Analiza

Seminar 6

1. Calc. derivata de ordinul $n \in \mathbb{N}$ a fc. de mai jos și precizați mulțimea pe care aceste fc. sunt indefinit derivabile

a) $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{IV}(x) = \sin x$

$$f^{n}(x) = \begin{cases} \sin x, & n = 4k \\ \cos x, & n = 4k+1 \\ -\sin x, & n = 4k+2 \\ -\cos x, & n = 4k+3 \end{cases}, \quad k \in \mathbb{N}$$

mulțimea pe care f este indefinit derivabilă este $\mathbb{R}$

b) $f(x) = \ln(x+1)$ C.E. $x+1 > 0 \Rightarrow x > -1 \Rightarrow x \in (-1, +\infty)$

$f'(x) = (x+1)^{-1}$

$f''(x) = -1 \cdot (x+1)^{-2}$

$$f'''(x) = -1 \cdot -2 \cdot (x+1)^{-3}$$

$$f^{n}(x) = (-1)^{n+1} \cdot (n-1)! \cdot (x+1)^{-n}$$

f indef derivabilă pe $(-1, +\infty)$

c) $f(x) = (x^2 - x) \cdot e^x$

f indef deriv. pe $\mathbb{R}$

$$(f \cdot g)^n = \sum_{k=0}^{n} C_n^k \cdot f^k \cdot g^{n-k}$$

$f = (x^2 - x)$

$g = e^x$

$$((x^2 - x) \cdot e^x)^{(n)} = C_n^0 \cdot (x^2 - x) \cdot (e^x)^{(n)} + C_n^1 \, (x^2 - x)' \cdot (e^x)^{(n-1)}$$

$$+ C_n^2 \cdot (x^2 - x)'' \cdot (e^x)^{(n-2)} + C_n^3 \cdot (x^2 - x)''' \cdot (e^x)^{(n-3)} + \ldots =$$

$$= (x^2 - x) \cdot e^x + \frac{\cancel{n!}\, n}{1!\,\cancel{(n-1)!}} \cdot (2x - 1) \cdot e^x + \frac{\cancel{n!}\,(n-1)(n)}{2!\,\cancel{(n-2)!}} \cdot 2 \cdot e^x \cdot \underbrace{\ldots}_{0}$$

pt $(x^2 - x)^{(k)} = 0$ pt $k \geq 3$

$$= (x^2 - x) \cdot e^x + n(2x - 1)\, e^x + n(n-1) \cdot 2e^x$$

d) $f(x) = \sqrt{1 - x}$, $x \leq 1 \Rightarrow x \in (-\infty, 1]$

$f(x) = (1 - x)^{\frac{1}{2}}$

$$f'(x) = \frac{1}{2} \cdot (1-x)^{-\frac{1}{2}} \cdot (-1)$$

$$f''(x) = -\frac{1}{4} \cdot (1-x)^{-\frac{3}{2}} \cdot (-1)$$

$$f'''(x) = -\frac{1 \cdot 3}{8} \cdot (1-x)^{-\frac{5}{2}} \cdot (-1)$$

$$f^{IV}(x) = -\frac{1 \cdot 3 \cdot 5}{16} \cdot (1-x)^{-\frac{7}{2}} \cdot (-1)$$

$$f^{n}(x) = -\frac{(2n-3)!!}{2^{n}} (1-x)^{-\frac{2n-1}{2}}$$

f indef deriv pe $(-\infty, 1]$

2. Pt fc de la ex anterior pct $x_0 = 0$ și $n \in \mathbb{N}$ dat.

Pol lui Taylor $\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^{k}$

a) $f(x) = \sin x$

$$T_n f_1(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^{k} = \underbrace{\frac{f(0)}{0!}}_{0} \cdot (x-0)^{0} +$$

$$+ \frac{f'(0)}{1!} \cdot (x-0)^{1} + \underbrace{\frac{f''(0)}{2!}}_{0} \cdot (x-0)^{2} + \frac{f'''(0)}{3!} \cdot (x-0)^{3} + \dots$$

$$+ \frac{f^{(n)}(0)}{n!} \cdot (x-0)^{k} = \frac{x}{1!} - \frac{x^{3}}{3!} + \frac{x^{5}}{5!} - \dots + \frac{f^{(n)}(0)}{n!} \cdot x^{n}$$

b) $f(x) = \ln(x+1)$

$$f^{(n)}(x) = (-1)^{n+1} \cdot (n-1)! \cdot (x+1)^{-n}, \quad n \geq 1$$

$$(T_n f)(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k = \frac{\ln 1}{0!} \cdot x^0 + \frac{x^1}{1} -$$

$$- \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + \frac{(-1)^{n+1} \cdot (n-1)! \cdot (x+1)^{-n}}{n!} \cdot x^n$$

c) $f(x) = (x^2 - x) \cdot e^x$

$$f^{(n)}(x) = e^x (x^2 + (2n-1)x + n^2 - 2n), \quad n \in \mathbb{N}$$

$$f^{(n)}(0) = n^2 - 2n$$

$$(T_n f)(x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k = \frac{1-2}{1!} x^1 + \frac{4-4}{2!} \cdot x^2 +$$

$$+ \frac{9-6}{3!} \cdot x^3 + \dots + \frac{n^2-2n}{n!} \cdot x^n = -x + \frac{x^3}{2} + \dots + \frac{n^2-2n}{n!} \cdot x^n$$

d) $f(x) = \sqrt{1-x}$

$$f^{(n)}(x) = \frac{-(2n-3)!!}{2^n} \cdot (1-x)^{-\frac{2n-1}{2}}, \quad n \geq 1$$

$$f^{(n)}(0) = \frac{-(2n-3)!!}{2^n}$$

$$(T_n f)(x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot (x-x_0)^k = x^0 - \frac{x}{2} - \frac{x^2}{2!} \cdot \frac{1}{2^2} -$$

$$- \frac{x^3}{3!} - \frac{3}{2^3} - \dots - \frac{\frac{-(2n-3)!!}{2^n}}{n!} \cdot x^n$$

b) Mulțimea de convergență a seriei Taylor corespunzătoare