1. Problems to try on your own - answers posted

1.1. Autoregressive processes

In the models we consider, there are random 'shock' or 'driving' processes that are exogenous to the model.[1] In our case these shocks (technology, time preference and monetary policy) will constitute the 'state' of the economy in the sense that all the endogenous variables (consumption, output, wages, interest rates) will in equilibrium be expressible as functions of these three variables (or subsets thereof, depending on the model). All the shocks we consider, when logged, follow an autoregressive process of order 1 or, for short, an $AR(1)$. Consider the technology shock A_t from the production function $Y_t = A_t N_t^{1-\alpha}$. When expressed in logs it follows this process[2]

$$\begin{aligned} a_t &= \rho_a a_{t-1} + \varepsilon_{a,t} \\ \varepsilon_{a,t} &\overset{iid}{\sim} N(0, \sigma_a^2) \\ a_t &\equiv \log(A_t) \end{aligned} \tag{1}$$

The random variable, $\varepsilon_{a,t}$ (what I will often refer to as an 'innovation'), is a Gaussian or 'Normal' variable with zero mean ($E_t[\varepsilon_{a,t+1}] = 0$) and variance, σ_a^2 ($E_t[(\varepsilon_{a,t+1} - 0)^2] = \sigma_a^2$). It is also independently and identically distributed (iid) which means that there is no dependence between its draws in different periods or between its draw and any other variables and the distribution from which it is drawn ($N(0, \sigma_a^2)$) is constant over time. The parameter ρ_a will be referred to as a 'persistence' parameter. We will always (in this course) consider cases where $|\rho_a| \in (0, 1)$ - in the language of stochastic processes, this ensures that it is a 'stationary' process.

- Show that[3]

$$a_t = \sum_{j=0}^{J-1} \rho_a^j \varepsilon_{a,t-j} + \rho_a^J a_{t-J} \tag{2}$$

Suppose data started in $t = 0$, so that a_0 is just given to us, then we just set $J = t$ in the above expression. If there is no explicit starting point then we can use the assumption on $|\rho_a| \in (0, 1)$ to

[1] You can brush up on random variables here, on Normal variables here and on expected value (or 'the mean') here.

[2] Note that is convenient to model it this way as it means that A_t, while random, will always be positive (as makes sense for a technology term that multiplies - some function of - labor to produce non-negative output). Frequently in economics or finance we use the exponential of a random variable to obtain a transformed random variable that is positive. In fact, exponentials have other nice properties when working with Normal distributions.

[3] HINT: Use equation (1) but for earlier periods (a_{t-1}, a_{t-2} etc.) to repeatedly replace the lagged values of a_t on the right hand side of equation (1).

state

$$a_t = \sum_{j=0}^{\infty} \rho_a^j \varepsilon_{a,t-j}$$

since $\rho_a^J a_{t-J} \to 0$ as $J \to \infty$. Intuitively, if the effects of shocks dies of to zero in the limit - and given that our shocks are well behaved - we can ignore the last term in expression (2) you just derived because it gets arbitrarily small.

- What is the effect of an innovation j periods ago on a_t (i.e. the effect of $\varepsilon_{a,t-j}$)?
- What is the effect of an innovation in t on a_{t+1}? On a_{t+2}? On a_{t+j}?

Any effect an innovation in t has on future values of a_t, say a_{t+j}, may be flooded by the effects of future innovations in later periods before period $t + j$. But it is still useful to talk about the effect the innovation has as this nevertheless does *contribute* to a_{t+j} (look back at the expression you derived above - a_t is made up of a weighted sum of all current and previous innovations, with those weights declining as the innovation period recedes into the distant past). Given the process we are considering and the iid assumptions made on $\varepsilon_{a,t}$, an innovation today does affect the expected value - from the perspective of today - of future values of the technology shock.[4]

- What is the expected value of a_{t+1} given information available in t (i.e. given you know a_t)?
- What is the expected value of a_{t+2} given information available in t
- What is the expected value of a_{t+j} given information available in t
- What is the expected value of $\Delta a_{t+1} \equiv a_{t+1} - a_t$ given information available in t
- How does today's (t) innovation affect your expectation of a_{t+j} relative to the expectation you held in $t - 1$ before you knew $\varepsilon_{a,t}$?

In the models we consider we will often express the equilibrium values of endogenous variables as functions of a_t (and other shocks). Suppose a variable s_t is expressed

$$s_t = \psi_0 + \psi_1 a_t$$

- What is s_{t+1} in terms of a_{t+1}?

[4] To answer questions involving expectations below, recall that the expectations operator is linear (in particular, that means $E_t[X + Y] = E_t[X] + E_t[Y]$), the expectation of a constant (or something already known when the expectation is being formed) is the constant itself and the expectation of a scalar constant times a random variable is the scalar constant times the expectation of the random variable.

- What is s_{t+1} in terms of a_t and $\varepsilon_{a,t+1}$?
- What is the expected value of s_{t+1} given information available in t (i.e. given you know a_t)?
- What is the expected value of $\Delta s_{t+1} \equiv s_{t+1} - s_t$ given information available in t

Answers

By repeatedly lagging and substituting the equation (1) we get (in what follows I just write ρ and ε_t instead of ρ_a and $\varepsilon_{a,t}$

$$\begin{aligned} a_t &= \rho a_{t-1} + \varepsilon_t \\ &= \rho(\rho a_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\ &= \rho(\rho(\rho a_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1}) + \varepsilon_t \\ &\dots \sum_{j=0}^{J-1} \rho^j \varepsilon_{t-j} + \rho^J a_{t-J} \end{aligned}$$

Since we have $\rho^J a_{t-J} \to 0$ as $J \to \infty$ we have $a_t = \sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j}$ so that the effect of ε_{t-j} on a_t is $\rho^j \varepsilon_{t-j}$. So if $\varepsilon_{t-j} = 1.7$ the contribution is $1.7\rho^j$. With similar logic, the effects of an innovation in t on a_{t+1}, a_{t+2} and a_{t+j} are $\rho\varepsilon_t$, $\rho^2\varepsilon_t$ and $\rho^j\varepsilon_t$, respectively.

The expected value of a_{t+1} given information in t is

$$E_t[a_{t+1}] = E_t[\rho a_t + \varepsilon_{t+1}] = E_t[\rho a_t] + E_t[\varepsilon_{t+1}] = \rho a_t + 0 = \rho a_t$$

The expected value of a_{t+2} given information in t is

$$E_t[a_{t+2}] = E_t[\rho a_{t+1} + \varepsilon_{t+2}] = E_t[\rho a_{t+1}] + E_t[\varepsilon_{t+2}] = \rho E_t[a_{t+1}] + 0 = \rho^2 a_t$$

The expected value of a_{t+2} given information in t is, by induction, $\rho^j a_t$ or, explicitly, by using equation (2) applied to a_{t+j}

$$E_t[a_{t+j}] = E_t[\sum_{k=0}^{j-1} \rho^k \varepsilon_{t+j-k} + \rho^j a_t] = \rho^j a_t + \sum_{k=0}^{j-1} \rho^k E_t[\varepsilon_{t+j-k}] = \rho^j a_t$$

The expected change in technology is given by

$$E_t[\Delta a_{t+1} \equiv E_t[a_{t+1} - a_t] = \rho a_t - a_t = (\rho - 1)a_t$$

or you could equivalently think of this as

$$E_t[\Delta a_{t+1} \equiv E_t[a_{t+1} - a_t] == \equiv E_t[(\rho - 1)a_t + \varepsilon_{t+1}] = (\rho - 1)a_t$$

Suppose you receive news (in the form of ε_t) between period $t-1$ and t. How would that change your expectation of a_{t+j}? Let us compare the following two expectations (look carefully at the first one and make sure you understand)

$$E_{t-1}[a_{t+j}] = \rho^{j+1} a_{t-1}$$
$$E_t[a_{t+j}] = \rho^j a_t$$

But the second can be re-expressed as $\rho^{j+1} a_{t-1} + \rho^j \varepsilon_t$ and thus

$$E_t[a_{t+j}] - E_{t-1}[a_{t+j}] = \rho^j \varepsilon_t$$

Sometimes we use the notation $\Delta E_t[a_{t+j}] \equiv E_t[a_{t+j}] - E_{t-1}[a_{t+j}]$ to denote the effect of news but this notation can be a bit confusing. The Δ refers to change in the expectation, not in a_{t+j}.

Finally, turning to variable s_t we have in terms of a_{t+1} or in terms of a_t and ε_{t+1}

$$\begin{aligned} s_{t+1} &= \Psi_0 + \Psi_1 a_{t+1} \\ s_{t+1} &= \Psi_0 + \rho \Psi_1 a_t + \Psi_1 \varepsilon_{t+1} \end{aligned}$$

The expected value of s_{t+1} given t information is

$$E_t[s_{t+1}] = E_t[\psi_0 + \psi_1 a_{t+1}] = \psi_0 + \psi_1 E_t[a_{t+1}] = \psi_0 + \psi_1 \rho a_t$$

And the expected change in s_{t+1} is

$$E_t[\Delta s_{t+1}] = E_t[\psi_0 + \psi_1 a_{t+1} - \psi_0 - \psi_1 a_t] = \psi_1 E_t[\Delta a_{t+1}] = \psi_1(\rho - 1)a_t$$

2. Problems to try on your own - Emil goes through in class and answers posted

2.1. Pricing with market power

In the NK model each good $i \in [0, 1]$ is produced by an individual firm in a framework that we call 'monopolistic competition'. Without delving into the details of monopolistic competition (see various lecture notes available online) what is important for our purposes is that the firms have some pricing power - they can vary their price and thereby induce changes in demand for their good. People sometimes refer to this as 'facing a downward sloping demand curve'.[5] If the firm wants to sell more, it must cut its price and if it wants to sell less, it must raise its price. Ultimately, then, their profit maximizing decision about the scale of operation (and implicitly employment etc.) comes down to choosing a price. They are price setters, not price takers.

In monopolistic competition, people normally have in mind a situation in which each monopolist produces a particular good that is somewhat differentiated but also somehow similar to goods produced by the other monopolists. The fact that the goods are different but somewhat substitutable means that raising the price may drive people towards the other goods. For example, a firm may have a monopoly in producing soft drink 'x' but if they raise the price, that partly leads people to substitute towards soft drink 'y' (a similar, if differentiated, good).[6]

In the NK model the particular structure of household preferences where the consumer gets utility from a Dixit-Stiglitz bundle of different consumption goods gives rise to a degree of substitutability across goods, which is reflected in the demand curves we discussed in class for each of the different goods. These curves show that demand is decreasing in the goods' relative price where the steepness of that decrease is controlled by the parameter, ε, which captures the cross price elasticity of substitution. In this homework example, we strip away all the NK background and consider a related but much simpler pricing problem for a single firm.

Let us assume that a firm faces a demand curve of the following form

$$Y_t = P_t^{-\varepsilon}$$

[5] This is in contrast to the perfect competition case where firms take prices as given and if their price - somehow - ever deviated from that price, they would lose all demand or absorb the whole market i.e. they effectively face a 'horizontal' demand curve at whatever the competitive market price is.

[6] An important element of 'competition' comes from firms entering or exiting the industry until expected profits from entry are zero - but we ignore this industrial organization stuff...

and that they have access to a production technology

$$Y_t = A_t N_t^{1-\alpha}$$

- The production function implies a function $\mathcal{N}$ that gives the amount of labor required to produce a certain amount of output (assuming A_t is given), i.e. $N_t = \mathcal{N}(Y_t)$. Derive that function.
- Using the demand curve, show how demand changes for a marginal change in price (i.e. $\frac{dY_t}{dP_t}$).
- What is the price elasticity of demand (i.e. $-\frac{dY_t}{dP_t}\frac{P_t}{Y_t}$)?
- Noting that total revenue is quantity times price, how does it change with a marginal change in price?
- Noting that total cost is wage times hours, how does it change with a marginal change in price?
- At an optimum, the change in profits from a marginal change in price should be zero - use this and your previous results to show the markup of price over marginal cost (i.e. the change in costs from a marginal change in output) at the optimum.[7]
- Briefly comment on how this differs conceptually from the price-taking perfectly competitive case?

Answers

If we rearrange the production function we can express N_t as a function of output, given technology

$$N_t = \mathcal{N}(Y_t) \equiv \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}}$$

The change in demand for a marginal change in price is

$$\frac{dY_t}{dP_t} = -\varepsilon P_t^{-\varepsilon-1}$$

which can be usefully re-expressed as (using the form of the demand curve)

$$\frac{dY_t}{dP_t} = -\varepsilon \frac{Y_t}{P_t}$$

[7]Hint: It should be $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon-1}$

The price elasticity of demand is thus

$$-\frac{dY_t}{dP_t}\frac{P_t}{Y_t} = \varepsilon$$

so the higher is $\varepsilon > 0$ the less pricing power the firm has (if, as we shall see will be the case, it tries to mark up prices, it very rapidly will lose demand).[8]

The change in total revenue associated with a marginal change in price is (dropping the time subscript now - not important)

$$\begin{aligned}
\frac{dR}{dP} &= \frac{\partial R}{\partial P} + \frac{\partial R}{\partial Y}\frac{dY}{dP} \\
&= Y + P(-\varepsilon P^{-\varepsilon-1}) \\
&= Y - \varepsilon P^{-\varepsilon} \\
&= (1-\varepsilon)Y
\end{aligned}$$

where we used $R \equiv PY$ and our earlier result on the response of demand to a change in price. The fact that firms face a downward sloping demand curve and are choosing prices makes this very different from the price taking behavior of a perfectly competitive model.

The change in total cost for a marginal change in price is given by

$$\begin{aligned}
\frac{dC}{dP} &= \overbrace{W \times \frac{d}{dY}\mathcal{N}(Y)}^{\text{Marginal cost } = \frac{dC}{dY}} \times \frac{dY}{dP} \\
&= \Psi\frac{dY}{dP} \\
&= -\varepsilon\frac{\Psi}{P}Y
\end{aligned}$$

where we denote nominal marginal cost with Ψ. Now, since optimality requires marginal cost = marginal revenue we equate $\frac{dR}{dP}$ and $\frac{dC}{dP}$ to obtain

$$-\varepsilon\frac{\Psi}{P}Y = (1-\varepsilon)Y$$

[8]Briefly referring back to the NK model in class, the parameter ε there operates in a similar way. While the consumer's Dixit-Stigltiz preferences (which are symmetric in all the goods i) mean that all else equal they prefer to consume the same amount of all the goods, if they face different prices across the goods, they are much more willing if ε is high to substitute away from (towards) the relatively expensive (cheap) goods, hence ε shows up in the demand curve derived in the NK model for each good i, which looks a lot like the demand curve in our simple example.

or

$$P = \mathcal{M}\Psi$$

where $\mathcal{M} \equiv \frac{\varepsilon}{\varepsilon-1}$ is the markup of price over nominal marginal cost. Note that this means that, equivalently, we could say that the markup is equal to *the inverse of* real marginal cost ($\Psi^{Real} \equiv \frac{\Psi}{P}$). This expression for the markup is what we will derive also in the case of the monopolistically competitive firms in the NK model and is the frictionless (flexible price) value of the markup that will prevail in steady state. Note that as pricing power deteriorates ($\varepsilon \to \infty$) the markup vanishes ($\mathcal{M} \to 1$) and price is equal to marginal cost.

If firms want to produce more and choose a higher level of output, this is equivalent to having to choose a (lower) price that induces the desired level of output - they pick a point further down the demand curve they face. We could have re-expressed this problem explicitly in choosing output rather than price and that would have perhaps made it more explicit that to sell additional output there needs to be a price response - whereas under perfect competition the change in total revenue from changing quantity simply is the prevailing (and unaffected by the quantity decision) price times the change in quantity. So setting marginal cost = marginal revenue is equivalent to setting marginal cost = price. But in our situation, because price is changing, marginal revenue is less than the price - the firm doesn't only end up reducing the price it charges for the marginal additional items it sells, but on all the other (infra-marginal) items it sells. As such, they do not expand output as much as if they took the price as given. In figure 1, we show this diagramatically (this is very rough - and doesn't correspond to the exact specification in the question - it is qualitative). Ultimately, output is pinned down at the point of marginal revenue = marginal cost and the price charged will be the value on the demand curve at this point and it will be a markup over marginal cost (and thus marginal revenue).

The fact that price is not equal to marginal cost indicates that there is an inefficiency. Effectively, the price reflects the marginal valuation of the output by the marginal consumer. Since the price is above marginal cost it means that at Y^* (the output level implied by the chosen price, P^*, given the demand curve) the 'consumer' values an extra marginal unit more than the cost to the firm of producing that marginal unit (the marginal cost), thus there are possible gains from trade that are not being exploited - the consumer would be willing to pay a price between $MC(Y^*)$ and $P(Y^*)$ for that good. The firm would be fine with that and so would the consumer - implying a Pareto improvement. This will be possible right up to the point, Y^{Eff} where we have price = marginal cost. Thus, since the price is 'too high' and the output 'too low' we face a 'deadweight loss' equal to the total loss of surplus (could be consumer or producer surplus) captured by the shaded (Harberger) triangle in the figure.

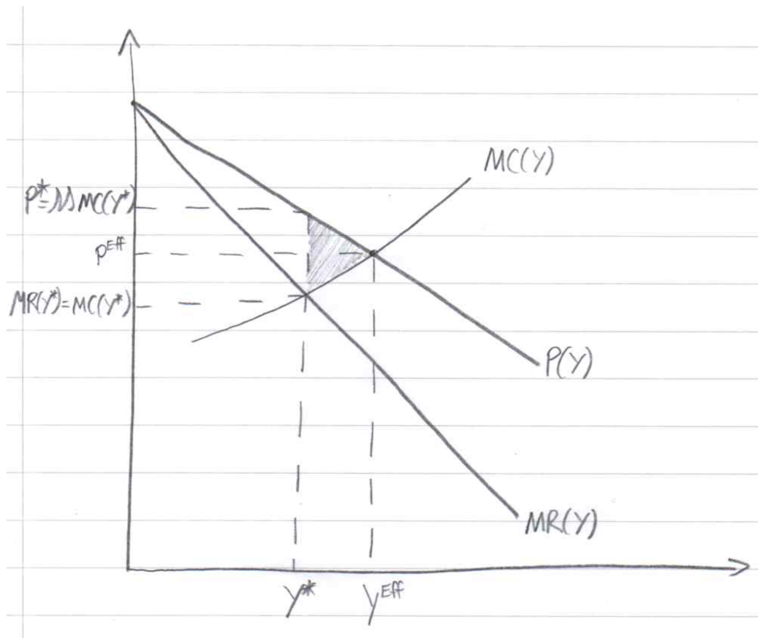

Figure 1: Price setting by a firm facing a downward sloping demand curve

2.2. Correcting an inefficiency with a Pigouvian subsidy

Consider the monopolistically competitive firm discussed above. The chosen output level (reflecting the chosen price) is such that price is a markup on marginal cost. This is inefficient. Effectively, the price reflects the marginal valuation of the output by the marginal consumer. Since the price is above marginal cost it means that at Y^* (the output level implied by the chosen price, P^*, given the demand curve) the 'consumer' values an extra marginal unit more than the cost to the firm of producing that marginal unit (the marginal cost), thus there are possible gains from trade that are not being exploited - the consumer would be willing to pay a price between $MC(Y^*)$ and $P(Y^*)$ for that good. The firm would be fine with that and so would the consumer - implying a Pareto improvement. This will be possible right up to the point, call it Y^{Eff}, where we have price = marginal cost.

For efficiency we would like to somehow induce the firm to produces $Y^{Eff} > Y^*$. Note that the firm produces to the point at which marginal cost = marginal revenue where the latter lies below the demand curve. One can offer a subsidy to the firm to change the 'private' marginal cost it faces. In particular, if the 'government' offers to pay fraction τ of the wage then for a given wage received by a worker, W_t, the firm only pays $(1-\tau)W_t$ - so from the firm's perspective this lowers their marginal cost. Call this private marginal cost $\widehat{MC}(Y_t) = \frac{(1-\tau)W_t}{MPN_t} = (1-\tau)MC(Y_t)$ (recall marginal cost is wage paid by firm divided by the marginal product of labor, at the firm's privately optimal choice).

What value should τ take to induce the firm to produce Y^{Eff}? HINT: The firm will still produce to the point where its *private* MC = MR

2.2.1. Answers

Since we want output to be produced to the point where $P_t = MC_t$ i.e. price equals the 'true' marginal cost and the firm will set $P_t = \mathcal{M}\widehat{MC}_t$ we require $1 - \tau = \mathcal{M}^{-1}$. Recalling the definition of $\mathcal{M}$, this implies $\tau = \varepsilon^{-1}$. Going back to diagramatic analysis, figure 2 illustrates the impact of confronting the firm with a lower effective marginal cost curve that induces it to produce Y^{Eff}. Given the appropriate choice of τ the firm sees its private marginal cost as equaling marginal revenue at a higher output level than before. At that output level the 'true' marginal cost is equal to price (and thus the marginal valuation of the consumer) and we have exhausted all gains from trade - we have a Pareto efficient situation (in the narrow partial equilibrium sense considered in this question).

As a final point note that by eliminating the deadweight loss (the Harberger triangle vanishes when $P = MC$) we have 'increased the size of the pie'. We have generated additional production that can be allocated so as to make everyone (firm and household) weakly better off and some agents strictly better off. In this case we achieved it by subsidizing the firm. Because the pie is bigger the government is able to fund this subsidy, which is often called a 'Pigouvian' susbidy after the eminent economist A. C. Pigou, who studied corrective taxes and subsidies to deal with market failures deriving from misaligned tradefoffs/implicit prices.

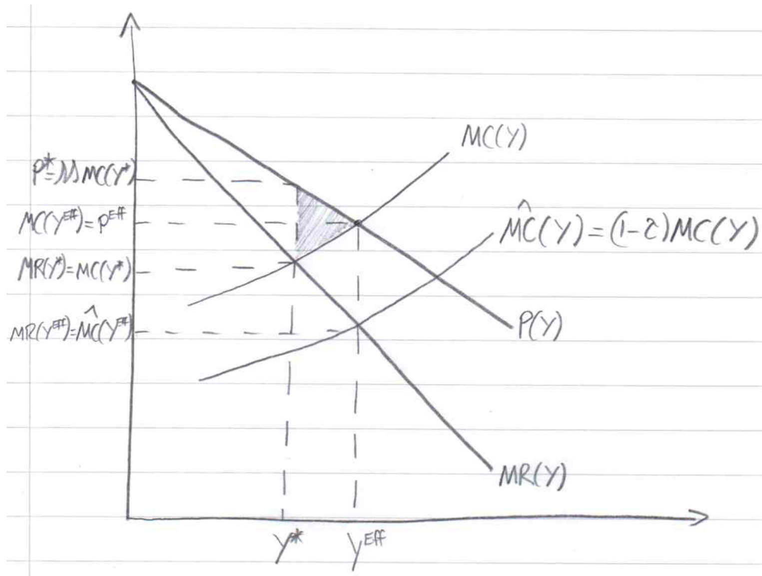

Figure 2: Price setting by a firm facing a downward sloping demand curve and a subsidized wage set to induce an efficient level of output