Analiza

$5+1$

1) a) $$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{1+xy}-1} = \lim_{(x,y)\to(0,0)} \frac{\cancel{xy}(\sqrt{1+xy}+1)}{\cancel{xy}} = 2$$

$\Rightarrow$ limita $\exists$

b) $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}, \quad f(x,y) = \frac{x^2-y^2}{x^2+y^2} \quad \forall (x,y) \quad \begin{matrix} x\to 0 \\ y\to 0 \end{matrix}$$

$$a^n = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}}, 0\right)$$

$$b^n = \lim_{n\to\infty}\left(0, \frac{1}{\sqrt{n}}\right)$$

$$\left.\begin{matrix} f(a^n) = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}-0}{\frac{1}{n}+0} = \dfrac{1}{n}\cdot\dfrac{n}{1} = 1 \\ f(b^n) = \lim\limits_{n\to\infty} \dfrac{0-\frac{1}{n}}{0+\frac{1}{n}} = -1 \end{matrix}\right\} \Rightarrow \nexists \lim_{x,y\to 0} f(x,y)$$

c) $$\lim_{(x,y)\to(\infty,\infty)} \frac{x^2+y^2}{x^4+y^4}, \quad \lim_{x,y\to(\infty,\infty)} f(x,y) = \frac{x^2+y^2}{x^4+y^4}$$

$$|f(x,y)-0| = \frac{x^2}{x^4+y^4} + \frac{y^2}{x^4+y^4} \leq \frac{x^2}{x^4} + \frac{y^2}{y^4} = \frac{1}{x^2} + \frac{1}{y^2} = 0$$

d) $$\lim_{x,y\to 0,0} \frac{x\sin(x^2-y^2)}{x^2+y^2} = \lim_{x,y\to 0,0} x \cdot \underbrace{\frac{\sin(x^2-y^2)}{(x^2-y^2)}}_{\to 1} \cdot \frac{(x^2-y^2)}{(x^2+y^2)}$$

$$= \lim_{x,y \to 0,0} \frac{x(x^2-y^2)}{(x^2+y^2)}$$

$$|f(x,y) - 0| = \left| \frac{x^3}{(x^2+y^2)} - \frac{xy^2}{(x^2+y^2)} \right| =$$

$$= \left| \frac{x \cdot x^2}{(x^2-y^2)} - \frac{x \cdot y^2}{(x^2+y^2)} \right| \leq \left| \frac{x \cdot x^2}{(x^2+y^2)} \right| + \left| \frac{x \cdot y^2}{x^2+y^2} \right|$$

$$\leq \left| x \left(\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} \right) \right| \leq \left| \frac{x}{\cancel{(x^2+y^2)}} \cdot \cancel{(x^2+y^2)} \right|$$

$$= 0$$

e) $$\lim_{x,y \to 0,0} \frac{x^3+y^3}{xy} = \lim_{x,y \to 0,0} \frac{x^{\cancel{3}\,2}}{\cancel{x}y} + \frac{y^{\cancel{3}\,2}}{x\cancel{y}} = \lim_{x,y \to 0,0} \frac{x^2}{y} + \frac{y^2}{x}$$

$$\left.\begin{aligned} a_n &= \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}}, \frac{1}{n} \right) \Rightarrow \lim_{n\to\infty} f(x) = \frac{1}{n} \cdot \frac{n}{1} + \frac{1}{\cancel{n^2}\, n\sqrt{n}} \cdot \frac{\cancel{\sqrt{n}}}{1} = 1 \\ b_n &= \lim_{n\to\infty} \left(\frac{1}{n^2}, \frac{1}{n^2} \right) \Rightarrow \lim_{n\to\infty} f(x) = \frac{1}{n^{\cancel{4}2}} \cdot \frac{\cancel{n}^2}{1} + \frac{1}{n^{\cancel{4}2}} \cdot \frac{\cancel{n}^2}{1} = 0 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \nexists \lim$$

f) $$\lim_{x,y \to 1,1} \frac{(x-1)(y-1)}{xy-1} = \lim_{\substack{u = x-1 \to 0 \\ v = y-1 \to 0}} \frac{uv}{(x-1+1)(v-1+1)-1} =$$

$$= \lim_{\substack{u \to x-1 \to 0 \\ v \to y-1 \to 0}} \frac{uv}{(u+1)(v+1)-1} = \lim_{\substack{u \to 0 \\ v \to 0}} \underbrace{\frac{uv}{uv+u+v\cancel{+1-1}}}_{f(x)} =$$

$$a_n = \left(\frac{1}{n}, \frac{1}{n}\right) \Rightarrow \lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} \frac{\frac{1}{n^2}}{\frac{1}{n^2} + \frac{^n 1}{n} + \frac{^n 1}{n}} =$$

$$= \lim_{n \to \infty} \frac{\frac{1}{n^2}}{\frac{1+n+n}{n^2}} = \lim_{n \to \infty} \frac{1}{\cancel{n^2}} \cdot \frac{\cancel{n^2}}{\underbrace{1+n+n}_{0}} = 0$$

$$b_n = \left(\frac{1}{n}, -\frac{1}{n}\right) \Rightarrow \lim_{n \to \infty} f(b_n) = \lim_{n \to \infty} \frac{-\frac{1}{n^2}}{-\frac{1}{n^2} + \cancel{\frac{1}{n}} - \cancel{\frac{1}{n}}} = 1$$

$\Rightarrow \nexists \lim$

$$g) \lim_{x,y,z \to 0_3} \frac{(x+y+z)^2}{x^2+y^2+z^2} = \lim_{x,y,z \to 0_3} \frac{x^2 + 2x(y+z) + (y+z)^2}{x^2+y^2+z^2} =$$

$$= \frac{x^2 + 2xy + 2xz + y^2 + 2yz + z^2}{x^2+y^2+z^2} = \cancel{\frac{x^2+y^2+z^2}{x^2+y^2+z^2}} + \frac{2(xy+xz+yz)}{x^2+y^2+z^2} =$$

$$= 1 + \frac{2(xy+xz+yz)}{x^2+y^2+z^2}$$

$$a_n = \left(0, \frac{1}{n}, \frac{1}{n}\right) \Rightarrow f(a_n) = 1 + \frac{\frac{2}{n^2}}{\frac{2}{n^2}} = 2$$

$$b_n = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}\right) \Rightarrow f(b_n) = 1 + \frac{\frac{6}{n^2}}{\frac{3}{n^2}} = 3$$

$\Rightarrow \nexists \lim$