Tracking Curved Paths

When tracking a curved path, several issues arise beyond those associated with the straight road. First, the coordinate system associated with the path rotates as the vehicle's projected point on the path changes. Second, the steady-state condition no longer has the entire vehicle following the path but rather a specific point. Both of these require some consideration.

Since things are getting a bit more complicated, let's start by collecting the general equations for our planar vehicle model moving along a path then add assumptions and simplify...

The vehicle equations are given by:

$$ma_x = m(\dot{U}_x - rU_y) = F_{xr} + F_{xf}\cos\delta - F_{yf}\sin\delta$$

$$ma_y = m(\dot{U}_y + rU_x) = F_{yf}\cos\delta + F_{yr} + F_{xf}\sin\delta$$

$$I_z\dot{r} = aF_{yf}\cos\delta + aF_{xf}\sin\delta - bF_{yr}$$

Here we have assumed the bicycle model but made no restrictions on angles, the linearity of tire forces or which wheels are driven.

The path coordinates evolve according to:

$$\dot{s} = \left(\frac{1}{1 - eK_r}\right)(U_x\cos\Delta\psi - U_y\sin\Delta\psi)$$

$$\dot{e} = U_y\cos\Delta\psi + U_x\sin\Delta\psi$$

$$\Delta\dot{\psi} = r - K_r\dot{s} \qquad \text{taking road curvature into account}$$

We also need second derivatives of e and $\Delta\psi$ to express our system in terms of the state variables $[e\ \dot{e}\ \Delta\psi\ \Delta\dot{\psi}]$ for analysis. Note that this is just rearranging the information above - I could simulate the whole system with the 6 equations above given the longitudinal forces, steer angle and a tire model.

$$\ddot{e} = \dot{U}_y\cos\Delta\psi + \dot{U}_x\sin\Delta\psi - U_y\sin\Delta\psi\,\Delta\dot{\psi} + U_x\cos\Delta\psi\,\Delta\dot{\psi}$$

$$\Delta\ddot{\psi} = \dot{r} - \dot{K}_r\dot{s} - K_r\ddot{s}$$

The expression for $\Delta\ddot{\psi}$ also contains the second derivative of s which gets a little complicated...

$$\ddot{s} = \left(\frac{1}{1-eK_1}\right)\left(\dot{U}_x\cos\Delta\psi - \dot{U}_y\sin\Delta\psi - U_x\sin\Delta\psi\,\Delta\dot{\psi} - U_y\cos\Delta\psi\,\Delta\dot{\psi}\right)$$
$$+ \frac{\dot{e}K_1 + e\dot{K}_1}{(1-eK_1)^2}\left(U_x\cos\Delta\psi - U_y\sin\Delta\psi\right)$$

Since curvature is solely a function of position along the path $K_1 = K_1(s)$ so

$$\dot{K}_1 = \frac{dK_1}{ds}\cdot\dot{s}$$

That got rather messy rather quickly! The various terms take on greater significance as the curvature gets larger and changes more rapidly, the speed of the vehicle changes rapidly and the vehicle deviates more from the path.

Let's resume analysis of the simpler case where these effects are minimal, confident that we can always simulate our resulting controller designs on the full set of equations.

So we will assume we are

* on a road with nonzero curvature
* staying relatively close to the path ($eK_1 \ll 1$)
* traveling at a constant speed U_x
* in a rear wheel drive vehicle
* in the linear region of the tires (or, in other words, our lateral acceleration is below about 0.6g on a dry road)
* able to use small angle approximations for steering, heading and slip angles

This leads directly to the fact that $\dot{U}_x = 0$ and we will further approximate

$$\dot{s} \approx U_x$$
$$\ddot{s} \approx 0$$

There are other ways to approximate these values if we want to capture some of the neglected effects but not others. This is the simplest approach.

Let's now rewrite the equations for our state vector with the curving road.

$$\dot{e} = U_y + U_x\Delta\psi$$
$$m\ddot{e} = m\dot{U}_y + mU_x\Delta\dot{\psi} \quad \text{(just as in previous lecture)}$$

$$m\ddot{e} = -C_{\alpha r}\left(\frac{U_y - br}{U_x}\right) - C_{\alpha f}\left(\frac{U_y + ar}{U_x}\right) + C_{\alpha f}\delta - m r U_x$$

$$+ m U_x (r - K_1 \dot{s})$$ a new term

$$= -\left(\frac{C_{\alpha f} + C_{\alpha r}}{U_x}\right) U_y - \left(\frac{aC_{\alpha f} - bC_{\alpha r}}{U_x}\right) r + C_{\alpha f}\delta - m U_x^2 K_1$$

$$= -\frac{(C_{\alpha f} + C_{\alpha r})}{U_x}\dot{e} + (C_{\alpha f} + C_{\alpha r})\Delta\psi - \frac{(aC_{\alpha f} - bC_{\alpha r})}{U_x}\Delta\dot{\psi}$$

$$- \frac{(aC_{\alpha f} - bC_{\alpha r})}{U_x} K_1 \dot{s} - m U_x^2 K_1$$

$$m\ddot{e} = -\frac{(C_{\alpha f} + C_{\alpha r})}{U_x}\dot{e} + (C_{\alpha f} + C_{\alpha r})\Delta\psi - \frac{(aC_{\alpha f} - bC_{\alpha r})}{U_x}\Delta\dot{\psi} + C_{\alpha f}\delta$$

$$- m U_x^2 K_1 + (bC_{\alpha r} - aC_{\alpha f}) K_1$$

$$I_z \Delta\ddot{\psi} = \left(\frac{-aC_{\alpha f} + bC_{\alpha r}}{U_x}\right) U_y - \left(\frac{a^2 C_{\alpha f} + b^2 C_{\alpha r}}{U_x}\right) r + aC_{\alpha f}\delta$$

$$- I_z \dot{K}_1 \dot{s} - \cancelto{0}{K_1 \ddot{s}}$$

$$= \frac{(bC_{\alpha r} - aC_{\alpha f})}{U_x}\dot{e} + (aC_{\alpha f} - bC_{\alpha r})\Delta\psi - \frac{(a^2 C_{\alpha f} + b^2 C_{\alpha r})}{U_x}\Delta\dot{\psi} + aC_{\alpha f}\delta$$

$$- \frac{(a^2 C_{\alpha f} + b^2 C_{\alpha r})}{U_x} K_1 \dot{s} - I_z \dot{K}_1 U_x$$

$$I_z \Delta\ddot{\psi} = \frac{(bC_{\alpha r} - aC_{\alpha f})}{U_x}\dot{e} + (aC_{\alpha f} - bC_{\alpha r})\Delta\psi - \frac{(a^2 C_{\alpha f} + b^2 C_{\alpha r})}{U_x}\Delta\dot{\psi} + aC_{\alpha f}\delta$$

$$- (a^2 C_{\alpha f} + b^2 C_{\alpha r}) K_1 - I_z \dot{K}_1 U_x$$

Moving back to state space form, this gives the same dynamic equations with the addition of a disturbance arising from curvature.

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \Delta\dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(C_{\alpha f}+C_{\alpha r})}{mU_x} & \frac{(C_{\alpha f}+C_{\alpha r})}{m} & \frac{(bC_{\alpha r}-aC_{\alpha f})}{mU_x} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{(bC_{\alpha r}-aC_{\alpha f})}{I_z U_x} & \frac{(aC_{\alpha f}-bC_{\alpha r})}{I_z} & -\frac{(a^2C_{\alpha f}+b^2C_{\alpha r})}{I_z U_x} \end{bmatrix} \begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \Delta\dot{\psi} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \frac{C_{\alpha f}}{m} \\ 0 \\ \frac{aC_{\alpha f}}{I_z} \end{bmatrix} \delta + \begin{bmatrix} 0 \\ -K_1\left(U_x^2 - \frac{(bC_{\alpha r}-aC_{\alpha f})}{m}\right) \\ 0 \\ -K_1\frac{(a^2C_{\alpha f}+b^2C_{\alpha r})}{I_z} - \dot{K}_1 U_x \end{bmatrix}$$ disturbance due to curvature

With the lookahead control scheme:

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \dot{\Delta\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_{la}}{m} & -\frac{(C_{\alpha f}+C_{\alpha r})}{mU_x} & \frac{(C_{\alpha f}+C_{\alpha r})}{m}-\frac{K_{la}x_{la}}{m} & \frac{(-aC_{\alpha f}+bC_{\alpha r})}{mU_x} \\ 0 & 0 & 0 & 1 \\ -\frac{K_{la}a}{I_z} & \frac{(bC_{\alpha r}-aC_{\alpha f})}{I_z U_x} & \frac{(aC_{\alpha f}-bC_{\alpha r})}{I_z}-\frac{K_{la}a x_{la}}{I_z} & -\frac{(a^2C_{\alpha f}+b^2C_{\alpha r})}{I_z U_x} \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \dot{\Delta\psi} \end{bmatrix}$$

$$+\begin{bmatrix} 0 \\ -\kappa_r\left(U_x^2-\frac{(bC_{\alpha r}-aC_{\alpha f})}{m}\right) \\ 0 \\ -\frac{\kappa_r(a^2C_{\alpha f}+b^2C_{\alpha r})}{I_z}-\dot{\kappa}_r U_x \end{bmatrix}+\begin{bmatrix} 0 \\ \frac{C_{\alpha f}}{m} \\ 0 \\ \frac{aC_{\alpha f}}{I_z} \end{bmatrix}\delta_{ff}$$

Since $\delta = \frac{-K_{la}}{C_{\alpha f}}(e+x_{la}\Delta\psi) + \delta_{ff}$

Notice that the curvature doesn't impact the stability of the system. The same stability results from the straight road case apply.

The issue with road curvature is that it produces a tracking error if left uncompensated. Let's look at this error and how we might compensate for it using a feedforward controller. So assume

$$\delta = \frac{-K_{la}}{C_{\alpha f}}(e+x_{la}\Delta\psi)+\delta_{ff}$$

In steady-state, $\dot{e}=\ddot{e}=\dot{\Delta\psi}=\ddot{\Delta\psi}=\dot{\kappa}_r=0$. Using the steering law above and zeroing these values in the state equations leaves two relationships:

$$0=-\frac{K_{la}}{m}e_{ss}+\left(\frac{C_{\alpha f}+C_{\alpha r}}{m}-\frac{K_{la}x_{la}}{m}\right)\Delta\psi_{ss}-\kappa_r\left(U_x^2-\frac{(bC_{\alpha r}-aC_{\alpha f})}{m}\right)+\frac{C_{\alpha f}}{m}\delta_{ff}$$

$$0=\frac{-K_{la}a}{I_z}e_{ss}+\left(\frac{aC_{\alpha f}-bC_{\alpha r}}{I_z}-\frac{K_{la}x_{la}a}{I_z}\right)\Delta\psi_{ss}-\kappa_r\left(\frac{a^2C_{\alpha f}+b^2C_{\alpha r}}{I_z}\right)+\frac{C_{\alpha r}}{I_z}\delta_{ff}a$$

or

$$0=-K_{la}e_{ss}+(C_{\alpha f}+C_{\alpha r}-K_{la}x_{la})\Delta\psi_{ss}-\kappa_r(mU_x^2-(bC_{\alpha r}-aC_{\alpha f}))+C_{\alpha f}\delta_{ff}$$

$$0=-K_{la}e_{ss}+\left(\frac{aC_{\alpha f}-bC_{\alpha r}}{a}-K_{la}x_{la}\right)\Delta\psi_{ss}-\kappa_r\left(\frac{a^2C_{\alpha f}+b^2C_{\alpha r}}{a}\right)+C_{\alpha f}\delta_{ff}$$

If we subtract the second equation from the first, both the lateral error and the feedforward steering terms cancel. This leaves one equation:

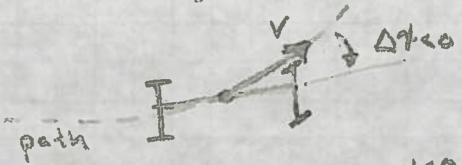
$$0=\left(C_{\alpha f}+C_{\alpha r}-C_{\alpha f}+\frac{b}{a}C_{\alpha r}\right)\Delta\psi_{ss}-\kappa_r\left(mU_x^2-bC_{\alpha r}+aC_{\alpha f}-aC_{\alpha f}-\frac{b^2}{a}C_{\alpha r}\right)$$

$$\frac{L}{a}C_{\alpha r}\,\Delta\psi_{ss}=\kappa_r\left(mU_x^2-bC_{\alpha r}-\frac{b^2}{a}C_{\alpha r}\right)$$

$$\frac{L}{a} C_{\alpha r} \Delta\psi_{ss} = K_1 \left(m U_x^2 - \frac{b(a+b)}{a} C_{\alpha r} \right)$$

$$\Rightarrow \quad \Delta\psi_{ss} = K_1 \left(\frac{m a U_x^2}{L C_{\alpha r}} - b \right)$$

What this means is that, regardless of the feedforward chosen, there will be a steady-state heading error we cannot compensate. This is a result of the fact that the vehicle itself has a nonzero lateral velocity in equilibrium. The sign on this error changes as a function of speed:



At low speed, $\Delta\psi_{ss} < 0$ in a left turn. The front axle tracks to the outside.

At higher speeds, this reverses and the front axle tracks to the inside.

We can't influence steady-state heading error with our feedforward but we can influence lateral error. Let's solve for the feedforward that gives zero steady-state lateral error so $e_{ss} = 0$ and when we add the two steady-state equations we get:

$$0 = \left(C_{\alpha f} + C_{\alpha r} + C_{\alpha f} - \frac{b}{a} C_{\alpha r} - 2 K_{1a} x_{1a} \right) \Delta\psi_{ss} - K_1 \left(m U_x^2 - b C_{\alpha r} + a C_{\alpha f} + a C_{\alpha f} + \frac{b^2}{a} C_{\alpha r} \right) + 2 C_{\alpha f} \delta_{ff}$$

$$0 = \left(1 + \frac{a-b}{2a} \frac{C_{\alpha r}}{C_{\alpha f}} - \frac{K_{1a} x_{1a}}{C_{\alpha f}} \right) \Delta\psi_{ss} - K_1 \left(\frac{m U_x^2}{2 C_{\alpha f}} + a + \frac{b(b-a)}{2a} \frac{C_{\alpha r}}{C_{\alpha f}} \right) + \delta_{ff}$$

$$0 = -\frac{K_{1a} x_{1a}}{C_{\alpha f}} \Delta\psi_{ss} + \left(1 + \frac{a-b}{2a} \frac{C_{\alpha r}}{C_{\alpha f}} \right) \left(\frac{m a U_x^2}{L C_{\alpha r}} - b \right) K_1 - K_1 \left(\frac{m U_x^2}{2 C_{\alpha f}} + a + \frac{b(b-a)}{2a} \frac{C_{\alpha r}}{C_{\alpha f}} \right) + \delta_{ff}$$

$$\delta_{ff} = \frac{K_{1a} x_{1a}}{C_{\alpha f}} \Delta\psi_{ss} - \left(\frac{m a U_x^2}{L C_{\alpha r}} + \frac{(a-b)}{2\cancel{a}} \frac{m \cancel{a} U_x^2}{L C_{\alpha f}} - b - \cancel{\frac{b(a-b)}{2a} \frac{C_{\alpha r}}{C_{\alpha f}}} \right) K_1 - K_1 \left(\frac{-m U_x^2}{2 C_{\alpha f}} - a - \frac{b(b-a)}{2a} \frac{C_{\alpha r}}{C_{\alpha f}} \right)$$

$$\delta_{ff} = \frac{K_{1a} x_{1a}}{C_{\alpha f}} \Delta\psi_{ss} + K_1 \left(L - \frac{m a U_x^2}{L C_{\alpha r}} - \frac{(a-b) U_x^2 m}{2 L C_{\alpha f}} - \frac{m U_x^2}{2 C_{\alpha f}} \right)$$

$$\delta_{ff} = \frac{K_{1a} x_{1a}}{C_{\alpha f}} \Delta\psi_{ss} + K_1 \left(L - U_x^2 \left\{ \frac{m (a C_{\alpha f} - b C_{\alpha r})}{L C_{\alpha f} C_{\alpha r}} \right\} \right)$$

this is just K!

$$\Rightarrow \delta_{ff} = \frac{K_{la} x_{la}}{C_{\alpha f}} \Delta\Psi_{ss} + \underbrace{\hat{\kappa}_r (L + K U_x^2)}_{\text{since } \hat{\kappa}_r = \frac{1}{R} \text{ this is just } \frac{L}{R} + K a_y}$$

So our feed forward is easily understood physically. The second term is just the steer angle we would expect we need to turn on a radius of $1/\hat{\kappa}_r$ based on our vehicle model.

The first term compensates for the steady-state heading error. Since our controller is trying to bring the lookahead error to zero, the steady-state heading error produces a steady-state lateral error:

$$e_{la} = e + x_{la} \Delta\Psi$$

$$e_{la,ss} = 0 = e_{ss} + x_{la} \Delta\Psi_{ss}$$

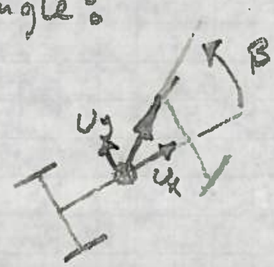
So the $\frac{K_{la} x_{la}}{C_{\alpha f}} \Delta\Psi_{ss}$ term in the feedforward results in a steady state lookahead error:

$$e_{la,ss} = x_{la} \Delta\Psi_{ss}$$

such that $e_{ss} = 0$.

If this term of the feedforward is changed to a distance other than x_{la} then a different point on the vehicle can be designed to have zero steady-state error. (at least if you ignore the curvature change within the vehicle's wheelbase).

To get a better sense of where the steady-state heading error comes from, recall that the dynamic bicycle model had two states, U_y and r. At a constant speed and small angles, the lateral velocity can be simply expressed in terms of the sideslip angle:



$$\beta = \tan^{-1}\left(\frac{U_y}{U_x}\right)$$

$$\beta \approx \frac{U_y}{U_x}$$

$$\dot{\beta} = \frac{\dot{U}_y}{U_x} \text{ for constant } U_x.$$

We can form a transfer function for $U_y(s)$ or $\beta(s)$ just as we did for $R(s)$. Unsurprisingly, it has the same denominator but different numerator:

$$\frac{B(s)}{\Delta(s)} = \frac{\frac{C_{\alpha f} I_z}{m U_x} s + \left(\frac{b L C_{\alpha f} C_{\alpha r}}{m U_x^2} - a C_{\alpha f} \right)}{I_z s^2 + \left[\frac{I_z (C_{\alpha f} + C_{\alpha r})}{m U_x} + \frac{a^2 C_{\alpha f} + b^2 C_{\alpha r}}{U_x} \right] s + \left[\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + b C_{\alpha r} - a C_{\alpha f} \right]}$$

Using the final value theorem, the steady state sideslip angle for a steer angle of δ is

$$\beta_{ss} = \frac{C_{\alpha f} \left[b L C_{\alpha r} - a m U_x^2 \right]}{L^2 C_{\alpha f} C_{\alpha r} + (b C_{\alpha r} - a C_{\alpha f}) m U_x^2} \delta$$

Remembering that $r_{ss} = \frac{U_x}{L + K U_x^2} \delta$ and that for steady cornering,

$$\frac{r_{ss}}{U_x} = \kappa_1$$

the steer angle required for curvature κ_1 at speed U_x is

$$\delta = \kappa_1 (L + K U_x^2)$$

This results in a steady-state sideslip error of

$$\beta_{ss} = \frac{C_{\alpha f} \left[b L C_{\alpha r} - a m U_x^2 \right] \kappa_1 (L + K U_x^2)}{L^2 C_{\alpha f} C_{\alpha r} + K L C_{\alpha f} C_{\alpha r} U_x^2}$$

$$\beta_{ss} = \frac{C_{\alpha f} \left[b L C_{\alpha r} - a m U_x^2 \right] \cancel{(L + K U_x^2)} \kappa_1}{L C_{\alpha f} C_{\alpha r} \cancel{(L + K U_x^2)}}$$

$$\beta_{ss} = \left(b - \frac{a m U_x^2}{C_{\alpha r} L} \right) \kappa_1$$

This is the same result we got for the steady-state heading error $\Delta\Psi_{ss}$ only with the opposite sign (since $\Delta\Psi = \Psi_{VE} - \Psi_{PE}$).