Analiza

1) Calculați limita șirului $(x_n)_{n\in\mathbb{N}}$:

a) $x_n = \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)^{\sqrt{\ }+\sqrt{\ }} = \lim_{n\to\infty} \sqrt{n}\left(\frac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}\right) =$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{1+\underbrace{\tfrac{1}{n}}_{\to 0}}+1\right)} = \frac{1}{2}$$

b) $x_n = \dfrac{n+\sin n}{n+\cos n}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{n+\sin n}{n+\cos n} = \lim_{n\to\infty} \frac{\cancel{n}\left(1+\overbrace{\frac{\sin n}{n}}^{\to 0}\right)}{\cancel{n}\left(1+\underbrace{\frac{\cos n}{n}}_{\to 0}\right)}$$

$$\left|\frac{\overbrace{\sin n}^{\in[-1,1]}}{n}\right| \le \frac{1}{|n|} \longrightarrow 0 \Rightarrow \frac{\sin n}{n} \longrightarrow 0$$

analog $\dfrac{\cos n}{n} \longrightarrow 0$

$\Rightarrow \lim_{n\to\infty} x_n = 1$

c) $x_n = \dfrac{(\sqrt{2}+1)^n}{(\sqrt{2})^n+1}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{\cancel{(\sqrt{2})^n}\left(1+\frac{1}{\sqrt{2}}\right)^n}{\cancel{(\sqrt{2})^n}\left(1+\underbrace{\frac{1}{\sqrt{2}^n}}_{\to 0}\right)} = \lim_{n\to\infty}\left(1+\frac{1}{\sqrt{2}}\right)^n = +\infty$$

2. Justificați cu definiția val. limitelor:

a) $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

Șirul (x_n), $x_n = \frac{1}{\sqrt{n}}$ are limita 0 dacă $\forall \varepsilon > 0$,

$\exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 : |x_n - x| < \varepsilon$

$|x_n - 0| < \varepsilon$

Fie $\varepsilon > 0$. Căutăm n_0

$\left|\frac{1}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} < \varepsilon \Rightarrow \sqrt{n} > \frac{1}{\varepsilon} \Rightarrow n > \frac{1}{\varepsilon^2}$, deci $n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1$, pt $n_0 \in \mathbb{N}$

Verificăm:

Pt $\varepsilon > 0$ arbitrar ales, $\exists n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1$ a.î. $\forall n \geq n_0$ $\left|\frac{1}{\sqrt{n}}\right| < \varepsilon$

$n \geq n_0 \Rightarrow$
$\Rightarrow \sqrt{n} \geq n_0 \Rightarrow$
$\Rightarrow \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{n_0}}$

$\left|\frac{1}{\sqrt{n}}\right| \leq \frac{1}{\sqrt{n_0}} < \varepsilon$

$$\Downarrow$$

$$n_0 > \frac{1}{\varepsilon^2}$$

$$\Downarrow$$

$$\sqrt{n_0} > \sqrt{\frac{1}{\varepsilon^2}} = \frac{1}{\varepsilon}$$

$$\Downarrow$$

$\frac{1}{\sqrt{n_0}} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon \Rightarrow$ Afirmația are loc pt $\forall \varepsilon > 0 \Rightarrow$

$\Rightarrow \lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

b) $\lim\limits_{n\to\infty} \frac{n^2}{n+1} = \infty$

Şirul $x_n = \frac{n^2}{n+1}$ are limita $+\infty$ doar dacă nu

$\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ a.î. $\forall\, n \geq n_0$. $x_n > \varepsilon$

Fie $\varepsilon > 0$. Căutăm n_0

$$\frac{n^2}{n+1} > \varepsilon$$

$$n^2 > \varepsilon(n+1)$$

$$n^2 > \varepsilon n + \varepsilon$$

$$n^2 - \varepsilon n - \varepsilon > 0$$

$$\Delta = \varepsilon^2 + 4\varepsilon > 0$$

$$n_1 = \frac{\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}}{2}$$

$$n_2 = \frac{\varepsilon - \sqrt{\varepsilon^2 + 4\varepsilon}}{2}$$

n	$-\infty$		n_1		n_2		$+\infty$
$n^2-\varepsilon n-\varepsilon$	+ + +	+	0	- - -	0	+ + +	

$$\Rightarrow n \in \left(\frac{\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}}{2}, +\infty \right)$$

Putem lua $n_0 = \left[\frac{\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}}{2} \right] + 1$ deci $n_0^2 - \varepsilon n_0 - \varepsilon > 0$

Verificăm:

Fie $\varepsilon > 0$ arbitrar ales. Atunci $\exists\, n_0 = \left[\frac{\varepsilon + \sqrt{\varepsilon^2 + 4\varepsilon}}{2} \right] + 1$ a.î. $\forall\, n \geq n_0$

$x_n = \frac{n^2}{n+1} > \varepsilon \Rightarrow \lim\limits_{n\to\infty} x_n = +\infty$

3. Studiați convergența șirului $(x_n)_{n\in\mathbb{N}}$ și calculați limita sa acolo unde este posibil (metode: monotonie, mărginire, criteriul cleștelui, subșiruri, șir fundamental)

a) $x_n = a^n$, $a \in \mathbb{R}$

$$\frac{x_{n+1}}{x_n} = \frac{a^{n+1}}{a^n} = a$$

$a = 1$, $a = 0 \Rightarrow$ șirurile sunt constante

$a < 1 \Rightarrow$ șir descrescător

$a > 1 \Rightarrow$ șir crescător

Cazul 1: $a > 1$

avem de arătat că $\forall\, \varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$, a.î. $\forall\, n \geq n_0$ $x_n = a^n > \varepsilon$

Căutăm n_0

$$a^n > \varepsilon \Rightarrow \ln a^n > \ln \varepsilon \Rightarrow n \ln a > \ln \varepsilon \Rightarrow n > \frac{\ln \varepsilon}{\ln a} \Rightarrow$$

$$\Rightarrow n_0 = \left[\frac{\ln \varepsilon}{\ln a}\right] + 1$$

ε arbitrar $\Rightarrow \forall\, \varepsilon > 0 \Rightarrow \lim\limits_{n\to\infty} a^n = \infty$, $a > 1$

Cazul 2: $a < -1$

$x_{2n} = a^{2n} = (a^2)^n \quad \lim_{n\to\infty} x_n = \begin{cases} +\infty, & a<-1 \\ 1, & a=-1 \end{cases}$

$x_{2n+1} = a^{2n+1} = a\cdot(a^2)^n \quad \lim_{n\to\infty} x_n = \begin{cases} -\infty, & a<-1 \\ -1, & a=-1 \end{cases}$

Cazul 3: $a \in (-1, 1)$, $a \neq 0$

Vom arăta că $\lim_{n\to\infty} a^n = 0 \iff \forall\, \varepsilon > 0, \exists\, n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0$

$|x_n| < \varepsilon$ adică $|a^n| < \varepsilon$

Căutăm n_0

$\varepsilon > 0$

$|a^n| = |a|^n \overset{a\neq 0}{\Rightarrow} |a|^n < \varepsilon \iff \ln |a|^n < \ln\varepsilon \Rightarrow n \ln |a| < \ln \varepsilon \Rightarrow$

$\Rightarrow n < \frac{\ln\varepsilon}{\ln|a|} \longrightarrow n_0 = \left[\frac{\ln\varepsilon}{\ln|a|}\right] + 1$

Verificăm: $|a|^n \leq |a|^{n_0}$, unde $|a|^{n_0} = |a|^{\left[\frac{\ln\varepsilon}{\ln|a|}\right]+1} \underset{-1<a<1}{\leq} |a|^{\left[\frac{\ln\varepsilon}{\ln|a|}\right]}$

$|a|^{\left[\frac{\ln\varepsilon}{\ln|a|}\right]} = e^{\ln|a|^{\frac{\ln\varepsilon}{\ln|a|}}} = e^{\ln|a|\cdot\frac{\ln\varepsilon}{\ln|a|}} = e^{\ln\varepsilon} = \varepsilon$

Deci: $\lim_{n\to\infty} a^n = 0$

Așadar, $\lim_{n\to\infty} a^n = \begin{cases} +\infty, & a>1 \\ 1, & a=1 \\ 0, & a\in(-1,1) \\ \nexists, & a\leq -1 \end{cases}$

b) $x_n = \frac{2^n}{n!}$

$$\frac{x_n}{x_{n+1}} = \frac{2^n}{n!} \cdot \frac{(n+1)!}{2^{n+1}} = \frac{n+1}{2} \geq 1, \ n \geq 1$$, deci $x_n \geq x_{n+1} \Rightarrow$

$\Rightarrow$ șir descrescător

E mărg superior de $x_1 = 2$ și fiind pozitiv, inferior de $0 \Rightarrow$

$\Rightarrow$ e convergent $\Rightarrow \exists \lim \in \mathbb{R}$

$x_{n+1} = \frac{n+1}{2} \cdot x_n$

$$\Rightarrow \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \Big(\overset{\nearrow 0}{\frac{n+1}{2}} \, x_n\Big) = 0$$

c) $x_n = \sqrt[n]{n}$

Arătăm: $\forall \varepsilon > 0$, $\exists n_0 \in \mathbb{N}$ a.î. $\forall n \geq n_0 : |\sqrt[n]{n} - 1| < \varepsilon$

$\forall n \geq 2 : \sqrt[n]{n} > 1$

Notăm $a_n = \sqrt[n]{n} - 1 > 0$

$\sqrt[n]{n} = 1 + a_n \ /()^n$

$$n = (1+a_n)^n \underset{(x+y)^n = \sum_{k=0}^{n} C_n^k x^{n-k} y^k}{\overset{\text{binomul}}{=\!=}} 1 + \frac{\cancel{n!}^{\,n}}{1!\,\cancel{(n-1)!}} \cdot a_n + \frac{\cancel{n!}^{\,(n-1)n}}{2!\,\cancel{(n-2)!}} \cdot a_n^2 + \ldots + a_n^n \Rightarrow$$

$\Rightarrow n > \frac{n(n-1)}{2} \cdot a_n^2 \ /:n$

$1 > \frac{n-1}{2} \cdot a_n^2 \ /()^{-1}$

$0 < 1 < \frac{2}{n-1} \cdot \frac{1}{a_n^2}$

$0 < a_n^2 < \frac{2}{n+1} \ /\sqrt{\ }$

$0 < |a_n| < \sqrt{\frac{2}{n+1}}$

$\Downarrow$

$$0 < |\sqrt[n]{n} - 1| < \sqrt{\frac{2}{n+1}}, \quad \forall n \geq 2, \quad \sqrt{\frac{2}{n+1}} < \varepsilon$$

$$\sqrt{\frac{2}{n+1}} \to 0$$

c.c

$$|\sqrt[n]{n} - 1| = 0 \Rightarrow \lim_{n \to \infty} \sqrt[n]{n} = 1$$

d) $x_n = \left(1 + \frac{1}{n}\right)^n$