Seminar Nr. 4, Markov Chains, Applications and Simulations

```
clear all;
format longg;

problem = input("Problem (1-5): ");
```

1. (Operating mode) A computer system can operate in two different modes. Every hour, it remains in the same mode or switches to a different mode according to the transition probability matrix

$$P = \begin{bmatrix} 0.4 & 0.6 \\ 0.6 & 0.4 \end{bmatrix}.$$

a) If the system is in Mode I at 5:30 pm, what is the probability that it will be in Mode I at 8:30 pm on the same day?
b) In the long run, in which mode is the system more likely to operate?

```
if problem==1
    % Transition matrix
    P = [0.4 0.6; 0.6 0.4]

    % a) Probability of being in Mode 1 at 8:30PM (3 hours later)
    P3 = P^3
    prob_mode_1 = P3(1, 1);
    fprintf("a) P(Mode I at 8:30pm | Mode I at 5:30pm) = %g", prob_mode_1);

    % b) Steady-state distribution: [pi1 pi2] * P = [pi1 pi2]
    % Equivalent to: (P' - I) * [pi1; pi2] = 0 with the constraint pi1 + pi2 = 1
    A = [P' - eye(2); 1 1]; % Augment with normalization condition
    b = [0; 0; 1];          % Right-hand side for the system
    steady_state = A\b;

    fprintf("b) P(Mode I Steady-State) = %g\n   P(Mode II Steady-State) = %g", steady_state(1),
steady_state(2));
end
```

2. (Genetics) An offspring of a black dog is black with probability 0.6 and brown with probability 0.4. An offspring of a brown dog is black with probability 0.2 and brown with probability 0.8. Rex is a brown dog. What is the probability that his grandchild is black?

```
if problem==2
    % Transition matrix
    P = [0.6 0.4; 0.2 0.8]

    % Probability of grandchild being black
    P2 = P^2
    prob_black = P2(2, 1);

    % Display result
    fprintf('P(Rex''s grandchild is black | Rex is brown) = %g', prob_black);
end
```

3. (Traffic lights) Every day, student A takes the same road from his home to the university. There are 4 street lights along his way, and he noticed the following pattern: if he sees a green light at an intersection, then 60% of the time the next light is also green (otherwise, red), and if he sees a red light, then 70% of the time the next light is also red (otherwise, green).
a) If the first light is green, what is the probability that the third light is red?
b) Student B has *many* street lights between his home and the university, but he notices the same pattern. If the first street light on his road is green, what is the probability that the last light is red?

```
if problem==3
    % Transition matrix
    P = [0.6 0.4; 0.3 0.7]

    % a) Probability that the third light is red if the first one is green
    P2 = P^2
    prob_third_red = P2(1, 2);
    fprintf('a) P(Third light red | First light green) = %g', prob_third_red);

    % b) Steady-state distribution: [pi1 pi2] * P = [pi1 pi2]
    % Equivalent to: (P' - I) * [pi1; pi2] = 0 with the constraint pi1 + pi2 = 1
    A = [P' - eye(2); 1 1]; % Augment with normalization condition
    b = [0; 0; 1];          % Right-hand side for the system
    steady_state = A\b;

    fprintf("b) P(Green Light Steady-State) = %g\n   P(Red Light Steady-State) = %g",
steady_state(1), steady_state(2));
end
```

4. (Shared device) A computer is shared by 2 users who send tasks to it remotely and work independently. At any minute, any connected user may disconnect with probability 0.5, and any disconnected user may connect with a new task with probability 0.2. Let $X(t)$ be the number of concurrent users at time t.
a) Find the transition probability matrix.
b) Suppose there are 2 users connected at 10:00 a.m. What is the probability that there will be 1 user connected at 10.02?
c) How many connections can be expected by noon?

```
if problem==4
    % a) Transition matrix
    % The number of concurrent users at time t, X(t), can take the values {0, 1, 2}.
    disp('a)')

    Prow1 = binopdf(0:2, 2, 0.2); % from 0 connections is the number of successes in n = 2 trials
    Prow21 = 0.5 * 0.8; % the connected user disconnects and the disconnected doesn't connect
    Prow22 = 0.5 * 0.8 + 0.5 * 0.2; % the connected user remains connected and the
    % disconnected remains disconnected or they swap
    Prow23 = 0.5 * 0.2; % the disconnected user connects and the connected remains connected
    Prow3 = binopdf(2:-1:0, 2, 0.5); % from 2 connections is the number of disconnections in n = 2
trials

    P = [Prow1; Prow21 Prow22 Prow23; Prow3]

    % b) Probability of 1 user connected at 10:02
    disp('b)')
```

```
    P2 = P^2
    prob_one_user = P2(3, 2);
    fprintf("P(1 user connected 10:02 | 2 user connected 10:00) = %g", prob_one_user);

    % c)
    disp('c)')
    A = [P' - eye(3); 1 1 1]
    b = [0; 0; 0; 1]
    steady_state = A\b

    % Expected number of connections
    expected_connections = steady_state' * (0:2)'
end
```

5. (Weather forecast) Recall the example at the lecture, about Rainbow City with sunny/rainy days (state 1 was "sunny" and state 2 was "rainy"), with transition probability matrix

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}.$$

a) If the initial forecast is 80% chance of rain, write a Matlab code to generate the forecast for the next 30 days.
b) In Rainbow City, if there are 7 days or more in a row of sunshine, there is the danger of drought, and if it rains for a week or more, there is the threat of flooding. Local authorities need to be prepared for each situation. Use the code from part a) to conduct a Monte Carlo study for estimating the probability of a water shortage and the probability of flooding.

```
if problem==5
    P0 = [0.2 0.8];
    P = [0.7 0.3; 0.4 0.6];

    % a) Forecast for the next 30 days
    disp('a)')
    forecast_30 = zeros(2, 30);
    forecast_30(:, 1) = P0;

    for i=2:30
        forecast_30(:, i) = P' * forecast_30(:, i-1);
    end

    % Display the forecast probabilities for 30 days
    disp('Forecast probabilities for the next 30 days (Sunny, Rainy):');
    disp(forecast_30');

    % b) Monte Carlo study for estimating water shortage and flooding
    disp('b)')
    N = input('Nr. of simulations: ');
    water_shortages = zeros(1, N);
    floodings = zeros(1, N);
    for sim=1:N
        rainy_streak = 0;
        sunny_streak = 0;
        for i=1:30
            if rand > forecast_30(1, i)
                if rainy_streak > 0
                    rainy_streak = rainy_streak + 1;
                else
                    if sunny_streak > 7
                        water_shortages(sim) = 1;
```

```
                    end
                    sunny_streak = 0;
                    rainy_streak = 1;
                end
            else
                if sunny_streak > 0
                    sunny_streak = sunny_streak + 1;
                else
                    if rainy_streak > 7
                        floodings(sim) = 1;
                    end
                    rainy_streak = 0;
                    sunny_streak = 1;
                end
            end
        end
    end
    fprintf("P(water shortage in the next 30 days): %g", sum(water_shortages) / N);
    fprintf("P(flooding in the next 30 days): %g", sum(floodings) / N);
end
```

```
a)
Forecast probabilities for the next 30 days (Sunny, Rainy):
                      0.2                      0.8
                     0.46                     0.54
                    0.538                    0.462
                   0.5614                   0.4386
                  0.56842                  0.43158
                 0.570526                 0.429474
                0.5711578                0.4288422
               0.57134734               0.42865266
              0.571404202              0.428595798
             0.5714212606             0.4285787394
            0.57142637818            0.42857362182
           0.571427913454           0.428572086546
          0.5714283740362          0.4285716259638
         0.57142851221086         0.42857148778914
        0.571428553663258        0.428571446336742
        0.571428566098977        0.428571433901023
        0.571428569829693        0.428571430170307
        0.571428570948908        0.428571429051092
        0.571428571284672        0.428571428715327
        0.571428571385401        0.428571428614598
         0.57142857141562        0.428571428584379
        0.571428571424686        0.428571428575314
        0.571428571427406        0.428571428572594
        0.571428571428221        0.428571428571778
        0.571428571428466        0.428571428571533
         0.57142857142854         0.42857142857146
        0.571428571428562        0.428571428571438
        0.571428571428568        0.428571428571431
         0.57142857142857        0.428571428571429
        0.571428571428571        0.428571428571429
b)
P(water shortage in the next 30 days): 0.099957
P(flooding in the next 30 days): 0.015819
```