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Odd period square roots

Problem 64

All square roots are periodic when written as continued fractions and can be written in the form:

$$\sqrt{N} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}$$

For example, let us consider $\sqrt{23}$:

$$\sqrt{23} = 4 + \sqrt{23} - 4 = 4 + \cfrac{1}{\cfrac{1}{\sqrt{23}-4}} = 4 + \cfrac{1}{1 + \cfrac{\sqrt{23}-3}{7}}$$

If we continue we would get the following expansion:

$$\sqrt{23} = 4 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{8 + \dots}}}}$$

The process can be summarised as follows:

$$a_0 = 4, \frac{1}{\sqrt{23}-4} = \frac{\sqrt{23}+4}{7} = 1 + \frac{\sqrt{23}-3}{7}$$
$$a_1 = 1, \frac{7}{\sqrt{23}-3} = \frac{7(\sqrt{23}+3)}{14} = 3 + \frac{\sqrt{23}-3}{2}$$
$$a_2 = 3, \frac{2}{\sqrt{23}-3} = \frac{2(\sqrt{23}+3)}{14} = 1 + \frac{\sqrt{23}-4}{7}$$
$$a_3 = 1, \frac{7}{\sqrt{23}-4} = \frac{7(\sqrt{23}+4)}{7} = 8 + \sqrt{23}-4$$
$$a_4 = 8, \frac{1}{\sqrt{23}-4} = \frac{\sqrt{23}+4}{7} = 1 + \frac{\sqrt{23}-3}{7}$$
$$a_5 = 1, \frac{7}{\sqrt{23}-3} = \frac{7(\sqrt{23}+3)}{14} = 3 + \frac{\sqrt{23}-3}{2}$$
$$a_6 = 3, \frac{2}{\sqrt{23}-3} = \frac{2(\sqrt{23}+3)}{14} = 1 + \frac{\sqrt{23}-4}{7}$$
$$a_7 = 1, \frac{7}{\sqrt{23}-4} = \frac{7(\sqrt{23}+4)}{7} = 8 + \sqrt{23}-4$$

It can be seen that the sequence is repeating. For conciseness, we use the notation $\sqrt{23} = [4;(1,3,1,8)]$, to indicate that the block (1,3,1,8) repeats indefinitely.

The first ten continued fraction representations of (irrational) square roots are:

√2=[1;(2)], period=1
√3=[1;(1,2)], period=2
√5=[2;(4)], period=1
√6=[2;(2,4)], period=2
√7=[2;(1,1,1,4)], period=4
√8=[2;(1,4)], period=2
√10=[3;(6)], period=1
√11=[3;(3,6)], period=2
√12= [3;(2,6)], period=2
√13=[3;(1,1,1,1,6)], period=5

Exactly four continued fractions, for N ≤ 13, have an odd period.

How many continued fractions for N ≤ 10000 have an odd period?