Analiza

1. $$\sum_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^{a}$$

D'alembert: $\lim\limits_{n\to\infty}\frac{x_n}{x_{n+1}} = \lim\limits_{n\to\infty}\left(\frac{\ln n}{n}\cdot\frac{n+1}{\ln n+1}\right)^{a} = \lim\limits_{n\to\infty}\left(\frac{\cancel{n}(1+\overset{\nearrow 0}{\frac{1}{n}})\ln n}{\cancel{n}\ln n+1}\right)^{a}$

$\lim\limits_{n\to\infty}\frac{\ln n}{\ln n+1}\underset{l'h}{\overset{\frac{\infty}{\infty}}{=}}\lim\limits_{n\to\infty}\frac{1}{n}\cdot\frac{n+1}{1} = \lim\limits_{n\to\infty}\frac{\cancel{n}(1+\overset{\nearrow 0}{\frac{1}{n}})}{\cancel{n}}$ $\Bigg\}\Rightarrow$

$$\lim_{n\to\infty}\frac{x_n}{x_{n+1}} = 1^{a} = 1$$

Raabe Duhamel: $\lim\limits_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim\limits_{n\to\infty} n\left(\left(\frac{(n+1)\ln n}{n\ \ln(n+1)}\right)^{a}-1\right) =$

$= \lim\limits_{n\to\infty} n(1-1) = 0 < 1 \Rightarrow \sum\limits_{n=2}^{\infty}\left(\frac{\ln n}{n}\right)^{a}$ divergenta $\forall\ a\in\mathbb{R}$

2. $f:(-1,\infty)^{2}\to\mathbb{R}$, $f(x,y)=\sqrt{(1+x)(1+y)^{\alpha}}$

$$\frac{\partial^2 f}{\partial x^2}(0,0)+\frac{\partial^2 f}{\partial y^2}(0,0)=2\frac{\partial^2 f}{\partial x\partial y}(0,0)$$

$$\frac{\partial f}{\partial x}=\left(\sqrt{1+x}\cdot\sqrt{(1+y)^{\alpha}}\right)'_x=\frac{1}{2\sqrt{1+x}}\cdot\sqrt{(1+y)^{\alpha}}+\sqrt{(1+y)^{\alpha}}\cdot 0=$$

$$=\frac{1}{2\sqrt{1+x}}\cdot\sqrt{(1+y)^{\alpha}}$$

$$\frac{\partial^2 f}{\partial x^2}=\left(\frac{1}{2\sqrt{1+x}}\cdot\sqrt{(1+y)^{\alpha}}\right)'_x=\frac{1}{2}\cdot\underset{1+x=u}{\left((1+x)^{-\frac{1}{2}}\right)'}\cdot\sqrt{(1+y)^{\alpha}}=\frac{1}{2}\cdot-\frac{1}{2}\cdot(1+x)^{-\frac{3}{2}}\sqrt{(1+y)^{\alpha}}=-\frac{\sqrt{(1+y)^{\alpha}}}{4(1+x)^{\frac{3}{2}}}$$

$$\frac{\partial f}{\partial y}=\left(\sqrt{1+x}\cdot\sqrt{(1+y)^{\alpha}}\right)'_y=\sqrt{1+x}\cdot\left((1+y)^{\frac{\alpha}{2}}\right)'=\sqrt{1+x}\cdot\frac{\alpha}{2}(1+y)^{\frac{\alpha-2}{2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \left(\sqrt{1+x} \cdot \frac{\alpha}{2}(1+y)^{\frac{\alpha-2}{2}}\right)'_y = \frac{\alpha}{2}\sqrt{1+x} \cdot \frac{\alpha-2}{2} \cdot (1+y)^{\frac{\alpha-4}{2}}$$

$$2\frac{\partial^2 f}{\partial x \partial y} = 2 \cdot \left(\frac{1}{2\sqrt{1+x}} \cdot \sqrt{(1+y)^{\alpha}}\right)_y = \cancel{2} \cdot \frac{1}{\cancel{2}\sqrt{1+x}} \cdot \frac{\alpha}{2}(1+y)^{\frac{\alpha-2}{2}} = \frac{1}{\sqrt{1+x}} \cdot \frac{\alpha}{2}(1+y)^{\frac{\alpha-2}{2}}$$

$$\left.\begin{aligned} &\frac{\partial^2 f}{\partial x^2}(0,0) = -\frac{\sqrt{1^{\alpha}}}{4} = -\frac{1}{4} \\ &\frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\alpha}{2} \cdot \frac{\alpha-2}{2} \cdot 1^{\frac{\alpha-4}{2}} = \frac{\alpha^2-2\alpha}{4} \end{aligned}\right\} \Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0) + \frac{\partial^2 f}{\partial y^2}(0,0) = \frac{\alpha^2-2\alpha-1}{4}$$

$$2\frac{\partial^2 f}{\partial x \partial y}(0,0) = \frac{\alpha}{2}$$

$$\frac{\alpha^2-2\alpha-1}{4} = \frac{\overset{2/}{\alpha}}{2}$$

$$\alpha^2 - 4\alpha - 1 = 0$$

$$\Delta = 16 + 4 = 20$$

$$\alpha_{1,2} = \frac{4 \pm \sqrt{20}}{2} \quad \begin{cases} \alpha_1 = \dfrac{4+\sqrt{20}}{2} \\ \alpha_2 = \dfrac{4-\overset{>4}{\sqrt{20}}}{2}, \ \alpha > 0 \Rightarrow \alpha_2 \text{ nu convine} \end{cases}$$

3. $\iint_A \frac{xy}{\sqrt{2-x^2}}\,dx\,dy$, $A = \{(x,y) \in \mathbb{R}^2 \mid x \geq y \geq 0,\ x^2+y^2 \leq 1\}$

4. a) Definiți noțiunea de rază de convergență a unei serii de puteri.

Raza de convergență e un număr pentru care seria de puteri (centrată în 0) este convergentă pe (x_0-r, x_0+r) și divergentă pe $(-\infty, x_0 - r) \cup (x_0 + r, +\infty)$.

Pentru o serie $\sum_{n=1}^{\infty} a_n (x - x_0)^n \Rightarrow r = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$

b) Exemplu:

$$a_n = n^n \quad \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \frac{n^n}{(n+1)^n \cdot n+1} = \lim_{n \to \infty} \left(\frac{n}{n+1} \right)^n \cdot \frac{1}{n+1} =$$

$$= \lim_{n \to \infty} \left(1 - \frac{1}{n+1} \right)^n \cdot \frac{1}{n+1} = \lim_{n \to \infty} \left(\left(1 - \frac{1}{n+1} \right)^{n+1} \right)^{\overset{\nearrow 1}{\frac{n}{n+1}}} \cdot \frac{1}{n+1} = \frac{e^{-1}}{n+1} = 0$$