EC956 - New Keynesian Modeling
Problem set 4

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1. Problems to try on your own - answers posted

1.1. Elasticities

As a quick refresher, if we have a function $f(x)$ then its **elasticity** with respect to x is defined as

$$\frac{x}{f(x)}\frac{df(x)}{dx}$$

which gives the percentage change in f for a percentage change in x (for small changes). In fact, one can alternatively calculate it as

$$\frac{d\log\left(f(x)\right)}{d\log\left(x\right)}$$

Now, consider the production function of firm i

$$Y_{i,t} = A_t N_{i,t}^{1-\alpha}$$

What is the elasticity of output with respect to labor?

Now, consider the Euler log-linearized Euler equation

$$\begin{aligned} c_t &= E_t[c_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - \rho) + \frac{1}{\sigma}(1-\rho_z)z_t \\ &= E_t[c_{t+1}] - \frac{1}{\sigma}(r_t - \rho) + \frac{1}{\sigma}(1-\rho_z)z_t \end{aligned}$$

where we have used the Fisher equation ($r_t = i_t - E_t[\pi_{t+1}]$). What is the elasticity of consumption with respect to the real interest rate (or nominal interest rate, for that matter)?

Answers

We have that

$$\frac{dY_{i,t}}{dN_{i,t}} = (1-\alpha)A_t N_{i,t}^{-\alpha}$$

so that

$$\begin{aligned}
\frac{N_{i,t}}{Y_{i,t}}\frac{dY_{i,t}}{dN_{i,t}} &= (1-\alpha)A_t N_{i,t}^{-\alpha}\frac{N_{i,t}}{Y_{i,t}} \\
&= (1-\alpha)A_t N_{i,t}^{1-\alpha}\frac{1}{A_t N_{i,t}^{1-\alpha}} \\
&= 1-\alpha
\end{aligned}$$

Alternatively, we could have used the derivation involving log versions (lower case) of the variables

$$\frac{d}{dn_{i,t}}\log\left(y_{i,t}\right) = \frac{d}{dn_{i,t}}(a_t + (1-\alpha)n_{i,t}) = 1-\alpha$$

In the case of the Euler equation we are already working in terms of log variables (r_t is the log of the 'gross' real interest rate) so we simply obtain the 'elasticity of intertemporal substitution', σ^{-1}.

1.2. Galí composite shock

Although it is natural to look for equilibrium functions for the endogenous variables in terms of the fundamental shocks, such as

$$\begin{aligned}
\pi_t &= \psi_{\pi,a}a_t + \psi_{\pi,z}z_t + \psi_{\pi,v}v_t \\
\tilde{y}_t &= \psi_{\tilde{y},a}a_t + \psi_{\tilde{y},z}z_t + \psi_{\tilde{y},v}v_t
\end{aligned}$$

Galí looks for expressions in terms of the 'composite' shock, u_t (in Ch. 3, P. 65 onward)

$$\begin{aligned}
\pi_t &= \psi_{\pi,u}u_t \\
\tilde{y}_t &= \psi_{\tilde{y},u}u_t \\
u_t &= -\psi_{yn,a}\left(\phi_y + \sigma(1-\rho_a)\right)a_t + (1-\rho_z)z_t - v_t
\end{aligned}$$

He then makes the assumption that u_t follows an AR(1) and then derives $\psi_{\pi,u}$ and $\psi_{\tilde{y},u}$ - i.e. he solves the model treating u_t as the only state.

Without going into detail, this approach relies very heavily on the linearity of the model and an assumption that u_t follows an AR(1) (which it isn't), rather than acknowledging that it is the sum of different AR(1) processes. Nevertheless, if one calculates the response to an innovation to u_t then one can in fact use that response to derive the correct responses to innovations to a_t, z_t and v_t.

- What values of impulses to the various fundamental shocks (i.e. δ_a, δ_z and δ_v) will induce a unit impulse in u_t?
- What innovation simultaneously to v_t will cancel out an innovation to z_t?

Answers

We obtain the values simply through inverting the coefficients in the equation defining u_t...

$$\begin{aligned}
\delta_a &= -\frac{1}{\psi_{yn,a}\left(\phi_y + \sigma(1-\rho_a)\right)} \\
\delta_z &= \frac{1}{1-\rho_z} \\
\delta_v &= -1
\end{aligned}$$

To offset an innovation to v_t, δ_v, the innovation to z_t would need to be $\delta_z = \frac{\delta_v}{1-\rho_z}$

2. Problems to try on your own - answers posted and Emil goes through in class

2.1. DIS and NKPC

The dynamic IS curve (DIS) and the New Keynesian Phillips curve (NKPC) are given by

$$\begin{aligned}
y_t &= E_t[y_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\pi_{t+1}] - \rho) + \frac{1}{\sigma}(1-\rho_z)z_t \\
\pi_t &= \beta E_t[\pi_{t+1}] + \kappa \tilde{y}_t
\end{aligned}$$

First we will consider the DIS curve and go through a few thought experiments to investigate some of its properties.

- Sketch a plot of the DIS with y_t on the horizontal axis and i_t on the vertical axis.
- What is the slope?

Note that implicitly the location of the curve depends on assumed values for the structural parameters (σ, ρ_z and $\rho \equiv -\log(\beta)$), the other endogenous variables that feature in the DIS ($E_t[y_{t+1}]$, $E_t[\pi_{t+1}]$) and the preference shock (z_t). Recall, also, that in our simple model $y_t = c_t$ (in richer models it might also feature investment and government expenditure).

- What happens to the curve in the following situations? Give intuition.
  - Suppose something shifts $E_t[y_{t+1}]$ up, holding all else equal.

  - Suppose something shifts $E_t[\pi_{t+1}]$ up, holding all else equal.
  - What if z_t increases, holding all else equal.
  - What if β increases, or σ?

- Why is 'holding all else equal' an unnatural assumption in this context?
- Redraw the DIS but now with the real interest rate on the vertical axis
  - Suppose, again, something shifts $E_t[\pi_{t+1}]$ up, holding all else equal. What movement is associated with this, now that we have re-drawn the DIS? Is it a curve shift? Comment.

Now we turn to the NKPC and consider some more thought experiments

- Sketch a plot of the NKPC with $\tilde{y}_t$ on the horizontal axis and π_t on the vertical axis.
- What is the slope?

Note that implicitly the location of the curve depends on assumed values for the structural parameters (β and recall κ is a function of various parameters) and expected inflation next period ($E_t[\pi_{t+1}]$).

- What happens to the curve in the following situations? Give intuition.
  - Suppose something shifts $E_t[\tilde{y}_{t+2}]$ up, holding all else equal.
  - Suppose something shifts $E_t[\tilde{y}_{t+3}]$ up, holding all else equal. Compare to your previous answer.
  - What if θ increases, or φ?
- Why is 'holding all else equal' an unnatural assumption in this context?

Answers

Before we get going on answering this question I just want to point out that 'shifting curves' is a very popular way of learning in undergraduate economics but, in my opinion, it is problematic. The DIS curve offers a good example (setting aside the 'unnatural' aspect of holding all else equal in equilibrium - see later part of question).

The DIS involves various endogenous variables, not just i_t and y_t. Specifically, it involves $E_t[y_{t+1}]$ and $E_t[\pi_{t+1}]$, which are no less a part of the DIS than i_t and y_t. Really, if we could draw in four (or, five, including z_t) dimensions we could completely sketch the DIS and then, when considering the first two 'curve shifting' experiments below, we would see that nothing is actually shifting - the relationship among all the variables is unchanged. Instead, when we choose to look at the relationship in 2-D space we can only look at a slice of the overall plot of the relationship - i.e. the relationship between i_t and y_t for a particular pair of $E_t[y_{t+1}]$ and $E_t[\pi_{t+1}]$. The first two experiments below are simply a question of looking at different slices - and in 2-D that looks like a curve shift.

If we wanted to imagine a 5-D relationship (adding z_t into the mix) then the same sort of logic applies, as discussed above. When it comes to changes in the deep parameters, though, I think it is fair to think of the relationship changing - and the 5-D association between i_t, y_t, $E_t[y_{t+1}]$, $E_t[\pi_{t+1}]$ and z_t being 'shifted' (though of course this is just a matter of taste, you could imagine an 8-D surface with axes for the parameters too - though that doesn't seem natural). Anyway...

As shown on the left hand side of figure 1 below, the DIS is a downward sloping relation in (y_t, i_t) space with slope $-\sigma$. Now, in terms of the experiments...

- If there is an increase in $E_t[y_{t+1}]$ then the curve 'shifts' (remember the previous discussion though) rightwards - i.e. at any given i_t, y_t will be higher. The intuition is that the consumer's optimality condition (on which this curve is effectively based since $c_t = y_t$ in equilibrium) reflects a desire for consumption smoothing.

- Higher expected inflation, for any given i_t implies a lower real interest rate - this makes consumption today more desirable, relative to consumption tomorrow (**which is being held equal**), so the curve shifts rightwards. In fact, it is perhaps more natural to think of it shifting vertically as now, a higher nominal interest rate is associated with a given y_t since that higher i_t, combined with the higher $E_t[\pi_{t+1}]$ yields the same r_t that was previously associated with that value of y_t (holding the other elements of the DIS constant).

- If z_t increases, all else equal, then it is as if the household becomes more impatient (recall, a higher z_t shows up a bit like a lower β) so that they want to pull consumption forward to the present. Hence there is more demand in the current period for a given i_t (or for a given r_t, which, since inflation expectations are being held fixed, is equivalent to a given i_t), which looks like a rightward shift in the the DIS, in (y_t, i_t) space.

- A higher β (as the previous answer suggests) would imply a leftward shift in the curve (less demand for current consumption relative to future consumption - which is being held fixed) for a given i_t due to the household's increased patience. Note that you need to recall that ρ is a transformed version of β. A change in σ actually changes the slope of the line in (y_t, i_t) space so that the slope becomes steeper (more steeply downward sloping). Recall that σ^{-1} is the elasticity of intertemporal substitution (i.e. how willing the agent is to substitute consumption over time in response to a change in the market terms of trade). The more willing you are to substitute (the larger is σ^{-1}, or the smaller is σ) the smaller a change in i_t is required for (or will be associated with) a given change in c_t, which in this case is equal to y_t.

Why is 'holding all else equal' unnatural in this context? We are only working with one of the equations that implicitly defines the equilibrium values of the endogenous variables in terms of the underlying state. In equilibrium, we use this and other (e.g. market clearing relations, the NKPC, the Taylor rule,...) to solve explicitly for the functions relating the endogenous variables to the state. The endogenous variables do not have direct causal influences on eachother (without making additional assumptions) and, most relevantly for this question, when they move, it must be because of some change in the underlying state. Generally, if one endogenous variable is changing (such as the changes considered in our questions) the state that is implicitly driving that change will, in equilibrium, also be changing the other variables. Thus, it is a bit unnatural to assume they *don't* change (which is what we mean by 'holding all else equal').

In the middle of figure 1 we see the DIS curve re-written in (r_t, y_t) space, where we recall $r_t = i_t - E_t[\pi_{t+1}]$. It looks pretty much as before - downward sloping with slope $-\sigma$. Note that once we re-express DIS explicitly in terms of r_t a change in inflation expectations, all else equal, is like a movement along the curve rather than a shift in the curve since it changes r_t for a given i_t where the latter is held fixed.

Now, turning to the NKPC, the right hand side of figure 1 shows that it is an upward sloping relations, with slope κ. Recall that κ is defined as follows

$$\begin{aligned}
\lambda &\equiv \theta^{-1}(1-\theta)(1-\beta\theta)\Theta > 0 \\
\Theta &\equiv \frac{1-\alpha}{1-\alpha+\alpha\varepsilon} \\
\kappa &\equiv \lambda\left(\sigma + \frac{\varphi+\alpha}{1-\alpha}\right)
\end{aligned}$$

Now, what if $E_t[\tilde{y}_{t+2}]$ increases, holding all else equal? Note that $E_t[\tilde{y}_{t+2}]$ doesn't figure explicitly but recall that we can re-express the relation as

$$\pi_t = \kappa \sum_{k=0}^{\infty} \beta^k E_t[\tilde{y}_{t+k}]$$

so we see that there will be a rightward shift in the curve, or perhaps it is more natural to think of it as an upward shift. Future output gap is expected to be higher and, thus, future marginal costs are expected to be higher, all else equal, so firms setting their prices today anticipate this (recall they are forward looking due to the fact the price they set today may prevail for several periods) and raise their prices more then they otherwise would have given the *current* output gap, implying higher inflation (hence the 'shift' upwards). The next part of the question is pretty much the same but the shift will be smaller for a given change in future expected output gap because it is weighted by $\kappa\beta^3 < \kappa\beta^2$. Again, holding everything else equal is somewhat unnatural in this context, perhaps most obviously because any news today that the output gap will (in expectation) be higher in two (or three) periods' time will 'likely' imply that expectations of the output gap will shift at other horizons too.

Regarding changes in parameters, as discussed in the previous homework, an increase in θ lowers λ and thus flattens the slope of the curve. In contrast, increasing φ steepens the Phillips curve.

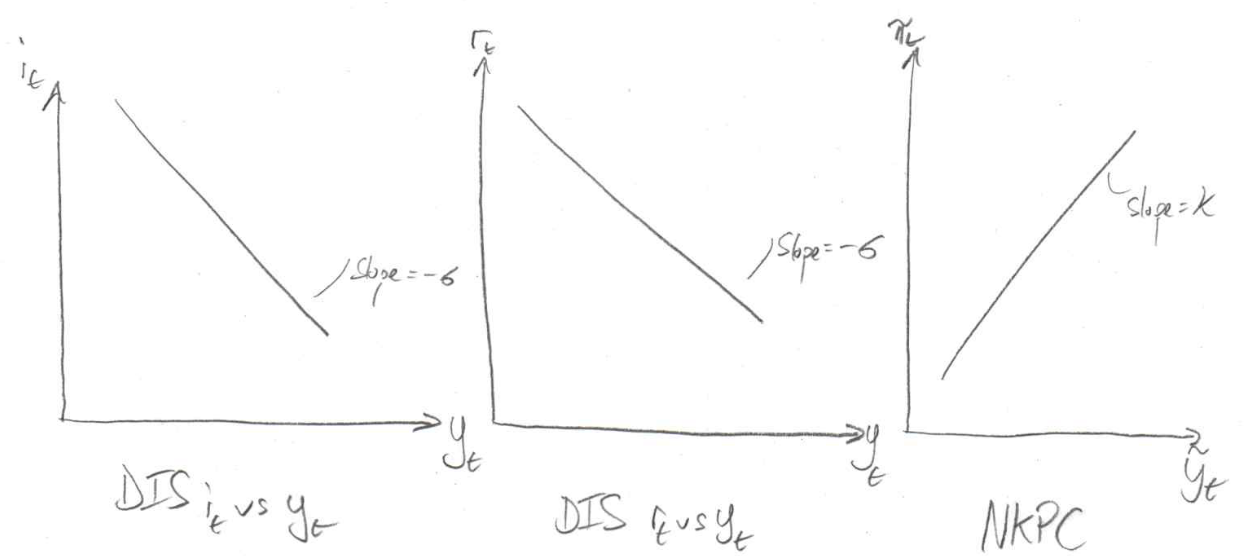

Figure 1: Sketch Dynamic IS and New Keynesian Phillips Curves

2.2. Definitions of IRFs

Our equilibrium implies variables, say y_t, can be expressed in terms of three shocks

$$y_t = \psi_y + \psi_{y,a}a_t + \psi_{y,z}z_t + \psi_{y,v}v_t$$

In class, we defined the impulse response of y_t to a_t as the difference between the paths of y_t under two different sequences of innovations. We called one, the 'baseline' path and the other, the 'shocked' path. The baseline path of the economy entails

$$\varepsilon_{a,t} = \varepsilon_{z,t} = \varepsilon_{v,t} = 0 \ \forall t$$

and the shocked path of the economy entails

$$\begin{aligned} \varepsilon_{a,1} &= \delta_a \\ \varepsilon_{a,t} &= 0 \ \forall t > 1 \\ \varepsilon_{z,t} = \varepsilon_{v,t} &= 0 \ \forall t \end{aligned}$$

As shown in class this yields the impulse response function

$$\Delta_{y,a}(t) = \psi_{y,a} \rho_a^{t-1} \delta_a$$

- What is the 'impact effect' (i.e. effect in $t = 1$ when the innovation hits)?
- Consider two sizes of impulse $\delta_a^{(B)} > \delta_a^{(S)} > 0$. Comment on the relationship between the impulse response to these two differently sized impulses - take their difference and their ratio. What does the ratio depend on (really, what does in *not* depend on)? *Hint: Try to distinguish magnitude from shape.*
- Consider two impulses, δ_a and $-\delta_a$. Comment on the relationship between the impulse response to these two differently signed impulses - take their ratio.
- In what way does the impulse response depend on the value of a_t prevailing in the period before the impulse?
- Do the properties you derived in the previous part of this question seem 'sensible'? Can you think of some simple examples/stories where the effect of an impulse might be dependent on the size/sign of the impulse and the state the economy is in on impact (in a more meaningful way than they do in our case)?

We defined an IRF by comparing two paths under a sequence of assumed future innovation realizations. Suppose *instead* someone (very reasonably) might want to think of an impulse response as how his/her **expectations** of the future might change, given an impulse today. In this case we will calculate the difference in expectations after the impulse vs without the impulse, acknowledging that future innovations are random variables.[1]

[1] We continue to assume $\varepsilon_{z,1} = \varepsilon_{v,1} = 0$

- What is the impulse response under this new approach of defining it as the difference in expected values of y_{t+j} conditional on $\varepsilon_{a,1} = \delta_a$ and conditional on $\varepsilon_{a,1} = 0$? NOTE: For this part of the question, eliminate z_t and v_t from the analysis, just to simplify the algebra slightly.

Answers

The impact effect in this case is $\psi_{y,a}\rho_a^0\delta_a = \psi_{y,a}\delta_a$. Now, under the two differently sized innovations we have the difference between the IRFs as

$$\psi_{y,a}\rho_a^{t-1}(\delta_a^{(B)} - \delta_a^{(S)})$$

and the ratio

$$\frac{\delta_a^{(B)}}{\delta_a^{(S)}}$$

The point here is that, obviously, the IRFs depend *in some sense* on the size of the initial impulse (bigger impulse means bigger response), but it is not a particularly 'interesting' dependence. In particular, the shape of the impulse response is effectively pinned down by the persistence parameter, ρ_a so that the two responses are scaled versions of eachother with, as the ratio shows, the scaling being determined by the relative size of the impulse. Similarly, if we compare impulses under a positive and negative impulse we find that the ratio is -1. Again, this is not a very meaningful difference as the responses are essentially the same but flipped, as we would expect. Also note that none of these responses make any reference to the value of a_0 - the technology prevailing prior to the innovation.

Thus, if you tell me one IRF in a model such as ours, you've essentially told me all IRFs (for the same shock). This means that when working with models such as this, it isn't that vital to specify the size or type of shock when plotting pictures of IRFs, as long as one is consistent. However, there is the convention that one picks meaningful sizes so the reader doesn't have to do the (easy but tedious) math to get a sense of what the model is saying. For example, people often pick the size of the innovation to be equal to a single standard deviation (in this case that would be σ_a) with the understanding that is the size of a 'typical' shock. Alternatively, one might scale the innovation size so that a given variable has an interpretable movement on impact (such as picking a monetary policy shock to induce a 25 basis point change in i_t - which might be though to be a relevant size of surprise, given Fed behavior).

Beware, however. This is an extremely special property of the linear models we have been dealing with (or at least they are linear after all our first order approximations). Nonlinear models (and the real world) will not exhibit this property - at least qualitatively (it is an empirical question how important nonlinearities are). Generally, we believe that a massive shock to, say, monetary policy, may have a different effect even beyond scaling and there is a lot of work discussing whether cuts in interest rates have the same effect as increases. Also, perhaps reflecting worse scope to borrow - and various other malfunctioning aspects of the economy - there is sometimes the belief that fiscal stimulus can be more effective when the 'state' of the economy is a recession - suggesting that the impact of a fiscal shock might be different, depending on what the economy looks like when it hits. Linear models do not allow this and traditional IRF analysis of the type we have been using is inadequate to model this.[2]

Another aspect of our assumption of linearity (and symmetric shocks, in fact) that is very special is that the redefined IRF (in terms of differences in expectations) gives the same answer as our originally defined IRF. This again reflects linearity of the model and the linearity of the expectations operator, as shown below...

First, follow the (small) simplification suggested in the question

$$y_t = \psi_y + \psi_{y,a} a_t$$

and (recalling problem set 2) note that

$$a_t = \sum_{j=0}^{t-1} \rho_a^j \varepsilon_{a,t-j} + \rho^t a_0$$

The innovation hits in $t = 1$ ($\varepsilon_{a,1} = \delta_a$) so we are looking for the difference in expectations of y_t conditional on $t = 1$ information where in the baseline case, part of that information is $\varepsilon_{a,1} = 0$ and in the shocked case, part of that information is $\varepsilon_{a,1} = \delta_a$. Thus we calculate

$$\begin{aligned} E_1[y_t | \varepsilon_{a,1} = 0] &= E_1[\psi_y + \psi_{y,a} a_t | \varepsilon_{a,1} = 0] \\ &= \psi_y + \psi_{y,a} E_1[a_t | \varepsilon_{a,1} = 0] \\ &= \psi_y + \psi_{y,a} E_1 \left[\rho_a^{t-1} \varepsilon_{a,1} + \sum_{j=0}^{t-2} \rho_a^j \varepsilon_{a,t-j} + \rho^t a_0 | \varepsilon_{a,1} = 0 \right] \\ &= \psi_y + \psi_{y,a} \rho^t a_0 \end{aligned}$$

[2] If you are interested see here, here and here.

and we also calculate

$$\begin{aligned}
E_1[y_t|\varepsilon_{a,1}=\delta_a] &= E_1[\psi_y+\psi_{y,a}a_t|\varepsilon_{a,1}=\delta_a] \\
&= \psi_y+\psi_{y,a}E_1[a_t|\varepsilon_{a,1}=\delta_a] \\
&= \psi_y+\psi_{y,a}E_1\left[\rho_a^{t-1}\varepsilon_{a,1}+\sum_{j=0}^{t-2}\rho_a^j\varepsilon_{a,t-j}+\rho^t a_0|\varepsilon_{a,1}=\delta_a\right] \\
&= \psi_y+\psi_{y,a}(\rho^t a_0+\rho_a^{t-1}\delta_a)
\end{aligned}$$

Taking the difference of these we obtain $\psi_{y,a}\rho_a^{t-1}\delta_a$ which is the same as under our previous definition.