Analiza

1. $\sum_{n=1}^{\infty} a^{1+\frac{1}{2}+\ldots+\frac{1}{n}}$, $a > 0$

D'alembert: $\lim_{n\to\infty} \frac{x_n}{x_{n+1}}$

$$\lim_{n\to\infty} \frac{a^{1+\frac{1}{2}+\ldots+\frac{1}{n}}}{a^{1+\frac{1}{2}+\ldots+\frac{1}{n+1}}} = \lim_{n\to\infty} a^{1+\frac{1}{2}+\ldots+\frac{1}{n}-1-\frac{1}{2}-\ldots-\frac{1}{n+1}} = \lim_{n\to\infty} \frac{1}{a^{\frac{1}{n+1}}} =$$

$$= \frac{1}{a^0} = \frac{1}{1} = 1 \text{ nu decide}$$

$$\frac{a^x - 1}{x} = \ln a$$

R.B.: $\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n \cdot \frac{a^{-\frac{1}{n+1}} - 1}{-\frac{1}{n+1}} \cdot -\frac{1}{n+1} =$

$$= \lim_{n\to\infty} -\frac{n \ln a}{n+1} = \lim_{n\to\infty} -\frac{\cancel{n} \ln a}{\cancel{n}(1+\underbrace{\frac{1}{n}}_{\searrow 0})} = -\ln a$$

$-\ln a > 1$

$\ln a < -1$

$\log_e a < -1$

$a < \frac{1}{e} \Rightarrow$ seria este convergentă

$a > \frac{1}{e} \Rightarrow$ seria este divergentă

pentru $a = \frac{1}{e} \Rightarrow \lim_{n\to\infty} \frac{1}{e}^{1+\frac{1}{2}+\ldots+\frac{1}{n}} = \frac{1}{e}^{\gamma \approx 0,577 \approx \frac{1}{2}} \approx \frac{1}{\sqrt{e}} < 1 \Rightarrow$

$\Rightarrow$ pentru $a = \frac{1}{e}$ seria este divergentă

2. $\int_0^1 (\ln x)^2\,dx = \lim_{u\searrow 0} \int_u^1 (\ln x)^2\,dx$

Fie $p \in \mathbb{R}$, $f(x) = (\ln x)^2$, $\lim_{x\searrow 0} (x-0)^p f(x) = \lambda$

pt. $p = \frac{1}{2} \Rightarrow \lim_{x\searrow 0} x^{\frac{1}{2}} \cdot (\ln x)^2 = \lim_{x\searrow 0} \frac{(\ln x)^2}{x^{-\frac{1}{2}}} \overset{\frac{\infty}{\infty}}{\underset{l'h}{=}}$

$= \lim_{x\searrow 0} \frac{2\ln x}{-\frac{1}{2}x^{-\frac{3}{2}}} = -4 \lim_{x\searrow 0} \frac{\ln x}{x^{-\frac{3}{2}}} \overset{\frac{\infty}{\infty}}{\underset{l'h}{=}} -4 \lim_{x\searrow 0} \frac{\frac{1}{x}}{-\frac{3}{2}x^{-\frac{5}{2}}} = \frac{8}{3} \lim_{x\searrow 0} \frac{1}{x} \cdot x^{\frac{5}{2}}$

$= \frac{8}{3} \cdot \lim_{x\searrow 0} \sqrt[5]{x^3} = 0$

Avem $p < 1$, $\lambda < \infty \Rightarrow$ integrala este convergentă

$\lim_{u\searrow 0} \int_u^1 (\ln x)^2 = \lim_{u\searrow 0} x \ln x \Big|_u^1 - 2\int_u^1 \ln x \cdot x\,dx =$

$g = (\ln x)^2 \quad g' = 2\ln x$

$f' = 1 \quad f = x$

$f' = x \quad f = \frac{x^2}{2}$

$g = \ln x \quad g' = \frac{1}{x}$

$= \lim_{u\searrow 0} u \ln u - \left(\frac{x^2}{2}\ln x\right)\Big|_u^1 + \int_u^1 x\,dx = \lim_{u\searrow 0} u\ln u + \frac{u^2}{2}\ln u + \frac{u^2}{2} + \frac{1}{2} =$

$= \lim_{u\searrow 0} \underset{\overset{\shortparallel}{0}}{\frac{u^2}{2}} \left(\frac{2}{u}\ln u + \ln u + 1 + \frac{1}{2}\right) = 0$

3. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = e^y (x \sin x + ay \cos x)$

$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0, \ \forall (x,y) \in \mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = e^{y}\,(\sin x + x\cos x - ay\sin x)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = e^{y}\,(\cos x - x\sin x + \cos x - ay\cos x) =$$

$$e^{y}(2\cos x - x\sin x - ay\cos x)$$

$$\frac{\partial f}{\partial y}(x,y) = e^{y}(x\sin x + ay\cos x + a\cos x)$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = e^{y}(x\sin x + ay\cos x + a\cos x + a\cos x)$$

$$= e^{y}(x\sin x + ay\cos x + 2a\cos x$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0$$

$$e^{y}(2\cos x \cancel{- x\sin x} \cancel{- ay\cos x}) + e^{y}(\cancel{x\sin x} + \cancel{ay\cos x} + 2a\cos x) = 0$$

$$e^{y}(2a\cos x + 2\cos x) = 0$$

$$e^{y}((\cos x(2a+2))) = 0$$

$$2a+2 = 0$$

$$2a = -2$$

$$a = -1$$

5 a) $f : A \to \mathbb{R}$

$x^{0} \in A$

$v \in \mathbb{R}^{m}$

dacă $\exists \lim\limits_{t \to 0} \dfrac{f(x^0+t\cdot v)-f(x^0)}{t}$ atunci se s.n. derivata lui f în x^0 după direcția vectorului v $\stackrel{\text{not}}{=} f'_v(x^0)$

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x+y+z$, $x^0 = (1,0,1)$, $v = (0,1,0)$

$$\lim_{t \to 0} \frac{f(x^0+t\cdot v)-f(x^0)}{t} = \lim_{t \to 0} \frac{f((1,0,1)+t\cdot(0,1,0))-2}{t} =$$

$$= \lim_{t \to 0} \frac{\cancel{2}+t-\cancel{2}}{t} = 1$$