1.

Given k standard dice,$(x+x^2...+x^6)$ is the generating function for each dice; So the whole generating function is h(n) = $(x(1+x...+x^5))^k$,we are looking for the coefficient of x^n.

$$(x(1+x...+x^5))^k = x^k(\frac{1-x^6}{1-x})^k$$
$$= x^k * (1-x^6)^k * (1-x)^{-k}$$
$$= x^k \sum_{l=0}^{k}(-1)^l * C(k,l) * x^{6l} * \sum_{m=0} C(k+m-1,m)x^m$$

We can see in order to make $n = k+6*l+m$,the coefficient is actually:

$$\sum_{l=0}^{k}(-1)^l * C(k,l) * C(n-6l-1, n-k-6l)$$

From $C(n-6l-1, n-k-6l)$, $n-k-6l > 0$ only when $n > k+6l$;it is not better than $C(n-6l-1, k-1)$

The answer:the coefficient of x^n is

$$\sum_{l=0}^{k}(-1)^l * C(k,l) * C(n-6l-1, k-1)$$

2.

By finding initial cases:

$a_0 = 1$

$a_1 = 1$

$a_2 = 2$

$a_3 = 5 = 2*2+1^2$

$a_4 = 14 = 5*2+2^2$

$a_5 = 37 = 14*2+3^2$

$a_n = 2*a_{n-1}+(n-2)^2$

Treat this whole sequence as combination with "+"'s and "-"'s. The rule is : there are n "+"'s and n "-"'s.

Also for last two characters: "$--$" or "$+-$" is the only option to choose. We have "$+-$",think it just as a intact material that won't affect any other sequence because the rest sequence just follow the last element rule as begining,So there is a a_{n-1};"$--$",we first take it as a_{n-1}, now total has $2*a_{n-1}$ so far;and then there has to be "$+-$" in somewhere from rest sequence to complement this "$-$" situation;so there are "n-2" left for both "+"'s and "-"'s,so there are total $(n-2)^2$ combinations; summing up all cases given : $a_n = 2*a_{n-1}+(n-2)^2$ for n ¿= 2. Since a_1 has only 1 "+" or "-" to play around,let's just treat them as base case.

By solving this relation:

$a_n = a_0 + a_1x + a_2x^2...$

$-2xa_n = 0 - 2a_0x - 2a_1x^2...$
$-\sum_{n\geq 2}(n-2)^2x^n = 0 - 0 - (3-2)x^3...$
Adding them up:
$(1-2x)a_n = 1 - x + \sum_{n\geq 2}(n-2)^2x^n$

$$a_n = \frac{\sum_{n\geq 0} C(n-2,n)x^n + \sum_{n\geq 2}(n-2)^2x^n}{(1-2x)}$$

$$a_n = \sum_{n\geq 2}(C(n-2,n) + (n-2)^2)x^n * \sum_{n\geq 0} 2^n * x^n$$

$$a_n = \sum_{n\geq 2} 2^n * (C(n-2,n) + (n-2)^2) * x^n$$

So the coefficient is :

$$2^n * (C(n-2,n) + (n-2)^2)$$

3. For distinct part: suppose each distinct partition can only be happened or not

k stands for value of each partition. The sum of those k's is equal n.

$$(1+x)(1+x^2) * (1+x^3) * ...(1+x^k)$$

$$= \frac{1-x^2}{1-x} * \frac{1-x^4}{1-x^2} ... \frac{1-x^{2k}}{1-x^k}$$

$$= \frac{1}{(1-x) * (1-x^3) * (1-x^5)...(1-x^l)}$$

note: l is odd here

Above equation turns out to be :

$$= (1 + x + x^{1+1} + x^{1+1+1} + ...) * (1 + x^3 + x^{3+3} + x^{3+3+3} + ...) * ...$$

The equation here is exactly the number of ways to partition n into k parts as k is odd.

So the distinct part generating function can be converted into odd part generating function, which proves the question.