Analiza

O introducere în nr reale

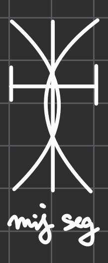
1. Clase de nr. reale

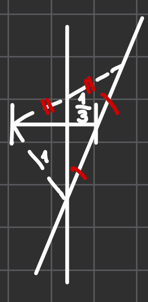
- raționale ($\frac{p}{q}$, $p, q \in \mathbb{Z}$)
- iraționale
- construcțibile (cu rigla și compasul)
- algebrice (exprimabile prin radical)
- transcendente
- reprezentabile (în mem. calculatorului)

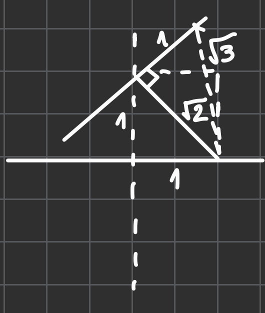
Un nr se numește construcțibil dacă reprezintă lungimea unui segment construit cu rigla și compasul plecând de la segmentul unitate

Ex: $\frac{1}{2}$, $\frac{1}{3}$, $\sqrt{2}$, $\sqrt{3}$ construcțibile

Ex: $\sqrt[3]{3}$, π nu sunt construcțibile







Obs: α este constructibil d.n.d α este rădăcina unei ec polinomiale cu coef întregi de grad o putere a lui 2.

$\alpha = \sqrt{2}$, $x^2 - 2 = 0$

$\alpha = \frac{1}{2}$, $2x - 1 = 0$

$\alpha = \frac{p}{q} \in \mathbb{Q}$ constructibil

Def: α algebric dacă este rădăcina unei ecuații polinomiale cu coef întregi

$$a_0 + a_1 x + \ldots + a_n x^n = 0$$

$$a_0, a_1, \ldots, a_n \in \mathbb{Z}, \ a_n \neq 0$$

raționale $\subseteq$ constructibile $\subseteq$ algebric

Def: α transcendent dacă nu este algebric

π transcendent (Lindemann, 1882)

transcendent $\subseteq$ iraționale

"Majoritatea nr reale sunt transcendente" depozit 1 leu

dobândă 100% / an

termen egal cu $\frac{1}{n}$ dintr-un an ($n \in \mathbb{N}^*$)

Ce sumă acumulăm după 1 an?

$n=1$: $1+1\cdot 1=2$

$n=2$: $1+\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\left(1+\frac{1}{2}\right)=\left(1+\frac{1}{2}\right)^2=2{,}25$

$n=3$: $1+\frac{1}{3}\cdot 1+\frac{1}{3}\left(1+\frac{1}{3}\right)+\frac{1}{3}\left(1+\frac{1}{3}\right)^2=\left(1+\frac{1}{3}\right)^3\simeq 2{,}37$

În general $S_n=\left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e \simeq 2{,}71$

e - transcendent (Hermite, 1873)

2. Reprezentări ale nr. reale

a) ca fracție zecimală

$a_k \in \{0,1,\dots,9\}$, $\forall k \in \mathbb{N}^*$, $a_0 \in \mathbb{Z}$

$x = a_0, a_1 a_2 \dots a_n \dots$ (infinită)

• $\exists\, n \in \mathbb{N}^*$ a.î. $a_k = 0$, $\forall k > n$

$x = a_0, a_1 a_2 \dots a_n$ (finită)

• $\exists\, n, p \in \mathbb{N}^*$ a.î. $a_{k+p} = a_k$, $\forall k > n$

$x = a_0, a_1 a_2 \dots a_n (a_{n+1} \dots a_{n+p})$ (periodică)

Fracțiile zecimale sau periodice reprezintă numere raționale.

$2 = 1,(9)$

$$x_n = 1,\underbrace{99\ldots9}_{n \text{ ori}}$$

$$l = \lim_{n\to\infty} x_n = 1,(9)$$

$$10 \cdot x_{n+1} - x_n = 19,\underbrace{9\ldots9}_{n \text{ ori}} - 1,\underbrace{9\ldots9}_{n \text{ ori}} = 18,\ n\to\infty$$

$10l - l = 18$

$9l = 18$

$l = 2$

Def: α reprezentabil dacă admite o scriere ca fracție zecimală finită cu un nr predeterminat de zecimale

Ex: $\frac{1}{3}, \frac{1}{7}, \sqrt{2}, \pi, e$ nu sunt reprezentabile

b) ca fracție continuă (recurentă, etajată)

$x \in \mathbb{R}$ calculăm succesiv

$$x_1 = \frac{1}{x-[x]},\quad x_2 = \frac{1}{x_1-[x_1]},\ \ldots,\ x_{n+1} = \frac{1}{x_n-[x_n]}$$

dacă $\exists n \in \mathbb{N}$ a.î. $x_n \in \mathbb{Z}$ atunci STOP

$$x = [x] + \frac{1}{x_1} = [x] + \frac{1}{[x_1] + \frac{1}{x_2}} = \dots = [x] + \frac{1}{[x_1] + \frac{1}{[x_2] + \dots}} =$$

$$\stackrel{\text{not}}{=} [x] + \frac{1}{[x_1]} + \frac{1}{[x_2] + \dots}$$

forma generală $a_0 + \frac{1}{a_1 +} \frac{1}{a_2 + \dots}$, $a_0 \in \mathbb{Z}$, $a_1, a_2, \dots \in \mathbb{N}^*$

Ex: $0 + \frac{1}{8+} \frac{1}{7+} \frac{1}{1+} \frac{1}{2} = \frac{123}{1000} = 0{,}123$

$$x = 1 + \frac{1}{2+} \frac{1}{2+} \dots = 1 + \boxed{\frac{1}{2+}}$$

$$x = 1 + \frac{1}{\underbrace{1 + 1 + \frac{1}{2+} \frac{1}{2+} \dots}} = 1 + \frac{1}{1 + x}$$

$$x - 1 = \frac{1}{x+1} \Rightarrow x^2 - 1 = 1,\ x^2 = 2 \Rightarrow x = \sqrt{2}$$

$$c_1 = 1 + \frac{1}{2} = \frac{3}{2} = 1{,}5$$

$$c_2 = 1 + \frac{1}{2+} \frac{1}{2} = 1 + \frac{1}{2 + \frac{1}{2}} = 1 + \frac{2}{5} = \frac{7}{5} = 1{,}4$$

$$c_3 = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{1}{2 + \frac{2}{5}} = 1 + \frac{5}{12} = \frac{17}{12} = 1{,}41(6)$$

$$c_n = 1 + \underbrace{\frac{1}{2+} \dots \frac{1}{2}}_{n \text{ ori}} \to \sqrt{2},\ n \to \infty \quad , \sqrt{2} = 1{,}414 \dots$$

$$F_k = \underbrace{\frac{1}{2+}\frac{1}{2+}\cdots\frac{1}{2}}_{k \text{ ori}} = \frac{1}{2+\underbrace{\frac{1}{2+}\cdots\frac{1}{2}}_{k-1 \text{ ori}}} = \frac{1}{2+F_{k-1}}, \quad \forall k \geq 1, \quad F_0 = 0$$

$$C_k = 1 + F_k$$

Obs 1) Orice nr. real se poate scrie ca o fracție continuă

2) Orice fracție continuă sau nu reprezintă un nr real

c) ca fracție continuă generalizată

$$a_0 + \frac{b_1}{a_1+}\frac{b_2}{a_2+}\cdots, \quad a_0, a_1, a_2, \ldots, b_1, b_2 \ldots \in \mathbb{Z}$$

Ex: $\pi = 3 + \frac{1^2}{6+}\frac{3^2}{6+}\frac{5^2}{6+}\cdots$

$$e = 2 + \frac{1}{1+}\frac{1}{2+}\frac{2}{3+}\frac{3}{4+}\frac{4}{5+}\cdots$$

Temă: Calculați constanta π folosind dezvoltarea în fracție continuă generalizată într-un program C++

Numere reale

① $x, y \in \mathbb{R}$,

$$\max\{x, y\} = \frac{|x-y| + (x+y)}{2}$$

$$\min\{x, y\} = ?$$

I) Dacă $x \leq y$

$$y = \max\{x, y\} = \frac{\cancel{x} + y \cancel{+x} + y}{2} = y \text{ (Adev.)}$$

II) Dacă $x > y$

$$x = \max\{x, y\} = \frac{x - y + x + y}{2} = x \text{ (Adev.)}$$

$$\min\{x, y\} + \max\{x, y\} = x + y$$

$$\min\{x, y\} = x + y - \max\{x, y\} = x + y - \frac{|x-y| + (x+y)}{2} =$$

$$= \frac{2x + 2y - |x-y| - (x+y)}{2} = \frac{x + y - |x-y|}{2}$$

② $x, y \in \mathbb{R}$ au loc

a) $|x+y| \leq |x| + |y| \Leftrightarrow |x+y|^2 \leq (|x| + |y|)^2 \Leftrightarrow \cancel{x^2} + 2xy + \cancel{y^2} \leq \cancel{x^2} + 2|x||y| + \cancel{y^2}$

$\Leftrightarrow x \cdot y \leq |x \cdot y|$

sau

$|x| \leq a \Leftrightarrow -a \leq x \leq a$

$|x+y| \leq \underbrace{|x| + |y|}_{a} \Leftrightarrow -|x| - |y| \leq x + y \leq |x| + |y|$

$-y \leq |y| \leq +y$

$-x \leq |x| \leq +x$

b) $|x-y| \geq |x| - |y| \Leftrightarrow |x-y| + |y| \geq |x|$

$|x| = |x - y + y| \overset{a)}{\leq} |x-y| + |y|$

③ Det $\inf A$, $\sup A$, $\min A$, $\max A$ pt

a) $A = [0,7) \cup [8,+\infty)$

b) $A = [-1,2] \setminus \mathbb{Q}$

c) $A = \{ \frac{1}{n} \mid n \in \mathbb{N}^* \}$

d) $A = \{ \frac{1}{x-[x]} \mid x \in \mathbb{R} \setminus \mathbb{Z} \}$

a) $MIN(A) = (-\infty, 0)$, $\inf(A) = \max(MIN(A)) = 0$

$MAJ(A) = \emptyset$, $\sup(A) = +\infty$

$\min(A) = 0$

$\max(A) \nexists$

b) $MIN(A) = (-\infty, -1] \Rightarrow \inf(A) = -1$

$MAJ(A) = [2, +\infty) \Rightarrow \sup(A) = 2$

$\min(A) = \nexists$

$\max(A) = \nexists$

c) $MIN(A) = (-\infty, 0] \Rightarrow \inf(A) = 0$

$\lim\limits_{n \to \infty} \frac{1}{n} = 0$

$MAJ(A) = [1, +\infty) \Rightarrow \sup(A) = 1$

$\min(A) = \nexists$

$\max(A) = 1$

4. $A \subseteq \mathbb{R}$ au loc

a) Dacă $\exists \min A \Rightarrow \min A = \inf A$

b) Dacă $\exists \max A \Rightarrow \max A = \sup A$

$$m = \inf A \Leftrightarrow \begin{cases} i) \ m \in MIN(A) \\ ii) \ \forall m' \in MIN(A) : m' \leq m \end{cases}$$

$$m = \min A \Leftrightarrow \begin{cases} m \leq a, \ \forall a \in A \\ m \in A \end{cases}$$

a) Fie $m = \min A$

Fie $m' \in MIN(A) \Rightarrow m' \leq a, \ \forall a \in A$
$m \in A$
$\Rightarrow m' \leq m$

Analog b) (temă)

5. $A, B \subseteq \mathbb{R}$ nevide, mărginite, $A \subseteq B$

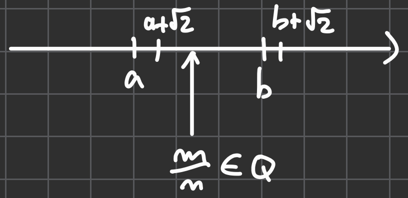
$\Rightarrow \inf B \leq \inf A \leq \sup A \leq \sup B$

$\inf A \leq a \leq \sup A \ \forall a \in A$

Fie $m = \inf B \Rightarrow m \in MIN(B) \Rightarrow m \leq b \ \forall b \in B$ $\xRightarrow{A \subseteq B}$ $m \leq a, \forall a \in A \Rightarrow m \in MIN(A) \Rightarrow$

Analog c) (temă) $\Rightarrow m \le \inf(A)$

⑥ Demonstrați că între $(\forall)\ x, y \in \mathbb{R}, x \neq y$ $\exists$ un număr rațional, respectiv irațional



$\frac{1}{b-a} > 0 \Rightarrow \exists\, n \in \mathbb{N}^*$ a.î. $n > \frac{1}{b-a}$

$\forall\, x \in \mathbb{R} : x - 1 < [x] \le x$

$[x] = \max\left\{ k \in \mathbb{Z} \mid k \le x \right\}$

Fie $m = [n \cdot a] + 1 \in \mathbb{Z}$

$\Rightarrow n \cdot a - 1 < [na] \le na \ /_{+1}$

$na < m \le na + 1 \ /_{:n}$

$a < \frac{m}{n} \le a + \frac{1}{n} < a + (b-a) = b$

$\Rightarrow \frac{m}{n} \in (a, b) \cap \mathbb{Q}$

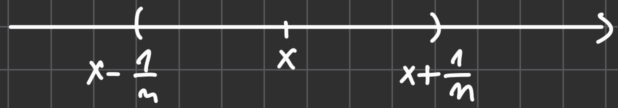
$\sqrt{2} \notin \mathbb{Q}$

$\Rightarrow \exists\, r \in (a+\sqrt{2}, b+\sqrt{2}) \cap \mathbb{Q}$

$\Rightarrow r - \sqrt{2} \in (a, b) \cap (\mathbb{R} \setminus \mathbb{Q})$

⑦ $(\forall)\ x \in \mathbb{R}$ este limita unui şir de numere raţionale respectiv iraţionale

$\forall n \in \mathbb{N}^*$



$\overset{(Q)}{\Rightarrow} \exists\, r_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \cap \mathbb{Q}$

$(r_n) \subseteq \mathbb{Q}$ şir, $x - \frac{1}{n} < r_n < x + \frac{1}{n}$, $n \to \infty \Rightarrow \lim\limits_{n \to \infty} r_n = x$

analog cu iraţionale

Exemple pt. $x = 2$, $y = \sqrt{2}$

$x = 2$

$r_n = 2 + \frac{1}{n}$, $n \in \mathbb{N}^*$

$i_n = 2 + \frac{\sqrt{7}}{n}$, $n \in \mathbb{N}^*$

$y = \sqrt{2}$

$i_n = \sqrt{2} + \frac{1}{n}$, $n \in \mathbb{N}^*$

$r_n = 1 + \underbrace{\frac{1}{2+}\ \frac{1}{2+} \cdots \frac{1}{2+}}_{n \text{ ori}}$

$\lim\limits_{n \to \infty} r_n = \sqrt{}$

$$x_{n+1} = 1 + \frac{1}{2+(x_n-1)} = 1 + \frac{1}{1+x_n} = \frac{2+x_n}{1+x_n}, \ \forall n \in \mathbb{N}, \ x_0 = 1$$

Alt exemplu

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}, \ \forall n \in \mathbb{N}, \ x_0 \in \mathbb{Q}, \ x_0 > 0$$

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \geq 2\sqrt{\frac{\cancel{x_n}}{2} \cdot \frac{1}{\cancel{x_n}}} \overset{x_0>0}{=} \sqrt{2}, \ \forall \in \mathbb{N}$$

$\Rightarrow x_n \geq \sqrt{2}, \ \forall n \geq 1 \Rightarrow x_n$ șir mărginit

$$x_{n+1} - x_n = \frac{x_n}{2} + \frac{1}{x_n} - x_n = \frac{1}{x_n} - \frac{x_n}{2} = \frac{2-x_n^2}{2x_n} \leq 0$$

$\Rightarrow (x_n)$ desc $\Rightarrow (x_n)$ conv, $\exists\, l = \lim_{n\to\infty} x_n \in \mathbb{R}$

$$l = \frac{l}{2} + \frac{1}{l}, \ l^2 = 2 \Rightarrow l = \sqrt{2}$$

⑧ $\alpha = \sqrt{2} + \sqrt[3]{3} \notin \mathbb{Q}$

$$\alpha - \sqrt{2} = \sqrt[3]{3} \quad /()^3$$

$$(\alpha - \sqrt{2})^3 = 3$$

$$(\alpha - \sqrt{2})(\alpha^2 - 2\alpha\sqrt{2} + 2) = 3$$

$$\alpha^3 - 3\sqrt{2}\alpha^2 + 6\alpha - 2\sqrt{2} = 3$$

$$\alpha^3 + 6\alpha - 3 = \sqrt{2}(3\alpha^2 + 2) \quad /()^2$$

$$(\alpha^3 + 6\alpha - 3)^2 = 2(3\alpha^2 + 2)^2$$

$\alpha^6 + \ldots + 1 = 0$

presupunem că $\alpha = \frac{p}{q} \Rightarrow p \vdots 1,\ q \vdots 1 \Rightarrow \frac{p}{q} \in \{-1, 1\}$ absurd $\Downarrow$ $\alpha \notin \mathbb{Q}$

Seminar 2

Șiruri de numere reale

① $\lim\limits_{n \to \infty} x_n$

a) $x_n = \sqrt{n} \cdot (\sqrt{n+1} - \sqrt{n})$

b) $x_n = \dfrac{n + \cos n}{n + \sin n}$

c) $x_n = \dfrac{(\sqrt{2}+1)^n}{(\sqrt{2})^n + 1}$

a) $\lim\limits_{n \to \infty} \sqrt{n} \cdot (\sqrt{n+1} - \sqrt{n}) = \lim\limits_{n \to \infty} \dfrac{\sqrt{n}(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} =$

$= \lim\limits_{n \to \infty} \dfrac{\cancel{\sqrt{n}}(\cancel{n}+1-\cancel{n})}{\cancel{\sqrt{n}}\left(\sqrt{\underbrace{\frac{1}{n}}_{\searrow 0}+1}+1\right)} = \dfrac{1}{2}$

b) $\lim\limits_{n \to \infty} \dfrac{n + \cos n}{n + \sin n} = \lim\limits_{n \to \infty} \dfrac{n\left(1 + \overbrace{\frac{\cos n}{n}}^{\nearrow 0}\right)}{n\left(1 + \underbrace{\frac{\sin n}{n}}_{\searrow 0}\right)} = \dfrac{1}{1} = 1$

$-1 \le \cos n \le 1 \,/ : n$

$-\dfrac{1}{n} \le \dfrac{\cos n}{n} \le \dfrac{1}{n}$

$\searrow 0 \swarrow$ c.d

c) $\lim_{n\to\infty} \frac{(\sqrt{2}+1)^n}{(\sqrt{2})^n+1} = \lim_{n\to\infty} \frac{\cancel{(\sqrt{2})^n}\left(1+\frac{1}{\sqrt{2}}\right)^n}{\cancel{(\sqrt{2})^n}\left(1+\underbrace{\left(\frac{1}{\sqrt{2}}\right)^n}_{\to 0}\right)} = \frac{\infty}{1} = +\infty$

② Justificați cu def val limitelor

a) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \iff \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0, \left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon$

$\frac{1}{\sqrt{n}} < \varepsilon \Rightarrow 1 < \sqrt{n} \cdot \varepsilon$

$\frac{1}{\varepsilon} < \sqrt{n} \,/ (\,)^2$

$n > \frac{1}{\varepsilon^2}$

Alegem $n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1$

$(\forall)\, \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $(\forall)\, n \geq n_0 : \frac{n^2}{n+1} > \varepsilon$

punem condiția

$\frac{n^2}{n+1} = \frac{n^2 - 1 + 1}{n+1} = \frac{(n-1)(n+1)+1}{n+1} = n - 1 + \frac{1}{n+1} > n - 1 > \varepsilon$

$n > \varepsilon + 1$

Alegem $n_0 = [\varepsilon] + 2$

Studiați convergența șirului și calc. lim

a) $x_n = a^n, a \in \mathbb{R}$

$$\lim_{n\to\infty} a^n = \begin{cases} \infty, & a>1 \\ 1, & a=1 \\ 0, & a\in(-1,1) \\ \nexists, & a\le -1 \end{cases},\quad \begin{aligned} & a=-2 \\ & x_n=(-2)^n \\ & x_{2k}=2^{2k}\to\infty \\ & x_{2k+1} \Rightarrow \nexists \lim_{n\to\infty} x_n \end{aligned}$$

b) $\dfrac{x_{n+1}}{x_n} = \dfrac{2^{n+1}}{(n+1)!}\cdot\dfrac{n!}{2^n} = \dfrac{2}{n+1} < 1,\ (\forall)\ n\ge 2$

$\Rightarrow x_n$ descrescător $\Rightarrow x_n > 0,\ \forall n\in\mathbb{N} \Rightarrow (x_n)$ mărginit inf $\Bigg\}\Rightarrow$

$\Rightarrow (x_n)$ convergent $\Rightarrow \exists \lim\limits_{n\to\infty} x_n = l\in\mathbb{R}$

$$\frac{x_{n+1}}{x_n} = \frac{2}{n+1} \quad / \cdot x_n$$

$$x_{n+1} = \frac{2}{n+1}\cdot x_n,\quad \forall n\in\mathbb{N}$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}\frac{2}{n+1}\cdot \lim_{n\to\infty} x_n \Rightarrow l = 0\cdot l = 0$$

c) $x_n = \sqrt[n]{n}$

$$y_n = \sqrt[n]{n} - 1 \ge 0,\quad \forall n\ge 1$$

$$\sqrt[n]{n} = 1 + y_n \quad /^n$$

$$n = (1+y_n)^n = 1 + C_n^1\cdot y_n + \underbrace{C_n^2\cdot y_n^2}+\ldots+C_n^n\cdot y_n^n >$$

$$> C_n^2\cdot y_n^2,\quad \forall n\ge 2 \Rightarrow$$

$$\Rightarrow n > \frac{n(n-1)}{2} \cdot y_n^2 \Rightarrow y_n^2 < \frac{2}{n-1} \Rightarrow \underbrace{0}_{\searrow 0} \le y_n < \underbrace{\sqrt{\frac{2}{n-1}}}_{\swarrow 0}, n \ge 2$$

$n \to \infty$

d) $X_n = \left(1 + \frac{1}{n}\right)^n$

$$X_n = 1 + C_n^1 \cdot \frac{1}{n} + C_n^2 \cdot \frac{1}{n^2} + \ldots + C_n^n \cdot \frac{1}{n^n} =$$

$$= 1 + n \cdot \frac{1}{n} + \frac{n \cdot (n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \ldots +$$

$$+ \frac{n(n-1) \cdot \ldots \cdot 2 \cdot 1}{n!} \cdot \frac{1}{n^n} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)$$

$$+ \ldots + \frac{1}{n!}\left(1 - \frac{1}{n}\right) \cdot \ldots \cdot \left(1 - \frac{n-1}{n}\right)$$

$$X_{n+1} = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \frac{1}{3!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right) + \ldots +$$

$$+ \frac{1}{n!}\left(1 - \frac{1}{n+1}\right) \cdot \ldots \cdot \left(1 - \frac{n-1}{n+1}\right) + \underbrace{\frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right) \cdot \ldots \cdot \left(1 - \frac{n}{n+1}\right)}_{>0}$$

$$\Rightarrow X_{n+1} > X_n, \ \forall n \ge 1 \Rightarrow (X_n) \text{ s. cresc.}$$

$$X_n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} =$$

$$\boxed{\frac{1}{n!} = \frac{1}{1\cdot 2\cdot 3\cdot \ldots \cdot n} < \frac{1}{1\cdot \underbrace{2\cdot 2\cdot \ldots \cdot 2}_{n-1 \text{ ori}}} = \frac{1}{2^{n-1}},\ n \geq 3}$$

$$= 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1 + 2\cdot\underbrace{\left(1-\frac{1}{2^n}\right)}_{<1} < 3,\ \forall n \geq 1$$

$\Rightarrow (x_n)$ mărg sup $\Rightarrow (x_n)$ convergent $\Rightarrow \exists \lim\limits_{n\to\infty} x_n \overset{not}{=} e \simeq 2{,}71$

e) din d) $\Rightarrow x_n < 3,\ \forall n \in \mathbb{N} \Rightarrow (x_n)$ mărg sup

(x_n) s. cresc $\Rightarrow (x_n)$ conv, $\exists \lim\limits_{n\to\infty} x_n = L \in \mathbb{R}$

notăm $y_n = \left(1+\frac{1}{n}\right)^n \overset{d)}{\Longrightarrow} y_n < x_n,\ \forall n \geq 1 \Rightarrow \lim\limits_{n\to\infty} y_n \leq \lim\limits_{n\to\infty} x_n$

$\Rightarrow e \leq L$

fie $p \in \mathbb{N}^*$ fixat și $n > p$

$y_n \quad 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \ldots + \frac{1}{p!}\left(1-\frac{1}{n}\right)\cdot \ldots \cdot\left(1-\frac{p-1}{n}\right), \forall n > p$

$n \to \infty \Rightarrow e \geq 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{p!} = x_p,\ \forall p \in \mathbb{N}^*,\ p \to \infty \Rightarrow$

$\Rightarrow \left[e \geq \lim\limits_{p\to\infty} = L\right] \Rightarrow L = e$

f) (x_n) fundamental $\Leftrightarrow \forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ a.î. $\forall p \in \mathbb{N}, \forall n \geq n_0$

$|x_{n+p} - x_n| < \varepsilon$

$$|x_{n+p}-x_n| = \left| \frac{\sin(1!)}{1\cdot 2} + \frac{\sin(2!)}{2\cdot 3} + \dots + \frac{\sin(n+p)!}{(n+p)(n+p+1)} - \frac{\sin 1!}{1\cdot 2} \right.$$

$$\left. - \dots - \frac{\sin(n!)}{n\cdot(n+1)} \right| = \left| \frac{\sin(n+1)!}{(n+1)(n+2)} + \dots + \frac{\sin(n+p)!}{(n+p)(n+p+1)} \right| \leq$$

$$\leq \frac{|\sin(n+1)!|}{(n+1)(n+2)} + \dots + \frac{|\sin(n+p)!|}{(n+p)(n+p+1)} \leq \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)} +$$

$$\dots + \frac{1}{(n+p)(n+p+1)} = \frac{1}{n+1} - \cancel{\frac{1}{n+2}} + \cancel{\frac{1}{n+2}} - \cancel{\frac{1}{n+3}} + \dots +$$

$$+ \cancel{\frac{1}{n+p}} - \frac{1}{n+p+1} = \frac{1}{n+1} - \frac{1}{n+p+1} < \frac{1}{n+1} \underset{\text{punem}}{<} \varepsilon \Rightarrow$$

$$\Rightarrow n+1 > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon} - 1$$

alegem $n_0 = \left[\frac{1}{\varepsilon}\right] \Rightarrow (x_n)$ fundamental $\Rightarrow (x_n)$ convergent $\Rightarrow$

$$\Rightarrow \exists\ l = \lim_{n\to\infty} x_n \in \mathbb{R}$$

4. Det. $\lim(x_n)$ pt.

a) $x_n = (-1)^n \cdot n \cdot \sin \frac{n\pi}{2}$

b) $x_n = \left(1 + \frac{\cos(n\pi)}{n}\right)^n$

a. $x_{2k} = (-1)^{2k} \cdot (2k) \cdot \sin \frac{2k\pi}{2}$

$= 2k \cdot \sin(k\pi) = 0 \ (\forall) n \in \mathbb{N}$

$x_{2k+1} = (-1)^{2k+1} \cdot (2k+1) \cdot \sin \frac{(2k+1)\pi}{2} = -(2k+1) \sin(k\pi + \frac{\pi}{2}) =$

$= \sin k\pi \cdot \cos \frac{\pi}{2} + \cos k\pi \cdot \sin \frac{\pi}{2} = \cos k\pi$

$k = 2m, \ m \in \mathbb{N}$

$x_{4m+1} = -(4m+1) \to -\infty$

$x_{4m+3} = -(4m+3) \cdot (-1) = 4m+3) \to +\infty$

$LIM(x_n) = \{0, -\infty, +\infty\}$

Seminar 3

Criteriul Stolz-Cesaro

(a_n) șir oarecare

(b_n) șir strict monoton și divergent

dacă $\exists \lim\limits_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = l \Rightarrow \exists \lim\limits_{n\to\infty} \frac{a_n}{b_n} = l$

① Fie x_n șir termeni strict pozitivi

$a_n = \ln x_n$

$b_n = n \nearrow +\infty$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \lim_{n\to\infty} \frac{\ln x_{n+1}-\ln x_n}{\cancel{n}+1-\cancel{n}} = \lim_{n\to\infty} \ln \frac{x_{n+1}}{x_n} =$$

$$= \ln\Big(\underbrace{\lim_{n\to\infty} \frac{x_{n+1}}{x_n}}_{L\in[0,+\infty]}\Big) = \ln L \quad , \text{cu conventia } \begin{array}{l} \ln 0 = -\infty \\ \ln \infty = +\infty \end{array}$$

Dacă $\exists \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = L$

$$\overset{S-C}{\Rightarrow} \ln L = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln x_n}{n} = \lim_{n\to\infty} \frac{1}{n}\cdot \ln x_n = \lim_{n\to\infty} \ln \sqrt[n]{x_n}$$

$$= \ln\big(\lim_{n\to\infty} \sqrt[n]{x_n}\big) \Rightarrow \lim_{n\to\infty} \sqrt[n]{x_n} = L = \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$$

② Calc $\lim_{n\to\infty} y_n$,

a) $y_n = \dfrac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}{\ln n}$

b) $y_n = \sqrt[n]{n!}$

c) $y_n = \dfrac{\sqrt[n]{n!}}{n}$

a. $a_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}$

$b_n = \ln n \nearrow \infty$

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\frac{1+\frac{1}{2}+\dots+\frac{1}{n+1}-\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)}{\ln(n+1)-\ln n}=\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\ln\frac{n+1}{n}}=$$

$$=\lim_{n\to\infty}\frac{1}{(n+1)\ln\frac{n+1}{n}}=\lim_{n\to\infty}\frac{1}{\ln\left(\frac{n+1}{n}\right)^{n+1}}=\lim_{n\to\infty}\frac{1}{\ln\left(1+\frac{1}{n}\right)^{n+1}}$$

$$=\lim_{n\to\infty}\frac{1}{\ln\left[\underbrace{\left(1+\frac{1}{n}\right)^{n}}_{e}\cdot\left(1+\frac{1}{n}\right)\right]}=1\Rightarrow\lim_{n\to\infty}y_n=1$$

b. $x_n=n!$

$$\lim_{n\to\infty}y_n=\lim_{n\to\infty}\sqrt[n]{x_n}=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{(n+1)!}{n!}=\lim_{n\to\infty}n+1=\infty$$

c. $$y_n=\frac{\sqrt[n]{n!}}{\sqrt[n]{n}}=\sqrt[n]{\frac{n!}{n^n}}$$

Fie $x_n=\frac{n!}{n}$ $\quad\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}=\lim_{n\to\infty}$

$$\frac{n!\,(n+1)\cdot n^n}{(n+1)(n+1)^n\cdot n!}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\left(1+\frac{n}{n+1}-1\right)^n=$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{-1}{n+1}\right)^{\frac{n+1}{-1}}\right]^{\frac{-n}{n+1}}=e^{\lim\limits_{n\to\infty}\frac{-n}{n+1}}=e^{-1}=\frac{1}{e}\Rightarrow\lim_{n\to\infty}y_n=\frac{1}{e}$$

③ Calc limitele cu S-C pt limitele

$$a_n = \sum_{k=1}^{n} \frac{1+(-1)^k}{2}, \quad b_n = n, \quad n \in \mathbb{N}^*$$

$$\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \frac{\frac{1+(-1)^{n+1}}{2}}{(n+1)-n} = \lim_{n\to\infty} \frac{1+(-1)^{n+1}}{2} \nexists$$

$$a_n \; 0+1+0+1+\dots+\frac{1+(-1)^n}{2} \begin{cases} k & , n=2k \\ k & , n=2k+1 \end{cases}$$

$$\left.\begin{aligned} &\lim_{k\to\infty} \frac{a_{2k}}{b_{2k}} = \lim_{k\to\infty} \frac{\cancel{k}}{2\cancel{k}} = \frac{1}{2} \\ &\lim_{k\to\infty} \frac{a_{2k+1}}{b_{2k+1}} = \lim_{k\to\infty} = \frac{k}{2k+1} = \frac{1}{2} \end{aligned}\right\} \Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}$$

Reciproca C-S nu este in general adev

⑤ Scrieți urm serii cu ajutorul Σ

a) $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots = \sum_{n=1}^{\infty} \frac{1}{2k-1}$

b) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots = \sum_{n=0}^{\infty} \frac{1}{2^n}$

c) $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\frac{1}{25}-\dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$

5. Calc suma seriilor

a) $\sum_{n=0}^{\infty} \frac{1}{n!} = \lim_{n \to \infty} \left(1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}\right) = e$

b) $\sum_{n=1}^{\infty} \frac{1}{5^n}$

c) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + \sqrt{n-1}}$

d) $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}$

e) $\sum_{n=2}^{\infty} \ln\left(1 - \frac{1}{n^2}\right)$

f) $\sum_{n=0}^{\infty} \frac{n \cdot 2^n}{(n+2)!}$

6. $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \ \forall \ a \in (-1, 1)$

$$a = \frac{1}{5} \Rightarrow \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = \frac{1}{1 - \frac{1}{5}} = \frac{5}{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=0}^{\infty} \frac{1}{5^n} - 1 = \frac{5}{4} - \overset{4/4}{1} = \frac{1}{4}$$

c. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n-1}}$

$S_n = \sum_{k=1}^{n} = \frac{1 \cdot \cancel{\sqrt{k}-\sqrt{k-1}}}{\sqrt{k}+\sqrt{k-1}} = \sum_{k=1}^{n} \frac{\sqrt{k}-\sqrt{k-1}}{k-(k-1)} = \sum_{k=1}^{n} \sqrt{k}-\sqrt{k-1} =$

$= \cancel{1}-0 + \cancel{\sqrt{2}} - \cancel{1} + \cancel{\sqrt{3}} - \cancel{\sqrt{2}} + \cancel{\ldots} + \cancel{\sqrt{n-2}} - \cancel{\sqrt{n-3}} + \cancel{\sqrt{n-1}} - \cancel{\sqrt{n-2}} + \sqrt{n} - \cancel{\sqrt{n-1}}$

$= \sqrt{n}$

$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sqrt{n} = \infty$

d. $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$, $S_n = \sum_{k=1}^{n} \frac{1}{4n^2-1} = \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right)$

$= \frac{1}{2}\left(1 - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\ldots} + \cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right) = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right)$

$S = \lim_{n\to\infty} S_n = \frac{1}{2}$

e. $\sum_{n=2}^{\infty} \ln\left(1-\frac{1}{n^2}\right)$, $S_n = \sum_{k=2}^{n} \ln\left(\frac{k^2}{\cancel{1}} - \frac{1}{k^2}\right) = \sum_{k=2}^{n} \ln\left(\frac{k^2-1}{k^2}\right) =$

$= \sum_{k=2}^{n} \ln \frac{(k-1)(k+1)}{k^2} = \sum_{k=2}^{n} \ln\left(\frac{k-1}{k} \cdot \frac{k+1}{k}\right) = \sum_{k=2}^{n}\left[\ln\left(\frac{k-1}{k}\right) +\right.$

$\left.+ \ln\left(\frac{k+1}{k}\right)\right] = \sum_{k=2}^{n}\left[\ln\frac{k-1}{k} - \ln\frac{k}{k+1}\right]$

$S_n = \ln\frac{1}{2} - \cancel{\ln\frac{2}{3}} + \cancel{\ln\frac{2}{3}} - \cancel{\ln\frac{3}{4}} + \cancel{\ldots} + \cancel{\ln\frac{n-1}{n}} - \ln\frac{n}{n+1}$

$= \ln\frac{1}{2} - \ln\frac{n}{n+1}$

$$S = \lim_{n\to\infty} \ln\frac{1}{2} - \ln\underbrace{\frac{n}{n+1}}_{\to 0} = \ln\frac{1}{2} = -\ln 2$$

f) $\sum_{n=0}^{\infty} \frac{n\cdot 2^n}{(n+2)!}$, $S_n = \sum_{k=0}^{n} \frac{k\cdot 2^k}{(k+2)!} = \sum_{k=0}^{n} \frac{(k+2-2)\cdot 2^k}{(k+2)!} =$

$$= \sum_{k=0}^{n}\left(\frac{(k+2)\cdot 2^k}{(k+2)!} - \frac{2\cdot 2^k}{(k+2)!}\right) = \sum_{k=0}^{n}\left(\frac{2^k}{(k+1)!} - \frac{2^{k+1}}{(k+2)!}\right) =$$

$$= \frac{2^0}{1!} - \frac{2^1}{2!} + \frac{2^1}{2!} - \frac{2^2}{3!} + \dots + \frac{2^n}{(n+1)!} - \frac{2^{n+1}}{(n+2)!} = 1 - \underbrace{\frac{2^{n+1}}{(n+2)!}}_{\to 0}$$

$$S = \lim_{n\to\infty} S_n = 1$$

Seminar 4

Studiați natura seriilor STP folosind criteriile indicate:

i) a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{5n^2-1}}$

(comparației)

b) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$

ii) cons crit. kummer

a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!}$

b) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n}}$

c) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{(2n)!!}{(2n+1)!!}\right)^2$

a) $$\frac{1}{\sqrt{4n^2-1}} > \frac{1}{\sqrt{4n^2}} = \frac{1}{2n}$$

$$\sum_{n=1}^{\infty} \frac{1}{2n} = \underbrace{\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}}_{\text{Divergentă}}$$

C.C.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^2-1}} \text{ divergentă}$$

b) $$\underbrace{\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)}_{x_n}$$

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{y_n}$$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\ln(1+\frac{1}{n^2})}{\frac{1}{n^2}} = \lim_{n\to\infty} n^2 \ln(1+\frac{1}{n^2})$$

$$= \lim_{n\to\infty} \ln(1+\frac{1}{n^2})^{n^2} = \ln e = 1 \in (0,\infty) \Rightarrow$$

$$\Rightarrow \sum x_n \sim \sum y_n \Rightarrow \sum x_n \text{ convergentă}$$

a) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

ii) $x_n = \frac{2^n}{n!}$

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\frac{2^n}{n!}}{\frac{2^{n+1}}{(n+1)!}} = \lim_{n\to\infty} \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}} =$$

$$= \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1 \Rightarrow \sum_{n=1}^{\infty} x_n \text{ - convergentă}$$

b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n}}$

$x_n = \left(\frac{1}{2}\right)^{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^{\sqrt{n}}}{\left(\frac{1}{2}\right)^{\sqrt{n+1}}} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{\sqrt{n}} \cdot \left(\frac{2}{1}\right)^{\sqrt{n+1}} = \lim_{n\to\infty} \frac{2^{\sqrt{n+1}}}{2^{\sqrt{n}}}$$

$$= \lim_{n\to\infty} 2^{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty} 2^{\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}}} = 2^0 = 1, \text{ nu decide}$$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}} - 1\right) =$$

$$\lim_{x\to 0} \frac{a^x - 1}{x} = \ln a, \quad \forall a > 0$$

$$= \lim_{n\to\infty} \frac{2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}} - 1}{\frac{1}{\sqrt{n+1}+\sqrt{n}}} \cdot \frac{n}{\sqrt{n+1}+\sqrt{n}} = \ln 2 \cdot \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{1+\frac{1}{n}}+1}$$

$$= \infty > 1 \Rightarrow \sum_{n=0}^{\infty} x_n \text{ convergent}$$

$$(2n)!! = 2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n$$

$$(2n+1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n+1)$$

c) $$x_n = \left(\frac{(2n)!!}{(2n+1)!!}\right)^2$$

$$\lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left(\frac{(2n)!!}{(2n+1)!!}\right)^2 \left(\frac{(2n+3)!!}{(2n+2)!!}\right)^2 =$$

$$= \lim_{n\to\infty} \left(\frac{2n+3}{2n+2}\right)^2 = 1 \text{ nu decide}$$

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{4n^2+12n+9}{4n^2+8n+4}-1\right)=$$

$$=\lim_{n\to\infty} n\left(\frac{4n+5}{4n^2+8n+4}\right)=\lim_{n\to\infty}\frac{4n^2+5n}{4n^2+8n+4}=1, \text{ nedecidabil}$$

$$\lim_{n\to\infty}(\ln n)\cdot\left(n\left(\frac{x_n}{x_{n+1}}-1\right)-1\right)=\lim_{n\to\infty}(\ln n)\left(\frac{4n^2+5n}{4n^2+8n+4}-1\right)$$

c) $$\lim_{n\to\infty}(\ln n)\left(\frac{-3n-4}{4n^2+8n+4}\right)=\lim_{n\to\infty}\frac{\ln n}{n}\cdot\left(\frac{-3n^2-4n}{4n^2+8n+4}\right)$$

$$=-\frac{3}{4}\cdot\lim_{n\to\infty}\frac{\ln n}{n}=-\frac{3}{4}\cdot\lim_{n\to\infty}\ln\sqrt[n]{n}=-\frac{3}{4}\cdot 0=0<1$$

$\Rightarrow \sum x_n$ divergentă

iii) radicalului

$$\sum_{n=1}^{\infty}\frac{n^2}{\left(2+\frac{1}{n}\right)^n},\quad x_n=\frac{n^2}{\left(2+\frac{1}{n}\right)^n}$$

$$\lim_{n\to\infty}\sqrt[n]{x_n}=\lim_{n\to\infty}\sqrt[n]{\frac{n^2}{\left(2+\frac{1}{n}\right)^n}}=\lim\frac{\sqrt[n]{n^2}}{\left(2+\frac{1}{n}\right)}=$$

$$=\lim_{n\to\infty}\frac{\left(\sqrt[n]{n}\right)^2}{2+\frac{1}{n}}=\frac{1}{2}<1\Rightarrow\sum x_n \text{ convergentă}$$

(iv) condensării

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p},\ p>0$$

$$x_n = \frac{1}{n(\ln n)^p} \searrow$$

$$|x_n| \text{ descresc} \overset{C.C.}{\Longrightarrow} \sum_{n=2}^{\infty} x_n \sim \sum_{n=1}^{\infty} 2^n \cdot x_{2^n} = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n(\ln 2^n)^p} =$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n \cdot \ln 2)^p} = \frac{1}{(\ln 2)^p} \cdot \sum_{n=1}^{\infty} \frac{1}{n^p} \text{ , s.t. conv} \Leftrightarrow p>1$$

caz particular $p=1 \Rightarrow \sum_{n=2}^{\infty} \frac{1}{n \cdot \ln n}$ diverg

② Studiați convergența și absolut convergența următoarelor serii:

a) $\sum_{n=0}^{\infty} (-1)^n \cdot \frac{2n+1}{3^n}$

$a_n = \frac{2n+1}{3^n}$, $\frac{a_n}{a_{n+1}} = \frac{\frac{2n+1}{3^n}}{\frac{2n+3}{3^{n+1}}} = \frac{6n+3}{2n+3} > 1 \Rightarrow$

$a_n \searrow 0$

$\Rightarrow a_n \searrow$

$\lim a_n = \lim \frac{2n+1}{3^n} = 0$

$$\left.\right\} \overset{CL}{\Longrightarrow} \sum_{n=1}^{\infty} (-1)^n \cdot \frac{2n+1}{3^n} \text{ converg}$$

$$\sum_{n=0}^{\infty} \left| (-1)^n \cdot \frac{2n+1}{3^n} \right| = \sum_{n=0}^{\infty} \underbrace{\frac{2n+1}{3^n}}_{a_n}$$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{6n+3}{2n+3} = 3 > 1 \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n \cdot a_n \text{ abs conv}$$

b) $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin n}{2^n} \right|$

$$\left.\begin{array}{l} \dfrac{|\sin n|}{2^n} \leq \dfrac{1}{2^n}, \forall n \in \mathbb{N} \\ \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} \text{ convergentă} \end{array}\right\} \overset{C.C}{\Rightarrow} \sum_{n=1}^{\infty} \frac{\sin n}{2^n} \text{ este abs. conv. deci și conv.}$$

3. Criteriul raportului pt. șiruri

(x_n) șir cu termeni strict pozitivi și $\exists \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = l$, are loc:

i) dacă $l > 1 \Rightarrow \lim_{n\to\infty} x_n = 0$

ii) dacă $l < 1 \Rightarrow \lim_{n\to\infty} x_n = +\infty$

Ex: 1) $x_n = \frac{2n+1}{3^n}$, $\lim_{x \to \infty} \frac{x_n}{x_{n+1}} = \lim_{n \to \infty} \frac{6n+3}{2n+3} = 3 > 1 \Rightarrow$

$\Rightarrow \lim_{n \to \infty} x_n = 0$

2) ct. $x_n = \frac{3^n \cdot n!}{n^n}$, limita sa?

i) fie seria termeni poz. $\sum x_n$, $\lim_{n \to \infty} \frac{x_n}{x_{n+1}} = l > 1 \Rightarrow$

$\Rightarrow \sum x_n$ conv $\Rightarrow \lim_{n \to \infty} x_n = 0$

ii) fie seria termeni poz. $\sum \frac{1}{x_n}$, $\lim_{n \to \infty} \frac{\frac{1}{x_n}}{\frac{1}{x_{n+1}}} = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} =$

$= \frac{1}{l} > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{x_n}$ convergentă $\Rightarrow \lim_{n \to \infty} \frac{1}{x_n} = 0_+ \Rightarrow \lim_{n \to \infty} x_n = +\infty$

① Justificați afirmațiile

a) $\frac{1}{n+1} < \ln(n+1) - \ln(n) < \frac{1}{n}$, $\forall n \in \mathbb{N}^*$

b) Șirul $c_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} - \ln n$ este convergent

c) Seria $\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}}$ este divergentă

d) Șirul $S_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{(-1)^{n+1}}{n}$ are limita 2

a) Fie $f : [n, n+1] \to \mathbb{R}$ $\quad f(x) = \ln x$

$\overset{TL}{\Rightarrow} \exists\, c \in (n, n+1)$ a. î. $\quad f'(c) = \dfrac{f(n+1) - f(n)}{n+1-n}$

$f'(x) = \dfrac{1}{x} \Rightarrow f'(c) = \dfrac{1}{c}$

$$\frac{1}{c} = \frac{\ln(n+1) - \ln(n)}{1} = \ln(n+1) - \ln(n)$$

$n < c < n+1 \Rightarrow \dfrac{1}{n} > c > \dfrac{1}{n+1} \Rightarrow \dfrac{1}{n} > \ln(n+1) - \ln n > \dfrac{1}{n+1}$

b) $c_{n+1} - c_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n} + \dfrac{1}{n+1} - \ln(n+1) - 1 - \dfrac{1}{2} - \dfrac{1}{3} - \ldots - \dfrac{1}{n} + \ln n =$

$= \ln n - \ln(n+1) + \dfrac{1}{n+1}$

$\dfrac{1}{n+1} + \ln n - \ln(n+1) = 0 \Rightarrow (c_n)$ - desc

$n = 1 \Rightarrow \ln_2 - \ln_1 < \dfrac{1}{1}$

$n = 2 \Rightarrow \ln_3 - \ln_2 < \dfrac{1}{2}$

$\vdots$

$n \in \mathbb{N} \Rightarrow \ln(n+1) - \ln n < \dfrac{1}{n}$

$-\ln 1 + \ln(n+1) < \dfrac{1}{1} + \dfrac{1}{2} + \ldots + \dfrac{1}{n} \quad / - \ln n$

$0 < \ln(n+1) - \ln(n) < c_n$

$\Rightarrow c_n > 0, \forall n \in \mathbb{N}^* \Rightarrow c_n$ marg inf

(1), (2) $\Rightarrow c_n$ conv.

$$\lim_{n\to\infty} c_n \overset{not}{=} \gamma \simeq 0{,}57 \text{ (constanta lui Euler)}$$

c) $\sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^{\ln n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverg

$$\lim_{n\to\infty} \frac{\left(\frac{1}{e}\right)^{1+\frac{1}{2}+\ldots+\frac{1}{n}}}{\left(\frac{1}{e}\right)^{\ln n}} = \lim_{n\to\infty} \left(\frac{1}{e}\right)^{c_n} = \left(\frac{1}{e}\right)^{\gamma} \in (0,+\infty) \Rightarrow \text{seriile au aceeași nat.}$$

d) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{n\to\infty} S_n = \ln 2$

$$S_{2n} = 1 - \frac{1}{2} + \frac{1}{3} - \ldots + \frac{1}{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} - 2\left(\frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n}\right) =$$

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} - \ln(2n) - \left[1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n\right] + \ln(2n) - \ln n =$$

$$= c_{2n} - c_n + \ln 2 \Rightarrow \lim_{n\to\infty} S_{2n} = \gamma - \gamma + \ln 2 = \ln 2$$

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} \left(S_{2n} + \frac{1}{2n+1}\right) = \ln 2 + 0 = \ln 2$$

② $A' = ?$

a) $A = \left\{ \frac{1}{2^n} \,\middle|\, n \in \mathbb{N} \right\}$

b) $A = \mathbb{Q}$

a) $x_n = \frac{1}{2^n} \to 0 \ (n \to \infty)$

$0 \in A' \Rightarrow A' = \{0\}$

b) $\forall x \in \mathbb{R}, \exists (r_n) \subseteq \mathbb{Q} \setminus \{x\}$ a.i. $\lim_{n\to\infty} r_n = x$

$\Rightarrow A' = \overline{\mathbb{R}}$

③ Det val extreme și dacă se ating pt

a) $f : (-1, 1) \to \mathbb{R}, f(x) = \ln \frac{1-x}{1+x}$

b) $f : [0, 1] \to \mathbb{R}, f(x) = \begin{cases} \frac{1}{2}, & x = 0 \\ x, & x \in (0, 1] \end{cases}$

c) $f : [-1, 1] \to \mathbb{R}$,

$f(x) = x \cdot \sqrt{1 - x^2}$

a) A -> interval Nu e interval închis nu se aplică

$A = (-1, 1)$

$\lim_{x \to -1} \ln \frac{1-x}{1+x} = \infty$

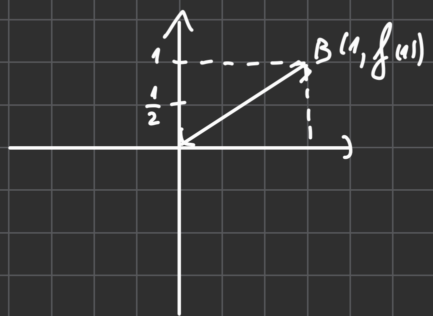
$x > -1$

$\lim_{\substack{x \to 1 \\ x < 1}} \ln \frac{1-x}{1+x} = -\infty \ (1)$

$\inf(f(A)) = -\infty$

$\sup(f(A)) = +\infty$

b) $A = [0, 1]$



f nu e cont în $x = 0$

$0 = \lim_{x \searrow 0} f(x) \neq f(0) = \frac{1}{2}$

$f(A) = (0, 1]$

$\inf(f(A)) = 0$ NU ATINGE

$\sup(f(A)) = 1 = f(1)$ ATINGE

c) f cont pe $[-1, 1] \overset{TW}{\Longrightarrow} f$ își atinge val extreme

$$f'(x) = \sqrt{1 - x^2} + x \cdot \frac{1}{2\sqrt{1-x^2}} \cdot (-2x) =$$

$$= \sqrt{1 - x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1 - x^2 - x^2}{\sqrt{1-x^2}} = \frac{1 - 2x^2}{\sqrt{1-x^2}}, \forall x \in (-1, 1)$$

$$f'(x) = 0 \Leftrightarrow 1 - 2x^2 = 0 \Leftrightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

x	-1		$-\frac{1}{\sqrt{2}}$		0		$\frac{1}{\sqrt{2}}$		1
$f'(x)$	$-$	$- - - -$	0	$+ + + +$		$+$	0	$- -$	
$f(x)$	0	$\longrightarrow$	$\frac{1}{2}$	$\nearrow$			$\frac{1}{2}$	$\searrow$	0

$f(A) = [-\frac{1}{2}, \frac{1}{2}]$

$$f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}} \cdot \sqrt[2]{1-\frac{1}{2}} = -\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = -\frac{1}{2}$$

$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$$

$$\inf f(A) = -\frac{1}{2}$$

$$\sup f(A) = \frac{1}{2}$$

5. Det punctele de extrem local ale f de la 3

$x = -\frac{1}{\sqrt{2}}$ punct de minim local

$x = \frac{1}{\sqrt{2}}$ punct de maxim local

$x = -1$ punct de maxim local

$x = 1$ punct de minim local

a) $f'(x) = (\ln(1-x))' - (\ln(1+x))' = -\frac{1}{1-x} - \frac{1}{1+x} =$

$= \frac{1+x-1-x}{(1-x)(1+x)} = \frac{-2}{(1-x)(1+x)} \neq 0, \forall x \in (-1, 1)$

Deci f nu are puncte de extrem

$f(1) = 1 \leq f(x),\ \forall x \in A \Rightarrow x = 1$ pct maxim local

$\forall x \in (0,1),\ f'(x) = 1 \neq 0 \Rightarrow f$ nu are pct de extrem în $(0,1)$

$x=0$ pct de maxim local $\overset{def}{\Leftrightarrow}$

$\exists\ \delta > 0$, a.î. $\forall x \in (-\delta, \delta) \cap [0,1] : f(0) \geq f(x)$

fie $\delta > \frac{1}{4}$, $\forall x \in [0, \frac{1}{4}) : \frac{1}{2} \geq f(x) \Rightarrow x = 0$ pct de maxim local

5 Calculați

a) $\lim\limits_{x \to 0} \dfrac{e - (1+x)^{\frac{1}{x}}}{x}$

$u = u(x)$, $v = v(x)$

$(u^v)' = (e^{\ln u^v})' = (e^{v \cdot \ln u})' = e^{v \cdot \ln u} \cdot (v \cdot \ln u)' = u^v \cdot (v' \cdot \ln u + v \cdot \frac{u'}{u})$

$$\lim_{x \to 0} \frac{e - (1+x)^{\frac{1}{x}}}{x} \underset{L'H}{\overset{\frac{0}{0}}{=}} \lim_{x \to 0} \frac{-\left[(1+x)^{\frac{1}{x}} \left[\frac{-\ln(1+x)}{x^2} + \frac{1}{x(x+1)}\right]\right]}{1}$$

$$= -e \lim_{x \to 0} \left(-\frac{^{(x+1)}\ln(1+x)}{x^2} + \frac{^{x)}1}{x(x+1)}\right) = -e \cdot \lim_{x \to 0} \frac{-(x+1)\ln(1+x) + x}{x^2(x+1)} \underset{L'H}{\overset{\frac{0}{0}}{=}}$$

$$= -e \cdot \lim_{x \to 0} \frac{-\ln(x+1) - 1 + 1}{3x^2 + 2x} = e \cdot \lim_{x \to 0} \frac{\ln(x+1)}{3x^2+2x} \underset{L'H}{\overset{\frac{0}{0}}{=}} e \cdot \lim_{x \to 0} \frac{\frac{1}{x+1}}{6x+2} = \frac{e}{2}$$

$$\left[(1+x)^{\frac{1}{x}}\right]' = (1+x)^{\frac{1}{x}} \left(\left(\frac{1}{x}\right)' \cdot \ln(1+x) + \frac{1}{x} \cdot \frac{1}{1+x}\right)$$

$$= (1+x)^{\frac{1}{x}} \left[-\frac{1}{x^2} \cdot \ln(1+x) + \frac{1}{x(1+x)}\right]$$

① Calc derivata de ordinul $n \in \mathbb{N}$ $f^{(n)}$ pe care funcțiile sunt indefinit derivabile

a) $f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$f'''(x) = -\cos(x)$

$f^{(4)}(x) = \sin(x)$

$D = \mathbb{R}$

$$f^{(n)}(x) = \begin{cases} \sin x, & n = 4k \\ \cos x, & n = 4k+1 \\ -\sin x, & n = 4k+2 \\ -\cos x, & n = 4k+3 \end{cases}$$

d) $$f^{(n)}(0) = -\frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-3)^{n+1}}{2^n}, \quad n \geq 2$$

$f^{(0)} = 1$

$f'(0) = -\frac{1}{2}$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 1 + \left(-\frac{x}{2}\right) - \sum_{n=2}^{\infty} \frac{(2n-3)!!}{2^n \cdot n!} \cdot x^n$$

$$a_n = \frac{(2n-3)!!}{2^n \cdot n!}$$

$$r = \lim_{n \to \infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n \to \infty} \frac{2n\left(1 + \frac{1}{n}\right)}{2n\left(1 - \frac{1}{2n}\right)} = 1$$

Justificați:

$$1+\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} x^n = \frac{1}{\sqrt{1-x}} \quad \forall x \in [-1,1)$$

1. Evaluați

a) $\displaystyle\int_0^3 \frac{1}{\sqrt{x^2+16}}\,dx = \underbrace{\ln(x+\sqrt{x^2+16})}_{F(x)}\Big|_0^3 = F3 - F(0)$

b) $\displaystyle\int_1^{\sqrt{3}} \frac{\operatorname{arctg} x}{x^2}\,dx = \int_1^{\sqrt{3}} \operatorname{arctg} x \cdot \frac{1}{x^2}\,dx =$

$\displaystyle\int_1^{\sqrt{3}} \operatorname{arctg} x \cdot \left(-\frac{1}{x}\right)' dx = -\frac{1}{x}\cdot \operatorname{arctg} x \Big|_1^{\sqrt{3}} - \int_1^{\sqrt{3}} (\operatorname{arctg} x)' \cdot \left(-\frac{1}{x}\right) dx$

$\displaystyle = -\frac{1}{\sqrt{3}}\cdot \operatorname{arctg}\sqrt{3} - \operatorname{arctg} 1 + \int_1^{\sqrt{3}} \frac{1}{1+x^2}\cdot\frac{1}{x}\,dx = -\frac{\pi}{3\sqrt{3}} + \frac{\pi}{4} +$

$\displaystyle \frac{1}{1+x^2}\cdot\frac{1}{x} = \frac{^{x)}Ax+B}{1+x^2} + \frac{^{1+x^2)}C}{x} = \frac{Ax^2+Bx+Cx^2+C}{x(x^2+1)} =$

$\displaystyle = \frac{x^2(A+C)+Bx+C}{x(x^2+1)} \Rightarrow \begin{cases} A+C=0 \\ B=0 \\ C=1 \end{cases} \Rightarrow A=-1$

$\displaystyle\int_1^{\sqrt{3}} \frac{-x}{1+x^2} + \frac{1}{x}\,dx = \int_1^{\sqrt{3}} \frac{-x}{1+x^2}\,dx + \int_1^{\sqrt{3}} \frac{1}{x}\,dx =$

$$= \frac{1}{2} \int_1^{\sqrt{3}} \frac{2x}{1+x^2} dx + \int_1^{\sqrt{3}} \frac{1}{x} dx$$

Substituții trigonometrice

$R(u, v)$ – funcții raționale, $a > 0$

$\int R(x, \sqrt{a^2 - x^2}) dx$, $x = a \cdot \sin t$, $x = a \cdot \cos t$

$\int R(x, \sqrt{a^2 + x^2}) dx$, $x = a \cdot tg\, t$, $x = a \cdot ctg\, t$

$\int R(x, \sqrt{x^2 - a^2}) dx$, $x = \frac{a}{\sin t}$, $x = \frac{a}{\cos t}$

c) $\int_{-1}^{1} \sqrt{1-x^2}\, dx$

$x = \sin t$

$dx = (\sin t)' \cdot dt = \cos t\, dt$

$x = -1 \Rightarrow \sin(t) = -1 \Rightarrow t = -\frac{\pi}{2}$

$x = 1 \Rightarrow \sin(t) = 1 \Rightarrow t = \frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos t| \cos t\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 t\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos 2t + 1}{2} dt =$$

$$= \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos 2t\, dt + \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt = \frac{1}{2} \ldots$$

d) $\int_{2}^{4} \frac{\sqrt{x^2-4}}{x}\,dx$

$x = \frac{2}{\sin t}$, $dx = \left(\frac{2}{\sin t}\right)' dt = \frac{-2\cos t}{\sin^2 t} \cdot dt$

Integrali impropri

① Calculati

a) $\int_{0}^{\infty} \frac{\operatorname{arctg} x}{1+x^2}\,dx = \lim_{v \to \infty} \int_{0}^{v} \frac{\operatorname{arctg} x}{1+x^2}\,dx = \lim_{x \to \infty} \int_{0}^{v} \operatorname{arctg} x \cdot (\operatorname{arctg} x)'$

$= \lim_{v \to \infty} \frac{1}{2} \operatorname{arctg}^2 x \Big|_{0}^{v} =$

$= \lim_{v \to \infty} \frac{1}{2} \operatorname{arctg}^2 v = \frac{1}{2} \cdot \left(\frac{\pi}{2}\right)^2$

b) $\int_{-1}^{1} \frac{x+1}{\sqrt{1-x^2}}\,dx$

c) $\int_{0}^{\infty} x^n \cdot e^{-x}\,dx$, $n \in \mathbb{N}$

d) $\int_{1}^{2} \frac{1}{\sqrt{x \cdot (2-x)}}\,dx$

b) $\int_{-1}^{0} \frac{x+1}{\sqrt{1-x^2}} + \int_{0}^{1-0} \frac{x+1}{\sqrt{1-x^2}}\,dx$

$\int \frac{x+1}{\sqrt{1-x^2}}\,dx = \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{1}{\sqrt{1-x^2}}\,dx =$

$= -\sqrt{1-x^2} + \arcsin x$

$$\lim_{u \searrow -1} \int_u^0 \frac{x+1}{\sqrt{1-x^2}}\,dx + \lim_{v \nearrow -1} \int_0^1 \frac{x+1}{\sqrt{1-x^2}}\,dx =$$

$$= \lim_{u \searrow -1} -\sqrt{1-x^2}\Big|_u^0 + \arcsin x\Big|_u^0 + \lim_{v \nearrow 1} -\sqrt{1-x^2}\Big|_0^v +$$

$$+ \arcsin x\Big|_0^v =$$

$$= \lim_{u \searrow 1} (-1 + \sqrt{1-u^2} + \arcsin u) + \lim_{v \nearrow 1} (-\sqrt{1-v^2} + 1 - \arcsin v)$$

$$= -1 + \frac{\pi}{2} + 1 + \frac{\pi}{2} = \pi$$

c) $\int_0^\infty x^n \cdot e^{-x}\,dx = -\int_0^\infty x^n \cdot (e^{-x})'\,dx = -x^n \cdot e^{-x}\Big|_0^\infty +$

$$+ \int_0^\infty n x^{n-1} \cdot e^{-x}\,dx = \lim_{v \to \infty} (-x^n \cdot e^{-x})\Big|_0^v + n\, I_{n-1} =$$

...

d) $\int_1^{2-0} \frac{1}{\sqrt{2x-x^2}}\,dx = \int_1^{2-0} \frac{1}{\sqrt{1-1+2x-x^2}}\,dx = \int_1^{2-0} \frac{1}{\sqrt{1-(1-2x+x^2)}}$

$$= \int_1^{2-0} \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \int_0^{1-0} \frac{1}{\sqrt{1-t^2}}\,dt = \arcsin t\Big|_0^{1-0} = \frac{\pi}{2}$$

$x-1=t$
$dx=dt$

$x = 1 \Rightarrow t = 0$
$x \nearrow 2 \Rightarrow t \nearrow 1$

Stud. convergența

a) $\int_0^3 \frac{x^3+1}{\sqrt{9-x^2}}\,dx$

b) $\int_0^\infty \frac{\operatorname{arctg} x}{x}\,dx$

c) $\int_0^{\pi} x \cdot \ln(\sin x)\,dx$

---

a) $f : [0,3) \to [0,+\infty)$, $f(x) = \frac{x^3+1}{\sqrt{9-x^2}}$

$\lambda = \lim_{x \nearrow 3} (3-x)^p \cdot f(x) = \lim_{x \nearrow 3} (3-x)^p \cdot \frac{x^3+1}{\sqrt{(3-x)(3+x)}} =$

· dacă $p<1$, $\lambda < +\infty \Rightarrow$ int conv

· dacă $p \geq 1$, $\lambda > 0 \Rightarrow$ int div

$= \lim_{x \nearrow 3} (3-x)^{p-\frac{1}{2}} \cdot \underbrace{\frac{x^3+1}{\sqrt{3+x}}}_{28/\sqrt{6}} \stackrel{p=\frac{1}{2}<1}{=} \frac{28}{\sqrt{6}} < +\infty \Rightarrow$ int conv

b) $\underbrace{\int_{0+0}^{\infty} \frac{\text{arctg } x}{x} dx}_{I_1} + \underbrace{\int_{1}^{\infty} \frac{\text{arctg } x}{x} dx}_{I_2}$

$I_1: f : (0,1] \to [0, +\infty), f(x) = \frac{\text{arctg } x}{x}$

$\lambda = \lim_{x \downarrow 0} (x-0)^p \cdot f(x) = \lim_{x \searrow 0} x$

$I_2: f : [1, \infty) \to [0, \infty), f(x) = \frac{\text{arctg } x}{x} \quad p < 0, x\text{-finit}$

$\lambda = \lim_{x \to \infty} x^p \frac{\text{arctg } x}{x} = \lim_{x \to \infty} x^{p-1} \frac{\pi}{2} = \frac{\pi}{2} > 0 \Rightarrow I_2 \text{ div} \overset{\Downarrow}{\underset{}{}} I_1 \text{ conv}$

③ Studiați conv int.

$I(\alpha) = \int_0^1 \left(\frac{x}{1-x}\right)^\alpha dx, \forall \alpha \in \mathbb{R}$ și calc $I(\frac{1}{2})$

Caz I $\alpha > 0$

$$I(\alpha) = \int_0^{1-0} \left(\frac{x}{1-x}\right)^\alpha dx$$

$f : [0,1) \to [0, \infty) \quad f(x) = \left(\frac{x}{1-x}\right)^\alpha$

$$\lambda = \lim_{x \nearrow 1} (1-x)^p f(x)$$

$$\lambda = \lim_{x \nearrow 1} (1-x)^p \left(\frac{x}{1-x}\right)^\alpha$$

$$\lambda = \lim_{x \nearrow 1} (1-x)^{p-\alpha} \cdot x^\alpha = \lim_{x \nearrow 1} (1-x)^{p-\alpha} \Longrightarrow$$

$$\xRightarrow{p=\alpha} \lambda = \lim_{x \nearrow 1} (1-x)^0 = 1$$

dacă $p < 1$ ($\alpha \leq 1$), $\lambda < \infty \Rightarrow I_\alpha$ conv

dacă $p \geq 1$ ($\alpha \geq 1$), $\lambda > 0 \Rightarrow I_\alpha$ div

I_2: $\alpha < 0$

$$I(\alpha) = \int_{0+0}^{1} \left(\frac{x}{1-x}\right)^\alpha dx$$

$$f : (0,1] \to [0,\infty), \quad f(x) = \left(\frac{x}{1-x}\right)^\alpha$$

$$\lambda = \lim_{x \searrow 0} (x-0)^p f(x) = \lim_{x \searrow 0} x^p \left(\frac{x}{1-x}\right)^\alpha$$

$$= \lim_{x \searrow 0} \frac{x^{p+\alpha}}{(1-x)^\alpha} = \lim$$

$$I_{\left(\frac{1}{2}\right)} = \int_0^1 \left(\frac{x}{1-x}\right)^{\frac{1}{2}} dx =$$

$$= \int_0^{1-0} \sqrt{\frac{x}{1-x}}\, dx$$

not $\sqrt{\frac{x}{1-x}} = t \Rightarrow t^2 = \frac{x}{1-x} \Rightarrow x = t^2 - t^2 x \Rightarrow$

$$x + t^2 x = t^2$$

$$x(1+t^2) = t^2$$

$$x = \frac{t^2}{1+t^2}$$

$$dx = \left(\frac{t^2}{1+t^2}\right)' dt = \frac{2t(1+t^2) - t^2 \cdot 2t}{(1+t^2)^2} = \frac{2t + \cancel{2t^3} - \cancel{2t^3}}{(1+t^2)^2} =$$

$$= \frac{2t}{(1+t^2)^2} \cdot dt$$

$x = 0 \Rightarrow t = 0$

$x \nearrow 1 \Rightarrow \lim\limits_{x \nearrow 1} \sqrt{\frac{x}{1-x}} = +\infty \Rightarrow t = \infty$

$$\int_0^{1-0} \sqrt{\frac{x}{1-x}}\, dx = \int_0^{\infty} t \cdot \frac{2t}{(1+t^2)^2}\, dt$$

$$\int_0^{\infty} t \cdot \left(\frac{-1}{1+t^2}\right)' dt = t \cdot \frac{-1}{1+t^2} \Big|_0^{\infty} + \int_0^{\infty} \frac{1}{1+t^2}\, dt =$$

$$= \lim_{v \to \infty} t \cdot \frac{1}{1+t^2} \Big|_0^v + \operatorname{arctg} t \Big|_0^\infty = \lim_{v \to \infty} \frac{-v}{1+v^2} +$$

$$+ \lim_{v \to \infty} \operatorname{arctg} t \Big|_0^v = 0 + \lim_{v \to \infty} \operatorname{arctg} v = \frac{\pi}{2}$$

Stud. existența limitelor

a) $$\lim_{(x,y)\to(0,0)} \frac{x \cdot y}{\sqrt{1+xy} - 1} \overset{x \cdot y = t}{=} \lim_{t \to 0} \frac{t^{\sqrt{t+1}}}{\sqrt{1+t} - 1} = 2$$

b) $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

$$\left.\begin{array}{l} \lim\limits_{x \to 0} \left(\lim\limits_{y \to 0} \frac{x^2 - y^2}{x^2 + y^2} \right) = \lim\limits_{x \to 0} \frac{x^2}{x^2} = 1 \\ \lim\limits_{y \to 0} \left(\lim\limits_{x \to 0} \frac{x^2 - y^2}{x^2 + y^2} \right) = -1 \end{array} \right\} \Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

$a^n, b^n \to (0,0)$ cu propr. $\lim\limits_{n \to \infty} f(a^n) \neq \lim\limits_{n \to \infty} f(b^n)$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

$$a^n = \left(\frac{1}{n}, \frac{1}{n} \right) \to (0,0), \; n \to \infty$$

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} 0 = 0$$

$$b^n = \left(\frac{1}{n}, 0\right) \to (0,0)$$

$$\lim_{n\to\infty} f(b^n) = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = 1$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

c) $$f(x,y) = \frac{x^2+y^2}{x^4+y^4}$$

$$|f(x,y) - 0| = \frac{x^2+y^2}{x^4+y^4} = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} < \frac{x^2}{x^4} + \frac{y^2}{y^4} =$$

$$= \frac{1}{x^2} + \frac{1}{y^2} \to 0 \Rightarrow \lim_{(x,y)\to(\infty,\infty)} f(x,y) = 0$$

d) $$\lim_{(x,y)\to(0,0)} \frac{x\cdot \sin(x^2-y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{\sin(x^2-y^2)}{(x^2-y^2)} \cdot$$

$$\cdot\ x^2-y^2 \cdot \frac{x}{x^2+y^2} \overset{x^2-y^2=t}{=\!=} \underbrace{\lim_{t\to 0} \frac{\sin t}{t}}_{1} \cdot \lim_{(x,y)\to(0,0)}$$

$$\frac{x(x^2-y^2)}{(x^2-y^2)} = \lim_{(x,y)\to(0,0)} \frac{x(x^2-y^2)}{(x^2-y^2)}$$

$$0 \le \left| \frac{x(x^2-y^2)}{x^2+y^2} - 0 \right| = \frac{|x(x^2-y^2)|}{x^2+y^2} = \frac{|x^3 - x\cdot y^2|}{x^2+y^2} \le$$

$$\leq \frac{|x^3| + |x \cdot y^2|}{x^2+y^2} = \frac{|x| \cdot (|x^2|+|y^2|)}{x^2+y^2} = \frac{|x| \cdot (x^2+y^2)}{x^2+y^2} =$$

$$= |x| \to 0, \quad x = 0$$

e) $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$

$$a^n = \left(\frac{1}{n}, \frac{1}{n^2}\right) \to (0,0)$$

$$\lim_{n\to\infty} f(a^n) = \lim_{n\to\infty} \frac{\frac{1}{n^3}+\frac{1}{n^6}}{\frac{1}{n^3}} = 1$$

$$b^n = \left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$$

$$\lim_{n\to\infty} f(b^n) = \lim \frac{\frac{1}{n^3}+\frac{1}{n^3}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{2}{n} = 0$$

$$\Rightarrow$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} f(x,y)$$

f) $\lim\limits_{(x,y)\to(1,1)} \frac{(x-1)(y-1)}{xy-1} = \lim\limits_{(t,z)\to(0,0)} \frac{(t\cdot z)}{(t+1)(z+1)-1} =$

$x-1=t$

$y-1=z$

$= \lim\limits_{(t,z)\to(0,0)} \frac{tz}{tz+t+z}$

$a^n = \left(\frac{1}{n}, 0\right)$

$\lim\limits_{n\to\infty} f(a^n) = \lim\limits_{n\to\infty} \frac{0}{\frac{1}{n}} = 0$

② $f: \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) = \begin{cases} x\cdot\cos\frac{1}{y^2} + y\cdot\cos\frac{1}{x^2}, & x\cdot y \neq 0 \\ 0, & x\cdot y = 0 \end{cases}$$

Este f cont in $(0,0)$? Dar in $(1,0)$?

f. cont in $(0,0) \Leftrightarrow \exists \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$

$$\lim_{(x,y)\to(0,0)}\left(x\cdot\cos\frac{1}{y^2}+y\cdot\cos\frac{1}{x^2}\right)=0$$

$$|f(x,y)-0| = \left|x\cdot\cos\frac{1}{y^2}+y\cdot\cos\frac{1}{x^2}\right| \leq$$

$$\leq |x|\cdot\underbrace{\left|\cos\frac{1}{y^2}\right|}_{\leq 1}+|y|\cdot\left|\cos\frac{1}{x^2}\right| \leq |x|+|y| \to 0$$

$$f(1,0)=0$$

$$\lim_{(x,y)\to(1,0)}\left(x\cdot\cos\frac{1}{y^2}+y\cdot\cos\frac{1}{x^2}\right)=$$

$$=\lim_{(x,y)\to(1,0)} x\cdot\cos\frac{1}{y^2}+\underbrace{\lim_{(x,y)\to(1,0)} y\cdot\cos\frac{1}{x^2}}_{0}$$

$$\frac{1}{y^2}=2n\pi \Rightarrow y=\frac{1}{\sqrt{2n\pi}},\ n\in\mathbb{N}$$

$$a^n=\left(1,\frac{1}{\sqrt{2n\pi}}\right)\to(1,0),\ n\to\infty$$

$$\lim_{n\to\infty} g(a^n)=1 \quad (1)$$

$$b^n = \left(1, \frac{1}{\sqrt{2n\pi + \frac{\pi}{2}}}\right)$$

$$\lim_{n\to\infty} g(b^n) = 0, (2)$$

$$(1) + (2) \Rightarrow \nexists \lim_{(x,y)\to(1,0)} g(x,y)$$

$\Rightarrow f$ nu e cont în $(1,0)$

Det val extreme și precizați dacă se ating pt:

a) $f:(0,+\infty)^2 \to \mathbb{R}$, $f(x,y) = \frac{x}{y} + \frac{y}{x}$

$$A = (0,+\infty)^2 \Rightarrow x, y > 0 \Rightarrow \frac{x}{y} + \frac{y}{x} > 0$$

$$\frac{x}{y} + \frac{y}{x} \geq 2 \Rightarrow \inf f(A) = 2 = f(1,1)$$

$$\lim_{x\to\infty} f(x,1) = \lim_{x\to\infty} \left(x + \frac{1}{x}\right) = +\infty$$

b) $A = B(0_2, 1) = \{(x,y) \in \mathbb{R}^2 / x^2+y^2<1\}$

$0 \leq x^2+y^2 < 1 \,\Big|_{+1} \Rightarrow 1 \leq x^2+y^2+1 \leq 2 \,\Big| \Rightarrow$

$\Rightarrow \frac{1}{2} < \frac{1}{1+x^2+y^2} \leq 1$

$\inf(f(A)) = \frac{1}{2}$ (nu se atinge)

Mulțimea A nu este mărg $\Rightarrow$ A nu este compactă

A nu este închisă $\Rightarrow$ A nu este compactă

c) $A = \{ f : A \to \mathbb{R},\ f(x,y) = x \cdot y \cdot (1-x-y)$

$A = \{(x,y) \in \mathbb{R}^2 / x \geq 0, y \geq 0, x+y \leq 1\}$

$\left.\begin{array}{l} A \text{ mărginită} \\ \text{fr } A \subseteq A \Rightarrow A \text{ închisă} \end{array}\right\} \Rightarrow A$ compactă $\overset{T.W.}{\Rightarrow} f$ își atinge val extreme

$f(x,y) \geq 0, \forall (x,y) \in A$

$0 = \min f(A) = f(0,0)$

$$\frac{u+v+w}{3} \geq \sqrt{u \cdot v \cdot w}$$

$$\Rightarrow \frac{1}{3} \geq \sqrt[3]{f(x,y)} \Rightarrow f(x,y) \leq \frac{1}{3^3}$$

$$\frac{1}{27} = \max f(A) = f\left(\frac{1}{3}, \frac{1}{3}\right)$$

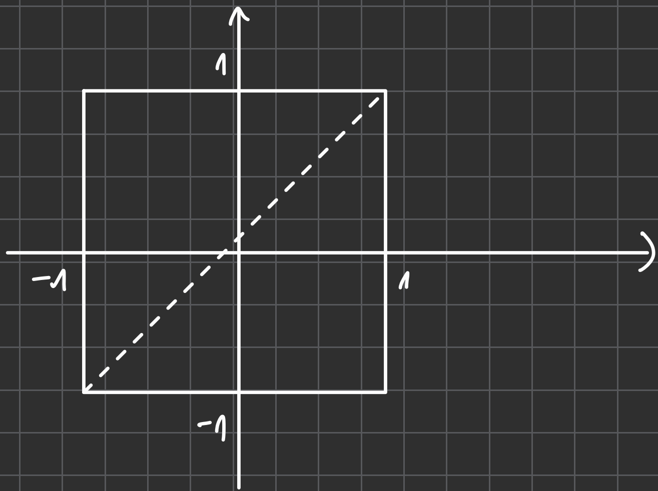
5) $A = \{(x,y) \in [-1,-1]^2 \mid x \neq y\}$ și

$f: A \to \mathbb{R}$, $f(x,y) = \dfrac{x^2 + y^2}{(x-y)^2}$

a) $\exists \lim\limits_{(x,y) \to (0,0)} f(x,y)$?

b) Este A compactă

c) Det $\inf f(A)$ și $\sup f(A)$. Se ating?



$(0,0) \notin A$

$(0,0) \in \operatorname{Fr} A$

$\operatorname{Fr} A \not\subseteq A \Rightarrow A$ nu este închisă $\Rightarrow A$ nu e compactă

$a^n \in A$ cu $\lim_{n \to \infty} f(a^n) = +\infty$

$a^n = \left(1 - \frac{1}{n}, 1\right) \in A, \forall n \in \mathbb{N}^*$

$\lim_{n \to \infty} f\left(1 - \frac{1}{n}, 1\right) = +\infty$

① Calculați derivatele parțiale, ∇f, df pt

a) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2 \cdot y^3 + y \cdot \sin x - 2z$

$$\frac{\partial f}{\partial x}(x,y,z) = 2xy^3 + y \cdot \cos x$$

$$\frac{\partial f}{\partial y} = 3x^2y^2 + \sin x$$

$$\frac{\partial f}{\partial z} = -2$$

$$\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$df(x,y,z)(v_1,v_2,v_3) = \nabla f(x,y,z) \cdot v =$$

$$= (2xy^3 + y\cos x) \cdot v_1 + (3x^2y^2 + \sin x) \cdot v_2 - 2v_3$$

b) $$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\left(\frac{x-y}{x+y}\right)^2 + 1} \cdot \left(\frac{x-y}{x+y}\right)'_x =$$

$$= \frac{1}{\frac{\cancel{x^2 - 2xy + y^2 + x^2 + 2xy + y^2}}{x^2 - 2y + y^2}} \cdot \frac{(x-y)'_x \cdot (x+y) - (x-y)(x+y)'_x}{(x+y)^2}$$

$$= \frac{x}{(x^2+y^2)}$$

$$\frac{\partial f}{\partial y} f(x,y) = \frac{1}{\left(\frac{x-y}{x+y}\right)^2+1} \cdot \left(\frac{x-y}{x+y}\right)'_y$$

$$= \frac{(x+y)^2}{2(x^2+y^2)} \cdot \frac{(x-y)'_y(x+y) - (x-y)(x+y)'_y}{(x+y^2)} =$$

$$= -\frac{x}{x^2+y^2}$$

c) $$\frac{\partial f}{\partial x} f(x,y) = \sqrt{x^2+y^2} + \frac{\cancel{2}x^2}{\cancel{2}\sqrt{x^2+y^2}} = \frac{y^2+2x^2}{\sqrt{x^2+y^2}} \quad (x,y)\neq(0,0)$$

$$\frac{\partial f}{\partial y} f(x,y) = x \cdot \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}, \quad (x,y)\neq(0,0)$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{\cancel{x}\cdot\sqrt{x^2}-0}{\cancel{x}}$$

$$= \lim_{x\to 0} |x| = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

② $f(x,y) = y \cdot \ln(x^2 - y^2)$

verifică

$$\frac{1}{x} \cdot \frac{\partial f}{\partial x}(x,y) + \frac{1}{y} \cdot \frac{\partial f}{\partial y}(x,y) = \frac{f(x,y)}{y^2}$$

$$\frac{\partial f}{\partial x} = y' \cdot \ln(x^2 - y^2) + y \cdot [\ln(x^2 - y^2)]' = \frac{2xy}{x^2 - y^2}$$

$$\frac{\partial f}{\partial y} = \ln(x^2 - y^2) + y \cdot \frac{2y}{x^2 - y^2}$$

3. $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \cancel{\frac{x^2 \cdot y}{x^2 + y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

④ Fie $f : \mathbb{R}^2 \to \mathbb{R}^3$

$f(x,y) = (x^2 - y,\ 3x - 2y,\ 2xy + y^2)$

și $g = g(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$

determinați:

a) $J(f)(1,1)$

b) D.P. ale funcției compuse $g \circ f$

$g \circ f : \mathbb{R}^2 \to \mathbb{R}$

$(g \circ f) = g(f(x,y)) = g(x^2 - y,\ 3x - 2y,\ 2xy + y^2)$

a) $J(f)(x,y) = \begin{pmatrix} \frac{d f_1}{d x} & \frac{d f_1}{d y} \\ \frac{d f_2}{d x} & \frac{d f_2}{d y} \\ \frac{d f_3}{d x} & \frac{d f_3}{d y} \end{pmatrix} = \begin{pmatrix} 2x & -1 \\ 3 & -2 \\ 2y & 2x+2y \end{pmatrix}$

$J(f)(1,1) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \\ 2 & 4 \end{pmatrix}$

b) $\nabla(g \circ f)(x,y) = \nabla g(f(x,y)) \cdot J(f)(x,y)$

$\Rightarrow \left(\frac{\partial (g \circ f)}{\partial x}(x,y),\ \frac{\partial (g \circ f)}{\partial y}(x,y) \right) =$

$= \left(\frac{\partial g}{\partial u}(f(x,y)),\ \frac{\partial g}{\partial v}(f(x,y)),\ \frac{\partial g}{\partial w}(f(x,y)) \right) \cdot$

$\cdot \begin{pmatrix} 2x & -1 \\ 3 & -2 \\ 2y & 2x+2y \end{pmatrix}$

Derivate parțiale de ordinul 2

$f: \mathbb{R} \times (0, +\infty) \to \mathbb{R}$, $f(x,y) = x \cdot y \cdot e^{\frac{x}{y}}$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial f}{\partial x} = y \cdot e^{\frac{x}{y}} + xy\, e^{\frac{x}{y}} \cdot \left(\frac{x}{y}\right)'_x$$

$$= y \cdot e^{\frac{x}{y}} + x \cdot e^{\frac{x}{y}}$$

$$= e^{\frac{x}{y}}(x+y)$$

$$\frac{\partial f}{\partial y} = x \cdot e^{\frac{x}{y}} + x \cdot y \cdot e^{\frac{x}{y}} \cdot \left(\frac{x}{y}\right)'_y$$

$$= x \cdot e^{\frac{x}{y}} + x \cdot y \cdot e^{\frac{x}{y}}\left(-\frac{x}{y^2}\right)$$

$$= e^{\frac{x}{y}}\left(x - \frac{x^2}{y}\right)$$

$$\frac{\partial f^2}{\partial x^2} = \frac{\partial}{\partial x} = \left[e^{\frac{x}{y}}(x+y)\right]'_x = \left(e^{\frac{x}{y}}\right)'_x (x+y) + (x+y)' \cdot e^{\frac{x}{y}} =$$

$$= e^{\frac{x}{y}} \cdot \left(\frac{x}{y}\right)'_x \cdot (x+y) + e^{\frac{x}{y}} = e^{\frac{x}{y}}\left(\frac{x+y}{y}+1\right) = e^{\frac{x}{y}}\left(\frac{x}{y}+2\right)$$

$$\frac{\partial f^2}{\partial y^2} = \left[e^{\frac{x}{y}}\left(x - \frac{x^2}{y}\right)\right]'_y = \left(e^{\frac{x}{y}}\right)'_y \cdot \left(x - \frac{x^2}{y}\right) + \left(x - \frac{x^2}{y}\right)'_y \cdot$$

$$\cdot e^{\frac{x}{y}}$$

① $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + 3xy^2 - 15x - 12y$

$a = (-2,-1)$ Precizați:

i) $\nabla f(a)$, $H(f)(a)$, $d^2 f(a)$

ii) natura punctului a

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}(x,y) + \frac{\partial f}{\partial y}(x,y)\right)$$

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 - 15$$

$$\frac{\partial f}{\partial y}(x,y) = 6xy - 12$$

$$\nabla f(a) = \nabla f(-2,-1) = 12 + 3 - 15 \cancel{+12} \cancel{-12} = 0$$

$$Hf(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} 6x & 6y \\ 6y & 6x \end{pmatrix}$$

$$Hf(a) = \begin{pmatrix} -12 & -6 \\ -6 & -12 \end{pmatrix}$$

$$d^2 f(a)(u_1, u_2) = -12u_1^2 - 12u_2^2 - 6u_1u_2 - 6u_1 v_2 =$$
$$= -12(u_1^2 + u_2^2 + v_1 v_2)$$

ii) $\nabla f(a) = 0_2 \Rightarrow a$ punct critic $\Rightarrow$ pct minim local / pct maxim local / pct șa

$\Delta_1 = -12 < 0$

$\Delta_2 = \begin{vmatrix} -12 & -6 \\ -6 & -12 \end{vmatrix} = -144 - 36 > 0 \overset{\text{Sylvester}}{\Longrightarrow} d^2 f(a)$ negativ def

$\Rightarrow a$ pct de maxim local

② Det. pct. critice și natura lor pt.

a) $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = 2x^2 - xy + 2xz - y + y^3 + z^2$

h) $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^4 + y^4 - 2x^2$

$$\frac{\partial f}{\partial x} = 4x - y + 2z$$

$$\frac{\partial f}{\partial y} = -1 + 3y^2 - x$$

$$\frac{\partial f}{\partial z} = 2x + 2z$$

$$\nabla f(x,y,z) = 0_3 = \begin{cases} 4x - y + 2z = 0 \\ -1 + 3y^2 - x = 0 \\ 2x + 2z = 0 \end{cases} \Rightarrow$$

$\Rightarrow 2x = y$

$z = -x$

$$12x^2 - x - 1 = 0$$

$$\Delta = 1 + 12x^2 = 49$$

$$x_{1,2} = \frac{1 \pm 7}{24}$$

$x_1 = \frac{1}{3}$, $y_2 = \frac{2}{3}$, $z_3 = -\frac{1}{3}$

$x = -\frac{1}{4}$, $y_2 = -\frac{1}{2}$, $z_2 = \frac{1}{4}$

$$a = \left(\frac{1}{3}, \frac{2}{3}, -\frac{1}{3}\right)$$

$$b = \left(-\frac{1}{4}; \frac{1}{2}, \frac{1}{4}\right)$$

$\Big\}$ puncte critice

$$H(f)(x,y,z) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 6y & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$H(f)(a) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\Delta_1 = 4 > 0$$

$$\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & 4 \end{vmatrix} = 16 - 1 = 15$$

$$\Delta_3 = \begin{vmatrix} 4 & -1 & 2 \\ -1 & 4 & 0 \\ 2 & 0 & 2 \end{vmatrix} = 32 - 16 - 2 = 14 > 0$$

$\Rightarrow D^2 f(a)$ pozitiv definit $\Rightarrow$ a punct de minim local

$$H(f)(b) = \begin{pmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 4 > 0$

$$\Delta_2 = \begin{vmatrix} 4 & -1 \\ -1 & -3 \end{vmatrix} = -13 < 0$$

$$\Delta_3 = \begin{vmatrix} 4 & -1 & 2 \\ -1 & -3 & 0 \\ 2 & 0 & 2 \end{vmatrix} = -14$$

$d^2 f(b)(v_1, v_2, v_3) = 4v_1^2 - 3v_2^2 + 2v_3^2 - 2v_1 v_2 + 4v_1 v_3$

$$\left.\begin{array}{l} d^2 f(b)(1,0,0) = 4 > 0 \\ d^2 f(b)(0,1,0) = -3 < 0 \end{array}\right\} \Rightarrow d^2 f(b) \text{ - nedefinită} \Rightarrow$$

$\Rightarrow b$ - pct șa

b) $\frac{\partial f}{\partial x} = 4x^3 - 4x$

$\frac{\partial f}{\partial y} = 4y^3$

$\nabla f(x,y) = (0,0) = \begin{cases} 4x(x^2-1) = 0 \\ 4y^3 = 0 \end{cases}$

Pct. critice sunt $(0,0)$ $(-1,0)$ $(1,0)$

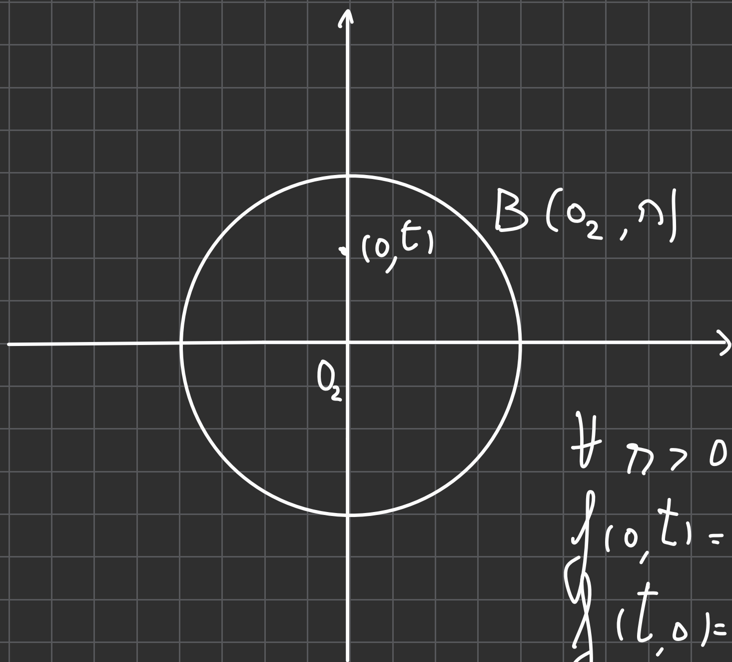
$H(f)(x,y) = \begin{pmatrix} 12x^2-4 & 0 \\ 0 & 12y^2 \end{pmatrix}$

$H(f)(0,0) = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$, $\Delta_1 = 4$, $\Delta_2 = 0$

$d^2 f(0,0)(v_1, v_2) = -4v_1^2 \leq 0$ negativ (semidefinită)

$\boxed{\text{Teoria nu se aplică}}$

$(0,0)$



$\forall r > 0$

$f(0,t) = t^2 > 0, \ t \neq 0$

$f(t,0) = t^4 - 2t^2 = t^2(t-\sqrt{2})(t+\sqrt{2}) < 0$

$\forall t \in (-\sqrt{2}, \sqrt{2})$

$f(0,0) = 0$

$\Rightarrow (0,0)$ punct şa

$f(1,0) = f(-1,0) = -1$

$f(x,y) = x^4 - 2x^2 + 1 + y^4 - 1$

$= \underbrace{(x^2-1)^2 + y^4}_{\geq 0} - 1 \geq -1 \ \forall (x,y) \in \mathbb{R}^2 \Rightarrow (1,0), (-1,0)$ pct. minim global

③ Det pct. de extrem condiționat și val extreme ale lui f relativ la S

a) $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = (1-x)(1-y)$

$S = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

b) $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x \cdot y \cdot z$

$S = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0,\ x^2+y^2+z^2=1\}$

$H_S : S \to \mathbb{R}$

a) $\left.\begin{array}{l} S \text{ compactă} \\ f \text{ cont} \end{array}\right| \xRightarrow{T.W.}$ f își atinge val extreme

$\Rightarrow$ f are pct de minim și maxim cond relativ la S

$F(x,y) = x^2 + y^2 - 1$

$L(x,y,\lambda) = f(x,y) + \lambda \cdot F(x,y) =$

$= (1-x)(1-y) + \lambda x^2 + \lambda y^2 - \lambda$

$\frac{\partial L}{\partial x} = -1 + y + 2\lambda x$

$$\frac{\partial L}{\partial y} = -1 + x + 2\lambda y$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1$$

$$\nabla L = 0_3 \Rightarrow \begin{cases} -1 + y + 2\lambda x = 0 \\ -1 + x + 2\lambda y = 0 \\ x^2 + y^2 - 1 = 0 \end{cases}$$

$$y - x + 2\lambda(x - y) = 0$$

$$(y - x)(1 - 2\lambda) = 0$$

I $y = x$

$x^2 + x^2 - 1 = 0$

$2x^2 = 1$

$x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}} = y$

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \lambda_1\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \lambda_2\right)$ pct critice pt L

II $2\lambda = 1 \Rightarrow \lambda = \frac{1}{2}$

$-1+y+x=0 \Rightarrow y=1-x$

$x^2+(1-x)^2-1=0$

$x^2+1-2x+x^2-1=0$

$2x^2-2x=0$

$2x(x-1)=0$

$x_1=0 \Rightarrow y_1=1$

$x_2=1 \Rightarrow y_2=0$

$(0,1,\frac{1}{2})$, $(1,0,\frac{1}{2})$ pct critice ptc

$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)=\left(1-\frac{1}{\sqrt{2}}\right)^2$

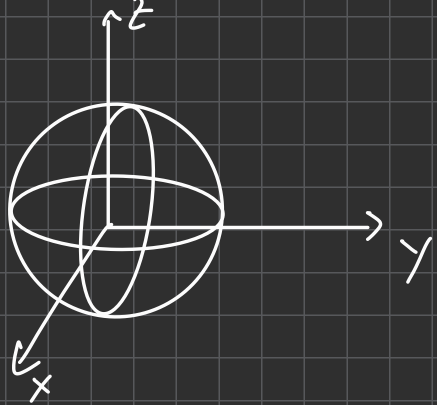
$f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)=\left(1+\frac{1}{\sqrt{2}}\right)^2=\max f(S)$

$f(1,0)=0$

$f(0,1)=0$ $=\min f(S)$

$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ pct de max cond

$(0,1)$ $(1,0)$ pct de min cond



$F_1(x,y,z) = x + y + z$

$F_2(x,y,z) = x^2 + y^2 + z^2 - 1$

$L(x, y, z, \lambda_1, \lambda_2) = f(x, y, z) + \lambda_1 F_1(x, y, z) +$
$+ \lambda_2 \cdot F_2(x, y, z)$

5) Det. val extreme ale lui f relativ la S

$f : \mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x + 2y + 3z$

$S = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1 \}$

$\left.\begin{array}{l} S \text{ compactă} \\ f \text{ cont.} \end{array}\right\} \Rightarrow$ f are pct de minim și maxim relativ la S

$S = \text{int } S \cup \text{fr } S$

I) $(x, y, z) \in \text{int} S$

$\nabla f = 0_3 \Rightarrow$ nicio sol.

II $(x, y, z) \in \text{fr} S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

Cu funcția L