Solving for the Dynamic Response

The models of systems we obtain from the modeling techniques described in the last lecture can be put in a form:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_n y = b_1 \frac{d^m u}{dt^m} + b_2 \frac{d^{m-1} u}{dt^{m-1}} + \dots + b_{m+1} u$$

These are linear, time-invariant, constant-coefficient ODEs describing a single input, single output (SISO) system.

While not everything can be put into this form, many systems can. There are so many analytical tools available for systems of this form that it often makes sense to try to fit the system into this form as a starting point.

In an ODE class, these equations are solved for two solutions:

Homogeneous	Particular
Input = 0	Depends upon input
Free response	Forced response
Natural response	

The homogeneous solution is easy -

Let $y = Ae^{st}$ (our basic building block)

$\dot{y} = Ase^{st}$

$\ddot{y} = As^2e^{st}$ $\Rightarrow$ substitute and solve for A and s

$\vdots$

Example

Mass-spring-damper

$$m\ddot{y} + b\dot{y} + ky = 0$$

$$\Rightarrow ms^2Ae^{st} + bsAe^{st} + KAe^{st} = 0$$

$$\Rightarrow ms^2 + bs + K = 0$$

$$s = \frac{-b \pm \sqrt{b^2 - 4mK}}{2m}$$

2 solutions for s: $s_1 = \frac{-b}{2m} + \frac{\sqrt{b^2-4mk}}{2m}$, $s_2 = \frac{-b}{2m} - \frac{\sqrt{b^2-4mk}}{2m}$

How do we get values for A_1 and A_2?

$$y = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

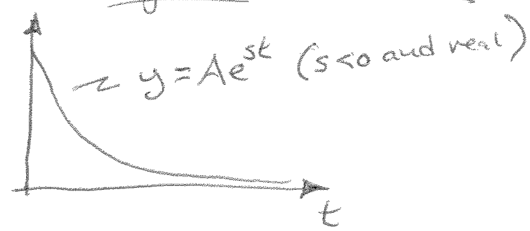
Look at initial conditions:

$$y(0) = A_1 + A_2$$
$$\dot{y}(0) = s_1 A_1 + s_2 A_2$$

$\Rightarrow$ 2 equations, 2 unknowns

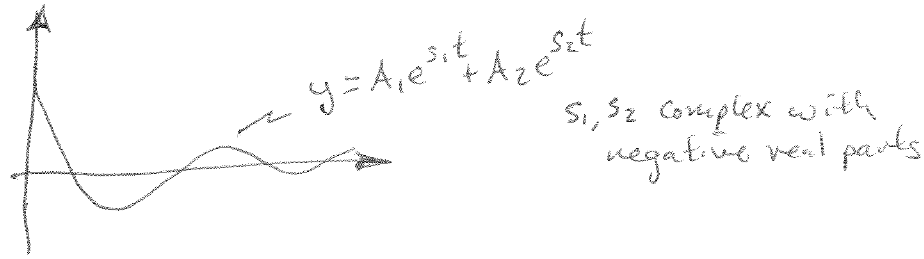
What do these solutions look like?

(a) if $b^2 > 4mk$, both s_1 and s_2 are <u>negative</u> and <u>real</u>



(b) if $b^2 < 4mk$, s_1 and s_2 are complex (in fact complex conjugates)

Since $e^{j\theta} = \cos\theta + j\sin\theta$, these will oscillate



Solutions to equations of this form are always real roots or complex conjugate pairs (why?)

There is another solution technique, however, that enables us to handle the homogeneous and particular solutions at once. It also provides greater insight about the system structure. This is the Laplace Transform.

Laplace Transforms

Laplace transforms transform

(1) a function of a real variable (like time) to a function of a complex variable

(2) problems with differential equations to algebra

(3) convolution to multiplication

The solution process for ODEs looks like:

$$\text{ODE} \xrightarrow{\mathcal{L}} \text{algebra problem} \xrightarrow{\mathcal{L}^{-1}} \text{solution of ODE}$$

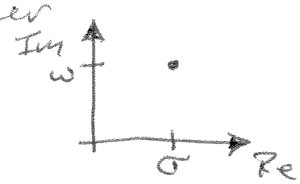
(sometimes we can stop here)

Given a function $f(t)$

$$\mathcal{L}[f(t)] \triangleq F(s) = \int_{0^-}^{\infty} f(\tau) e^{-s\tau} d\tau$$

s is a complex number

$s = \sigma + j\omega$



It naturally follows that:

$$\mathcal{L}[f_1(t) + a f_2(t)] = F_1(s) + a F_2(s)$$

Since the exponential is such an important building block in our ODEs, it makes sense to look at the Laplace transform of the exponential

$$f(t) = A e^{-\alpha t} \qquad t \geq 0$$

$$\mathcal{L}[f(t)] = \int_0^{\infty} A e^{-\alpha\tau} e^{-s\tau} d\tau$$

$$= A \int_0^{\infty} e^{-(\alpha+s)\tau} d\tau$$

$$= \frac{-A}{\alpha+s} e^{-(\alpha+s)\tau} \Big|_0^{\infty}$$

$$= \frac{-A}{\alpha+s} \left(\cancelto{0}{e^{-(\alpha+s)\infty}} - \cancelto{1}{e^{-(\alpha+s)\cdot 0}} \right) = \frac{A}{\alpha+s}$$

<u>Step function</u> $\quad f(t) = A$

$$\mathcal{L}[f(t)] = \int_0^\infty A e^{-s\tau} d\tau = \frac{A}{-s} e^{-s\tau}\Big|_0^\infty = \frac{A}{s}$$

Notice that the exponential converges to a step as $\alpha \to 0$.

<u>Differentiation</u>

This is a great example of integration by parts:

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

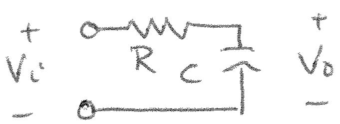
$$F(s) = \int_0^\infty \underbrace{f(\tau)}_{u} \underbrace{e^{-s\tau} d\tau}_{dv} = f(\tau)\frac{e^{-s\tau}}{-s}\Big|_0^\infty - \int_0^\infty \left[\frac{d}{dt} f(\tau)\right] \frac{e^{-s\tau}}{-s} d\tau$$

$$= -\frac{1}{s}\left[f(\infty)\cdot 0 - f(0) - \mathcal{L}\left\{\frac{d}{dt} f(t)\right\}\right]$$

$$\Rightarrow \quad sF(s) = f(0) + \mathcal{L}\left[\frac{df(t)}{dt}\right]$$

So multiplication by s is equivalent to differentiation.

<u>RC Circuit Example</u>



$$\dot{V}_o = \frac{1}{RC}\left[V_i - V_o\right]$$

$$sV_o(s) - V_o(0) = \frac{1}{RC}\left[V_i(s) - V_o(s)\right]$$

$$V_o(s)(RCs + 1) = V_i(s) + V_o(0)\cdot RC$$

$$\Rightarrow \quad V_o(s) = \underbrace{\frac{RC\,V_o(0)}{RCs+1}}_{\text{free (homogeneous)}} + \underbrace{\frac{V_i(s)}{RCs+1}}_{\text{forced (particular)}}$$

$$V_o(t) = \mathcal{L}^{-1}\left[V_o(s)\right] = \mathcal{L}^{-1}\left[\frac{RC\,V_o(0)}{RCs+1}\right] + \mathcal{L}^{-1}\left[\frac{V_i(s)}{RCs+1}\right]$$

$$\qquad \qquad \qquad \qquad \qquad \| \\ V_o(0)\, e^{-t/RC}$$

Particular solution for a step input $\quad V_i(s) = \frac{A}{s}$

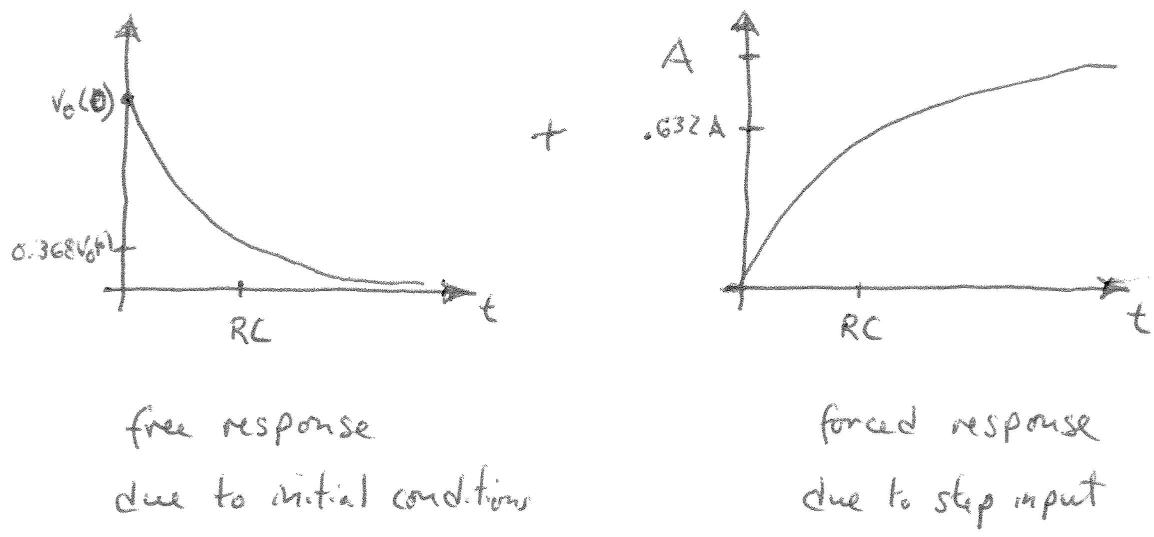
$$V_o(t) = V_o(0) e^{-t/RC} + \mathcal{L}^{-1}\left[\frac{A/RC}{s(s + \frac{1}{RC})}\right]$$

$$\| $$

$$\mathcal{L}^{-1}\left[\frac{A}{s} - \frac{A}{s + \frac{1}{RC}}\right] = A - Ae^{-t/RC}$$

$$\Rightarrow \quad V_o(t) = V_o(0) e^{-t/RC} + A\left(1 - e^{-t/RC}\right)$$

The output voltage is the sum of two exponential responses:



At $t = RC$ (the time constant of the system) the initial conditions have decayed to $1/e$ (or ~36.8%) of their initial values. The step input has caused the output to reach $1 - 1/e$ or about 63.2% of its final value.

A useful representation of the system is its <u>transfer function</u> - the ratio of the Laplace transform of the output to the Laplace transform of the input, assuming zero initial conditions.

Here the output is $V_o(s)$ and the input is $V_i(s)$ so the transfer function is

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{RCs + 1}$$

The time constant of a first order system is very easy to spot in its transfer function.

There are many other uses of the transfer function as well...