Analiza

Gr. 216 - R1

1. $a \in (0, \infty)$ a.î. $\sum\limits_{n=1}^{\infty} C_{2n}^{n} \cdot a_n$ să fie convergentă

Kummer

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}}, \quad x_n = C_{2n}^{n} \cdot a^n = \lim_{n\to\infty} \frac{C_{2n}^{n} \cdot \cancel{a^n}}{C_{2n+1}^{n+1} \cdot \cancel{a^{n+1}}\, a} =$$

$$\lim_{n\to\infty} \frac{\frac{\cancel{(2n)!}}{n! \cdot \cancel{(2n-n)!}}}{\frac{\cancel{(2n+2)!}\,(2n+1)(2n+2)}{\cancel{(n+1)!}_{n+1} \cdot \cancel{(n+1)!}_{n+1}} \cdot a} = \lim_{n\to\infty} \frac{1}{\frac{(2n+1)(2n+2)}{(n+1)^2} \cdot a} = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} \cdot \frac{1}{a} \overset{\frac{\infty}{\infty}}{=}$$

$$= \lim_{n\to\infty} = \frac{1}{4a}$$

pt $\sum$ conv $\Rightarrow \frac{x_n}{x_{n+1}} > 1$

$\Rightarrow \frac{1}{4a} > 1 \;/()^{-1}$

$4a < 1$

$a < \frac{1}{4}$

$a \in (0, \infty)$

$\Rightarrow a \in (0, \frac{1}{4})$

2. $f^{(n)}(x)$, $n \in \mathbb{N}$, $f : \mathbb{R} \setminus \{-\frac{b}{a}\} \to \mathbb{R}$, $f(x) = \frac{1}{ax+b}$

$$f'(x) = ((ax+b)^{-1})' \overset{u^{n\prime} = n \cdot u^{n-1} \cdot u'}{=\!=\!=\!=} -(ax+b)^{-2} \cdot a$$

$$f''(x) = (-(ax+b)^{-2})'' = (-1) \cdot (-2) \cdot (ax+b)^{-3} \cdot a^2$$

...

$$f^{(n)}(x) = (-1)^n \cdot n! \cdot (ax+b)^{-n-1} \cdot a^n, \ \forall\, n \in \mathbb{N}$$

Inducție matematică:

P_p $f^{(k)}(x) = (-1)^k \cdot k! \cdot (ax+b)^{-k-1} \cdot a^k$, $\forall k \in \mathbb{N}$

Dem $f^{(k+1)}(x) = (-1)^{(k+1)} \cdot (k+1)! \cdot (ax+b)^{-k-2} \cdot a^{k+1}$

$(f^{(k)}(x))' = (-1)^k \cdot k! \cdot (ax+b)^{-k-1-1} \cdot a^k \cdot a \cdot -(k+1) =$

$= (-1)^{k+1} \cdot (k+1)! \cdot (ax+b)^{-k-2} \cdot a^{k+1}$

3. $\lim\limits_{n\to\infty} \sqrt[n]{\dfrac{(n!)^2}{8^n \cdot (2n)!}}$, fie $x_n = \dfrac{(n!)^2}{8^n \cdot (2n)!}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}\,\cancel{(n+1)!}^{\,n+1}}{8^{\cancel{n+1}} \cdot \underbrace{\cancel{(2n+2)!}}_{(2n+1)(2n+2)}} \cdot \frac{\cancel{8^n} \cdot \cancel{(2n)!}}{\cancel{n!} \cdot \cancel{n!}} = \lim_{n\to\infty} \frac{n^2+2n+1}{8(4n^2+6n+2)} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}{8\cancel{n^2}\left(4+\frac{6}{n}+\frac{2}{n^2}\right)} = \frac{1}{32} \in [0,\infty] \xRightarrow{\text{Stolz Cesaro}} \lim_{n\to\infty} \sqrt[n]{x_n} = \frac{1}{32}$$

Gr. 216 - R2

1. $a \in (0,\infty)$ a.î. $\sum\limits_{n=1}^{\infty} \dfrac{1}{C_{2n}^n \cdot a^n}$ conv

Fie $x_n = \dfrac{1}{C_{2n}^n \cdot a^n} \overset{K.D.}{\Longrightarrow} \dfrac{x_n}{x_{n+1}} > 1$ pt $\sum\limits_{n=1}^{\infty} x_n$ conv

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{1}{C_{2n}^n\, a^n} \cdot C_{2n+2}^{n+1} \cdot a^{n+1} = \lim_{n\to\infty} \frac{1}{\frac{(2n)!}{n! \cdot n!} \cdot a^n} \cdot \frac{(2n+2)!}{(n+1)!\,(n+1)!} a^{n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\cdot\cancel{n!}}{\cancel{(2n)!}\cdot\cancel{a^n}} \cdot \frac{\overset{(2n+1)(2n+2)}{\cancel{(2n+2)!}}\cdot\overset{a}{\cancel{a^{n+1}}}}{\underset{n+1}{\cancel{(n+1)!}}\cdot\underset{n+1}{\cancel{(n+1)!}}} = \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)(n+1)} \cdot a =$$

$$= \lim_{n\to\infty} \frac{4n^2+4n+2}{n^2+2n+1} \cdot a = 4a$$

$$4a > 1$$

$$\left.\begin{array}{l} a > \frac{1}{4} \\ a \in (0,\infty) \end{array}\right\} \Rightarrow a \in \left(\frac{1}{4},\infty\right)$$

$a = \frac{1}{4}$

$$R = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \left(\frac{n^2+n+\frac{1}{2}}{n^2+2n+1} - 1\right) =$$

$$= \lim_{n\to\infty} n \cdot \left(\frac{-n-\frac{1}{2}}{n^2+2n+1}\right) = \lim_{n\to\infty} \cancel{n} \cdot \left(\frac{\cancel{n}(-1-\frac{1}{2n})}{\cancel{n^2}(1+\frac{2}{n}+\frac{1}{n^2})}\right) = -1 < 1$$

2. $f^{(n)}(x) = ?$, $n \in \mathbb{N}$, $f : \left(-\frac{b}{a}, \infty\right) \to \mathbb{R}$, $f(x) = \sqrt{ax+b}$

$$f(x) = \sqrt{ax+b} = (ax+b)^{\frac{1}{2}} \qquad (u^n)' = n\,u^{n-1}\cdot u'$$

$$f'(x) = \frac{1}{2}(ax+b)^{-\frac{1}{2}} \cdot a$$

$$f''(x) = \frac{1}{2}\cdot -\frac{1}{2}\cdot(ax+b)^{-\frac{3}{2}} \cdot a^2$$

$$f'''(x) = \frac{(-1)(-3)}{2^3} \cdot (ax+b)^{-\frac{5}{2}} \cdot a^3$$

...

$$f^{(n)}(x) = \frac{(-1)^{n+1}\cdot(2n-3)!!}{2^n} \cdot (ax+b)^{-\frac{(2n-1)}{2}} \cdot a^n, \ n \geq 1$$

Inducție. PP $f^{(n)}(x) = \frac{(-1)^{n+1} \cdot (2n-3)!!}{2^n} \cdot (ax+b)^{-\frac{(2n-1)}{2}} \cdot a^n, \; n \geq 1$

Dem $f^{(n+1)}(x) = \frac{(-1)^{n+2} \cdot (2n-1)!!}{2^{n+1}} \cdot (ax+b)^{-\frac{(2n+1)}{2}} \cdot a^{n+1}$

$$\left(f^{(n)}(x)\right)' = \frac{(-1)^{n+1} \cdot (2n-3)!!}{2^n} \cdot -\frac{(2n-1)}{2} (ax+b)^{-\frac{(2n-1)}{2}+1} \cdot a^n \cdot a$$

$$= \frac{(-1)^{n+2} \cdot (2n-1)!!}{2^{n+1}} \cdot (ax+b)^{-\frac{(2n+1)}{2}} a^{n+1}$$

3. $\lim\limits_{n\to\infty} \frac{\sqrt{1+2^2}+\sqrt{1+3^2}+\ldots+\sqrt{1+n^2}}{1+n^2}$

fie $x_n = \sqrt{1+2^2}+\sqrt{1+3^2}+\ldots+\sqrt{1+n^2}$

$y_n = 1+n^2$ cresc și div

$$\lim_{n\to\infty} \frac{x_{n+1}-x_n}{y_{n+1}-y_n} = \lim_{n\to\infty} \frac{\sqrt{1+(n+1)^2}}{\cancel{1}+(n+1)^2-\cancel{1}-n^2} = \lim_{n\to\infty} \frac{\sqrt{n^2+2n+2}}{2n+1}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}}{\cancel{n}\left(2+\frac{1}{n}\right)} = \frac{1}{2} \in [0,\infty) \overset{S.C.}{\Rightarrow} \lim_{n\to\infty} \frac{x_n}{y_n} = \frac{1}{2}$$

Gr. 217 - R1

1. $a \in (0,\infty)$, $\sum\limits_{n=1}^{\infty} \frac{5^n}{n\sqrt{n}\cdot a^n}$ - conv

fie $x_n = \frac{5^n}{n\sqrt{n}\cdot a^n}$

$$\Delta = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{4^n}{n\sqrt{n}\cdot a^n} \cdot \frac{(n+1)\sqrt{n+1}\cdot a^{n+1}}{4^{n+1}} = \lim_{n\to\infty} \frac{a}{4}\cdot \frac{(n+1)\sqrt{n+1}}{n\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{a}{4}\cdot \frac{\sqrt{(n+1)^2(n+1)}}{\sqrt{n^3}} = \lim_{n\to\infty} \frac{a}{4}\sqrt{\frac{(n^2+2n+1)(n+1)}{n^3}} = \lim_{n\to\infty} \frac{a}{4}\sqrt{\underbrace{\frac{n^3+3n^2+3n+1}{n^3}}_{\to 1}}$$

$$\left.\begin{array}{l} = \frac{a}{4} \\ \Delta > 1 \end{array}\right\} \Rightarrow \begin{array}{l} \frac{a}{4} > 1 \\ a > 4 \end{array}$$

pt $a = 4$

$$x_n = \frac{\cancel{4^n}}{n\sqrt{n}\cdot \cancel{4^n}} = \frac{1}{n\sqrt{n}} = \frac{1}{n\cdot n^{\frac{1}{2}}} = \frac{1}{n^{\frac{3}{2}}}, \ \sum_{n=1}^{\infty} x_n$$ convergentă (seria armonică generalizată cu $p = \frac{3}{2} > 1$)

2. $f:\mathbb{R}\to\mathbb{R}$, $f(x) = \ln(1+x^2) - 2\,\mathrm{arctg}\, x$

$$\left.\begin{array}{l} f'(x) = \frac{2x}{1+x^2} - \frac{2}{1+x^2} = \frac{2(x-1)}{1+x^2} \\ f'(x) = 0 \end{array}\right\} \Rightarrow x = 1$$

x	$-\infty$		1		∞
$f'(x)$		$- - - - - -$	0	$+ + + + + +$	
$f(x)$	∞	$\searrow$	$\ln 2 - \frac{\pi}{2}$	$\nearrow$	∞

$$f(1) = \ln 2 - \frac{\cancel{2}\pi}{\cancel{4}\,2}$$

$$\lim_{x\to-\infty} f(x) = \infty - \cancel{2}\cdot -\frac{\pi}{\cancel{2}} = \infty$$

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
tg	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-$
ctg	$-$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0

$$\lim_{x\to\infty} f(x) = \infty - \cancel{2}\cdot\frac{\pi}{\cancel{2}} = \infty$$

$\inf = \ln 2 - \frac{\pi}{2}$ (se atinge)

$\sup = +\infty$ (nu se atinge)

3, $$\lim_{x\to 0} \frac{\ln(1-\cos x)}{\ln(\sin^2(x))} \overset{\frac{-\infty}{-\infty}}{\underset{l'h}{=}} \lim_{x\to 0} \frac{\frac{\sin x}{1-\cos x}}{\frac{1}{\sin^2 x}\cdot((\sin x)^2)'} = \lim_{x\to 0} \frac{\frac{\sin x}{1-\cos x}}{\frac{1}{\cancel{\sin^2 x}_{\sin x}}\cdot 2\cancel{\sin x}} =$$

$$\lim_{x\to 0} \frac{\frac{\sin x}{1-\cos x}}{\frac{2}{\sin x}} = \lim_{x\to 0} \frac{\sin^2 x}{2-2\cos x} = 1$$

$\sin^2 x \to 0$, $2-2\cos x \to 0$

$\lim_{x\to 0}\frac{\sin x_n}{x_n} = 1$

Gen 217 - R2

1. $a\in(0,\infty)$, $\sum_{n=1}^{\infty} \frac{a^n}{n\sqrt{n}\cdot 3^n}$ conv

Fie $x_n = \frac{a^n}{n\sqrt{n}\cdot 3^n}$

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\cancel{a^n}}{n\sqrt{n}\cdot\cancel{3^n}} \cdot \frac{(n+1)(\sqrt{n+1})\cdot\cancel{3^{n+1}}^{\,3}}{\cancel{a^{n+1}}_{\,a}} = \frac{3}{a}$$

$\frac{3}{a} > 1$

$a < 3$

pt $a = 3$

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ convergentă, din seria armonică generalizată

$p = \frac{3}{2} > 1$

$$S \text{ este } \begin{cases} C, & a \in (0,3] \\ D, & a \in (3,\infty) \end{cases}$$

2. $f:(0,\infty) \to \mathbb{R}, \quad f(x) = \ln x - \operatorname{arctg} x$

$$\left.\begin{array}{l} f'(x) = \overset{1+x^2)}{\dfrac{1}{x}} - \overset{x)}{\dfrac{1}{1+x^2}} = \dfrac{x^2-x+1}{x(1+x^2)} \\ f'(x) = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} x^2-x+1=0 \\ \Delta = 1-4<0 \end{array}\right\} \Rightarrow \nexists \text{ răd}$$

x	0	∞
$f'(x)$	$+ \; + \; + \; + \; + \; + \; + \; + \; +$	
$f(x)$	$-\infty \nearrow$	∞

$$\lim_{x \to 0} f(x) = -\infty$$

$$\lim_{x \to \infty} f(x) = \infty - \frac{\pi}{2} = \infty$$

nu se ating limitele

3. $$\lim_{x \to 1} \frac{x \cdot (\ln x - 1) + 1}{(x-1) \cdot \ln x} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x \to 1} \frac{\ln x - 1 + 1}{\frac{x-1}{x} + \ln x} = \lim_{x \to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} \overset{\frac{0}{0}}{\underset{l'H}{=}}$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{+\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{2}$$

$(m+1)\cdot\ldots\cdot(2m)$

1. $A = \left\{ \frac{\cancel{(2m)!}}{3^m \cdot \cancel{m!}}, \ m \in \mathbb{N} \right\}$

$A = \left\{ \frac{(m+1)\cdot\ldots\cdot(2m)}{3^m}, \ m \in \mathbb{N} \right\}$

fie $x_m = \frac{(m+1)\cdot\ldots\cdot(2m)}{3^m}, \ m \in \mathbb{N}$

$\lim\limits_{x\to\infty} \frac{x_m}{x_{m+1}} = \frac{\cancel{(m+1)\cdot\ldots\cdot(2m)}}{\cancel{3^m}} \cdot \frac{\cancel{3^{m+1}}\ 3}{\cancel{(m+1)\cdot\ldots}\cdot(2m+1)} = \lim\limits_{x\to\infty} \frac{3}{2m+1}$

$= 0 \Rightarrow x_m$ divergent $\Rightarrow \lim\limits_{x\to\infty} x_m = +\infty$

pt $m=1$, $\frac{(2m)!}{3^m\cdot m!} = \frac{2}{3}$

$\inf = \min = \frac{2}{3}$

$\max \nexists$, $\sup = +\infty$

2. $\lim\limits_{m\to\infty} \sqrt[m]{\frac{(m!)^2}{(2m)!\cdot 8^m}}$

Notăm $x_m = \frac{(m!)^2}{(2m)!\cdot 8^m}$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}\,\cancel{(n+1)!}^{\,n+1}}{\cancel{(2n+2)!}\cdot \cancel{8^{n+1}}\,8} \cdot \frac{\cancel{(2n)!}\cdot\cancel{8^n}}{\cancel{n!}\,\cancel{n!}} =$$

$$(2n+1)(2n+2)$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{8(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{n^2+2n+1}{8(4n^2+6n+2)} = \frac{1}{32} \in \mathbb{R} \Rightarrow$$

$$\Rightarrow \sqrt[n]{x_n} = \frac{1}{32}$$

3. $$\lim_{x\searrow 0} \sqrt{x}\cdot \ln(\sin x) = \lim_{x\searrow 0} \sqrt{x}\cdot \ln(x) =$$

$$= \lim_{x\searrow 0} \frac{1}{x^{-\frac{1}{2}}}\cdot \ln(x) = \lim_{x\searrow 0} \frac{\ln(x)}{x^{-\frac{1}{2}}} = \lim_{x\searrow 0} \frac{\ln(x)}{\frac{1}{\sqrt{x}}} \overset{\frac{\infty}{\infty}}{\underset{l'H}{=}}$$

$$= \lim_{x\searrow 0} \frac{\frac{1}{x}}{-\frac{1}{2}\cdot x^{-\frac{3}{2}}} = \lim_{x\searrow 0} \frac{1}{x}\cdot -\frac{2}{\frac{1}{\sqrt{x^3}}} = \lim_{x\searrow 0} \frac{1}{\cancel{x}}\cdot(-2\cancel{x}\sqrt{x}) = 0$$