (x, y)
(x, y)	U	DL, UUR	$(x, y+1)$
(x, y)	D	DL, DL, DR, UUR	$(x, y-1)$
(x, y)	L	DL, DL, UUR	$(x-1, y)$
(x, y)	R	DL, DR, UUR	$(x+1, y)$
(x, y)	UR	DL, UUR, DL, DR, UUR	$(x+1, y+1)$

The robot can travel from one square to any orthogonally or diagonally adjacent square by a finite sequence of moves. Therefore, it can reach any square (m, n) on the infinite grid by a finite sequence of moves.

Exercise 6.2

Claim: $P(n) : n^3 < 2^n \quad \forall n \geq 10.$

Proof:

Base case: $P(10)$ is T.

Induction: assume $P(n)$, prove $P(n+1) : (n+1)^3 < 2^{n+1} \quad \forall n \geq 10.$

$n^3 < 2^n$

$(n+1)^3 = n^3 + 3n^2 + 3n + 1$

$n^3 + 3n^2 + 3n + 1 < 2^n + 3n^2 + 3n + 1$

$(n+1)^3 < 2^n + 3n^2 + 3n + 1$

Stuck; we must create a new hypothesis: $Q(n) : n^3 < 2^n \wedge 3n^2 + 3n + 1 < 2^n \quad \forall n \geq 10.$

Base case: $Q(10)$ is T.

Induction: assume $Q(n)$, prove $Q(n+1) : (n+1)^3 < 2^{n+1} \wedge 3n^2 + 9n + 7 < 2^{n+1} \quad \forall n \geq 10.$

$n^3 < 2^n$

$3n^2 + 3n + 1 < 2^n$

$n^3 + 3n^2 + 3n + 1 < n^3 + 2^n < 2^n + 2^n = 2^{n+1}$

$(n+1)^3 < 2^{n+1}$

We proved the first half of $Q(n)$ but we're stuck on the second; we must create a new hypothesis: $R(n) : n^3 < 2^n \wedge 3n^2 + 3n + 1 < 2^n \wedge 6n + 6 < 2^n \quad \forall n \geq 10.$

Base case: $R(10)$ is T.

Induction: assume $R(n)$, prove $R(n+1) : (n+1)^3 < 2^{n+1} \wedge 3n^2 + 9n + 7 < 2^{n+1} \wedge 6(n+1) + 6 < 2^{n+1} \quad \forall n \geq 10.$

$(n+1)^3 < 2^{n+1}$ (proven already)

$6n + 6 < 2^n$

$3n^2 + 3n + 1 < 2^n$

$3n^2 + 3n + 1 + 6n + 6 < 2^n + 6n + 6 < 2^n + 2^n = 2^{n+1}$

$3n^2 + 9n + 7 < 2^{n+1}$

$6n + 6 < 2^n$

$6 < 6n + 6$

$6 < 2^n$

$6n + 6 + 6 < 2^n + 6 < 2^n + 2^n = 2^{n+1}$

$6n + 12 < 2^{n+1}$

$R(n+1)$ is T.

$R(n) \rightarrow R(n+1)$

$\therefore Q(n) \rightarrow Q(n+1)$

$\therefore P(n) \rightarrow P(n+1)$

Therefore, the claim $P(n)$ is true for all $n \geq 10$.

Exercise 6.4

Claim: if the missing square is at position (n, n) in the $2^n \times 2^n$ grid, the patio can still be L-tiled.

Stronger claim: $P(n)$: if there is a missing square on the $2^n \times 2^n$ patio at any location, the patio can still be L-tiled.

Proof:

Base case: a 2×2 grid can be L-tiled for every possible location of the black square.

Induction: assume $P(n)$, prove $P(n+1)$.

Take a $2^{n+1} \times 2^{n+1}$ grid and split it into four sections. L-tile the center such that you do not tile the section containing the missing square. Each of these sections is now a $2^n \times 2^n$ grid with a missing square. Because $P(n)$ is T, we know that we can L-tile each of these sections and thereby the whole grid. $P(n+1)$ is T, so $P(n) \to P(n+1)$, therefore $P(n)$ is true for all $n \geq 1$.