1.

a)

Assume that G has no cycle, and consider the longest path P in G. Let v be one of the endpoint vertex in P since v has degree at least 2, it must have at least two edges $e_1$ and $e_2$(or more like $e_3...e_n$) incident on it.

Let $e_1$ be the last edge of the path P. Then $e_2$ and other edges cannot be incident on any other vertex of P since that would create a cycle. So $e_2, e_3...e_n$ are not part of P, and can be appended to P to give a strictly longer path. Because of this contradicts our choice of P. This would be the same if we choose another endpoint or choose both

It is contradiction,hence G must contain a cycle.

b)

proof of a) can be rightfully applied to prove this.Since it is a graph,and there must be a longest path existing.And given every vertex with at least 2 degree, that can be conducted as both endpoints have at least 2 degree,we can exactly use the same contradiction to prove there is cycle.

2.

According to definition, a perfect matching of $K_{n,n}$ can be informally conducted as the number of ways to partition the $2n$ vertices into n sets of two vertices each:

It is generally a multinomial case(or say labelled balls in unlabelled bins): $C(2n; a_1, a_2...a_n) = \frac{2n!}{(2!)^n * n!}$,where $A(n) = 2, 2, 2, 2...2$ with n terms.

3.

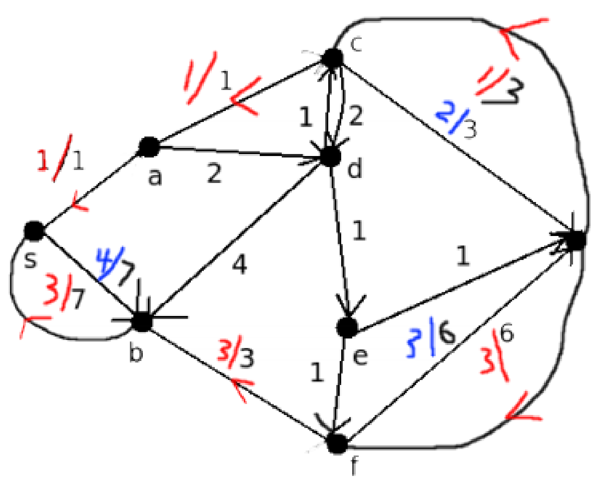
a)

min-cut=s,b

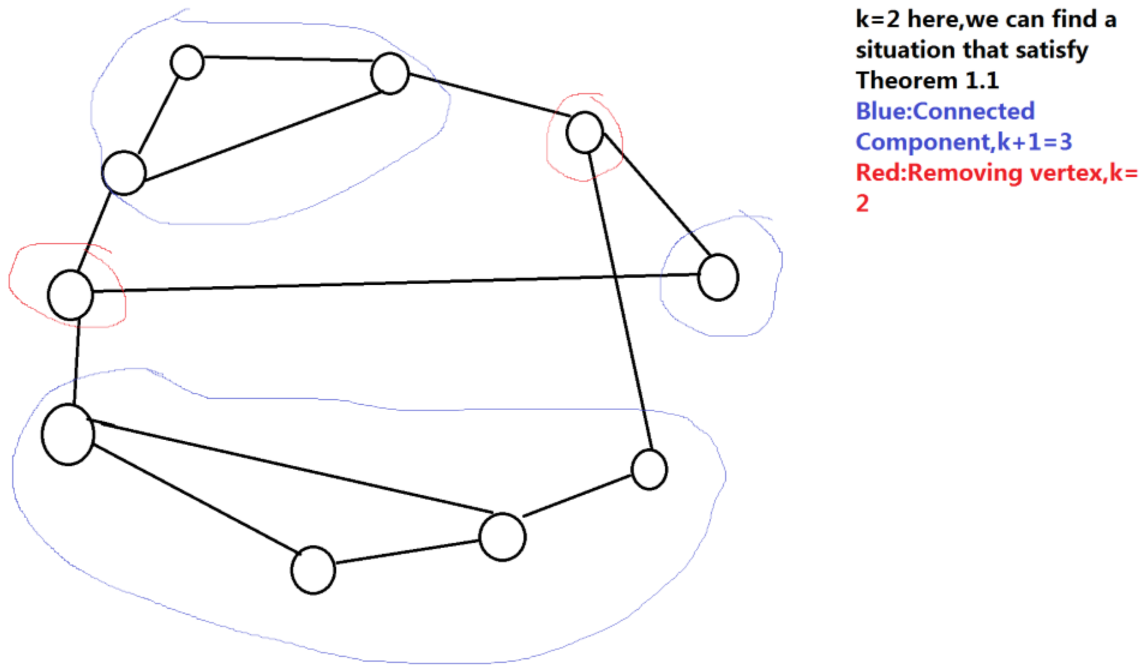
The capacity of min-cut is 4, since

$$d(s- > a) = 1, d(b- > f) = 3$$

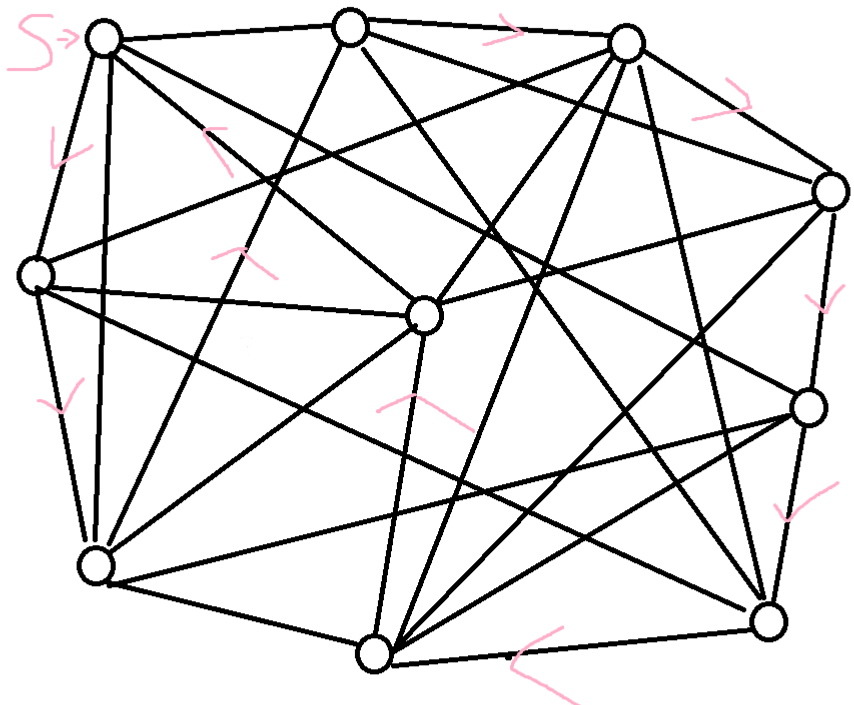
b) The max flow is not unique,the value of this flow is 4 as it is equal to the capacity of min-cut.



4.
a)




b) Graph is on next page:



sum of degrees = 5+5+6+6+5+5+5+6+5+6 = 2|E|
so |E| = 27, |V| = 10. Every distinct non-adjacent vertex will satisfy ore's property. And by ore's theorem,the above graph must have a Hamiltonian cycle

I still bother finding it by marking with pink color.
:)