Analiza

Seminar 7

① Evaluați integralele:

a) $\mathcal{I} = \int_0^1 \frac{e^x}{\sqrt{e^{2x}+1}}\,dx = \int_1^e \frac{1}{\sqrt{t^2+1}}\,dt = \ln(t+\sqrt{t^2+1})\Big|_1^e =$

$e^x = t \,/(\,)'$
$e^x = dt$
$x=0 \Rightarrow t=1$
$x=1 \Rightarrow t=e$

$= \ln(e+\sqrt{e^2+1}) - \ln(1+\sqrt{2}) = \ln\frac{e+\sqrt{e^2+1}}{1+\sqrt{2}}$

c) $I = \int_1^{\sqrt{3}} \frac{\operatorname{arctg} x}{x^2}\,dx = -\frac{1}{x}\cdot \operatorname{arctg} x\Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} -\frac{1}{x(x^2+1)} =$

$f' = \frac{1}{x^2} \qquad f = -\frac{1}{x}$

$g = \operatorname{arctg} x \qquad g' = \frac{1}{x^2+1}$

$= -\frac{1}{\sqrt{3}}\cdot\frac{\pi}{3} + \frac{\pi}{4} = -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} + \int_1^{\sqrt{3}} \frac{1}{x(x^2+1)}$

$\frac{1}{x(x^2+1)} = \frac{\overset{x^2+1)}{A}}{x} + \frac{\overset{x)}{Bx+C}}{x^2+1} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)} = \frac{1}{x(x^2+1)}$

$Ax^2+A+Bx^2+Cx = 1$

$x^2(A+B) + x(C) + A = 1$

$A+B=0 \Rightarrow A=-B$?

$C = 0$

$A = 1$

$\Big\} \Rightarrow B = -1$

$$\int_1^{\sqrt{3}} \frac{1}{x(x^2+1)} = \int_1^{\sqrt{3}} \frac{1}{x} + \frac{-x}{x^2+1} = \ln x \Big|_1^{\sqrt{3}} - \frac{1}{2}\int \frac{1}{t}\,dt = \ln\sqrt{3} - \frac{1}{2}\ln(x^2+1)\Big|_1^{\sqrt{3}}$$

$x^2+1 = t \;/'$

$2x = dt$

$$= \ln\sqrt{3} - \frac{1}{2}\ln 4 + \frac{1}{2}\ln 2 = \ln\sqrt{3} - \ln 2 + \ln\sqrt{2} = \ln\frac{\sqrt{6}}{2}$$

d) $$y = \int_{-1}^{1} \sqrt{1-x^2}\,dx = x\sqrt{1-x^2}\Big|_{-1}^{1} - \int_{-1}^{1} \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx =$$

$f' = 1 \quad f = x$

$g = \sqrt{1-x^2} \quad g' = -\frac{\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}$

$$= -\left(\int_{-1}^{1} \frac{\overset{\sqrt{1-x^2}}{\cancel{1-x^2}}}{\cancel{\sqrt{1-x^2}}}\,dx - \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx\right) = -\left(\int_{-1}^{1}\sqrt{1-x^2}\,dx - \arcsin x\Big|_{-1}^{1}\right) =$$

$$= -\Big(y - (\underbrace{\arcsin 1}_{\frac{\pi}{2}} - \underbrace{\arcsin(-1)}_{\frac{3\pi}{2}})\Big) = -y + \frac{\pi}{2} - \frac{3\pi}{2}$$

$$2y = \frac{\pi}{2} - \frac{3\pi}{2} \quad /:2$$

$$y = \frac{\pi}{4} - \frac{3\pi}{4} = -\frac{2\pi}{4} = -\frac{\pi}{2}$$

e) $$y = \int_2^4 \frac{\sqrt{x^2-4}}{x}\,dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{\sqrt{x^2-2^2}}{x}\,dx = \int_2^4 \frac{\left(\frac{2}{\sin t}\right)^2 - 2^2}{\frac{\cancel{2}}{\sin t}} \cdot \frac{-\cancel{2}\cos t}{\overset{\sin t}{\cancel{\sin^2 t}}}$$

$x = \frac{2}{\sin t} \;/'$

$dx = \frac{-2\cos t}{\sin^2 t}$

$$= \int_2^4 \sqrt{\frac{4}{\sin^2 t} - \frac{\sin^2 t}{4}} \cdot (-\operatorname{ctg} t)\, dt = \int_2^4 \sqrt{\frac{4 - 4\sin^2 t}{\sin^2 t}} \cdot (-\operatorname{ctg} t)\, dt$$

$$= \int_2^4 \sqrt{\frac{4(1-\sin^2 t)}{\sin^2 t}} \cdot (-\operatorname{ctg} t)\, dt = \int_2^4 2\sqrt{\frac{\cos^2 t}{\sin^2 t}} \cdot (-\operatorname{ctg} t)\, dt$$

$$= \int_2^4 2 \operatorname{ctg} t \cdot -\operatorname{ctg} t \; dt = -2\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \operatorname{ctg}^2 t =$$

$$= -2\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{\cos^2 x}{\sin^2 x} = -2\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{1-\sin^2 x}{\sin^2 x} = -2\int_{\frac{\pi}{2}}^{\frac{\pi}{6}} \frac{1}{\sin^2 x} - 1 =$$

$$= -2\left[-\operatorname{ctg} x \Big|_{\frac{\pi}{2}}^{\frac{\pi}{6}} - x \Big|_{\frac{\pi}{2}}^{\frac{\pi}{6}} \right] = 2\left(\frac{\cos x}{\sin x} \Big|_{\frac{\pi}{2}}^{\frac{\pi}{6}} - x \Big|_{\frac{\pi}{2}}^{\frac{\pi}{6}} \right)$$

$$= 2\left(\sqrt{3} - \left(\frac{\pi}{6} - \frac{\cancel{\pi}^{3}}{2}\right)\right) = 2\left(\sqrt{3} - \frac{2\pi}{\cancel{6}^{3}}\right) = 2\sqrt{3} - \frac{2\pi}{3}$$

$$= \frac{6\sqrt{3} - 2\pi}{3}$$