Analiza

Seminarul 4

1. Studiați natura următoarelor S.T.P utilizând criteriile indicate:

i) criteriul comparației

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{4n^2-1}} = x_n$

$\sum_{n=1}^{\infty} \frac{1}{n} = y_n$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{n}{\sqrt{4n^2-1}} = \lim_{n\to\infty} \frac{n}{n\sqrt{4-\underbrace{\frac{1}{n^2}}_{\to 0}}} = \frac{1}{4} \in (0,\infty) \Rightarrow$$

$\Rightarrow x_n \sim y_n \Rightarrow x_n$ divergentă

b) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right) = x_n$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = y_n$

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{x_n\to 0} \frac{\ln(1+x_n)}{x_n} = 1 \in (0,\infty)$$

$$\left.\begin{array}{l} \Rightarrow x_n \sim y_n \\ y_n \text{ conv}, p>1 \end{array}\right\} \Rightarrow x_n \text{ convergentă}$$

ii) consecințe ale criteriului lui Kummer

a) $\sum_{n=0}^{\infty} \frac{2^n}{n!} = x_n$

$$D: \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\cancel{2^n}}{\cancel{n!}} \cdot \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{2^{n+1}}_{\,2}} = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1$$

$\Rightarrow x_n$ convergentă

b) $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{\sqrt{n}} = x_n$

$$D. \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{\sqrt{n}} \cdot 2^{\sqrt{n+1}} = \lim_{n\to\infty} \frac{2^{\sqrt{n+1}}}{2^{\sqrt{n}}} =$$

$$= \lim_{n\to\infty} 2^{\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}} = \lim_{n\to\infty} 2^{\frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} \nearrow 0} = 1$$

$$R: \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}} - 1\right)$$

$$\lim_{x_n\to 0} \frac{a^{x_n}-1}{x_n} = \ln a$$

$$\lim_{n\to\infty} n\left(\frac{2^{\frac{1}{\sqrt{n+1}+\sqrt{n}}}-1}{\frac{1}{\sqrt{n+1}+\sqrt{n}}} \cdot \frac{1}{\sqrt{n+1}+\sqrt{n}}\right) = \lim_{n\to\infty} \frac{\ln 2 \cdot n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty}$$

$$\frac{\cancel{n}^{\sqrt{n}}}{\cancel{\sqrt{n}}\left(1+\sqrt{1+\frac{1}{n}}\right)} \ln 2 = +\infty > 1 \Rightarrow x_n \text{ convergentă}$$

c) $\sum_{n=1}^{\infty} \left[\frac{(2n)!!}{(2n+1)!!}\right]^2 = x_n$

$$D: \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \left[\frac{(2n)!!}{(2n+1)!!} \cdot \frac{(2n+3)!!}{(2n+2)!!}\right]^2 = \lim_{n\to\infty} \left[\frac{2n+3}{2n+2}\right]^2 =$$

$$= \lim_{n\to\infty} \left[\frac{\cancel{n}\left(2+\frac{3}{n}\right)}{\cancel{n}\left(2+\frac{2}{n}\right)}\right]^2 = 1$$

$$R: \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{(2n+3)^2-(2n+2)^2}{(2n+2)^2}\right) =$$

$$= \lim_{n\to\infty} n\left(\frac{\cancel{4n^2}+12n+9-\cancel{4n^2}-8n-4}{4n^2+8n+4}\right) = \lim_{n\to\infty} n\left(\frac{4n+5}{4n^2+8n+4}\right)$$

$$= \lim_{n\to\infty} n\left(\frac{\cancel{n}(4+\frac{5}{n})}{n^{\cancel{2}}(4+\frac{8}{n}+\frac{4}{n^2})}\right) = \lim_{n\to\infty} \frac{4n}{4n} = 1$$

$$B.\ \lim_{n\to\infty} \ln n\left[n\left(\frac{x_n}{x_{n+1}}-1\right)-1\right] = \lim_{n\to\infty} \ln n\left(\frac{4n^2+5n}{4n^2+8n+4}-1\right)$$

$$= \lim_{n\to\infty} \ln n\left(\frac{\cancel{4n^2}+5n-\cancel{4n^2}-8n-4}{4n^2+8n+4}\right) = \lim_{n\to\infty} \ln n\left(\frac{-3n-4}{4n^2+8n+4}\right)$$

$$= \lim_{n\to\infty} \frac{\ln n}{n}\left(\frac{-3n^2-4n}{4n^2+8n+4}\right) = \lim_{n\to\infty} -\frac{3}{4n}\cdot \ln n = \lim_{n\to\infty} \ln n^{-\frac{3}{4n}} =$$

$$= \ln 1 = 0 < 1 \Rightarrow x_n \text{ div.}$$

iii) criteriul radicalului

$$\sum_{n=1}^{\infty} \frac{n^2}{(2+\frac{1}{n})^n} = x_n$$

$$\lim_{n\to\infty} \sqrt[n]{\frac{n^2}{(2+\frac{1}{n})^n}} = \lim_{n\to\infty} \frac{\sqrt[n]{n^2}}{(2+\frac{1}{n})} = \lim_{n\to\infty} \frac{(\sqrt[n]{n})^2}{(2+\frac{1}{n})} = \frac{1}{2} < 1 \Rightarrow$$

$\Rightarrow x_n$ convergentă

iv) criteriul condensării

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} = x_n$$

$$\sum x_n \sim \sum 2^n x_{2^n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p} \sim \sum_{n=2}^{\infty} \cancel{2^n} \frac{1}{\cancel{2^n}(\ln 2^n)^p} = \sum_{n=2}^{\infty} \frac{1}{n^p \cdot \ln 2^p} = \frac{1}{\ln 2^p} \underbrace{\sum_{n=2}^{\infty} \frac{1}{n^p}}_{\substack{\text{conv pt } p>1 \\ \text{div pt } p\in[0,1]}}$$

2. Studiați convergența și absolut convergența următoarelor serii cu termeni oarecare

a) $\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{3^n}$

Seria $\sum_{n=1}^{\infty} x_n$ este absolut convergentă dacă seria $\sum_{n=1}^{\infty} |x_n|$ este convergentă

Studiem convergența seriei $\sum_{n=1}^{\infty} \frac{2n+1}{3^n}$:

$$D: \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{2n+1}{\cancel{3^n}} \cdot \frac{\cancel{3^{n+1}}\,3}{2n+3} = \lim_{n\to\infty} \frac{6n+3}{2n+3} =$$

$= \frac{6}{2} = 3 > 1 \Rightarrow$ seria este absolut convergentă $\Rightarrow$ seria e conv.

b) $\sum_{n=1}^{\infty} \frac{\sin n}{2^n}$

Seria $\sum_{n=1}^{\infty} \left|\frac{\sin n}{2^n}\right|$ are numărătorul $(\sin n) \in [0,1]$, $2^n > 1$ $\forall n \in \mathbb{N}^* \Rightarrow \frac{\sin n}{2^n} \in [0,1]$

$$\frac{\sin n}{2^n} \leq \frac{1}{2^n} \quad \forall n \in \mathbb{N}^*$$

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ = progresie geometrică cu rația $\frac{1}{2} \Rightarrow$ Suma este:

$$S = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{\frac{1}{2}^n - 1}{\frac{1}{2} - 1} = \lim_{n\to\infty} \cancel{\frac{1}{2}} \cdot \frac{\overset{0}{\frac{1}{2^n}} - 1}{-\cancel{\frac{1}{2}}} = 1 \Rightarrow \text{convergentă}$$

$\lim_{n\to\infty} \frac{\frac{\sin}{2^n}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{\sin}{2^n} \in [0,1] \Rightarrow \sum \frac{\sin}{2^n} \sim \sum \frac{1}{2^n} \Rightarrow \sum \left|\frac{\sin}{2^n}\right|$ absolut convergentă, deci $\sum \frac{\sin}{2^n}$ convergent

3. criteriul raportului pentru șiruri

Fie x_n un STP pentru care $\exists \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = l$, au loc afirmațiile:

i) Dacă $l > 1$ atunci $\lim_{n\to\infty} x_n = 0$

ii) Dacă $l < 1$ atunci $\lim_{n\to\infty} x_n = +\infty$

Exerciții suplimentare

1. Stud. natura STP:

a) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{n}}}, \quad \frac{1}{n} < 1 \Rightarrow \text{seria este div.}$$

b) $$\sum_{n=0}^{\infty} \frac{2^n}{n+3^n}$$

$$\sum_{n=0}^{\infty} \frac{2^n}{n+3^n} < \sum_{n=0}^{\infty} \frac{2^n}{3^n}$$

$\sum_{n=0}^{\infty} \frac{2^n}{3^n}$ este o sumă geometrică cu rația $\frac{2}{3}$, unde suma

$$S = \lim_{n\to\infty} \frac{2}{3} \cdot \frac{\overset{0}{\left(\frac{2}{3}\right)^n} - 1}{\frac{2}{3} - 1} = \frac{2}{\cancel{3}} \cdot \cancel{3} = 2 \Rightarrow \sum_{n=0}^{\infty} \frac{2^n}{3^n} \text{ convergentă}$$

$$\left.\begin{array}{l} \sum_{n=0}^{\infty} \frac{2^n}{3^n} \text{ convergentă} \\ \sum_{n=0}^{\infty} \frac{2^n}{n+3^n} < \sum_{n=0}^{\infty} \frac{2^n}{3^n} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{2^n}{n+3^n} \text{ convergentă}$$

c) $$\sum_{n=1}^{\infty} \sin^3 \frac{1}{n} = \sum_{n=1}^{\infty} \left(\sin \frac{1}{n}\right)^3$$

$$\frac{\left(\sin \frac{1}{n}\right)^3}{\left(\frac{1}{n}\right)^3} = \left(\underbrace{\frac{\sin \frac{1}{n}}{\frac{1}{n}}}_{\to 1}\right)^3 = 1 \in (0, \infty) \Rightarrow \sum_{n=1}^{\infty} \sin^3 \frac{1}{n} \sim \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^3}}_{\text{convergentă}}$$

d) $$\sum_{n=1}^{\infty} \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2n+1}$$

$$D = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \lim_{n\to\infty} \frac{\cancel{(2n-1)!!}}{\cancel{(2n)!!}} \cdot \frac{\overset{2n+2}{\cancel{(2n+2)!!}}}{\underset{2n+1}{\cancel{(2n+1)!!}}} \cdot \frac{1}{2n+1} \quad \frac{2n+3}{1} =$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+3)}{(2n+1)^2} = \lim_{n\to\infty} \frac{4n^2+6n+4n+6}{4n^2+4n+1} =$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(4+\overset{\nearrow 0}{\frac{10}{n}}+\overset{\nearrow 0}{\frac{6}{n^2}})}{\cancel{n^2}(4+\underset{\searrow 0}{\frac{4}{n}}+\underset{\searrow 0}{\frac{1}{n^2}})} = 1$$

$$R = \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - \overset{4n^2+4n+1}{1}\right) = \lim_{n\to\infty} n\left(\frac{6n+5}{4n^2+4n+1}\right) =$$

$$= \lim_{n\to\infty} \cancel{n}\left(\frac{\cancel{n}(6+\overset{\nearrow 0}{\frac{5}{n}})}{\cancel{n^2}(4+\underset{\searrow 0}{\frac{4}{n}}+\underset{\searrow 0}{\frac{1}{n^2}})}\right) = \frac{6}{4} > 1 \Rightarrow \text{serie convergentă}$$

e) $\sum_{n=1}^{\infty} a^{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}$

$$D: \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \frac{a^{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}}}{a^{1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\frac{1}{n+1}}} = \lim_{n\to\infty} a^{\overset{\nearrow 0}{-\frac{1}{n+1}}} = 1$$

$$R: \lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{a^{-\frac{1}{n+1}} - 1}{-\frac{1}{n+1}} \cdot -\frac{1}{n+1}\right) =$$

$$= \lim_{n\to\infty} \cancel{n}\left(\ln a \cdot -\frac{1}{\cancel{n}(1+\underset{\searrow 0}{\frac{1}{n}})}\right) = -\ln a = \ln a^{-1} = \ln \frac{1}{a} \Rightarrow$$

$\Rightarrow$ pt. $a \in (0, \frac{1}{e}) \Rightarrow \ln \frac{1}{a} > 1 \Rightarrow \Sigma$ conv.

pt. $a \in (\frac{1}{e}, \infty) \Rightarrow \ln \frac{1}{a} < 1 \Rightarrow \Sigma$ div

pt $a = \frac{1}{e} \Rightarrow \ln \frac{1}{a} = 1$

B: $\lim_{n\to\infty} \ln n \cdot \left[n\left(\frac{x_n}{x_{n+1}} - 1\right) - 1\right] = \lim_{n\to\infty} \ln n \quad (1-1) = 0 < 1 \Rightarrow$

$\Rightarrow \sum$ div

f) $\sum_{n=0}^{\infty} \left(\frac{n+1}{n+2}\right)^{n^2}$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n+1}{n+2}\right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^{n} = \lim_{n\to\infty} \left(\frac{n+2-1}{n+2}\right)^{n} =$$

$$= \lim_{n\to\infty} \left(\left(1 + \left(-\frac{1}{n+2}\right)\right)^{-\frac{n+2}{1}}\right)^{-\frac{n}{n+2}} = e^{-1} = \frac{1}{e} < 1 \Rightarrow \sum \text{conv}$$

h) $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^2}}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{\ln n}{\cancel{n^2}} \cdot \cancel{n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{\ln n}{n^{\frac{1}{2}}} \stackrel{l'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{2\sqrt{n}} =$$

$= 0$

2. Stud. conv și abs conv:

a) $\sum_{n=1}^{\infty} (-1)^n n$

$$\sum_{n=1}^{\infty} (-1)^n n = \underbrace{-1+2}_{1} \underbrace{-3+4}_{1} \underbrace{-5+6}_{1} - \ldots \underbrace{-(n-1)+n}_{1} \Rightarrow \text{suma}$$

este divergentă

b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}}{n+\sqrt{2}}$

$$\frac{\sqrt{n}}{n+\sqrt{2}} > \frac{\sqrt{n}}{n+\sqrt{n}} = \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}(\sqrt{n}+1)} > \frac{1}{n}$$

$\frac{1}{n}$ divergentă $\Rightarrow \frac{\sqrt{n}}{n+\sqrt{2}}$ divergent, deci nu e abs conv.

Stud convergența seriei:

Leibniz:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{n+\sqrt{2}} = \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}}{\cancel{\sqrt{n}}\left(\sqrt{n}+\frac{\sqrt{2}}{\sqrt{n}}\right)} = 0$$

$$a_n - a_{n+1} > 0 \iff \frac{\sqrt{n}}{n+\sqrt{2}} > \frac{\sqrt{n+1}}{n+1+\sqrt{2}}$$