Stanford University
AA222: Engineering Design Optimization
Spring 2020
Prof. Mykel J. Kochenderfer • Durand 255 • email: *mykel@stanford.edu*

PROJECT 3 **Due date: June 5th, noon PDT**

You have until Friday June 5, noon PDT to complete this project. The late policy will be no late days. **Assignments submitted past the deadline will receive 0 credit.** You may consult any material (e.g., books and online resources), and you may discuss the questions with others at a conceptual level, but you cannot share solutions or code with anyone else, inside or outside of the class. Although you may discuss with others, we encourage you to try to answer the questions on your own first. We would like you to leave this class feeling comfortable enough to apply an optimization method you have learned to a real-world project (e.g. for research or work). Questions may be asked during office hours and on Piazza. Please choose a platform, e.g. Julia/Python/MATLAB, that you are comfortable with and will be able to debug in because there will be very minimal code support from the course staff.

Answer either question 1 or question 2 to receive full credit. Indicate which question you have chosen to be your "main question" by tagging the correct pages on gradescope. Answer the other question to recieve up to a 20% bonus. Indicate that this question is your "bonus question" by tagging the correct pages on gradescope.

Question 1. A manufacturer of vitamin supplements wants to launch a new multivitamin supplement and needs help defining an ideal recipe. The product is composed of two ingredients and a filler. One kilogram of the product must contain a minimum of the four vitamins and minerals described below:

Nutrient	Vitamin A	Vitamin B	Vitamin C	Minerals
grams	90	50	20	2

Each ingredient contains a different amount of vitamins and minerals. The following table describes the nutritional content in grams per kilogram and cost per kilogram of each ingredient.

	Vitamin A	Vitamin B	Vitamin C	Minerals	cost/kg
Ingredient 1	200	130	15	0	60
Ingredient 2	100	70	35	7	40
Filler	0	0	0	0	0

Q1 Problem 1 What is the cheapest combination of ingredients and filler to produce one kg of the product?

- **Task 1** (5pts) Formulate an optimization problem to solve Problem 1.
- **Task 2** (5pts) Solve the optimization problem using any approach you want. Include how you solved the problem (code, pen and paper or any other method you used) and the optimal solution and value of the cost function. We suggest that you consider using one of the following optimization packages: CVXPY (Python), MATLAB, and JuMP/convex.jl (Julia), but you are free to use anything else.
- **Task 3** (5pts) What is the ideal recipe for the product? What is the cost of manufacturing it? What is its nutritional composition?

Q1 Problem 2 What is the cheapest combination of ingredients and filler to produce one kilogram of product with the requirements below:

- If we use Ingredient 2 we incur in a fixed cost of 25 as storing the ingredient requires a refrigerated warehouse.
- We only need to satisfy at least 2 of the nutritional requirements.

  - **Task 4** (5pts) Formulate an optimization problem to solve Problem 2.

    Hint: Note that for some binary variable x_b and bounded real-valued variable $x \in [l, u] \subset \mathbb{R}$, the constraint $x_b \geq \frac{x}{u}$ forces

$$x_b = \begin{cases} 1 & \text{if } x > 0 \\ 0 \text{ or } 1 & \text{otherwise} \end{cases}$$

Think of how this approach can be used when x_b forms part of the objective function with a positive coefficient.

- **Task 5** (5pts) Solve the optimization problem using any approach you want. Include how you solved the problem (code, pen and paper or any other method you used) and the optimal solution and value of the cost function. We suggest that you consider using one of the following optimization packages: CVXPY (Python), SciPy (Python), MATLAB, and JuMP (Julia), but you are free to use what you choose.

- **Task 6** (5pts) Provide an intrerpretation of the results in the context of the problem, explain what the ideal recipe for the product is, what the cost of manufacturing it is and its nutritional composition. Explain which nutritional requirements are satisfied with the resulting recipe.

Question 2. *Expression Optimization.*

In this question, you will use expression optimization to produce approximate closed-form expressions for the motion of stars in a trinary star system.

Background: The three body problem is a problem in classical mechanics that describes the motion of three celestial bodies moving under the influence of gravity. We can write down the differential equations that govern the motion of the three bodies, i.e. how their positions $r_i(t)$ change in time,[1] but solving this system of equations for the positions of the bodies $r_i(t)$ is entirely another matter. There are, however, solutions to the two-body problem (think the Moon orbiting around Earth), but for the three body problem, researchers have only discovered, usually numerically, initial conditions that correspond to periodic motion.[2] There is no general closed form solution as a function of time, and many of the periodic orbits that have been discovered do not have associated closed form expressions. So, we are going to try to find one of these closed form expressions!

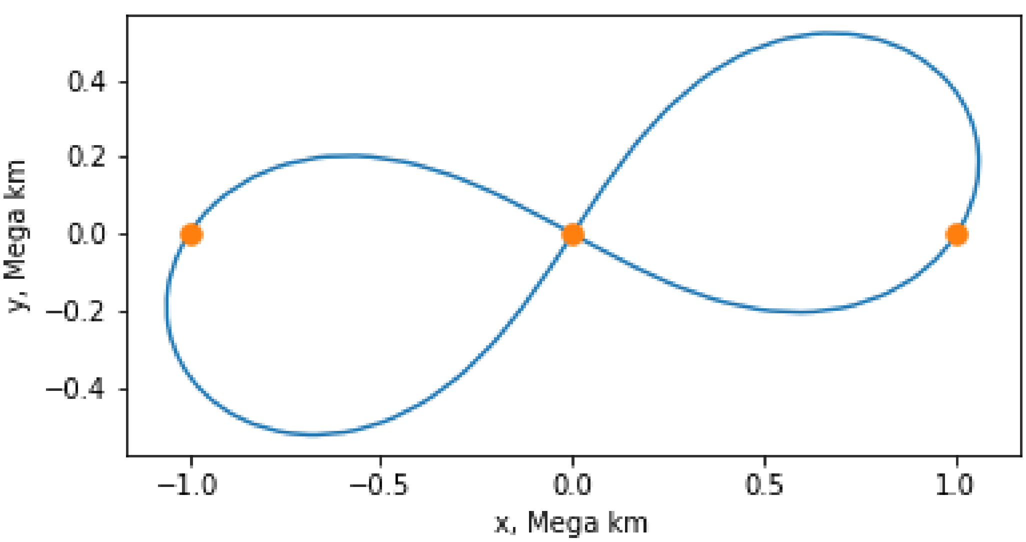

Figure 1: The trajectories of the stars in the trinary star system all follow this curve shown here, but begin at different points on the curve. For a gif of the orbits see this website: http://three-body.ipb.ac.rs/sol.php?id=1

[1]The position of each of the three bodies is described by a position vector $r_i \in \mathbb{R}^3$ that points from the origin point x_0 in the global frame to the center of mass of each body. Each body has a distinct mass m_i. The differential equations governing motion are stated as follows:

$$\ddot{r}_1 = -Gm_2 \frac{r_1 - r_2}{|r_1 - r_2|^3} - Gm_3 \frac{r_1 - r_3}{|r_1 - r_3|^3} \tag{1}$$

$$\ddot{r}_2 = -Gm_3 \frac{r_2 - r_3}{|r_2 - r_3|^3} - Gm_1 \frac{r_2 - r_1}{|r_2 - r_1|^3} \tag{2}$$

$$\ddot{r}_3 = -Gm_1 \frac{r_3 - r_1}{|r_3 - r_1|^3} - Gm_2 \frac{r_3 - r_2}{|r_3 - r_2|^3} \tag{3}$$

Where $G = 6.674E - 11$ [m^3/(kg s^2)] is the gravitation constant.

[2]See http://three-body.ipb.ac.rs/ for cool animations.

We have observed a trinary star system and have put these observed trajectories into the file traj.txt in the 222 Box drive.[3] A plot of the orbits is show in Fig. 1.

Q2 Problem 1: Use expression optimization to produce an approximate expression for the closed form solution of the three body problem observed in the trinary star system. The orbit is 2D and all stars follow the same path (offest in phase), so we will solve for only two expressions: $x(t)$ and $y(t)$.

Tasks:

1. (5pts) Write down a reasonable objective function for this optimization problem for $x(t)$. You can denote the true closed form expression that generated our trajectory data $x(t)$ and our approximated closed form expression $\hat{x}(t)$ where t is time. You can denote the time range $t_i \in [t_0, t_{max}]$.
2. (5pts) Write down a grammar that you think is appropriate for solving this problem with expression optimization.
3. (5pts) Describe a particular algorithm for expression optimization that could be used for this problem.
4. (7pts) Write code that uses the algorithm that is descibed in (3) to find a closed form equation of motion for the orbit. You can use any software package that you like, including:
   - gplearn in python
   - ExprOptimization.jl in Julia[4]
   - Anything else!

   For maximum learning, you can try implementing an expression optimization algorithm yourself instead of calling a library. For this, you may find 2018 Midterm 3 useful.
5. (8pts) Run and evaluate your code implemented in (4). Try to get your orbit to capture qualitative features of the ground truth orbit. Plot/print the following items:
   (a) Plot each fitted function, evaluated at the same time points as the ground truth trajectory data, compared to ground truth:
      i. as a function of time: e.g. $\hat{x}(t)$ and $x(t)$, and then on a separate plot $\hat{y}(t)$ and $y(t)$
      ii. in the x-y plane: $(x(t),\ y(t))$ and $(\hat{x}(t),\ \hat{y}(t))$
   (b) Use the objective function that you defined in (1) to demonstrate that your $\hat{x}(t)$ is quantitatively better than simply predicting the mean of the $x(t)$ trajectory, and that your $\hat{y}(t)$ is quantitatively better than simply predicting the mean of the $y(t)$ trajectory.
   (c) Print the best fit expression that your optimization algorithm produced.
      - If you are using python, you can use graphviz to pretty print
      - If you are using Julia, ExprOptimization.jl has a built-in display method that you can use, or you can try TikzGraphs.jl[5]
   (d) Extra challenge plot (not graded): Plot the value of the objective function as a function of the number of iterations (using whatever definition of iteration is used by your algorithm).
6. Extra challenge task (not graded). Try to fit the butterfly orbit trajectories: butterfly.txt, also in the 222 Box drive.

(Optional, no credit activity)
Write a little rhyme about something you learned in this class. A selection of clever ones will be compiledand shared with the class. By default they will be shared anonymously, but if you would like your name associated with it, please indicate that.

[3]This files has 3 columns of data: t, x, y

[4]If you are using ExprOptimization.jl and encounter domain errors or divide by 0 errors, try using a try-catch statement in your objective function!

[5]This is also a good directory of packages for plotting graphs in Julia: https://juliagraphs.org/LightGraphs.jl/stable/plotting/