EE270
Large scale matrix computation, optimization and learning

Instructor : Mert Pilanci

Stanford University

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Randomized Linear Algebra
Lecture 9: High-dimensional Problems, Least-norm Solutions and Randomized Methods

Faster Least Squares Optimization: Random Projection

- **Left-sketching**

  Form SA and Sb where $S \in \mathbb{R}^{m \times n}$ is a random projection matrix
- Solve the smaller problem

$$\min_{x \in \mathbb{R}^d} \|SAx - Sb\|_2^2$$

- using any classical method.

  Direct method complexity md^2

Gaussian Sketch

- Let S be $\frac{1}{m}\times$ i.i.d. Gaussian. $\mathbb{E}[S^T S] = I$

$$\tilde{x} = \arg\min_{x \in \mathbb{R}^d} \|SAx - Sb\|_2^2$$

- Unbiased $\mathbb{E}\left[\tilde{x}\right] = x_{LS}$
  since $\tilde{x} = x_{LS} + \underbrace{(A^T S^T SA)^{-1} A^T S^T S b^{\perp}}_{\text{zero mean}}$

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- Variance

  $$\mathbb{E}\|A(\tilde{x} - x_{LS})\|_2^2 = f(x_{LS})\frac{d}{m-d-1}$$

  valid for $m > d+1$ where $f(x) = \|Ax - b\|_2^2$

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valid for $m > d+1$ where $f(x) = \|Ax - b\|_2^2$
- Function value
$f(\tilde{x}) = \|A\tilde{x} - b\|_2^2 = \|A(\tilde{x} - x_{LS})\|_2^2 + \|Ax_{LS} - b\|_2^2$
- $\mathbb{E}f(\tilde{x}) - f(x_{LS}) = f(x_{LS})\frac{d}{m-d-1}$

Variance Reduction by Averaging

- Let $S_1, ..., S_r$ be $\frac{1}{m}\times$ i.i.d. Gaussian. $\mathbb{E}[S^T S] = I$

$$\tilde{x}_i = \arg\min_{x \in \mathbb{R}^d} \|S_i A x - S_i b\|_2^2$$

- let $\tilde{x} = \frac{1}{r}\sum_{i=1}^{r} x_i$
- Unbiased $\mathbb{E}\left[\tilde{x}\right] = x_{LS}$
- Variance is reduced by $\frac{1}{r}$
- $\mathbb{E}\|A(\tilde{x} - x_{LS})\|_2^2 = f(x_{LS})\frac{1}{r}\frac{d}{m-d-1}$

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- no unique solution
- minimum (ℓ_2) norm solution is unique

$$x_{\text{min-norm}} = \arg\min_{Ax=b} \|x\|_2^2$$

Minimum norm solution and SVD

$$x_{\text{min-norm}} = \arg\min_{Ax=b} \|x\|_2^2$$

Random projection to reduce dimension: Right Sketch

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- We can right multiply A and form AS where $S \in \mathbb{R}^{d \times m}$ and solve

$$\arg\min_{ASz=b} \|z\|_2^2$$

Random projection to reduce dimension: Right Sketch

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- How do we use $z \in \mathbb{R}^m$?

Right Sketch

$$x_{\text{min-norm}} = \arg\min_{Ax=b} \underbrace{\|x\|_2^2}_{f(x)}$$

$$\text{approximation} \qquad \tilde{x} = S\tilde{z}$$

$$\text{where } \tilde{z} := \arg\min_{ASz=b} \|z\|_2^2$$

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- Let S be i.i.d. Gaussian $N(0, \frac{1}{\sqrt{m}})$
- Is $\tilde{x}$ unbiased, i.e., $\mathbb{E}\tilde{x} \stackrel{?}{=} x_{\text{min-norm}}$

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- Is $\tilde{x}$ unbiased, i.e., $\mathbb{E}\tilde{x} =^? x_{\text{min-norm}}$
- Yes, conditioned on SA

$$\tilde{x} \sim N(x_{\text{min-norm}}, VV^T b^T (AS^T SA^T)^{-1} b)$$

- VV^T is the projection onto the null space of A
- error $\tilde{x} - x_{\text{min-norm}} \in \text{Null}(A)$

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- Using $\mathbb{E}(AS^T SA^T)^-1 = AA^T \frac{m}{m-n-1}$
  $\mathbb{E}\|\tilde{x} - x_{\text{min-norm}}\|_2^2 = \frac{d-n}{m-n-1} f(x_{\text{min-norm}}) = \frac{d-n}{m-n-1} \|x_{\text{min-norm}}\|_2^2$

Left Sketch vs Right Sketch Summary

- Both are unbiased using Gaussian projections
- A is $n \times d$
- Left sketch $n \geq d$

$$\tilde{x} = \arg\min_{x \in \mathbb{R}^d} \|SAx - Sb\|_2^2$$

  Variance: $\mathbb{E}\|A(\tilde{x} - x_{LS})\|_2^2 = f(x_{LS})\frac{d}{m-d-1}$

- Right sketch $d > n$

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Back to Left Sketch: Which sketching matrices are good?

- We need to find conditions to guarantee approximate optimality
- Let $A = U\Sigma V^T$ SVD in compact form

some deterministic options

- $S = U^T$ is $d \times n$
- $S = A^T$

- For random S matrices $A^T S^T S A$ needs to be invertible
  we want it to be close to $A^T A$

Approximate Matrix Multiplication

- Let the approximate product of AB be $C = AS^TSB$

$$\mathbb{P}\left[\|AB - C\|_F > \epsilon\|A\|_F\|B\|_F\right] \leq \delta$$

- Follows from JL Moment property
- $S \in \mathbb{R}^{m\times n} \sim \frac{1}{\sqrt{m}}\times$random i.i.d. sub-Gaussian, e.g., ± 1, or $N(0,1)$ with $m = \frac{c_1}{\epsilon^2}\log\frac{1}{\delta}$
- $S \in \mathbb{R}^{m\times n} \sim \frac{1}{\sqrt{m}}\times$CountSketch matrix (one nonzero per column, which is ± 1 at a uniformly random location) with $m = \frac{c_2}{\epsilon^2\delta}$
- $S \in \mathbb{R}^{m\times n} \sim \frac{1}{\sqrt{m}}\times$Fast JL Transform with $m = \frac{c_3}{\epsilon}\log\frac{1}{\delta}$

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- $S \in \mathbb{R}^{m\times n} \sim \frac{1}{\sqrt{m}}\times$Fast JL Transform with $m = \frac{c_3}{\epsilon}\log\frac{1}{\delta}$
- Sparse JL and Fast JL are more efficient
- advantages: doesn't require any knowledge about matrices A and B (**oblivious)**
- optimal sampling probabilities depend on the column/row norms of A and B

Basic Inequality Method

- We minimize $\tilde{x} = \arg\min \|S(Ax - b)\|_2^2$
- x_{LS} minimizes $\|Ax - b\|_2^2$
- How far is $\tilde{x}$ from x_{LS}?
- **Step 1**. Establish two optimality (in)equalities for these variables
- $\|Ax_{LS} - b\|_2^2 \leq \|Ax' - b)\|_2^2$ for any x', i.e., $A^T(Ax_{LS} - b) = 0$
- $\|S(A\tilde{x} - b)\|_2^2 \leq \|S(Ax_{LS} - b)\|_2^2$

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- $\|S(A\tilde{x} - b)\|_2^2 \le \|S(Ax_{LS} - b)\|_2^2$
- **Step 2**. Define error $\Delta = \tilde{x} - x_{LS}$ and re-write these inequalities in terms of δ
- $\|SA\Delta\|_2^2 \le 2{b^{\perp}}^T(S^TS - I)A\Delta$
- **Step 3**. Argue $S^TS \approx I$

Leverage Scores

Questions?