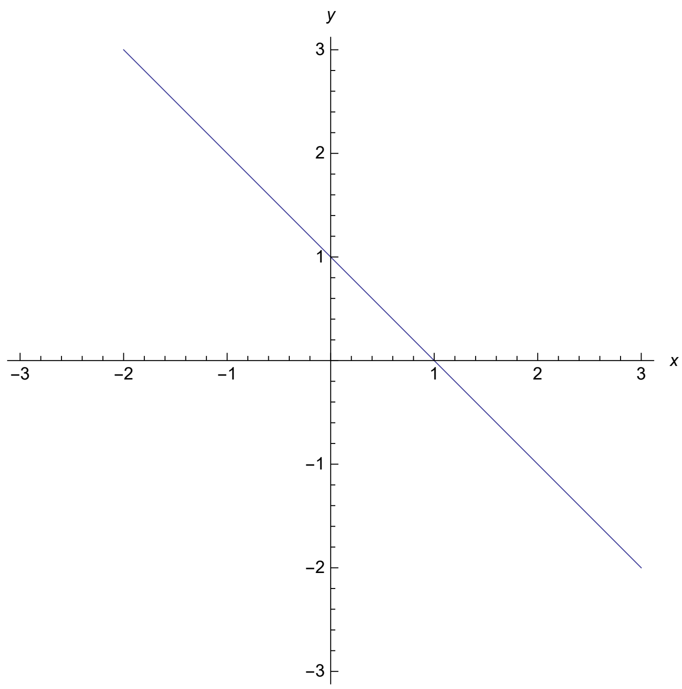
Multimi remarcabile in plan

$\mathbb{R}^2 = \{(\mathbf{x}, \mathbf{y}) \mid \mathbf{x}, \mathbf{y} \in \mathbb{R}\}$

Dreapta in plan

$D = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^2 \mid \mathbf{ax} + \mathbf{by} + \mathbf{c} = 0\}$, **unde a, b, c** $\in \mathbb{R}$ **constante**

```
ContourPlot[x + y - 1 == 0, {x, -3, 3}, {y, -3, 3}, Frame → None, Axes → True, AxesLabel → Automatic]
```



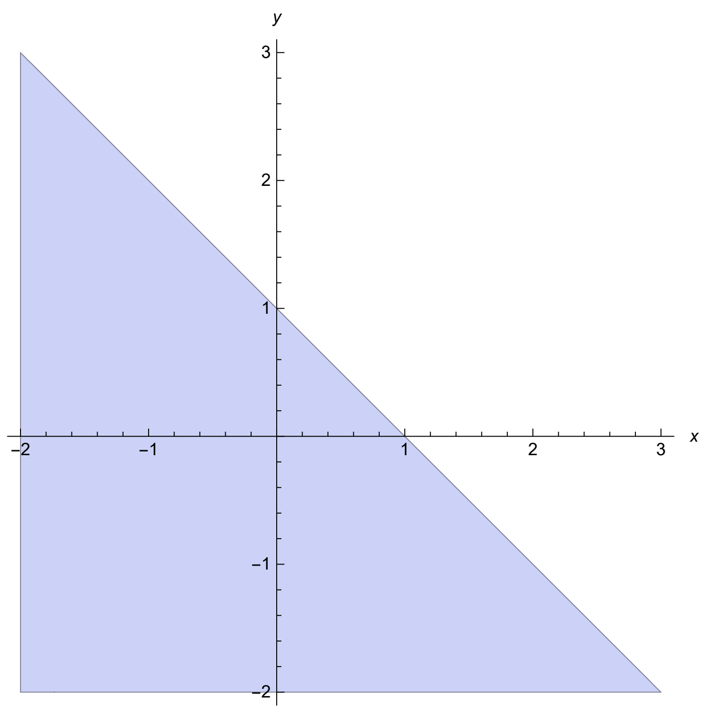
Semiplane inchise delimitate de o dreapta

$D_- = \{ (x, y) \in \mathbb{R}^2 \mid ax + by + c \leq 0 \}$

$D_+ = \{ (x, y) \in \mathbb{R}^2 \mid ax + by + c \geq 0 \}$, unde a, b, c $\in \mathbb{R}$ constante

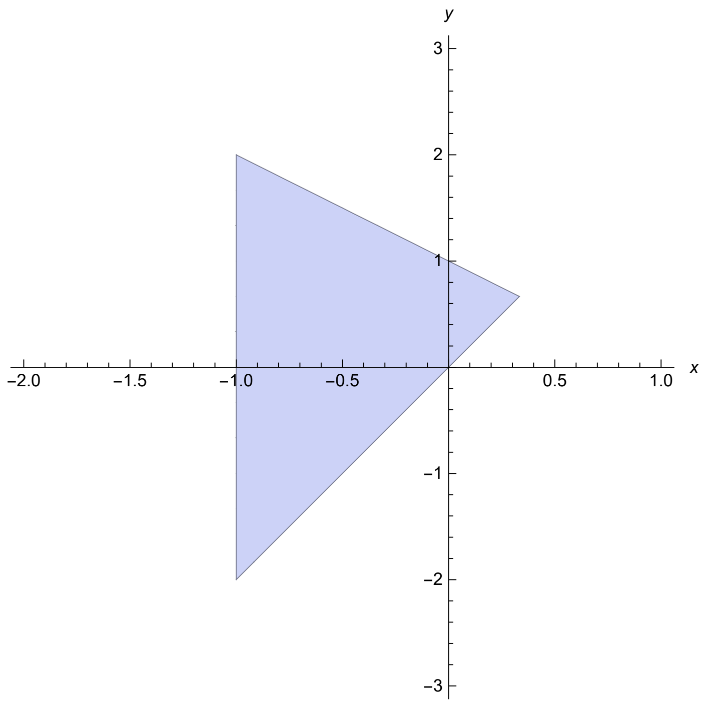
$D_- \cap D_+ = D$

```
RegionPlot[x + y - 1 ≤ 0, {x, -2, 3}, {y, -2, 3}, Frame → None, Axes → True, AxesLabel → Automatic]
```



Ex : $T = \{(x, y) \in \mathbb{R}^2 \mid x + y \leq 1,\ y - 2x \geq 0,\ x + 1 \geq 0\}$

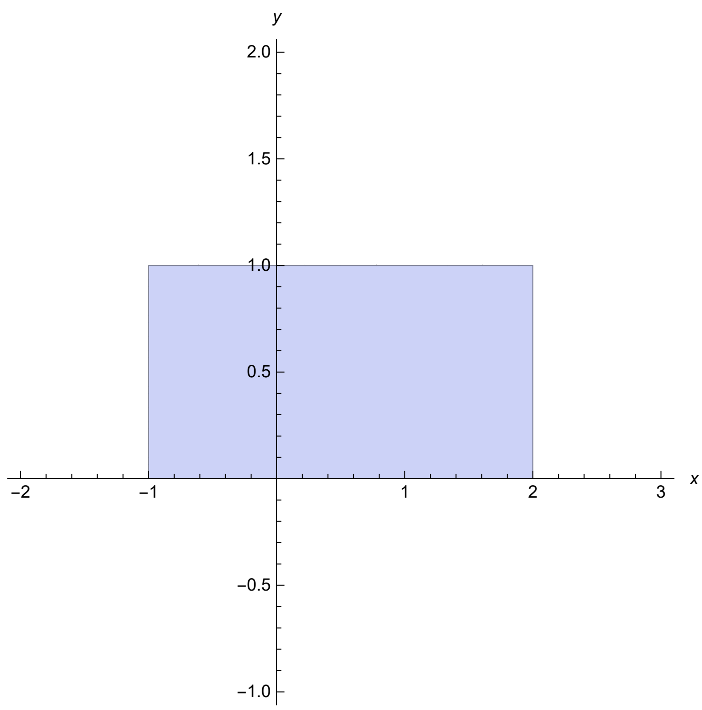
```
RegionPlot[x + y ≤ 1 && y - 2 x ≥ 0 && x + 1 ≥ 0, {x, -2, 1}, {y, -3, 3},
 PlotPoints → 10, Frame → None, Axes → True, AxesLabel → Automatic]
```



Regiunea dreptunghiulara [a, b] x[c, d]

$R = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$

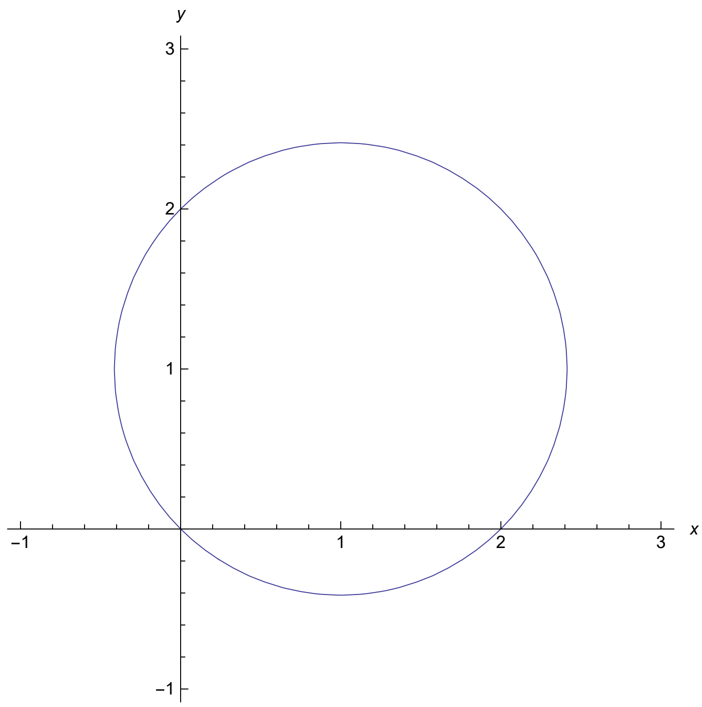
```
RegionPlot[ -1 ≤ x ≤ 2 && 0 ≤ y ≤ 1, {x, -2, 3}, {y, -1, 2},
 PlotPoints → 10, Frame → None, Axes → True, AxesLabel → Automatic]
```



Cerc de centru (x_0, y_0) si raza r

$$C = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; (x - x_0)^2 + (y - y_0)^2 = r^2 \right\}$$

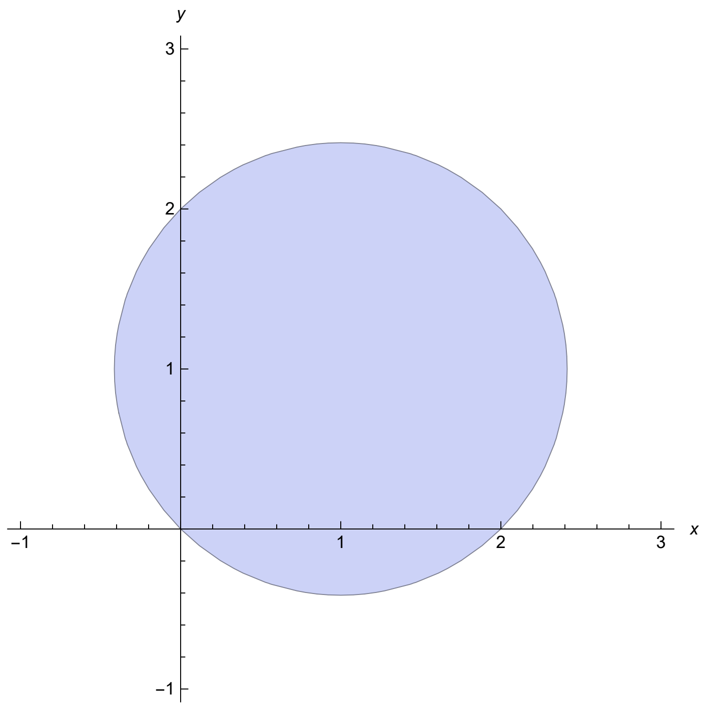
```
ContourPlot[(x - 1)^2 + (y - 1)^2 == 2, {x, -1, 3},
 {y, -1, 3}, Frame → None, Axes → True, AxesLabel → Automatic]
```



Disc inchis de centru (x_0, y_0) si raza r

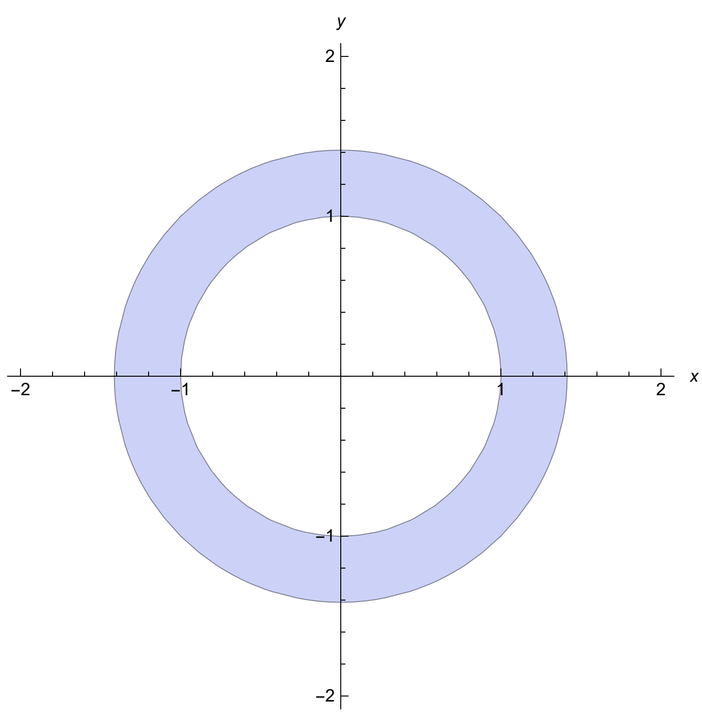
$$C_- = \{ (x, y) \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 \leq r^2 \}$$

```
RegionPlot[(x - 1)^2 + (y - 1)^2 ≤ 2, {x, -1, 3}, {y, -1, 3},
 PlotPoints → 10, Frame → None, Axes → True, AxesLabel → Automatic]
```



Ex : $C = \{ (x, y) \in \mathbb{R}^2 \mid 1 \leq x^2 + y^2 \leq 2 \}$

```
RegionPlot[1 ≤ x^2 + y^2 ≤ 2, {x, -2, 2}, {y, -2, 2},
 PlotPoints → 10, Frame → None, Axes → True, AxesLabel → Automatic]
```



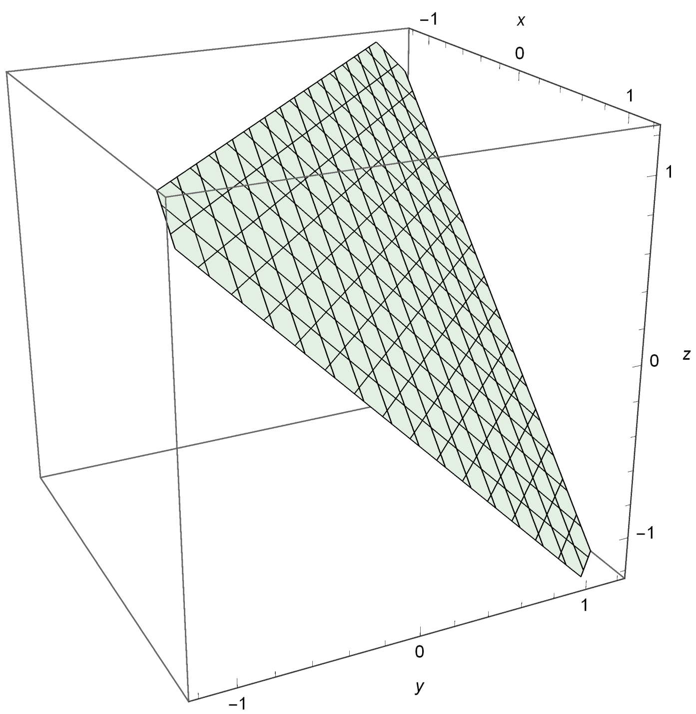
Multimi remarcabile in spatiu

$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$

Plan in spatiu

$P = \left\{(x, y, z) \in \mathbb{R}^3 \;\middle|\; ax + by + cz + d = 0\right\}$, unde a, b, c, d ∈ ℝ constante

```
ContourPlot3D[x + y + z - 1 == 0, {x, -1.2, 1.2}, {y, -1.2, 1.2},
 {z, -1.2, 1.2}, AxesLabel → Automatic, BoxRatios → Automatic]
```

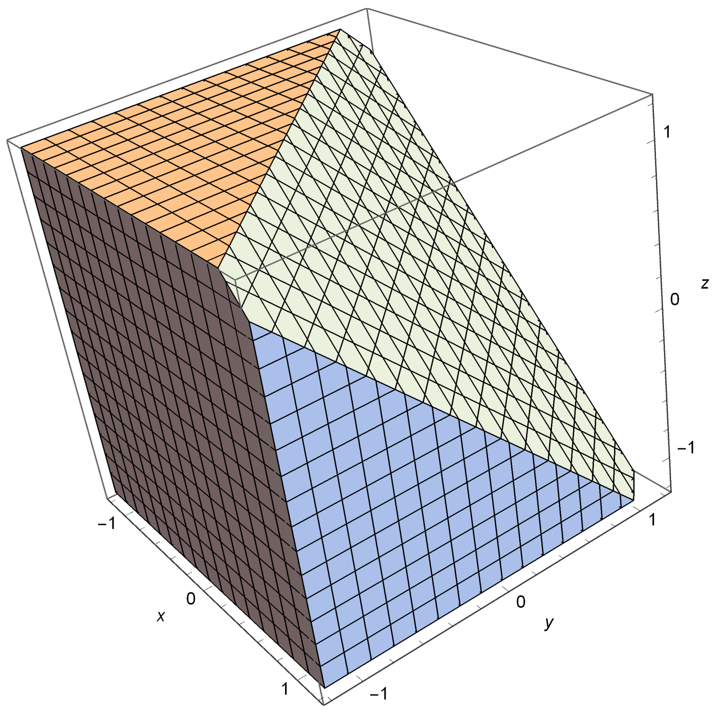


Semispatii inchise delimitate de un plan

$$P_- = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; ax + by + cz + d \le 0 \right\}$$

$$P_+ = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; ax + by + cz + d \ge 0 \right\}$$

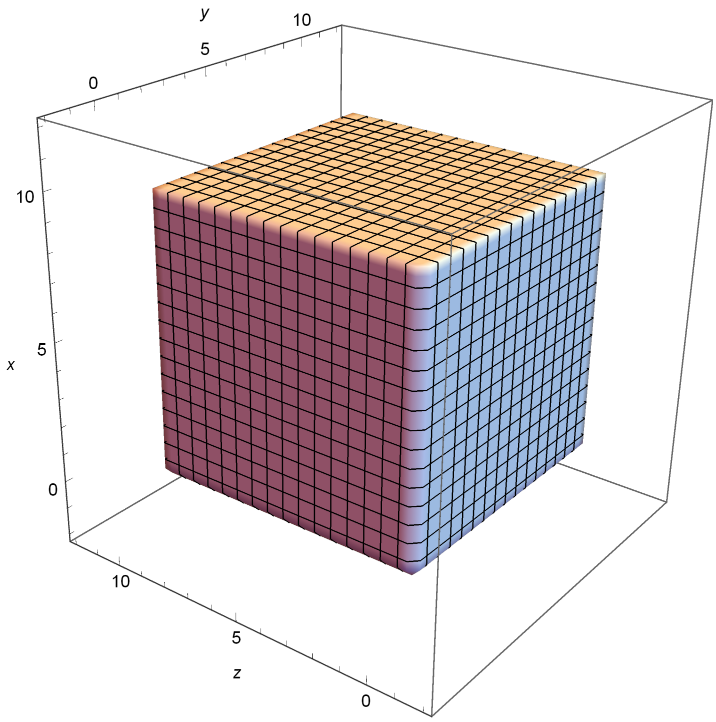
```
RegionPlot3D[x + y + z - 1 ≤ 0, {x, -1.2, 1.2},
 {y, -1.2, 1.2}, {z, -1.2, 1.2}, AxesLabel → Automatic]
```



Regiunea paralelipipedica [a, b] x[c, d] x[e, f]

$$R = \{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$$

```
RegionPlot3D[0 ≤ x ≤ 10 && 0 ≤ y ≤ 10 && 0 ≤ z ≤ 10, {x, -2, 12},
 {y, -2, 12}, {z, -2, 12}, AxesLabel → Automatic, PlotPoints → 50]
```



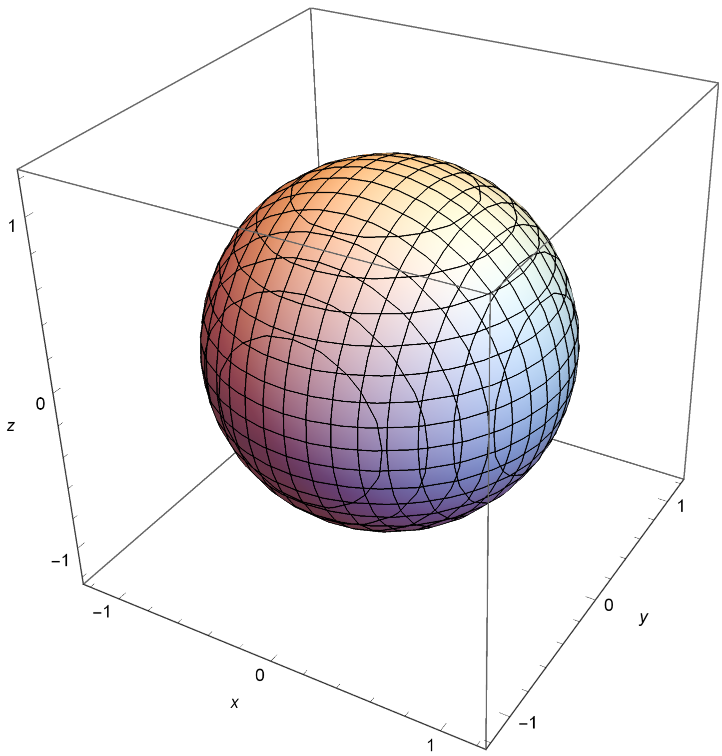
Sfera de centru (x_0, y_0, z_0) si raza r

$$S = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \right\}$$

Bila inchisa de centru (x_0, y_0, z_0) si raza r

$$\overline{B} = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 \le r^2 \right\}$$

```
ContourPlot3D[x^2 + y^2 + z^2 == 1, {x, -1.2, 1.2},
 {y, -1.2, 1.2}, {z, -1.2, 1.2}, AxesLabel → Automatic]
```

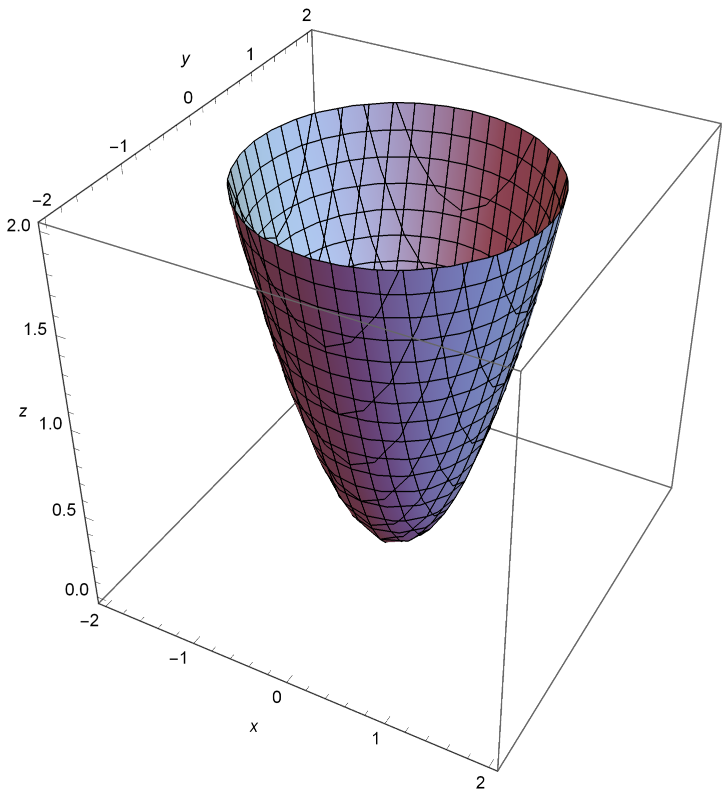


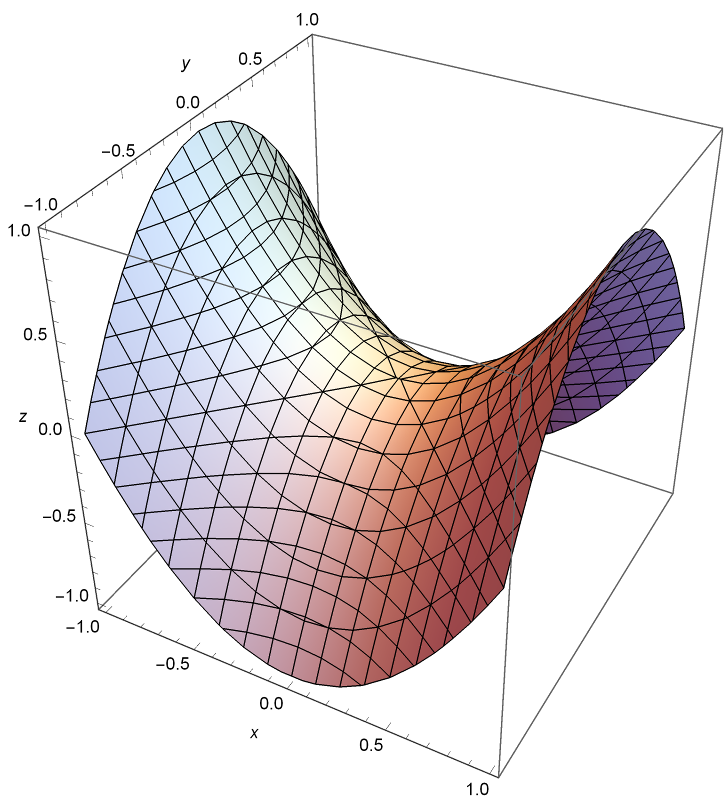
Paraboloid eliptic : $P_e = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = \frac{x^2}{p} + \frac{y^2}{q} \right\}$

Paraboloid hiperbolic : $P_h = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = \frac{x^2}{p} - \frac{y^2}{q} \right\}$,

unde $p, q > 0$ constante

```
ContourPlot3D[z == x^2 + y^2, {x, -2, 2},
 {y, -2, 2}, {z, 0, 2}, AxesLabel → Automatic, BoxRatios → 1]
ContourPlot3D[z == x^2 - y^2, {x, -1, 1}, {y, -1, 1},
 {z, -1, 1}, AxesLabel → Automatic, BoxRatios → 1]
```





Multimi remarcabile in $\mathbb{R}^m$ si esantionarea Monte Carlo

$\mathbb{R}^m = \left\{(x_1, x_2, \ldots, x_m) \mid x_i \in \mathbb{R},\ i = \overline{1, m}\right\}$

Hiperplanul

$P = \{ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid a_1 x_1 + a_2 x_2 + \ldots + a_m x_m + a_0 = 0 \}$,

unde $a_i \in \mathbb{R}$ sunt constante

Hipercubul inchis de centru O_m si latura $2r$

$\overline{H}(O_m, r) = \{ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid -r \le x_1 \le r,\ -r \le x_2 \le r,\ \ldots,\ -r \le x_m \le r\} \overset{not}{=} [-r, r]^m$

Cazuri particulare

$\overline{H}(O_1, r) = [-r, r]$, lungime $= 2r$

$\overline{H}(O_2, r) = [-r, r] \times [-r, r]$, arie $= (2r)^2$

$\overline{H}(O_3, r) = [-r, r] \times [-r, r] \times [-r, r]$, volum $= (2r)^3$

Care este "volumul" unui hipercub?

$\text{vol}\,\overline{H}(O_m, r) = (2r)^m, \quad m \in \mathbb{N},\ m \ge 1$

Bila inchisa de centru O_m si raza r

$$\bar{B}(O_m, r) = \left\{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_1^2 + x_2^2 + \ldots + x_m^2 \le r^2\right\}$$

Cazuri particulare

$$\bar{B}(O_1, r) = \left\{x_1 \in \mathbb{R} \mid x_1^2 \le r^2\right\} = [-r, r], \text{ lungime} = 2\,r$$

$$\bar{B}(O_2, r) = \left\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le r^2\right\}, \text{ arie} = \pi r^2$$

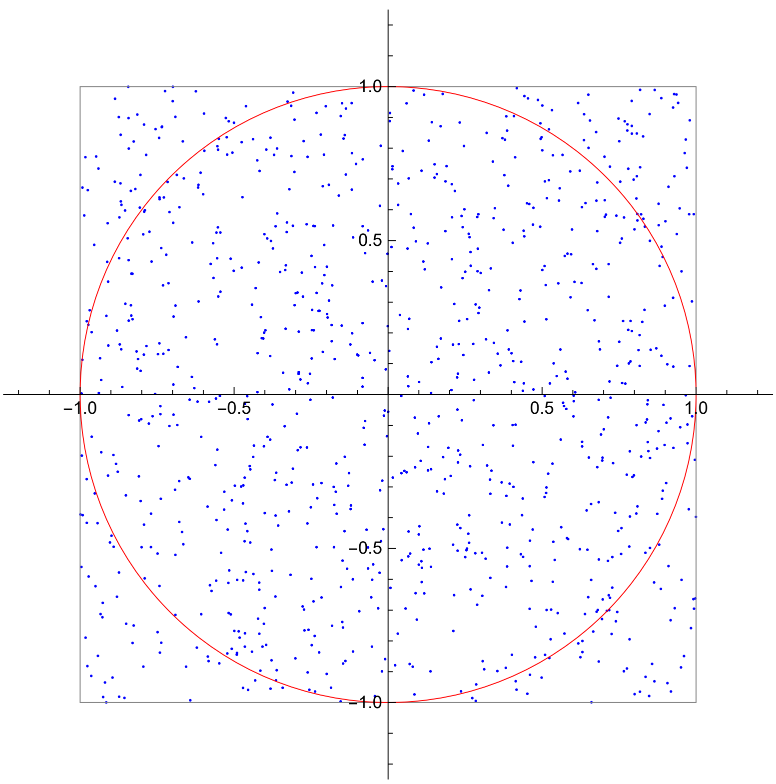
$$\bar{B}(O_3, r) = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \le r^2\right\}, \text{ volum} = \frac{4\,\pi}{3} r^3$$

Care este "volumul" unei (hiper) bile?

Vom da o definitie probabilistica a "volumului" bilei

Cazul plan

```
g = Graphics[{Gray, Line[{{1, 1}, {-1, 1}, {-1, -1}, {1, -1}, {1, 1}}], Red, Circle[{0, 0}, 1]}];
l = ListPlot[Table[{2 * RandomReal[] - 1, 2 * RandomReal[] - 1}, {n, 1000}],
   PlotStyle -> {Blue, PointSize[0.003]}, AspectRatio → Automatic];
Show[l, g, PlotRange → {{-1.2, 1.2}, {-1.2, 1.2}}]
```



Din cele "n" puncte generate aleatoriu in patratul $\overline{H}$ (O_2, 1),
un numar "B(n)" vor fi in discul $\overline{B}$ (O_2, 1)

$$\lim_{n\to\infty} \frac{B(n)}{n} = \frac{\text{aria } \overline{B}(O_2, 1)}{\text{aria } \overline{H}(O_2, 1)} = \frac{\pi}{4} \simeq 0.7853$$

```
n = 1000000;
For[k = 1; b = 0, k ≤ n, k++, b = If[RandomReal[]^2 + RandomReal[]^2 ≤ 1, b + 1, b]]
N[b / n]
```

0.785322

Cazul general

Cum generam puncte in hipercubul $\overline{H}$ (O_m, 1) ?

$x_1, x_2, \ldots, x_m \in [-1, 1]$

Cum testam apartenenta punctelor la bila $\overline{B}$ (O_m, 1) ?

$x_1^2 + x_2^2 + \ldots + x_m^2 \leq 1$

Din cele "n" puncte generate aleatoriu in hipercubul $\overline{H}$ (O_m, 1),
un numar "B(n)" vor fi in bila $\overline{B}$ (O_m, 1)

$$\lim_{n\to\infty} \frac{B(n)}{n} = \frac{\text{vol } \overline{B}(O_m, 1)}{\text{vol } \overline{H}(O_m, 1)} = \frac{\text{vol } \overline{B}(O_m, 1)}{2^m}, \text{ deci}$$

$$\text{vol } \overline{B}(O_m, 1) \overset{\text{def}}{=} 2^m \lim_{n\to\infty} \frac{B(n)}{n}$$

Sa calculam vol $\overline{B}$ (O_4, 1)

```
n = 1000000;
f[x1_, x2_, x3_, x4_] = x1^2 + x2^2 + x3^2 + x4^2;
For[k = 1;
 b = 0, k ≤ n, k++,
 b = If[f[RandomReal[], RandomReal[], RandomReal[], RandomReal[]] ≤ 1, b + 1, b]]
2^4 * N[b / n]
```

4.93968

Are loc formula

$$\operatorname{vol} \overline{B}\,(O_m, r) = \frac{\pi^{m/2}}{\Gamma\left(1 + \frac{m}{2}\right)} r^m, \; m \in \mathbb{N}, \; m \geq 1$$

unde Γ este functia Gama a lui Euler

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, dx, \; \forall\, t > 0$$

$$\operatorname{vol} \overline{B}\,(O_4, r) = \frac{\pi^{4/2}}{\Gamma\left(1 + \frac{4}{2}\right)} r^4 = \frac{\pi^2}{\Gamma(3)} r^4 = \frac{\pi^2}{2!} r^4 \simeq 4.934\, r^4$$

$$\operatorname{vol} \overline{B}\,(O_3, r) = \frac{\pi^{3/2}}{\Gamma\left(1 + \frac{3}{2}\right)} r^3 = \frac{\pi^{3/2}}{\frac{3}{2^2}\,\Gamma\left(\frac{1}{2}\right)} r^3 = \frac{4\,\pi}{3} r^3$$

$$\operatorname{vol} \overline{B}\,(O_2, r) = \frac{\pi^{2/2}}{\Gamma\left(1 + \frac{2}{2}\right)} r^2 = \frac{\pi}{\Gamma(2)} r^2 = \pi r^2$$

EXERCITIU

Calculati valoarea lui $\Gamma\left(1 + \frac{m}{2}\right), \; m \in \mathbb{N}$