Laboratory 1: Modelling Change with Difference Equations

1. Write the first five terms of the following sequences:

   (a) $a_{n+1} = 3a_n$, $a_0 = 1$;

   (b) $a_{n+1} = 3a_n\,(a_n + 1)$, $a_0 = 0$;

   (c) $a_{n+1} = 2a_n^2$, $a_0 = -1$;

```
> restart;
> n := 5;
```

$$n := 5 \tag{1}$$

Exercise (1.a)

```
> a[0] := 1;
```

$$a_0 := 1 \tag{2}$$

```
> for i from 1 to n do
   a[i] := 3 * a[i-1];
  end do:
> seq(a[i], i=0..n);
```

$$1, 3, 9, 27, 81, 243 \tag{3}$$

Exercise (1.b)

```
> a[0] := 0;
```

$$a_0 := 0 \tag{4}$$

```
> for i from 1 to n do
    a[i] := 3 * a[i-1] * (a[i-1] + 1);
  end do:
> seq(a[i], i=0..n);
```

$$0, 0, 0, 0, 0, 0 \tag{5}$$

Exercise (1.c)

```
> a[0] := -1;
```

$$a_0 := -1 \tag{6}$$

```
> for i from 1 to n do
    a[i] := 2 * a[i-1]^2;
  end do:
> seq(a[i], i=0..n);
```

$$-1, 2, 8, 128, 32768, 2147483648 \tag{7}$$

2. Write the first ten terms of the sequence satisfying the following difference equations and draw the corresponding graph of the generated dynamical system:

   (a) $\Delta a_n = \frac{1}{3} a_n$, $a_0 = 1$;

   (b) $\Delta a_n = 0.01 \cdot (200 - a_n)$, $a_0 = 10$;

   (c) $\Delta a_n = 1.5 \cdot (100 - a_n)$, $a_0 = 200$;

```
> restart;
> n := 10;
```

$$n := 10 \tag{8}$$

Exercise (2.a)

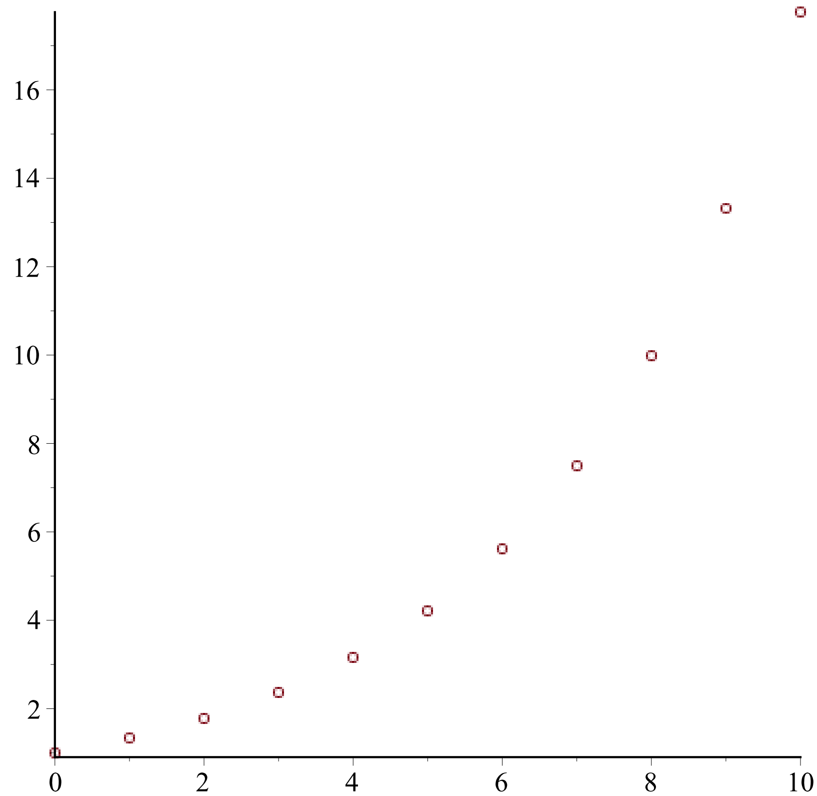
```
> a[0] := 1;
```

$$a_0 := 1 \tag{9}$$

```
> for i from 1 to n do
    a[i] := a[i-1] + 1/3 * a[i-1];
  end do:
> seq(a[i], i=0..n);
```

$$1, \frac{4}{3}, \frac{16}{9}, \frac{64}{27}, \frac{256}{81}, \frac{1024}{243}, \frac{4096}{729}, \frac{16384}{2187}, \frac{65536}{6561}, \frac{262144}{19683}, \frac{1048576}{59049} \tag{10}$$

```
> plot([[j, a[j]]$j=0..n], style=point, symbol=circle);
```



Exercise (2.b)

```
> a[0] := 10;
```

$$a_0 := 10 \tag{11}$$

```
> for i from 1 to n do
    a[i] := a[i-1] + 0.01 * (200 - a[i-1])
  end do:
```

$$a_1 := 11.90$$

$$a_2 := 13.7810$$

$$a_3 := 15.643190$$

$$a_4 := 17.48675810$$

$$a_5 := 19.31189052$$

$$a_6 := 21.11877161$$

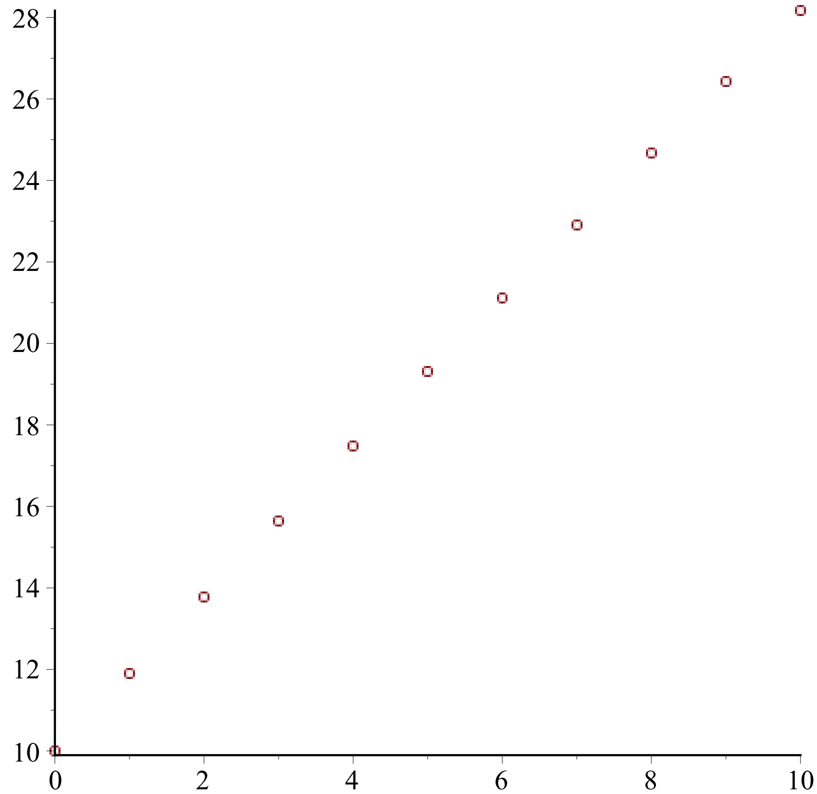
$$a_7 := 22.90758389$$

$$a_8 := 24.67850805$$

$$a_9 := 26.43172297$$

$$a_{10} := 28.16740574 \quad (12)$$

```
> plot([[j, a[j]]$j=0..n], style=point, symbol=circle);
```



Exercise (2.c)

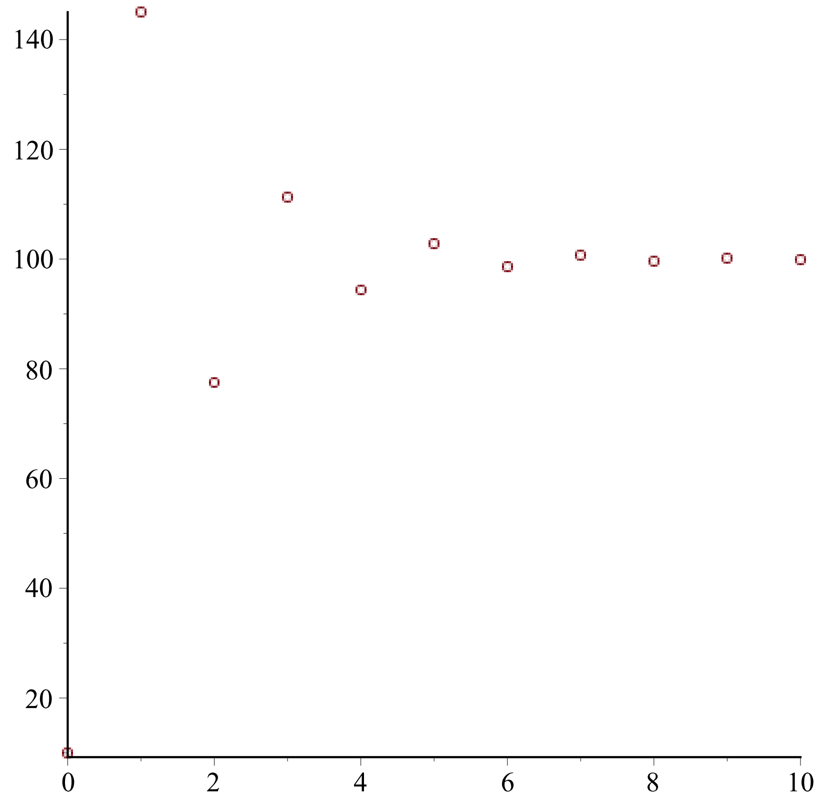
```
> a[0] := 200;
```

$$a_0 := 200 \quad (13)$$

```
> for i from 1 to n do
    a[i] := a[i-1] + 1.5 * (100 - a[i-1]);
  end do:
> seq(a[i], i=1..n);
```

145.0, 77.50, 111.250, 94.3750, 102.81250, 98.593750, 100.7031250, 99.64843750, 100.1757812, 99.91210940 **(14)**

```
> plot([[j, a[j]]$j=0..n], style=point, symbol=circle);
```



3. (Decay of Digoxin in the Bloodstream) Digoxin is used in treatment of heart disease. Doctors must prescribe an amount of medicine that keeps the concetration of digoxin in the bloodstream above an *effective level* without exceeding the *safe level*. For an initial dosage of 0.5 mg in the bloodstream , table shows the amount of digoxin a_n remaining in the bloodstream of a particular pacient after n days.

n	0	1	2	3	4	5	6	7	8
a_n	0.5	0.345	0.238	0.164	0.113	0.078	0.054	0.037	0.026

(a) Get Δa_n and plot Δa_n versus a_n

(b) Find a proportional constant for Δa_n and a_n.

```
> restart;
> n := 0..8;
  a := [0.5, 0.345, 0.238, 0.164, 0.113, 0.078, 0.054, 0.037, 0.026];
```

$$n := 0\,..8$$

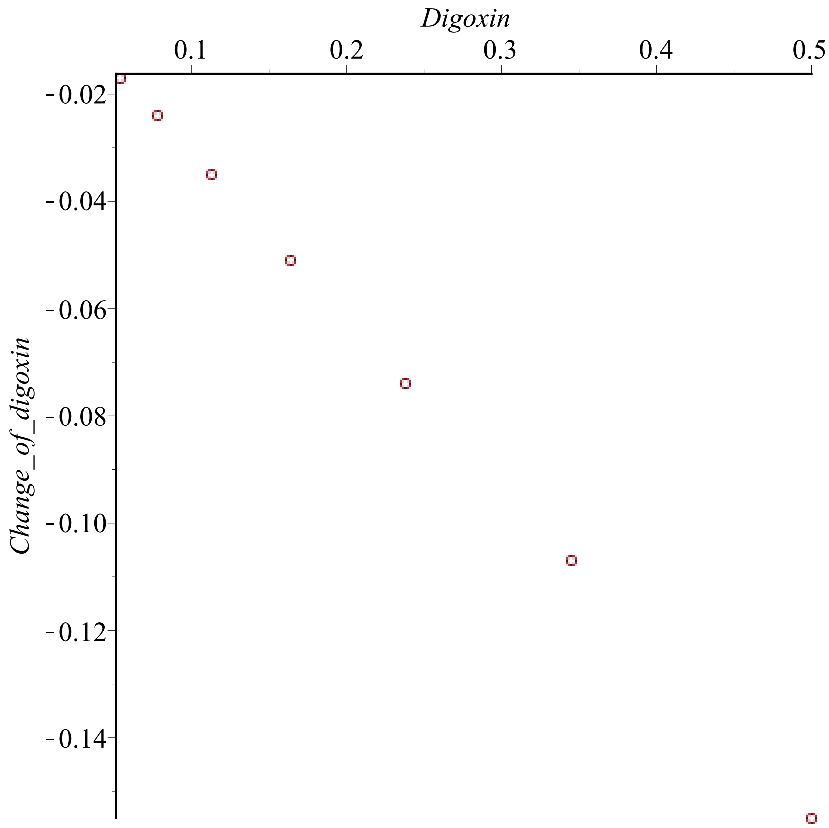
$$a := [0.5, 0.345, 0.238, 0.164, 0.113, 0.078, 0.054, 0.037, 0.026] \quad \textbf{(15)}$$

Exercise (3.a)

```
> for i from 1 to 7 do
    delta_a[i] := a[i+1] - a[i];
  end do:
> seq(delta_a[i], i=1..7);
```

$$-0.155, -0.107, -0.074, -0.051, -0.035, -0.024, -0.017 \quad (16)$$

```
> plot([[a[j], delta_a[j]]$j=1..7], style=point, symbol=circle,
  labels=[Digoxin, Change_of_digoxin], labeldirections=[HORIZONTAL,
  VERTICAL]);
```



Exercise (3.b)

```
> k = (delta_a[6] - delta_a[4]) / (a[6] - a[4]);
```

$$k = -0.3139534884 \quad (17)$$

4. The following data were obtained for the growth of a sheep population introduced into new enviroment on the island of Tasmania:

Year	1814	1824	1834	1844	1854	1864
Population	125	275	830	1200	1750	1650

(a) Plot the data. Is there a trend?

(b) Formulate discrete dynamical system that resonable approximate the change observed. Compare real data with estimated data obtained from the model.

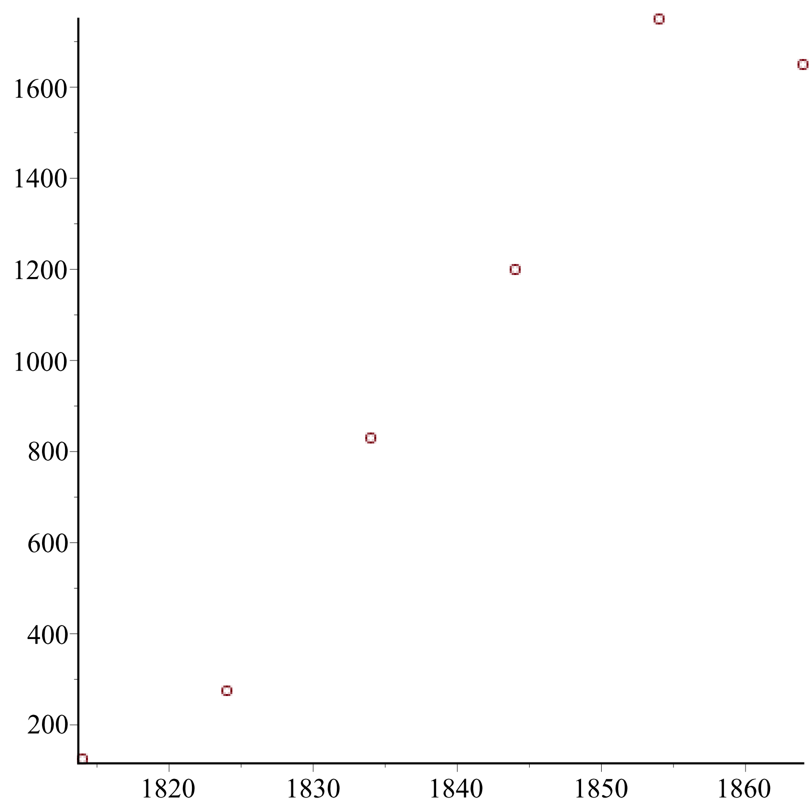
```
> restart;
> y := [1814, 1824, 1834, 1844, 1854, 1864];
  p := [125, 275, 830, 1200, 1750, 1650];
```

$$y := [1814, 1824, 1834, 1844, 1854, 1864]$$

$$p := [125, 275, 830, 1200, 1750, 1650] \quad (18)$$

Exercise (4.a)

```
> plot([[y[j], p[j]]$j=1..6], style=point, symbol=circle);
```

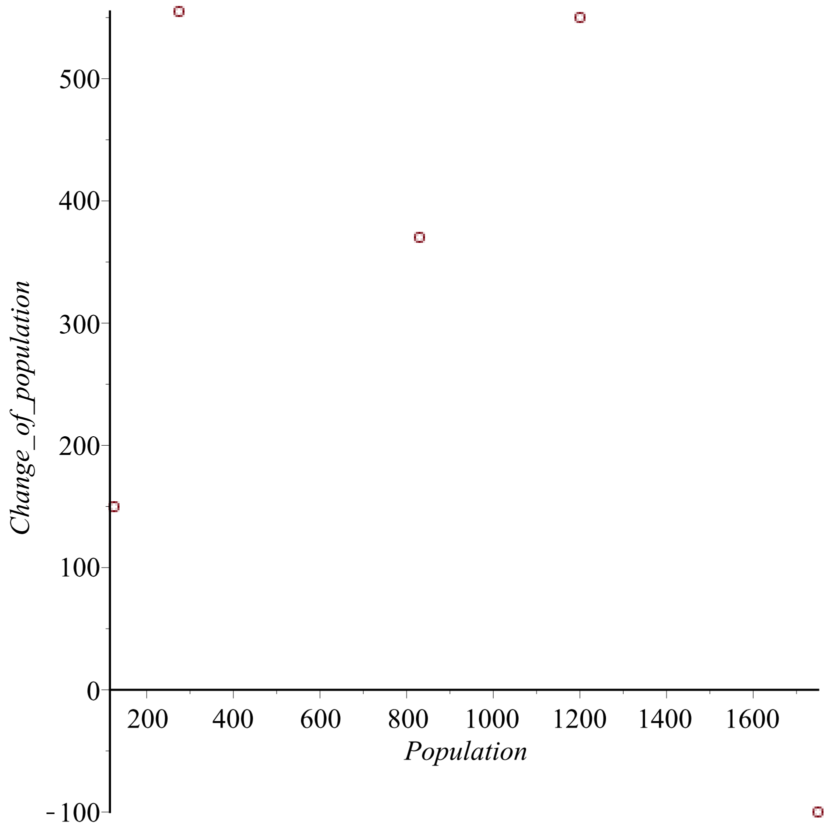


We cannot really notice a trend between the year and the population, we'll try to check for a trend between the population and delta population.

```
> for i from 1 to 5 do
    delta_p[i] := p[i+1] - p[i];
  end do:
> seq(delta_p[i], i=1..5);
```

$$150, 555, 370, 550, -100 \tag{19}$$

```
> plot([[p[j], delta_p[j]]$j=1..5], style=point, symbol=circle,
  labels=[Population, Change_of_population], labeldirections=
  [HORIZONTAL, VERTICAL]);
```



There seems to be a proportionality between populations 1, 3 and 4 and between populations 2, 3 and 5. We'll plot both to see which model is better.

Exercise (4.b)

```
> real_data := [[y[j], p[j]]$j=1..6]:
> k := (delta_p[4] - delta_p[3]) / (p[4] - p[3]);
```

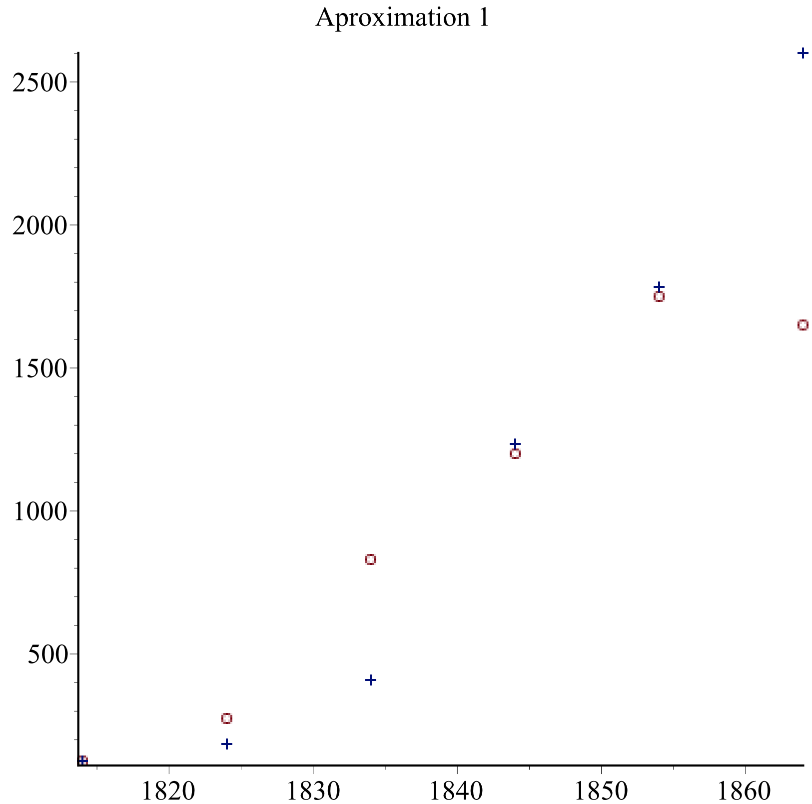
$$k := \frac{18}{37} \tag{20}$$

```
> p_est[1] := p[1]:
  for i from 2 to 6 do
    p_est[i] := p[i-1] * (k + 1);
  end do:
```

```
> seq(p_est[i], i=1..6);
```

$$125, \frac{6875}{37}, \frac{15125}{37}, \frac{45650}{37}, \frac{66000}{37}, \frac{96250}{37} \quad \textbf{(21)}$$

```
> est_data := [[y[j], p_est[j]]$j=1..6]:
> plot([real_data, est_data], style=point, symbol=[circle, cross],
  title="Aproximation 1");
```



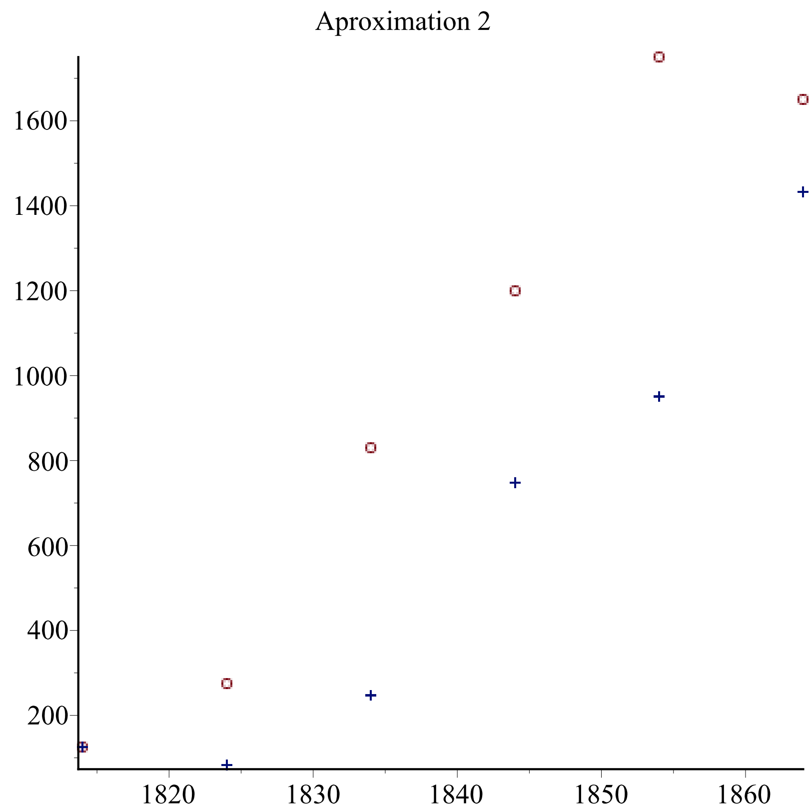

```
> k2 := (delta_p[3] - delta_p[2]) / (p[3] - p[2]);
```

$$k2 := -\frac{1}{3} \quad \textbf{(22)}$$

```
> p_est[1] := p[1]:
  for i from 2 to 6 do
    p_est[i] := p[i-1] + p_est[i-1] * k2;
  end do:
> seq(p_est[i], i=1..6);
```

$$125, \frac{250}{3}, \frac{2225}{9}, \frac{20185}{27}, \frac{77015}{81}, \frac{348235}{243} \quad \textbf{(23)}$$

```
> est_data := [[y[j], p_est[j]]$j=1..6]:
> plot([real_data, est_data], style=point, symbol=[circle, cross],
  title="Aproximation 2");
```



It seems like the first model is a bit optimistic because k1 > 0, so it will always predict a larger value than the previous one, while model 2 is very pesimistic because k2 < 0, so it will always predict a smaller value than the previous one, which from the graph it's not available in most of the cases. Because we have a few available data points, it's hard to find a better model.