Analiza

1.

a) $f(x,y,z) = x^2 y^3 + y\sin x - 2z$

$$\frac{\partial f}{\partial x}(x,y,z) = 2xy^3 + y\cos x$$

$$\frac{\partial f}{\partial y}(x,y,z) = 3y^2x^2 + \sin x$$

$$\frac{\partial f}{\partial z}(x,y,z) = -2$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2xy^3 + y\cos x,\ 3y^2x^2 + \sin x,\ -2)$$

$$df = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} x^0 \cdot u_i = \frac{\partial f}{\partial x}(x,y,z)\cdot u_1 + \frac{\partial f}{\partial y}(x,y,z)\cdot u_2$$

$$+ \frac{\partial f}{\partial z}(x,y,z)\cdot u_3 =$$

$x^0 = (x, y, z)$

x_1^0

x_2^0

x_3^0

$$= (2xy^3 + y\cos x)u_1 + (3y^2x^2 + \sin x)u_2$$

$$-2u_3$$

b) $f(x,y) = \operatorname{arctg}\left(\frac{x-y}{x+y}\right)$

$$\operatorname{arctg} u' = \frac{1}{u^2+1} \cdot u'$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{\left(\frac{x-y}{x+y}\right)^{2(x+y)^2} + 1} \cdot \frac{x+y-x+y}{(x+y)^2} = \frac{1}{\frac{(x-y)^2 + (x+y)^2}{\cancel{(x+y)^2}}} \cdot \frac{2y}{\cancel{(x+y)^2}}$$

$$= \frac{2y}{(x-y)^2+(x+y)^2} = \frac{2y}{x^2 \cancel{-2xy} + y^2 + x^2 \cancel{+2xy} + y^2} = \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)} =$$

$$= \frac{y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{1}{\left(\frac{x-y}{\cancel{x+y}}\right)^2 + 1} \cdot \frac{-x \cancel{-y} - x \cancel{+y}}{\cancel{(x+y)^2}} = \frac{-2x}{(x-y)^2+(x+y)^2}$$

$$= \frac{-2x}{x^2 \cancel{-2xy} + y^2 + x^2 \cancel{+2xy} + y^2} = \frac{-\cancel{2}x}{\cancel{2}(x^2+y^2)} = -\frac{x}{x^2+y^2}$$

$$\nabla f = \left(\frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}\right)$$

$$df(x,y)(v_1,v_2) = v_1\left(\frac{y}{x^2+y^2}\right) - v_2\left(\frac{x}{x^2+y^2}\right)$$

c) $f(x,y) = x\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x}(x,y) = \sqrt{x^2+y^2} + \frac{\cancel{2}x^2}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}} =$$

$$= \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial f}{\partial y}(x,y) = x \cdot \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\nabla f(x,y) = \left(\frac{2x^2+y^2}{\sqrt{x^2+y^2}}, \frac{xy}{\sqrt{x^2+y^2}}\right)$$

$$df(x,y)(v_1, v_2) = \frac{2x^2+y^2}{\sqrt{x^2+y^2}} \cdot v_1 + \frac{xy}{\sqrt{x^2+y^2}} \cdot v_2$$

2. $f(x,y) = y \ln(x^2 - y^2)$

$$\frac{\partial f}{\partial x}(x,y) = \frac{2xy}{x^2-y^2}$$

$$\frac{\partial f}{\partial y}(x,y) = \ln(x^2-y^2) - \frac{2y^2}{x^2-y^2}$$

$$\frac{1}{\cancel{x}} \cdot \frac{2\cancel{x}y}{x^2-y^2} + \frac{1}{y}\left(\ln(x^2-y^2) - \frac{2y^2}{x^2-y^2}\right) = \frac{\ln(x^2-y^2)}{y}$$

$$\cancel{\frac{2y}{(x^2-y^2)}} + \frac{\ln(x^2-y^2)}{y} - \cancel{\frac{2y}{x^2-y^2}} = \frac{\ln(x^2-y^2)}{y}$$

3. $f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

$$\frac{\partial f}{\partial x} = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \frac{0-0}{x} = 0$$

$$\frac{\partial f}{\partial y} = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y-0} = \frac{0-0}{y-0} = 0$$

5. $f(x,y) = (x e^{y} + x e^{-y}, \; x e^{y} - x e^{-y})$

$$\frac{\partial f_1}{\partial x} = e^{y} + e^{-y}$$

$$\frac{\partial f_1}{\partial y} = x e^{y} - x e^{-y}$$

$$\frac{\partial f_2}{\partial x} = e^{y} - e^{-y}$$

$$\frac{\partial f_2}{\partial y} = x e^{y} + x e^{-y}$$

$$J(f)(x,y) = \begin{pmatrix} e^{y}+e^{-y} & x e^{y} - x e^{-y} \\ e^{y} - e^{-y} & x e^{y} + x e^{-y} \end{pmatrix}$$

$$\frac{\partial g \circ f}{\partial x}(x,y) = \frac{\partial g}{\partial x}(e^{y}+e^{-y}) + \frac{\partial g}{\partial x}(x e^{y} - x e^{-y})$$

$$\frac{\partial g \circ f}{\partial y}(x,y) = \frac{\partial g}{\partial y}(e^{y}-e^{-y}) + \frac{\partial g}{\partial y}(x e^{y} - x e^{-y})$$

6. a) $f : (1,\infty) \times \mathbb{R} \to \mathbb{R}$, $f(x,y) = \ln(x + y^2 - 1)$

$$\frac{\partial^2 f}{\partial x^2} \quad \frac{\partial f}{\partial x \partial y} \quad \frac{\partial f}{\partial y \partial x} \quad \frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial f}{\partial x}(x,y) = \frac{1}{x+y^2-1} \cdot 1 = (x+y^2-1)^{-1}$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = -(x+y^2-1)^{-2}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -2y(x+y^2-1)^{-2}$$

$$\frac{\partial f}{\partial y}(x,y) = \frac{2y}{x+y^2-1}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{2(x+y^2-1) - 4y^2}{(x+y^2-1)^2} = \frac{2}{(x+y^2-1)} - \frac{4y^2}{(x+y^2-1)^2}$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{-2y}{(x+y^2-1)^2}$$

b) $f(x,y) = xye^{\frac{x}{y}}$

$$\frac{\partial f}{\partial x}(x,y) = x' \cdot y \cdot e^{\frac{x}{y}} + x \cdot y' \cdot e^{\frac{x}{y}} + x \cdot y \cdot e^{\frac{x}{y}'} =$$

$$= ye^{\frac{x}{y}} + xy \cdot \frac{1}{y} \cdot e^{\frac{x}{y}} = ye^{\frac{x}{y}} + xe^{\frac{x}{y}} = e^{\frac{x}{y}}(x+y)$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{e^{\frac{x}{y}}}{y} \cdot (x+y) + e^{\frac{x}{y}} = e^{\frac{x}{y}}\left(\frac{x}{y}+2\right)$$

$$\frac{\partial f}{\partial y}(x,y) = xe^{\frac{x}{y}} + xy \cdot -\frac{x}{y^2} \cdot e^{\frac{x}{y}} = xe^{\frac{x}{y}} - \frac{x^2}{y} e^{\frac{x}{y}} =$$

$$= e^{\frac{x}{y}}\left(x - \frac{x^2}{y}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \left(e^{\frac{x}{y}}\left(x - \frac{x^2}{y}\right)\right)'_y = \left(e^{\frac{x}{y}}\right)' \cdot \left(x - \frac{x^2}{y}\right) + \left(e^{\frac{x}{y}}\right) \cdot \left(x - \frac{x^2}{y}\right)'$$

$$= -\frac{x}{y^2} \cdot e^{\frac{x}{y}} \left(x - \frac{x^2}{y}\right) + \left(e^{\frac{x}{y}}\right)\left(0 - \frac{x^2}{y^2}\right) = e^{\frac{x}{y}}\left(-\frac{x}{y^2}\left(x - \frac{x^2}{y}\right) - \frac{x^2}{y^2}\right)$$

$$= e^{\frac{x}{y}}\left(-\frac{x^2}{y^2} + \frac{x^3}{y^3} - \frac{x^2}{y^2}\right) = e^{\frac{x}{y}}\left(-\frac{2x^2}{y^2} + \frac{x^3}{y^3}\right)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \left(e^{\frac{x}{y}}(x+y)\right)'_y = \left(-\frac{x}{y^2} \cdot e^{\frac{x}{y}}\right)(x+y) + x e^{\frac{x}{y}} =$$

$$= e^{\frac{x}{y}}\left(-\frac{x}{y^2}(x+y) + x\right) =$$

$$= e^{\frac{x}{y}}\left(-\frac{x^2}{y^2} - \frac{x}{y} + x\right)$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \left(e^{\frac{x}{y}}\left(x - \frac{x^2}{y}\right)\right)'_x = \frac{1}{y} \cdot e^{\frac{x}{y}}\left(x - \frac{x^2}{y}\right) +$$

$$+ e^{\frac{x}{y}} \cdot \left(1 - \frac{2x}{y}\right) =$$

$$= e^{\frac{x}{y}}\left(\frac{1}{y}\left(x - \frac{x^2}{y}\right) + 1 - \frac{2x}{y}\right)$$

$$= e^{\frac{x}{y}}\left(\frac{x}{y} - \frac{x^2}{y^2} + 1 - \frac{2x}{y}\right)$$