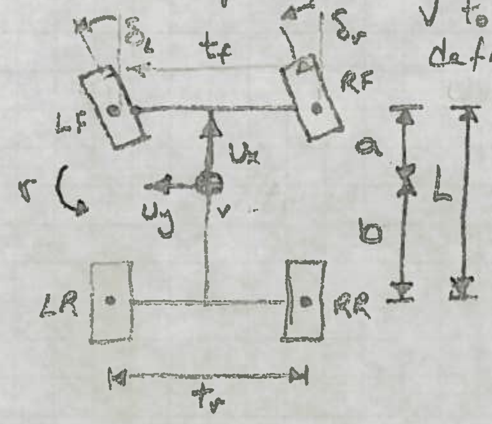
Dynamic Bicycle Model

The tire model relates slip angles to lateral tire forces. To get equations of motion, we need to calculate the slip angles at each tire, use these to generate forces and use the resulting forces with Newton's laws.

The velocities, and therefore the slip angles, at each wheel can be determined in terms of the velocity equations developed previously. We will take the reference point V to be the center of gravity and define the velocity components there as:



$$\underset{\sim}{V}_{OV,V} = \begin{bmatrix} U_x \\ U_y \\ 0 \end{bmatrix}$$

Define $\dot{\psi}_{VE} \triangleq r$ - the vehicle's <u>yaw rate</u>

The tires are located relative to the vehicle c.g. by the vectors:

$$r_{VRR,V} = \begin{bmatrix} -b \\ -t_r/2 \\ 0 \end{bmatrix} \quad r_{VLR,V} = \begin{bmatrix} -b \\ t_r/2 \\ 0 \end{bmatrix} \quad r_{VRF,V} = \begin{bmatrix} a \\ -t_f/2 \\ 0 \end{bmatrix} \quad r_{VLF,V} = \begin{bmatrix} a \\ t_f/2 \\ 0 \end{bmatrix}$$

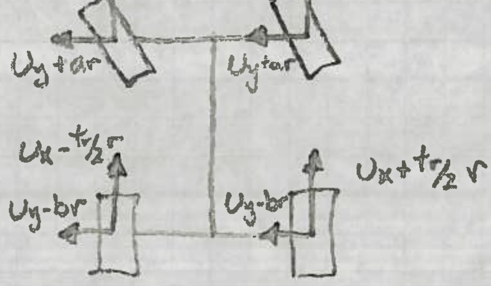
Since $\underset{\sim}{\omega}_{OV,V} = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix}$

$$\underset{\sim}{V}_{OLR,V} = \underset{\sim}{V}_{OV,V} + \underset{\sim}{\omega}_{OV,V} \times \underset{\sim}{r}_{VLR,V} = \begin{bmatrix} U_x \\ U_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \times \begin{bmatrix} -b \\ t_r/2 \\ 0 \end{bmatrix} = \begin{bmatrix} U_x - \frac{t_r}{2} r \\ U_y - br \\ 0 \end{bmatrix}$$

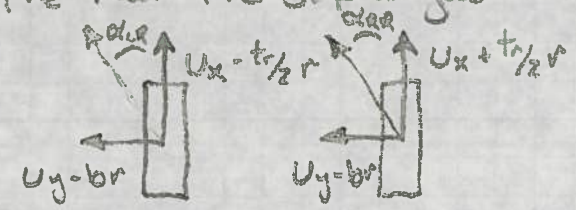
$$\underset{\sim}{V}_{ORR,V} = \underset{\sim}{V}_{OV,V} + \underset{\sim}{\omega}_{OV,V} \times \underset{\sim}{r}_{VRR,V} = \begin{bmatrix} U_x \\ U_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \times \begin{bmatrix} -b \\ -t_r/2 \\ 0 \end{bmatrix} = \begin{bmatrix} U_x + \frac{t_r}{2} r \\ U_y - br \\ 0 \end{bmatrix}$$

Similarly, $\underset{\sim}{V}_{OLF,V} = \begin{bmatrix} U_x - \frac{t_r}{2} r \\ U_y + ar \\ 0 \end{bmatrix} \quad \underset{\sim}{V}_{ORF,V} = \begin{bmatrix} U_x + \frac{t_r}{2} r \\ U_y + ar \\ 0 \end{bmatrix}$

Giving the picture



The rear tire slip angles are given by:



$$\tan \alpha_{LR} = \frac{U_y - br}{U_x - {}^{tr}/_2 r}$$

$$\tan \alpha_{RR} = \frac{U_y - br}{U_x + {}^{tr}/_2 r}$$

In general, these angles are not very different. They tend to show a greater difference as longitudinal velocity U_x decreases and yaw rate r increases.

Take a low vehicle speed of $10\,m/s$ (22.4 mph) and a lateral acceleration of $9\,m/s^2$. This is near the limit for most cars and will produce the max yaw rate

$$r = \frac{a_y}{U_x} = 0.9\ rad/s$$

(If it isn't clear why this equation holds, wait 2 pages)

Track width for a car is about 1.5m so:

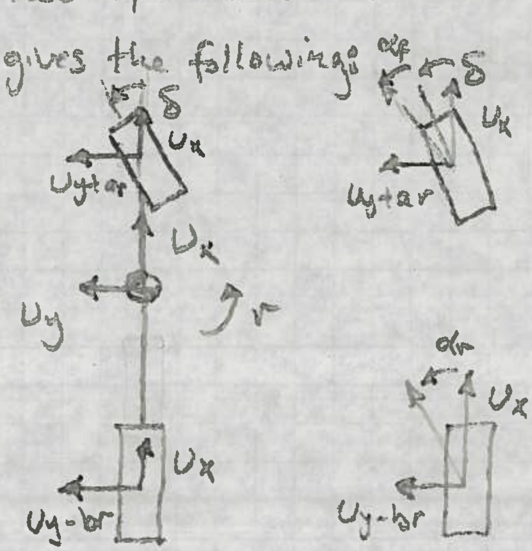
$$U_x - {}^{tr}/_2 r = 10 - \tfrac{1.5}{2}(0.9) = 9.325\,m/s$$

$$U_x + {}^{tr}/_2 r = 10 + \tfrac{1.5}{2}(0.9) = 10.68\ m/s$$

These are less than 7% different from the value of $U_x = 10\,m/s$ calculated along the centerline despite the low speed and high yaw.

Just as with the kinematic model, the simplicity and insight that comes from looking at a model with two wheels is often worth the loss of accuracy involved (and, in situations where that is not the case, we can look at two separate wheels on each axle).

This gives the following:



$$\tan(\alpha_f + \delta) = \frac{U_y + ar}{U_x}$$

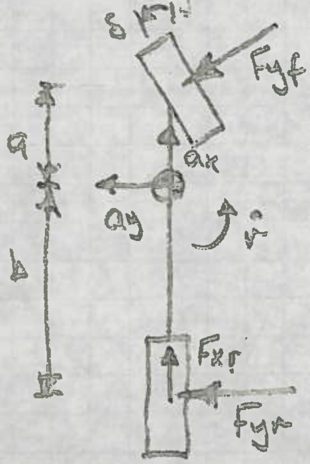
with small angles:

$$\alpha_f + \delta \approx \frac{U_y + ar}{U_x}$$

$$\tan \alpha_r = \frac{U_y - br}{U_x}$$

$$\alpha_r \approx \frac{U_y - br}{U_x}$$

The slip angles produce forces which result in linear and angular accelerations. Assume we have lateral forces F_{yf} and F_{yr} and a longitudinal force F_{xf} that we will use to keep speed constant for now.



Force balances give:

$$F_{xr} - F_{yf}\sin\delta = m a_x$$

$$F_{yf}\cos\delta + F_{yr} = m a_y$$

A moment balance gives:

$$a F_{yf}\cos\delta - b F_{yr} = I_z \dot{r}$$

moment of inertia about z-axis

We just need expressions for the vehicle's acceleration (relative to the inertial system) expressed in body-fixed coordinates.

$$\underset{\sim}{a}_{ov,v} = A_{vo}\,\underset{\sim}{a}_{ov,o} = A_{vo}\,\dot{\underset{\sim}{V}}_{ov,o} \qquad \underset{\sim}{V}_{ov,o} = A_{ov}\,\underset{\sim}{V}_{ov,v}$$

$$= A_{vo}\left[\dot{A}_{ov}\,\underset{\sim}{V}_{ov,v} + A_{ov}\,\dot{\underset{\sim}{V}}_{ov,v}\right]$$

So $\quad \underset{\sim}{a}_{ov,v} = \underset{\sim}{\omega}_{ov,v} \times \underset{\sim}{V}_{ov,v} + \dot{\underset{\sim}{V}}_{ov,v}$

$$\underset{\sim}{a}_{ov,v} = \begin{bmatrix} a_x \\ a_y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ r \end{bmatrix} \times \begin{bmatrix} U_x \\ U_y \\ 0 \end{bmatrix} + \begin{bmatrix} \dot{U}_x \\ \dot{U}_y \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{U}_x - rU_y \\ \dot{U}_y + rU_x \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad a_x = \dot{U}_x - rU_y \qquad a_y = \dot{U}_y + rU_x$$

So our equations of motion are:

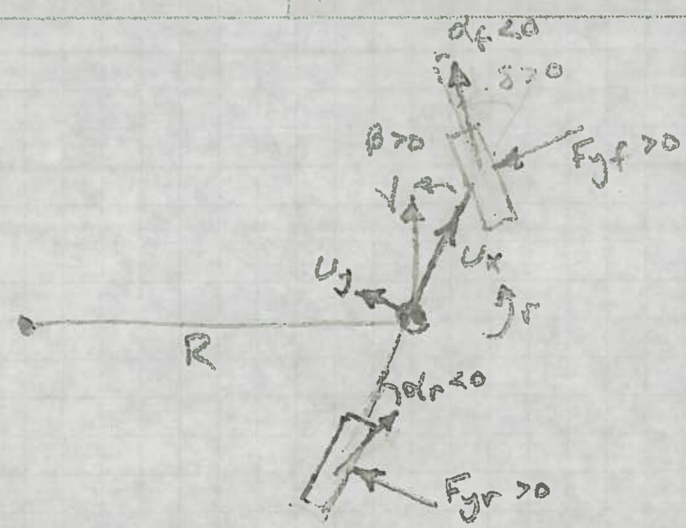
$$F_{xr} - F_{yf}\sin\delta = m\left[\dot{U}_x - rU_y\right]$$

$$F_{yf}\cos\delta + F_{yr} = m\left[\dot{U}_y + rU_x\right]$$

$$a F_{yf}\cos\delta - b F_{yr} = I_z \dot{r}$$

This describes a system with states U_x, U_y and r and inputs δ and F_{xr}. Slip angles can be found from the states, forces follow from slip angles and the the state derivatives are functions of states, inputs and forces.

Let's look at these equations in a steady corner...

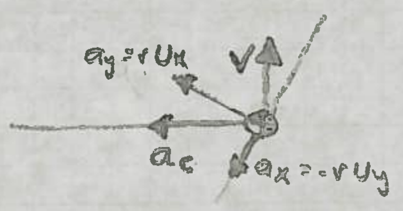


This is an example of a low radius turn at low speed. The vehicle has a positive lateral velocity meaning that it is pointed outside its turning circle.

Since the vehicle is in steady cornering, $\dot{U}_y = 0$ and $\dot{U}_x = 0$

$$a_y = r U_x \quad \text{(aka centripetal acceleration)}$$

$$a_x = -r U_y$$



$$U_y^2 + U_x^2 = V^2$$

$$a_c^2 = a_y^2 + a_x^2 = r^2 U_x^2 + r^2 U_y^2 = r^2 V^2$$

$$a_c = rV = \frac{V^2}{R} \implies \frac{r}{V} = \frac{1}{R}$$

In steady cornering, the acceleration is perpendicular to the velocity vector. Since the vehicle itself is not parallel to the velocity vector, the acceleration will have lateral and longitudinal components.

If I assume small angles and steady-state,

$$\left.\begin{aligned} F_{yf} + F_{yr} &= m a_y = m r U_x \\ a F_{yf} - b F_{yr} &= 0 \end{aligned}\right\} \text{ Force balances}$$

$$\left.\begin{aligned} U_x &= V \\ \alpha_f + \delta &= \frac{U_y + a r}{U_x} \\ \alpha_r &= \frac{U_y - b r}{U_x} \end{aligned}\right\} \text{ Velocities and geometry}$$

These combine to give some interesting results:

$$\frac{U_y}{U_x} = \alpha_r + \frac{b r}{U_x} = \alpha_f + \delta - \frac{a r}{U_x}$$

or $\delta = \alpha_r - \alpha_f + (a+b)\dfrac{r}{U_x}$ where $\dfrac{r}{U_x} = \dfrac{1}{R}$

$$\Rightarrow \quad \delta = \underset{\substack{\uparrow \\ \text{our kinematic} \\ \text{model}}}{\frac{L}{R}} + \alpha_r - \alpha_f$$

↖ difference due to slip angles

This relationship assumes steady-state cornering and small angles.

Combining the lateral force and moment balances:

$$F_{yf} + F_{yr} = m a_y \qquad a F_{yf} = b F_{yr}$$

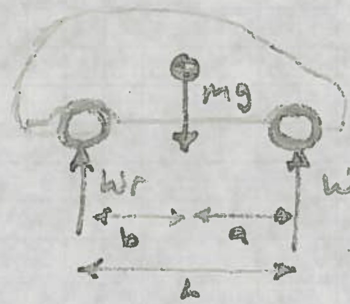
$$\Rightarrow F_{yr} + \frac{b}{a} F_{yr} = m a_y$$

$$(a+b) F_{yr} = a m a_y$$

$$\Rightarrow \quad F_{yr} = \frac{a}{L} m a_y = \frac{W_r}{g} a_y$$

$$F_{yf} = \frac{b}{L} m a_y = \frac{W_f}{g} a_y$$

The lateral forces are therefore in direct proportion to the weight carried on each axle. If 60% of the vehicle load is supported by the front axle, 60% of the total lateral force is required there in steady turning.



$$W_f = \frac{b}{L} m g$$

$$W_r = \frac{a}{L} m g$$

If we further assume linear tires

$$F_{yf} = \frac{W_f}{g} a_y = -C_{\alpha f} \alpha_f \Rightarrow \alpha_f = \frac{-W_f}{C_{\alpha f}} \frac{1}{g} a_y$$

$$F_{yr} = \frac{W_r}{g} a_y = -C_{\alpha r} \alpha_r \Rightarrow \alpha_r = \frac{-W_r}{C_{\alpha r}} \frac{1}{g} a_y$$

$$\Rightarrow \quad \delta = \frac{L}{R} - \frac{W_r}{C_{\alpha r}} \frac{1}{g} a_y + \frac{W_f}{C_{\alpha f}} \frac{1}{g} a_y$$

$$\delta = \underset{\substack{\nearrow \\ \text{Kinematic model or} \\ \text{Ackerman angle}}}{\frac{L}{R}} + \underbrace{\left[\left(\frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}} \right) \frac{1}{g} \right]}_{\text{the understeer gradient, } K} a_y$$

If a vehicle has

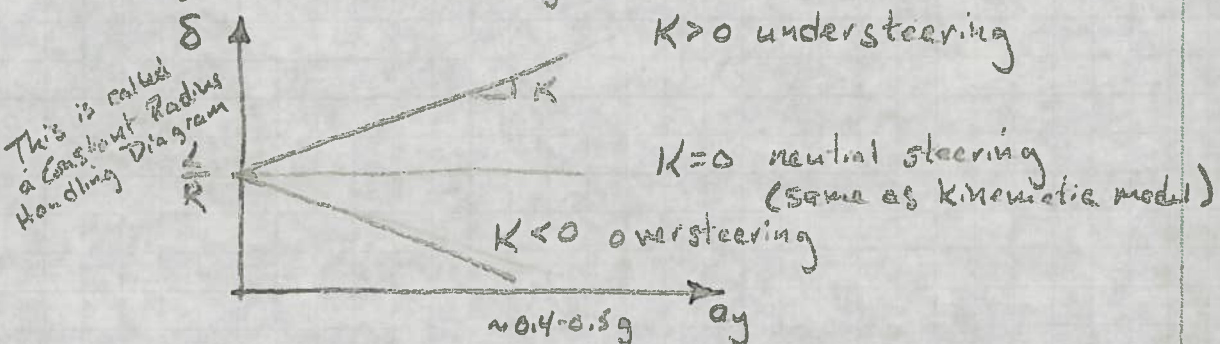
$K > 0$ it is <u>understeering</u>

$K > 0$ it is <u>oversteering</u>

$K = 0$ it is <u>neutral steering</u>

"Under" and "over" here can be viewed in terms of the Ackerman angle or kinematic model on a constant radius turn. For an understeering car to hold the radius, the driver must turn the steering wheel further into the turn as speed increases. The car is therefore "understeering" the kinematic model since steering must be added. Conversely, the driver must reduce the steer angle to track the constant radius in an oversteering vehicle.

This gives the following picture for a radius R



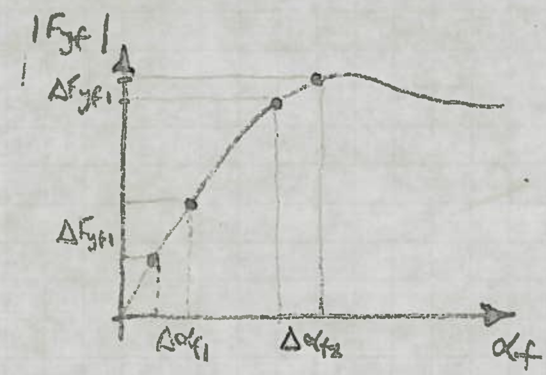
Cars often (but not always) show linear understeer or oversteer characteristics up to about 0.4-0.5g. Beyond that, tire nonlinearities begin to show up.

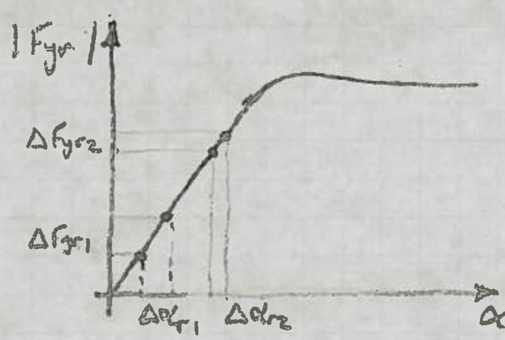
The primary influence on the understeer gradient is the weight distribution. Small front-engine cars will tend to have a large front weight bias and be understeering. Cars with a rear weight bias will tend towards oversteer. Oversteer is undesirable for reasons discussed in the next lecture so a rear weight bias will be compensated by tires (such as on a Porsche 911) or suspension. Many manufacturers such as BMW or Mercedes strive for a 50/50 front/rear weight distribution to give more neutral steering characteristics. Achieving this may require things like moving the battery to the very back of the car.

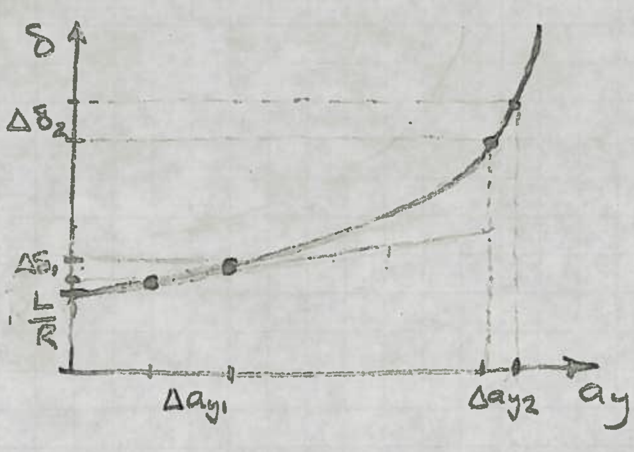
Cars with a front weight bias require more lateral force at the front axle: Since the coefficient of friction for rubber decreases at higher loads, the available friction at the front axle tends to be less than that at the rear when the front is more heavily loaded.

As a result, understeering cars will tend to also be limit understeering. In other words, they reach the friction limit on the front axle before the rear in a steady turn.

Since the tire curves tend to flatten near the peak, the change in slip angles at the front and rear axles is no longer linear as the car approaches the limits. Producing the necessary slip angles to hold the turn requires a nonlinear change in steer angle.







$$\delta = \frac{L}{R} + \alpha_r - \alpha_f$$

Left turn: $\delta > 0$

$\alpha_f, \alpha_r < 0$

60/40 weight balance

As the front axle nears the peak of the tire curve, the incremental slip angle required for an increase in lateral acceleration becomes greater in magnitude. This requires increasingly larger steering angles in order to hold the path. Eventually, the tire force reaches its peak and turning the steering further has no effect.