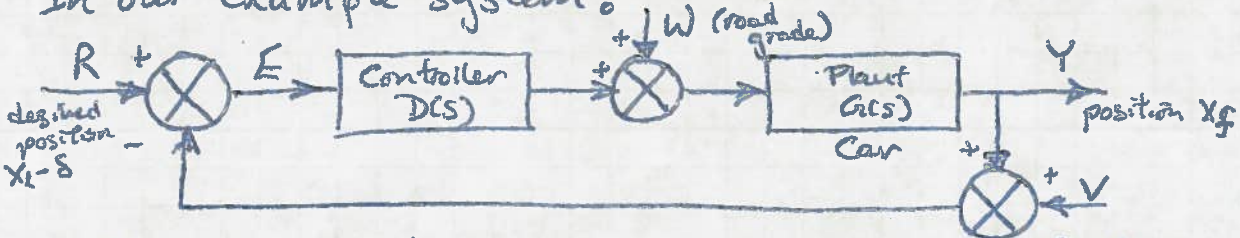
Error Response and System Type

In our example system:



the error, $e(t)$ or $E(s)$, is extremely important. If this value gets too large, the cars may collide. We therefore want to look not only at the transfer function from reference to output but also at the transfer function from reference to error:

$$\frac{E(s)}{R(s)} = \frac{1}{1+DG}$$

For the simple case of proportional control $D(s) = K$ and

$$G(s) = \frac{1}{ms^2+bs}$$

So $$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{K}{ms^2+bs}} = \frac{ms^2+bs}{ms^2+bs+K}$$

Since the system is stable, we can use the final value theorem to look at the response to different reference inputs:

<u>Step change</u>: $R(s) = \frac{1}{s}$

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{ms^2+bs}{ms^2+bs+K} = 0$$

$\Rightarrow$ No steady-state error to a step reference

<u>Ramp</u>: $R(s) = \frac{1}{s^2}$ (step change in lead vehicle speed)

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{ms+b}{ms^2+bs+K} = \frac{b}{K}$$

$\Rightarrow$ finite error. Error reduces as K is increased

<u>Parabola</u>: $R(s) = \frac{2}{s^3}$ (step change in lead vehicle acceleration)

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{2(ms+b)}{ms^3+bs^2+Ks}$$

$\Rightarrow$ Error grows to infinity!

This is a problem! If the lead vehicle starts to brake, the spacing error continues to grow!

In discussing the steady-state response with regards to a particular input, control engineers use the term "System Type"

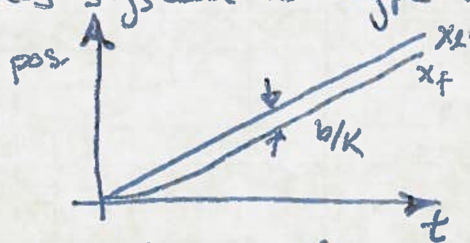
System Type

Type 0: Finite error to a step (position error)

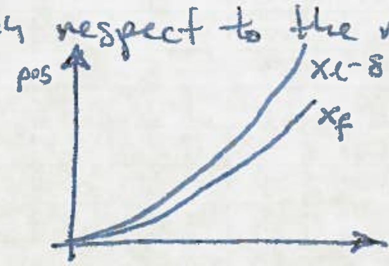
Type 1: Finite error to a ramp (velocity error)

Type 2: Finite error to a parabola (acceleration error)

This system is Type 1 with respect to the reference



Ramp input



Parabolic input

To determine system type for a unity feedback connection (no blocks in the feedback loop), you can look at the loop gain or loop transfer function

$$L(s) = D(s)\,G(s)$$

$$= \frac{L_0(s)}{s^n}$$

n integrators mean a system type of n

$$\frac{E(s)}{R(s)} = \frac{1}{1+L(s)} \qquad R(s) = \frac{k!}{s^{k+1}} \text{ for } r(t) = t^k$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} \frac{1}{1+L(s)} \cdot s \cdot \frac{k!}{s^{k+1}}$$

$$= \lim_{s\to 0} \frac{1}{1+\frac{L_0(s)}{s^n}} \cdot \frac{k!}{s^k}$$

$$= \lim_{s\to 0} \frac{s^n}{s^n + L_0(s)} \cdot \frac{k!}{s^k} = 0 \text{ if } n > k$$

$\Longrightarrow$ n is the system type

To be on the safe side, you can always use the final value theorem to check instead of remembering a particular formula. The key thing to understand from this is that system type is related to the number of integrators in the loop transfer function — the more integrators, the higher the system type.

In addition to looking at the reference input, it is also possible to characterize the system type with regards to a disturbance. This gives some measure of the disturbance rejection capability of the system.

$$\frac{E(s)}{W(s)} = \frac{G(s)}{1 + D(s)G(s)} = \frac{1}{ms^2+bs+K}$$

$$e_{ss} = \lim_{s \to 0} s \frac{1}{ms^2+bs+K} \cdot \frac{A}{s} = \frac{A}{K}$$ for a step change A in road grade.

$\Rightarrow$ The system is type 0 with respect to disturbance.

System type does not depend upon the control gains (in this case, K) chosen. It is instead related to the structure of the plant and controller. If we want to change system type, we need to change the controller structure.

How can we change it to track an accelerating / decelerating vehicle and reject disturbances?

Integral Control

The system is Type 1 with regards to reference inputs. We want a Type 2 system.

$\Rightarrow$ Add an integrator to $L(s) = D(s)G(s)$

PI controller: $D(s) = K_p + \frac{K_I}{s} = \frac{K_p s + K_I}{s}$

This means $U(s) = D(s)E(s)$ or

$$u(t) = K_p e(t) + K_I \int_0^t e\,dt$$

If the initial error is quite large, the integral of the error can be huge, resulting in a large overshoot. This effect is known as integrator wind-up and is something to be careful about when using integral control.

With the integral controller, the loop gain is

$$L(s) = \frac{K_p s + K_I}{s} \cdot \frac{1}{s(ms+b)} = \frac{K_p s + K_I}{s^2(ms+b)}$$

two integrators in the denominator

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{K_p s + K_I}{s^2(ms+b)}} = \frac{s^2(ms+b)}{ms^3+bs^2+K_p s+K_I}$$

For a ramp, the steady-state error is given by:

$$E(s) = \lim_{s\to 0} s \cdot \frac{s^2(ms+b)}{ms^3+bs^2+K_p s+K_I} \cdot \frac{1}{s^2} = 0$$

So the system will have zero error to a ramp. For a parabola characteristic of an accelerating or decelerating lead vehicle:

$$E(s) = \lim_{s\to 0} s \cdot \frac{s^2(ms+b)}{ms^3+bs^2+K_p s+K_I} \cdot \frac{2}{s^3} = \frac{b}{K_I}$$

K_I can be tuned to give acceptable error so the cars don't collide.

For the disturbance:

$$\frac{E(s)}{W(s)} = \frac{\frac{1}{s(ms+b)}}{1+\frac{K_p s + K_I}{s^2(ms+b)}} = \frac{s}{ms^3+bs^2+K_p s+K_I}$$

So for a step change in road grade:

$$E(s) = \lim_{s\to 0} s \cdot \frac{s}{ms^3+bs^2+K_p s+K_I} \cdot \frac{1}{s} = 0$$

This all looks good but remember that the final value theorem is only valid for stable systems. Do we have a stable system? The characteristic equation is:

$$1 + D(s)G(s) = 0$$

$$ms^3+bs^2+K_p s+K_I = 0$$

Is it enough to say that $m, b, K_p, K_I > 0$? It turns out it isn't. We can always solve for the poles and check if they are in the left half plane but there is an analytical way based on the characteristic equation.