EC956 - New Keynesian Modeling
2-period Household Decision Problem

Setting up the household's decision problem

I was asked in class to go through the origins of the optimality conditions (7) and (8) on P. 21 in the textbook. I agree that it is good background, even though the heuristic arguments in the book/slides for why those conditions are necessary for optimality should be fairly clear: 'marginal benefit' = 'marginal cost' of any affordable/feasible change in labor/consumption/savings at the optimum. It turns out that in an infinite horizon setting the tools you need to do this 'properly' are a bit more involved than I would like so we will instead consider a simplified finite horizon (2-period) version of the problem that still gets the main point across. As we shall see, if labor decisions are involved, then even this simplified case can throw up some extra issues...

Endowment income example (no labor)

In this case, instead of the agent earning income from supplying hours of labor, we assume that she simply is supplied with an endowment in each period. Her utility function is defined only with respect to consumption and she chooses her consumption/savings to maximize the expected discounted (with β) sum of utilities over her lifetime (of two periods). In this case, this comes down to choosing C_1 and C_2 or, equivalently in this case, period 1 savings, in the form of bond holdings.

We assume that in period 1 she only has resources from her endowment, M_1 and in period 2 has resources from her endowment M_2 plus the payoffs of the B_1 riskless nominal 'discount' bonds she purchased in $t = 1$ as savings. Each bond in $t = 1$ costs Q_1 dollars and pays off \$1 for sure in $t = 2$. We could introduce bond purchases in the second period, B_2, but since the agent does not live beyond $t = 2$ we know that in an optimum she will not choose to save in the second period - thus we set $B_2 = 0$ from the outset.

Thus the problem then is

$$\max_{C_1,C_2} \frac{C_1^{1-\sigma} - 1}{1-\sigma} + \beta E_1\left[\frac{C_2^{1-\sigma} - 1}{1-\sigma}\right] \tag{1}$$

subject to

$$P_1C_1 + Q_1B_1 = M_1 \tag{2}$$
$$P_2C_2 = B_1 + M_2 \tag{3}$$

Note that equation (3) implies that we can replace C_2 in the objective function and eliminate a constraint, allowing us to re-express the problem in terms of choosing C_1 and B_1

$$\max_{C_1,B_1} \frac{C_1^{1-\sigma}-1}{1-\sigma} + \beta E_1\left[\frac{\left(\frac{B_1+M_2}{P_2}\right)-1}{1-\sigma}\right] \tag{4}$$

subject to

$$P_1C_1 + Q_1B_1 = M_1 \tag{5}$$

This representation has the (minor) advantage notationally that we are choosing objects both of which are known in $t = 1$, savings and current consumption.[1] Letting the Lagrange multiplier on the remaining constraint be given by λ we get first order conditions ('differentiate and set equal to zero') with respect to C_1 and B_1 of[2]

$$C_1^{-\sigma} - \lambda P_1 = 0 \tag{6}$$
$$\beta E_1\left[\frac{1}{P_2}\left(\frac{B_1+M_2}{P_2}\right)^{-\sigma}\right] - \lambda Q_1 = 0 \tag{7}$$

Then, using these two conditions together to eliminate λ and recalling the definition of C_2 in terms of savings, endowments and P_2, we obtain an optimality condition of the sort we see in the textbook

[1] We could also use the remaining constraint to substitute out C_1 and then maximize solely with respect to B_1 but it's useful to see what the nature of the Lagrange multiplier is - specifically its connection to marginal utility of consumption, adjusting for the price level. Ultimately, once one variable - say B_1 is chosen in $t = 1$ consumption in the two periods is pinned down.

[2] The Lagrangian would be

$$\mathcal{L}(C_1, B_1, \lambda) = \frac{C_1^{1-\sigma}-1}{1-\sigma} + \beta E_1\left[\frac{\left(\frac{B_1+M_2}{P_2}\right)-1}{1-\sigma}\right] - \lambda(P_1C_1 + Q_1B_1 - M_1)$$

for consumption-savings decisions (it is typically known as an 'Euler' equation)

$$Q_1 = E_1 \left[\beta \left(\frac{C_2}{C_1} \right)^{-\sigma} \frac{P_1}{P_2} \right] \tag{8}$$

Earned income example (involves labor decision)

Now we add labor. The complication here is that the agent must consider her choice of hours supplied in $t = 2$ when taking decisions in $t = 1$. Since the next period's wage and price level are random, as of $t = 1$, the optimal labor supply in $t = 2$ will also be random from the perspective of $t = 1$. In this case, the agent is best thought of as choosing a *rule* for labor supply (and, implicitly, consumption) in $t = 2$ as a function of the next period's 'state' (whatever that may be), which we will denote by s, where s is distributed according to a distribution function F. Effectively, we will think of the agent as choosing C_1, N_1, B_1 - all of which are known at $t = 1$ and do not depend on s - plus a value of N_2 *for each realization of* s *next period.*[3] This effectively amounts to choosing a function, $\mathcal{N}$ such that $N_2 = \mathcal{N}(s)$ where s takes values from the set $\mathcal{S}$.[4]

The maximization problem is thus written as

$$\max_{C_1, B_1, N_1, \{N_2(s)\}_{s \in \mathcal{S}}} \frac{C_1^{1-\sigma} - 1}{1 - \sigma} - \frac{N_1^{1+\varphi}}{1 + \varphi} + \beta E_1 \left[\frac{\left(\frac{B_1 + W_2 N_2}{P_2} \right) - 1}{1 - \sigma} - \frac{N_2^{1+\varphi}}{1 + \varphi} \right] \tag{9}$$

or, making the dependence on the state explicit

$$\max_{C_1, B_1, N_1, \{N_2(s)\}_{s \in \mathcal{S}}} \frac{C_1^{1-\sigma} - 1}{1 - \sigma} - \frac{N_1^{1+\varphi}}{1 + \varphi} + \beta \int_{s \in \mathcal{S}} \frac{\left(\frac{B_1 + W_2(s) N_2(s)}{P_2(s)} \right) - 1}{1 - \sigma} - \frac{N_2(s)^{1+\varphi}}{1 + \varphi} dF(s) \tag{10}$$

subject to

$$P_1 C_1 + Q_1 B_1 = W_1 N_1 \tag{11}$$

[3]Note that once these variables are decided then $C_2(s)$ can be calculated using the budget constraint.

[4]The formal analysis here is very loose but it gets the point across.

The first order conditions with respect to C_1, B_1 and N_1 are

$$\begin{aligned}
C_1^{-\sigma} - \lambda P_1 &= 0 \qquad (12)\\
\beta E_1 \left[C_2^{-\sigma} \frac{1}{P_2} \right] - \lambda Q_1 & \qquad (13)\\
-N_1^{\varphi} + \lambda W_1 = 0 & \qquad (14)
\end{aligned}$$

from which we can derive the Euler equation pretty much as before and by combining the first and third condition, we obtain the intratemporal optimality condition from class, for $t = 1$

$$\frac{N_1^{\varphi}}{C_1^{-\sigma}} = \frac{W_1}{P_1} \qquad (15)$$

Finally, for each s we have the FOC for $N_2(s)$ given by

$$\frac{W_2(s)}{P_2(s)} \left(\overbrace{\frac{B_1 + W_2(s) N_2(s)}{P_2(s)}}^{C_2(s)} \right)^{-\sigma} - N_2(s)^{\varphi} = 0 \qquad (16)$$

which implies, for every s

$$\frac{N_2(s)^{\varphi}}{C_2(s)^{-\sigma}} = \frac{W_2(s)}{P_2(s)} \qquad (17)$$

This is part of the optimal plan of the agent, as envisaged at $t = 1$. The agent must anticipate this in order to make the right decision today, as it will influence (via labor income next period) what resources will be available for consumption tomorrow, given savings carried over from today. It is not enough that the intratemporal condition holds in expectation - it must hold under every state realization tomorrow.