Seminar 8

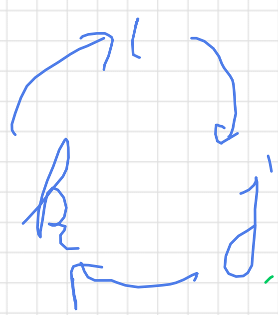
17. XI. 2021

Temă: 1.4.40, 1.4.41, 1.4.42
1.4.53, 1.4.54 } !

$\boxed{2.1.52}$ (Grupul quaternionilor)

Fie mulțimea $H=\{1,-1,i,-i,j,-j,k,-k\}$. Pe H definim o operație de înmulțire a.î.:

- 1 este elem. n.
- Regula semnelor $(-x)y = x(-y) = -x\cdot y$.
- $i^2 = j^2 = k^2 = -1$
- $ij = k = -ji$
  $jk = i = -kj$
  $ki = j = -ik$



Sac $(H,\cdot)$ este un grup.

Calculăm tabla Cayley / op.

$\cdot$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	$\boxed{1}$	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	$\boxed{1}$	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	$\boxed{1}$	k	$-k$	$-j$	j
$-i$	$-i$	i	$\boxed{1}$	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	$\boxed{1}$	i	$-i$
$-j$	$-j$	j	k	$-k$	$\boxed{1}$	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	$\boxed{1}$
$-k$	$-k$	k	$-j$	j	i	$-i$	$\boxed{1}$	-1

Pt. ca $(H, \cdot)$ să fie grup. rămâne să verif asociativitatea „ $\cdot$ ”.

Metoda 1 Computațional.

$$\forall x, y, z \in H \quad \underline{x \cdot (y \cdot z)} \stackrel{?}{=} \underline{(x \cdot y) \cdot z}$$

$512 = 8^3$ calcule.

discuție despre normă: $4^3 = 64$ calcule

Metoda 2: Considerăm matricea din $\mathcal{M}_2(\mathbb{C})$

$1 \longrightarrow I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$i \longrightarrow I = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$j \longrightarrow J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$k \longrightarrow K = I \cdot J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

Fie mult. $\{I_2, -I_2, I, -I, J, -J, K, -K\}$

$I^2 = J^2 = K^2 = -I_2$ $\quad$ $\cong (H, \cdot)$

$IJ = K = -JI$

$KI = J = -IK$

$JK = I = -KJ$

Înmulțirea în $\mathcal{M}_2(\mathbb{C})$ este asoc. $\Rightarrow$

„ $\cdot$ ” în H este asoc.

Met 3 $M_4(\mathbb{R})$ $I_4=\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}$

$$I=\begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{pmatrix}$$

$$K=\begin{pmatrix}0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$$

$$J=\begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$$

$\boxed{2.1.53}$ Sac grupurile $(\mathbb{R},+)$ și $(\mathbb{R}_+^*,\cdot)$ sunt izomorfe.

Fie $f:\mathbb{R}\to\mathbb{R}_+^*$ $f(x)=a^x$, $a\neq 1$, $a>0$

1) Vrem f morfism

$f(x+y)=f(x)\cdot f(y),\ \forall x,y\in\mathbb{R}$

$f(x+y)=a^{x+y}$

$f(x)\cdot f(y)=a^x\cdot a^y$

2) Vrem f injectivă

$$\forall x, y \in \mathbb{R} \quad f(x) = f(y) \overset{?}{\Longrightarrow} x = y$$

$$a^x = a^y \;\Big|\; \log_a$$

3) Vrem f surjectivă

$$\forall y \in \mathbb{R}_+^*, \; \overset{?}{\exists}\, x \in \mathbb{R} \text{ a.î. } f(x) = y$$

$$a^x = y \;\Big|\; \log_a$$

$$x = \log_a y$$

$$\log_a y$$

Așadar f este ~~izomorf~~ $\Longrightarrow$

$$(\mathbb{R}, +) \simeq (\mathbb{R}_+^*, \cdot)$$

$$f^{-1} : \mathbb{R}_+^* \to \mathbb{R}, \quad f^{-1}(x) = \log_a x$$

2.1.54 Spac $f: \mathbb{C}^* \to \mathbb{R}$

$f(x) = \arg x$

este un morf. de gr. între $(\mathbb{C}^*, \cdot)$ și $(\mathbb{R}, +)$. Det Ker f, Im f.

GREȘIT: $f(x \cdot y) = f(x) + f(y)$

$$x = y = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}$$

$$x \cdot y = \cos 3\pi + i \sin 3\pi$$

$$f(xy) = \pi$$

$$f(x) + f(y) = \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi$$

$f: \mathbb{R} \to \mathbb{C}^*$

$f(x) = \cos x + i \sin x$ morf

$f: (\mathbb{R}, +) \to (\mathbb{C}^*, \cdot)$

Vrem $f(x+y) = f(x) \cdot f(y), \forall x, y \in \mathbb{R}$

$f(x+y) = \cos(x+y) + i\sin(x+y)$

$$f(x) \cdot f(y) = (\cos x + i\sin x) \cdot (\cos y + i\sin y) = \cos(x+y) + i\sin(x+y)$$

$\ker f = \{x \in \mathbb{R} \mid f(x) = 1\}$

elem. neutru din $(\mathbb{C}^*, \cdot)$ codom

$f(x) = 1 \Rightarrow \cos x + i\sin x = 1 \Rightarrow$

$$\Rightarrow \begin{cases} \cos x = 1 \\ \sin x = 0 \end{cases} \Rightarrow x = 2k\pi, k \in \mathbb{Z}$$

$\ker f = \{2k\pi \mid k \in \mathbb{Z}\}$

$\operatorname{Im} f = f(\mathbb{R}) = \{f(x) \mid x \in \mathbb{R}\}$

$f(x) = \boxed{\cos x + i\sin x}$ $\quad r = |f(x)| =$

Recap: $z \in \mathbb{C} \to z = r(\cos\alpha + i \sin\alpha)$

$\alpha \in [0, 2\pi)$

$r = |z| = d(\underline{O}, \underline{z})$

$= \sqrt{\underbrace{\cos^2 x + \sin^2 x}} = \sqrt{1} = \underline{1}$

raza

$\underline{\mathrm{Im}\, f} = \boxed{\mathcal{C}(O, 1)} \leq (\mathbb{C}^*, \cdot)$

$ \| \qquad \|$

$(0,0) \qquad \le (\mathbb{R}\times\mathbb{R} \setminus \{(0,0)\}, \cdot)$

$\boxed{R = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}}$

$\boxed{f = (\mathbb{R}, \mathbb{R}, R)}$

$\boxed{2.157}$ Să se găsească toate subgrupurile lui $(\mathbb{Z}, +)$.

$\underline{\text{Indicație:}}$ $\mathrm{Sub}(\mathbb{Z}, +) = \{n\mathbb{Z} \mid n \in \mathbb{N}\}$

$\underline{n\mathbb{Z}} = M_n = \{n x \mid x \in \mathbb{Z}\}$

Pas. 1 $\quad n\mathbb{Z} \leq (\mathbb{Z},+) \quad n \in \mathbb{N}$

- $\boxed{0 = n \cdot 0 \in n\mathbb{Z}}$

- $\forall x \in n\mathbb{Z}$ atunci $-x \in n\mathbb{Z}$

Fie $x \in n\mathbb{Z} \Rightarrow \exists x' \in \mathbb{Z}$ a.î. $x = nx'$

$x' \in \mathbb{Z} \Rightarrow -x' \in \mathbb{Z}$

$$-x = n \cdot \underbrace{(-x')}_{\in \mathbb{Z}} \in n\mathbb{Z}$$

- $\forall x, y \in n\mathbb{Z}$ vrem $x+y \in n\mathbb{Z}$

$x \in n\mathbb{Z} \Rightarrow \exists x' \in \mathbb{Z}$ a.î. $x = nx'$

$y \in n\mathbb{Z} \Rightarrow \exists y' \in \mathbb{Z}$ a.î. $y = ny'$

$$x + y = nx' + ny' = n\underbrace{(\underset{\in \mathbb{Z}}{x'} + \underset{\in \mathbb{Z}}{y'})}_{\in \mathbb{Z}} \Rightarrow$$

$\Rightarrow x + y \in n\mathbb{Z}$

Apoi des $n\mathbb{Z} \leq (\mathbb{Z}, +)$

Pas 2 Dacă $H \leq (\mathbb{Z}, +)$

Vrem $\exists\, n \in \mathbb{N}$ a.î. $H = n\mathbb{Z}$

$H \leq (\mathbb{Z}, +) \Rightarrow 0 \in H$

Caz I $H = \{0\} = 0 \cdot \mathbb{Z}$

Caz II $H \neq \{0\} \Rightarrow \exists\, x \neq 0$ / $\boxed{x \in H}$

$\left.\begin{array}{l} H \text{ grup} \\ x \in H \end{array}\right| \Rightarrow \boxed{-x \in H}$

$\Rightarrow$ H conține nr. întregi pozitive nenule

$\Rightarrow$ H conține nr. nat. nenule

$(\mathbb{N}, \leq)$ orice submulțime admite un el. mai mic element $\Big| \Rightarrow$

Putem alege un cel mai mic nr. nat. nenul din H ~~nat~~ n

Dem că $H = n\mathbb{Z}$

1) Verif. $n\mathbb{Z} \subseteq H$

$$n \in H \Rightarrow \underbrace{n + n + \ldots + n}_{x \text{ ori}} \in H, \forall x \in \mathbb{N}^*$$

$$\Rightarrow n \cdot x \in H, \forall x \in \mathbb{N}^*$$

$$n \cdot 0 = 0 \in H$$

$$n \cdot x \in H \Rightarrow -n \cdot x \in H, \forall x \in \mathbb{N}^*$$

$\Rightarrow$

$$n\mathbb{Z} \subseteq H.$$

2) Verif. $H \subseteq n\mathbb{Z}$.

Pp. $\exists\, x \in H$ a.î. $x \notin n\mathbb{Z}$

$\Downarrow$

$n \nmid x$

$\rightarrow \exists q, r \in \mathbb{Z}$ ai $\underline{x = nq + r}$

$0 < r < n$

$\left.\begin{array}{l} x \in H \\ n \in H \end{array}\right| \Rightarrow \underbrace{x - n - n - \ldots - n}_{q \text{ ori}} \in H$

$\Rightarrow x - nq \in H \rightarrow r \in H$

contradictie cu algoritmul lui n

$\left.\begin{array}{l} \Rightarrow H \subseteq n\mathbb{Z} \\ n\mathbb{Z} \subseteq H \end{array}\right| \Rightarrow H = n\mathbb{Z}$

2.158, 2.159, 2.161, 2.162