MFE Economics
Problem set 3

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## 1. Stochastic growth model

- Technology process

$$Y_t = K_t^{\alpha}\left(\theta_t L_t\right)^{1-\alpha}$$
$$\theta_t = e^{z_t} > 0$$
$$z_t = \mu(1-\rho)t + \rho z_{t-1} + \varepsilon_t$$

- Technology has trend component and persistent shocks

$$\log\theta_t = \mu(1-\rho)t + \rho\log\theta_{t-1} + \varepsilon_t$$

- Firm maximises profits

$$\max_{K_t, L_t}\left(K_t^{\alpha}\left(\theta_t L_t\right)^{1-\alpha} - w_t L_t - r_t K_t\right)$$

- Representative household maximizes lifetime utility

$$\begin{aligned}\max_{L_{t+s}, C_{t+s}, K_{t+s}} \quad & E_t \quad \sum_{s=0}^{\infty}\beta^s(\log C_{t+s} - \chi L_{t+s}) \\ s.t. & \\ K_{t+s+1} = r_{t+s}K_{t+s} + w_{t+s}L_{t+s} - C_{t+s}\ \forall s \geq 0 & \\ K_t\ given & \\ \lim_{T\to\infty} E_t[\Lambda_{t,T}K_{t+1}] \geq 0 &\end{aligned}$$

- Derive the firm and household optimality conditions.[1]
- Show that the only stable solution has (like in the lecture 2 model) implies $\frac{Y_t}{C_t} = \frac{1}{1-\alpha\beta}$

[1]Note this is very similar to the model discussed in lecture 2 but with an expectation operator $E_t$ floating around, which doesn't hugely change the appearance of the expressions - the main difference is allowing for non-trivial labor supply decisions.

- Using the fact that (as you should have shown in the first part of the question) firm optimality implies the real wage is equal to the marginal product of labor, show that $\frac{Y_t}{C_t} = \frac{1}{1-\alpha\beta}$, combined with the household's labor-consumption optimality condition, yields an expression for equilibrium $L_t$ in terms of deep parameters $(\alpha, \beta, \chi)$
- Does equilibrium labor vary over time?
- Payments to capital (or capitalists that own them) are $rK_t$. Show that the capital share of income ($\frac{r_t K_t}{Y_t}$) equals $\alpha$ - so the labor share is $1-\alpha$. What number should we set $\alpha$ to be for the US, on this basis? Consult here and here. Briefly discuss/critique your answer.
- Given the previous answers and assuming $\beta = 0.95$, what should $\chi$ be for steady state $L$ to be 1/3 of the time endowment (8 hours out of a 24 hour day spent working)

**2. Correcting an inefficiency with a Pigouvian subsidy**

Consider the monopolistically competitive firm discussed in the previous problem set. The chosen output level (reflecting the chosen price) is such that price is a markup on marginal cost. This is inefficient. Effectively, the price reflects the marginal valuation of the output by the marginal consumer. Since the price is above marginal cost it means that at $Y^*$ (the output level implied by the chosen price, $P^*$, given the demand curve) the 'consumer' values an extra marginal unit more than the cost to the firm of producing that marginal unit (the marginal cost), thus there are possible gains from trade that are not being exploited - the consumer would be willing to pay a price between $MC(Y^*)$ and $P(Y^*)$ for that good. The firm would be fine with that and so would the consumer - implying a Pareto improvement. This will be possible right up to the point, call it $Y^{Eff}$, where we have price = marginal cost.

For efficiency we would like to somehow induce the firm to produces $Y^{Eff} > Y^*$. Note that the firm produces to the point at which marginal cost = marginal revenue where the latter lies below the demand curve. One can offer a subsidy to the firm to change the 'private' marginal cost it faces. In particular, if the 'government' offers to pay fraction $\tau$ of the wage then for a given wage received by a worker, $W_t$, the firm only pays $(1-\tau)W_t$ - so from the firm's perspective this lowers their marginal cost. Call this private marginal cost $\widehat{MC}(Y_t) = \frac{(1-\tau)W_t}{MPN_t} = (1-\tau)MC(Y_t)$ (recall marginal cost is wage paid by firm divided by the marginal product of labor, at the firm's privately optimal choice).

What value should $\tau$ (a 'Pigouvian' tax) take to induce the firm to produce $Y^{Eff}$? HINT: The firm will still produce to the point where its *private* MC = MR

**3. Optional: Calvo Pricing**

**RB: See the Gali textbook for discussions of Calvo pricing. This question is to get you to do a bit of thinking about this pricing protocol before I cover it in class. If you find it tricky, don't worry, as I will be going over it.**

Under the Calvo (1983) pricing protocol a firm faces a probability, $\theta$, of being unable to reset its price in any given period and this probability is independent of how long any given price has so far been prevailing. Sometimes people imagine a 'Calvo fairy' who, in any given period, will wave a wand over the firm's head (with probability $1-\theta$) and allow them to reset their price.

- What is the probability of a price prevailing $j$ periods after it has been chosen? *HINT: Think about the sequence of fairy arrivals or non arrivals and remember the probabilities are independent over time so you can multiply appropriate probabilities together...*
- Show that the expected duration of a price (i.e. how many periods on average will it prevail before being reset) is $(1-\theta)^{-1}$? Note: Don't spend more than 20 mins on this bit - either it will pop into your head how to rearrange the algebra, or it won't - but try for 20mins...
- What value of $\theta$ would I pick my for my (quarterly) model to match real world data that suggests prices typically prevail for 3 quarters?

**4. Autoregressive processes**

In the models we consider, there are random 'shock' or 'driving' processes that are exogenous to the model.[2] In our case these shocks (technology, time preference and monetary policy) will constitute the 'state' of the economy in the sense that all the endogenous variables (consumption, output, wages, interest rates) will in equilibrium be expressible as functions of these three variables (or subsets thereof, depending on the model). All the shocks we consider, when logged, follow an autoregressive process of order 1 or, for short, an $AR(1)$. Consider the technology shock $A_t$ from the production function $Y_t = A_t N_t^{1-\alpha}$. When expressed in logs it follows this process[3]

$$\begin{aligned} a_t &= \rho_a a_{t-1} + \varepsilon_{a,t} \\ \varepsilon_{a,t} &\overset{iid}{\sim} N(0, \sigma_a^2) \\ a_t &\equiv \log(A_t) \end{aligned} \tag{1}$$

The random variable, $\varepsilon_{a,t}$ (what I will often refer to as an 'innovation'), is a Gaussian or 'Normal' variable with zero mean ($E_t[\varepsilon_{a,t+1}] = 0$) and variance, $\sigma_a^2$ ($E_t[(\varepsilon_{a,t+1} - 0)^2] = \sigma_a^2$). It is also independently and identically distributed (iid) which means that there is no dependence between its draws in different periods or between its draw and any other variables and the distribution from which it is drawn ($N(0, \sigma_a^2)$) is constant over time. The parameter $\rho_a$ will be referred to as a 'persistence' parameter. We will always (in this course) consider cases where $|\rho_a| \in (0,1)$ - in the language of stochastic processes, this ensures that it is a 'stationary' process.

[2] You can brush up on random variables here, on Normal variables here and on expected value (or 'the mean') here.

[3] Note that is convenient to model it this way as it means that $A_t$, while random, will always be positive (as makes sense for a technology term that multiplies - some function of - labor to produce non-negative output). Frequently in economics or finance we use the exponential of a random variable to obtain a transformed random variable that is positive. In fact, exponentials have other nice properties when working with Normal distributions.

- Show that[4]

$$a_t = \sum_{j=0}^{J-1} \rho_a^j \varepsilon_{a,t-j} + \rho_a^J a_{t-J} \tag{2}$$

Suppose data started in $t = 0$, so that $a_0$ is just given to us, then we just set $J = t$ in the above expression. If there is no explicit starting point then we can use the assumption on $|\rho_a| \in (0,1)$ to state

$$a_t = \sum_{j=0}^{\infty} \rho_a^j \varepsilon_{a,t-j}$$

since $\rho_a^J a_{t-J} \to 0$ as $J \to \infty$. Intuitively, if the effects of shocks dies of to zero in the limit - and given that our shocks are well behaved - we can ignore the last term in expression (**??**) you just derived because it gets arbitrarily small.

- What is the effect of an innovation $j$ periods ago on $a_t$ (i.e. the effect of $\varepsilon_{a,t-j}$)?
- What is the effect of an innovation in $t$ on $a_{t+1}$? On $a_{t+2}$? On $a_{t+j}$?

Any effect an innovation in $t$ has on future values of $a_t$, say $a_{t+j}$, may be flooded by the effects of future innovations in later periods before period $t + j$. But it is still useful to talk about the effect the innovation has as this nevertheless does *contribute* to $a_{t+j}$ (look back at the expression you derived above - $a_t$ is made up of a weighted sum of all current and previous innovations, with those weights declining as the innovation period recedes into the distant past). Given the process we are considering and the iid assumptions made on $\varepsilon_{a,t}$, an innovation today does affect the expected value - from the perspective of today - of future values of the technology shock.[5]

- What is the expected value of $a_{t+1}$ given information available in $t$ (i.e. given you know $a_t$)?
- What is the expected value of $a_{t+2}$ given information available in $t$
- What is the expected value of $a_{t+j}$ given information available in $t$
- What is the expected value of $\Delta a_{t+1} \equiv a_{t+1} - a_t$ given information available in $t$
- How does today's ($t$) innovation affect your expectation of $a_{t+j}$ relative to the expectation you held in $t-1$ before you knew $\varepsilon_{a,t}$?

In the models we consider we will often express the equilibrium values of endogenous variables as functions of $a_t$ (and other shocks). Suppose a variable $s_t$ is expressed

$$s_t = \psi_0 + \psi_1 a_t$$

[4]HINT: Use equation (**??**) but for earlier periods ($a_{t-1}$, $a_{t-2}$ etc.) to repeatedly replace the lagged values of $a_t$ on the right hand side of equation (**??**).

[5]To answer questions involving expectations below, recall that the expectations operator is linear (in particular, that means $E_t[X+Y] = E_t[X] + E_t[Y]$), the expectation of a constant (or something already known when the expectation is being formed) is the constant itself and the expectation of a scalar constant times a random variable is the scalar constant times the expectation of the random variable.

- What is $s_{t+1}$ in terms of $a_{t+1}$?
- What is $s_{t+1}$ in terms of $a_t$ and $\varepsilon_{a,t+1}$?
- What is the expected value of $s_{t+1}$ given information available in $t$ (i.e. given you know $a_t$)?
- What is the expected value of $\Delta s_{t+1} \equiv s_{t+1} - s_t$ given information available in $t$

**5. Very optional: Non-separable consumption-labor preferences**

**RB: This question involves quite a tedious amount of algebra but ut's a good exercise - especially for being confident in your linearizations/log-linearizations. Have a go - but know that you'll soon have answers to refer to if it gets a bit much. If you work through it systematically though, it should be fine.**

Do question 2.1 in Galí (first question in exercises at end of chapter 2). This entails deriving the equivalent of equations (7), (8) and then (10) in the text (based on the 2nd edition of the textbook). Note that $Z_t$ is dropped from the preferences and the utility function is given by...

$$U(C_t, N_t) = \frac{\left[C_t \left(1 - N_t\right)^v\right]^{1-\sigma} - 1}{1 - \sigma}$$