Seminar 9

24.XI.2021

$\boxed{2.1.58}$ Să se găsească un exemplu de 2 subgrupuri ale unui grup a căror reuniune NU este subgrup.

$2\mathbb{Z} = \{2k \mid k \in \mathbb{Z}\} \leq (\mathbb{Z}, +)$

$3\mathbb{Z} = \{3k \mid k \in \mathbb{Z}\} \leq (\mathbb{Z}, +)$

$2\mathbb{Z} \cup 3\mathbb{Z} \not\leq (\mathbb{Z}, +)$ pt că

$$\begin{array}{ccccc} 4 & + & 3 & = & 7 \\ \in 2\mathbb{Z} & & \in 3\mathbb{Z} & & \notin 2\mathbb{Z} \wedge \notin 3\mathbb{Z} \\ \in 2\mathbb{Z}\cup 3\mathbb{Z} & & \in 2\mathbb{Z}\cup 3\mathbb{Z} & & \Downarrow \\ & & & & \notin 2\mathbb{Z}\cup 3\mathbb{Z} \end{array}$$

$\boxed{2.1.59}$ Fie $(G, +)$ grup abelian $H, K \leq G$. Să se arate că $\langle H \cup K \rangle = H + K$, unde $H + K = \{x + y \mid x \in H, y \in K\}$

$\langle H \cup K \rangle = H + K \Leftrightarrow$ 1) $H + K \leq G$
2) $H \cup K \subseteq H + K$
3) dacă $L \leq G$ a.î. $H \cup K \subseteq L$ atunci $H + K \leq L$

1) $\underline{H + K \leq G}$

I $H, K \leq G \Rightarrow 0 \in H, K$

$$\left.\begin{array}{l} 0 = \underset{\in H}{0} + \underset{\in K}{0} \end{array}\right| \Rightarrow 0 \in H + K$$

II $\forall x, y \in H + K$. Vrem $x + y \in H + K$

$\Downarrow$

$\exists a \in H, b \in K$ a.î. $x = a + b$
$\exists c \in H, d \in K$ a.î. $y = c + d$

$$x + y = (a + b) + (c + d) \quad \underline{\underline{+ \text{ com, asoc}}}$$

$$= \underbrace{(\underset{\in H}{a} + \underset{\in H}{c})}_{\in H \leq G} + \underbrace{(\underset{\in K}{b} + \underset{\in K}{d})}_{\in K \leq G} \in H + K.$$

$//$

III $\forall x \in H+K$ vrem $-x \in H+K$

$\Downarrow$

$\exists a \in H, b \in K$ a.î. $x = a+b \Rightarrow$

$-x = (-a)+(-b)$

$a \in H \xrightarrow{H \leq G} -a \in H$

$b \in K \xrightarrow{K \leq G} -b \in K$

$\Rightarrow -x \in H+K$

2) $H \cup K \subseteq H+K$

Fie $x \in H \cup K$ vrem $x \in H+K$

$\Downarrow$

$x \in H$ sau $x \in K$

Caz I $x \in H$ vrem $x \in H+K$

$$x = \underbrace{x}_{\in H} + \underbrace{0}_{\in K \leq G}$$

Caz II $x \in K$

$$x = \underbrace{0}_{\in H} + \underbrace{x}_{\in K} \Rightarrow x \in H+K$$

[1] Dacă $L \leq G$ a.î. $H \cup K \subseteq L$.

Vrem. $\underbrace{H+K}_{\leq G} \leq L \Leftrightarrow H+K \subseteq L$

Fie $x \in H+K$ Vrem $x \in L$

$\exists h \in H$, $k \in K$ a.î. $x = \underline{h} + \underline{k}$

$h \in H \subseteq H \cup K \subseteq L$
$k \in K \subseteq H \cup K \subseteq L$ $\Rightarrow h+k \in L$ (L e sbgr.) ($L \leq G$)

$\Rightarrow x \in L$

$\boxed{2.161}$ Fie $n, m \in \mathbb{Z}$. Arăt.

(a) $n\mathbb{Z} \subseteq m\mathbb{Z} \Leftrightarrow m \mid n$

(b) $n\mathbb{Z} \cap m\mathbb{Z} = k\mathbb{Z}$ unde $k = \text{lcm}(n, m)$

(c) $n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z}$, unde $d = \gcd(n, m)$

(a) "$\Rightarrow$" Știm $\underline{n\mathbb{Z}} \subseteq \underline{m\mathbb{Z}}$ Vrem $m \mid n$

$n = n \cdot 1 \in n\mathbb{Z} \xRightarrow{n\mathbb{Z} \subseteq m\mathbb{Z}} n \in m\mathbb{Z} \Rightarrow$

$\Rightarrow \underline{\exists k \in \mathbb{Z} \text{ a.î. } n = m \cdot k \Rightarrow} m/n$

"$\Leftarrow$" Știm m/n. Vrem $\underbrace{n\mathbb{Z}} \subseteq \underbrace{m\mathbb{Z}}$

Fie $x \in n\mathbb{Z}$ Vrem $x \in m\mathbb{Z}$

$\Downarrow$

$\exists k \in \mathbb{Z}$ a.î. $x = n \cdot k$

$m/n \Rightarrow \exists t \in \mathbb{Z}$ a.î. $n = m \cdot t$ $\Bigg| \Rightarrow$

$\Rightarrow x = m \cdot \underbrace{(\underset{\in \mathbb{Z}}{t} \cdot \underset{\in \mathbb{Z}}{k})}_{\in \mathbb{Z}} \Rightarrow \underline{x \in m\mathbb{Z}}$

(b) Vrem $\underline{n\mathbb{Z} \cap m\mathbb{Z}} = \underline{b\mathbb{Z}}$, $b = \text{lcm}(n, m)$;

dem prin dublă incluziune.

Dem că $n\mathbb{Z} \cap m\mathbb{Z} \subseteq k\mathbb{Z}$

Fie $x \in n\mathbb{Z} \cap m\mathbb{Z}$. Vrem $x \in k\mathbb{Z}$

$\Downarrow$

$x \in n\mathbb{Z}$ și $x \in m\mathbb{Z}$

$\Downarrow \qquad\qquad \Downarrow$

$n \mid x \qquad\qquad m \mid x$

$\Rightarrow x$ este multiplu comun pt n și m

$k = [m(n,m)] \Rightarrow n \mid k$ și $m \mid k$

$\Rightarrow$ dacă $\exists\, t$ a.î. $n \mid t$, $m \mid t$ atunci $k \mid t$

$\Rightarrow$

$k \mid x \Rightarrow x \in k\mathbb{Z}$

Vrem $k\mathbb{Z} \subseteq n\mathbb{Z} \cap m\mathbb{Z}$

Fie $x \in k\mathbb{Z}$. Vrem $x \in n\mathbb{Z} \cap m\mathbb{Z}$

$\Downarrow$

$k \mid x$

$k = \operatorname{lcm}(n, m) \Rightarrow n \mid k$ și $m \mid k$

$n \mid k$ și $k \mid x \Rightarrow n \mid x \Rightarrow \exists t \in \mathbb{Z}$ a.î. $x = n t \Rightarrow$

$m \mid k$ și $k \mid x \Rightarrow m \mid x \Rightarrow \exists t' \in \mathbb{Z}$ a.î. $x = m \cdot t'$

$$\left.\begin{array}{l} \Rightarrow x \in n\mathbb{Z} \\ \Rightarrow x \in m\mathbb{Z} \end{array}\right| \Rightarrow x \in n\mathbb{Z} \cap m\mathbb{Z}$$

(c) $n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z}$, $d = \gcd(n, m)$

Dublă incluziune

"$\subseteq$" Vrem $n\mathbb{Z} + m\mathbb{Z} \subseteq d\mathbb{Z}$

Fie $x \in \underbrace{\boxed{n\mathbb{Z}}}_{A} + \underbrace{\boxed{m\mathbb{Z}}}_{B}$ Vrem $x \in d\mathbb{Z}$

$\Downarrow$

$\exists a \in n\mathbb{Z}, b \in m\mathbb{Z}$ a.î. $x = a + b$

$\exists a' \in \mathbb{Z}, b' \in \mathbb{Z}$ a.î. $a = n \cdot a'$

$b = m \cdot b'$ $\Rightarrow$

$$\boxed{x = n \cdot a' + m\, b'}$$

$$d = \gcd(n, m) \Rightarrow d \mid n \wedge d \mid m$$

$$\Downarrow$$

$$\exists\, u, v \in \mathbb{Z} \text{ a.î. } \boxed{\begin{aligned} n &= d \cdot u \\ m &= d \cdot v \end{aligned}}$$

$$\Rightarrow x = d \cdot u \cdot a' + d \cdot v \cdot b'$$

$$= d\underbrace{(ua' + vb')}_{\in \mathbb{Z}} \Rightarrow x \in d\mathbb{Z}$$

„$\supseteq$" Vrem $d\mathbb{Z} \subseteq n\mathbb{Z} + m\mathbb{Z}$

Fie $x \in d\mathbb{Z}$

$$\Downarrow$$

$$\exists\, t \in \mathbb{Z} \text{ a.î. } x = d \cdot t$$

$$d = \gcd(n, m) \Rightarrow \begin{cases} d \mid n \wedge d \mid m \\ \text{dacă } \exists\, d' \in \mathbb{Z} \end{cases}$$

a.î. $d' \mid n \wedge d' \mid m \Rightarrow$
$d' \mid d$.

$d|n \overset{a)}{\Rightarrow} n\mathbb{Z} \subseteq d\mathbb{Z}$ („$\Leftarrow$")

$d|m \overset{a)}{\Rightarrow} m\mathbb{Z} \subseteq d\mathbb{Z}$

$n\mathbb{Z} \le (\mathbb{Z},+)$ $\quad m\mathbb{Z} \le (\mathbb{Z},+)$ $\Big| \Rightarrow$

$n\mathbb{Z} + m\mathbb{Z} \le \boxed{(\mathbb{Z},+)}$

$\Downarrow$

$\exists k \in \mathbb{Z}$ a.î. $\underline{n\mathbb{Z} + m\mathbb{Z} = k\mathbb{Z}}$

Vrem $d\mathbb{Z} \subseteq k\mathbb{Z}$

$\Updownarrow a)$

$k|d$

$\left\{\begin{array}{l} \underline{n\mathbb{Z}} \subseteq n\mathbb{Z} + m\mathbb{Z} = \underline{k\mathbb{Z}} \overset{a)}{\Rightarrow} k|n \\ m\mathbb{Z} \subseteq n\mathbb{Z} + m\mathbb{Z} = k\mathbb{Z} \overset{a)}{\Rightarrow} k|m \end{array}\right| \Rightarrow$

k divizor comun pt n, m)

$d = \gcd(n, m)$) $\Rightarrow$ $\boxed{R/Q}$.

$\boxed{R.1.62}$ Să se arate că pt $n, m \in \mathbb{N}$ cu $d = \gcd(n, m)$, $\exists\, s, t \in \mathbb{Z}$ ai $d = s \cdot n + t\, m$.

În particular s.rec. $1 = \gcd(n, m) \Leftrightarrow \exists\, s, t \in \mathbb{Z}$ ai $1 = s \cdot n + t\, m$

Fie $n, m \in \mathbb{N}$, $d = \gcd(n, m)$

$\Rightarrow n\mathbb{Z} + m\mathbb{Z} = d\mathbb{Z}$

$d \in d\mathbb{Z} \Rightarrow d \in n\mathbb{Z} + m\mathbb{Z}$

$d \cdot 1$

$\Downarrow$

$\exists\, s, t \in \mathbb{Z}$ ai

$d = n \cdot s + m \cdot t$.

În particular $\gcd(n, m) = 1 \Rightarrow$

$\exists\, s, t \in \mathbb{Z}$ ai $1 = n\, s + m\, t$

„$\Leftarrow$": Știm că $\exists\, s, t \in \mathbb{Z}$ a.î.

$$1 = m \cdot s + n \cdot t$$

Vrem că $\gcd(m, n) = 1$.

$$\gcd(m, n) = d \Rightarrow d \mid m,\ d \mid n$$

$$\Downarrow \qquad \Downarrow$$

$$d \mid m \cdot s \qquad d \mid n \cdot t$$

$$\Downarrow$$

$$d \mid m \cdot s + n \cdot t$$

$$\Downarrow$$

$$d \mid 1 \Rightarrow d = 1$$

$$\Rightarrow \gcd(m, n) = 1. \qquad \square$$