Machine Learning

Empirical and Structural Risk. Error Decomposition. Model Selection. Underfitting and overfitting

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ML Research

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SOFIA UNIVERSITY

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1. Structural Risk and its Minimization
2. Overfittning and underfitting

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1. Structural Risk and its Minimization
2. Overfittning and underfitting
3. Model Selection overview
4. Bias-variance tradeoff
5. Recent results: Double Descent

Instance-based learning

- X – set of objects descriptions, Y – set of objects labels
- Unknown target dependency: mapping $y : X \to Y$
- Finite training set: $X^m = \{(x_1, y_1), \ldots, (x_m, y_m)\}$, so as $y_i = y(x_i)$

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- **Empirical Risk Minimization** (ERM) – the common approach to solve the broad range of tasks of inductive learning (e.g., classification / regression tasks)

Empirical risk: definitions

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Empirical Risk (ER)

Performance metric reflecting the average error made by an algorithm a upon the set X^m: $R(a, X^m) = \frac{1}{m}\sum_{i=1}^{m} L(a(x_i), y(x_i))$

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ERM cons

Overfitting on the training set X^m. Happens almost always when using ERM, because the performance criteria is the error **on the very same set** (solution: to measure the performance it makes sense to change the set)

Loss functions examples

Classification task

- Classification error: $L(a, x) = L(\hat{y}, y) = [\hat{y} \neq y] = 1 - \delta_y(\hat{y})$
- The function is discontinuous $\Rightarrow$ ERM is a task of combinatorial optimization $\Rightarrow$ in many practical applications can be reduced to the search of maximal consistent subsystem of inequality system (number of inequalities is equal to the number of training examples m) $\Rightarrow$ NP-hard

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Regression task

- Squared error: $L(\hat{y}, y) = (\hat{y} - y)^2$

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Hard to guess in advance what is the right form of the regularization term $C(a)$ and what should be the regularization weight λ

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Definition

Overfitting is an undesirable phenomenon that occurs when solving problems of learning by precedents, when the probability of the error of the trained algorithm on the objects of the test sample is significantly higher than the average error on the training sample.
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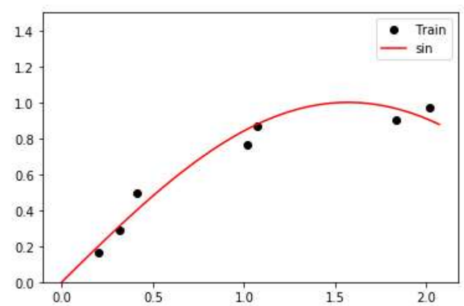
Train error observation

Overfitting and Underfitting: examples



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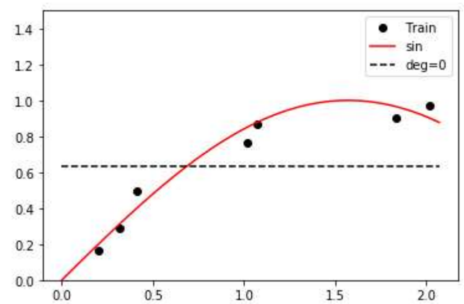

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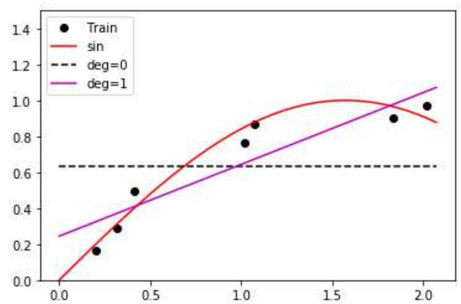

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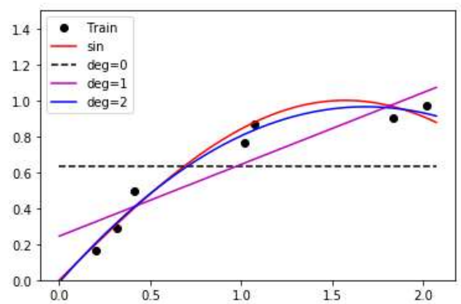

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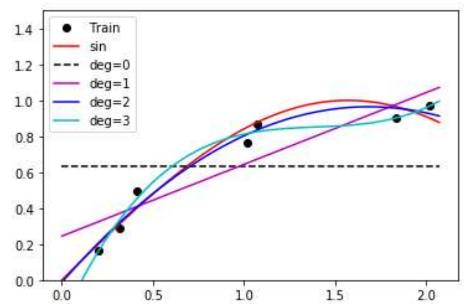

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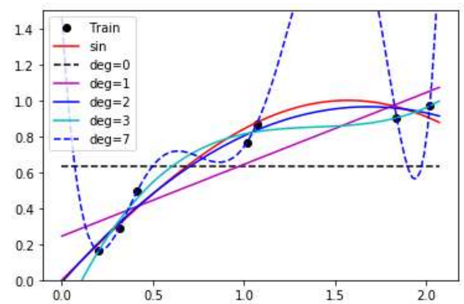

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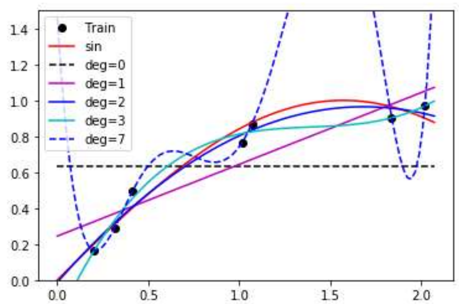

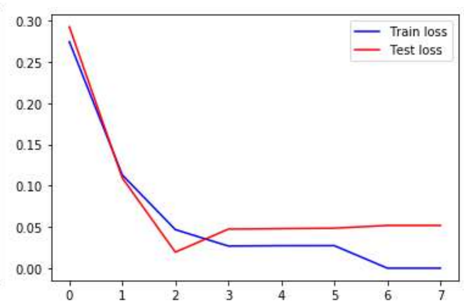

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- **Hyperparameters**: the degree of the polynomial n, which is chosen before training starts; then chosen from the set of hyperparameters tested on the validation set

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- Explainability (tradeoff between good and interpretable model)

Derivation of mean squared error expression

Definitions

Let $y = y(x) = f(x) + \varepsilon$ be the target dependence, where $f(x)$ is the deterministic function, $\varepsilon \sim N(0, \sigma^2)$ and $a(x)$ is the machine learning algorithm.

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= Dy + Da + (E(f-a))^2 = \sigma^2 + variance(a) + bias^2(f,a)
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The mean squared error decomposition in the example above is called the **bias-variance tradeoff**

Model of Optimal Complexity: Classic View

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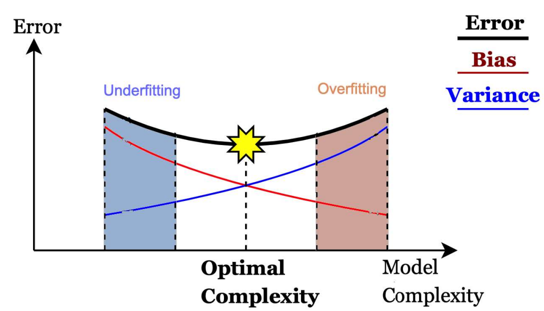
- The optimal complexity of the model is somewhere between



Model of Optimal Complexity: Recent Empirical Evidence

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- This behavior is called **double descent**

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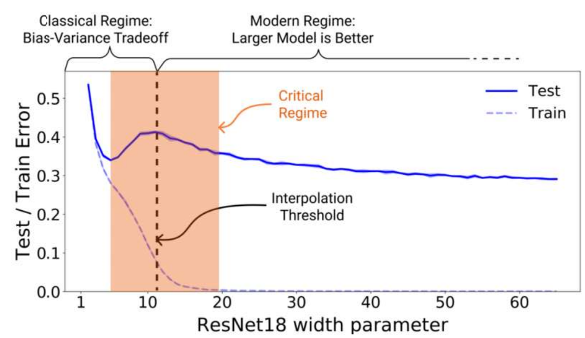
Model of Optimal Complexity: Double Descent

- Example of double descent in practice[2]:



[2] Image source: `https://arxiv.org/pdf/1912.02292.pdf`

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4. A lot of different considerations should be taken into account while thinking of the most appropriate model choice
5. In case of a huge amount of data and parameters ($\approx$billions), classical estimates stop working

Thank you!