A Graph Theoretical Approach for Network Coding in Wireless Body Area Networks

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***Abstract*— Modern medical wireless systems, such as wireless body area networks (WBANs), are applications of wireless networks that can be used as a tool of data transmission between patients and doctors. Accuracy of data transmission is an important requirement for such systems. In this paper, we will propose a WBAN which is robust against erasures and describe its properties using graph theoretic techniques.**

I. Introduction

Network coding has been widely studied in the last decade since the publication of the seminal paper [1] in which it was shown that significant gains could be achieved in a multicast transmission if coding of data is used in addition to simply routing. One of the applications of network coding is to *wireless body area networks* (WBANs) [2], which could offer valuable support to monitoring a person's physiological data. Such systems are now more practical with the advent of new generation miniature, low-power wireless devices. A WBAN is a network that sends a person's relevant health information from attached or implanted miniature sensors, via relays, to a monitoring station (MS). Such systems allow continuous remote updating, which has the potential to offer many advantages in modern medical care, allowing greater patient freedom and improved response to acute situations.

An important consideration of WBAN design is that the MS can retrieve all information sent in spite of packet loss. At the same time, power-consumption and communication range of sensors should be also taken into consideration, since these sensors must be small and light, with small batteries and antennae. An efficient WBAN must be operable under very low transmission power compared with general wireless networks, so any coding scheme for a given WBAN should have low computational complexity. Another requirement is to minimise the number of re-transmissions requested due to errors.

In [3] a simple WBAN coding scheme robust to packet erasures was presented. In this paper, we generalize that scheme, taking a graph theoretic perspective. More precisely, we consider a graph which represents a coding scheme for a WBAN, and use it to analyze the robustness of the scheme against packet loss.

We present preliminaries in Section II, and describe a graph representation of a WBAN coding scheme. In Section III we give a necessary and sufficient condition for a given WBAN coding scheme to be able to retrieve all data at its MS after the erasure of some packets, in terms of its corresponding graph representation, in which case we call the graph decodable. In Section IV, we give an expression for the *decoding probability* of the given WBAN scheme and provide a partial characterization of those with a high decoding probability. In Section V, we present simulation data for a given WBAN.

II. Basic Background

We begin with some background on WBANs (cf. [2], [3]). A WBAN consists of *sensors* S_i, *relays* R_j and a *monitoring station* (MS). These sensors might be implanted, attached to a persons skin or clothing or in the proximity of the body. Each sensor S_i sends a packet P_i (a vector over $GF(2)$) to one or more relays where packets are encoded by taking linear combinations of them. The relays then send the encoded packets to the MS. A *coding scheme* for a given WBAN is a collection of $GF(2)$-linear vectorial functions $f_a(P_1, ..., P_n)$, corresponding to packet encodings at each relay.

We consider schemes with *redundancy* r, in which case each packet is sent to r different relays. Now let n and k be the number of sensors and relays, respectively, and assume that each relay receives and sends t packets. Then for a WBAN with redundancy r, observe that rn, the total number of packets sent from sensors to relays, is equal to tk, the total number of packets sent from relays to the MS. For the sake of simplicity, we assume the following throughout this paper.

- $t = sr$ for some $s \in \mathbb{N}$ and therefore, $n = sk$ holds. Since $n > k$ in general, we have $s \geq 2$.
- Relay R_j receives and encodes t packets $P_{js+1}, P_{js+2}, \ldots, P_{js+t}$ where $js + \ell$ is computed modulo n for each $j = 0, 1, \ldots, k-1$.
- Each encoded outgoing packet (to be sent to the MS) has the form P_i or $P_i \oplus P_{i'}$, $i \neq i'$, where as usual $\oplus$ denotes addition of binary vectors. In other words, $f_a(P_1, ..., P_n)$ is either P_i or $P_i \oplus P_{i'}$ for some i, i'.

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- Erasures do not occur in communication from sensors to relays but some may occur in the communication from relays to the MS.

We next present some preliminaries on graphs, (see [5] for further reading). Let G be a finite graph with vertex set $V(G)$ and edge set $E(G)$, respectively. We write L_G to denote the number of loops of G. We define the *incidence degree* of a vertex v, expressed $d_I(v)$, as the number of edges incident with v (each loop at v contributes a count of one to this number; of course this is different to the standard definition of the degree of v, in which loops contribute a count of two to the degree). We denote by $\delta_I(G)$ the minimum incidence degree of G, that is, $\delta_I(G) = \min\{d_I(v) : v \in V(G)\}$. We write $\delta_\ell(v)$ to denote the number of loops incident with a given vertex v, and we let $\Delta_\ell(G) := \max\{\delta_\ell(v) : v \in V(G)\}$. For any graph G, it is well known that the sum of degrees of vertices in G is equal to $2|E(G)|$. On the other hand, the sum of the incidence degrees $S_I(G)$ is given by $S_I(G) = 2|E(G)| - L_G$ since each loop is counted as one edge.

A graph G is called *connected* if there is a path connecting each pair of vertices, otherwise G is called *disconnected*. Given a connected graph G, the *edge-connectivity* κ_G of G is the smallest number of edges such that the resulting graph formed by deleting those edges is disconnected. Observe that $\kappa_G \le \delta_I(G)$ since deleting all edges attached to a vertex v with incidence degree $d_I(v) = \delta_I(G)$ makes v isolated. A *subgraph* H of a graph G is a graph such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. Especially, H is called a subgraph of G *induced by* vertices in $V(H)$ when any edge in G whose endpoints are both in $V(H)$ is an edge in H.

We now describe a decoding scheme for a WBAN via graph theory. Given a WBAN *coding scheme* $\mathcal{C} = \{f_1, ..., f_{rn}\}$ (a coding scheme consists of rn functions since there are rn packets to be sent to the MS), we generate a (multi)graph representation $G = G_{\mathcal{C}}$ for $\mathcal{C}$ as follows.

1) G has as vertices $1, 2, \ldots, n$.
2) For $i \ne i'$, (i, i') is an edge of G if $P_i \oplus P_{i'} = f_a(P_1, ..., P_n)$ for some f_a (*i.e.*, if $P_i \oplus P_{i'}$ is sent to the MS).
3) G has a loop at i if $P_i = f_a(P_1, ..., P_n)$ for some f_a (*i.e.*, if P_i is sent to the MS).

The erasure of packets during a transmission can be identified with deletions of edges in G. Clearly, for a WBAN with n sensors and redundancy r, any graph representation of a corresponding coding scheme must have n vertices and rn edges.

III. Decodable Graphs

We provide a necessary and sufficient condition for full data retrieval at the MS.

Theorem III.1 *Let $\mathcal{C} = \{f_1, ..., f_{rn}\}$ be a WBAN coding scheme, where each f_a is an encoding of packets $P_1, ...P_n$, and $G = G_{\mathcal{C}}$ be the graph representation for $\mathcal{C}$. Now let H be a subgraph of G formed by deleting edges of G corresponding to packet erasures occurring in the transmission. The MS can retrieve (i.e., decode) all packets $P_1, ...P_n$ if and only if each connected component in H has at least one loop.*

Proof: We first show that having a loop in each component of H is sufficient to retrieve all packets $P_1, P_2, \ldots, P_n$. Clearly, it is enough to show that for a component with a loop, all packets P_i, where i is a vertex in the component, can be retrieved at the MS.

Let C be a connected component with a loop in H. A vertex i in C with a loop signifies that P_i has been received at the MS. Now pick another vertex j in C. Since i and j are in the same connected component, there exists a path $\pi = i, a_1, \cdots, a_\ell, j$ from i to j in H, which means that $P_i \oplus P_{a_1}$, $P_{a_m} \oplus P_{a_{m+1}}$ $(1 \le m \le \ell - 1)$ and $P_{a_\ell} \oplus P_j$ have been received at the MS. Then the MS decoder can retrieve P_j from P_i and $P_i \oplus P_j$.

We prove the converse by contradiction. Let C' be a connected component of H with no loops. Then, for the ℓ' vertices $i_1, i_2, \ldots, i_{\ell'}$ in C', the MS can compute only $P_{i_j} \oplus P_{i_s}$, $1 \le j < s \le \ell'$. Since these correspond to a system of equations of rank at most $\ell' - 1$ over $GF(2)$ the decoder cannot uniquely determine all $P_{i_1}, P_{i_2}, \ldots, P_{i_{\ell'}}$. ■

Therefore, the existence of loops at each component plays an important role in selecting a graph for a WBAN coding scheme. For the remainder, we call a graph G *decodable* if each of its connected components has a loop, and denote by $\mathcal{D}(n, m)$ the set of decodable graphs with n vertices and m edges. Otherwise we say that G is called *undecodable*.

Given $G \in \mathcal{D}(n, m)$, we define a *loop cut* to be a subset $\mathcal{L}$ of $E(G)$ such that $G - \mathcal{L}$ is undecodable. We write $m(G)$ to denote the smallest cardinality of any loop cut of G.

Remark III.2 *For $m(G)$ of a graph G, we note the following.*

1) *$m(G) \le \min(L_G, \delta_I(G))$ since deleting all loops in G or deleting all edges attached to a vertex $\mathbf{v}$ with incidence degree $d_I(v) = \delta_I(G)$ yields an undecodable graph.*
2) *If $L_G \ge \kappa_G$, then $\kappa_G \le m(G)$ since a resulting graph $\tilde{G}$ of G after deletion of some edges cannot be undecodable, if $L_{\tilde{G}} \ne 0$, unless $\tilde{G}$ is disconnected.*

The robustness of a WBAN coding scheme to packet loss can be measured as a function of the number of decodable subgraphs found upon deleting some edges.

IV. The Decoding Probability of a WBAN

Given a graph $G \in \mathcal{D}(n, m)$, we denote by c_x^G the number of decodable subgraphs of G formed by deleting x edges of G and we write $k_x^G = \binom{m}{x} - c_x^G$ to denote the number of undecodable subgraphs of G found by deleting some x-set of its edges. We define the *decoding probability* of G by

$$\mathcal{P}_G := \sum_{x=0}^{m} c_x^G p^{m-x} q^x,$$

where p is the probability that an edge is not deleted (*i.e.*, the probability that a packet is successfully transmitted to the MS) and $q = 1 - p$ the probability that an edge is deleted (*i.e.*, the probability that a packet is erased during the transmission).

Our interest is to construct a coding scheme $\mathcal{C}$ for a fixed WBAN whose corresponding graph $G = G_{\mathcal{C}}$ has a high decoding probability.

Lemma IV.1 *Let $G \in \mathcal{D}(n, m)$. Then $m \geq n$.*

Proof: It is well known that a connected graph T with z vertices has least $z - 1$ edges, with equality if and only if T is a tree. Therefore, a connected graph with z vertices is decodable only if it has at least z edges since it must contain loops. Now let G have connected components C_i, $i = 1, ..., h$. Then $|E(C_i)| \geq |V(C_i)|$ for each i and hence $m = \sum_{i=1}^{h} |E(C_i)| \geq \sum_{i=1}^{h} |V(C_i)| = n$. ∎

From Lemma IV.1, we immediately deduce that $c_x^G = 0$ for $x \geq m - n + 1$ and so $\mathcal{P}_G = \sum_{x=0}^{m-n} c_x^G p^{m-x} q^x$. Moreover,

$$\mathcal{P}_G = \prod_{i=1}^{h} \mathcal{P}_{C_i},$$

when G consists of h components $C_1, ..., C_h$.

The question of a graph G having optimal decoding probability is related to $m(G)$, which is the smallest number x for which $c_x^G < \binom{m}{x}$. The next lemma gives an upper bound on $m(G)$.

Lemma IV.2 *Let $G \in \mathcal{D}(n, m)$. Then $\delta_I(G) \leq 2m/n - 1$ and $m(G) \leq 2m/(n+1)$. In particular, $m(G) \leq \min(\lfloor 2m/n - 1 \rfloor, \lfloor 2m/(n+1) \rfloor)$*

Proof: Since G is decodable, $L_G \geq 1$, and we have $n\delta_I(G) \leq S_I(G) = 2m - L_G < 2m$. Furthermore, since $m(G) \leq L_G$, we have $nm(G) \leq n\delta_I(G) \leq S_I(G) = 2m - L_G < 2m - m(G)$. ∎

It follows that for a WBAN $\mathcal{W}$ with n packets and redundancy r, any graph representation G of a coding scheme for $\mathcal{W}$ satisfies $m(G) \leq \min(2r - 1, \lfloor \frac{2rn}{n+1} \rfloor) = \lfloor \frac{2rn}{n+1} \rfloor$, which is simply $2r - 1$ whenever $r \leq \frac{n+1}{2}$. The following proposition shows that it is indeed possible to generate some G for which equality in the above holds. Note that the subscripts i of P_i are computed modulo n in what follows, if not indicated explicitly.

Relay	Inter-encoding					
R_0	P_1	P_2	$P_3 \oplus P_4$	$P_4 \oplus P_5$	$P_5 \oplus P_6$	$P_6 \oplus P_1$
R_1	P_4	$P_5 \oplus P_8$	$P_6 \oplus P_7$	$P_7 \oplus P_8$	$P_8 \oplus P_9$	$P_9 \oplus P_4$
R_2	P_7	$P_8 \oplus P_{11}$	$P_9 \oplus P_{10}$	$P_{10} \oplus P_{11}$	$P_{11} \oplus P_{12}$	$P_{12} \oplus P_7$
R_3	P_{10}	$P_{11} \oplus P_2$	$P_{12} \oplus P_1$	$P_1 \oplus P_2$	$P_2 \oplus P_3$	$P_3 \oplus P_{10}$

TABLE I

THE CODING SCHEME UNDER ALGORITHM 1 FOR 12 SENSORS, 4 RELAYS AND REDUNDANCY 2 WHEN $L_G = 5$

Proposition IV.3 *Let $\mathcal{C}$ be the coding scheme for a WBAN with n packets, k relays and redundancy r, where $k, r \geq 2$, defined as in Algorithm 1. Let $s = n/k$ and let the graph representation G of $\mathcal{C}$ satisfy $L_G \geq 2r - 1$. If $k \leq L_G \leq (s-1)k$ then it holds that $m(G) = \delta_I(G) = 2r - 1$.*

Proof: Observe that the graph G satisfies the following.

1) Each vertex i with $i \equiv 1 \pmod{s}$ has a loop.
2) For each $1 \leq i \leq n$, the number of edges between vertices i and $i + 1$ is $r - 1$.

Algorithm 1 : A coding scheme for a WBAN $\mathcal{W}$ with n packets, k relays and redundancy r.

Require: Suppose $L_G = yk + z$, $0 \leq z \leq k - 1$.
while $0 \leq j \leq k - 1$ **do**
let $f_{jt+1}, f_{jt+2}, \ldots, f_{(j+1)t}$ be $t(= nr/k = rs)$ packet encodings of relay R_j.
for t packets $P_{js+1}, P_{js+2}, \ldots, P_{js+t}$ received by R_j **do**
if $b \leq s - 1$ **then**
set $f_{jt+b} := P_{js+b} \oplus P_{(j+1)s+b}$.
else
if $s \leq b \leq t - 1$ **then**
set $f_{jt+b} := P_{js+b} \oplus P_{js+b+1}$.
else
if $b = t$ **then**
$f_{jt+b} := P_{js+t} \oplus P_{js+1}$.
end if
end if
end if
if $b \leq y$ **then**
reset f_{jt+b} to be $f_{jt+b} := P_{js+b}$.
else
if $j + 1 \leq z$ **then**
reset f_{js+y+1} to be $f_{js+y+1} := P_{js+y+1}$.
end if
end if
end for
end while
return $\mathcal{C} = \{f_1, f_2, \ldots, f_{rn}\}$ as a coding scheme.

3) G cannot be disconnected without deleting the (multi)edges $(a, a+1)$ and $(a', a'+1)$ for $a \neq a'$.
4) If a vertex i does not have a loop, then
   - it is adjacent to the vertex $i+s$ when $i \not\equiv 0 \pmod{s}$.
   - it is adjacent to the vertex $i - t + 1$ otherwise.

It is straightforward to see that G is connected and $\delta_I(G) = 2r - 1$ holds, so the edge-connectivity κ_G of G satisfies $\kappa_G \leq 2r - 1 \leq L_G$. Also, we can obtain from Properties 2) and 3) that $\kappa_G \geq 2(r - 1) = 2r - 2$, which automatically implies $m(G) \geq 2r - 2$ from 2) in Remark III.2.

Now suppose that edges described in Property 3) are deleted from G, and call the resulting graph $\hat{G}$. Denote by H_a the subgraph of $\hat{G}$ induced by the vertices $a + 1, a + 2, ..., a'$, and by $H_{a'}$ the one induced by the vertices $a' + 1, a' + 2, ..., a$. If both H_a and $H_{a'}$ contain loops, then we can conclude that $m(G) \geq 2r - 1$. Furthermore, it cannot happen that neither H_a nor $H_{a'}$ have loops since $L_G = L_{\hat{G}} \geq 1$. Therefore, we need only to consider the case (without loss of generality) when H_a contains a loop but $H_{a'}$ does not. In this case, we will see H_a and $H_{a'}$ within $\hat{G}$ and show the existence of an edge in $E(\hat{G})$ joining them, which implies that $m(G) \geq 2r - 1$.

As $H_{a'}$ does not contain loops, $|V(H_{a'})| < s$ since otherwise, at least one of the vertices i in $H_{a'}$ satisfies $i \equiv 1 \pmod{s}$, and therefore, $H_{a'}$ contains a loop from Property 1).

If $H_{a'}$ contains a vertex i with $i \equiv 0 \pmod{s}$, i is adjacent to the vertex $i-t+1$, where $i-t+1 \equiv 1 \pmod{s}$ as $t = sr$. Since $i-t+1$ has a loop from Property 1), it is in H_a. If each vertex i in $H_{a'}$ satisfies $i \not\equiv 0 \pmod{s}$, then i is adjacent to $i+s$. Since $|V(H_{a'})| < s$ and $|V(H_a)| > n-s = (k-1)s \geq s$, $i+s$ is in H_a. In each case, there exists an edge in $E(\hat{G})$ joining H_a and $H_{a'}$ as required. ■

We now present some upper bounds on c_x^G for x. The following lemma is a sharp upper bound on c_x^G when $x = m(G)$.

Lemma IV.4 *Let G be a decodable graph with n vertices and m edges. Then* $c_{m(G)}^G \leq \binom{m}{m(G)} - m(G)(n+1) - n + 2m - 1$

Proof: First recall that $m(G) \leq \min(L_G, \delta_I(G))$. Let α be the number of vertices of G with incidence degree $m(G)$ and let β be the number of vertices with incidence degree at least $m(G)+2$. Then

$$m(G)\alpha + (m(G)+1)(n-\alpha-\beta) \leq 2m - L_G - (m(G)+2)\beta,$$

which implies that

$$\alpha \geq L_G + nm(G) + n - 2m + \beta \geq m(G)(n+1) + n - 2m,$$

since $L_G \geq m(G)$ and $\beta \geq 0$. Clearly $c_{m(G)}^G \leq \binom{m}{m(G)} - \alpha$, since deleting any $m(G)$ edges incident with a vertex of incidence degree $m(G)$ results in an undecodable graph. Since G is decodable, no vertex of incidence degree $m(G)$ is incident with all loops of G. Therefore, if $\alpha = m(G)(n+1)+n-2m$, then $L_G = m(G)$ and so $c_{m(G)}^G \leq \binom{m}{m(G)} - \alpha - 1$ since we also have to count the case of deleting all $m(G)$ loops from G. If $\alpha > m(G)(n+1)+n-2m$, the result follows trivially. ■

Clearly, $k_{m(G)}^G \geq m(G)(n+1) - n + 2m - 1$ for any $G \in \mathcal{D}(n,m)$. We can also show a tight upper bound when x is close to $m(G)$.

Lemma IV.5 *Let $G \in \mathcal{D}(n,m)$ satisfy $k_{m(G)}^G = m(G)(n+1) + n - 2m + 1$. Then, with the same notation as in Lemma IV.4, $\beta = 0$ and either*

1) *$\alpha = m(G)(n+1) + n - 2m$ and $L_G = m(G)$, or*
2) *$\alpha = m(G)(n+1) + n - 2m + 1$ and $L_G = m(G) + 1$.*

Proof: Let $\theta = m(G)(n+1) + n - 2m$. Recall that, as in the proof of Lemma IV.4,

$$\alpha \geq L_G + nm(G) + n - 2m + \beta \geq m(G)(n+1) + n - 2m = \theta. \tag{1}$$

Therefore,

$$\theta + 1 = k_{m(G)}^G \geq \alpha \geq \theta,$$

so either $\alpha = \theta$, in which case $k_{m(G)}^G = \alpha + 1$, or $\alpha = \theta + 1$ and $k_{m(G)}^G = \alpha$. For the case $\alpha = \theta$ from (1), we must have $L_G = m(G)$ and $\beta = 0$. In the latter case we have $\alpha = \theta + 1 \geq \theta - m(G) + L_G + \beta$, which gives $m(G) + 1 \geq L_G + \beta \geq m(G) + \beta$. Therefore, either $\beta = 1$ and $L_G = m(G)$ or $\beta = 0$ and $L_G = m(G)+1$. Since for $\alpha = \theta + 1$ we have $k_{m(G)}^G = \alpha$, every undecodable subgraph of G found by deleting $m(G)$ edges is constructed by deleting the $m(G)$ edges that meet a vertex of incidence degree $m(G)$. If $L_G = m(G)$ then G has a vertex of incidence degree $m(G)$ that is incident with every loop of G, contradicting the decodability of G, so we deduce that $\beta = 0$ and $L_G = m(G) + 1$. ■

Lemma IV.6 *Let G be a graph with $m > 2$ and satisfying the hypothesis of Lemma IV.5. Let $\theta = m(G)(n+1) + n - 2m$. Then $\Delta_\ell(G) \leq m(G) - 1$ and*

$$k_{m(G)+x}^G \geq (\theta+1)\binom{m - m(G)}{x} + (n-\theta)\binom{m - m(G) - 1}{x - 1},$$

for any x satisfying $1 \leq x \leq m(G) - \Delta_\ell(G)$.

Proof: Clearly, since G is decodable, $\delta_\ell(v) \leq m(G) - 1$ for any $v \in V(G)$ such that $\delta_I(v) = m(G)$, and $\delta_\ell(v) \leq m(G)$ for any $v \in V(G)$ satisfying $\delta_I(v) = m(G) + 1$.

Suppose that $\alpha = \theta$ (*i.e.*, that 1) of Lemma IV.5 holds). Then $L_G = m(G)$ and $k_{m(G)}^G = \alpha + 1$. Suppose that $v \in V(G)$ has incidence degree $m(G) + 1$. If $\delta_\ell(v) = m(G)$ then v has exactly one neighbour, so an undecodable subgraph results by deleting the only non-loop edge incident with v and we deduce that $m(G) = 1$. Then $0 \leq \alpha = 2(n-m)+1 \leq 1$, since $n \leq m$, which forces $n = m$ and $\alpha = 1$. It follows that G is a path graph with a single loop at a vertex of incidence degree 2 and one leaf (a vertex of degree 1). Therefore, $k_{m(G)}^G = k_1^G = m = n$. On the other hand, $k_1^G = \alpha + 1 = 2$, contradicting our assumption that $m > 2$. We deduce that $\Delta_\ell(G) \leq m(G) - 1$.

For the case $\alpha = \theta + 1$ (*i.e.*, if 2) of Lemma IV.5 holds), we have $L_G = m(G)+1$ and $k_{m(G)}^G = \alpha$. If v is a vertex of G with $\delta_I(v) = m(G) + 1$ and $\delta_\ell(v) = m(G)$, then an undecodable subgraph with $m - 2$ edges results by deleting the only non-loop edge of v and the single loop not incident with v. Then $m(G) \leq 2$. If $m(G) = 2$ then $\alpha = k_{m(G)}^G = k_2^G \geq \alpha + 1$. If $m(G) = 1$ then $0 \leq \alpha = 2(n-m) + 2$ so that either $n = m$ and $\alpha = 2$ or $n = m - 1$ and $\alpha = 0$. In the former case, G must have exactly two connected components, each of which is a path graph with exactly one loop at a vertex of incidence degree 2 and one leaf. Then $2 = \alpha = k_{m(G)}^G = m = n$, giving a contradiction to $m > 2$. In the latter case, G is a path graph with exactly 2 loops and no leaves, so deleting a single edge never results in an undecodable subgraph, contradicting $m(G) = 1$. It follows that $\Delta_\ell(G) \leq m(G) - 1$.

Let $x \in \{1, ..., m(G) - \Delta_\ell(G)\}$. Consider the following operations, each of which results in an undecodable subgraph of G with $m - m(G) - x$ edges.

1) Delete $m(G)$ edges incident with a vertex of incidence degree $m(G)$ and delete a further x edges arbitrarily.
2) Delete $m(G) + 1$ edges incident with a vertex of incidence degree $m(G) + 1$ and delete a further $x - 1$ edges arbitrarily.
3) Delete all L_G loops of G, and then delete a further $m(G) + x - L_G$ edges arbitrarily.

Observe first that no two distinct vertices of incidence degree d are coincident with d edges, since G is decodable, so there are exactly $\alpha\binom{m-m(G)}{x}$ (respectively $(n-\alpha)\binom{m-m(G)-1}{x-1}$) ways to produce an undecodable subgraph by the operation

1) (respectively, by the operation 2)). The operations 1) and 2) are mutually exclusive, since in 1) at most $x \le m(G) - 1$ edges are deleted from a vertex of incidence degree $m(G)+1$. Moreover, the operations 2) and 3) are exclusive to each other, since in 2) at most

$$\delta_\ell(v) + x - 1 \le m(G) - (\Delta_\ell(G) - \delta_\ell(v)) - 1 \le m(G) - 1$$

loops are deleted, for any vertex v of incidence degree $m(G)+1$.

For the case $\alpha = \theta + 1$, 1) and 3) are exclusive, since $m(G) < L_G$, and at most $\delta_\ell(v) + x \le m(G) - (\Delta_\ell(G) - \delta_\ell(v)) \le m(G)$ loops are deleted for any given vertex v of degree $m(G)$.

Now suppose that $\alpha = \theta$ and let $v \in V(G)$ such that $\delta_I(v) = m(G)$ and $\delta_\ell(v) \ge 1$. The following actions result in an undecodable subgraph by deleting some $m(G)$ edges of G.

(a) Delete $m(G)$ edges incident with a vertex of incidence degree $m(G)$.
(b) Delete all $m(G)$ loops.
(c) Delete the $m(G) - \delta_\ell(v)$ non-loops edges incident with v and delete the remaining $L_G - \delta_\ell(v)$ loops of G that are not incident with v.

Clearly under the assumption $\delta_\ell(v) \ge 1$, the operations (a),(b) and (c) are pairwise exclusive and so $\alpha + 1 = k^G_{m(G)} \ge \alpha + 2$, giving a contradiction, so we deduce that no vertex of incidence degree $m(G)$ is incident with a loop. Then in 1), for a given vertex v satisfying $\delta_I(v) = m(G)$, at most $\delta_\ell(v) + x = x \le m(G) - 1$ loops are deleted, which means 1) and 3) are mutually exclusive.

It follows that

$$\begin{aligned} k^G_{m(G)+x} &\ge \alpha\binom{m - m(G)}{x} + (n-\alpha)\binom{m - m(G) - 1}{x-1} \\ &+ \binom{m - L_G}{m(G) + x - L_G}, \end{aligned}$$

which yields

$$k^G_{m(G)+x} \ge (\theta+1)\binom{m - m(G)}{x} + (n-\theta)\binom{m - m(G) - 1}{x-1},$$

for any $G \in \mathcal{D}(n, m)$. ■

For given c^G_x we can compute an upper bound on c^G_{x+z} for $z \ge 0$ by using the following easy result.

Lemma IV.7 *Let G be a graph with n vertices and m edges. Then*

$$k^G_{x+z} \ge k^G_x \binom{m-x}{z} \Big/ \binom{x+z}{z}$$

for $z \ge 0$.

The following corollary is now immediate.

Corollary IV.8 *Let $G \in \mathcal{D}(n, m)$ satisfy the hypothesis of Lemma IV.5. Then for each $z \ge 0$*

$$k^G_{2m(G)-\Delta_\ell(G)+z} \ge k^G_{2m(G)-\Delta_\ell(G)} \frac{\binom{m-2m(G)+\Delta_\ell(G)}{z}}{\binom{2m(G)-\Delta_\ell(G)+z}{z}}.$$

V. Simulation Results of WBANs

In this section, we will provide simulation results and see the correspondence between simulation results and theoretical results that have been discussed in this paper. We focus on coding schemes of WBANs with 9 sensors (which implies 9 packets), 3 relays and redundancy 2. More precisely, we follow the coding scheme introduced in Algorithm 1 for $1 \le L_G \le 9$ and the one with no inter-encoded packets, as presented in Table II.

Relay	Inter-encoding					
R_0	P_1	P_2	P_3	P_4	P_5	P_6
R_1	P_4	P_5	P_6	P_7	P_8	P_9
R_2	P_7	P_8	P_9	P_1	P_2	P_3

TABLE II

No inter-encoded packets

Now let G_i and G the graph representations of the coding scheme with $L_G = i$ and the one with no inter-encoded packets, respectively. Since each representation consists of 9 vertices and 18 edges, we have from Lemma IV.1 that $c^{G_i}_x = c^G_x = 0$ for any i whenever $x \ge 10$. The detailed information on $c^{G_i}_x$ and c^G_x for $1 \le x \le 9$ is given in Table III. The table also contains the information on D_x's, which are the upper bounds of c^H_x's for $H \in \mathcal{D}(9, 18)$ obtained from Lemmas IV.4, IV.6 and IV.7.

For $3 \le i \le 9$, observe that $\binom{18}{x} = c^{G_i}_x$ for $x = 1, 2$, which implies $m(G_i) = 3 (= 2r - 1)$. Furthermore, $c^{G_3}_x$ is the largest one amongst all examples for any $1 \le x \le 9$. In addition, $c^{G_3}_3$ and $c^{G_3}_4$ meet the upper bounds obtained from Lemmas IV.4 and IV.6.

x	1	2	3	4	5	6	7	8	9
$\binom{18}{x}$	18	153	816	3060	8568	18564	31824	43758	48620
D_x	18	153	812	2994	8064	17472	29952	41184	45760
$c^{G_1}_x$	17	136	677	2333	5842	10803	14540	13297	10340
$c^{G_2}_x$	18	152	797	2889	7603	14769	20880	20073	12365
$c^{G_3}_x$	18	153	812	2994	8052	16053	23388	23277	12500
$c^{G_4}_x$	18	153	812	2993	8042	16008	23273	23101	12365
$c^{G_5}_x$	18	153	811	2979	7952	15660	22402	21731	11273
$c^{G_6}_x$	18	153	810	2964	7851	15260	21405	20232	10192
$c^{G_7}_x$	18	153	809	2948	7736	14779	20135	18161	8532
$c^{G_8}_x$	18	153	808	2932	7621	14299	18886	16199	7053
$c^{G_9}_x$	18	153	807	2916	7506	13821	17667	14373	5776
c^G_x	18	144	672	2016	4032	5376	4608	2034	512

TABLE III

The number of decodable graphs

Using the information, we can derive the decoding probabilities $\mathcal{P}_{G_i}$ and $\mathcal{P}_G$. We provide the decoding probabilities, together with the probabilities obtained from simulations in Table IV. As for the simulation results, we computed the probabilities P as

$$P = \frac{\text{the number of success simulations}}{\text{the total number of simulations}},$$

where success simulations mean the ones in which all packets are retrieved. We ran the programme by setting the total number of simulations to be 5000000. We can see that applying coding scheme increases the decoding probability remarkably.

	Decoding probability	Simulation results
G_1	$\mathcal{P}_{G_1} = 0.7728010935$	0.77262
G_2	$\mathcal{P}_{G_2} = 0.9257409618$	0.92564
G_3	$\mathcal{P}_{G_3} = 0.9558104057$	0.95578
G_4	$\mathcal{P}_{G_4} = 0.9551821038$	0.95518
G_5	$\mathcal{P}_{G_5} = 0.9493923505$	0.94944
G_6	$\mathcal{P}_{G_6} = 0.9429367740$	0.94272
G_7	$\mathcal{P}_{G_7} = 0.9353111111$	0.93524
G_8	$\mathcal{P}_{G_8} = 0.9277553360$	0.92766
G_9	$\mathcal{P}_{G_9} = 0.9202926069$	0.92018
G	$\mathcal{P}_{G} = 0.6924597789$	0.69254

TABLE IV

THE DECODING PROBABILITIES WITH $p = 0.8$

For any probability p, $\mathcal{P}_{G_3}$ has been the optimal (in terms of decoding probability) amongst all graphs in $\mathcal{C}(9, 18)$ at this moment. Indeed, if there exists $H \in \mathcal{C}(9, 18)$ such that $c_x^H = D_x$ we have $\mathcal{P}_H - \mathcal{P}_{G_3} = 0.02087697704$ (***resp.*** 0.0007125786313) when $p = 0.8$ (***resp.*** $p = 0.9$).

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