The Transfer Function

The equations resulting from system modeling in this class take the form:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^m u}{dt^m} + b_2 \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_{m+1} u$$

Laplace transforming gives

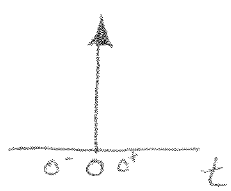
$$\left[s^n + a_1 s^{n-1} + \ldots + a_n \right] Y(s) = \left[b_1 s^m + b_2 s^{m-1} + \ldots + b_{m+1} \right] U(s)$$

When initial conditions are set to zero. This can be arranged as a Transfer Function or ratio of two polynomials, $H(s)$:

$$\frac{Y(s)}{U(s)} = H(s) = \frac{b_1 s^m + b_2 s^{m-1} + \ldots + b_{m+1}}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

This can tell us many useful things about the system and has several interpretations.

The first relates to the impulse response. An impulse is a signal that is nonzero only at one point in time.



Unit impulse $\quad \delta(t) = 0$ except when $t = 0$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt \equiv 1$$

What is the Laplace transform of an impulse?

$$\mathcal{L}\{\delta(t)\} = \int_0^{\infty} \delta(t) e^{-st} dt = \int_{0^-}^{0^+} \delta(t) e^{0} dt = \int_{0^-}^{0^+} \delta(t)\, dt = 1$$

Remember the step

$$1(t) = \begin{cases} 1 & t > 0 \\ 0 & \text{otherwise} \end{cases} \qquad \frac{d}{dt} 1(t) = \delta(t)$$

$$\mathcal{L}\{1(t)\} = \frac{1}{s} \qquad \mathcal{L}\left\{\frac{d}{dt} 1(t)\right\} = s\, \mathcal{L}\{1(t)\} - 1(0)$$

$$= s \cdot \frac{1}{s} - 0 = 1$$

The impulse is thus the derivative of the step and everything we know about differentiation checks out.

Back to the transfer function

If $\frac{Y(s)}{U(s)} = H(s)$ then $Y(s) = H(s)$ for an impulse

$$\mathcal{L}^{-1}\{H(s)\} = h(t) = y(t) \text{ for an impulse}$$

$\Rightarrow$ The transfer function is the Laplace transform of the system response to an impulse.

Think of the impulse response as figuratively (and sometimes literally) hitting the system with a hammer. You can see a lot of things about a system in this way including its stability, frequency of any resonances, decay time, etc. We can similarly see all of these in the transfer function if we know where to look.

Why do we focus on $G(s)$ and not $g(t)$ directly?

It is much easier to solve problems in the Laplace domain since multiplication in s is convolution in t.

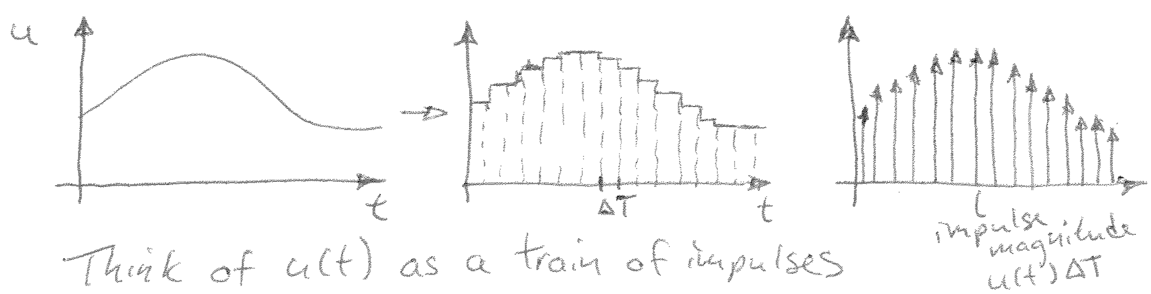
Laplace

$$y(t) = \mathcal{L}^{-1}\left[H(s) \cdot U(s)\right]$$

Time domain

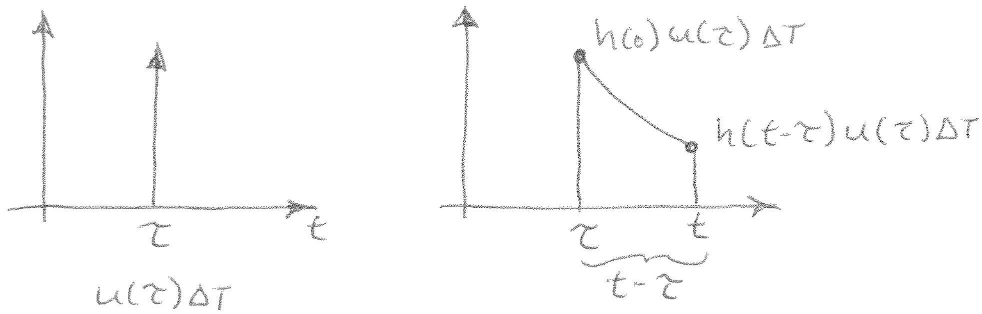
$$y(t) = \int_0^t h(t-\tau)\, u(\tau)\, d\tau$$

what this says is that we can think of the system response as a composition of impulse responses.



Think of u(t) as a train of impulses

What is the response to one of these impulses?

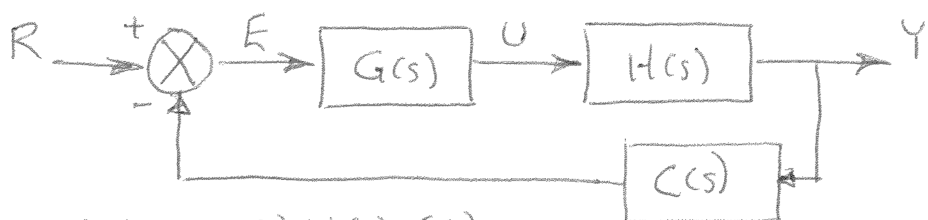


At any time t, the output y is a result of all past impulses so:

$$y(t) = \sum \Big[\underbrace{h(t)}_{\substack{\text{Response to an}\\ \text{impulse after } t \text{ seconds}}} u(0) + h(t-\Delta T)u(\Delta T) + h(t-2\Delta T)u(2\Delta T) + \dots\Big]\Delta T$$

In the limit: $y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau = u(t) * h(t)$

Clearly, we can do this but it is harder than multiplication and may need to repeated a number of times:



$$Y(s) = G(s)H(s)E(s)$$

$$E(s) = R(s) - C(s)Y(s)$$

$$E(s) = R(s) - \underbrace{C(s)G(s)H(s)E(s)}_{\text{quite a lot of convolution!}}$$

$$\Rightarrow \frac{E(s)}{R(s)} = \frac{1}{1 + C(s)G(s)H(s)}$$ Much easier with multiplication

Since the transfer function is the Laplace transform of the impulse response, it should have the same information contained in it. How do we extract that information?

Poles and the Impulse Response

One way to understand the impulse response is to do a partial fraction expansion of the transfer function

The transfer function can be written as

$$H(s) = \frac{K\prod^{m}(s-z_i)}{\prod^{n}(s-p_i)}$$

zeros

poles

$m \le n$ for a physical system

It can also be written as

$$H(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \dots + \frac{c_n}{s-p_n}$$

(This form requires that all of the poles be distinct - we will look at repeated poles shortly).

The impulse response is therefore given by:

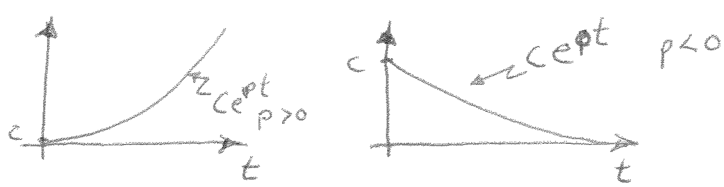
$$y(t) = \mathcal{L}^{-1}[H(s)] = \mathcal{L}^{-1}\left[\frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \dots + \frac{c_n}{s-p_n}\right]$$

$$y(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \dots + c_n e^{p_n t}$$

Just a sum of our exponential building blocks! We can tell a lot about the response just by knowing the poles of the system.

Real and Complex Poles

If a pole is real, it must be negative for the system to be stable. The exponential can only grow or decay



What about a pole $p = -\sigma + j\omega$

$$ce^{(-\sigma+j\omega)t} = ce^{-\sigma t}\cdot e^{j\omega t} = \underbrace{ce^{-\sigma t}}_{\text{decays or grows}}[\cos\omega t + j\sin\omega t]$$

oscillates

how do we handle this?

If there is a pole $p_1 = -\sigma + j\omega$, its complex conjugate $p_2 = -\sigma - j\omega$ must also be a pole.

Then $(s-p_1)(s-p_2) = s^2 - (-\sigma+j\omega)s + (-\sigma-j\omega)s + \sigma^2+\omega^2$

$= s^2 + 2\sigma s + \sigma^2 + \omega^2$ All real coefficients!

This is also true of the coefficients in the partial fraction expansion: if $c_1 = \alpha - \beta j$ then $c_2 = \alpha + \beta j$ must also be a coefficient so that the numerator polynomial has real coefficients.

This means that a complex pair of poles appears in the impulse response as:

$$H(s) = \frac{\alpha - \beta j}{s + \sigma - \omega j} + \frac{\alpha + \beta j}{s + \sigma + \omega j} + \ldots$$

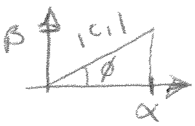
so

$$y(t) = (\alpha - \beta j)\, e^{-\sigma t}\left[\cos\omega t + j\sin\omega t\right] + (\alpha + \beta j)\, e^{-\sigma t}\left[\cos\omega t - j\sin\omega t\right] + \ldots$$

$$= e^{-\sigma t}\left[2\alpha\cos\omega t + 2\beta\sin\omega t\right]$$

$$= 2|c_1|\, e^{-\sigma t}\cos(\omega t - \phi)$$

where $|c_1| = \sqrt{\alpha^2 + \beta^2}$ and $\tan\phi = \beta/\alpha$

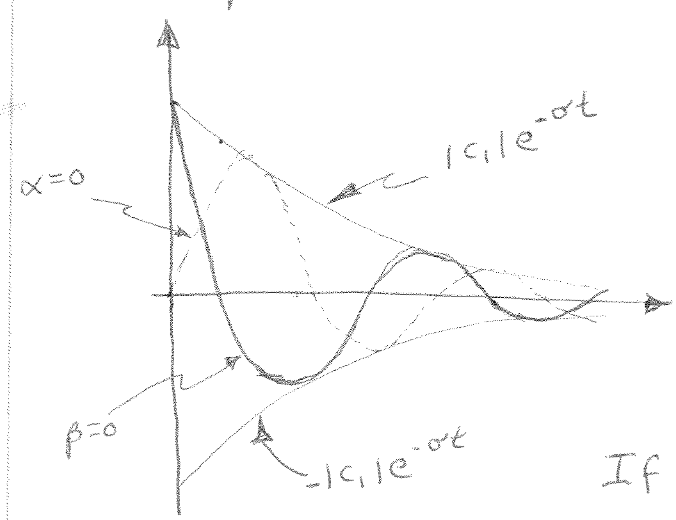


(Why? $\alpha = |c_1|\cos\phi$ and $\beta = |c_1|\sin\phi$ so

$$\alpha\cos\omega t + \beta\sin\omega t = |c_1|\cos\phi\cos\omega t + |c_1|\sin\phi\sin\omega t$$
$$= |c_1|\cos(\omega t + \phi) \quad)$$

Each value has some meaning:

- σ represents the rate of exponential decay
- ω represents the frequency of oscillation
- α represents how much cosine term exists
- β represents how much sine term exists



Real poles give a stable response when they are negative

Complex poles give a stable response when they have negative real parts.

If $\sigma = 0$, $s = \pm j\omega$ and

$$y(t) = 2|c_1|\cos(\omega t - \phi)$$

$\Rightarrow$ a non-decaying sinusoid