Linearizations and log-linearizations
Technical Appendix

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Disclaimer

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Linearizations and log-linearizations

We will require no sophisticated maths in this course

- Phew. . .

But there will unavoidably be some maths that can confuse even good students from time to time

- There is a lot to learn at once
- The volume of topics makes each one more difficult than in isolation
- Most (probably all) of it is high school level, but we all get rusty. . .

Some of the (slightly) more tricky maths need only be broadly understood and implemented - rather than getting the deep nitty-gritty

- Some/many of you will find it trivial
- All of you will be capable of it

Unofficial maths 'requirements'

Most of the maths we use will entail...

- Basic algebra
  - C_t will represent consumption in time t
- Basic probability
  - Mean/expectation and maybe standard deviation
- Collecting coefficients / factorization
  - $ax + bx = (a + b)x$
- Summations
  - $\sum_{j=0}^{J} f(x_j) \equiv f(x_0) + f(x_1) + \ldots + f(x_J)$
- Calculus
  - You will need to differentiate very simple functions
  - You will probably only need to understand what an integral *means*
  - You will need to be able to linearize and log-linearize

Solving an economic model

What does it mean to ‘solve’ an economic model?

- Models involve a lot of ‘variables’ (consumption, unemployment, output, wages,. . .)
- Accounting and technological constraints imply relationships among these variables
- The assumption that people and firms are optimizing also implies relationships among these variables
- There is a core set of variables that are needed to describe ‘the current situation’ (all the relevant info.)
- We call these variables ‘the state’
- **Solving a model $\Leftrightarrow$ finding functions that relate all the variables in the economy to the state**

Taylor approximations

Consider consumption in time t, C_t

- A solved model implies

$$C_t = f(tax\ rate, income, assets, monetary\ policy, \ldots)$$

- Or let's just call the state, s_t

$$C_t = f(s_t)$$

- Sadly, it is rare that the function f can actually be calculated
- Happily, we can more often calculate its derivatives
- Remember Taylor approximations from high school (e.g. 2^{nd} order)

$$f(s_t) \approx f(\bar{s}) + f'(\bar{s})(s_t - \bar{s}) + \frac{1}{2!} f''(\bar{s})(s_t - \bar{s})^2$$

Taylor approximations

In this course, we don't even need second order!

- We will only work with first order approximations
- **In fact, at 1^{st} order we proceed simply by linearizing the equations describing technological constraints and firm/household optimality**
- If we solve those linear equations (like in high school) we will obtain a first order approximation to f
- For more info on 'higher-order asymptotic approximations' and 'perturbation methods' see...
  - `https://www.nber.org/econometrics_minicourse_2011/Chapter_2_Pertubation.pdf`

So we need to be reminded how to take a linear approximation

Linearization - scalar case

Under various assumptions (that I won't describe here but which hold for the models we consider)

$$f(x) \approx f(\bar{x}) + f'(\bar{x})(x - \bar{x})$$

where

$$f'(\bar{x}) \equiv \frac{df}{dx}(\bar{x})$$

This is a first order approximation of f with respect to x, around the point $x = \bar{x}$

Linearization - scalar case

A linear approximation will be exact if f is linear to begin with

- Consider $f(x) = \alpha x$
- $f'(x) = \alpha$ for all x
- $f(x) \overset{Exact}{=} f(\bar{x}) + \alpha(x - \bar{x}) = \alpha x = f(x)$ for all $\bar{x}$
- Clearly, this is pointless

Consider a more general case of a quadratic f

- Consider $f(x) = \frac{\alpha}{2}x^2$
- $f'(x) = \alpha x$
- $f(x) \approx f(\bar{x}) + \alpha\bar{x}(x - \bar{x}) \equiv \hat{f}(x)$ for arbitrary x
- $f(x) \overset{Exact}{=} f(\bar{x}) + \alpha\bar{x}(x - \bar{x}) = f(\bar{x})$ only for $x = \bar{x}$ (trivially)

We take a slope at a point and then using the linear function with *that slope* from *that point*, to approximate the function of interest *at other points*

Linearization - multivariate case

Linear approximation of a scalar valued function with many arguments is essentially the same deal...

- $f(x,y) \approx f(\bar{x},\bar{y}) + \frac{\partial f}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial f}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) \equiv \hat{f}(x,y)$
- Consider $f(x,y) = x^2 y^3$
- $\frac{\partial f}{\partial x}(\bar{x},\bar{y}) = 2\bar{x}\bar{y}^3$
- $\frac{\partial f}{\partial y}(\bar{x},\bar{y}) = 3\bar{x}^2\bar{y}^2$
- $f(x,y) \approx \bar{x}^2\bar{y}^3 + 2\bar{x}\bar{y}^3(x-\bar{x}) + 3\bar{x}^2\bar{y}^2(y-\bar{y})$

We are making a new function, $\hat{f}$, that will be $= f$ at the approximation point, $(\bar{x},\bar{y})$, but which will only be an approximation for other x and y, by extrapolating the 'slope' of f at $(\bar{x},\bar{y})$.

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Linearization - multivariate case

Continuing with the $f(x,y) = x^2y^3$ example...

- Recognize the expression as a function of variables (here x and y)
- Figure out the value each variable takes at the approximation point (here $\bar{x}$ and $\bar{y}$)
- Find the first partial derivative of each function in terms of each variable (here $2xy^3$ and $3x^2y^2$)
  - Note: At this point we have not evaluated those derivative **at** $\bar{x}$ and $\bar{y}$
- Build the approximation for each function as
  1. The value of the original function at the approximation point
  2. Plus each of the first derivatives **evaluated at the approximation point** $\times$ the deviation of the relevant variable from the approximation point

$$\hat{f}(x,y) \equiv \bar{x}^2\bar{y}^3 + 2\bar{x}\bar{y}^3(x-\bar{x}) + 3\bar{x}^2\bar{y}^2(y-\bar{y})$$

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$$\hat{f}(x, y) \equiv \bar{x}^2\bar{y}^3 \;+\; 2\bar{x}\bar{y}^3(x - \bar{x}) + 3\bar{x}^2\bar{y}^2(y - \bar{y})$$

Linearization - common format in our applications

Frequently we will encounter situations where we have (for some functions f and g)

$$f(x,y) = g(x,y)$$

We know that if this relationship is true, then $f(\bar{x},\bar{y}) = g(\bar{x},\bar{y})$ (why?) so our first order linear approximation to this relation

$$\begin{aligned} & f(\bar{x},\bar{y}) + \frac{\partial f}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial f}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) \\ \approx\ & g(\bar{x},\bar{y}) + \frac{\partial g}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial g}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) \end{aligned}$$

implies

$$\frac{\partial f}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial f}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) \approx \frac{\partial g}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial g}{\partial y}(\bar{x},\bar{y})(y-\bar{y})$$

Logs and exponentials

A **logarithm** of y 'to base x' is the value to which x must be raised to make it equal y

$$x^{\log_x(y)} = y$$

We will typically be working with the 'natural' logarithm which has the exponential constant, e, as its base

- $e = \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.71828$
- $\log_e(z)$ is sometimes written $\ln(z)$ but we will typically use $\log(z)$ in this course
- The exponential function $\exp(z) \equiv e^z$
- See `https://people.duke.edu/~rnau/411log.htm`

Useful properties of the log function

Log of product = sum of logs

$$\log(xy) = \log(x) + \log(y)$$

Exponents become coefficients

$$\log(x^y) = y \log(x)$$

Log of ratio = difference in logs (implied by results above)

$$\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$$

Log of unity = 0 (anything raised to 0 equals unity)

$$\log(1) = 0$$

Useful properties of the log function

Differentiation of logs

$$\frac{d}{dx}\log\left(f(x)\right) = \frac{f'(x)}{f(x)}$$

Differentiation of exponentials

$$\frac{d}{dx}\exp\left(f(x)\right) = f'(x)\exp\left(f(x)\right)$$

Product of exponentials = exponential of sum

$$\exp\left(g(x)\right)\exp\left(f(y)\right) \equiv \exp\left(g(x) + f(y)\right)$$

Useful properties of the log function

$\log(1+i) \approx i$ for small i (useful for gross and net interest rates)

- To see this, take a linear approximation of $\log(1+i)$ around $i = 0$
  - Define $f(i) \equiv \log(1+i)$
  - Then $f(i) \approx \log(1+0) + \frac{1}{1+0}(i-0) = i$
- Note we used $\log(1) = 0$ and $\frac{d}{dx}\log(f(x)) = \frac{f'(x)}{f(x)}$

Difference in logs $\approx$ percentage difference (for small differences)

- To see this note that (recall earlier results)

$$\log(x) - \log(y) = \log\left(\frac{x}{y}\right) = \log\left(1 + \frac{x-y}{y}\right) \approx \frac{x-y}{y}$$

- So today's log minus yesterday's $\approx$ the percentage *growth rate*
- Percent changes are 'unit free' (talk about GDP growth in % not \$)

Log linearizations

Log linearization $\Rightarrow$ approximating a function where the slopes and deviations are taken with respect to the *logs of the variables*, rather than the variables themselves

- Often analytically convenient and more natural to think in terms of percent deviations
- For small changes, log deviations are approximately percent deviations

Log linearizations

For a simple way of obtaining a log-linearization I prefer the following:

- Let $x \equiv \log(X)$ and $y \equiv \log(Y)$ (lower case for logs)
- Suppose you're asked to log-linearize $f(X, Y)$ around $(\bar{X}, \bar{Y})$
- Go through $f(X, Y)$ replacing X with $\exp(x)$ and Y with $\exp(y)$
- This effectively defines a new function $\tilde{f}$ such that $\tilde{f}(x, y) \equiv f(X, Y)$
- Then linearize $\tilde{f}$ in terms of x and y around $(\bar{x}, \bar{y})$ where $\bar{x} \equiv \log(\bar{X})$ and $\bar{y} \equiv \log(\bar{Y})$

Log linearizations

Let us go back to our previous example where $f(X, Y) = X^2 Y^3$

- Define a new function in terms of the logs (note the use of $e^{2x} e^{3y} = e^{2x+3y}$)

$$\tilde{f}(x, y) = \exp(2x + 3y)$$

- Linearize $\tilde{f}$ around $(\bar{x}, \bar{y})$

$$\tilde{f}(x, y) \approx \tilde{f}(\bar{x}, \bar{y}) + 2\exp(2\bar{x} + 3\bar{y})(x - \bar{x}) + 3\exp(2\bar{x} + 3\bar{y})(y - \bar{y})$$

We may not be interested in talking in terms of deviations but simply in obtaining expressions in terms of x and y

- Then all the terms involving $\bar{x}$ and $\bar{y}$ will be coefficients on x and y and/or constants

Log linearizations - worked example

Consider a more elaborate example

- The trick is to stay calm and convert the language of the question into the framework discussed above

See Galí p. 21 and p. 44 (equations (8), (10) and (51))

- We have the 'Bond pricing Euler equation' (equation (8))

$$Q_t = \beta E_t \left[\left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left(\frac{Z_{t+1}}{Z_t} \right) \left(\frac{P_t}{P_{t+1}} \right) \right]$$

- He says to use a log-linearization to show this implies equation (10)

$$c_t = E_t[c_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - \rho - (1 - \rho_z)z_t)$$

Log linearizations - worked example

We can rewrite (8) as

$$\begin{aligned}1 &= E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\left(\frac{P_t}{P_{t+1}}\right)Q_t^{-1}\right]\\ &\equiv E_t\left[\beta G_{C,t+1}^{-\sigma}G_{Z,t+1}\Pi_{t+1}^{-1}\mathcal{I}_t\right]\end{aligned}$$

- Consider the expression in the expectation: $\beta G_{C,t+1}^{-\sigma}G_{Z,t+1}\Pi_{t+1}^{-1}\mathcal{I}_t$
- Think of $f(G_{C,t+1}, G_{Z,t+1}, \Pi_{t+1}, \mathcal{I}_t) \equiv \beta G_{C,t+1}^{-\sigma}G_{Z,t+1}\Pi_{t+1}^{-1}\mathcal{I}_t$
- $G_{C,t+1}$, $G_{Z,t+1}$, Π_{t+1} and $\mathcal{I}_t$ are like X and Y in our earlier examples
- Now do a log linearization...

Log linearizations - worked example

Define

$$\tilde{f}(g_{c,t+1}, g_{z,t+1}, \pi_{t+1}, i_t) \equiv \exp\left(-\rho - \sigma g_{c,t+1} + g_{z,t+1} - \pi_{t+1} + i_t\right)$$

where lower case denotes logs and $\rho \equiv -\log(\beta)$

What are the '$\bar{x}$ variables'?

- We are told that inflation and growth are constant (at $\bar{\pi}$ and $\bar{g}_c$)
- In a steady state / perfect foresight situation, $\bar{g}_z = 0$ (see text)
- In perfect foresight situation with constant inflation and growth, we are told that $\bar{i} = \rho + \bar{\pi} + \sigma \bar{g}_c$

What is $\tilde{f}(\bar{g}_c, \bar{g}_z, \bar{\pi}, \bar{i})$?

- Given what we've been told: $\exp\left(-\rho - \sigma \bar{g}_c + \bar{g}_z - \bar{\pi} + \bar{i}\right) = e^0 = 1$

Log linearizations - worked example

$$\tilde{f}(g_{c,t+1}, g_{z,t+1}, \pi_{t+1}, i_t) \equiv \exp\left(-\rho - \sigma g_{c,t+1} + g_{z,t+1} - \pi_{t+1} + i_t\right)$$

What are the 'first order' terms in the linearization of $\tilde{f}$?

- Recall rule for differentiating exponentials (applies to partial derivatives too)

$$\frac{d}{dx}e^{g(x)} = g'(x)e^{g(x)}$$

- For example, the term corresponding to $g_{c,t+1}$

$$-\sigma \underbrace{\exp\left(-\rho - \sigma\bar{g}_c + \bar{g}_z - \bar{\pi} + \bar{i}\right)}_{e^0=1}(g_{c,t+1} - \bar{g}_c) = -\sigma(g_{c,t+1} - \bar{g}_c)$$

- For example, the term corresponding to π_{t+1}

$$-1 \times \underbrace{\exp\left(-\rho - \sigma\bar{g}_c + \bar{g}_z - \bar{\pi} + \bar{i}\right)}_{e^0=1}(\pi_{t+1} - \bar{\pi}) = -(\pi_{t+1} - \bar{\pi})$$

Log linearizations - worked example

Thus the linearization of $\tilde{f}$ (equivalent to log-linearization of f) is

$$\begin{aligned}\tilde{f}(g_{c,t+1}, g_{z,t+1}, \pi_{t+1}, i_t) &\approx 1 - \sigma(g_{c,t+1} - \bar{g}_c) - (\pi_{t+1} - \bar{\pi}) \\ &\quad + (g_{z,t+1} - 0) + (i_t - \bar{i}) \\ &= 1 + i_t - \sigma g_{c,t+1} - \pi_{t+1} - \rho + g_{z,t+1}\end{aligned}$$

Recalling that we were attempting to approximate

$$1 = E_t\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\sigma}\left(\frac{Z_{t+1}}{Z_t}\right)\left(\frac{P_t}{P_{t+1}}\right)Q_t^{-1}\right]$$

we then have

$$0 = E_t[i_t - \sigma g_{c,t+1} - \pi_{t+1} - \rho + g_{z,t+1}]$$

Log linearizations - worked example

We then rewrite and rearrange

$$0 = E_t[i_t - \sigma g_{c,t+1} - \pi_{t+1} - \rho + g_{z,t+1}]$$

to obtain

$$c_t = E_t[c_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - \rho - (1 - \rho_z)z_t)$$

where we have used...

- ...the fact (will be explained in class) that $E_t[g_{z,t+1}] = (\rho_z - 1)z_t$
- ...by properties of logs, $g_{c,t+1} \equiv \log\left(\frac{C_{t+1}}{C_t}\right) \equiv \log(C_{t+1}) - \log(C_t)$
- ...variables dated t and constants are known at t and thus the E_t (expectation) can be dropped