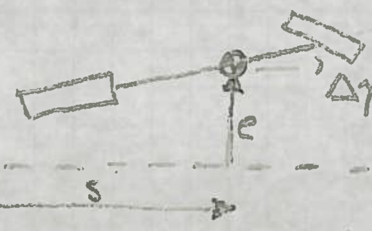
Lanekeeping

The bicycle model captures many of the essential dynamics of an automobile in terms of the velocity states U_y and r (assuming constant U_x). When considered relative to a road or path, the position states also come into play.

Let's start by considering a straight road. The path fixed coordinates are then straightforward:



s - distance along the path
e - lateral distance from c.g. to path or lane centerline
Δψ - heading error

The vehicle equations assuming constant speed, small angles and linear tires are:

$$m\dot{U}_y = -C_{\alpha r}\left(\frac{U_y - br}{U_x}\right) - C_{\alpha f}\left(\frac{U_y + ar}{U_x}\right) - mrU_x + C_{\alpha f}\delta$$

$$I_z\dot{r} = -aC_{\alpha f}\left(\frac{U_y + ar}{U_x}\right) + bC_{\alpha r}\left(\frac{U_y - br}{U_x}\right) + aC_{\alpha f}\delta$$

The position coordinates are:

$$\dot{s} = \left(\frac{1}{1 - e\kappa}\right)(U_x \cos\Delta\psi - U_y \sin\Delta\psi)$$

$$\dot{s} = U_x - U_y\Delta\psi \approx U_x \quad \text{since } \kappa = 0,\ \Delta\psi \text{ small and } U_y \ll U_x$$

$$\dot{e} = U_x\Delta\psi + U_y$$

$$\Delta\dot{\psi} = r$$

We can write a new set of equations in terms of the state vector:

$$\begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \Delta\dot{\psi} \end{bmatrix} \qquad U_y = \dot{e} - U_x\Delta\psi$$

by differentiating the position state equations and substituting for the rates of change of U_y and r...

$$m\ddot{e} = m\dot{U}_y + mU_x\Delta\dot{\psi}$$

$$= -C_{\alpha r}\left(\frac{U_y - br}{U_x}\right) - C_{\alpha f}\left(\frac{U_y + ar}{U_x}\right) + C_{\alpha f}\delta - mrU_x + mrU_x$$

$$= -\left(\frac{C_{\alpha f} + C_{\alpha r}}{U_x}\right)U_y - \left(\frac{aC_{\alpha f} - bC_{\alpha r}}{U_x}\right)r + C_{\alpha f}\delta$$

$$m\ddot{e} = -\frac{(C_{\alpha f} + C_{\alpha r})}{U_x}\dot{e} + (C_{\alpha f} + C_{\alpha r})\Delta\psi - \frac{(aC_{\alpha f} - bC_{\alpha r})}{U_x}\Delta\dot{\psi} + C_{\alpha f}\delta$$

$$I_z\Delta\ddot{\psi} = \left(\frac{-aC_{\alpha f} + bC_{\alpha r}}{U_x}\right)U_y - \left(\frac{a^2C_{\alpha f} + b^2C_{\alpha r}}{U_x}\right)r + aC_{\alpha f}\delta$$

$$I_z\Delta\ddot{\psi} = \left(\frac{-aC_{\alpha f} + bC_{\alpha r}}{U_x}\right)\dot{e} + (aC_{\alpha f} - bC_{\alpha r})\Delta\psi - \left(\frac{a^2C_{\alpha f} + b^2C_{\alpha r}}{U_x}\right)\Delta\dot{\psi} + aC_{\alpha}$$

Combining these gives:

"damping" terms

$$\begin{bmatrix} m & 0 \\ 0 & I_z \end{bmatrix}\begin{bmatrix} \ddot{e} \\ \Delta\ddot{\psi} \end{bmatrix} = \frac{1}{U_x}\begin{bmatrix} -C_{\alpha f} - C_{\alpha r} & bC_{\alpha r} - aC_{\alpha f} \\ bC_{\alpha r} - aC_{\alpha f} & -a^2C_{\alpha f} - b^2C_{\alpha r} \end{bmatrix}\begin{bmatrix} \dot{e} \\ \Delta\dot{\psi} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & C_{\alpha f} + C_{\alpha r} \\ 0 & aC_{\alpha f} - bC_{\alpha r} \end{bmatrix}\begin{bmatrix} e \\ \Delta\psi \end{bmatrix} + \begin{bmatrix} C_{\alpha f} \\ aC_{\alpha f} \end{bmatrix}\delta$$

mass matrix

what is the significance of this zero?

"spring terms"

This form shows we have a coupled pair of mass-spring-damper systems (for the second order yaw dynamics and the second order lateral dynamics). There is no "spring" term on e in the lateral dynamics, which makes sense if you pause to consider that we have merely rewritten the state equations.

This can also be put in the form $\dot{\underset{\sim}{x}} = A\underset{\sim}{x} + Bu$:

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \Delta\dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{(C_{\alpha f}+C_{\alpha r})}{mU_x} & \frac{(C_{\alpha f}+C_{\alpha r})}{m} & \frac{(bC_{\alpha r}-aC_{\alpha f})}{mU_x} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{(bC_{\alpha r}-aC_{\alpha f})}{I_zU_x} & \frac{(aC_{\alpha f}-bC_{\alpha r})}{I_z} & -\frac{(a^2C_{\alpha f}+b^2C_{\alpha r})}{I_zU_x} \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \Delta\dot{\psi} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ \frac{C_{\alpha f}}{m} \\ 0 \\ \frac{aC_{\alpha f}}{I_z} \end{bmatrix}\delta$$

With no steering input, the state equations take the form $\dot{\underset{\sim}{x}} = A\underset{\sim}{x}$

The system poles are just the eigenvalues of A. The characteristic equation is $|\lambda I - A| = 0$ which gives for this case:

$$\lambda^2(\lambda^2 + a_1\lambda + a_2) = 0$$

$$a_1 = \frac{(C_{\alpha f} + C_{\alpha r}) I_z + (a^2 C_{\alpha f} + b^2 C_{\alpha r}) m}{I_z m U_x}$$

$$a_2 = \frac{C_{\alpha f} C_{\alpha r} (a+b)^2 + (b C_{\alpha r} - a C_{\alpha f}) m U_x^2}{I_z m U_x^2}$$

These might look familiar - this is the same characteristic equation we had for the bicycle model alone, together with two pure integrators. That makes sense - we have so far done nothing to alter the dynamics of the system. We have merely changed coordinates and added position states to the problem.

We could just as well have calculated the transfer function:

$$s\underset{\sim}{X}(s) = A\underset{\sim}{X}(s) + BU(s)$$

$$(sI - A)\underset{\sim}{X}(s) = BU(s)$$

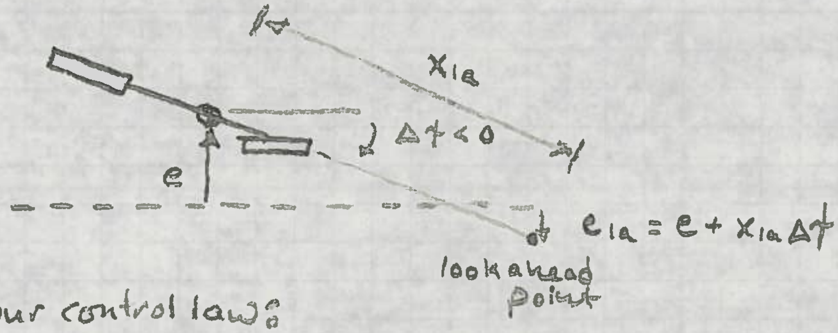
$$\Rightarrow \quad \frac{\underset{\sim}{X}(s)}{U(s)} = (sI - A)^{-1} B$$

This gives a vector of transfer functions with the term $(sI-A)$ in the denominator, demonstrating why the poles are found by evaluating the expression:

$$|\lambda I - A| = 0$$

The system dynamics change and the poles shift when we use a control law to change the steering in response to lateral and heading error.

One way to do this is to use the steering to produce a force proportional to the lookahead error - the error at a point projected out in front of the vehicle...



our control law:

$$C_{\alpha f}\,\delta = -K_{la} e_{la} = -K_{la}(e + x_{la}\Delta\psi)$$

$$\text{or} \quad \delta = -\frac{K_{la}(e + x_{la}\Delta\psi)}{C_{\alpha f}}$$

With this control input, the new system matrix is:

$$\frac{d}{dt}\begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \Delta\dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_{la}}{m} & -\frac{(C_{\alpha f}+C_{\alpha r})}{mU_x} & \frac{(C_{\alpha f}+C_{\alpha r})}{m} - \frac{K_{la}x_{la}}{m} & \frac{(-aC_{\alpha f}+bC_{\alpha r})}{mU_x} \\ 0 & 0 & 0 & 1 \\ -\frac{K_{la}a}{I_z} & \frac{(bC_{\alpha r}-aC_{\alpha f})}{I_z U_x} & \frac{(aC_{\alpha f}-bC_{\alpha r})}{I_z} - \frac{K_{la}a x_{la}}{I_z} & -\frac{(a^2C_{\alpha f}+b^2C_{\alpha r})}{I_z U_x} \end{bmatrix}\begin{bmatrix} e \\ \dot{e} \\ \Delta\psi \\ \Delta\dot{\psi} \end{bmatrix}$$

and $\lambda^4 + d_1\lambda^3 + d_2\lambda^2 + d_3\lambda + d_4 = 0$

$$d_1 = \frac{(C_{\alpha f}+C_{\alpha r})I_z + (a^2C_{\alpha f}+b^2C_{\alpha r})m}{I_z m U_x}$$

$$d_2 = \frac{C_{\alpha f}C_{\alpha r}L^2 + (bC_{\alpha r}-aC_{\alpha f})mU_x^2 + K_{la}U_x^2(I_z + m a x_{la})}{I_z m U_x^2}$$

$$d_3 = \frac{K_{la}L\,C_{\alpha r}(a + x_{la})}{I_z m U_x}$$

$$d_4 = \frac{K_{la}L\,C_{\alpha r}}{I_z m}$$

Only one of the coefficients in the characteristic equation can become negative – d_2. This can only become negative for the oversteering vehicle. It might be tempting to think that the system is stable for any understeering vehicle but that would be a mistake.

Having all coefficients positive is necessary and sufficient for stability of a second order system but only necessary for higher order systems. There are two additional criteria for sufficiency:

$$d_1 d_2 > d_3$$

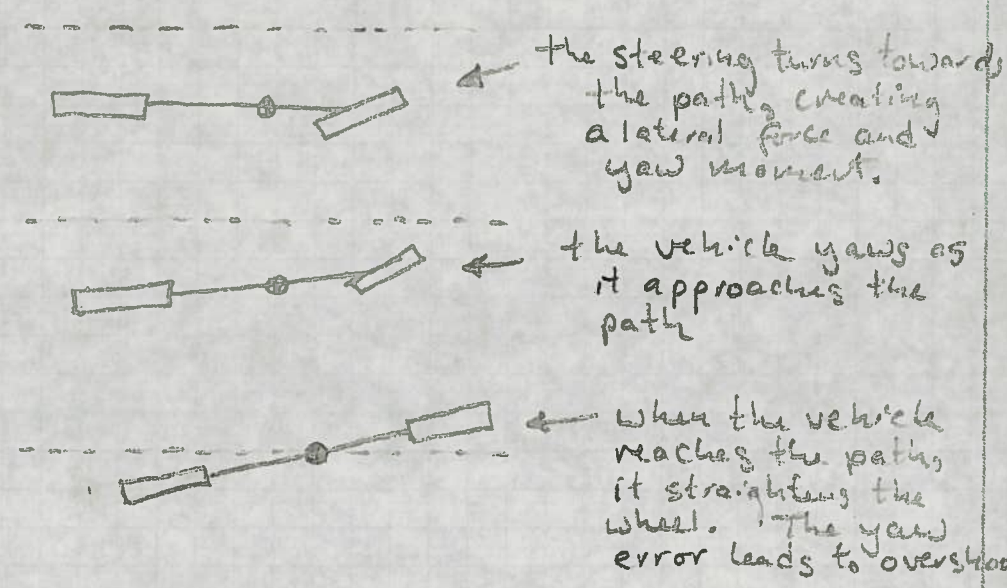
$$d_1 d_2 d_3 > d_3^2 + d_1^2 d_4$$

These conditions follow from the Routh Array.

Translating this into specific requirements on the parameters can be an algebraic nightmare. But there are several ways of looking at this problem to gain intuition.

First, notice that K_{1a} and x_{1a} appear in different combinations in d_2, d_3 and d_4. Thus they represent separate knobs to turn in meeting all of the criteria simultaneously.

Second, the need for the lookahead can be seen by considering a controller acting on the lateral error alone. Suppose the vehicle starts with a lateral error.



This is a fundamental difference between the dynamic model and the kinematic model. In the kinematic model, steering led to yaw and yaw changed the lateral error. In the dynamic model, steering produces both a lateral force and yaw moment that drive the coupled mass-spring-damper system. This requires a bit more care to maintain stability.