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Author(s): Rhys Bidder and Ian Dew-Becker

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Long-Run Risk Is the Worst-Case Scenario[†]

By Rhys Bidder and Ian Dew-Becker*

We study an investor who is unsure of the dynamics of the economy. Not only are parameters unknown, but the investor does not even know what order model to estimate. She estimates her consumption process nonparametrically—allowing potentially infinite-order dynamics—and prices assets using a pessimistic model that minimizes lifetime utility subject to a constraint on statistical plausibility. The equilibrium is exactly solvable and the pricing model always includes long-run risks. With risk aversion of 4.7, the model matches major facts about asset prices, consumption, and dividends. The paper provides a novel link between ambiguity aversion and nonparametric estimation. (JEL D11, D12, D81, G11, G12)

Economists do not agree on the dynamic properties of the economy. There has been a long debate in the finance literature over how risky consumption growth is in the long run (e.g., Bansal, Kiku, and Yaron 2012 and Beeler and Campbell 2012), and it is well known that long-run forecasting is econometrically difficult (Müller and Watson 2013). It is likely that the average investor is also unsure of the true model driving the world. This paper studies the behavior of such an investor.

With exactly solved results, we show that a model in which investors have Epstein-Zin preferences and uncertainty about consumption dynamics generates high and volatile risk premia, excess volatility in stock returns, a large degree of predictability in stock returns, low and stable interest rates, and an estimated elasticity of intertemporal substitution from interest rate regressions of zero as measured in Hall (1988) and Campbell and Mankiw (1989). Moreover, variation over time in risk or model uncertainty is not required to generate predictability in returns.

We argue that investors consider a set of models of the economy that is only weakly constrained. People face pervasive ambiguity: no one can say they know the exact specification to estimate when forecasting economic activity. So rather

than just allowing uncertainty about the parameters in a specific model, or putting positive probability on a handful of models, we treat investors as considering an infinite-dimensional space of autoregressive moving average (ARMA) specifications of unrestricted order. They therefore face a nonparametric problem in the sense that the number of parameters to be estimated is potentially infinite.[1]

Infinite-dimensional estimation problems are well known to be difficult or impossible to approach with standard likelihood-based methods.[2] But people must estimate *some* model. So, following the literature on nonparametric time series estimation, we assume that in a finite sample, they estimate a relatively small-scale model, which they view as an approximation to the truth. But the true model may be infinite-dimensional. So investors face a highly nonstandard estimation problem, and, as Sims (1972) shows, the consequences for welfare of using a misspecified model can be severe.

The uncertainty due to estimating consumption dynamics when the true model is unknown and potentially of infinite order is clearly very different from that due to standard sources of risk, such as future innovations to the consumption process; in many cases a valid posterior distribution *cannot* be placed on the space of models. It is simply not always possible to be a Bayesian (in particular, when the model order is unknown). It is reasonable to think that people view model uncertainty fundamentally differently from other sources of risk, in the sense of Knight (1921) and Ellsberg (1961).

There are multiple potential ways of modeling attitudes toward such ambiguity. We draw upon the insights of Gilboa and Schmeidler's (1989) work on choice under ambiguity in modeling people as making choices under a "worst-case" process for consumption that is chosen to minimize lifetime utility.[3] Such analysis is highly tractable, it is a plausible and parsimonious description of how people might approach ambiguity, and it yields results that have a natural economic interpretation because they point directly to the specific model that is most painful to agents.

Our headline theoretical result is that for an ambiguity-averse agent whose point estimate is that consumption growth is white noise, the worst-case model used for decision-making, chosen from the entire space of ARMA models, is an ARMA(1,1) with a highly persistent trend—literally the homoskedastic version of Bansal and Yaron's (2004) long-run risk model. More generally, whatever the investor's point estimate, the worst-case model always adds a long-run risk component to it.

The low-frequency fluctuations, that people in our model fear, die out at precisely the rate of time preference. In a sense, then, they are shocks that effectively last the rest of the investor's life. So a way of interpreting our results is that they say that what people fear most, and what makes them averse to investing in equities, is that growth rates or asset returns are going to be persistently lower over the rest of their

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[1] This is a typical definition of nonparametric estimation; see, e.g., Chen (2007).

[2] Diaconis and Freedman (1986) note that Doob's (1948) theorem on the consistency of Bayesian estimates only applies to finite-dimensional parameter spaces. In infinite-dimensional settings, Bayesian estimators need not be consistent for even apparently reasonable priors. Sims (1971, 1972) shows that it is generally impossible to create accurate confidence intervals in such settings. See also Chow and Grenander (1985); Faust (1999); and Hansen and Sargent (2007) (who note the links between Sims (1971); Diaconis and Freedman (1986); and robust control theory).

[3] Alternative models of ambiguity include Hansen and Sargent (2001); Epstein and Schneider (2003); Klibanoff, Marinacci, and Mukerji (2005); and Maccheroni, Marinacci, and Rustichini (2006).

lives than they have been on average in the past. Our specific formulation of model uncertainty allows us to formalize that intuition.

Our results are derived in the frequency domain, which allows strikingly clear conceptual and analytical insights. Two factors determine the behavior of the worst-case model at each frequency: estimation uncertainty and how utility is affected by fluctuations at that frequency. Growth under the worst-case model has larger fluctuations at frequencies about which there is more uncertainty or that are more painful. Quantitatively, we find that differences in estimation uncertainty across frequencies are relatively small. Instead, since people with Epstein-Zin preferences are highly averse to low-frequency fluctuations (for parameterizations such as ours where they prefer an early resolution of uncertainty), persistent shocks play the largest role in robust decision-making.

A criticism of the long-run risk model has always been that it depends on a process for consumption growth that is difficult to test for.[4] We turn that idea on its head and argue that it is the difficulty of testing for and rejecting long-run risk that actually makes it a sensible model for investors to focus on. If anything, our result is more extreme than that of Bansal and Yaron (2004): whereas they posit a consumption growth trend with shocks that have a half-life of 3 years, the endogenous worst-case model that we derive features trend shocks with a half-life of 70 years.

In a calibration of the model, we show that it explains a wide range of features of financial markets that have been previously viewed as puzzling. Similar to the intuition from Bansal and Yaron (2004), equities earn high average returns in our model because low-frequency fluctuations in consumption growth induce large movements in both marginal utility and equity prices. In our setup, though, long-run risk need not actually exist; it only needs to be plausible.

The results that we obtain on excess volatility, forecasting, and interest rate regressions all follow from the fact that the pricing model that our investor uses always involves more persistence than her point estimate (i.e., the model used by an econometrician with access to the same data sample as the investor). Since the pricing model has excess persistence, investors overextrapolate recent news relative to what the point estimate would imply. Following positive shocks, then, stock prices are relatively high and econometric forecasts of returns are low. We are thus able to generate predictability without any changes in risk or risk aversion over time. Significantly, we obtain this result in a model in which agents are rational, if uncertain. The lowercase of the "r" in rational here is important. Agents behave as if optimizing under a "worst-case" distribution that is different from the true data generating process, putting us outside the standard rational expectations paradigm. As discussed in Hansen and Sargent (2007), this is characteristic of models of ambiguity.

In generating all these results we have no more free parameters than other standard models of consumption and asset prices. We link the parameter that determines how the agent penalizes deviations from her point estimate for consumption dynamics directly to the coefficient of relative risk aversion. There is thus a single free parameter that determines risk preferences, and it corresponds to a coefficient

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[4] See Beeler and Campbell (2012) and Marakani (2009).

of relative risk aversion of only 4.7. We also take no extreme liberties with beliefs; the investor's pricing model is essentially impossible to distinguish from the true model in a 100-year sample. Using a correctly specified likelihood ratio test, the null hypothesis that the pricing model is true is rejected at the 5-percent level in, at most, only 6 percent of samples.

Our analysis directly builds on a number of important areas of research. First, the focus on a single worst-case outcome is closely related to Gilboa and Schmeidler's (1989) work on ambiguity aversion that provides an axiomatic basis for decision making under a worst-case distribution (though here we do not start from an axiomatic foundation). Second, we build on the analysis of generalized recursive preferences to allow for the consideration of multiple models, especially Hansen and Sargent (2010) and Ju and Miao (2012).[5] The work of Hansen and Sargent (2010) is perhaps most comparable to ours, in that they study an investor who puts positive probability on both a white noise model and a parameterized long-run risk model for consumption growth. The key difference here is that we consider a nonparametric setting in which the agent considers *all* ARMA models, instead of only two. The emergence of the long-run risk model as the one that she focuses on is entirely endogenous.[6] We also obtain analytic results that provide economic insights into precisely what model is most painful to agents, whereas Hansen and Sargent's (2010) analysis is numerically solved.

Since the worst-case model is more persistent than the point estimate, pricing behavior is similar to the extrapolation implied by the "natural expectations" studied by Fuster, Hebert, and Laibson (2011). Our results differ from theirs, though, in that we *always* obtain excess extrapolation, whereas in their setting it results from the interaction of suboptimal estimation on the part of investors with a specific data-generating process. Cecchetti, Lam, and Mark (2000) examine a setting in which investors use a rule-of-thumb estimation technique and are subject to random belief shocks to obtain excess volatility and predictability. In our analysis, there is no assumption that the estimation method is suboptimal, and excess volatility and predictability are natural consequences of the pessimistic model that agents choose. Our paper complements the literature on belief distortions and extrapolative expectations by deriving them as a natural response to model uncertainty.[7]

The remainder of the paper is organized as follows. Section I discusses the agent's estimation method. Section II describes the basic structure of the agent's preferences, and Section III then derives the worst-case model. We examine asset prices in general under the preferences in Section IV. Section V then discusses the calibration

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[5] See, e.g., Kreps and Porteus (1978); Weil (1989); Epstein and Zin (1991); Maccheroni, Marinacci, and Rustichini (2006); and Hansen and Sargent (2005), among many others. There is also a large recent literature in finance that specializes models of ambiguity aversion to answer particularly interesting economic questions, such as Liu, Pan, and Wang (2004) and Drechsler's (2013) work with tail risk and the work of Uppal and Wang (2003); Maenhout (2004); Sbuelz and Trojani (2008); and Routledge and Zin (2009) on portfolio choice. Recent papers on asset pricing with learning and ambiguity aversion include Veronesi (2000); Brennan and Xia (2001); Epstein and Schneider (2007); Cogley and Sargent (2008); Leippold, Trojani, and Vanini (2008); Ju and Miao (2012); and Collin-Dufresne, Johannes, and Lochstoer (2016).

[6] Bidder and Smith (2015) also develop a model in which the worst-case process of an agent with multiplier preferences also features extra persistence that arises from the interaction of ambiguity aversion and stochastic volatility.

[7] See also Barsky and De Long (1993); Abel (2002); Brandt, Zeng, and Zhang (2004); and Hirshleifer, Li, and Yu (2015).

and Section VI analyzes the quantitative implications of the model. Section VII concludes.

I. Measuring Model Plausibility

We begin by describing the set of possible models that investors consider and the estimation method they use to measure the relative plausibility of different models.

A. Economic Environment

We study a pure endowment economy.

ASSUMPTION 1: *Investors form expectations for future log consumption growth,* Δc, *using models of the form*

$$(1) \qquad\qquad \Delta c_t = \mu + a(L)(\Delta c_{t-1} - \mu) + \varepsilon_t$$

$$(2) \qquad\qquad \varepsilon_t \sim N(0, \sigma^2),$$

where μ *is mean consumption growth,* $a(L)$ *is a power series in* L, *the lag operator, with roots inside the unit circle, and* ε_t *is an innovation.*

The change in log consumption on date t, Δc_t, is a function of past consumption growth and a shock. We restrict our attention to models with purely linear feedback from past to current consumption growth. It seems reasonable to assume that people use linear models for forecasting, even if consumption dynamics are not truly linear, given that the economics literature focuses almost exclusively on linear models. For our purposes, the restriction to the class of linear processes is a description of the agent's modeling method, rather than an assumption about the true process driving consumption growth. We make the further assumption that ε_t is i.i.d. normal. While the assumption of normality is not necessary, it simplifies the exposition; the key assumption is that ε_t is serially independent.[8]

In much of what follows, it will be more convenient to work with the moving average (MA) representation of the consumption process (1),

$$(3) \qquad\qquad \Delta c_t = \mu + b(L)\varepsilon_t$$

$$(4) \qquad\qquad \text{where } b(L) \equiv \left(1 - La(L)\right)^{-1}$$

$$(5) \qquad\qquad = 1 + \sum_{j=1}^{\infty} b_j L^j.$$

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[8] It is straightforward to solve the model when ε_t has an arbitrary distribution but remains serially independent. While time varying volatility in innovations is an important area of analysis (e.g., Drechsler and Yaron 2011) we avoid it here for simplicity.

We can thus express the dynamics of consumption equivalently as depending on a or just on the power series $b(L)$, with coefficients b_j.[9] The two different representations are each more convenient than the other in certain settings, so we will refer to both in what follows. They are directly linked to each other through equation (4), so that a particular choice of a is always associated with a distinct value of b and vice versa (as long as b is invertible, which we impose).

There are no latent state variables. When a model $a(L)$ has infinite order we assume that the agent knows all the necessary lagged values of consumption growth for forecasting (or has dogmatic beliefs about them) so that no filtering is required. We discuss necessary constraints on the models below. For now simply assume that they are sufficiently constrained that any quantities we derive exist.

We set the notation $\Theta \equiv \{b, \mu, \sigma^2\}$ to represent the set of parameters that defines a model of consumption growth. Note that Θ induces a joint Gaussian density over sequences of consumption growth realizations.

B. Model Distance

For the purpose of forecasting consumption growth, the agent in our model chooses among specifications for consumption growth, Θ, partly based on their statistical plausibility. As is common in the literature, the plausibility of a model, or its distance from the agent's point estimate, is measured by the Kullback-Leibler divergence, or relative entropy, which is the expected value of a log likelihood ratio statistic comparing a pair of models.[10]

We denote the divergence between two models by $g(\Theta; \overline{\Theta})$, where $\overline{\Theta}$ is the benchmark or point estimate, and Θ is some alternative model. Suppose the agent has a point estimate $\overline{\Theta}$ and she compares it to an alternative model Θ based on their relative log likelihoods. Then as the number of observations grows to infinity, $g(\Theta; \overline{\Theta})$ is the limit of the expectation of that likelihood ratio statistic if the data is generated by the model Θ. The likelihood ratio is a natural choice to measure the difference between a pair of models because the Neyman-Pearson lemma shows that such a test is the most powerful way to discriminate between a pair of models.

Any sample of observed consumption growth of length T from the model (3) has a multivariate normal distribution with mean μ and a covariance matrix determined by $\{b, \sigma\}$, denoted $\Sigma_{\Theta, T}$. The log likelihood for a sample of length T is then

$$(6) \qquad LL_T(\Theta) = -\frac{1}{2}\log|\Sigma_{\Theta, T}| - \frac{1}{2}(\Delta c_{1,\dots,T} - \mu)'\Sigma_{\overline{\Theta}, T}^{-1}(\Delta c_{1,\dots,T} - \mu),$$

where $\Delta c_{1,\dots,T}$ denotes a column vector containing the sample of observed consumption growth between dates 1 and T.

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[9] In working with finite order regressive or moving average representations, a and b can be regarded as polynomials. When, as will be necessary below, we deal with potentially infinite order representations, we continue to use the term polynomial for ease of expression, though in that case $b(L)$ and $a(L)$ are, formally, power series.

[10] See Hansen and Sargent (2001); Maccheroni, Marinacci, and Rusticini (2006); Sbuelz and Trojani (2008); Strzalecki (2011); and Drechsler (2013), among others.

Our analysis of the model takes place in the frequency domain because it will allow us to obtain a tractable and interpretable solution. The analysis centers on the *transfer function*,

$$(7) \qquad B(\omega) \equiv b(e^{i\omega}),$$

for $i \equiv \sqrt{-1}$. The transfer function measures how the filter $b(L)$ transfers power at each frequency, ω, from the white noise innovations, ε, to consumption growth.

Now suppose consumption growth is generated by the model Θ. One may show that as $T \to \infty$, the expected difference between the log likelihoods for the models Θ and $\overline{\Theta}$ converges to

$$(8) \quad \lim_{T \to \infty} T^{-1} E_\Theta \big[LL_T(\overline{\Theta}) - LL_T(\Theta) \big] = -\frac{1}{2} \frac{\sigma^2}{\overline{\sigma}^2} \int \frac{\big| B(\omega) - \overline{B}(\omega) \big|^2}{\big| \overline{B}(\omega) \big|^2} d\omega$$

$$-\frac{1}{2} \frac{(\mu - \overline{\mu})^2}{\overline{\sigma}^2 \big| \overline{B}(0) \big|^2} + \frac{1}{2} \left(\log\left(\frac{\sigma^2}{\overline{\sigma}^2} \right) - \frac{\sigma^2 - \overline{\sigma}^2}{\overline{\sigma}^2} \right),$$

where $|\cdot|$ denotes the norm of a complex number, the notation E_Θ indicates an unconditional expectation taken over the probability measure induced by Θ, and the integral sign with no limits denotes $\frac{1}{2\pi} \int_{-\pi}^{\pi}$. This result is a simple application of Whittle's (1953) limiting formula for the log likelihood (see, e.g., Dahlhaus 1996).

The relative entropy between Θ and $\overline{\Theta}$ depends on three terms. The first measures the difference between the dynamics (in terms of autocorrelations) implied by the two models. A simple way to interpret it is that $\big| \overline{B}(\omega) \big|^2$ measures the uncertainty about the size of fluctuations at frequency ω, so the entropy distance penalizes squared deviations between the models, $\big| B(\omega) - \overline{B}(\omega) \big|^2$, less strongly at frequencies at which there is more uncertainty. This is the major term that will drive our results; it determines how the agent quantifies deviations of dynamics from the benchmark.

The second term incorporates the differences in the means of the distributions under the two models.

$$(9) \qquad \overline{f}(\omega) \equiv \overline{\sigma}^2 \big| \overline{B}(\omega) \big|^2$$

is the *spectral density* (or spectrum) of consumption growth under the model $\overline{\Theta}$. The spectral density decomposes the total variance of a time series into components coming from fluctuations at different frequencies. We obtain the typical result that the uncertainty about the mean depends on the spectrum at frequency zero. In other words, deviations in the mean of consumption growth between the two models are penalized with a similar scaling to deviations in the dynamics, but they are essentially infinitely low frequency differences—i.e., infinitely long lived shocks to consumption growth—so they are scaled by $\overline{\sigma}^2 \big| \overline{B}(0) \big|^2$.

Finally, there is a contribution from the deviation of the innovation variance, σ^2, from the benchmark, which we find to have quantitatively minimal effects in our calibration.

As a divergence measure, (8) is precisely the limit of the Kullback-Leibler (KL) divergence between the models Θ and $\overline{\Theta}$—the relative entropy between the joint distributions for consumption growth implied by Θ and $\overline{\Theta}$ as the sample length grows to infinity. We therefore define the agent's measure of model plausibility in the following assumption:

ASSUMPTION 2: *Given a point estimate* $\overline{\Theta}$, *investors measure the statistical plausibility of an alternative model* Θ *through the divergence measure*,

$$(10) \qquad g(\Theta; \overline{\Theta}) \equiv \frac{1}{2} \frac{\sigma^2}{\overline{\sigma}^2} \int \frac{\left| B(\omega) - \overline{B}(\omega) \right|^2}{\left| \overline{B}(\omega) \right|^2} \, d\omega$$

$$+ \frac{1}{2} \frac{(\mu - \overline{\mu})^2}{\overline{\sigma}^2 |\overline{B}(0)|^2} - \frac{1}{2} \left(\log\left(\frac{\sigma^2}{\overline{\sigma}^2} \right) - \frac{\sigma^2 - \overline{\sigma}^2}{\overline{\sigma}^2} \right).$$

While the KL divergence is widely studied and has a prominent place in the ambiguity literature, one could ask how an investor, in reality, might adopt such a measure of statistical discrimination. Is it reasonable to think that an investor armed with a simple estimation toolkit would behave as if she were using this distance measure?

As noted above, it is the first term in the definition of $g(\Theta; \overline{\Theta})$ that drives our results. That component also arises if the investor uses the nonparametric estimation methods for AR and MA models described by Berk (1974) and Brockwell and Davis (1988b). Specifically, one may model agents as estimating AR or MA models whose lag orders grow with the sample size. They thus have nonparametric confidence intervals. We show in the online Appendix that if they follow such an estimation process, then as the sample size grows, a Wald test of the difference in the MA coefficients between the point estimate and any alternative approaches $\frac{1}{2\pi} \int_{-2\pi}^{2\pi} \frac{\left| B(\omega) - \overline{B}(\omega) \right|^2}{\left| \overline{B}(\omega) \right|^2} d\omega$. So the key part of the KL divergence that addresses dynamics may also be derived from an explicit nonparametric estimation method.[11]

II. Preferences

Given a particular model of consumption dynamics, the agent has Epstein-Zin (1991) preferences. We augment those preferences with a desire for a robustness against alternative models. The desire for robustness induces the agent to form expectations, and hence calculate utility and asset prices, under a pessimistic but plausible model, where plausibility is quantified using the estimation approach described above.

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[11] Intuitively, this result is related to the fact that Wald and likelihood ratio tests are asymptotically equivalent. We thank Lars Hansen for pointing out the connection between the method based on the Wald test—which is how we originally derived our results—and the KL divergence.

A. Utility Given a Model

ASSUMPTION 3: *Given a forecasting model* $\Theta \equiv \{b, \mu, \sigma^2\}$, *the investor's util-ity is described by Epstein-Zin (1991) preferences. The coefficient of relative risk aversion is* α, *the time discount parameter is* β, *and the elasticity of intertemporal substitution (EIS) is equal to* 1. *Lifetime utility,* v, *for a fixed model* Θ, *is*

$$(11) \quad v(\Delta c^t; \Theta) = (1 - \beta) c_t + \frac{\beta}{1 - \alpha} \log E_t \left[\exp\left(v(\Delta c^{t+1}; \Theta)(1 - \alpha) \right) | \Theta \right]$$

$$(12) \qquad = c_t + \sum_{k=1}^{\infty} \beta^k E_t[\Delta c_{t+k} | \Theta] + \frac{\beta}{1 - \beta} \frac{1 - \alpha}{2} \sigma^2 b(\beta)^2,$$

where $E_t[\cdot \, | \Theta]$ *denotes the expectation operator conditional on the history of con-sumption growth up to date* t, Δc^t, *assuming that consumption is driven by the model* Θ.

$\frac{\beta}{1 - \beta} \frac{1 - \alpha}{2} \sigma^2 b(\beta)^2$ is an adjustment to utility for risk. The investor's utility is lower when risk aversion or the riskiness of the endowment is higher. The relevant measure of the risk of the endowment is $\sigma^2 b(\beta)^2$, which measures the variance of the shocks to lifetime utility in each period. $b(\beta)$ measures the total discounted effect of a unit innovation to ε_{t+1} on consumption growth, and hence utility, in the future. It is the term involving $b(\beta)$ that causes people with Epstein-Zin preferences to be averse to long-run risk.[12] $b(\beta)$ can be written in terms of the transfer function as

$$(13) \qquad\qquad b(\beta) = \int Z(\omega)^* B(\omega) \, d\omega$$

$$(14) \qquad \text{where } Z(\omega) \equiv \sum_{j=0}^{\infty} \beta^j e^{i\omega j}$$

and * denotes a complex conjugate. The parameter μ enters through the discounted expectation of future consumption growth, and hence appears as $\frac{\beta}{1 - \beta}\mu$.

B. Robustness over Dynamics

Equation (12) gives lifetime utility conditional on consumption dynamics. We now discuss the investor's consideration of alternative models of dynamics.

The investor entertains a set of possible values for Θ and can associate with any model a measure of its plausibility, g (from Assumption 2). Seeking robustness, the investor makes decisions that are optimal in an unfavorable world, specifi-cally, as though consumption growth is driven by worst-case dynamics, denoted

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[12] We focus on the case of a unit EIS to ensure that we can derive analytic results. The precise behavior of interest rates is not our primary concern, so a unit EIS is not particularly restrictive. The unit EIS also allows us to retain the result that Epstein-Zin preferences are observationally equivalent to a robust control model, as in Barillas, Hansen, and Sargent (2009), which will be helpful in our calibration below.

$\Theta^w \equiv \{b^w, \sigma_w^2, \mu^w\}$. These dynamics are not the worst in an unrestricted sense but, rather, are the worst among statistically plausible models. So the investor does not fear completely arbitrary models—she focuses on models that are not too far from her point estimate in terms of KL distance (expected log likelihood).

ASSUMPTION 4: *Investors use a worst-case model to form expectations—for both calculating utility and pricing assets—that is obtained as the solution to a penalized minimization problem:*

$$(15) \qquad \Theta^w = \arg\min_{\Theta}\Big\{E\big[v(\Delta c^t; b, \sigma^2, \mu)|\Theta\big] + \lambda g\big(\Theta; \overline{\Theta}\big)\Big\}.$$

$\{b^w, \sigma_w^2, \mu^w\}$ is the model that gives the agent the lowest unconditional expected lifetime utility, subject to the penalty g.[13] λ is a parameter that determines how much weight the penalty receives. As usual, λ can either be interpreted directly as a parameter or as a Lagrange multiplier on a constraint on the KL divergence g. The KL divergence can be large for three reasons: deviations in the dynamics, b; deviations in the mean, μ; and deviations in the conditional variance, σ^2.

The agent's assessment of plausibility is based on our statistical measure of distance and controlled by λ. We are modeling the agent's beliefs about potential models by assuming that she compares possible models to a point estimate $\overline{\Theta}$. The role of g in our analysis is similar to that of the KL divergence used in the robust control model of Hansen and Sargent (2001), in that it imposes discipline on what models the investor considers.

There are three important differences between our analysis and the multiplier preferences of Hansen and Sargent (2001). First, in Hansen and Sargent's (2001) model, the deviation between the worst-case model and the benchmark is purely in the distribution of innovations, ε_{t+1}. Here we explicitly focus on uncertainty about dynamics. As we showed above, shifts in the mean of the distribution of ε_{t+1} (as obtained in Barillas, Hansen, and Sargent 2009) represent deviations in the model only at frequency zero. Our nonparametric analysis allows for deviations at all frequencies. The allowance of models with alternative dynamics is central to our results—it is what generates predictability in returns and excess volatility in asset prices.

Second, Hansen and Sargent (2001) model agents as having log utility over fixed models, while we start from the assumption that agents have Epstein-Zin (1991) preferences over fixed models. Because Epstein-Zin preferences imply that the timing of resolution of uncertainty affects utility, our agents view more persistent processes as less favorable (for $\alpha > 1$).

Finally, under multiplier preferences, the state variables in the worst-case model are the same as the state variables in the benchmark model. So, for example, if consumption growth is an AR(1) under the benchmark model, then lagged consumption growth is also the only state variable under the worst-case model. In our

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[13] Since consumption can be nonstationary, this expectation does not always exist. In that case, we simply rescale lifetime utility by the level of consumption yielding $E\big[v(\Delta c^t; b) - c_t|\Theta\big]$, which does exist. Scaling by consumption is a normalization that has no effect other than to ensure the expectation exists.

setting, though, the state variables in the worst-case model are typically different from those under the benchmark (and in general, include the entire history of consumption growth). Our setting thus allows agents to consider potentially infinitely richer economic dynamics.

A natural question is why we analyze a worst case instead of allowing the agent to average as a Bayesian across all the possible models. Our answer is that people may not actually be Bayesians, or they may not be able to assign priors to all models. Machina and Siniscalchi (2014) discuss the extensive experimental evidence that people make choices consistent with ambiguity aversion.

Ambiguity aversion is particularly compelling in our context because, as we will see, it is ultimately dynamics at the very lowest frequencies that drive our results. And direct estimation of, say, 50- or 100-year autocorrelations is, for practical purposes with realistic data sources, impossible. So investors face a situation where they simply do not have data that directly measures all features of consumption dynamics. They do not face a standard estimation problem—rather, they must make decisions in the face of model uncertainty that cannot be resolved by the data at hand, a problem akin to that discussed by Knight (1921) and Ellsberg (1961). Moreover, as Hansen and Sargent (2007) note, almost any model is necessarily just an approximation, even a high-order one. If the true model has infinite order, then it can never be fully characterized in any finite sample.

Finally, (again, following Hansen and Sargent 2007), it is well understood that when the parameter space is infinite, likelihood-based methods are difficult to implement at best, and often impossible. Specifically, frequentist methods with an infinite-dimensional parameter space lead to degenerate estimates (in the context of spectral estimation, for example, see Chow and Grenander 1985), while constructing priors over such a space that lead to accurate posterior confidence intervals is difficult or impossible (Sims 1971 and Diaconis and Freedman 1986). So when an investor does not know the true order of the model driving the endowment process, she may literally not be able to assign likelihoods to different specifications. Instead, she uses $g(\Theta; \bar{\Theta})$ to measure the "plausibility" of potential models, even though she has no probabilities on models over which to integrate.

Ultimately, our ambiguity-averse investor's utility takes the form of that of an Epstein-Zin agent but using Θ^w to form expectations about future consumption growth,[14]

$$(16) \quad v^w(\Delta c^t) = v(\Delta c^t; \Theta^w) = c_t + \frac{\beta}{1-\beta} \frac{1-\alpha}{2} b^w(\beta)^2 + \sum_{k=1}^{\infty} \beta^k E_t[\Delta c_{t+k}|\Theta^w].$$

In modeling investors as choosing a single worst-case Θ^w, we obtain a setup similar to Gilboa and Schmeidler (1989); Maccheroni, Marinacci, and Rustichini (2006); and Epstein and Schneider (2007) in the limited sense that we are essentially constructing a set of models and minimizing over that set. Our worst-case model is,

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[14]Note that since utility is recursive, the agent's preferences are time-consistent, but under a pessimistic probability measure. Furthermore, the assumption that b^w is chosen unconditionally means that b^w is essentially unaffected by the length of a time period, so the finding in Skiadas (2013) that certain types of ambiguity aversion become irrelevant in continuous time does not apply here.

however, chosen once and for all and is not state- or choice-dependent. The choice of Θ^w is timeless—it is invariant to the time-series evolution of consumption—so what it represents is an unconditional worst-case model: if an agent had to choose a worst-case model to experience prior to being born into the world, it would be Θ^w. The worst-case analysis is certainly not the only way to model behavior under ambiguity, but it is plausible and generates economically interpretable results.

Unlike in some recent related papers, the investor in this model does not change her probability weights every day or adjust the worst case according to a learning process.[15] She chooses a single pessimistic model to protect against. An added benefit of the assumption that the worst-case model is chosen timelessly is that it ensures time-consistency. The agent sets the model used for forming expectations once and for all, and then uses a recursive utility specification conditional on that single probability measure. In other words, since the model is chosen a single time, our agent's preferences inherit the time-consistency of Epstein-Zin preferences.

III. The Worst-Case Scenario

Our analysis above leads us to a simple quadratic optimization problem. The solution is summarized in the following proposition:

PROPOSITION 1: *Under Assumptions 1–4, for an agent who chooses a model* $\Theta^w \equiv \left\{ b^w, \mu^w, \sigma_w^2 \right\}$ *to minimize the unconditional expectation of lifetime utility subject to the loss function* $g(\Theta; \overline{\Theta})$, *that is,*

$$(17) \qquad \Theta^w = \arg\min_{\Theta} \frac{\beta}{1 - \beta} \frac{1 - \alpha}{2} \sigma^2 b(\beta)^2 + \frac{\beta}{1 - \beta} \mu + \lambda g(\Theta; \overline{\Theta}),$$

the worst-case model is determined by the set of equations

$$(18) \qquad \textit{Dynamics:} \qquad \sigma_w^2 \left| B^w(\omega) \right|^2 = \bar{f}(\omega)$$

$$+ \lambda^{-1} \beta (1 + \beta)(\alpha - 1) \sigma_w^2 b^w(\beta)^2 \bar{f}(\omega) \left| Z(\omega) \right|^2$$

$$(19) \qquad \textit{Mean:} \qquad \mu_w = \overline{\mu} - \lambda^{-1} \frac{\beta}{1 - \beta} \bar{f}(0),$$

where $B(\omega) = b(e^{i\omega})$, $Z(\omega) \equiv \sum_{j=0}^{\infty} \beta^j e^{i\omega j}$, $\sigma_w^2 \left| B^w(\omega) \right|^2$ *is the spectrum under the worst-case model, and $\bar{f}$ is the benchmark spectrum. The time-domain model $b^w(L)$ has coefficients b_j^w that are obtained as the Wold representation associated with* $\left| B^w(\omega) \right|^2$ *(see Priestley 1981, section 10.1.1).*

We thus have a closed form expression for the worst-case model. The spectral density under the worst case, $f^w(\omega) = \sigma_w^2 \left| B^w(\omega) \right|^2$, in (18) is equal to the point estimate

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[15] See, for example, Hansen and Sargent (2010); Ju and Miao (2012); and Collin-Dufresne, Johannes, and Lochstoer (2016).

Panel A. Weighting function $|Z|^2$

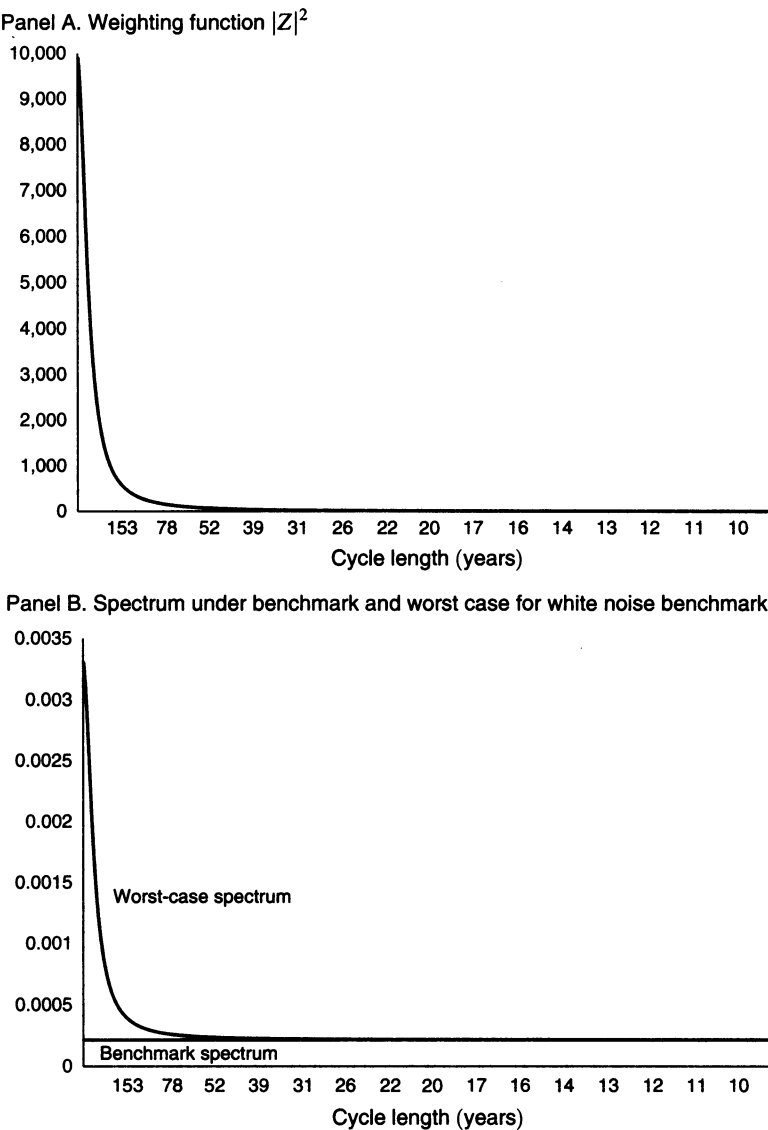Panel B. Spectrum under benchmark and worst case for white noise benchmark

FIGURE 1

plus a term that depends on three factors. First, $\lambda^{-1}\beta(1+\beta)(\alpha-1)\sigma_w b^w(\beta)$ represents the ratio of the utility losses from a marginal increase in $\sigma_w b^w(\beta)$ to the cost of deviations from the point estimate, λ. When risk aversion, α, is higher or the cost of deviating from the point estimate, λ, is lower, the worst-case model is farther from the point estimate.

Second, $\bar{f}(\omega)$ represents the amount of uncertainty the agent has about consumption dynamics at frequency ω. Where the spectral density, $\bar{f}(\omega)$, is high, there is relatively more uncertainty and the worst-case model is farther from the point estimate.

Finally, $|Z(\omega)|^2$ determines how much weight the lifetime utility function places on frequency ω. The top panel of Figure 1 plots $|Z(\omega)|^2$ for $\beta = 0.99$, a standard

annual calibration. It is strikingly peaked near frequency zero; in fact, the x-axis does not even show frequencies corresponding to cycles lasting less than ten years because they carry essentially zero weight. Since the mass of $|Z(\omega)|^2$ lies very close to frequency 0, the worst case shifts power to very low frequencies. In that sense, the worst-case model *always* includes long-run risk.

The mean of consumption growth also differs from the benchmark in a natural way: when people are willing to consider more extreme models, they are more risk averse, or they have more uncertainty about the mean (through $\bar{f}(0)$), μ_w is farther below μ.

When $\alpha = 1$, so that the investor's preferences reduce to time-separable log utility, the shift in the dynamics is set to zero: $f^w(\omega) = \bar{f}(\omega)$. That is because under time separable preferences, the dynamics of consumption do not affect average utility. But the mean growth rate obviously still matters, so even for $\alpha = 1$, $\mu^w < \bar{\mu}$.

Finally, people in principle also consider deviations of σ_w^2 from the point estimate $\bar{\sigma}^2$. We find below that this effect is extremely small in a typical calibration.

An important feature of the results is that the worst-case model depends on preferences. In particular, when β is higher—the investor is more patient—the worst-case model is more persistent. At the same time, when the investor is more risk averse— α is higher—the amount of long-run risk in the worst-case model is higher. The model thus has implications for disagreement across agents that is tightly linked to preferences or investment horizons.

Proposition 1 represents the completion of the solution to the model. To summarize, given a point estimate, $\bar{\Theta}$ (estimated from a finite-order model that we need not specify here), the agent selects a worst-case model Θ^w. She then uses the worst-case model when calculating expectations and pricing assets.

A. Long-Run Risk is the Worst-Case Scenario

COROLLARY 1: *Suppose the agent's point estimate is that consumption growth is white noise, with $\bar{b}(L) = 1$. The worst-case model is then an ARMA(1,1), and consumption growth has the following representation under the worst-case dynamics:*

$$(20) \qquad \Delta c_t = (1 - \beta)\mu_w + \beta\Delta c_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$$

$$(21) \qquad \varepsilon_t \sim N(0, \sigma_w^2).$$

The above ARMA(1,1) process also has an equivalent state-space representation,

$$(22) \qquad \Delta c_t = \mu_w + x_{t-1} + \eta_t$$

$$(23) \qquad x_t = \beta x_{t-1} + v_t,$$

where

$$(24) \qquad \eta_t \sim N(0, \bar{\sigma}^2) \quad \text{and} \quad v_t \sim N(0, \bar{\sigma}^2\varphi(1 - \beta^2))$$

$$(25) \qquad \varphi \equiv \lambda^{-1}\frac{\beta}{1 - \beta}(\alpha - 1)\sigma_w^2 b^w(\beta)^2.$$

The state-space form in equations (22) and (23) is observationally equivalent to the process (20) and (21) in the sense that they have identical autocovariances for consumption growth, and (22) and (23) is exactly case I from Bansal and Yaron (2004), the homoskedastic long-run risk model.

The worst-case process exhibits a small but highly persistent trend component, and the persistence is exactly equal to the time discount factor. Intuitively, since β^j determines how much weight in lifetime utility is placed on consumption j periods in the future, a shock that decays with β spreads its effects as evenly as possible across future dates, scaled by their weight in utility. And spreading out the effects of the shock over time minimizes its detectability. The worst-case/long-run risk model is thus the departure from pure white noise that generates the largest increase in risk prices (and decrease in lifetime utility) for a given level of statistical distinguishability.

There is a difference between the worst-case model derived here and the long-run risk model, which is that in this setting, the long-run trend, x_t, is unobservable. Asset prices therefore carry no ability to forecast future consumption growth beyond what is available in the history of lagged consumption. However, one may show that the volatility of the pricing kernel, and hence the price of risk, is actually higher under the process (20) and (21) than under (22) and (23) (due to the later arrival of information). The key intuition behind the long-run risk model is that a small persistent component in consumption growth can induce large fluctuations in the pricing kernel, and that force (or fear of it) is also present in our model.

The composite parameter φ determines the degree to which the worst-case dynamics differ from the point estimate and also the relative volatility of η and ν. When the worst-case model differs from the benchmark by a greater degree, the worst-case model displays more volatile trend-type shocks, ν_t, and the white noise shocks, η_t, become relatively smaller.

The bottom panel of Figure 1 plots the spectrum for the white noise benchmark and the worst-case model. The spectrum for white noise is totally flat, while the worst case has substantial power at the very lowest frequencies, exactly as we would expect from the top panel of Figure 1.

Hansen and Sargent (2010) also study a setting in which investors price assets as though the long-run risk model might be driving the consumption process. The key difference between their analysis and ours, though, is that they start from the assumption that agents put a nonzero probability on the possibility that the long-run risk model is actually true. We, on the other hand, start from a simple white noise benchmark and obtain the long-run risk model entirely endogenously.

IV. The Behavior of Asset Prices

The investor's Euler equation is calculated under the worst-case dynamics. For any available return R_{t+1},

$$(26) \qquad 1 = E_t[R_{t+1} M_{t+1} | \Theta^w],$$

$$(27) \quad \text{where } M_{t+1} \equiv \beta \exp(-\Delta c_{t+1}) \frac{\exp\left(\nu(\Delta c^{t+1}; \Theta^w) \times (1 - \alpha)\right)}{E_t\left[\exp\left(\nu(\Delta c^{t+1}; \Theta^w) \times (1 - \alpha)\right) | \Theta^w\right]}.$$

M_{t+1} is the stochastic discount factor (SDF). The SDF is identical to what is obtained under Epstein-Zin preferences, except that here expectations are calculated under Θ^w. The key implication of that change is that expected shocks to $v(\Delta c^{t+1}; \Theta^w)$ have a larger standard deviation since the worst-case model features highly persistent shocks that affect lifetime utility much more strongly than the less persistent point estimate.

A. Consumption and Dividend Claims

It is straightforward, given that log consumption follows a linear Gaussian process, to derive approximate expressions for prices and returns on levered consumption claims. We consider an asset whose dividend is C_t^γ in every period, where γ represents leverage. Denote the return on that asset on date $t+1$ as r_{t+1} and the real risk-free rate as $r_{f,t+1}$. We will often refer to the levered consumption claim as an equity claim, and we view it as a simple way to model equity returns (Abel 1999).[16]

From the perspective of an econometrician who has the same point estimate for consumption dynamics as the investor, $\overline{\Theta}$, the expected excess log return on the levered consumption claim is

$$(28) \quad E_t\big[r_{t+1} - r_{f,t+1}|\overline{\Theta}\big] + \frac{1}{2}\mathrm{var}_t(r_{t+1}) = -\mathrm{cov}^w(r_{t+1}, m_{t+1}) + \frac{1}{2}\big(\mathrm{var}_t(r_{t+1}) - \mathrm{var}_t^w(r_{t+1})\big)$$

$$+ \frac{\gamma - \delta a^w(\delta)}{1 - \delta a^w(\delta)}\big[\big(1 - a^w(1)\big)(\mu - \mu^w) + \big(\overline{a}(L) - a^w(L)\big)(\Delta c_t - \mu)\big],$$

where δ is a linearization parameter from the Campbell-Shiller approximation that depends on the steady-state price/dividend ratio. The addition of the variance is a correction to make the left-hand side approximately the arithmetic mean return.

The first term, $-\frac{1}{2}\mathrm{cov}_t^w(r_{t+1}, \log M_{t+1})$ (i.e., the conditional covariance of log returns with the log SDF measured under the worst-case dynamics), is the standard risk premium, and it is calculated under the worst-case model. The primary way that the model increases risk premia compared to standard Epstein-Zin preferences is that the covariance of the return with the SDF is more negative. That covariance, in turn, is more negative for two reasons. First, since the agent behaves as though shocks to consumption growth are highly persistent, they have large effects on lifetime utility, thus making the SDF very volatile. Second, again because of the persistence of consumption growth under the worst case, shocks to consumption have large effects on expected long-run dividend growth, so the return on the levered consumption claim is also very sensitive to shocks.[17] These two effects cause the consumption claim to strongly negatively covary with the SDF and generate a high risk premium.

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[16]The model of dividends here is very simplistic and does not match the dynamics of dividend growth particularly well. The online Appendix examines a more realistic model of dividends that matches the standard deviation of dividend growth and the correlation between dividend and consumption growth in a setting in which dividends and consumption are cointegrated. The behavior of equity returns is almost identical in that setting to what we obtain with the simple model $D_t = C_t^\gamma$.

[17]We have $\mathrm{cov}^w(r_{t+1}, m_{t+1}) = \big(-1 + (1 - \alpha)b^w(\beta)\big)\frac{\gamma - \delta a^w(\delta)}{1 - \delta a^w(\delta)}\sigma_w^2$. The volatility of the SDF is determined by $\big(-1 + (1 - \alpha)b^w(\beta)\big)\sigma_w$, while the volatility of the return is $\frac{\gamma - \delta a^w(\delta)}{1 - \delta a^w(\delta)}\sigma_w$.

The second term is a quantitatively trivial (under our benchmark calibration) adjustment to the arithmetic mean due to the difference between the variance of returns under the benchmark and worst-case models.

The second line is the part of the expected return that comes from differences in forecasts of consumption under the models of the econometrician and investors. As long as $\gamma > \delta a^w(\delta)$ (which is always satisfied if $\gamma > 1$), the coefficients on $(\mu - \mu^w)$ and $(\overline{a}(L) - a^w(L))(\Delta c_t - \mu)$ are positive. The former term implies that average equity returns are higher when investors are more pessimistic about mean consumption (and hence dividend) growth, μ^w.

The term involving $(\Delta c_t - \mu)$ is zero on average, but it induces predictability in returns. When the worst case implies higher future consumption growth, investors pay relatively more for equity compared to riskless assets, thus lowering expected excess returns. This channel leads to procyclical asset prices and countercyclical expected returns when $a^w(L)$ implies more persistent dynamics than $\overline{a}(L)$, similarly to Fuster, Hebert, and Laibson (2011).

The procyclicality of asset prices also depends on leverage, γ. Expected returns for assets more risky than consumption ($\gamma > 1$) are countercyclical, and we model equity as having $\gamma > 1$. But for assets safer than consumption, prices will in fact be countercyclical, because variation in the risk-free rate dominates variation in the risk premium for them.

We also note that since risk aversion and conditional variances are constant, the excess return on a levered consumption claim has a constant conditional expectation from the perspective of investors. That is, while returns are predictable from the perspective of an econometrician, investors believe that they are unpredictable. So if investors in this model are surveyed about their expectations of excess returns, their expectations will not vary, even if econometric evidence implies that returns are predictable. This model therefore generates extrapolative expectations of the type discussed by Greenwood and Shleifer (2014) (and many references therein), but that extrapolation applies to consumption growth rather than equity returns.

Finally, it is worth noting that since asset prices depend purely on the history of consumption growth, there is no way to improve to the investor's estimates of consumption dynamics by including information on asset prices. The price of a consumption claim is completely redundant given data on the history of consumption.

B. *Interest Rates*

The risk-free rate follows

$$(29) \quad r_{f,t+1} = -\log\beta + \mu^w + a^w(L)(\Delta c_t - \mu^w) - \frac{1}{2}\sigma_w^2 + (1 - \alpha)b^w(\beta)\sigma_w^2.$$

With a unit EIS, interest rates move one-for-one with expected consumption growth. In the present model, the relevant measure of expected consumption growth is $\mu^w + a^w(L)(\Delta c_t - \mu^w)$, which is the expectation under the worst-case model.

The online Appendix derives analytic expressions for the prices of long-term zero-coupon bonds, which we discuss in our calibration below.

V. Calibration

We now parameterize the model to analyze its quantitative implications. Most of our analysis is under the assumption that the agent's point estimate implies that consumption growth is white noise and that the point estimate is also the true dynamic model. Despite this parsimony, we obtain striking empirical success in terms of matching important asset pricing moments.

Many of the required parameters are standard. We use a quarterly calibration of $\beta = 0.99^{1/4}$, implying a pure rate of time preference of 1 percent per year. Setting β smaller would imply that the worst-case model is less persistent, but it also implies that interest rates are higher. We will see below that the equilibrium real interest rate in the model ends up being still too large relative to the data, which implies that β should not be lower.

The steady-state dividend/price ratio used in the Campbell-Shiller approximation is 5 percent per year, as in Campbell and Vuolteenaho (2004), so $\delta = 0.95^{1/4}$.[18] Other parameters are calibrated to match moments reported in Bansal and Yaron (2004). The agent's point estimate is that consumption growth is i.i.d. with a quarterly standard deviation of 1.47 percent, which we also assume is the true data-generating process. Finally, the leverage parameter for the consumption claim, γ, is set to 4.806 to generate mean annualized equity returns of 6.33 percent.

We calibrate λ to equal 106.8 to match the observed Sharpe ratio on equities. The calibration of α is slightly more difficult. Its effects on asset prices are highly similar to those of λ, so it is not straightforward to calibrate both α and λ from aggregate data.

It is possible to draw a link between α and λ if we interpret α as in the literature on multiplier preferences. Hansen and Sargent (2005) and Barillas, Hansen, and Sargent (2009) show that the coefficient of relative risk aversion in Epstein-Zin preferences—our α—can be interpreted as a measure of the agent's willingness to consider alternative distributions of the innovations ε, where the measure of distance between distributions is again the KL divergence. That is, our combination of model uncertainty over Θ with Epstein-Zin preferences can alternatively be viewed as a combination of two layers of model uncertainty—over Θ and the distribution of ε separately. The Mathematical Appendix formalizes this argument and shows that if agents use the same penalty on the KL divergence over the distribution of ε and also the model Θ, then α is linked to λ through the formula

$$(30) \qquad\qquad \alpha = 1 + \frac{1}{\lambda(1 - \beta)},$$

yielding $\alpha = 4.73$.

The link between α and λ in equation (30) is useful because it eliminates a degree of freedom from our calibration. In order to justify it, though, one must accept the assumptions of multiplier preferences, which include that agents have unit risk aversion over unambiguous risks. So while (30) is helpful, it is also only a very special

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[18] Setting this parameter to a higher value (e.g., $0.975^{1/4}$ as in Bansal and Yaron 2004) increases the mean and standard deviation of equity returns but has no effect on the Sharpe ratio.

case. Alternatively, α can simply be taken as another free preference parameter, rather than as a measure of robustness (in the multiplier preference interpretation). So it is important that our calibration of α be objectively reasonable as a measure of risk aversion. Risk aversion of 4.73 is extraordinarily small in the context of the consumption-based asset pricing literature with Gaussian innovations. It is well within the upper bound of ten proposed by Mehra and Prescott (1985), and almost precisely equal to average risk aversion measured by Barsky et al. (1997).[19] α therefore takes on a plausible value in its own right, separate from any connection it may have to λ.

To further investigate how reasonable λ is, in the next section we show that it implies a worst-case model that is rarely rejected by statistical tests on data generated by the true model. An investor with the true model as her point estimate might reasonably believe the worst case could have actually generated the data that led to that point estimate.

VI. Quantitative Implications

A. *The White Noise Case*

We report the values of the parameters in the worst-case consumption process in Table 1. As noted above, the autocorrelation of the predictable part of consumption growth under the worst-case model is β, implying that trend shocks have a half-life of 70 years, as opposed to the 3-year half-life in the original calibration in Bansal and Yaron (2004).[20] $b^w(\beta) \sigma_w$, the relevant measure of the total risk in the economy, is 0.036 at the quarterly frequency in our model, compared to 0.031 in theirs.[21] Our model thus has more persistence and more total risk.

Note also that σ_w is only 0.4 percent larger than $\overline{\sigma}$. So the conditional variance of consumption growth under the worst-case model is essentially identical to that under the benchmark. However, because the worst-case model is so persistent, $b^w(\beta)$ is 2.45 times higher than $\overline{b}(\beta)$, thus implying that the worst-case economy is far riskier than the point estimate.

Unconditional Moments.—Table 1 reports key asset pricing moments. The first column shows that the model can generate a high standard deviation for the pricing kernel (and hence a high maximal Sharpe ratio), high and volatile equity returns, and low and stable real interest rates, as in the data. The equity premium and its

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[19] Average here is the harmonic mean—the inverse of the arithmetic mean of risk tolerance. In an update and extension of Barsky et al.'s (1997) analysis, Kimball, Sahm, and Shapiro (2008) estimate similar values.

Our value for α is less than half that used by Bansal and Yaron (2004), who themselves are notable for using a low value, and it is similar to that of Barro (2006), who uses values of two to four in a model with rare disasters.

[20] The 70-year half-life is sensitive to the choice of β. It is equal to $\log(1/2)/\log(\beta)$. If $\beta = 0.975^{1/4}$, the half-life falls to 27 years. Nevertheless, the worst case is robustly far more persistent than Bansal and Yaron's (2004) trend—to obtain a half-life of 3 years, we would need an annual rate of time preference of 21 percent. Online Appendix Table A3 reports results under a calibration with $\beta = 0.95^{1/4}$.

[21] Using the notation for the long-run risk model from above, and denoting the AR(1) coefficient for x as ρ, we calculate the equivalent of $b(\beta)$ in Bansal and Yaron's (2004) model as $b(\beta) = \sqrt{\frac{\text{var}(\nu)}{(1 - \beta\rho)^2} + \text{var}(\eta)}$ which yields $b(\beta) = 0.029$. Dew-Becker (forthcoming) discusses this calculation and its relation with risk premia in more detail.

Consumption dynamics		Benchmark	Worst-case	
$\bar{\sigma}$	Consumption conditional volatility	1.47%	1.47%	
$\bar{b}(\beta)$	Long-run volatility	1	2.449	
$\bar{\mu}$	Mean consumption growth	0.45%	0.37%	
Asset pricing moments (annualized)		Model	Standard EZ	Data
Standard deviation of pricing kernel		0.30	0.14	N/A
Mean excess equity return		6.34	1.95	6.33
Standard deviation of equity return		19.44	14.09	19.42
Mean risk-free rate		1.89	2.44	0.86
Standard deviation of risk-free rate		0.31	0	0.97
One-year autocorrelation of P/D		0.96	N/A	0.81
Standard deviation of P/D		0.19	0	0.29
Mean ten-year/one-quarter term spread		−13.3bp	0	N/A
Implied estimate of EIS		0	N/A	0.14

Notes: Moments from the model with a white noise benchmark process for consumption growth. The standard Epstein-Zin results are for where the agent is sure of the consumption process. P/D is the price/dividend ratio for the levered consumption claim. The values in the data treat the aggregate equity market as analogous to the levered consumption claim. The EIS estimate is based on a regression of consumption growth on interest rates. In the second column interest rates are constant, so the regression is degenerate.

volatility are 6.34 and 19.44 percent respectively, identical to the data. The real risk-free rate has a mean of 1.89 percent and a standard deviation of 0.31 percent.

The second column in the bottom section of Table 1 shows what would happen if we set $\lambda = \infty$ but held α fixed at 4.73, so that we would be back in the standard Epstein-Zin setting where there is no uncertainty about dynamics. The equity premium then falls from 6.34 to 1.95 percent, since the agent exhibits no concern for long-run risk. Furthermore, because the agent no longer behaves as if consumption growth is persistent, a shock to consumption has far smaller effects on asset prices. The standard deviation of returns falls from 19.4 to 14.1 percent and the standard deviation of the price/dividend ratio falls from 19 percent to exactly zero. The agent's fear of a model with long-run risk thus raises the mean of returns by a factor of more than 3 and the volatility by a factor of 1.4.

Recall from equation (28) that the mean equity premium is due both to the investor's belief about the covariance of returns with the pricing kernel—which depends on the worst-case dynamics, b^w—and also the difference between the true and worst-case mean levels of consumption growth, $\bar{\mu} - \mu^w$. Of the 634 basis point equity premium, 54 basis points come from the mean effect, while 579 come from the dynamics. So, uncertainty about mean consumption growth is relatively unimportant in driving the equity premium. That result is driven by the fact that the worst-case dynamics have two compounding effects on the equity premium—they make both the pricing kernel and equity returns more volatile, interacting to generate a large increase in the equity premium.[22]

Going back to the first column, we see that there are large and persistent movements in the price/dividend ratio in our model. The one-year autocorrelation of the price/dividend ratio at 0.96 is somewhat higher than the empirical autocorrelation,

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[22]Collin-Dufresne, Johannes, and Lochstoer (2016) also find that uncertainty about dynamics is relatively more important for the equity premium than uncertainty about mean consumption growth.

while the standard deviation is 0.19, similar to the empirical value of 0.29. These results are particularly notable given that there is no free parameter that allows us to directly match the behavior of prices.

Volatility in the price/dividend ratio has the same source as the predictability in equity returns discussed above: the agent prices assets under a model where consumption growth has a persistent component. So following positive shocks, she is willing to pay relatively more, projecting that dividends will continue to grow in the future. From the perspective of an econometrician, these movements seem to be entirely due to discount-rate effects: dividend growth is entirely unpredictable, since dividends are a multiple of consumption, and consumption follows a random walk. On the other hand, from the perspective of the investor (or her worst-case model), there is almost no discount-rate news. Rather, she prices the equity claim differently over time due to variation in forecasts of future cash flows.

The bottom row of Table 1 reports the average gap between the yields on real 1- and 10-year zero-coupon bonds. The term structure is very slightly downward-sloping in the model, a feature it shares with Bansal and Yaron's (2004) results. The downward slope is consistent with the long sample of inflation-indexed bonds from the United Kingdom reported in Evans (1998). A thorough analysis of the implications of our model for the term structure of interest rates is beyond the scope of this paper, but we simply note that the implications of the model for average yields are not wildly at odds with the data and are consistent with past work.

A final feature of the data that papers often try to match is the finding that interest rates and consumption growth seem to be only weakly correlated, suggesting that the EIS is very small. Since consumption growth in this model is unpredictable by construction, standard regressions of consumption growth on lagged interest rates that are meant to estimate the EIS, such as those in Hall (1988) and Campbell and Mankiw (1989), will generate EIS estimates of zero on average.

Return Predictability.—To quantify the degree of predictability in returns, Figure 2 plots percentiles of sample R^2s from regressions of returns on price/dividend ratios in 360-quarter samples (the length of the empirical sample). The gray line is the set of corresponding values from the data between 1926 and 2015. We report R^2s for horizons of one quarter to ten years. At both short and long horizons the model matches the data well. The median R^2 from the predictive regressions at the 10-year horizon is 31 percent, while in the data it is 25 percent. This is in contrast with the complete lack of predictability (reflecting the i.i.d. nature of the true data generating process) of consumption and dividend growth.

Beeler and Campbell (2012) argue that cash flows are excessively predictable in Bansal and Yaron's (2004) calibration. In our setting, there is precisely zero cash-flow predictability by construction. Our results are thus consistent with evidence from various sources on the excess volatility in aggregate asset prices (LeRoy and Porter 1981; Shiller 1981; Cochrane 2008).

In fact, our results are consistent with all three of Beeler and Campbell's (2012) criticisms of the long-run risk model: consumption and dividends are not predicted with asset prices, asset return volatility is not predicted by asset prices, and consumption growth is not predicted by interest rates. In other words, we obtain the major intuition and results of the long-run risk model—that investors fear low-frequency
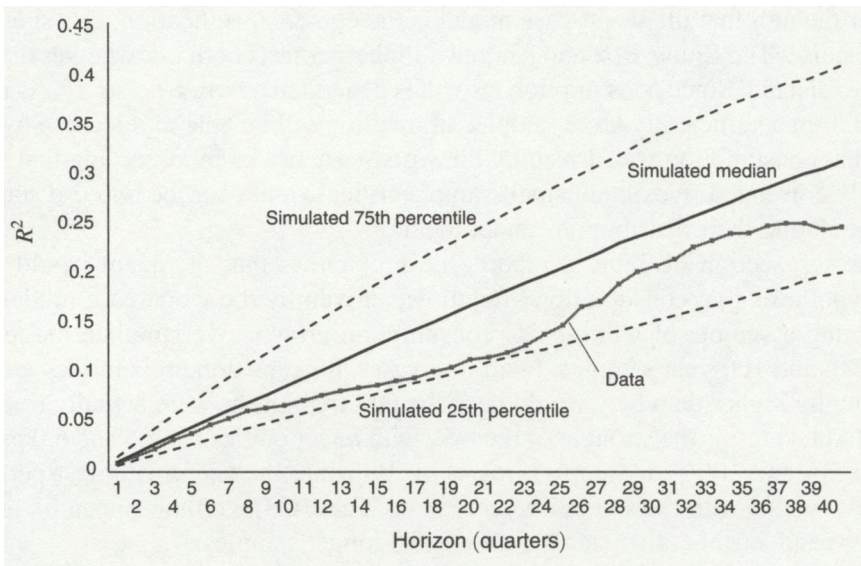
FIGURE 2. EMPIRICAL AND MODEL-IMPLIED R^2S FROM RETURN FORECASTING REGRESSIONS

Notes: Black lines give results from simulated regressions on 90-year samples. The gray line plots R^2s from regressions of aggregate equity returns on the price/dividend ratio between 1926 and 2015.

fluctuations in consumption growth and thus demand large risk premia on stocks—but without relying on significant predictability in consumption, dividends, or volatility.

Of course, we obtain these results by exploiting the wedge between the true data generating process and the one that informs investors' evaluation of risky payoffs. But, as we show below, this wedge is empirically reasonable, in the sense that the worst-case model is a perfectly reasonable process for a person to believe generated the data, given, say, a century of data.

Probability of Rejecting the Worst-Case Dynamics.—For our calibration of λ to be intuitively reasonable, the worst-case model should be thought plausible by the agent. One way of interpreting this statement is that the worst-case model should fit a sample of data generated by the true model nearly as well as the true model itself.

We consider two tests of the fit of the worst-case model to the true white noise consumption process: Ljung and Box's (1978) portmanteau test and the likelihood-based test of an ARMA(1,1) suggested by Andrews and Ploberger (1996).[23] The likelihood-based test is in fact a correctly specified likelihood-ratio test and thus should be asymptotically most powerful. To test that the worst-case model is the correct specification, we take a simulated sample of consumption growth, Δc_t, and construct artificial residuals,

$$(31) \qquad \varepsilon_t^{\Theta^w} \equiv \left(\Delta c_t - \mu_w - a^w(L) \left(\Delta c_{t-1} - \mu_w \right) \right) \left(\sigma_w \right)^{-1}.$$

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[23] The intuition behind this approach is similar to that underpinning the detection error probability (DEP) calculations of Barillas, Hansen, and Sargent (2009), which are widely used to calibrate robustness models. Although we do not report them here, the DEPs in our case also indicate that the worst-case and benchmark models are difficult to distinguish.

Under the null that the worst-case model is the correct specification, $\varepsilon_t^{\Theta^w}$ should be white noise. The Ljung-Box and Andrews-Ploberger tests both ask whether that null can be rejected. Since consumption growth is generated as white noise, $\varepsilon_t^{\Theta^w}$ is in fact not i.i.d. In a sufficiently large sample, an investor will be able to reject the hypothesis that consumption was driven by the worst-case model by observing that $\varepsilon_t^{\Theta^w}$ is serially correlated. We obtain small-sample critical values for the two test statistics by simulating their distributions under the null.

The top section of Table 2 reports the probability that the agent would reject the hypothesis that consumption growth was driven by the worst-case model after observing a sample of white noise consumption growth. We simulate the tests in both 50- and 100-year samples. In all four cases, the rejection probabilities are only marginally higher than they would be if the null hypothesis were actually true. The $ARMA(1,1)$ test is the stronger of the two, with rejection rates of 5.6 and 6.0 percent in the 50- and 100-year samples, respectively, while the Ljung-Box test performs slightly worse, with rates of 5.1 percent in both cases. The online Appendix reports further results using other statistical tests and longer samples.[24]

Table 2 thus shows that the worst-case model, while having economically large differences from the point estimate in terms of its asset pricing implications, can barely be distinguished from the point estimate in long samples of consumption growth. From a statistical perspective, it is entirely plausible that an investor would be concerned that the worst-case model could be what drives the data. Thus both λ and α (which were calibrated jointly with only a single degree of freedom) take on independently reasonable values.

Furthermore, the extreme difficulty of distinguishing the benchmark and worst-case models with long samples of data also suggests that our decision not to model learning explicitly is likely relatively innocuous. Although allowing for learning is conceptually desirable, any learning about low frequency properties of the model would still apply to frequencies far shorter than those that our investor emphasizes under the worst case, as determined by $Z(\omega)$.

Since risk aversion and the conditional volatility of consumption growth are constant, stock returns should not be predictable if the worst-case model is true. So another potential way to test the worst-case model would be to test for return predictability. Since returns are a linear function of current and past consumption growth, though, they contain no information not contained in the history of consumption growth. We focus on formal statistical tests, like the likelihood ratio, because they should use that information efficiently.

Stock prices and returns depend on the low-frequency characteristics of consumption growth, though—exactly where the worst-case deviates from the benchmark. So we might expect them to provide a stronger test. As an example, consider simulations of the R^2 for forecasts of five-year excess equity returns in 100-year samples. The average R^2 when the data is generated by the white noise model but agents price under the worst-case is 0.19. The average R^2 when the data is generated by the worst-case model—in which case the equity premium is constant—is

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[24] The under-rejection in small samples comes from the fact that $\varepsilon_t^{\Theta^w}$ is negatively autocorrelated when the data is driven by the white noise benchmark. The online Appendix explores rejection probabilities for a larger set of test statistics and with samples up to 1,000 years long.

TABLE 2—PROBABILITY OF REJECTING THE PRICING MODEL (*Percent*)

	50-year sample	100-year sample
Rejection Probs. (5 Percent Critical Value, $H_0 =$ Worst-Case Model)		
Ljung-Box	5.1	5.1
ARMA(1,1)	5.6	6.0

ARMA(1,1) rejection probabilities for alternative persistence in pricing model

Persistence	100-year sample	
0.9975	6.0	(Our worst case)
0.995	8.7	
0.99	13.5	(Hansen and Sargent 2010)
0.98	28.3	
0.94	82.7	(Bansal and Yaron 2004)

Notes: Rejection probabilities are obtained by simulating the distributions of the test statistics in 50- and 100-year simulations of the cases where consumption growth is generated by the worst-case and white noise models and asking how often the statistics in the latter simulation are outside the 95 percent range in the former simulation. In the bottom section, persistence is reduced but the price of risk in the pricing model is held constant.

0.13. So our ambiguity aversion substantially increases predictability (to the point that the model matches the data well). But the difference is small relative to the dispersion in R^2: in only 14.7 percent of samples does the R^2 allow the agent to reject the worst-case at the 95-percent level.[25] So even with a test that focuses on the frequencies where the worst-case model deviates from the benchmark most, with a century of data, the agent would still statistically reject the worst-case very infrequently.

Alternative Calibrations of the Pricing Model.—We derive the worst-case model endogenously, but similar models have also been assumed for investor expectations. Bansal and Yaron (2004) argue that a model with trend shocks with a quarterly persistence 0.94 fits the data well. Hansen and Sargent (2010) consider a setting where investors focus on two models for consumption growth, the more persistent of which has a trend component with an autocorrelation of 0.99. Due to ambiguity aversion in their model, asset prices are primarily driven by the more persistent model.

The bottom section of Table 2 examines how rejection probabilities change if we modify the pricing model to use a less persistent trend. In all rows we hold $b^w(\beta)$ fixed and simply modify the persistence of consumption growth under the pricing (null) model. In other words, we ask how easy different models are to distinguish from white noise, holding constant the relevant measure of risk and varying the persistence of shocks.

The top row is the calibration from the main analysis, where persistence is equal to the time discount factor. As the degree of persistence falls, the investor's ability to reject the pricing model in a century-long sample rapidly improves. When the persistence is 0.99, as in Hansen and Sargent (2010), the pricing model is rejected 13.5 percent of the time—twice as often as our endogenous worst-case model. When the persistence falls to 0.94 as in Bansal and Yaron (2004), the pricing model

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[25] That is, the R^2 for data generated under the benchmark is above the ninety-fifth percentile of the R^2 for data generated by the worst-case in 14.7 percent of the samples.

is rejected 82.7 percent of the time. The result that the persistence of the worst-case model should be equal to β is clearly key to ensuring that the model is difficult to reject in simulated data.

B. Estimated Consumption Dynamics

We now examine the worst-case scenario associated with an estimated small-scale model of consumption growth. The above analysis assumes that the point estimate for consumption dynamics is that consumption growth is i.i.d. We now relax that assumption and examine a more sophisticated estimation framework.

We study quarterly data on per capita nondurables and services consumption in the United States. The Bayesian information criterion leads us to the choice of an ARMA(1,1) from a range of small-scale ARMA models (up to an ARMA(5,5)). A key feature of the estimate is that the spectral density $\bar{f}(\omega)$ varies across frequencies, which allows us to ask whether variation in estimation uncertainty across frequencies is qualitatively or quantitatively relevant in determining the worst-case model.

We concentrate on the worst-case choice of dynamics and abstract from distortions to the innovation mean and variance. We first consider what worst-case model the agent would derive if she were constrained to minimize utility with respect to a transfer function implied by an ARMA(1,1). That is, utility is minimized by choosing a worst-case $\{\rho, \theta\}$ in the model

$$(32) \qquad \Delta c_t = \mu + \rho(\Delta c_{t-1} - \mu) + \varepsilon_t - \theta\varepsilon_{t-1}$$

in order to minimize lifetime utility subject to the penalty $\lambda g(b; \bar{b})$. We denote this "parametric" worst case as b^p. The restriction that the worst case only allows changes in the parameters ρ and θ is typical in the literature; it assumes that investors know the model driving consumption growth and they need only estimate its parameters.[26] We compare b^p (which depends only on $\{\rho^p, \theta^p\}$) to the b^w obtained following the main analysis above (i.e., allowing an arbitrary b^w, instead of just an ARMA(1,1)).

In Figure 3 we plot the real part of the transfer functions implied under the benchmark and the two worst cases. Except at low frequencies, the parametric worst case is essentially indistinguishable from the point estimate, while the nonparametric worst case differs much more dramatically. Intuitively, since there are only two free parameters in the restricted parametric problem, it is impossible to generate the deviations very close to frequency zero that have both high utility cost and low detectability. So, instead of large deviations on a few frequencies, as in the nonparametric case, the parametric worst case puts very small deviations on a wide range of frequencies.

The specific parameters in the benchmark and parametric worst-case models are $\{\bar{\rho}, \bar{\theta}\} = \{0.789, 0.587\}$ and $\{\rho^p, \theta^p\} = \{0.803, 0.600\}$ respectively, implying $\bar{b}(\beta)$ = 1.94 and $b^p(\beta) = 2.02$. The parametric worst-case model is thus nearly identical

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[26] See Collin-Dufresne, Johannes, and Lochstoer (2016) for a deep analysis of the case of parameter uncertainty with Epstein-Zin preferences. Andrei, Carlin, and Hasler (2013) study a model in which investors disagree about the persistence of trend growth rates.
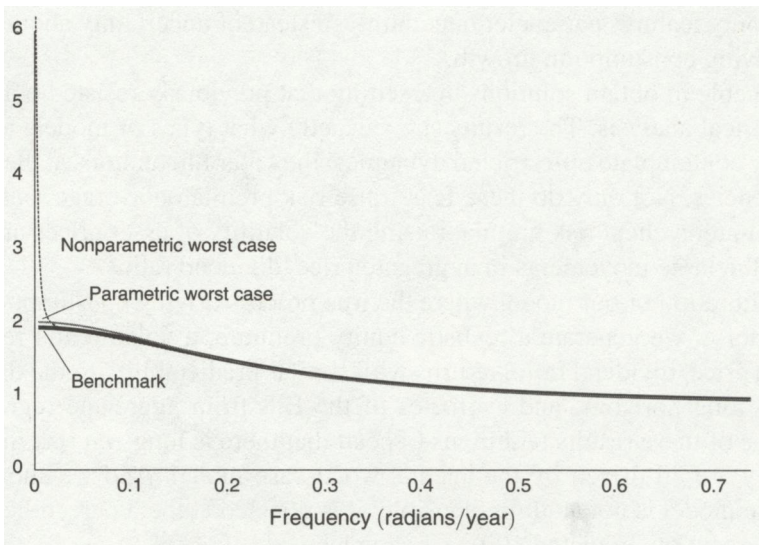
FIGURE 3. BENCHMARK AND WORST-CASE TRANSFER FUNCTION

Notes: Benchmark, parametric, and nonparametric transfer functions for the ARMA$(1,1)$ specification estimated on postwar US consumption growth. The parametric worst case is the ARMA$(1,1)$ that minimizes lifetime utility subject to the same KL divergence as is used for the nonparametric worst case.

to the benchmark model. The relevant measure of risk in the economy is thus essentially identical under the two models, meaning that the equity premium is almost completely unaffected by parameter uncertainty of this sort. In contrast, under the unconstrained nonparametric worst case $b^w(\beta) = 4.40$, more than two times higher than $\bar{b}(\beta)$.

Thus, when we allow the agent to choose an unrestricted worst-case model, the outcome is very similar to what we obtained for the white noise benchmark. The worst-case deviates from the benchmark at *very* low frequencies. This is true even though in this setting the estimation uncertainty $\left(\text{through } \bar{f}(\omega)\right)$ varies across frequencies.

VII. Conclusion

This paper studies asset pricing when agents are unsure about the endowment process. The fundamental insight is that the long-run risk model, precisely because it is difficult to test for empirically and yet has important welfare implications, represents a natural model for investors who are unsure of the true data-generating process to use for pricing assets. More technically, for an agent with Epstein-Zin preferences who estimates consumption dynamics nonparametrically, the model that leads to the lowest lifetime utility for a given level of plausibility displays large amounts of long-run risk in consumption growth. In fact, when the agent's point estimate is that consumption growth is i.i.d., the worst-case model is literally the homoskedastic long-run risk model of Bansal and Yaron (2004). Furthermore, the nonparametric worst-case model can differ substantially from a parametric worst

case that only features parameter uncertainty, instead of uncertainty about the actual model driving consumption growth.

We are able to obtain solutions in a setting that previously resisted both analytic and numerical analysis. The results show exactly what types of models agents fear when they contemplate unrestricted dynamics: they fear fluctuations at the very lowest frequencies. Not only do these fears raise risk premia on average, but they also induce countercyclical risk premia, raising the volatility of asset prices and helping to match the large movements in aggregate price/dividend ratios.

In a calibration of our model where the true process driving consumption growth is white noise, we generate a realistic equity premium, a volatile and realistically persistent price/dividend ratio, returns with similar predictability to the data at both short and long horizons, and estimates of the EIS from aggregate regressions of zero. None of these results require us to posit that there is long-run risk in the economy. They are all driven by the agent's worst-case model. And we show that the worst-case model is not at all implausible: it is rejected at the 5 percent level in less than 10 percent of simulated 100-year samples.

Economists have spent years arguing over what the consumption process is. We argue that a reasonable strategy, and one that is tractable to solve, for an investor facing that type of uncertainty, would be to make plans for a worst-case scenario. The message of this paper is that worst-case planning is able to explain a host of features of the data that were heretofore viewed as puzzling and difficult to explain in a setting that was even remotely rational.

MATHEMATICAL APPENDIX

A. *Minimization*

This Appendix provides a derivation of Proposition 1 and the worst-case scenario for the white-noise benchmark. The full set of results is available in the online Appendix. The investor's optimization problem to find the worst-case model is

$$\text{(A1)} \qquad \min_{b,\mu,\sigma^2} \frac{\beta}{1-\beta} \frac{1-\alpha}{2} b(\beta)^2 \sigma^2 + \frac{\beta}{1-\beta}\mu$$

$$\text{(A2)} \qquad + \frac{\lambda}{2}\left[\int \frac{f(\omega)}{\bar{f}(\omega)} - \log\frac{f(\omega)}{\bar{f}(\omega)}\,d\omega + \frac{(\mu-\bar{\mu})^2}{\bar{f}(0)}\right],$$

where the integral sign without limits denotes $\frac{1}{2\pi}\int_{-\pi}^{\pi}$.

The first-order condition for μ yields

$$\text{(A3)} \qquad \mu_w = \bar{\mu} - \lambda^{-1}\frac{\beta}{1-\beta}\bar{f}(0).$$

This is the typical result that the uncertainty about the mean depends on the spectral density at frequency zero.

The spectral density $f(\omega)$ can be expressed as

$$(A4) \qquad f(\omega) = \exp\left(2\sum_{j=0}^{\infty} c_j \cos(\omega j)\right),$$

for a set of real coefficients c_j (Priestley 1981). The coefficients c_j are simply the Fourier coefficients of the log of the spectrum; we only include coefficients for non-negative j because the spectrum is a real and even function. Furthermore, setting $\sigma = \exp(c_0)$, we have

$$(A5) \qquad \sigma B(\omega) = \exp\left(\sum_{j=0}^{\infty} c_j e^{i\omega j}\right)$$

$$(A6) \qquad b_m = \int e^{-i\omega m} \exp\left(\sum_{j=1}^{\infty} c_j e^{i\omega j}\right) d\omega,$$

where the b_j are the coefficients from the Wold representation for the spectrum $f(\omega) = \sigma^2 |B(\omega)|^2$ (Priestley 1981). Since $\sigma = \exp(c_0)$, $b_0 = 1$. Furthermore, $b_j = 0$ for all $j < 0$. (A5) is known as the canonical factorization of the spectrum. We solve the optimization problem by directly choosing the c_j. Since the Fourier transform is one-to-one, choosing the c_j is equivalent to optimizing over the spectrum directly. Since $B^w(\omega)$ is obtained from the Wold representation, it is guaranteed to be causal, invertible, and minimum-phase. Last, the innovation variance associated with the spectrum $f(\omega)$ is $\sigma^2 = \exp(2c_0)$.

We first calculate derivatives involved in the optimization

$$(A7) \qquad \frac{d}{dc_j}[\sigma b(\beta)] = \frac{d}{dc_j} \sum_{m=0}^{\infty} \beta^m \int \exp\left(\sum_{j=0}^{\infty} c_j e^{i\omega j}\right) e^{-i\omega m} d\omega$$

$$(A8) \qquad = \sum_{m=0}^{\infty} \beta^m \int \exp\left(\sum_{j=0}^{\infty} c_j e^{i\omega j}\right) e^{-i\omega(m-j)} d\omega$$

$$(A9) \qquad = \sigma \sum_{m=0}^{\infty} \beta^m b_{m-j} = \sigma b(\beta) \beta^j,$$

where the derivative can be passed inside the integral because $B(\omega)$ is continuous and differentiable with respect to the c_j.

Next, the derivative of the ratio of the spectra is

$$(A10) \qquad \frac{d}{dc_j} \int \frac{f(\omega)}{\bar{f}(\omega)} d\omega = \frac{d}{dc_j} \int \frac{\exp\left(2\sum_{j=0}^{\infty} c_j \cos(\omega j)\right)}{\bar{f}(\omega)} d\omega$$

$$(A11) \qquad = 2\int \frac{f(\omega)}{\bar{f}(\omega)} \cos(\omega j) \, d\omega.$$

And last,

(A12)
$$\frac{d}{dc_0} \int \log \frac{f(\omega)}{\bar{f}(\omega)} \, d\omega = 2.$$

The first-order condition (FOC) for each $j > 0$ is

(A13)
$$0 = 2\frac{\beta}{1-\beta}\frac{1-\alpha}{2}\sigma_w^2 b^w(\beta)^2 \beta^j + \frac{\lambda}{2} \int \frac{f^w(\omega)}{\bar{f}(\omega)} 2\cos(\omega j) \, d\omega.$$

For $j = 0$,

(A14)
$$0 = 2\frac{\beta}{1-\beta}\frac{1-\alpha}{2}\sigma_w^2 b^w(\beta)^2 + \frac{\lambda}{2} \int \frac{f^w(\omega)}{\bar{f}(\omega)} 2 \, d\omega - \lambda.$$

Now multiply each of the FOCs by $\cos(j\kappa)$ for some κ. We then sum over the FOCs, with the FOC for $j = 0$ multiplied by $1/2$, to obtain

(A15)
$$0 = \frac{\beta}{1-\beta}\frac{1-\alpha}{2}\sigma_w^2 b^w(\beta)^2 \left(1 + \sum_{j=1}^{\infty} 2\cos(j\kappa)\beta^j\right)$$

(A16)
$$+ \frac{\lambda}{2} \int \frac{f^w(\omega)}{\bar{f}(\omega)} \left(1 + \sum_{j=1}^{\infty} 2\cos(j(\kappa+\omega))\right) d\omega - \frac{\lambda}{2}.$$

We have

(A17)
$$1 + \sum_{j=1}^{\infty} 2\cos(j(\kappa+\omega)) = \delta(\kappa+\omega),$$

where $\delta(\cdot)$ is the Dirac delta function. Furthermore, notice that $Z(\omega)$ is the transfer function of an AR(1) model with autocorrelation of β and unit innovation variance. It then follows that

(A18)
$$(1 - \beta^2)|Z(\kappa)|^2 = 1 + \sum_{j=1}^{\infty} 2\cos(j\kappa)\beta^j,$$

(A15) finally becomes

(A19)
$$0 = \frac{\beta}{1-\beta}\frac{1-\alpha}{2}\sigma_w^2 b^w(\beta)^2 (1 - \beta^2)|Z(\omega)|^2 + \frac{\lambda}{2}\left(\frac{f^w(\omega)}{\bar{f}(\omega)} - 1\right)$$

(A20)
$$f^w(\omega) = \bar{f}(\omega) + \lambda^{-1}\beta(1+\beta)(\alpha-1)\bar{f}(\kappa)\sigma_w^2 b^w(\beta)^2 |Z(\omega)|^2,$$

which is the result in Proposition 1.

B. The White Noise Benchmark

In the white noise case, $\bar{b}(L) = 1$. The mean is then

$$\mu_w = \bar{\mu} - \lambda^{-1}\frac{\beta}{1-\beta}\bar{\sigma}^2.$$

For the dynamics, we have

(A21) $f^w(\omega) = \bar{\sigma}^2 + \lambda^{-1}\beta(1 + \beta)(\alpha - 1)\bar{\sigma}^2\sigma_w^2 b^w(\beta)^2 |Z(\omega)|^2.$

Denote the autocovariances under the worst-case model as γ_j^w. Then, since the spectrum is

(A22) $$f^w(\omega) = \gamma_0^w + 2\sum_{j=1}^{\infty} \gamma_j^w \cos(\omega j),$$

we have

(A23) $\gamma_0^w + 2\sum_{j=1}^{\infty} \gamma_j^w \cos(\omega j) = \bar{\sigma}^2\left(1 + \varphi\left(1 + 2\sum_{j=1}^{\infty} \cos(\omega j)\beta^j\right)\right)$

(A24)   where $\varphi \equiv \lambda^{-1}\dfrac{\beta}{1 - \beta}(\alpha - 1)b^w(\beta)^2\sigma_w^2.$

Matching coefficients on each side yields

(A25) $\gamma_0^w = \bar{\sigma}^2(1 + \varphi)$

(A26) $\gamma_j^w = \bar{\sigma}^2\varphi\beta^j$ for $|j| > 0.$

These may be recognized as the autocovariances of an ARMA(1,1) process. Specifically, set

(A27) $\Delta c_t = x_t + v_t$

(A28) $x_t = \beta x_{t-1} + \mu_t$

(A29) $\sigma_v^2 = \bar{\sigma}^2$

(A30) $\sigma_\mu^2 = \bar{\sigma}^2\varphi(1 - \beta^2),$

(where x_t is unobservable). Then one may confirm that Δc_t has autocovariances γ_j^w. Furthermore, note that given the above, $\Delta c_t - \beta\Delta c_{t-1}$ follows an MA(1) process. The online Appendix shows how to derive the univariate ARMA(1,1) representation. One may solve for φ numerically, for example through an iteration: initially guess that $\varphi = \lambda^{-1}\dfrac{\beta}{1 - \beta}(\alpha - 1)\bar{\sigma}^2$, then solve for $\sigma_w b^w(\beta)$ as a function of that guess and update φ.

C. Multiplier Preference Interpretation

In our main analysis, we model agents as having Epstein-Zin preferences. Such preferences are observationally equivalent (in the sense that they rank all consumption streams identically) to Hansen and Sargent's (2001) multiplier preferences. In that model, agents have log utility over consumption, but they form expectations

using a worst-case model over *innovations* to the consumption process. Specifically, their preferences are obtained through

$$(A31) \qquad v_t = \min_{h_{t+1}} c_t + \beta\left(E_t[h_{t+1} v_{t+1}] + \psi E_t[h_{t+1} \log h_{t+1}]\right),$$

where h_{t+1} is a change of measure with $E[h_{t+1}] = 1$. h_{t+1} represents an alternative distribution of the innovations to the state variables at date $t + 1$. In this model, agents select an alternative distribution for *innovations* (instead of a full distribution over consumption growth) penalizing alternative distributions based on their KL divergence $(E_t[h_{t+1} \log h_{t+1}])$.

Inserting the value of h_{t+1} that solves the minimization problem yields

$$(A32) \qquad v_t = c_t - \beta\psi \, \log E_t \exp\left(-\psi^{-1} v_{t+1}\right).$$

That is, the Epstein–Zin preferences used in the main text can be interpreted as multiplier preferences with $-\psi^{-1} = (1 - \alpha)$.

We can thus interpret the model described in the paper as involving two layers of robustness, or two evil agents. First, there is an evil agent who, in a timeless manner, selects a full worst-case process for consumption growth. Next, taking that process as given, a second evil agent causes further deviations in each period's innovations.

The second evil agent's minimization problem is (A31), and the minimized value function is (A32), which is exactly the preference specification that is minimized in the main text.

In other words, both the minimization problem over the full models for consumption growth that we study and also the minimization over one-step deviations—which induces Epstein–Zin preferences—depend on a KL divergence penalty.

A natural benchmark is to equalize the penalty on the KL divergence that is involved in both minimization problems. Since the entropy penalty for the second agent is applied in every period, we naturally scale it up by the discount rate. That is,

$$(A33) \qquad \lambda = \psi/(1 - \beta).$$

Which immediately yields a connection between λ and α,

$$(A34) \qquad \lambda = \frac{1}{1 - \beta} \frac{1}{\alpha - 1}$$

$$(A35) \qquad \alpha = 1 + \frac{\lambda^{-1}}{1 - \beta}.$$

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