System Modeling

The analysis and design tools in this class apply to dynamic system models that can represent a range of different physical systems. While developing models of mechanical or electrical systems can be a course in and of itself, there are a few basic concepts needed for this class

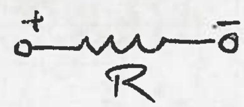
Electrical Systems

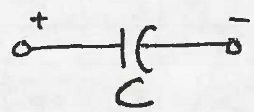
* Definitions of Elements
* Kirchoff's Laws
* Golden Rules of Op Amps

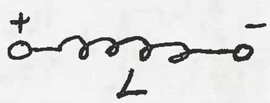
Mechanical Systems

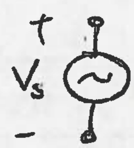
* Free body diagrams
* $F = ma$

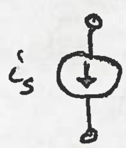
Electrical System Elements:

Resistor  $V = iR$

Capacitor:  $i = C\frac{dV}{dt}$

Inductor:  $V = L\frac{di}{dt}$

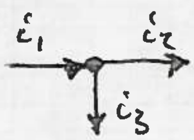
Voltage Source:  $V = V_s$

Current Source:  $i = i_s$

Each capacitor and inductor add one derivative to the model and thus add one state (this connection should get clearer in the next couple of lectures). Resistors do not involve derivatives and do not add states to the dynamic model.

Kirchoff's Laws

Current law

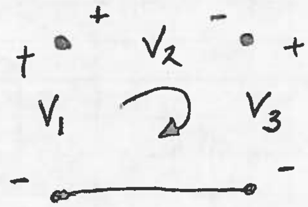


$i_1 = i_2 + i_3$

The sum of currents flowing into a node is zero.

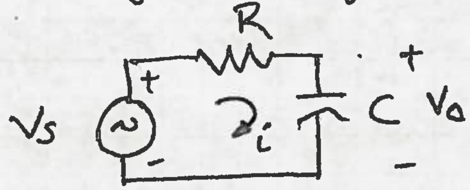
Voltage law

Sum of voltages taken around a closed path is zero



$$V_1 = V_2 + V_3$$

Putting it all together for a simple RC circuit...



$$V_s - iR = V_o$$

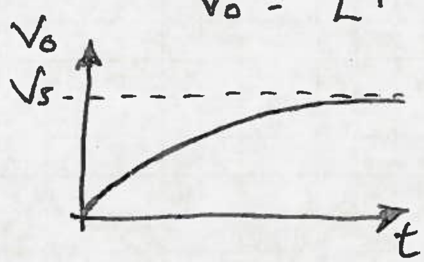
$$i = C\,\frac{dV_o}{dt}$$

$$V_s - RC\,\frac{dV_o}{dt} = V_o$$

$$\Rightarrow \quad \frac{dV_o}{dt} = \frac{1}{RC}\left[V_s - V_o\right]$$

For zero initial conditions, this has the solution

$$V_o = \left[1 - e^{-t/RC}\right] V_s$$



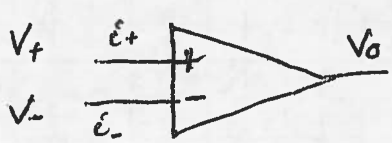
A first order system response

Ideal Op Amps

Ideal Op-amps satisfy two rules:

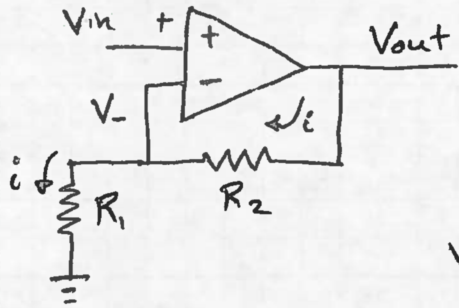
(1) V_o does what it takes to make $V_+ = V_-$

(2) $i_+ = i_- = 0$ so no current is drawn



With these two rules, Kirchoff's laws and the definition of the elements, equations can be derived for basic Op-amp circuits.

Example 1



$$V_- = V_{out} - iR_2 = iR_1 = V_{in}$$

$$\Rightarrow V_{in} = V_{out} - iR_2$$

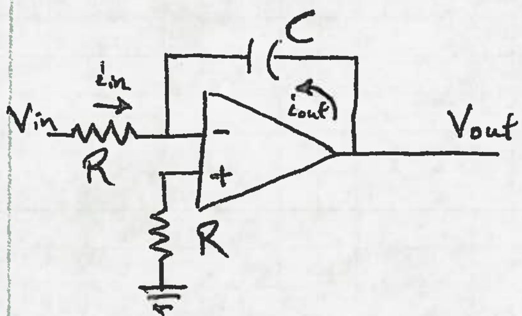
$$i = \frac{V_{out} - V_{in}}{R_2}$$

$$V_{in} = iR_1 = \frac{R_1}{R_2}\left[V_{out} - V_{in}\right]$$

$$\Rightarrow V_{out} = V_{in}\left[1 + \frac{R_2}{R_1}\right]$$

This is just a static gain. There are no dynamics in this system since no derivatives are involved (no capacitors or inductors).

Example 2



$$i_{in} = -i_{out}$$

$$\frac{V_{in}}{R} = -C\frac{dV_{out}}{dt}$$

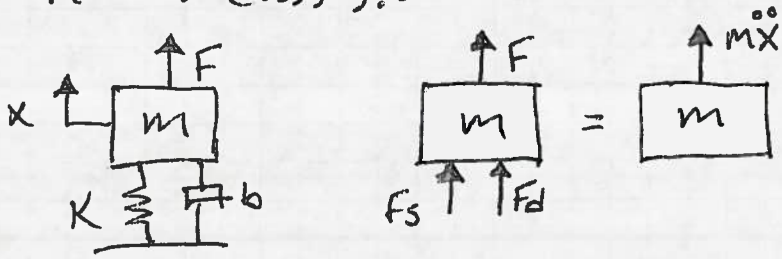
$$\frac{dV_{out}}{dt} = -\frac{1}{RC}V_{in}$$

For zero initial conditions, $V_{out} = -\frac{1}{RC}\int_0^t V_{in}(\tau)\,d\tau$

This is an integrator circuit. It does have dynamics due to the addition of the capacitor.

Free Body Diagrams

The key to working with mechanical systems is to draw a free body diagram for each mass in the system which shows all of the forces acting on that mass. The equations of motion follow from $F = ma$ (we won't do anything with more complicated 3D dynamics in this class).



spring: $F_s = -Kx$

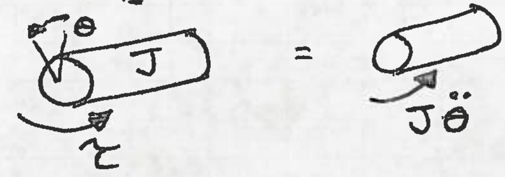
damper: $F_d = -b\dot{x}$

$$\Sigma F = ma = m\ddot{x} \quad \Rightarrow \quad F + f_s + f_d = m\ddot{x}$$

$$F - Kx - b\dot{x} = m\ddot{x}$$

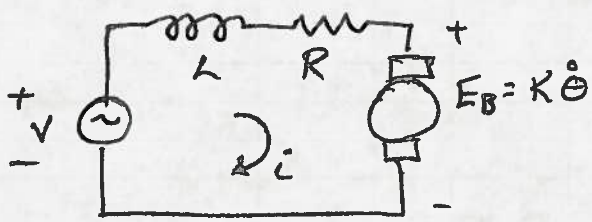
$$\Rightarrow \quad m\ddot{x} + b\dot{x} + Kx = F$$

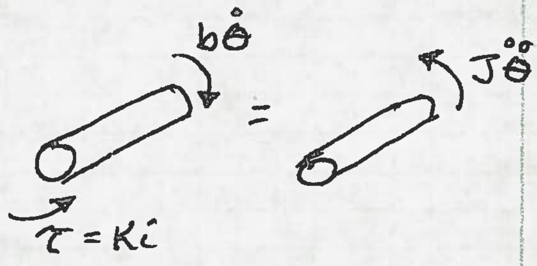
The rotational form is pretty much the same - just think in terms of torque and inertia instead of force and mass. Angular acceleration replaces linear acceleration.



$$\Sigma \tau = J\ddot{\theta}$$

Combining mechanical and electrical modeling, we can get a simple motor model:





$$V - L\frac{di}{dt} - iR - K\dot{\theta} = 0 \qquad\qquad \tau = b\dot{\theta} + J\ddot{\theta}$$

$$L\frac{di}{dt} + iR + K\dot{\theta} = V \qquad\qquad J\ddot{\theta} + b\dot{\theta} = Ki$$

These two equations define the dynamics of the motor. Often L is small relative to R and K. In this case, the electrical dynamics are approximately

$$iR + K\dot{\theta} = V$$

So $$J\ddot{\theta} + b\dot{\theta} = K\left[\frac{V}{R} - \frac{K\dot{\theta}}{R}\right]$$

$$J\ddot{\theta} + \left(b + \frac{K^2}{R}\right)\dot{\theta} = \frac{K}{R}V$$

In this case, we have a second order system (two derivatives). If we are interested in positions, we get two states or two derivatives from each mass in the system.

If we only care about angular velocity $\omega = \dot{\theta}$

$$\Rightarrow \quad J\dot{\omega} + \left(b + \frac{K^2}{R}\right)\omega = \frac{K}{R}V$$

This is a first order system

General Form of Models

The models we will examine in this class can be put in a general form:

$$\frac{d^n y}{dt^n} + a_1 \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_n y = b_1 \frac{d^m u}{dt^m} + b_2 \frac{d^{m-1} u}{dt^{m-1}} + \ldots + b_{m+1} u$$

Where u is the input
y is the output
n is the system order or number of states
$y, \frac{dy}{dt}, \ldots, \frac{d^{n-1}}{dt^{n-1}} y$ are the states
$a_1 \ldots a_n$ and $b_1 \ldots b_{m+1}$ are system parameters

For example, the electric motor model

Speed control $\quad \frac{d\omega}{dt} = \left(\frac{b}{J} + \frac{K^2}{JR}\right)\omega = \frac{K}{JR} V$

V is the input
ω is the output and the state
$n = 1$ so the system is first order
parameters are b, J, K, R or $\left(\frac{b}{J} + \frac{K^2}{JR}\right)$ and $\frac{K}{JR}$

Position control $\quad \frac{d^2\theta}{dt^2} + \left(\frac{b}{J} + \frac{K^2}{JR}\right)\frac{d\theta}{dt} = \frac{K}{JR} V$

V is the input
θ is the output
$n = 2$ so the system is second order
θ and $\frac{d\theta}{dt}$ are the states

Position Control with Motor Inductance

This one at first looks a little different since our states as we have modeled the system are θ, $\frac{d\theta}{dt}$ and the current, i, with equations

$$\frac{d^2\theta}{dt^2} + \left(\frac{b}{J}\right)\frac{d\theta}{dt} = \frac{K}{J} i$$

$$L \frac{di}{dt} + iR + K \frac{d\theta}{dt} = V$$

The choice of states is not unique, however, and we can rearrange this into the canonical form above. Differentiating the first equation

$$\frac{d^3\theta}{dt^3} + \frac{b}{J}\frac{d^2\theta}{dt^2} = \frac{K}{J}\frac{di}{dt} = \frac{K}{JL}\left(V - K\frac{d\theta}{dt} - iR\right)$$

where $i = \frac{J}{K}\frac{d^2\theta}{dt^2} + \frac{b}{K}\frac{d\theta}{dt}$

So we can completely remove i and write:

$$\frac{d^3\theta}{dt^3} + \frac{b}{J}\frac{d^2\theta}{dt^2} = \frac{K}{JL}V - \frac{K^2}{JL}\frac{d\theta}{dt} - \frac{KR}{JL}\cdot\frac{J}{K}\frac{d^2\theta}{dt^2} - \frac{KR}{JL}\frac{b}{K}\frac{d\theta}{dt}$$

$$\Rightarrow \frac{d^3\theta}{dt^3} + \left(\frac{b}{J}+\frac{R}{L}\right)\frac{d^2\theta}{dt^2} + \left(\frac{K^2}{JL}+\frac{bR}{JL}\right)\frac{d\theta}{dt} = \frac{K}{JL}V$$

In this form,

V is the input
θ is the output
$n=3$ so the system is third order
θ, $\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt^2}$ are the states

When we work with Laplace Transforms, we will generally not need to algebraically rearrange the system like this. The point is that we can, so this is a general form for our models.

Torque and Electrical Constants

Some references (like your book) use separate values for the motor torque constant and electrical constant so

$$\tau = K_t i \qquad E_B = K_e\dot{\theta}$$

I used the same value K for both - why? A motor can convert electrical power to mechanical power and vise versa (when it acts as a generator). This power conversion neither creates or dissipates power (the resistor and friction do the dissipation).

$$\text{Power in} = E_B \cdot i = K_e\dot{\theta} i$$
$$\text{Power out} = \tau\cdot\dot{\theta} = K_t\dot{\theta} i$$

if $K_e \neq K_t$ $P_{in} \neq P_{out}$

So if the motor constants are not the same, our model fails since it will magically create power either as a motor or a generator.

To see that these are exactly the same number, they need to be in standard SI units of Nm/A and V-s. If you use units like oz-in/mA and mV/RPM you won't immediately see the equality (so don't use strange motor units regardless of what you see in a spec sheet).