1.
We can conclude that $\phi(p^k) = p^k - p^{k-1} = p^k * (1 - \frac{1}{p})$.Since p is prime,there is p^{k-1} numbers can be divisible to p^k,like $p, 2p...p^{k-1} * p$ in total.Subtracting those numbers from p^k will give other numbers which are prime to p^k.

Since for $p_1...p_i$ are distinct prime numbers dividing n(1 doesn't count).There is always an equation existing: $n = p_1^{j_1} * p_2^{j_2} ... * p_i^{j_i}$. Applying $\phi(n)$, will give

$$p_1^{j_1} * (1 - \frac{1}{p_1}) * p_2^{j_2} * (1 - \frac{1}{p_2}) * ... * p_i^{j_i} * (1 - \frac{1}{p_i})$$

equals:$p_1^{j_1} * p_2^{j_2} * ...p_i^{j_i} * (1 - \frac{1}{p_1}) * (1 - \frac{1}{p_2}) * ...(1 - \frac{1}{p_i})$
$= n * (1 - \frac{1}{p_1}) * (1 - \frac{1}{p_2}) * ...(1 - \frac{1}{p_i})$, thus proves the equation.

2.
Given $f_n - 1 + f_n = f_{n+1}$
$f_0 = 0, f_1 = 1, f_2 = 1, f_3 = 2, f_4 = 3, f_5 = 5$
$f_0^2 = 0, f_1^2 = 1, f_2^2 = 1, f_3^2 = 4, f_4^2 = 9$
We observe that $S(n) = f_0^2 + f_1^2 + f_2^2 ... + f_n^2$,$S(3) = 6 = 2*3; S(4) = 15 = 3*5$
Make a assumption about the formula : $f_n * f_{n+1} = S(n), n \geq 0$,n is integer.
By induction: Base case:$S(0) = 0 = 0 * 1$, satisfies the case.
Now we want to prove $S(n+1) = f_{n+1} * f_{n+2}$ by giving $f_n * f_{n+1} = S(n)$.
Both sides add f_{n+1}^2
RHS:$S(n+1)$
LHS:$f_n * f_{n+1} + f_{n+1}^2 = f_{n+1} * (f_{n+1} + f_n)$.By the basic formula given at the beginning, $f_{n+1} * (f_{n+1} + f_n) = f_{n+1} * f_{n+2}$
RHS = LHS,now it proves the equation.

3.
By trying small n values, there are :
$t_1 = 1, t_2 = 3, t_3 = 7, t_4 = 15$, assumption about the recurrence formula is : $a_n = 2 * a_{n-1} + 1, n > 1$.Unfortunately,it is not a linear recurrence relationship, the guess method won't apply here.Instead of that ,we do a simple expansion:
$a_0 = 1; a_1 = 1; a_2 = (2*(a_1)+1); a_3 = (2*(a_2)+1) = (2*((2*(a_1)+1)+1)$;
for $a_n = 2 * a_{n-1} + 1$

$$= 2^{n-1} * a_1 + (1 + 2 + 2^2 + ...2^{n-2})$$

$$= 2^{n-1} + 2^{n-1} - 1 = 2^n - 1$$

.By plug t_1, t_2, t_3 in, it satisfies the equation.
By induction, we want to prove $t_{n+1} = 2^n - 1$,we start with $t_n = 2^{n-1} - 1$. Adding $t_n + 1$ on both sides.
LHS: t_{n+1}
RHS: $(2^{n-1} - 1) + (2^{n-1} - 1) + 1 = 2^n - 1$,thus proves the equation.

4.
Total cases:6^k
There are k standard dices(value 1,2,3,4,5,6) sum up to n.
Generating function :$f_1(x) = x + x^2 ... + x^6$,$f_2(x) = x + x^2 ... + x^6 ... f_k(x) = x + x^2 ... + x^6$.

According to Newtons binomial theorem:(no restriction on dice)

$$f(x) = \sum_{k\geq 0} C(n+k-1,k)x^k$$

Suppose $a_1 + a_2 + ...a_k = n$,$1 \leq a_k \leq 6$
By substitute $a_k = b_k + 1$,then $b_1 + b_2 + ...b_k = n - k$,$0 \leq b_k$,

$$f(x) = \sum_{k\geq 0} C(n-k+k-1,k)x^k = \sum_{k\geq 0} C(n-1,k)x^k$$

However this case will include dice has value over 7.
That is we want to find $c_1 + c_2 + ...c_k = n - k$,$7 \leq c_k$, substitute $c_k = d_k + 7$,$d_1 + d_2 + ...d_k = n - k - 7k = n - 8k$,thus the case over 7 is

$$\sum_{k\geq 0} C(n-7k-1,k)x^k$$

.

By exclusion rule,the probability is

$$\frac{\sum_{k\geq 0} C(n-1,k)x^k - \sum_{k\geq 0} C(n-7k-1,k)x^k}{6^k}$$