Zeros

Up to this point, most of our discussion of system responses has been focused on the poles in the denominator of the transfer function. The poles represent the basic "building blocks" of the response and determine the system stability.

The zeros are also important in determining the system response, though they do not impact the stability of the open-loop transfer function. Open-loop zeros do impact the stability of the closed-loop transfer function. Furthermore, the zeros can give a lot of insight into the structure of a system.

Most fundamentally, zeros determine how much of each "building block" appears in the system response. To see this, consider three transfer functions with the same characteristic equation:

$$s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3) = 0$$

(a) $$H(s) = \frac{3s^2 + 12s + 11}{s^3 + 6s^2 + 11s + 6} = \frac{1}{s+1} + \frac{1}{s+2} + \frac{1}{s+3}$$

(b) $$H(s) = \frac{6s + 7}{s^3 + 6s^2 + 11s + 6} = \frac{1}{s+1} + \frac{1}{s+2} - \frac{2}{s+3}$$

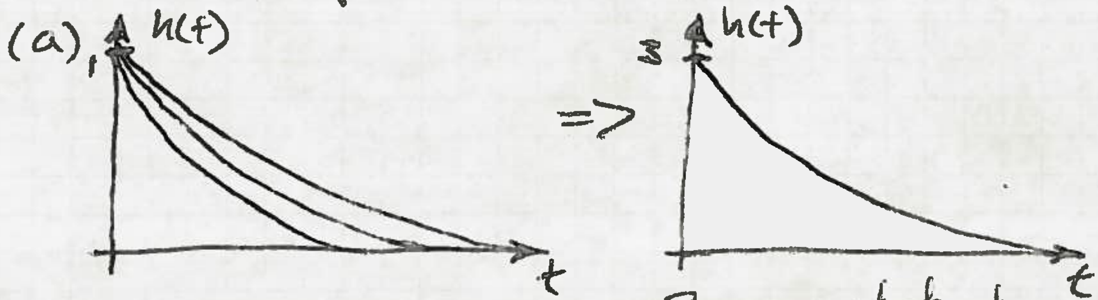
(c) $$H(s) = \frac{2s - 1}{s^3 + 6s^2 + 11s + 6} = \frac{-1.5}{s+1} + \frac{5}{s+2} - \frac{3.5}{s+3}$$

The impulse response of each transfer function consists of some combination of the three basic exponentials:

$$e^{-t} \qquad e^{-2t} \qquad e^{-3t}$$

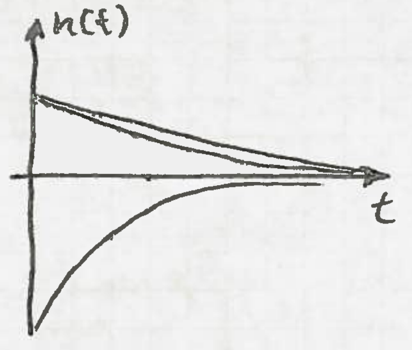
However, the different weightings on each term gives dramatically different responses

(a)

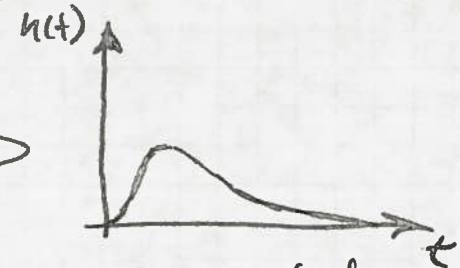


Response starts at 3 then decays.

(b)

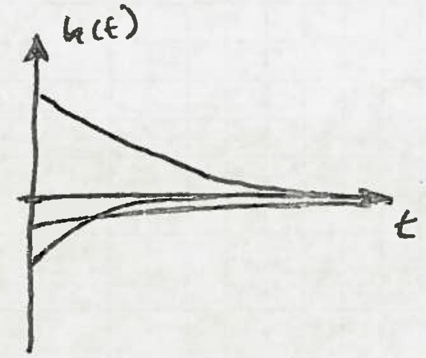


$\Rightarrow$

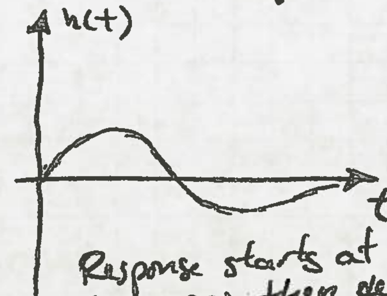


Response starts at zero, increases then decays.

(c)



$\Rightarrow$

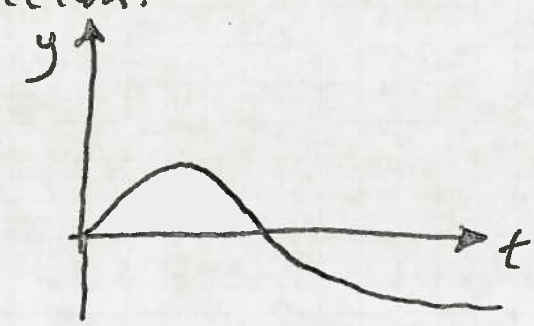


Response starts at zero, increases then decreases, changing sign.

The response in (c) is characteristic of a non-minimum phase system which has zeros located in the right half plane. If a system has an odd number of right half plane zeros, its step response will initially move in the opposite direction.

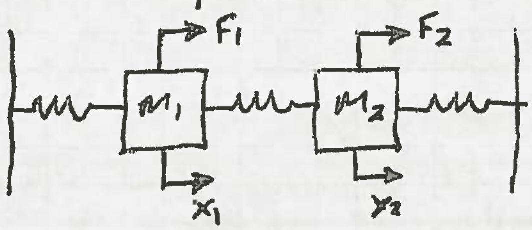
$$Y(s) = H(s)\frac{1}{s}$$

for system (c)



- This characteristic can be seen in controlling a drifting automobile or in controlling the altitude of an aircraft with the elevators.

Zeros can also tell a lot about the structure of the system input in relation to the output. For example, consider the two mass system:



Force balances:

mass 2 $F_2 - K_3 x_2 + K_2 (x_1 - x_2) = m_2 \ddot{x}_2$

$$\Rightarrow F_2(s) - K_3 X_2(s) + K_2 X_1(s) - K_2 X_2(s) = m_2 s^2 X_2(s)$$

$$F_2(s) + K_2 X_1(s) = (m_2 s^2 + K_{23}) X_2(s) \qquad K_{23} = K_2 + K_3$$

mass 1 $f_1 - K_1 x_1 + K_2 (x_2 - x_1) = m_1 \ddot{x}_1$

$$F_1(s) + K_2 X_2(s) = (m_1 s^2 + K_{12}) X_1(s) \qquad K_{12} = K_1 + K_2$$

$$X_1(s) = \frac{K_2}{m_1 s^2 + K_{12}} X_2(s) + \frac{1}{m_1 s^2 + K_{12}} F_1(s)$$

$$\Rightarrow \frac{K_2^2 X_2(s)}{m_1 s^2 + K_{12}} + \frac{K_2 f_1(s)}{m_1 s^2 + K_{12}} + F_2(s) = (m_2 s^2 + K_{23}) X_2(s)$$

$$X_2(s) = G_1 F_1(s) + G_2 F_2(s)$$

$$G_1(s) = \frac{K_2}{m_1 m_2 s^4 + (m_1 K_{23} + m_2 K_{12}) s^2 + (K_{12} K_{23} - K_2^2)}$$

The input F_1 appears in the 4^{th} derivative of the position x_2. The system has a relative degree of 4 between input and output.

$$G_2(s) = \frac{m_1 s^2 + K_{12}}{m_1 m_2 s^4 + (m_1 K_{23} + m_2 K_{12}) s^2 + (K_{12} K_{23} - K_2^2)}$$

The input F_2 appears in the 2^{nd} derivative of the position x_2. The system has a relative degree of 2 corresponding to the two integrations needed to go from force to position.

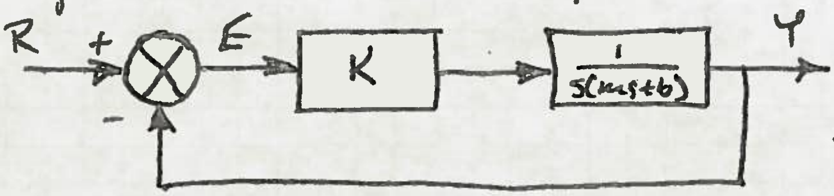
$\Rightarrow$ Relative degree of the numerator and denominator polynomials gives a measure of how "far" (in terms of integrators) the input is from the output.

The role of relative degree can also be seen in the <u>Initial Value Theorem</u>

$$\lim_{s \to \infty} s F(s) = f(0^+) \qquad \text{(just after time zero)}$$

If the system has at least one more pole than zero, the initial value of the system in response to a step is equal to zero. In other words, the input must go through at least one integrator to reach the output so the output does not change instantly.

If the system has the same number of poles and zeros, the output will change instantaneously in response to a step input. Such direct feedthrough is not common in plants but does occur often in the system error in closed-loop:



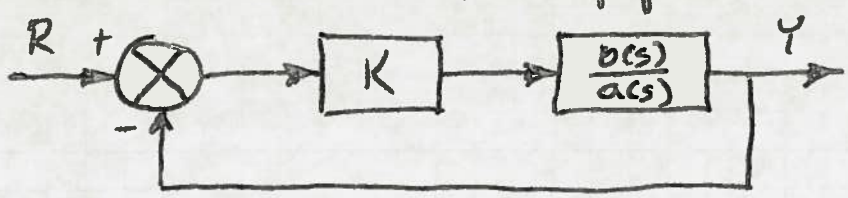
$$E(s) = R(s) - Y(s)$$

$$\frac{E(s)}{R(s)} = \underbrace{1}_{\text{direct feedthrough}} - \frac{Y(s)}{R(s)} = \frac{ms^2 + bs}{ms^2 + bs + K}$$

relative degree zero

This makes sense – a step change in the reference value will produce a step change in the error since the plant cannot respond instantly (it has two integrators between input and output).

Finally, the open-loop poles of a transfer function help to determine the closed-loop poles



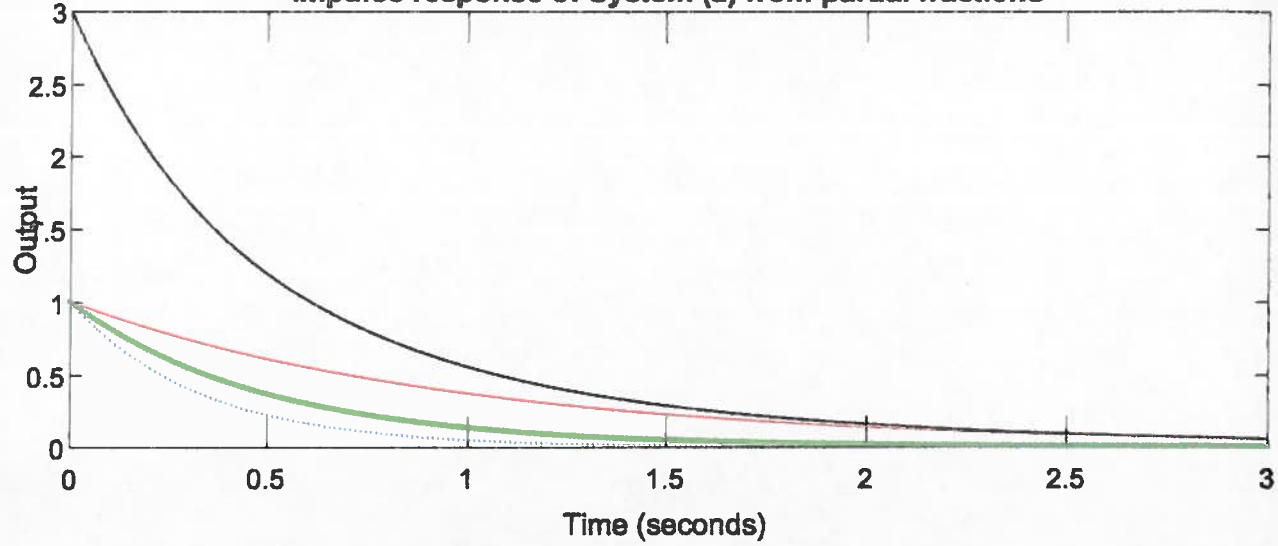
$$\frac{Y(s)}{R(s)} = \frac{K\frac{b(s)}{a(s)}}{1 + K\frac{b(s)}{a(s)}} = \frac{K\,b(s)}{a(s) + K\,b(s)}$$

Characteristic equation giving closed-loop poles is

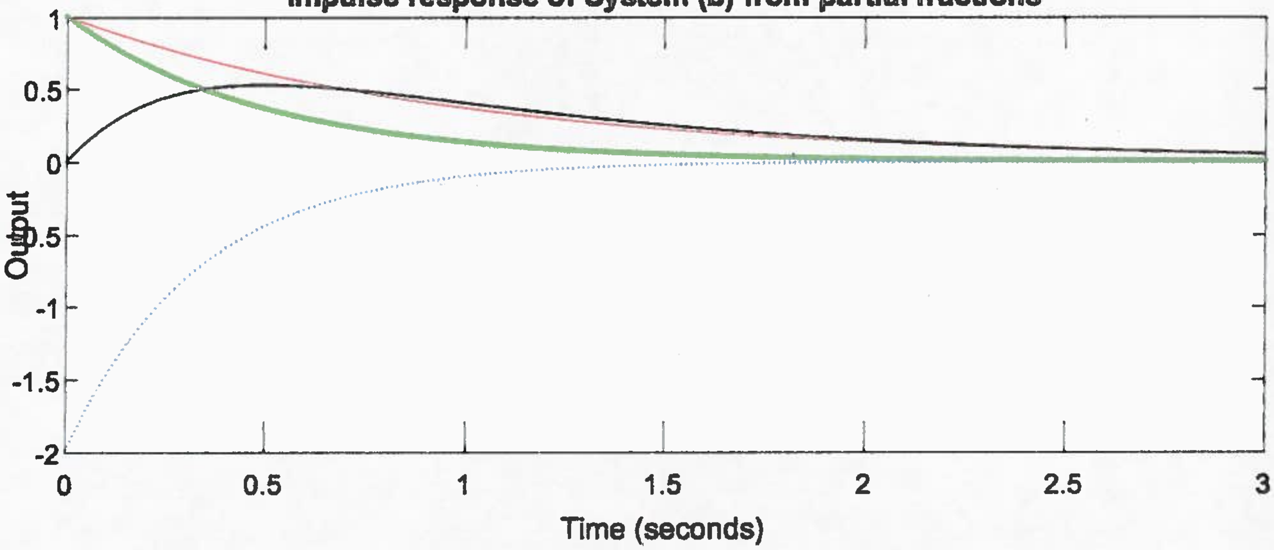
$$1 + K\frac{b(s)}{a(s)} = 0 \quad \text{or} \quad \underbrace{a(s)}_{\text{open-loop poles}} + K\underbrace{b(s)}_{\text{open-loop zeros}} = 0$$

The closed-loop poles can be described in terms of the open-loop poles, open-loop zeros and the gain K. This is the concept behind the root locus.

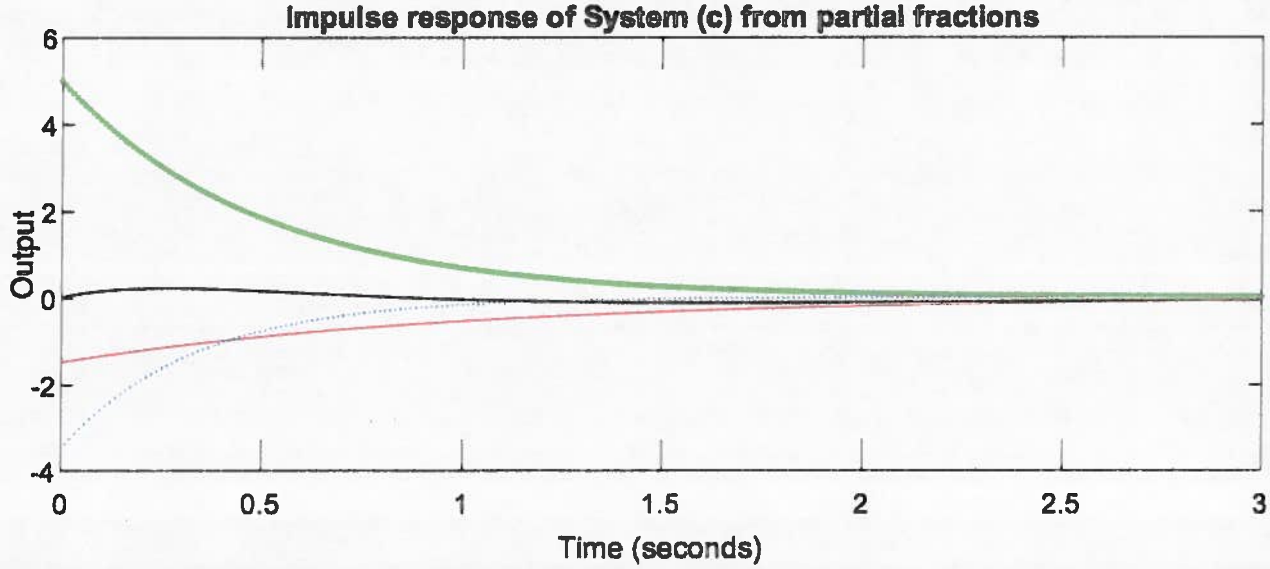
Impulse response of System (a) from partial fractions

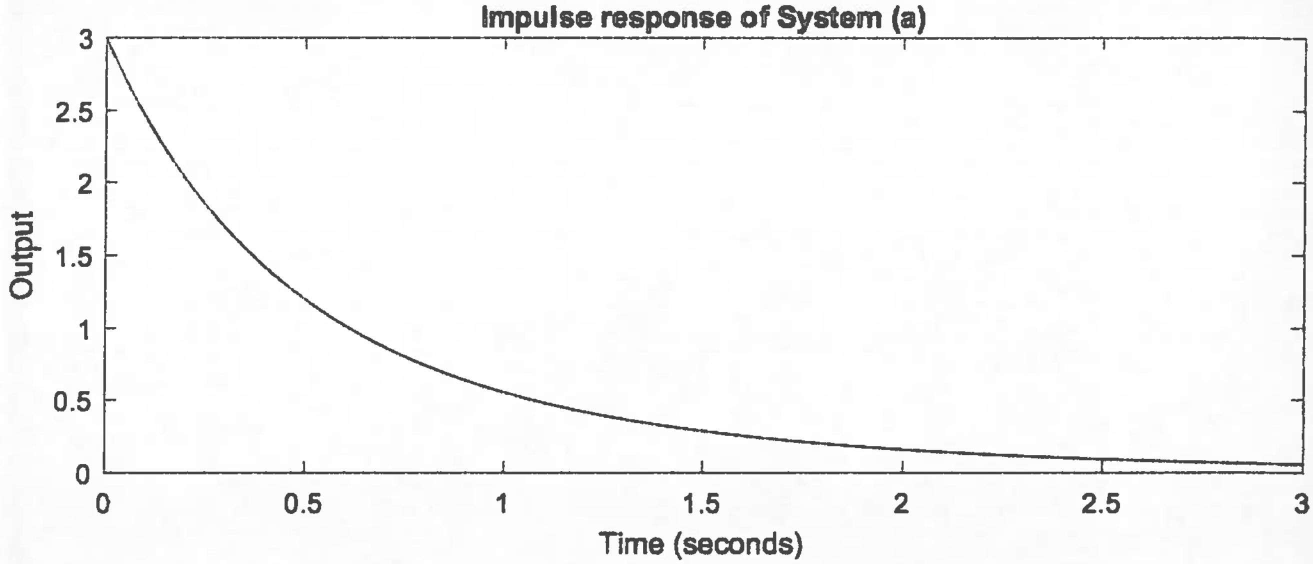


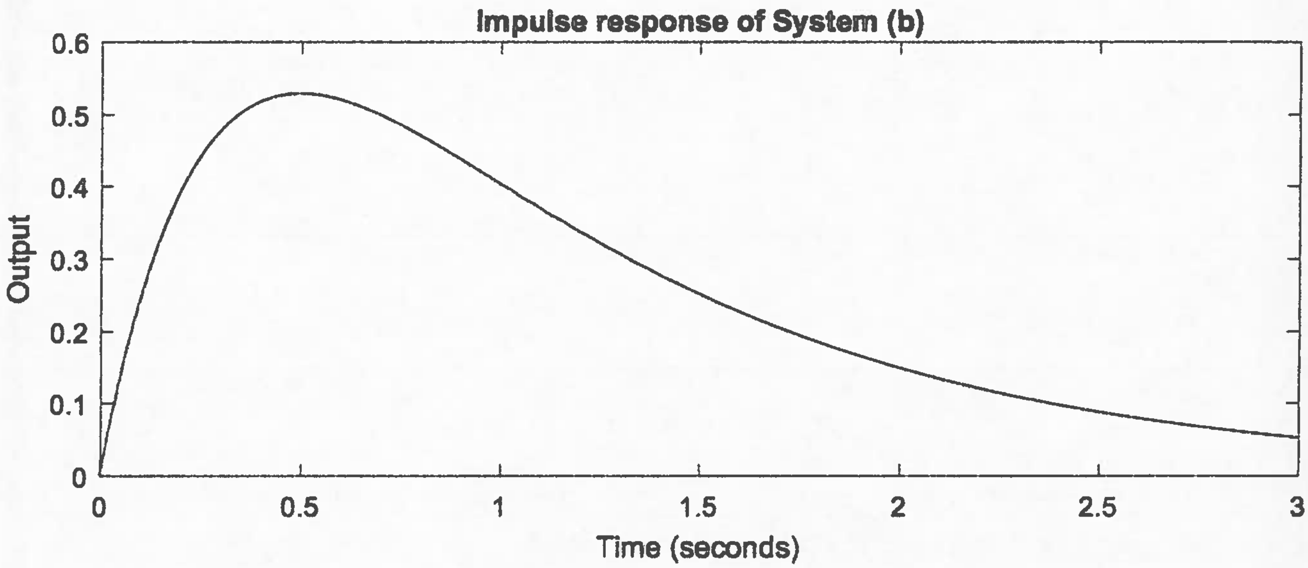
Impulse response of System (b) from partial fractions

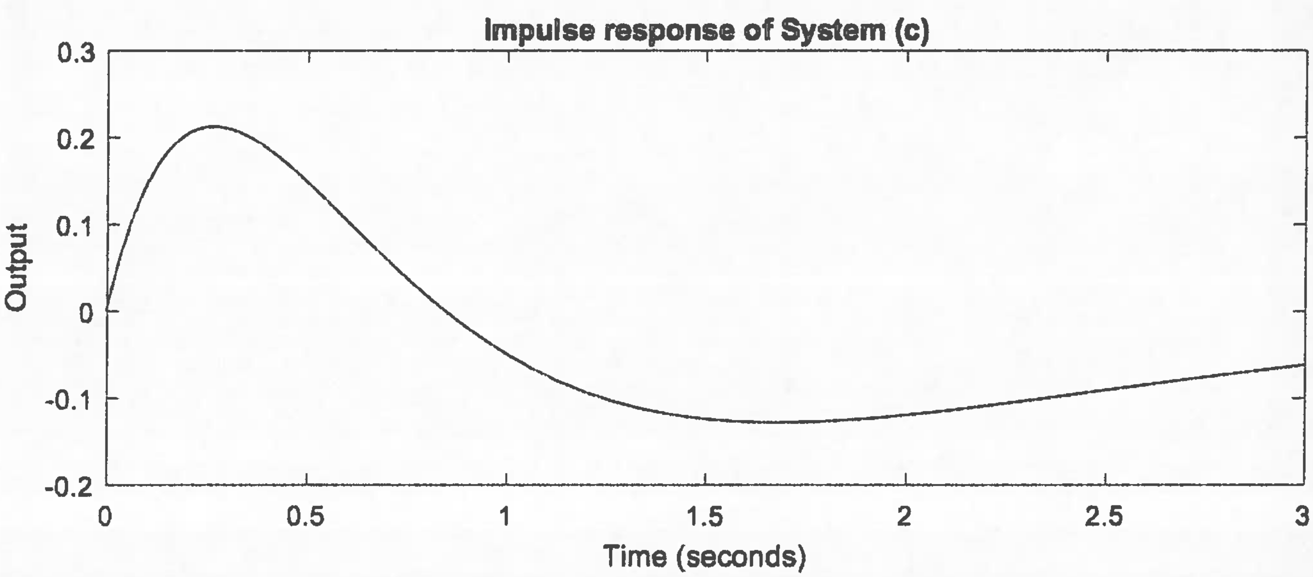


Impulse response of System (c) from partial fractions




Impulse response of System (a)
Output
Time (seconds)
0
0.5
1
1.5
2
2.5
3


Impulse response of System (b)
Output
Time (seconds)
0.6
0.5
0.4
0.3
0.2
0.1
0
0.5
1
1.5
2
2.5
3


Impulse response of System (c)
Output
Time (seconds)
0.3
0.2
0.1
0
-0.1
-0.2
0.5
1
1.5
2
2.5
3