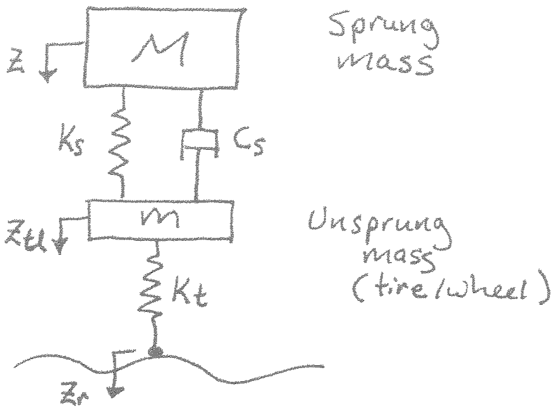
Simple quarter-car Model



$\Sigma F_{\text{sprung mass}}$:

$$\text{I.} \quad M\ddot{z} = k_s(z_u - z) + c_s(\dot{z}_u - \dot{z})$$

$\Sigma F_{\text{unsprung mass}}$:

$$\text{II.} \quad m\ddot{z}_u = k_s(z - z_u) + c_s(\dot{z} - \dot{z}_u) + k_t(z_r - z_u)$$

Input is z_r (or $\dot{z}_r$ or $\ddot{z}_r$)

Output might be: $\ddot{z}$ - sprung mass acceleration

$z - z_u$ - rattlespace deflection

$z_u - z_r$ - tire deflection

Let's look at $\ddot{z}$ using Laplace transforms

$$\text{I} \rightarrow \quad Ms^2 Z(s) = k_s\,(Z_u(s) - Z(s)) + c_s\,(s Z_u(s) - sZ(s))$$

$$[Ms^2 + c_s s + k_s]\,Z = [c_s s + k_s]\,Z_u$$

$$\text{II} \rightarrow \quad [ms^2 + c_s s + (k_s + k_t)]\,Z_u = [c_s s + k_s]\,Z + k_t Z_r$$

$$Z_u = \frac{c_s s + k_s}{ms^2 + c_s s + (k_s + k_t)}\,Z + \frac{k_t}{ms^2 + c_s s + (k_s + k_t)}\,Z_r$$

$$[Ms^2 + c_s s + k_s][ms^2 + c_s s + (k_s + k_t)]\,Z = [c_s s + k_s]^2 Z + k_t[c_s s + k_s]\,Z_r$$

$$\begin{aligned}[Mms^4 + c_s(M+m)s^3 + \{k_s m + k_s M + k_t M + c_s^2 - c_s^2\}\,s^2 \\ + \{k_s c_s + (k_s + k_t)c_s - 2c_s k_s\}\,s + k_s(k_s + k_t) - k_s^2]\,Z = \\ k_t\,[c_s s + k_s]\,Z_r\end{aligned}$$

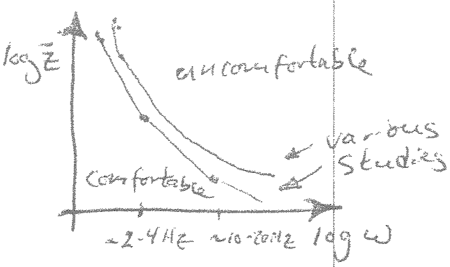
$$\Rightarrow \quad \frac{Z(s)}{Z_r(s)} = \frac{k_t c_s s + k_s k_t}{Mms^4 + c_s(M+m)s^3 + (k_s M + (k_s + k_t)M)s^2 + c_s k_t s + k_s k_t}$$

$$\frac{\ddot{Z}(s)}{\ddot{Z}_r(s)} = \frac{s^2\,Z(s)}{s^2\,Z_r(s)} = \frac{Z(s)}{Z_r(s)}$$

Human Response to Vibration

A lot of studies have examined human response to periodic motion in different directions and at different frequencies. These studies have displayed decreased tolerance at higher frequencies with regards to the amplitude of motion.

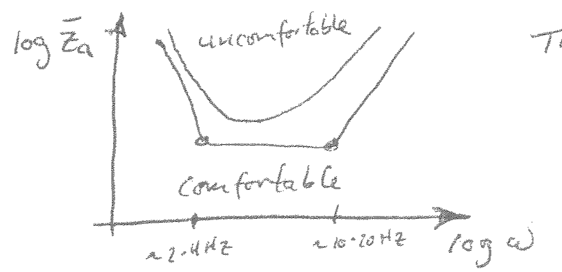
If $z = \bar{Z}\sin\omega t$, then the resulting curves have the shape indicated at left. While they show a continually negative slope there do tend to be "kinks" or inflections around 2-4 Hz and 10-20 Hz.



Viewed in terms of acceleration, the picture changes a bit. If $z = \bar{Z}\sin\omega t$

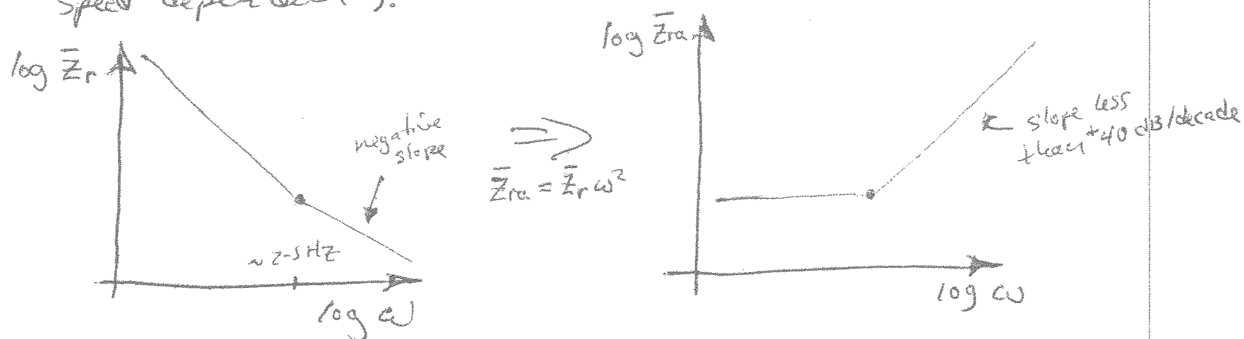
$$\Rightarrow \ddot{z} = -\omega^2 \bar{Z}\sin\omega t = -\bar{Z}_a \sin\omega t$$

Therefore, viewed in terms of acceleration amplitudes the discomfort curves become:



Thus the curves show a bowl-like shape with maximum sensitivity around 4-8 Hz (approximately). This corresponds physiologically with resonances in the body cavity, so tends to make sense.

What accelerations does a road produce? At highway speeds a plot of acceleration versus frequency can be derived from the position of the road versus frequency (which is speed dependent).

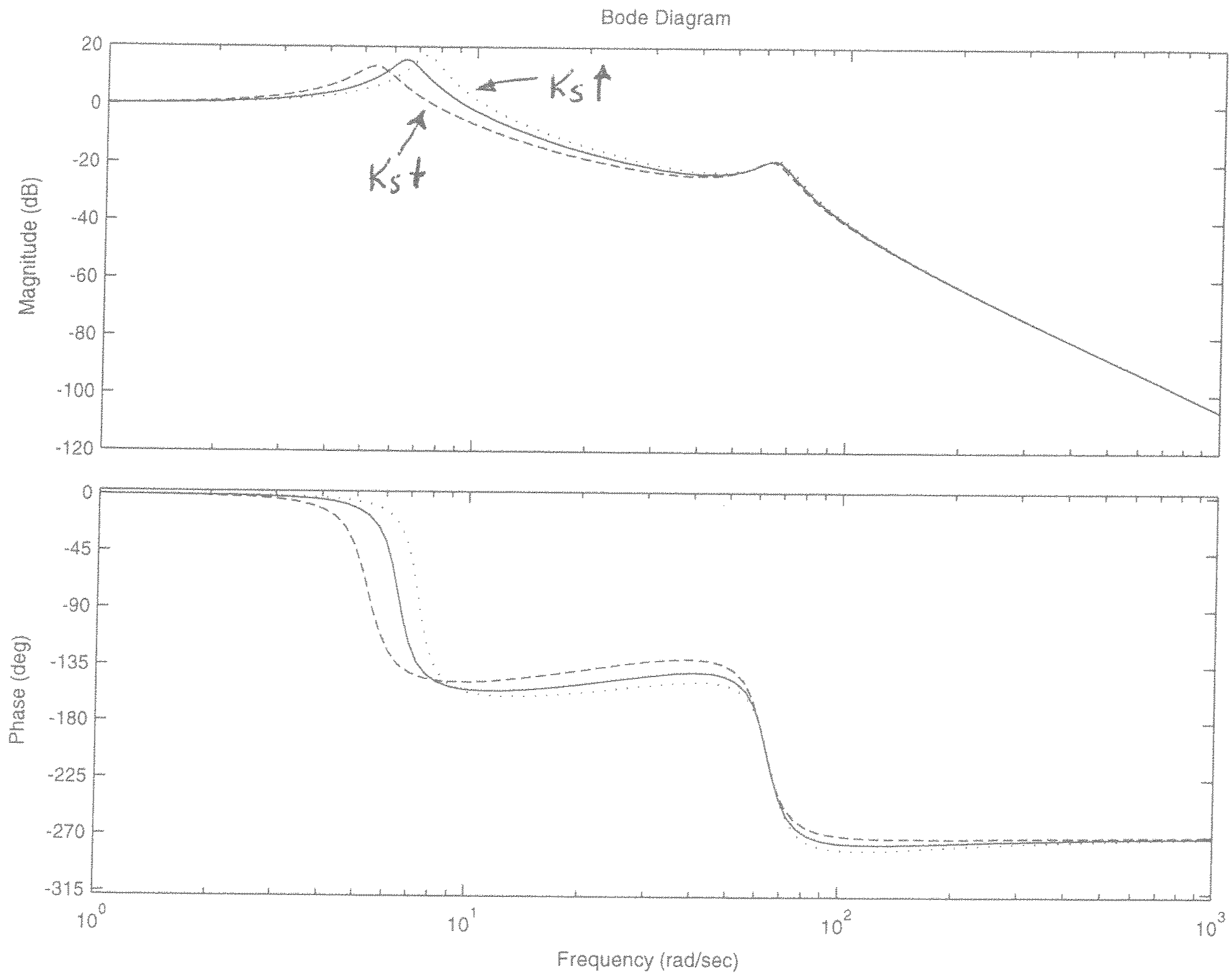


Amplitude of road height (decrease w/ frequency)

Acceleration input from road (increase w/ frequency)

This is just like adding 2 zeros at $s=0$ to transfer function, so the slope shifts by 40 dB/decade.

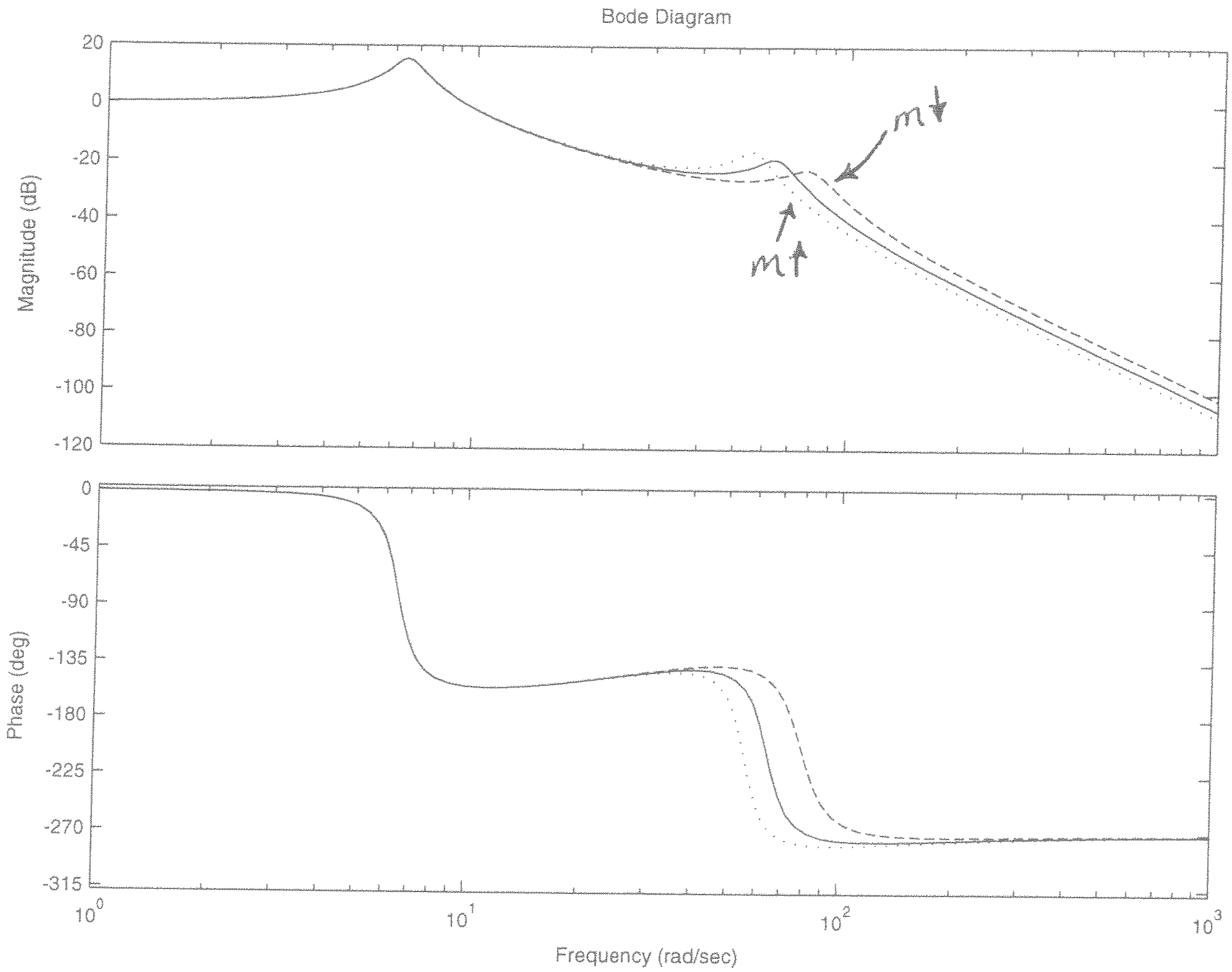
Since our two mass model rolls off at -60 dB/decade, we can handle very high frequencies. Very low frequencies can be tolerated, so the range between the sprung and unsprung mass resonances becomes the key to ride quality!

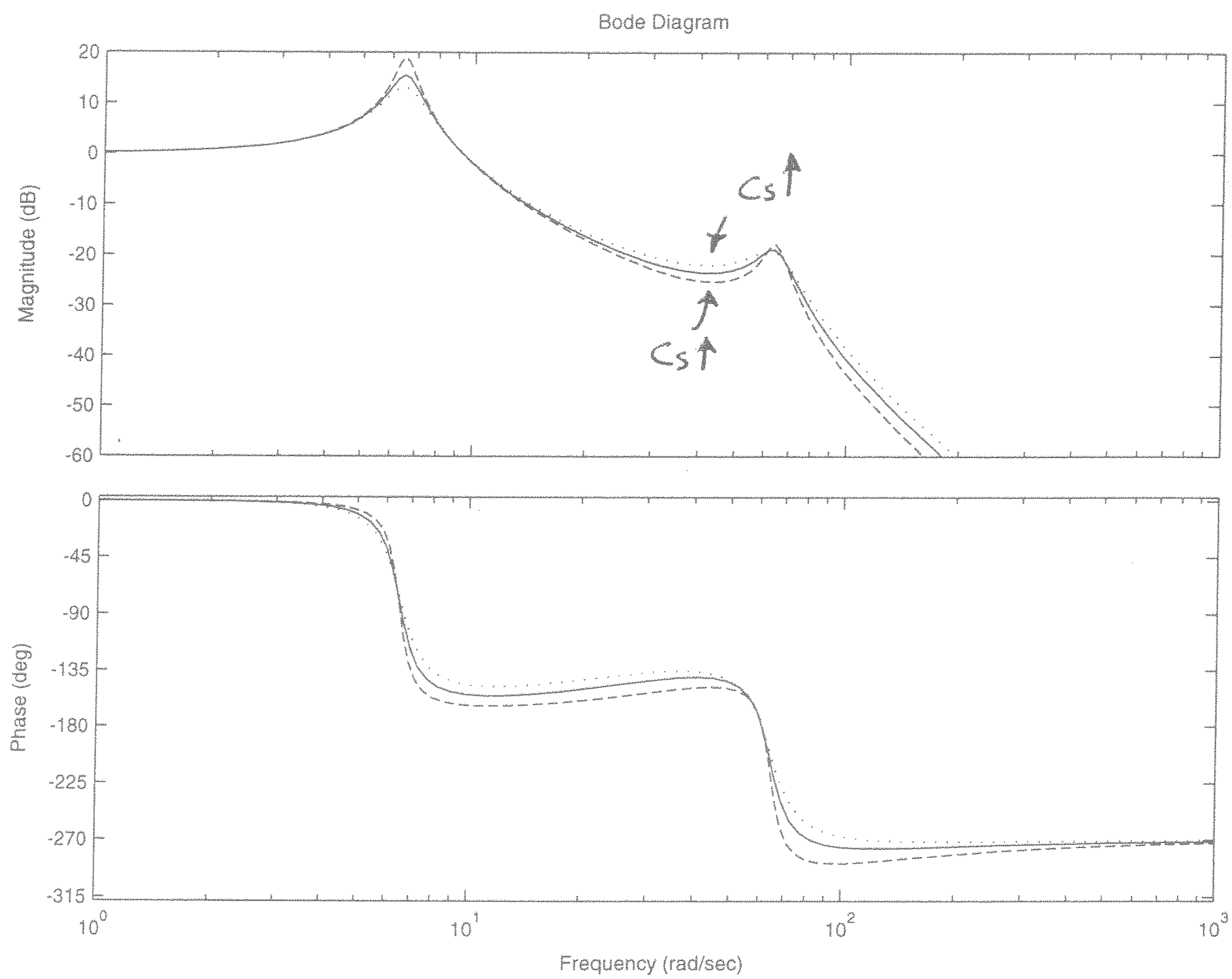


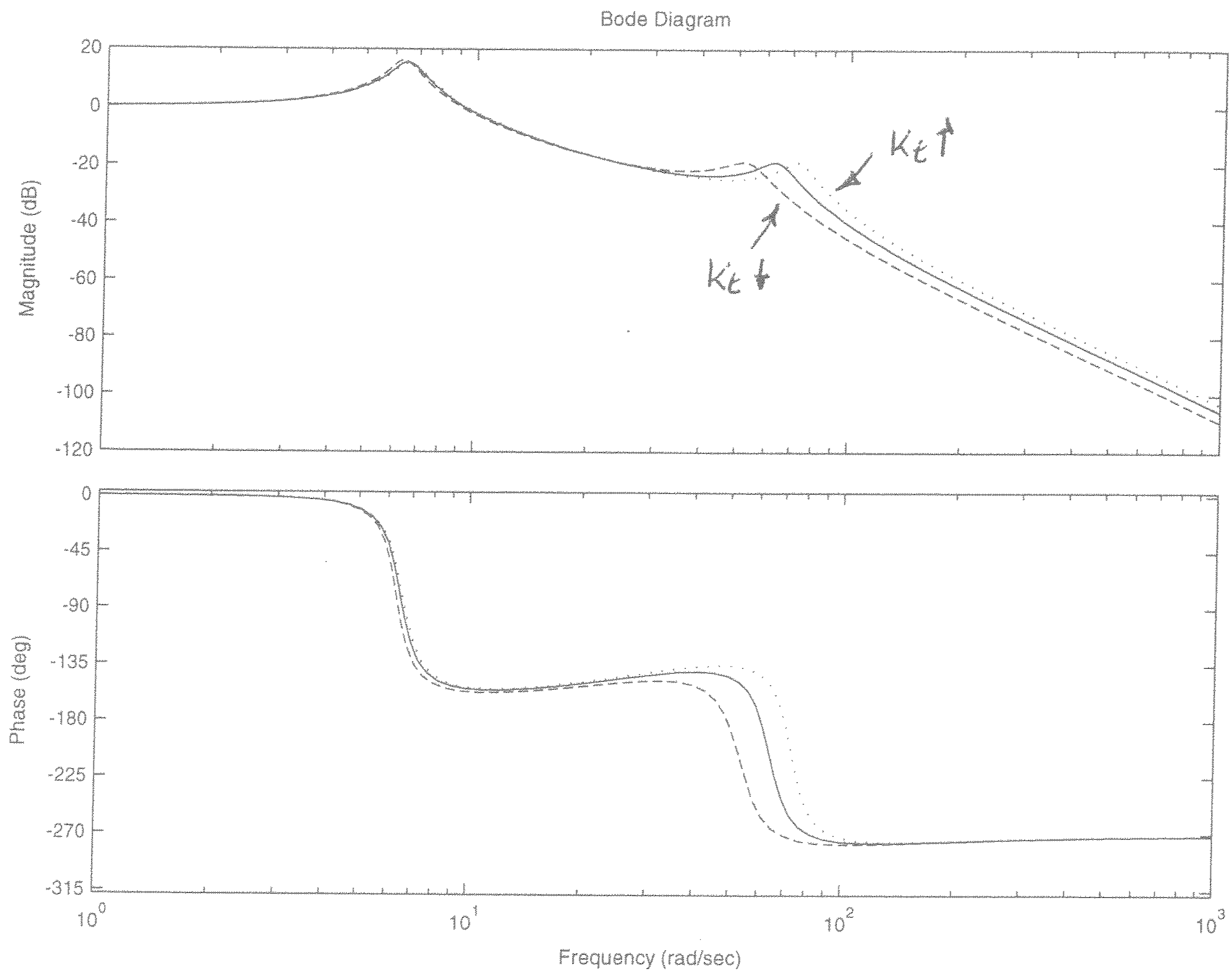
Nominal parameters for all plots:

$M = 360$ kg
$m = 40$ kg
$K_s = 17000$ N/m
$K_t = 150000$ N/m
$C_s = 500$ N/(m/s)

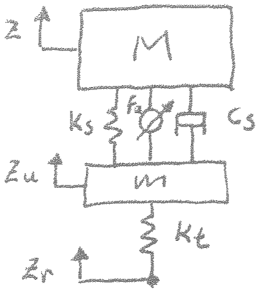
↑ denotes 30% increase in parameter
↓ " " decrease " "


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
20
0
-20
-40
-60
-80
-100
-120
0
-45
-90
-135
-180
-225
-270
-315
10^0
10^1
10^2
10^3
m↓
m↑


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
Cs↑
Cs↑
20
10
0
-10
-20
-30
-40
-50
-60
0
-45
-90
-135
-180
-225
-270
-315
10^0
10^1
10^2
10^3



Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/sec)
$K_t \uparrow$
$K_t \downarrow$



Here is an active suspension with an actuator force fa. Note that the sign convention is different from what we used before (z positive up here) to match Hedrick and Butsuen instead of Gillespie. The equations come out exactly the same however (whew!).

Let's rewrite in terms of passive forces (springs + dampers) and active force's (the actuator) between the two masses.

$$M\ddot{z} = f_p + f_a$$

$$m\ddot{z}_u = -f_p - f_a + k_t(z_r - z_u)$$

where $f_p \triangleq -k_s(z - z_u) - c_s(\dot{z}_s - \dot{z}_u)$

$f_a \triangleq$ actuator force (what we can control)

Adding these two gives:

$$M\ddot{z} + m\ddot{z}_u = k_t(z_r - z_u)$$

In other words, no matter what the spring, damper and actuator are doing, the total inertial force in the system comes from the tire and its deflection. This physical intuition is the key to deriving a frequency-based design constraint.

Taking the Laplace transform

$$M\ddot{Z}(s) + (ms^2 + k_t)Z_u(s) = k_t Z_r(s)$$

$$M\ddot{Z}(s) + (ms^2 + k_t)Z_u(s) = (k_t + ms^2)Z_r(s) - ms^2 Z_r(s)$$

$$M\ddot{Z}(s) + (k_t + ms^2)(Z_u(s) - Z_r(s)) = -ms^2 Z_r(s)$$

$$M\ddot{Z}(s) + (k_t + ms^2)(Z_u(s) - Z_r(s)) = -m\ddot{Z}_r(s)$$

$$\Rightarrow \quad M\underbrace{\left(\frac{\ddot{Z}(s)}{\ddot{Z}_r(s)}\right)}_{\substack{G_A(s) \\ \text{our acceleration} \\ \text{transfer function}}} + (k_t + ms^2)\underbrace{\left(\frac{Z_u(s) - Z_r(s)}{\ddot{Z}_r(s)}\right)}_{\substack{G_T(s) \\ \text{our tire deflection} \\ \text{transfer function}}} = -m$$

From a frequency perspective, this means that there is a relationship between the vertical acceleration felt by the driver and the tire deflection (and thus road holding). Our choice of suspension components (springs, dampers and actuation) enables us to choose how we handle this tradeoff, but does not change the basic relationship.

In the frequency domain:

$$M G_A(j\omega) + (k_t - m\omega^2) G_T(j\omega) = -m$$

What does this tell us?

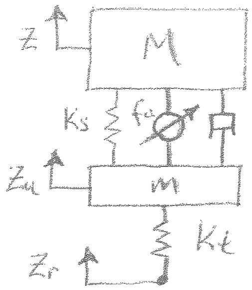
$$M G_A(j\omega) = -m \qquad \text{when} \quad k_t - m\omega^2 = 0$$

At a frequency $\omega = \sqrt{\frac{k_t}{m}}$, $G_A(j\omega) = \frac{-m}{M}$

This means that a road acceleration of x m/s^2 produces a body acceleration of $\frac{m}{M}x$ m/s^2 regardless of the suspension design when the road excitation comes at $\sqrt{\frac{k_t}{m}}$ rad/s.

Where is this? Out near the wheel hop mode which we approximated as $\omega = \sqrt{\frac{k_t + k_s}{m}}$ with $k_t >> k_s$.

Active Suspensions



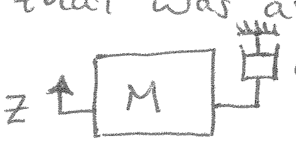
Some of the trade-offs associated with choice of springs and damping values can be eliminated by adding active control. This can be accomplished in several ways.

A <u>semi-active</u> suspension enables selection of various amounts of damping. This may be selectable in discrete steps and rely on the driver changing a setting or may vary continuously in response to what frequencies the road is currently exciting.

Semi-active suspensions can be viewed roughly as giving the driver or designer a set of passive suspensions from which to choose. Each member of the set experiences the tradeoffs we have discussed with passive suspensions.

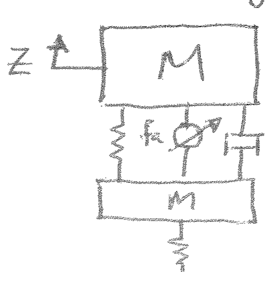
<u>Active suspensions</u> add an actuator (usually a hydraulic actuator) that can apply a force between the sprung and unsprung masses. While there are still limitations on what this actuator can influence, there are some new possibilities that open up. These particularly relate to reducing sprung mass oscillation.

Many algorithms have been proposed for active suspension control. Perhaps the simplest is the idea of <u>skyhook damping</u>. With this control algorithm, the actuator applies the same force to the sprung mass as would a damper that was attached to the sky.

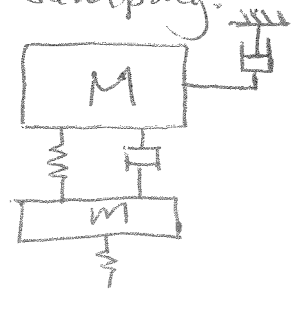


$$\text{force} = -C_{sky}\dot{z} \triangleq f_a$$

Note that since we cannot really apply a force from the sky and the force must instead be applied against the unsprung mass, this control law does not give ideal skyhook damping.



with $f_a = -C_{sky}\dot{z}$ $\neq$



With skyhook damping, the equations of motion become:

$$M\ddot{Z} = -K_s(Z - Z_u) - C_s(\dot{Z}_s - \dot{Z}_u) - C_{sky}\dot{Z}_s$$

$$m\ddot{Z}_u = -K_s(Z_u - Z) - C_s(\dot{Z}_u - \dot{Z}_s) + C_{sky}\dot{Z}_s + K_t(Z_r - Z_u)$$

Laplace transforming gives:

$$[Ms^2 + (C_s + C_{sky})s + K_s]\,Z(s) = [C_s s + K_s]\,Z_u(s)$$

$$[ms^2 + C_s s + (K_s + K_t)]\,Z_u(s) = [(C_s + C_{sky})s + K_s]\,Z(s) + K_t Z_r(s)$$

Combining and simplifying results in a transfer function

$$\frac{\ddot{Z}(s)}{\ddot{Z}_r(s)} = \frac{K_t C_s s + K_s K_t}{Mms^4 + [C_s(M+m) + \underbrace{C_{sky} m}]s^3 + [K_s m + (K_s + K_t)M]s^2 + (C_s + \underbrace{C_{sky}})K_t s + K_s}$$

2 added terms

Skyhook damping therefore changes the transfer function of the system in a manner that is different from merely changing the damping value, though the difference is subtle. From the discussion of active suspension limitations, we might expect that our active control would be more successful near the sprung mass since we have a fixed point near the unsprung mass resonance. In fact this is the case as the attached plots indicate.

Are there other choices for algorithms? Absolutely. Most of the research work in active suspensions has proposed using some form of optimal control to handle the different objectives. This can be formulated mathematically as minimizing the performance index J

$$J = \lim_{T\to\infty} \frac{1}{2T}\int_0^T \{y^T Q y + u^T R u\}\,dt$$

where u is our input (how much force we use) and y are outputs we want to keep small. These can be chosen to be things like acceleration of the sprung mass, tire or suspension deflection and - in more detailed models - suspension pitch and roll. Q and R are weighting matrices that enable the designer to make different objectives more or less important.