1. Problems to try on your own - answers posted

As discussed in class and shown in the textbook (P. 60-63) we have

$$\pi_t = \beta E_t[\pi_{t+1}] - \lambda \hat{\mu}_t \tag{1}$$
$$= \beta E_t[\pi_{t+1}] + \kappa \tilde{y}_t \tag{2}$$

where

$$\begin{aligned}
\lambda &\equiv \theta^{-1}(1-\theta)(1-\beta\theta)\Theta > 0 \\
\Theta &\equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon} \\
\kappa &\equiv \lambda\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)
\end{aligned}$$

and where

$$\begin{aligned}
\hat{\mu}_t &\equiv \mu_t - \mu \\
\mu_t &\equiv p_t - \psi_t \\
\tilde{y}_t &\equiv y_t - y_t^n
\end{aligned}$$

- Use equation (1) to show that
$$\pi_t = -\lambda \sum_{k=0}^{\infty} \beta^k E_t[\hat{\mu}_{t+k}]$$
NOTE: For this you will need to use the 'law of iterated expectations' which means that $E_t[E_{t+j}[X]] = E_t[X]$ for a random variable X and $j \geq 0$. Basically it means your expectation of 'X' now (with your limited information available at t) of your expectation in the future of 'X' (with greater information at $t+j$) is simply equal to your expectation now - which is intuitive.

- Show that λ is decreasing in the 'Calvo parameter', θ and briefly discuss how a greater degree of price stickiness influences the value of inflation associated with a given expected path of markup deviations?[1]

[1] For this you will need to recall that

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x) \text{ and } \frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$$

- As discussed in class and shown in the textbook (P. 62) we have

$$
\begin{aligned}
\mu_t &= p_t - \psi_t \\
&= -(w_t - p_t) + (a_t - \alpha n_t + \log(1-\alpha)) \\
&= -(\sigma y_t + \varphi n_t) + (a_t - \alpha n_t + \log(1-\alpha)) \\
&= -\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t + \left(\frac{1+\varphi}{1-\alpha}\right) a_t + \log(1-\alpha) \qquad (3)
\end{aligned}
$$

  Briefly give intuition for the dependence of the coefficient on y_t in equation (3) on σ, φ and α. Hint: Note that the negative of the log markup is log real marginal cost. The intuition may be easier if you discuss the coefficient in terms of real marginal cost and its association with the output level...

$$
mc_t \equiv \psi_t - p_t = \left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right) y_t - \left(\frac{1+\varphi}{1-\alpha}\right) a_t - \log(1-\alpha)
$$

Answers
We start with the recursive relation in equation (1)

$$
\begin{aligned}
\pi_t &= \beta E_t[\pi_{t+1}] - \lambda \hat{\mu}_t \\
&= \beta E_t[\beta E_{t+1}[\pi_{t+2}] - \lambda \hat{\mu}_{t+1}] - \lambda \hat{\mu}_t \\
&= \beta^2 E_t[E_{t+1}[\pi_{t+2}]] - \lambda \beta E_t[\hat{\mu}_{t+1}] - \lambda \hat{\mu}_t
\end{aligned}
$$

Now we use the law of iterated expectations to obtain

$$
\pi_t = \beta^2 E_t[\pi_{t+2}] - \lambda \beta E_t[\hat{\mu}_{t+1}] - \lambda \hat{\mu}_t
$$

Continuing, accordingly we can write this as

$$
\begin{aligned}
\pi_t &= \beta^2 E_t[\beta E_{t+2}[\pi_{t+3}] - \lambda \hat{\mu}_{t+2}] - \lambda \beta E_t[\hat{\mu}_{t+1}] - \lambda \hat{\mu}_t \\
&= \beta^3 E_t[\pi_{t+3}] - \lambda \beta^2 E_t[\hat{\mu}_{t+2}] - \lambda \beta E_t[\hat{\mu}_{t+1}] - \lambda \hat{\mu}_t \\
&\cdots \\
&= \beta^J E_t[\pi_{t+J}] - \lambda \sum_{j=0}^{J-1} \beta^j E_t[\hat{\mu}_{t+j}]
\end{aligned}
$$

and since $\beta \in (0,1)$ and our inflation process is well behaved we obtain, as $J \to \infty$

$$\pi_t = -\lambda \sum_{j=0}^{\infty} \beta^j E_t[\hat{\mu}_{t+j}]$$

since $\beta^J E_t[\pi_{t+J}] \to 0$ as $J \to \infty$.

Now, since $\Theta \in (0,1]$ and is a constant, to show that λ is decreasing in θ we need only show that

$$\frac{(1-\theta)(1-\beta\theta)}{\theta}$$

is decreasing in θ.

If we take the derivative with respect to θ (using the rules in the footnote in the question) we obtain

$$\frac{(-(1-\beta\theta)-\beta(1-\theta))-(1-\theta)(1-\beta\theta)}{\theta^2}$$

which is proportional to the numerator, as $\theta^2 > 0$. Thus we need to show that the numerator is negative. If we rearrange it, then we find that it is equal to $\beta\theta^2 - 1$ which is negative since β and θ are both assumed to be positive and less than 1.

The insight that, for a given expected path of deviations of markups from desired, the change in the price level is less extreme, may initially seem obvious. θ being larger naturally suggests more inertia, all else equal, since fewer firms can change prices in a given period. Note however, that it is not *completely* obvious *a priori* as one might have wondered if, in equilibrium, the changes made by the firms that *can* reset their prices might be larger to 'compensate' for a larger θ - possibly to the extent that the net effect would be to make the change in the price level bigger for a given expected path of markups. However, it turns out that is not the case here.

Note that the definition of κ in the New Keynesian Phillips curve means that a lower λ arising from a higher θ makes the slope of the Phillips curve 'flatter'. The (apparent) flattening of the Phillips curve relationship has been a matter of intense debate in recent years (in the context of statistical models and much richer structural New Keynesian models).[2]

[2] One implication of a flattening Phillips Curve is, to the extent monetary policy is thought to operate on inflation via its impact on activity, more dramatic shifts policy are required to influence the price level since they must induce more dramatic effects on activity (note this argument is often very loose and frequently takes a 'non-equilibrium' flavor in the sense of ignoring what shocks might have caused the lack of co-movement.). Additionally, a flattening Phillips Curve has been suggested as a reason why dramatic real weakness in the last recession did not result in as low inflation as pre-crisis models would have suggested (again, these arguments are often made very loosely).

Turning to the coefficient on y_t in equation (3) we first note that (as mentioned in the hint) $-\mu_t$ is equal to (log) real marginal cost. The discussion below is most natural if we explain the association between marginal cost and output. Thus

$$mc_t \equiv \psi_t - p_t = \left(\sigma + \frac{\varphi + \alpha}{1-\alpha}\right) y_t - \left(\frac{1+\varphi}{1-\alpha}\right) a_t - \log(1-\alpha)$$

Let us first consider it under the assumption of constant returns to scale in the production technology ($\alpha = 0$).[3] In that case the coefficient simply reflects the household's intratemporal optimality condition (where this has entered via the wage's role in marginal cost) plus the fact that we have substituted y_t for c_t. These two elements reflect two features of equilibrium - optimality and market clearing. The higher is φ the more an additional hour costs a household in disutility. Thus to obtain the greater labor supply associated with greater output, the wage must be higher (raising marginal cost, all else equal).

Similarly, if σ increases, the marginal utility of consumption is, all else equal, reduced, implying that the value of an additional hour to the household is reduced on the margin (since the consumption purchased with the additional wages yields less utility). Again, all else equal, the wage - and thus the marginal cost - must increase more within any increase in output. Note that we aren't making causal statements - these are simply statements about associations between variables that must hold in equilibrium.

Interestingly, in equilibrium, there is an association between marginal cost and the scale of output (for a given a_t) even if there are constant returns to scale ($\alpha = 0$). Under a constant returns to scale assumption, from the perspective of an individual price taking firm, marginal cost is independent of their scale of production since they treat the wage as given and marginal product of labor is simply A_t (recall, marginal cost is wage over marginal product of labor). The positive association arises in equilibrium however because wages must adjust in equilibrium in order to make the changed level of output consistent with optimal labor supply decisions by the household. If there are *decreasing* returns to scale ($\alpha > 0$), marginal cost is not constant with scale from the firm's perspective - it is positively related to scale - and this feeds through in the aggregate also. Hence α appearing in the coefficient on y_t.

[3] All the statements below take a_t as given. Note that it may be a shock to a_t that drives changes in y_t (and μ_t).

2. Problems to try on your own - Emil goes through in class and answers posted

2.1. Calvo Pricing

Under the Calvo pricing protocol a firm faces a probability, θ, of being unable to reset their price in any given period and this probability is independent of how long any given price has so far been prevailing. All firms within the economy face the same probability and their probabilities of being able to reset are mutually independent. We assume, as in class, that there is a continuum of firms, indexed by $i \in [0, 1]$ (so there is a unit mass of infinitesimally small firms). As such, a fraction θ of firms are unable to reset their prices each period.[4]

- What is the probability of a price prevailing j periods after it has been chosen?
- Show that the expected duration of a price (i.e. how many periods on average will it prevail before being reset) is $(1-\theta)^{-1}$? Note: Don't spend more than 20 mins on this bit - either it will pop into your head how to rearrange the algebra, or it won't - but try for 20mins...
- What value of θ would I pick my for my (quarterly) model to match real world data that suggests prices typically prevail for 3 quarters?
- In period t, what fraction of prices are prevailing that were last reset in period $t - j$?

Answers

The probability that a price lasts for only one period is $1-\theta$ - that is, the probability that in the first period after setting the price, the firm gets to set it again. The probability that the price lasts for two periods is the probability that it is fixed for one period multiplied by the probability that it will be reset in the following period, conditional on it having not been reset after one period. But since the Calvo probability is *independent of how long a price has been prevailing*, this latter probability is always simply $1 - \theta$. This logic then extends...

[4] Imagine there instead are 10 firms, then the expected number of firms that are unable to reset is 10θ but in any given period, there will be sampling variability and the *fraction* could likely be quite different from θ. Now imagine there are 1 million firms, in this case the fraction will be approximately θ but with a very small proportional deviation (this is simply the 'law of large numbers'). Imagine the number of firms becoming infinitely large while each firm becomes infinitely small - effectively that's what we achieve by working with a continuum, so there is no 'sampling variability' around the expected fraction of firms unable to reset their price. It's a bit like tossing a fair coin a massive (in the limit, infinite) number of times and keeping track of the fraction of heads. This number will $\to 0.5$. Whenever you see something like

$$\int_0^1 f(i)di$$

then all this really means, from our perspective, is that we are adding up (effectively averaging) $f(i)$ for all the values of i. In our applications i typically indexes firms/goods.

- Pr[Prevails for 1 period] = $1 - \theta$
- Pr[Prevails for 2 periods] = Pr[No reset through second period] $\times(1-\theta) = \theta(1-\theta)$
- Pr[Prevails for 3 periods] = Pr[No reset through third period] $\times(1-\theta) = \theta^2(1-\theta)$
- etc
- Pr[Prevails for j periods] = Pr[No reset through j^{th} period] $\times(1-\theta) = \theta^{j-1}(1-\theta)$

Thus, the probability is $\theta^{j-1}(1-\theta)$

As to the expected duration of the price, we note that the duration is a random variable. We have just found the probability associated with each of its possible values so to find the expected duration we simply calculate the mean, which is a probability-weighted average:

$$\begin{aligned} E[Duration] &= \sum_{j=1}^{\infty} Pr[\text{Prevails for j periods}]j \\ &= \sum_{j=1}^{\infty}(1-\theta)\theta^{j-1}j \\ &= (1-\theta)\sum_{j=1}^{\infty}\theta^{j-1}j \end{aligned}$$

Thus, we need to figure out the value of $\sum_{j=1}^{\infty}\theta^{j-1}j$. If we write it out 'carefully' we see that it has a clean structure...

$$\begin{aligned} \sum_{j=1}^{\infty}\theta^{j-1}j &= 1 + 2\theta + 3\theta^2 + 4\theta^3 + \ldots \\ &= 1 + \theta + \theta^2 + \theta^3 + \ldots \\ &\quad +\theta + \theta^2 + \theta^3 + \ldots \\ &\quad +\theta^2 + \theta^3 + \ldots \\ &\quad + \ldots \end{aligned}$$

but this can be re-written as follows

$$\sum_{j=1}^{\infty}\theta^{j-1}j = 1+\theta+\theta^2+\theta^3+\ldots$$
$$+\theta(1+\theta+\theta^2+\theta^3+\ldots)$$
$$+\theta^2(1+\theta+\theta^2+\theta^3+\ldots)$$
$$+\ldots$$

which, in turn, can be re-written as

$$\begin{aligned}\sum_{j=1}^{\infty}\theta^{j-1}j &= \sum_{k=0}^{\infty}\theta^k\left(\sum_{j=0}^{\infty}\theta^j\right)\\ &= \frac{1}{1-\theta}\sum_{j=0}^{\infty}\theta^j\\ &= \frac{1}{1-\theta}\frac{1}{1-\theta}\end{aligned}$$

Thus we have that

$$E[Duration] = (1-\theta)\sum_{j=1}^{\infty}\theta^{j-1}j = \frac{1}{1-\theta}$$

Thus, as one would expect, the higher the degree of price stickiness (the higher the probability, θ, of not being able to adjust) the longer is the expected duration of a price. Now, if one has data on prices from the real world, then one can get a sense of what θ should be by calculating the average duration in the data and setting θ to match it. Of course, this model is too simple to capture the many other properties of the distribution of price duration, but it is a useful simplification.[5]

If we want to pick θ to match the given real world 'moment' we require $(1-\theta)^{-1} = 3$. This implies that, after rearranging, $\theta = \frac{2}{3}$.

Finally, in period t a fraction $1-\theta$ of prices are reset, so the remaining fraction θ of prices are not.

- Among this θ fraction of prices carried over from $t-1$, a fraction $1-\theta$ were reset in $t-1$ and a fraction θ were not. This implies, that a $(1-\theta)\theta$ fraction of prices in t were reset in $t-1$ and a fraction θ^2 were reset before $t-1$.

[5]Once θ is pinned down by matching the mean duration, we have no other degrees of freedom to match the variance, skew or whatever other moments of the price duration distribution observed in the real world.

- Among this θ^2 fraction of prices in t that are carried over from $t-2$, a fraction $1-\theta$ were reset in $t-2$ and a fraction θ were not. This implies that a $(1-\theta)\theta^2$ fraction of prices in t were reset in $t-2$ and a fraction θ^3 were reset before $t-2$.

Following this logic, a fraction $(1-\theta)\theta^j$ of prices prevailing in t were reset in period $t-j$. Note that this fraction goes to zero as $j \to \infty$ (since the magnitude of θ is < 1) but there is a very (infinitely) long tail to the distribution of how long prices have been prevailing (there is some positive - but tiny - fraction of prices that haven't been reset for a century). Obviously this is unrealistic (in the real world, eventually prices get adjusted within some sensible amount of time - or the firm goes bust or whatever…) but as long as θ is set to approximately match the frequency of price changes (or to hit the average or median duration in the data, say) then the model can still be a reasonable approximation.[6]

Also, just to confirm/reassure that our fractions are correct, we check that they sum to 1 (so they account for all the different lengths of time for which prices in t have been prevailing)

$$\sum_{k=0}^{\infty}(1-\theta)\theta^j = (1-\theta)\sum_{k=0}^{\infty}\theta^j = (1-\theta)\frac{1}{1-\theta} = 1$$

3. Problems to try if you're keen (and want valuable practice at log-linearizations) - not discussed in seminar but answers posted

3.1. Optimal price setting

Use a first order approximation of

$$\sum_{k=0}^{\infty}\theta^k E_t\left[\Lambda_{t,t+1}Y_{t+k|t}\frac{1}{P_{t+k}}\left(P_t^* - \mathcal{M}\Psi_{t+k|t}\right)\right] = 0 \tag{4}$$

around the zero inflation steady state, to obtain (as claimed in the text p. 57 2nd ed.)

$$p_t^* = \mu + (1-\beta\theta)\sum_{k=0}^{\infty}(\beta\theta)^k E_t[\psi_{t+k|t}]$$

[6] As mentioned in Galí chapter 1 Nakamura and Steinsson (2008) suggest the median duration of certain prices in the U.S. may be around 8 to 11 months.

Answers

We define $\Omega_{t,t+k} \equiv \frac{1}{\beta^k}\Lambda_{t,t+k}$ (noting that $\Omega_{t,t+k} = 1$ in steady state, for all k) and rewrite equation (4) as[7]

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t \left[\Omega_{t,t+1} Y_{t+k|t} \frac{1}{P_{t+k}} \left(P_t^* - \mathcal{M}\Psi_{t+k|t}\right)\right] = 0$$

We the re-express in terms of log versions (lower case) of the variables in preparation for linearization in terms of the log variables

$$\sum_{k=0}^{\infty}(\beta\theta)^k E_t \left[e^{\omega_{t,t+k}+y_{t+k|t}-p_{t+k}} \left(e^{p_t^*} - e^{\mu+\psi_{t+k|t}}\right)\right] = 0$$

where $\mu \equiv \log(\mathcal{M})$ is the log 'desired' or 'flexible price' markup.

Now, note that in steady state $P_t - \mathcal{M}\Psi_{t+k|t}$ (in the steady state, the desired markup holds) - so the steady state of the term $\left(e^{p_t^*} - e^{\mu+\psi_{t+k|t}}\right)$ is 0. Let us also call the steady state of the term $e^{\omega_{t,t+k}+y_{t+k|t}-p_{t+k}}$, ζ, which is > 0. Then we can take a first order approximation and obtain

$$0 \approx \sum_{k=0}^{\infty}(\beta\theta)^k E_t \left[\zeta\left((p_t^* - \bar{p}^*)e^{\bar{p}^*} - (\psi_{t+k|t} - \bar{\psi})e^{\mu+\bar{\psi}}\right)\right]$$

but since $\zeta > 0$ we can divide through and get

$$\begin{aligned}
0 &\approx \sum_{k=0}^{\infty}(\beta\theta)^k E_t \left[\left((p_t^* - \bar{p}^*)e^{\bar{p}^*} - (\psi_{t+k|t} - \bar{\psi})e^{\mu+\bar{\psi}}\right)\right] \\
&= \sum_{k=0}^{\infty}(\beta\theta)^k E_t \left[\left(p_t^* e^{\bar{p}^*} - \psi_{t+k|t}e^{\mu+\bar{\psi}} - \bar{p}^* e^{\bar{p}^*} + \bar{\psi}e^{\mu+\bar{\psi}}\right)\right] \\
&= \sum_{k=0}^{\infty}(\beta\theta)^k E_t \left[\left(p_t^* e^{\bar{p}^*} - \psi_{t+k|t}e^{\bar{p}^*} - \bar{p}^* e^{\bar{p}^*} + \bar{\psi}e^{\bar{p}^*}\right)\right]
\end{aligned}$$

where in the last line we also used the fact that $\bar{p}^* = \mu + \bar{\psi}$ (this is just the steady state markup

[7] $\Omega_{t,t+k} = 1$ in steady state, for all k since consumption is constant in steady state, $\Lambda_{t,t+k} \equiv \beta^k \frac{U_{c,t+k}}{U_{c,t}}$ and we have taken out the β element of the multi-step SDF in defining Ω.

relation). Now we can divide through by $e^{\bar{p}^*}$ to obtain

$$\begin{aligned} 0 &\approx \sum_{k=0}^{\infty} (\beta\theta)^k E_t \left[\left(p_t^* - \psi_{t+k|t} - \bar{p^*} + \bar{\psi} \right) \right] \\ &= \sum_{k=0}^{\infty} (\beta\theta)^k E_t \left[p_t^* - \psi_{t+k|t} - \mu \right] \end{aligned}$$

where we used the definition of the log markup. Now we are almost done as we simply need to rearrange. Specifically, we have

$$\sum_{k=0}^{\infty} (\beta\theta)^k E_t \left[p_t^* - \mu \right] \approx \sum_{k=0}^{\infty} (\beta\theta)^k E_t \left[\psi_{t+k|t} \right]$$

but the terms on the LHS are known at t so we can drop the expectation operator and, recalling that $\sum_{k=0}^{\infty} (\beta\theta)^k = (1 - \beta\theta)^{-1}$ (since $\beta\theta \in (0, 1)$) we get

$$\frac{p_t^* - \mu}{1 - \beta\theta} \approx \sum_{k=0}^{\infty} (\beta\theta)^k E_t \left[\psi_{t+k|t} \right]$$

or, finally, as required

$$p_t^* \approx \mu + (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k E_t \left[\psi_{t+k|t} \right]$$

So that price is a markup of a particular weighted average of current and future marginal costs.