Algebra

Seminar 10

Aplicații

Un K-spațiu vectorial este format dintr-un grup abelian $(V, \boxplus)$ și o operație externă $\odot : K \times V \to V$ care respectă următoarele axiome:

① $\alpha \odot (x \boxplus y) = \alpha \odot x \boxplus \alpha \odot y$

② $(\alpha + \beta) \odot x = \alpha \odot x \boxplus \beta \odot x$

③ $(\alpha\beta) \odot x = \alpha \odot (\beta \odot x)$

④ $1 \odot x = x$

pt. orice $x, y \in V$ și $\alpha, \beta \in K$

① Să se arate că $\mathbb{R}_+^* = (0, \infty)$ este un $\mathbb{R}$ spațiu vectorial în raport cu adunarea vectorilor:

$\boxplus : \mathbb{R}_+^* \times \mathbb{R}_+^* \to \mathbb{R}_+^*$

$x \boxplus y = x \cdot y, \forall x, y \in \mathbb{R}_+^*$

și cu înmulțirea cu scalari:

$\boxdot : \mathbb{R} \times \mathbb{R}_+^* \to \mathbb{R}_+^*$

$\alpha \boxdot x = x^\alpha, \forall x \in \mathbb{R}_+^*, \alpha \in \mathbb{R}$

pup you very much pookie :*

Soluție:

I $(\mathbb{R}_+^*, \boxplus)$ grup abelian?

1. p.s.

$\forall x, y \in \mathbb{R}_+^* \Rightarrow x \boxplus y \in \mathbb{R}_+^*$

$x \boxplus y = \underbrace{x}_{\in \mathbb{R}_+^*} \cdot \underbrace{y}_{\in \mathbb{R}_+^*} \in \mathbb{R}_+^*$

2. comutativitate

$\forall x, y \in \mathbb{R}_+^* \Rightarrow x \boxplus y = y \boxplus x$

$xy = yx$

3. asociativitate

$\forall x, y, z \in \mathbb{R}_+^* \Rightarrow x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$

$x \boxplus yz = xy \boxplus z$

$xyz = xyz$

4. element neutru

$\forall x \in \mathbb{R}_+^* \ \exists!\ e \in \mathbb{R}_+^*$ a.î. $x \boxplus e = x$

$x \boxplus e = x$

$xe = x \Rightarrow e = 1 \in \mathbb{R}_+^*$

⑤ elemente simetrizabile

$\forall\, x \in \mathbb{R}_+^*,\ \exists\, x' \in \mathbb{R}_+^*$ a.î. $x \boxplus x' = e$

$x \boxplus x' = e$

$x x' = e$

$x' = \frac{e}{x} = \frac{1}{x}$

①, ..., ⑤ $\Rightarrow (\mathbb{R}_+^*, \boxplus)$ este grup abelian

II 1) $\alpha \boxdot (x \boxplus y) = (\alpha \boxdot x) \boxplus (\alpha \boxdot y)$

$\alpha \boxdot (x \boxplus y) = \alpha \boxdot (xy) = (xy)^{\alpha} = x^{\alpha} \cdot y^{\alpha} = (\alpha \boxdot x) \boxplus (\alpha \boxdot y)$

2) $(\alpha \boxplus \beta) \boxdot x = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$

$(\alpha + \beta) \boxdot x = x^{(\alpha+\beta)} = x^{\alpha} \cdot x^{\beta} = (\alpha \boxdot x) \boxplus (\beta \boxdot x)$

3) $(\alpha \beta) \boxdot x = \alpha \boxdot (\beta \boxdot x)$

$(\alpha \beta) \boxdot x = x^{\alpha\beta} = x^{\alpha\beta} = x^{\beta\alpha} = (\beta \boxdot x)^{\alpha} = \alpha \boxdot (\beta \boxdot x)$

4) $1 \boxdot x = x$

$x^1 = x$

I, II $\Rightarrow \mathbb{R}_+^*$ este un $\mathbb{R}$-spațiu vectorial împreună cu $\boxplus$ și op. externă $\boxdot$

② Să se verifice că operațiile

$\boxplus : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$x \boxplus y = \sqrt[5]{x^5 + y^5}, \quad \forall x, y \in \mathbb{R}$

$\boxdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$\alpha \boxdot x = \sqrt[5]{\alpha} \cdot x, \quad \forall \alpha, x \in \mathbb{R}$

I $(\mathbb{R}, \boxplus)$ este grup abelian?

① p.s.

$\forall x, y \in \mathbb{R} \Rightarrow x \boxplus y \in \mathbb{R}$

$x \boxplus y = \sqrt[5]{x^5 + y^5}$

$$\left.\begin{array}{l} x \in \mathbb{R} \Rightarrow x^5 \in \mathbb{R} \\ y \in \mathbb{R} \Rightarrow y^5 \in \mathbb{R} \end{array}\right\} \Rightarrow \sqrt[5]{x^5 + y^5} \in \mathbb{R}$$

② comutativitate

$\forall x, y \in \mathbb{R} \Rightarrow x \boxplus y = y \boxplus x$

$$\left.\begin{array}{l} x \boxplus y = \sqrt[5]{x^5 + y^5} \\ y \boxplus x = \sqrt[5]{y^5 + x^5} = \sqrt[5]{x^5 + y^5} \end{array}\right\} \Rightarrow x \boxplus y = y \boxplus x$$

③ asociativitate

$\forall x, y, z \in \mathbb{R} \Rightarrow x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$

$$x \boxplus (\sqrt[5]{y^5+z^5}) = (\sqrt[5]{x^5+y^5}) \boxplus z$$

$$\sqrt[5]{x^5+(\sqrt[5]{y^5+z^5})^5} = \sqrt[5]{(\sqrt[5]{x^5+y^5})^5+z^5}$$

$$\sqrt[5]{x^5+y^5+z^5} = \sqrt[5]{x^5+y^5+z^5}$$

④ el. neutru

$$\forall x \in \mathbb{R} \quad \exists!\, e \in \mathbb{R} \quad \text{a.î.} \quad x \boxplus e = x \overset{\text{com}}{=} e \boxplus x$$

$$x \boxplus e = x$$

$$\sqrt[5]{x^5+e^5} = x \quad /()^5$$

$$x^5+e^5 = x^5$$

$$e^5 = x^5 - x^5$$

$$e^5 = 0$$

$$e = 0 \in \mathbb{R}$$

⑤ inversa

$$\forall x \in \mathbb{R} \quad \exists\, x' \in \mathbb{R} \quad \text{a.î.} \quad x \boxplus x' = e \overset{\text{com}}{=} x' \boxplus x$$

$$x \boxplus x' = e$$

$$\sqrt[5]{x^5+x'^5} = 0 \quad /()^5$$

$$x^5 + x'^5 = 0$$

$$x' = -x \in \mathbb{R}$$

Din ①, ..., ⑤ $\Rightarrow$ $(\mathbb{R}, \boxplus)$ grup abelian

II 1) $\alpha \odot (x \boxplus y) = \alpha \odot x \boxplus \alpha \odot y$

$$\alpha \odot \sqrt[5]{x^5 + y^5} = \sqrt[5]{\alpha} \cdot \sqrt[5]{x^5 + y^5} = \sqrt[5]{\alpha (x^5 + y^5)} =$$

$$= \sqrt[5]{\alpha x^5 + \alpha y^5} = \sqrt[5]{(\sqrt[5]{\alpha}\, x)^5 + (\sqrt[5]{\alpha}\, y)^5} = \alpha \odot x \boxplus \alpha \odot y$$

2) $(\alpha + \beta) \odot x = \alpha \odot x \boxplus \beta \odot x$

$$\sqrt[5]{(\alpha + \beta)} \cdot x = \sqrt[5]{x^5 (\alpha + \beta)} = \sqrt[5]{\alpha x^5 + \beta x^5} = \sqrt[5]{(\sqrt[5]{\alpha}\, x)^5 + (\sqrt[5]{\beta} x)^5}$$

$$= \alpha \odot x \boxplus \beta \odot x$$

3) $(\alpha \beta) \odot x = \alpha \odot (\beta \odot x)$

$$\left.\begin{aligned} &(\alpha \beta) \odot x = \sqrt[5]{\alpha\beta} \cdot x \\ &\alpha \odot (\beta \odot x) = \alpha \odot (\sqrt[5]{\beta} \cdot x) = \sqrt[5]{\alpha} \cdot \sqrt[5]{\beta} \cdot x = \sqrt[5]{\alpha\beta} \cdot x \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow (\alpha \beta) \odot x = \alpha \odot (\beta \odot x)$$

4) $1 \odot x = x$

$$\sqrt[5]{1} \cdot x = x$$

$$x = x$$

I, II $\Rightarrow$ $\mathbb{R}$ este un $\mathbb{R}$-spațiu vectorial împreună cu op $\boxplus$ și cea externă $\odot$

Fie V un K-spațiu vectorial, $U \subseteq V$. U.A.S.E:

(1) $U \leq_K V$

(2) a) $0 \in U$

b) $x, y \in U \Rightarrow x + y \in U$

(3) a) $0 \in U$

b) $\left.\begin{matrix} x, y \in U \\ \alpha, \beta \in K \end{matrix}\right\} \Rightarrow \alpha x + \beta y \in U$

③ Care dintre următoarele submulțimi ale mulțimii $\mathbb{R}^3$ sunt $\mathbb{R}$ subspații:

a) $A = \{ x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0 \}$

Soluție:

$0 \in \mathbb{R}^3$, ptc $0 = [0, 0, 0] = 2 \cdot 0 + 0 - 0 = 0 \Rightarrow 0 \in A$ (1)

$x, y \in A$

$x = [x_1, x_2, x_3]$, $2x_1 + x_2 - x_3 = 0$

$y = [y_1, y_2, y_3]$, $2y_1 + y_2 - y_3 = 0$

$\alpha, \beta \in \mathbb{R}$

$x + y = [x_1 + y_1, x_2 + y_2, x_3 + y_3]$

$= 2(x_1+y_1) + x_2+y_2 - x_3 - y_3$

$= 2x_1 + 2y_1 + x_2 + y_2 - x_3 - y_3$

$= \underbrace{2x_1 + x_2 - x_3}_{0} + \underbrace{2y_1 + y_2 - y_3}_{0}$

$= 0 + 0 = 0 \in A$

$\forall \alpha \in \mathbb{R}$

$\forall x \in \mathbb{R}^3$

$\alpha x = [\alpha(2x_1 + x_2 - x_3)] = \alpha \cdot 0 = 0 \in A \Rightarrow \alpha x \in A$ (2)

Din (1), (2) $\Rightarrow A \leq_{\mathbb{R}} \mathbb{R}^3$

b) $B = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 1\}$

$0 \in \mathbb{R}^3 \Rightarrow [0,0,0] = 2 \cdot 0 + 0 - 0 = 1$

$0 = 1$ Fals $\Rightarrow B \nleq_{\mathbb{R}} \mathbb{R}^3$

c) $C = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 = x_2 = x_3\}$

$0 \in \mathbb{R}^3 \Rightarrow [0,0,0] \Rightarrow 0 = 0 = 0$ Adev

$\forall x, y \in C$

$x = [x_1, x_2, x_3] \quad x_1 = x_2 = x_3$

$y = [y_1, y_2, y_3] \quad y_1 = y_2 = y_3$

$x + y = [x_1+y_1, x_2+y_2, x_3+y_3]$

$x_1+y_1 = x_2+y_2 = x_3+y_3 \in C$ (1)

$\forall \alpha \in \mathbb{R}$

$$\left.\begin{array}{l}\alpha x = [\alpha x_1, \alpha x_2, \alpha x_3], \quad x_1 = x_2 = x_3 \,/\cdot\alpha \\ \alpha x_1 = \alpha x_2 = \alpha x_3\end{array}\right\} \Rightarrow \alpha x \in C \ (2)$$

(1), (2) $\Rightarrow C \leq_{\mathbb{R}} \mathbb{R}^3$

d) $D = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1^2 + x_2 = 0\}$

$0 \in \mathbb{R}^3$

$0 = [0,0,0] = 0^2 + 0 = 0$ A

$\forall x, y \in D \Rightarrow x+y \in D$

$x = [x_1, x_2, x_3], \quad x_1^2 + x_2 = 0 \Rightarrow x_1^2 = -x_2 \Rightarrow \pm x_1 = \sqrt{-x_2}$

$y = [y_1, y_2, y_3], \quad y_1^2 + y_2 = 0 \Rightarrow y_1^2 = -y_2 \Rightarrow \pm y_1 = \sqrt{-y_2}$

$x+y = [x_1+y_1, x_2+y_2, x_3+y_3]$

$(x_1+y_1)^2 + x_2 + y_2 = 0$

$x_1^2 + 2x_1y_1 + y_1^2 + x_2 + y_2 = 0$

$\cancel{-x_2} + 2x_1y_1 + \cancel{-y_2} \cancel{+x_2} \cancel{+y_2} = 0$

$2x_1y_1 = 0 \,/:2$

$x_1y_1 = 0$

$\forall\, x_1, y_1 \in \mathbb{R}$

$D \not\leq_{\mathbb{R}} \mathbb{R}^3$

e) $E = \mathbb{R}^3 \setminus A$

$A = \{ [x_1, x_2, x_3] \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 = 0 \}$

$0 \in A$

$\Rightarrow 0 \notin \mathbb{R}^3 \setminus A \Rightarrow 0 \notin E \Rightarrow E \not\leq_{\mathbb{R}} \mathbb{R}^3$

V un K-sp. vectorial

O aplicație f este liniară dacă $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$

$x, y \in V$, $\alpha, \beta \in K$

④ Care dintre următoarele aplicații sunt liniare:

a) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f[x_1, x_2, x_3] = [x_1 - x_2, x_2 - x_3, x_3 - x_1]$

$\alpha, \beta \in \mathbb{R}$, $x, y \in \mathbb{R}^3$

$$\left.\begin{array}{l} \alpha x = [\alpha x_1, \alpha x_2, \alpha x_3] \\ \beta y = [\beta y_1, \beta y_2, \beta y_3] \end{array}\right\} \Rightarrow \alpha x + \beta y = [\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3]$$

$f(\alpha x + \beta y) = [\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, \alpha x_2 + \beta y_2 - \alpha x_3 - \beta y_3,$
$\alpha x_3 + \beta y_3 - \alpha x_1 - \beta y_1]$ (1)

$\alpha f(x) = [\alpha x_1 - \alpha x_2, \alpha x_2 - \alpha x_3, \alpha x_3 - \alpha x_1]$

$\beta f(y) = [\beta y_1 - \beta y_2, \beta y_2 - \beta y_3, \beta y_3 - \beta y_1]$

$\alpha f(x) + \beta f(y) = [\alpha x_1 - \alpha x_2 + \beta y_1 - \beta y_2, \alpha x_2 - \alpha x_3 + \beta y_2 - \beta y_3,$
$\alpha x_3 - \alpha x_1 + \beta y_3 - \beta y_1]$ (2)

(1) = (2) $\Rightarrow$ f este aplicație liniară

$\text{Ker} f = \{ x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid f(x) = 0 \}$

$f(x) = 0$

$f[x_1, x_2, x_3] = 0$

$[x_1 - x_2, x_2 - x_3, x_3 - x_1] = 0$

$$\left.\begin{array}{l} x_1 - x_2 = 0 \Rightarrow x_1 = x_2 \\ x_2 - x_3 = 0 \Rightarrow x_2 = x_3 \\ x_3 - x_1 = 0 \Rightarrow x_3 = x_1 \end{array}\right\} \Rightarrow x_1 = x_2 = x_3 \Rightarrow$$

$\Rightarrow \text{Ker} f = \{ x = [x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1 = x_2 = x_3 \}$

b) $f: \mathbb{R}^3 \to \mathbb{R}^3$, $f[x_1, x_2, x_3] = [x_1 - 1, x_2 + 2, x_3 + 1]$

$\forall \alpha, \beta \in \mathbb{R}$

$\forall\ x, y \in \mathbb{R}^3$

$x = [x_1, x_2, x_3]$

$y = [y_1, y_2, y_3]$

$\alpha f(x) + \beta f(y) = [\alpha x_1 - \alpha, \alpha x_2 + 2\alpha, \alpha x_3 + \alpha] + [\beta y_1 - \beta, \beta y_2 + 2\beta, \beta y_3 + \beta] = [\alpha x_1 + \beta y_1 - \alpha - \beta, \alpha x_2 + \beta y_2 + 2\alpha + 2\beta, \alpha x_3 + \alpha + \beta y_3 + \beta]$ (1)

$\alpha x + \beta y = [\alpha x_1, \alpha x_2, \alpha x_3] + [\beta y_1, \beta y_2, \beta y_3] =$
$= [\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3]$

$f(\alpha x + \beta y) = [\alpha x + \beta y - 1, \alpha x_2 + \beta y_2 + 2, \alpha x_3 + \beta y_3 + 1]$ (2)

(1) $\neq$ (2) $\Rightarrow$ f nu e aplicație liniară

c) $f: \mathbb{R}^2 \to \mathbb{R}^3$

$f[x_1, x_2] = [x_1 + x_2, x_1 - x_2, 2x_1 + x_2]$

$x, y \in \mathbb{R}^2$

$\alpha, \beta \in \mathbb{R}$

$x = [x_1, x_2]$

$y = [y_1, y_2]$

$$\alpha f(x) + \beta f(y) = [\alpha x_1 + \alpha x_2, \alpha x_1 - \alpha x_2, 2\alpha x_1 + \alpha x_2]$$
$$+ [\beta y_1 + \beta y_2, \beta y_1 - \beta y_2, 2\beta y_1 + \beta y_2] =$$
$$= [\alpha x_1 + \alpha x_2 + \beta y_1 + \beta y_2, \alpha x_1 - \alpha x_2 + \beta y_1 - \beta y_2, 2\alpha x_1 + \alpha x_2$$
$$+ 2\beta y_1 + \beta y_2] \quad (1)$$

$$f(\alpha x + \beta y) = f([\alpha x_1, \alpha x_2] + [\beta y_1, \beta y_2]) =$$
$$= f([\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2]) = [\alpha x_1 + \alpha x_2 + \beta y_1 + \beta y_2,$$
$$\alpha x_1 + \beta y_1 - \alpha x_2 - \beta y_2, 2\alpha x_1 + 2\beta y_1 + \alpha x_2 + \beta y_2] \quad (2)$$

$(1) = (2) \Rightarrow f$ aplicație liniară