Seminar 3

13 oct 2021

1.3.45 Fie $f : A \to B$, $Y_1, Y_2 \subseteq B$.

(6) $f^{-1}(Y_1 \cap Y_2) = \boxed{f^{-1}(Y_1) \cap f^{-1}(Y_2)}$

"$\supseteq$" Fie $a \in \underline{f^{-1}(Y_1) \cap f^{-1}(Y_2)}$

Vrem $a \in f^{-1}(Y_1 \cap Y_2)$

$a \in f^{-1}(Y_1) \cap f^{-1}(Y_2) \Rightarrow$

$a \in f^{-1}(Y_1)$ și $a \in f^{-1}(Y_2) \Rightarrow$

$(\exists b_1 \in Y_1$ a.î. $f(a) = b_1)$ și

$(\exists b_2 \in Y_2$ a.î. $f(a) = b_2$

$\Downarrow$

$b = b_1 = b_2$

$\Rightarrow b \in Y_1$ și $b \in Y_2 \Rightarrow \underline{b \in Y_1 \cap Y_2}$

$\underline{f(a) = b}$ $\Rightarrow$

$f(a) \in Y_1 \cap Y_2 \Rightarrow a \in f^{-1}(Y_1 \cap Y_2)$

$\square$

(4) $f(f^{-1}(Y)) \subset Y$

Fie $b \in f(f^{-1}(Y))$

$(b \in f(M))$

$\Rightarrow$

$A \subseteq B$

Fie $a \in A \Rightarrow \ldots \Rightarrow a \in B$

$f : A \to B$

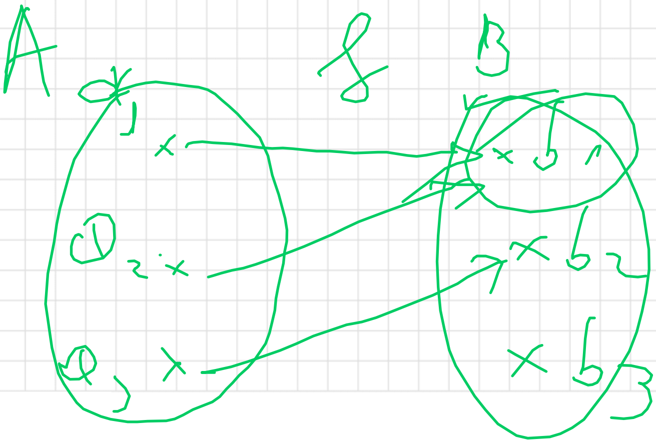
$M \subseteq A$

$f(M) = \{ f(x) \mid x \in M \}$

$b \in f(M) \Rightarrow \exists a \in M \subseteq A$ a.î. $f(a) = b$

$\exists a \in f^{-1}(Y)$ a.î. $f(a) = b$

$a \in f^{-1}(Y) \Rightarrow \exists b' \in Y$ a.î. $f(a) = b'$



$f(\{a_1, a_2\}) = \{b_1\}$

$f^{-1}(\{b_1\}) = \{a_1, a_2\}$

$f^{-1}(\{b_3\}) = \emptyset$

$\Rightarrow b = b' = f(a)$ / $b' \in Y$ $\Rightarrow b \in Y$.

$\boxed{1.3.48}$ Fie A, B mult. finite cu $|A| = m$ și $|B| = n$. Să se obț. $|B^A|$.

Recap $B^A = \{ f \mid f : A \to B$ funcții $\}$

Notație: $|B^A| = n^m$

$|A| = m \Rightarrow A = \{a_1, a_2, \dots, a_m\}$.

$|B| = n \Rightarrow B = \{b_1, b_2, \dots, b_n\}$

$f(a_1) \in B = \{b_1, b_2, \dots, b_n\} \longrightarrow n$ pos.

$f(a_2) \in \{b_1, \dots, b_n\} \longrightarrow n$ pos.

...

$f(a_m) \in \{b_1, \dots, b_n\} \longrightarrow n$ pos.

Total: $\underbrace{n \cdot n \cdot \ldots \cdot n}_{m \text{ ori}} = n^m$ funcții

Var II

x	a_1	a_2	$\ldots$	a_n
$f_1(x)$	b_1	b_1		b_1
$f_2(x)$	b_2	b_1	$\ldots$	b_1
$f_3(x)$	b_3	b_1	$\ldots$	b_1
$\vdots$				
$f_m(x)$	b_m	b_1	$\ldots$	b_1
	b_1	b_2	b_1 $\ldots$	b_1
	b_2	b_2	b_1 $\ldots$	b_1
	b_3	b_2	b_1 $\ldots$	b_1
	$\vdots$			
$f_{m^n}(x)$	b_m	b_m	b_m $\ldots$	b_m

Var III Inductie după $n \in \mathbb{N}^*$.

$P(n): \quad |B^A| = m^n$, unde $n = |A|$

$\boxed{n=1}$ $A = \{a_1\}$

$f_i : A \to B$ } m funcții

$f_i(a_i) = b_i$ $\quad |B^A| = m = m^1$

Pp $P(k)$: $\underline{|B^A| = m^k}$, unde $k = |A|$.

Dem $P(k+1)$

$|A'| = k+1$ $\quad$ Vrem $|B^{A'}| = m^{k+1}$

$\Downarrow$

$A' = A \cup \{a_{k+1}\} = \{a_1, \ldots, a_{k+1}\}$.

$\underline{f : A' \longrightarrow B}$

$f|_A : A \longrightarrow B \rightsquigarrow m^k$ posibilit.

$f(a_{k+1}) \in \{b_1, \ldots, b_m\} \rightsquigarrow m$ pos.

Total $|B^{A'}| = m^k \cdot m = m^{k+1}$.

$\boxed{13.49}$ Fie A și B mult finite cu $|A| = n$ și $|B| = m$. Să se det nr-ul tuturor fct injective de la A la B [redacted]

$\left(R: A_m^n = \frac{m!}{(m-n)!}\right)$

$A = \{a_1, \ldots, a_n\}$

$B = \{b_1, \ldots, b_m\}$

$\underbrace{\{f(a_1), f(a_2), \ldots, f(a_n)\}}_{\operatorname{Im} f} \subseteq B$

f inj $\Rightarrow f(a_1), \ldots, f(a_n)$ obținute

2 câte 2 $\Rightarrow |\operatorname{Im} f| = n$

Obs Trebuie să avem $n \leq m$ ($\operatorname{Im} f \subseteq B$)

Astfel dc $n > m$ vor exista 0 funcții injective: $A \to B$.

Pp. $n \leq m$

$f(a_1) \in \{b_1, b_2, \ldots, b_m\}$ → m pos

$f(a_2) \in \{b_1, b_2, \ldots, b_m\} \setminus \{f(a_1)\}$ → $m-1$ pos

$f(a_3) \in \{b_1, b_2, \ldots, b_m\} \setminus \{f(a_1), f(a_2)\}$ → $m-2$ pos

...

$f(a_n) \in \{b_1, b_2, \dots, b_m\} \setminus \{f(a_1), f(a_2), \dots, f(a_{n-1})\} \longrightarrow \underline{m-n+1}$ pos

Total: $m \cdot (m-1)(m-2) \cdot \ldots \cdot (m-n+1)$

$$= \frac{m(m-1) \cdot \ldots \cdot (m-n+1) \cdot (m-n)(\ldots \cdot 2 \cdot 1)}{(m-n)(m-n-1) \cdot \ldots \cdot 2 \cdot 1}$$

$$= \frac{m!}{(m-n)!}$$

1.350 Fie A o mulț. finită cu $|A| = n$. Să se det. nr-ul tuturor fct. bijective $f: A \to A$ (adică nr-ul tuturor permutărilor lui A)

---

f bij $\Longleftrightarrow$ $\underline{f \text{ inj}}$ și $\underline{f \text{ surj}}$

$A = \{a_1, a_2, \dots, a_n\}$?

$f(a_1) \in \{a_1, a_2, \ldots, a_n\} \rightsquigarrow n$ pos.

$f(a_2) \in \{a_1, a_2, \ldots, a_n\} \setminus \{f(a_1)\} \rightsquigarrow$

$n-1$ pos.

...

$f(a_n) \in \{a_1, a_2, \ldots, a_n\} \setminus \{f(a_1), f(a_2), \ldots, f(a_{n-1})\} \rightsquigarrow 1$ pos

Total: $n(n-1) \cdot \ldots \cdot 1 = n!$

funcții injective.

Analog problema anterioară

f inj $\Rightarrow |\operatorname{Im} f| = |A| = n$

$f : A \to \ldots$

$\operatorname{Im} f \subseteq A$

codomeniul lui f.

$\operatorname{Im} f = A \Rightarrow f$ surj $\Rightarrow$

funcțiile numărate sunt bijective

$R: n!$

Obs. Dacă $f: A \to B$ este o funct. inj. (surj.), a.î. $|A| = |B| \in \mathbb{N}^*$ (finite) $\Rightarrow$ f bijectivă

$$\frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

13.51 Fie B o mult. finită cu $|B| = m$. Să se obț. nr-ul tuturor submulțimilor lui B cu n elemente.

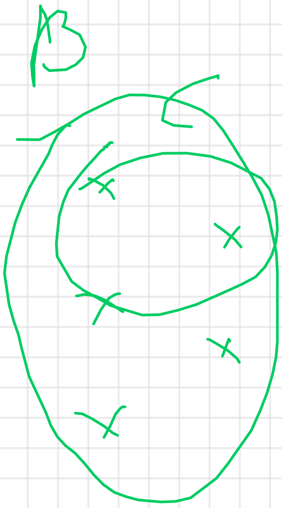
(Ind: $R = \boxed{\binom{m}{n}} = C_m^n = \frac{m!}{n!\,(m-n)!}$)

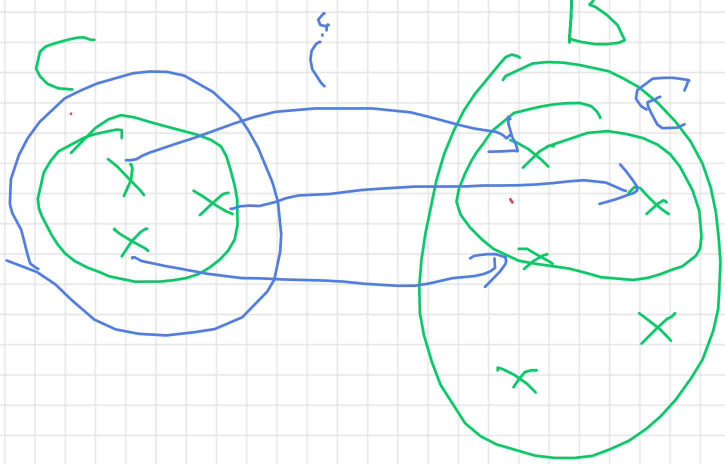
$C \subseteq B = \{b_1, b_2, \ldots, b_m\}$

$C = \{c_1, \dots, c_n\}$ f.t. de incluziune

$C \subseteq B \iff \exists\, i : C \to B$

$\boxed{i(x) = x}$





Clar i este injectivă

Nr-ul tuturor funcțiilor injective de la C la B este $A_m^n = \frac{m!}{(m-n)!}$

$\Rightarrow$ Nr-ul funcțiilor i este $\frac{m!}{n!(m-n)!}$

1.352

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

$$\boxed{C_n^0 + C_n^1 + \ldots + C_n^n = 2^n}$$

$\boxed{V_1}$ Submult (temă)

$\boxed{V_2}$ Ind după n

$$C_{n+1}^0 + C_{n+1}^1 + \ldots + C_{n+1}^{n+1} = 2^{n+1}$$

$$\boxed{C_{n+1}^k = C_n^k + C_n^{k-1}} \text{ (temă)}$$

$C_2^1 = C_1^0 + C_1^1$

$1 = C_1^0 \quad 1 = C_1^1$

$1 = C_2^0 \quad 2 = C_2^1 \quad 1 = C_2^2$