Seminar Nr. 5, Counting Processes

```
format longg;
```

1. On the average, 6 airplanes per minute land at a certain international airport. Assume the number of landings is modeled by a Binomial counting process.
a) What frame length should be used to guarantee that the probability of a landing does not exceed 0.1?
b) Using the chosen frames, compute the probability of no landings during the next half a minute;
c) Using the chosen frames, compute the probability of more than 170 landed airplanes during the next 30 minutes.

```
% a) Frame length to guarantee some probability
lambda = 6; % planes per minute
p_max = 0.1;
delta = p_max / lambda; % Frame length in minutes
delta_sec = delta * 60;
fprintf("a) Frame length in seconds: %g", delta_sec);
```

```
a) Frame length in seconds: 1
```

```
% b) Probability of no landings during the next half a minute
n_b = 30; % number of frames for half a minute
p = 0.1; % probability of landing per frame
prob_no_landings = binopdf(0, n_b, p);
fprintf("b) P(no landings during the next half minute) = %g", prob_no_landings);
```

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b) P(no landings during the next half minute) = 0.0423912
```

```
% c) Probability of more than 170 landed airplanes during the next 30 minutes
n_b = 30 * 60; % number of frames for 30 minutes
p = 0.1; % probability of landing per frame
prob_at_most_170 = binocdf(170, n_b, p);
prob_more_than_170 = 1 - prob_at_most_170;
fprintf("c) P(more than 170 landed airplanes during next 30 minutes) = %g", prob_more_than_170);
```

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c) P(more than 170 landed airplanes during next 30 minutes) = 0.770884
```

2. Messages arrive at a communications center according to a Binomial counting process with 30 frames per minute. The average arrival rate is 40 messages per hour. How many messages can be expected to arrive between 10 a.m. and 10:30 a.m.? What is the standard deviation of that number of messages?

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lambda = 40/60; % messages per minute
delta = 1/30; % frame length in minutes
p = lambda * delta % probability of a message arriving at a certain frame
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p =
      0.0222222222222222
```

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interval = 30; % time frame to check the expected messages
n = interval/delta; % number of frames in the given time frame
expected_messages = interval/delta * p;
fprintf("E(X) = %g", expected_messages);
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E(X) = 20
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```
standard_deviation = sqrt(n * p * (1-p));
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fprintf("sigma(X) = %g", standard_deviation);
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sigma(X) = 4.42217
```

3. An internet service provider offers special discounts to every third connecting customer. Its customers connect to the internet according to a Poisson process with the rate of 5 customers per minute. Compute
a) the probability that no offer is made during the first 2 minutes;
b) the probability that no customers connect for 20 seconds;
c) expectation and standard deviation of the time of first offer.

```
% a) Probability that no offer is made during the first 2 minutes
lambda = 5; % per minute
lambda_t = lambda * 2; % the first two minutes
% P(no offer) = P(X < 3) = P(X <= 2)
p_no_offer = poisscdf(2, lambda_t);
fprintf("a) P(no offer in 2 minutes) = %g", p_no_offer)
```

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a) P(no offer in 2 minutes) = 0.0027694
```

```
% b) No customer connects for 20 seconds
lambda_t = lambda * 20/60;
p_no_customer = poisspdf(0, lambda_t);
fprintf("b) P(no customer in 20 seconds) = %g", p_no_customer)
```

```
b) P(no customer in 20 seconds) = 0.188876
```

```
% c) expectation and standard deviation of the first time of first order
alpha = 3; % we need 3 events to happen in total
expected_first_offer = alpha * 1/lambda;
fprintf("c) E(first offer) = %g", expected_first_offer);
```

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c) E(first offer) = 0.6
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```
std_first_offer = sqrt(alpha * 1/lambda^2);
fprintf("   std(first offer) = %g", std_first_offer)
```

```
   std(first offer) = 0.34641
```

4. On the average, Mr. X drinks and drives once in 4 years. He knows that

- every time he drinks and drives, he is caught by the police;
- according to the law of his state, the third time he is caught drinking and driving, he loses his driver's license;
- a Poisson counting process models such "rare events" as drinking and driving.

What is the probability that Mr. X will keep his driver's license for at least 10 years?

```
lambda = 1/4; % times a year
lambda_t = lambda * 10; % times 10 years
prob_keeping_license = poisscdf(2, lambda_t);
fprintf("P(Mr. X keeps his license) = %g", prob_keeping_license);
```

```
P(Mr. X keeps his license) = 0.543813
```

5. Simulation and illustration of Binomial and Poisson counting processes.
a) Given sample path size N_B and probability of arrival p, simulate a Binomial counting process $X(t)$.
Application: For a frame size of 1 second, simulate the number of airplane landings from Problem 1., for 1 minute.
b) Given frequency λ and a time frame $[0, T_{max}]$, simulate a Poisson counting process $X(t)$.
Application: Simulate the number of internet connections from Problem 3., for a period of half an hour.

```
NB = 60 % simulation time frame seconds
```

```
NB =
    60
```

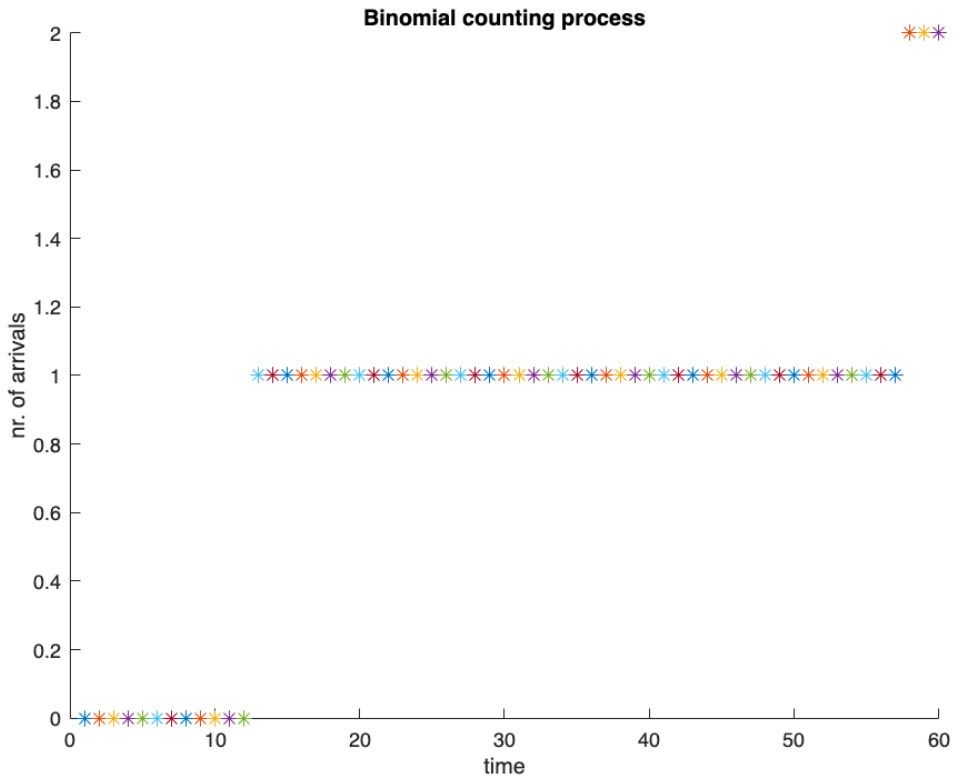
```
p = 0.1; % probability of success (arrival)


X = zeros(1, NB); % allocate memory for X
X(1) = (rand < p); % first Bernoulli trial; X nr. of successes
for t = 2 : NB
    X(t) = X(t - 1) + (rand < p); % count the nr. of successes
end
X
```

```
X = 1×60
     0     0     0     0     0     0     0     0     0     0     0     0     1···
```

```

clf
axis([0 NB 0 max(X)]); % allocate the box for the entire simulated segment
hold on
title('Binomial counting process')
xlabel('time');
ylabel('nr. of arrivals')
for t = 1:NB
    plot(t, X(t), '*', 'MarkerSize', 8); hold on
    % plot each point with a '*'
end
hold off
```

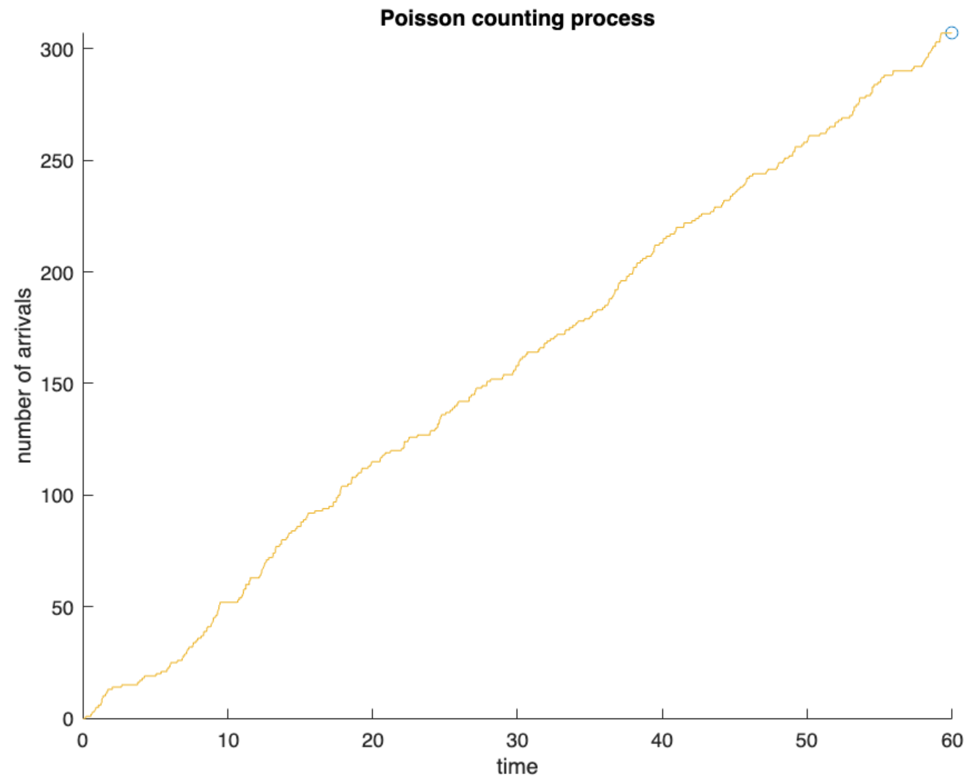



```
lambda = 5; % given frequency lambda
Tmax = 60; % given time frame period (in minutes)
arr_times = -1/lambda * log(rand); % array containing arrival times
                                   % each interarriv. time is Exp(lambda)
last_arrival = arr_times;
while last_arrival <= Tmax
    last_arrival = last_arrival - 1/lambda * log(rand);
    arr_times = [arr_times, last_arrival];
end

arr_times = arr_times(1 : end - 1); % nr. of arrivals during time Tmax
% last last_arrival should not be included

%%% Graph the trajectory %%%
step = 0.01; % small step size, simulate continuity
t = 0 : step: Tmax; % time variable
Nsteps = length(t);
X = zeros(1, Nsteps); % Poisson process X(t)
for s = 1 : Nsteps
    X(s) = sum(arr_times <= t(s));
end % X(s) is the number of arrivals by the time t(s)
% X

% illustration
clf
axis([0 max(t) 0 max(X)]); hold on
title('Poisson counting process')
xlabel('time'); ylabel('number of arrivals')
comet(t, X);
```


Poisson counting process
number of arrivals
time
0
10
20
30
40
50
60
0
50
100
150
200
250
300