EC956 - New Keynesian Modeling
Problem set 1 - With solutions

1. Problems to try on your own - answers posted

1.1. Linearizations and log-linearizations - basic examples

On the course webpage there is a note on linearizations and log-linearizations that you should read before attempting questions in this section. I assume that you are comfortable with simple calculus (basic differentiation) and the use of logarithms (logs) and exponential functions, though there is a refresher of some of their most relevant properties in the note on the webpage.

- Linearize $5 + 3x^2$ around $\bar{x}$. Now linearize it around $x = 1$ and $x = 2$ (i.e. set $\bar{x} = 1$ and $\bar{x} = 2$). Roughly sketch a picture of the exact function and the approximating functions in the vicinity of the two specified approximation points. What is the value of the exact function and approximated functions at $x = 1$, $x = 2$ and $x = 3$? Comment on the approximation errors.
- Log linearize $15 - 3xy^2$ around $(\bar{x}, \bar{y})$. Now log-linearize it around $(x, y) = (2, 3)$. What is the value of the exact function at $(1, 4)$ and what is the value of the approximating function.

Answers

We have that $f'(x) = 6x$ so the linearization is

$$\begin{aligned} f(x) &\approx f(\bar{x}) + f'(\bar{x})(x - \bar{x}) \\ &= 5 + 3\bar{x}^2 + 6\bar{x}(x - \bar{x}) \\ &= \psi_0 + \psi_1 x \end{aligned}$$

where $\psi_0 \equiv 5 - 3\bar{x}^2$ and $\psi_1 \equiv 6\bar{x}$. Since the approximation depends on the $\bar{x}$ chosen we can make this explicit by writing the approximating function as

$$\hat{f}(x; \bar{x}) = \psi_0(\bar{x}) + \psi_1(\bar{x})x$$

If we take $\bar{x}$ to be 1 and 2 in turn, we have (see figure 1 for sketches)

$$
\begin{aligned}
\hat{f}(x;1) &= 2 + 6x \\
\hat{f}(x;2) &= -7 + 12x
\end{aligned}
$$

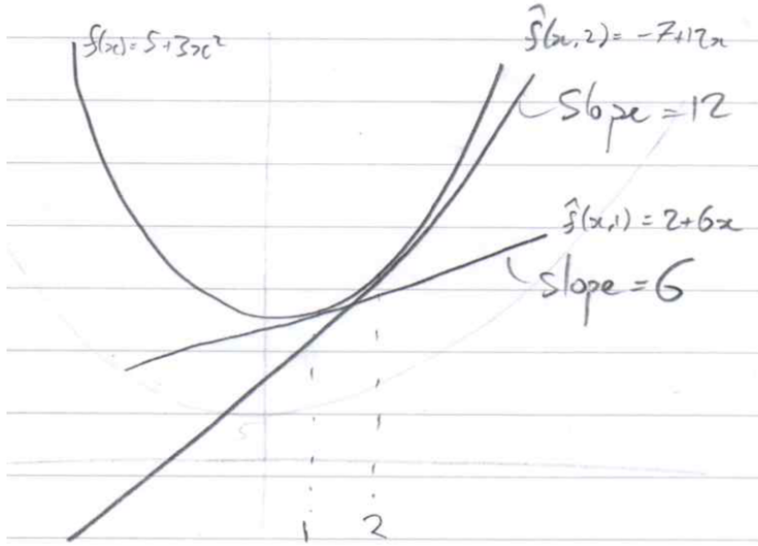

Figure 1: Sketch of exact and approximating functions

Table 1 shows that the choice of approximation point ($\bar{x}$) can affect the quality of approximation at various evaluation points (x). In *this* case we see that the closer we are to the approximation point, the smaller the errors.[1] In practice, the selection of an approximation point is very important as many economic models require some sort of approximation for us to be able to solve them. If the type of approximation used is as discussed in class (a 'local' solution method such as linearizing or using higher order Taylor approximations) then the selection of the approximation point will typically depend on the problem at hand.

Imagine a situation (outside the scope of our course) where maybe we are interested in approximating an agent's consumption function, where consumption depends on asset holdings, a_t. In this case we would be approximating $c_t = f(a_t)$ around some $\bar{a}$ for some complicated function, f. The model may imply that, over time, a_t evolves randomly but spends 'most of its time' somewhat near its average value $E[a_t]$ and only occasionally drops to extremely low levels, say, in the vicinity of $a_{low} << E[a_t]$.

[1] This doesn't hold for *arbitrary* points in the more general case (with more general functions, say) but the basic idea that the approximation point matters is what I am trying to convey.

$\mathbf{x}$	$\mathbf{f(x)}$	$\mathbf{\hat{f}_1(x)}$	**Error**	$\mathbf{\hat{f}_2(x)}$	**Error**
1	8	8	0	5	-3
2	17	14	-3	17	0
3	32	20	-12	29	-3

Table 1: Approximation quality and dependence on approximation point, $\bar{x}$

Assuming we want our approximation to perform well 'most of the time' we would likely want to choose an approximation point near the average value, rather than near a_{low}, whereas if we are particularly concerned with behavior at low wealth levels, we might actually set $\bar{a} = a_{low}$.
Now let us consider the second example, where $f(X, Y) = 15 - 3XY^2$. In this case we are required to do a log-linearization which, as discussed in the notes, can be obtained by re-expressing in terms of log versions of the variables and then doing a linearization in terms of these transformed variables. Thus, we define a function g such that $g(x, y) \equiv f(X, Y)$ (for $x \equiv \log(X)$ and $y \equiv \log(Y)$ and then linearize g. We have that

$$\begin{aligned} g(x,y) &\equiv 15 - 3e^{x+2y} \\ g_x(x,y) &= -3e^{x+2y} \\ y_y(x,y) &= -6e^{x+2y} \end{aligned}$$

The approximation is then given by

$$\begin{aligned} f(X,Y) &\equiv g(x,y) \approx g(\bar{x},\bar{y}) + g_x(\bar{x},\bar{y})(x-\bar{x}) + g_y(\bar{x},\bar{y})(y-\bar{y}) \\ &= 15 - 3e^{\bar{x}+2\bar{y}} - 3e^{\bar{x}+2\bar{y}}(x-\bar{x}) - 6e^{\bar{x}+2\bar{y}}(y-\bar{y}) \end{aligned}$$

If we take the approximation point for f to be $(\bar{X}, \bar{Y}) = (2, 3)$ then the approximation point for g is $(\bar{x}, \bar{y}) = (\log(2), \log(3))$ and we obtain

$$\hat{g}(x,y) \approx 117.08 - 54x - 108y$$

which implies that at $(X, Y) = (1, 4)$ (equivalently $(x, y) = (0, \log(4))$) we have the exact value of the function of -33 whereas the approximating value is -32.64.

1.2. Linearizations and log-linearizations - economics/financial example

Campbell and Shiller (1988) propose a very useful approximation to the return on an asset that enables the price of that asset to be expressed in terms of its future stream of returns and dividends. For a more extensive analysis (on which this is based) see the notes on Simon Gilchrist's website.

The realized return on an asset from t to $t+1$ is given by

$$R_{t+1} \equiv \frac{P_{t+1} + D_{t+1}}{P_t}$$

where P_t is *ex dividend* price of the asset in t and, D_t is the dividend paid in t. The return is made up of any dividend payment next period, plus the value of holding the asset next period, relative to the price paid today (i.e. capital gains). If we take logs of both sides we can express log returns as follows (you should understand these steps)

$$\begin{aligned} \log(R_{t+1}) &\equiv \log(P_{t+1} + D_{t+1}) - \log(P_t) \\ &\equiv \log(P_{t+1}(1 + \frac{D_{t+1}}{P_{t+1}})) - \log(P_t) \\ &\equiv p_{t+1} - p_t + \log(1 + \exp(d_{t+1} - p_{t+1})) \end{aligned} \tag{1}$$

Note that at this point we have not yet taken any approximations but, as I mention in the notes, it has been setup in a way that is amenable to doing a linearization in the (lower case $p_t \equiv \log(P_t)$ etc.) log forms of the variables, which is equivalent to a log-linearization in terms of the original versions of the variables.

- Linearize $\log(1 + \exp(d_{t+1} - p_{t+1}))$ around $(\bar{p}, \bar{d})$ and show that[2]

  $$\log(1 + \exp(d_{t+1} - p_{t+1})) \approx k + (1-\rho)(d_{t+1} - p_{t+1})$$

  where

  $$\begin{aligned} \rho &\equiv (1 + \exp(\bar{d} - \bar{p}))^{-1} \\ k &\equiv -\log(\rho) - (1-\rho)\log(\rho^{-1} - 1) \end{aligned}$$

- Use the above approximation, along with equation (1), to obtain

  $$p_t = \rho p_{t+1} + k + (1-\rho)d_{t+1} - r_{t+1}$$

  and, therefore,

  $$p_t - d_t = \rho(p_{t+1} - d_{t+1}) + k + \Delta d_{t+1} - r_{t+1}$$

[2] Really you just need to show $\log(1 + \exp(d_{t+1} - p_{t+1})) \approx \log(1 + e^{d-p}) + \frac{e^{d-p}}{1+e^{d-p}}(d_{t+1} - d) - \frac{e^{d-p}}{1+e^{d-p}}(p_{t+1} - p)$ and then try and rearrange terms appropriately. In fact, we should really be doing this linearization in terms of the (log) dividend-price ratio, $dp_{t+1} \equiv d_t - p_{t+1}$ as there may be no natural constant steady state value of dividends or prices, due to growth (and/or inflation, if these are expressed in nominal terms) but let us ignore that for now.

- Denoting $\delta_t \equiv p_t - d_t$, show that this implies (hint: keep substituting for δ_{t+1}, δ_{t+2},... on the RHS)

$$\delta_t = \frac{k}{1-\rho} + \sum_{j=0}^{\infty} \rho^j (\Delta d_{t+1+j} - r_{t+1+j}) \tag{2}$$

Though you don't need to know this for the course, what is useful about this relation is that (with error only arising from the linear approximation) it shows that variation in the price dividend ratio must be accompanied by variations in future dividend growth, or future returns, or both. This holds *ex post* (subject only to approximation error from the linearization) but, in addition, the relationship holds *ex ante*. That is, one can take expectations of each side to say (note that $E_t[p_t - d_t] = p_t - d_t$ as the price-dividend ratio in t is known at t):

$$\delta_t = \frac{k}{1-\rho} + \sum_{j=0}^{\infty} \rho^j E_t[\Delta d_{t+1+j} - r_{t+1+j}]$$

Answers

Let $f(p_{t+1}, d_{t+1}) \equiv \log(1 + e^{d_{t+1} - p_{t+1}})$ then we have

$$\begin{aligned} f_p(p_{t+1}, d_{t+1}) &= -\frac{e^{d_{t+1}-p_{t+1}}}{1+e^{d_{t+1}-p_{t+1}}} \\ f_d(p_{t+1}, d_{t+1}) &= \frac{e^{d_{t+1}-p_{t+1}}}{1+e^{d_{t+1}-p_{t+1}}} \end{aligned}$$

and we can construct a first order approximation as (let's just use p and d instead of $\bar{p}$ and $\bar{d}$ here)

$$\begin{aligned} f_d(p_{t+1}, d_{t+1}) &\approx f(p,d) + f_p(p,d)(p_{t+1} - p) + f_d(p,d)(d_{t+1} - d) \\ &= \log(1 + e^{d-p}) - \frac{e^{d-p}}{1+e^{d-p}}(p_{t+1} - p) + \frac{e^{d-p}}{1+e^{d-p}}(d_{t+1} - d) \qquad (3) \\ &= -\log(\rho) - \frac{e^{d-p}}{1+e^{d-p}}(d - p) + \frac{e^{d-p}}{1+e^{d-p}}(d_{t+1} - p_{t+1}) \qquad (4) \end{aligned}$$

where in going from equation (3) to equation (4) we used the definition of ρ. Now, we note that

$$1 - \rho \equiv 1 - \frac{1}{1+e^{d-p}} = \frac{e^{d-p}}{1+e^{d-p}}$$

so we can rewrite equation (4) as

$$f_d(p_{t+1}, d_{t+1}) = -\log(\rho) - (1-\rho)(d-p) + (1-\rho)(d_{t+1} - p_{t+1}) \tag{5}$$

Further, we note that

$$\rho^{-1} - 1 = 1 + e^{d-p} - 1 = e^{d-p}$$

which implies $\log\left(\rho^{-1} - 1\right) = d - p$. Using this in equation (5) we obtain the desired expression. We then have

$$r_{t+1} \approx p_{t+1} - p_t + k + (1-\rho)(d_{t+1} - p_{t+1})$$

Or, rearranging (and dropping the $\approx$ symbol)...

$$\begin{aligned} p_t &= p_{t+1} + k + (1-\rho)(d_{t+1} - p_{t+1}) - r_{t+1} \\ &= \rho p_{t+1} + k + (1-\rho)d_{t+1} - r_{t+1} \end{aligned}$$

Then, subtracting d_t from each side (and using the notation $\Delta Z_t \equiv Z_t - Z_{t-1}$) we have

$$p_t - d_t = \rho(p_{t+1} - d_{t+1}) + k + \Delta d_{t+1} - r_{t+1} \tag{6}$$

Finally, letting $\delta_t \equiv p_t - d_t$ equation (6) implies

$$\begin{aligned} \delta_t &= \rho\delta_{t+1} + k + \Delta d_{t+1} - r_{t+1} \\ &= \rho(\rho\delta_{t+2} + k + \Delta d_{t+2} - r_{t+2}) + k + \Delta d_{t+1} - r_{t+1} \\ &= \rho(\rho(\rho\delta_{t+3} + k + \Delta d_{t+3} - r_{t+3}) + k + \Delta d_{t+2} - r_{t+2}) + k + \Delta d_{t+1} - r_{t+1} \\ &= k\sum_{j=0}^{J}\rho^j + \sum_{j=0}^{J}\rho^j(\Delta d_{t+1+j} - r_{t+1+j}) + \rho^{J+1}\delta_{t+1+J} \end{aligned}$$

Then if we let $J \to \infty$ and assume (a 'transversality condition') $\rho^{J+1}\delta_{t+1+J} \to 0$ then we obtain the desired result (noting that $\sum\limits_{j=0}^{\infty}\rho^j = \frac{1}{1-\rho}$)

$$\delta_t = \frac{k}{1-\rho} + \sum_{j=0}^{\infty}\rho^j(\Delta d_{t+1+j} - r_{t+1+j})$$

2. Problems to try on your own - Emil works through answers in class

2.1. L'Hopital's rule

L'Hopital's rule states (roughly) that, if...

- $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$
- $g'(x) \neq 0$

- $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists

then

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

We can use this result to explain why for $\sigma > 1$ we use the CRRA utility specification used in class, while for $\sigma = 1$ we 'use' log utility (where σ does not seem to appear in any meaningful way). In fact, log utiliy is a limiting case of CRRA as $\sigma \to 1$. Use L'Hopital's rule to show this. *Hint: rewrite* C_t *as* $e^{\log(C_t)}$ *in the CRRA utility function discussed in class.*

Answers

Think of the utility function as being a function of σ (for any given C). In fact, think of it being a ratio of functions, f and g where

$$\begin{aligned} f(\sigma) &\equiv C^{1-\sigma} - 1 \\ g(\sigma) &\equiv 1 - \sigma \end{aligned}$$

We see that both $f(\sigma)$ and $g(\sigma)$ go to zero as $\sigma \to 1$. Furthermore, $g'(\sigma) = -1$ which is non-zero for all values of σ. Thus, two of the three conditions for use of L'Hopital's rule hold here. Now consider the ratio of derivatives, $f'(\sigma)/g'(\sigma)$. Before we do this, it is useful to re-express f as follows

$$f(\sigma) = C^{1-\sigma} - 1 \equiv e^{(1-\sigma)\log(C)} - 1$$

Thus the derivative of f with respect to σ is (remember the notes on how to differentiate exponentials) as follows[3]

$$f'(\sigma) = -\log(C) e^{(1-\sigma)\log(C)}$$

Consequently, we have the ratio of derivatives

$$\frac{f'(\sigma)}{g'(\sigma)} = \log(C) e^{(1-\sigma)\log(C)}$$

The limit of this ratio as $\sigma \to 1$ exists and is equal to $\log(C)$ (since $e^0 = 1$) but by L'Hopital this means that the limit of our utility function is, in fact, a log utility function.

[3] The relevant rule here is

$$\frac{d}{dx} e^{f(x)} = f'(x) e^{f(x)}$$

2.2. Non-separable consumption-labor preferences

Do question 2.1 in Galí (first question in exercises at end of chapter 2). This entails deriving the equivalent of equations (7), (8) and then (10) in the text (based on the 2nd edition of the textbook). Note that Z_t is dropped from the preferences and the utility function is given by...

$$U(C_t, N_t) = \frac{[C_t\,(1-N_t)^v]^{1-\sigma} - 1}{1-\sigma}$$

Answers

Consider the utility function

$$C(C_t, N_t) = \frac{(C_t(1-N_t)^v)^{1-\sigma} - 1}{1-\sigma}$$

In this case the marginal utility derived from consumption and the marginal (dis)utility derived from hours spent working are dependent on hours and consumption, respectively. The preferences are not 'separable' as in the case considered in class. This question involves a fair amount of algebra but is basically the same as the long example at the end of the linearization notes (and in Galí p. 21 and p. 24). First we derive the marginal utilities

$$\begin{aligned}
U_c(C_t, N_t) &= C_t^{-\sigma}(1-N_t)^{(1-\sigma)v} \\
U_n(C_t, N_t) &= -vC_t^{-\sigma}(1-N_t)^{(1-\sigma)v}\frac{C_t}{1-N_t}
\end{aligned}$$

Thus the 'intratemporal optimality' condition will be (real wage = marginal rate of substitution)

$$W_t^R \equiv \frac{W_t}{P_t} = -\frac{U_n(C_t, N_t)}{U_c(C_t, N_t)} = v\frac{C_t}{1-N_t}$$

From now on, let W_t be the *real* wage, just to simplify notation (rather than use W_t^R). One approach (taken in class by Emil) is to first simply take logs of this condition (lower case variables mean logs)

$$\log(v) + c_t - \log(1-e^{n_t}) = w_t$$

and then simply approximate $\log(1-e^{n_t})$ as that is the only bit that isn't already linear in logs to begin with. Another (unnecessarily) more thorough approach (but which shows more working) is as follows. First, re-express in terms of log variables

$$ve^{c_t} = e^{w_t}(1-e^{n_t})$$

Then taking first order approximations of both sides (treat both sides as functions to be approximated - the LHS a function of c_t and the RHS a function of w_t and n_t) we have (where variables without a time subscripts are the approximation pint - we're dropping the $\bar{c}$ notation)

$$ve^c + ve^c(c_t - c) \approx e^w(1-e^n) + e^w(1-e^n)(w_t - w) - e^n e^w(n_t - n)$$

But we know that $ve^c = e^w(1-e^n)$ (since that's just the optimality condition evaluated at the approximation point) so we can drop the associated constants from the two sides, leaving

$$ve^c(c_t - c) \approx e^w(1-e^n)(w_t - w) - e^n e^w(n_t - n)$$

Again, we can use $ve^c = e^w(1-e^n)$ to divide both sides through by $e^w(1-e^n)$ leaving

$$c_t - c \approx w_t - w - \frac{e^n}{1-e^n}(n_t - n)$$

Define $\psi_1 \equiv \frac{e^n}{1-e^n}$ then we have

$$c_t + \psi_1 n_t + w - c - \psi_1 n = w_t$$

Then (to match up with Emil from class), define $\psi_0 \equiv w - c - \psi_1 n$ to obtain[4]

$$\psi_0 + c_t + \psi_1 n_t \approx w_t$$

Now we turn to the intertemporal optimality condition (the 'Euler equation') and what we will find is that because consumption and 'leisure' ($1 - n_t$ can be thought of as leisure) are not separable, the expected path of consumption, given interest rates, will be affected by expectations regarding the path of hours - as hours in t and in $t+1$ affect the marginal utility of consumption and thus the tradeoffs that the household considers desirable when deciding on savings and on evaluating risky assets. We being with

$$Q_t = E_t\left[\beta \frac{U_c(C_{t+1}, N_{t+1})}{U_c(C_t, N_t)} \frac{P_t}{P_{t+1}}\right]$$

or

$$1 = E_t\left[\beta \frac{U_c(C_{t+1}, N_{t+1})}{U_c(C_t, N_t)} I_t \Pi_{t+1}^{-1}\right] \tag{7}$$

where $I_t \equiv Q_t^{-1}$ and $\Pi_t \equiv \frac{P_t}{P_{t-1}}$. Now, in a perfect foresight zero inflation steady state (with no

[4]Again, using the optimality condition evaluated at the approximation point we can show $\psi_0 = \log(v) - \log(1-e^n) - \psi_1 n$

growth - which we assume here) we have

$$\frac{C_{t+1}}{C_t} = 1 \tag{8}$$
$$\Pi_{t+1} = 1 \tag{9}$$
$$N_t = \bar{N} \tag{10}$$
$$N_{t+1} = \bar{N} \tag{11}$$
$$I_t = \bar{I} \tag{12}$$
$$\tag{13}$$

and, clearly, we also have, therefore

$$\frac{1-N_{t+1}}{1-N_t} = \frac{1-\bar{N}}{1-\bar{N}} = 1$$

Now, let us consider the expression inside the expectation in equation (7). Using the utility function from the question we have

$$f(G_{c,t+1}, N_t, N_{t+1}, \Pi_{t+1}, I_t) \equiv \beta G_{c,t+1}^{-\sigma} \left(\frac{1-N_{t+1}}{1-N_t}\right)^{(1-\sigma)v} I_t \Pi_{t+1}^{-1}$$

where $G_{c,t+1} \equiv \frac{C_{t+1}}{C_t}$ and, thus, $\bar{G}_c = 1$. Note also that $\bar{I} = \beta^{-1}$.[5] Since we want a log-linearization we re-express in terms of log (lower-case) versions of the variables...

$$\exp\left(-\rho - \sigma g_{c,t+1}(1-\sigma)v\left(\log\left(1-e^{n_{t+1}}\right) - \log\left(1-e^{n_{t+1}}\right)\right) + i_t - \pi_{t+1}\right)$$

where we recall that $\rho \equiv -\log(\beta)$. Note that in steady state this expression is equal to $e^0 = 1$ (since $\bar{i} = \rho$ and all the other steady states of the variables (in logs) are zero or, in the case of the hours variables, cancel out. Thus a linear approximation yields

$$1 - \sigma(g_{c,t+1} - 0) - (1-\sigma)v\frac{e^{\bar{n}}}{1-e^{\bar{n}}}((n_{t+1} - \bar{n}) - (n_t - \bar{n})) + (i_t - \rho) - (\pi_{t+1} - 0)$$

or, without making the steady state value explicit and/or canceling

$$1 - \sigma g_{c,t+1} - (1-\sigma)v\frac{e^{\bar{n}}}{1-e^{\bar{n}}}(n_{t+1} - n_t) + (i_t - \pi_{t+1} - \rho)$$

[5] We can see that $\bar{I} = \beta^{-1}$ by imposing all the other steady state values in the Euler equation and noting that in a perfect foresight case expected values are equal to realized, so we have $1 = \beta\bar{I}$.

Inserting this approximation into equation (7) we obtain (defining $g_{n,t} \equiv n_t - n_{t-1}$ and recalling the definition of r_t)

$$0 \approx E_t \left[-\sigma g_{c,t+1} - (1-\sigma) v \frac{e^{\bar{n}}}{1-e^{\bar{n}}} g_{n,t+1} + r_t - \rho \right]$$

Then we can rearrange to obtain

$$c_t = E_t \left[c_{t+1} \right] - \frac{1}{\sigma}(r_t - \rho) - \left(1 - \frac{1}{\sigma} \right) v \frac{e^{\bar{n}}}{1-e^{\bar{n}}} E_t[g_{n,t+1}]$$

Because the amount of leisure (or hours worked) affects the marginal utility of consumption in any given period (unlike in the separable case discussed in the lectures / main text of the chapter) expectations about the path of hours influences the optimal allocation of consumption over time. If $\sigma > 1$ (so the EIS is 'low') then higher expected hours growth acts in the same direction as a higher r_t (all else equal), implying lower current consumption relative to tomorrow's.