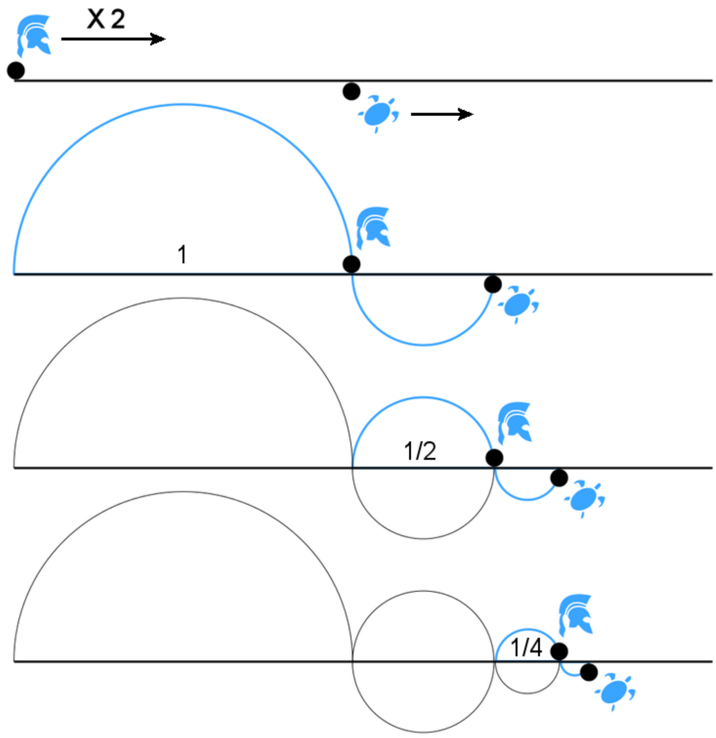
Aplicatii nostime ale seriilor de numere

Paradoxul lui Zeno



$$1 + \frac{1}{2} + \frac{1}{4} + \ldots = \sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2$$

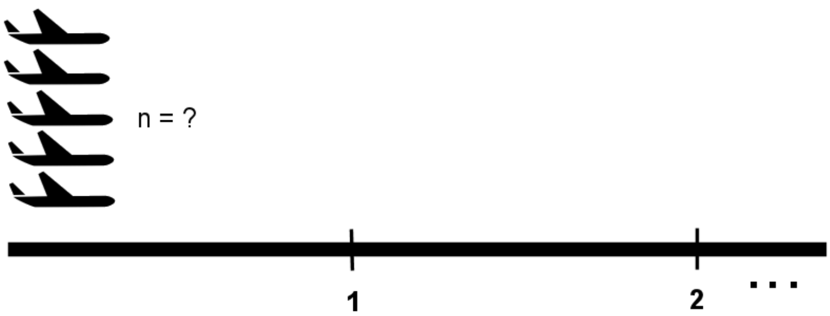
O problema de logistica militara

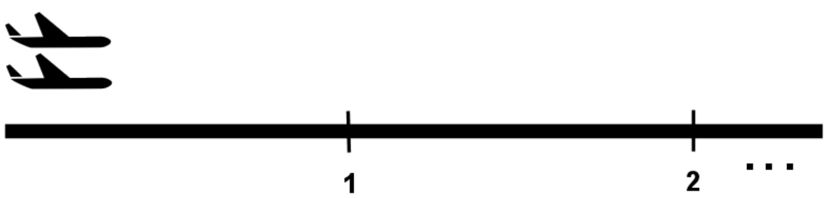
- "n" aparate de zbor pornesc simultan in aceeasi directie cu scopul de a acoperi o distanta maxima

- autonomie de zbor : o unitate de lungime

- abilitate transfer combustibil intre aparate

- se cere numarul initial "n" pentru a putea acoperi distanta de 10 unitati





In general, cele "n" aparate parcurg distanta maxima

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} \rightarrow +\infty \qquad (n \rightarrow +\infty)$$

$$H_n = 10 \implies n = ?$$

```
N[HarmonicNumber[12 366], 10]
N[HarmonicNumber[12 367], 10]
```

```
9.999962148
10.00004301
```

$$\implies n = 12\,367$$

<u>Fractalul lui Koch</u>

Se porneste cu un triunghi echilateral de latura unitate

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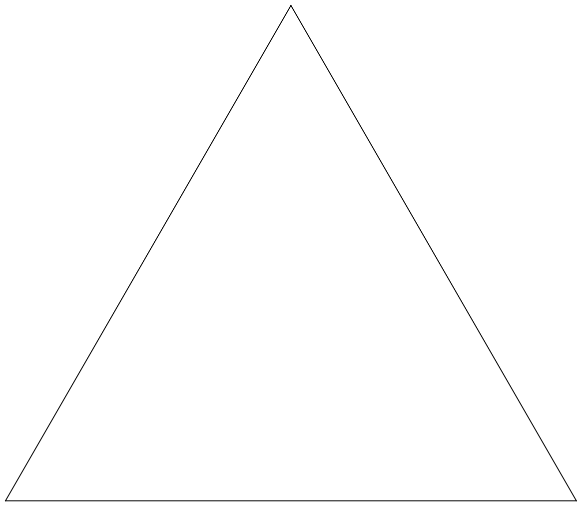
KochSnowflake[0]
```



Latura = 1

Aria = $\dfrac{\sqrt{3}}{4} \overset{\text{not}}{=} \alpha$

Perimetrul = 3

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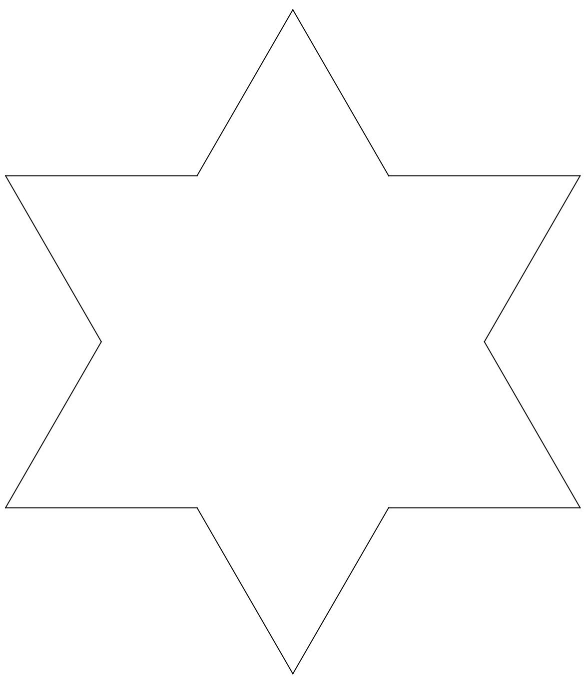
KochSnowflake[1]
```



Latura $= \frac{1}{3}$

Aria $= \alpha + 3 \times \frac{1}{3^2} \alpha$

Perimetrul $= 3 \times 4 \times \frac{1}{3}$

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KochSnowflake[2]
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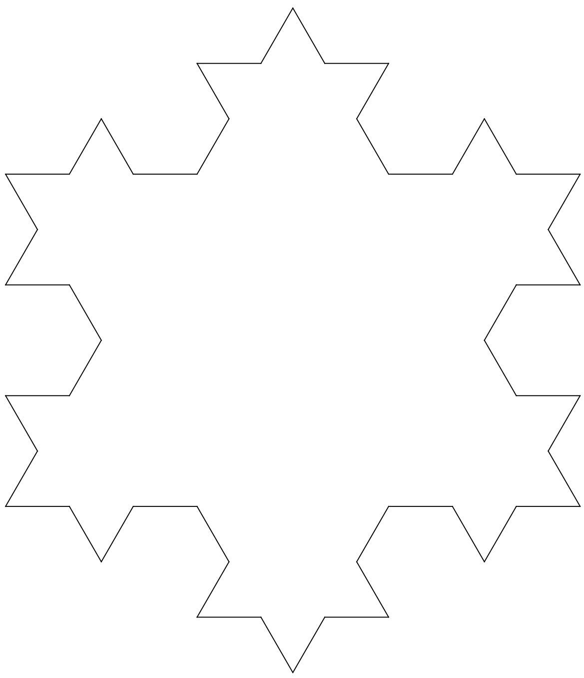

Latura $= \frac{1}{3^2}$

Aria $= \alpha + 3 \times \frac{1}{3^2} \alpha + 3 \times 4 \times \frac{1}{3^4} \alpha$

Perimetrul $= 3 \times 4^2 \times \frac{1}{3^2}$

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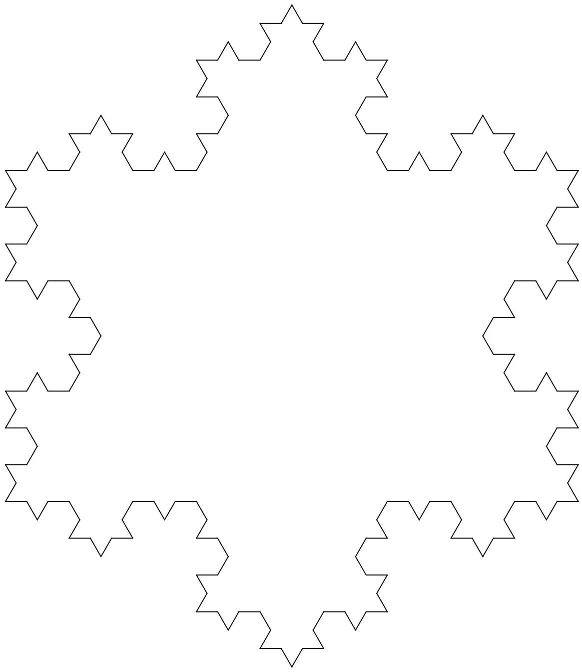
KochSnowflake[3]
```

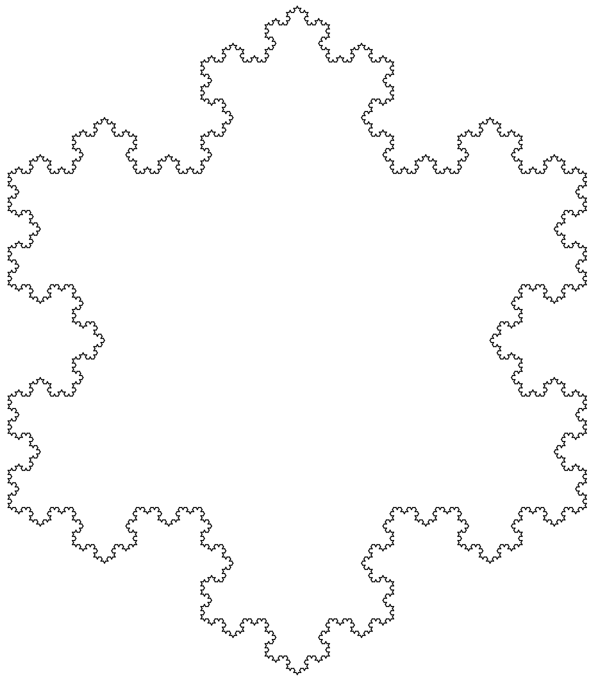


Latura = $\frac{1}{3^3}$

Aria = $\alpha + 3 \times \frac{1}{3^2}\alpha + 3 \times 4 \times \frac{1}{3^4}\alpha + 3 \times 4^2 \times \frac{1}{3^6}\alpha$

Perimetrul = $3 \times 4^3 \times \frac{1}{3^3}$

```
KochSnowflake[5]
```



In general, dupa "n" iteratii avem

Latura = $\frac{1}{3^n}$

Aria = $\alpha + 3 \times \frac{1}{3^2}\alpha + 3 \times 4 \times \frac{1}{3^4}\alpha + \ldots + 3 \times 4^{n-1} \times \frac{1}{3^{2n}}\alpha =$

$$= \alpha + \frac{3}{4}\sum_{k=1}^{n}\left(\frac{4}{9}\right)^k \alpha$$

Perimetrul = $3\left(\frac{4}{3}\right)^n$

La limita se obtine fractalul Koch ($n \to \infty$)

$$\text{Aria} \to \alpha + \frac{3}{4} \sum_{n=1}^{\infty} \left(\frac{4}{9}\right)^{n} \alpha =$$

$$= \alpha + \frac{3}{4} \left[-1 + \sum_{n=0}^{\infty} \left(\frac{4}{9}\right)^{n}\right] \alpha = \alpha + \frac{3}{4}\left[-1 + \frac{1}{1 - \frac{4}{9}}\right] \alpha = \frac{8}{5} \alpha$$

Perimetrul $\to \infty$

Dati un alt exemplu de figura geometrica plana care sa fie simultan

- **marginita**
- **de arie finita (eventual nula)**
- **de perimetru infinit**

(fractalul Vicsek - vezi Wikipedia)

Functia zeta a lui Riemann

ζ = "zeta"

$\zeta : (1, \infty) \longrightarrow \mathbb{R}$,

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}, \quad x > 1$$

$\zeta(1) = +\infty$

$\zeta(p) = ?$, $p \in \mathbb{N}$, $p \geqslant 2$

"Demonstratia lui Euler" pentru valoarea $\zeta(2)$

OBS : Daca se cunosc cele n radacini reale $r_1, r_2, \ldots, r_n$ ale unui polinom $P(x)$ de gradul n, atunci

$$P(x) = A\,(x - r_1)\,(x - r_2) \cdot \ldots \cdot (x - r_n), \quad A \in \mathbb{R} \text{ const.}$$

sau

$$P(x) = B\left(1 - \frac{x}{r_1}\right)\left(1 - \frac{x}{r_2}\right) \cdot \ldots \cdot \left(1 - \frac{x}{r_n}\right), \quad B \in \mathbb{R} \text{ const.}$$

Consideram polinomul de grad "infinit"

$$P(x) = \frac{\sin(x)}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots$$

avand toate radacinile de forma $\pm k\pi$, $k \in \mathbb{N}^*$,

deoarece $P(\pm k\pi) = \frac{\sin(\pm k\pi)}{\pm k\pi} = 0$, $k \in \mathbb{N}^*$

$$\begin{aligned} P(x) &= B\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{3\pi}\right)\left(1 + \frac{x}{3\pi}\right) \cdot \ldots \\ &= B\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdot \ldots \end{aligned}$$

Egalam cele doua polinoame

$$\begin{aligned} 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \ldots &= B\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right) \cdot \ldots \\ &= B\left(1 - \left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \ldots\right) x^2 + \ldots\right) \end{aligned}$$

si identificam termenul liber si coeficientul lui x^2

$$B = 1$$

$$-\frac{1}{3!} = -\frac{1}{\pi^2}\left(\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \ldots\right)$$

$$\Longrightarrow$$

$$\frac{\pi^2}{6} = 1 + \frac{1}{4} + \frac{1}{9} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)$$

Alte valori remarcabile ale functiei ζ

$$\zeta(2) = \frac{\pi^2}{6}, \ \zeta(4) = \frac{\pi^4}{90}, \ \zeta(6) = \frac{\pi^6}{945}, \ \ldots$$

Exista formula pentru $\zeta(2n)$, $n \in \mathbb{N}^*$

Nu exista formula pentru $\zeta(2n+1)$, $n \in \mathbb{N}^*$

$\zeta(3) = ?$, $\zeta(5) = ?$, …

$\zeta(3) \approx 1.202\ldots \notin \mathbb{Q}$ **(constanta lui Apery)**

Legatura cu functia eta a lui Dirichlet

$$\eta(p) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}, \quad p \geq 1$$

$$\eta(1) = \ln(2)$$

$$\eta(p) = \left(1 - 2^{1-p}\right) \zeta(p), \quad p > 1$$

$$\eta(2) = 1 - \frac{1}{n^2} + \frac{1}{n^4} - \frac{1}{n^6} + \ldots = \left(1 - 2^{-1}\right) \zeta(2) = \frac{\pi^2}{12}$$

Legatura functiei zeta cu numerele prime

$$\prod_{n \in \text{prim}} \left(1 - \frac{1}{n^p}\right) = \frac{1}{\zeta(p)}, \quad p > 1$$

OBS : Probabilitatea ca doua numere naturale alese la intamplare sa fie prime intre ele este $\frac{6}{\pi^2}$

ALGORITM

p = 0

Pentru n = 1 pana la 10^7 executa
a = AlegeAleator$[1, 10^{10}]$
b = AlegeAleator$[1, 10^{10}]$
Daca CoPrime[a, b] atunci p = p + 1

Afiseaza $(p/10^7)$

```
t = 10.^7;
For[n = 1;
 p = 0, n ≤ t, n++,
 p = If[ CoprimeQ[RandomInteger[{1, 10^10}], RandomInteger[{1, 10^10}]], p + 1, p]]
p / t
```

```
0.607992
```

```
6./Pi^2
```

```
0.607927
```

TEMA

Realizati un algoritm pentru a calcula probabilitatea ca alegand la intamplare o pereche de numere reale (x, y) cu $-1 \le x \le 1$, $-1 \le y \le 1$, acestea sa verifice inegalitatea $x^2 + y^2 \le 1$

Raspuns : $\approx \frac{\pi}{4}$