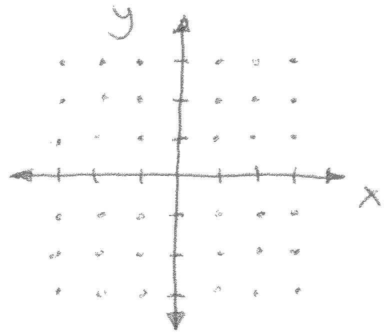
Phase Plane Analysis

One very effective method for studying the dynamics of nonlinear systems is phase plane analysis. This is especially useful for second order dynamic systems and provides a graphical method for visualizing the behavior of such systems. Around equilibrium points, the nonlinear system will resemble a linear system (not every nonlinear system can be linearized but our vehicle models can). Phase portraits show the local behavior around equilibria and how the system trajectories are attracted to or repelled by the equilibria.

As a motivating example, consider the following second-order system:

$$\dot{x} = -x$$
$$\dot{y} = -y$$



* Draw the flow field (vectors representing $\dot{x}$ and $\dot{y}$ at each point)

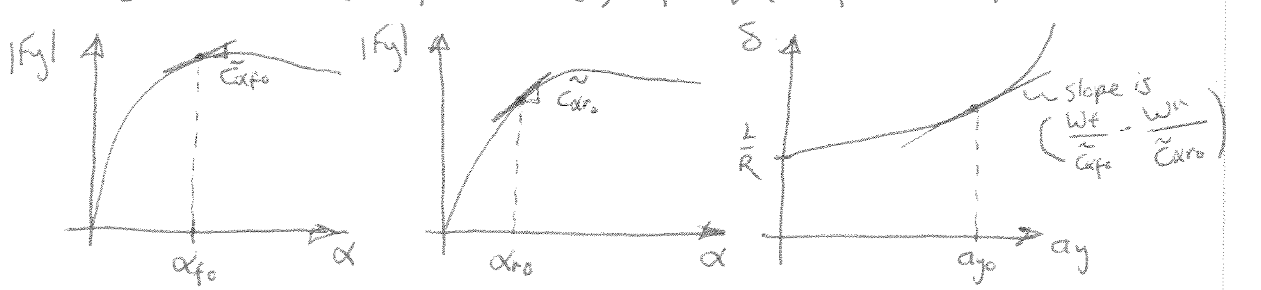
* Draw trajectories

Often visualizing differential equations in this way gives a lot of insight into the system behavior.

We are primarily interested in understanding the behavior of our two state bicycle model. This model can be linearized around different operating points, giving:

$$mV(\Delta\dot{\beta} + \Delta r) = -(\tilde{C}_{\alpha f_0} + \tilde{C}_{\alpha r_0})\Delta\beta - \frac{1}{V}(a\tilde{C}_{\alpha f_0} - b\tilde{C}_{\alpha r_0})\Delta r + \tilde{C}_{\alpha f_0}\Delta\delta$$

$$I_z\,\Delta\dot{r} = -(a\tilde{C}_{\alpha f_0} - b\tilde{C}_{\alpha r_0})\Delta\beta - \frac{1}{V}(a^2\tilde{C}_{\alpha f_0} + b^2\tilde{C}_{\alpha r_0})\Delta r + a\tilde{C}_{\alpha f_0}\Delta\delta$$



Front tire curve   Rear tire curve   Constant radius handling diagram

$$\begin{bmatrix}\Delta\dot{\beta} \\ \Delta\dot{r}\end{bmatrix} = \begin{bmatrix} \frac{-1}{mV}(\tilde{C}_{\alpha f_0} + \tilde{C}_{\alpha r_0}) & \frac{-1}{mV^2}(a\tilde{C}_{\alpha f_0} - b\tilde{C}_{\alpha r_0}) - 1 \\ \frac{-1}{I_z}(a\tilde{C}_{\alpha f_0} - b\tilde{C}_{\alpha r_0}) & \frac{-1}{I_z}(a^2\tilde{C}_{\alpha f_0} + b^2\tilde{C}_{\alpha r_0}) \end{bmatrix}\begin{bmatrix}\Delta\beta \\ \Delta r\end{bmatrix} + \begin{bmatrix}\frac{\tilde{C}_{\alpha f_0}}{mV} \\ \frac{a\tilde{C}_{\alpha f_0}}{I_z}\end{bmatrix}\Delta\delta$$

in state space form

If steering and longitudinal velocity are constant, these dynamics take the general form of constant matrices:

$$\dot{x}_1 = a x_1 + b x_2 \qquad \dot{\underset{\sim}{x}} = A \underset{\sim}{x} \qquad \underset{\sim}{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\dot{x}_2 = c x_1 + d x_2$$

To find the poles of this system, we can rearrange into a single second-order system:

$$\ddot{x}_1 = a\dot{x}_1 + b\dot{x}_2 \qquad b\dot{x}_2 = bc x_1 + bd x_2$$
$$= bc x_1 + d(\dot{x}_1 - a x_1)$$

$$\ddot{x}_1 = a\dot{x}_1 + bc x_1 + d(\dot{x}_1 - a x_1)$$

$$\Rightarrow \ddot{x}_1 - (a+d)\dot{x}_1 + (-cb + ad)x_1 = 0$$

The poles are therefore solutions to the characteristic equation:

$$s^2 - (a+d)s + (-cb + ad) = 0$$

The state space form is particularly useful when looking at analysis in the phase plane. To see this, recall that if λ is an eigenvalue of matrix A

$$A\bar{\underset{\sim}{x}} = \lambda \bar{\underset{\sim}{x}} \quad \text{for an eigenvector } \bar{\underset{\sim}{x}}$$

Eigenvalues of the matrix A are solutions to

$$\det(\lambda I - A) = 0$$

This gives $\det \begin{bmatrix} \lambda - a & -b \\ -c & \lambda - d \end{bmatrix} = (\lambda - a)(\lambda - d) - cb = 0$

$$\Rightarrow \lambda^2 - (a+d)\lambda + (-cb + ad) = 0$$

The eigenvalues are just the poles of the system!

So for each pole (eigenvalue), λ, there is an associated direction $\bar{\underset{\sim}{x}}$ which can be found by solving

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \lambda \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

These directions tell us a lot about the system since along an eigenvector:

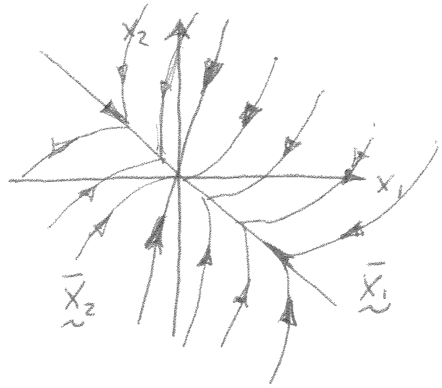
$$\dot{\underset{\sim}{x}} = A\underset{\sim}{x} = \lambda \underset{\sim}{x}$$

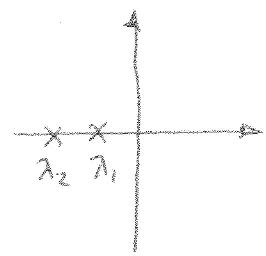
The derivative is along the vector to the equilibrium at $(0,0)$!

If we solve for the two eigenvalues and associated eigenvectors of our system, we can start to sketch the behavior around the equilibrium (globally for a linear system and locally for a linearized nonlinear system).

The system will move directly along eigenvectors either towards the equilibrium (if the eigenvalue is negative) or away from the equilibrium (if the eigenvalue is positive). Since other points in the plane represent a linear combination of eigenvectors, they have derivatives that are a combination of the motion along the two eigenvectors.

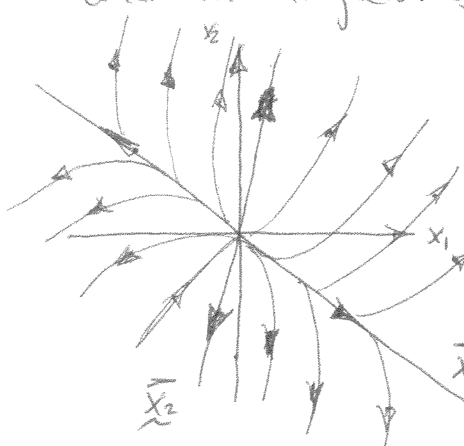
If the system has real eigenvalues, it will have real eigenvectors that define important directions in the phase plane of x_1 and x_2.

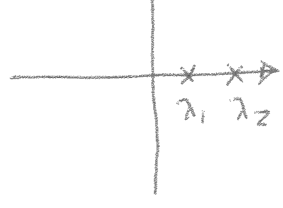




Since both eigenvalues are negative, all trajectories approach the equilibrium. This system has an equilibrium known as a <u>stable node</u>. With knowledge of the eigenvectors and associated relative magnitudes of the eigenvalues, it is easy to sketch trajectories around the equilibrium.

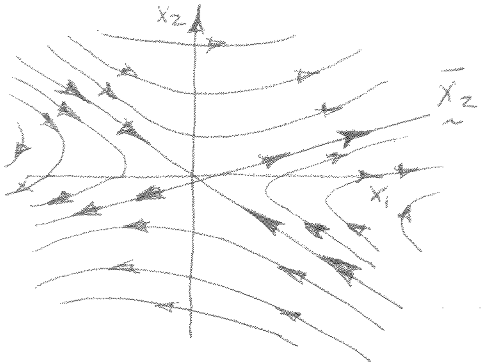
If both eigenvalues are positive, this is an <u>unstable node</u> and all trajectories lead away from the equilibrium.

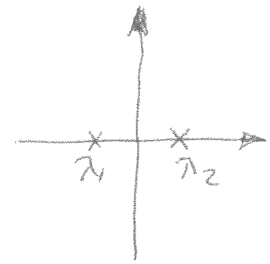




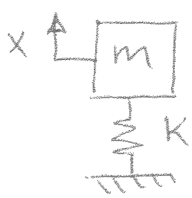
Again, sketching is simple knowing only eigenvectors and relative magnitudes of the eigenvalues.

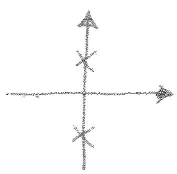
If one eigenvalue is positive and one is negative, the system is still unstable. The behavior around the equilibrium is much different than that of the unstable node, however. The combination of one stable and one unstable eigenvalue gives rise to the saddle point:

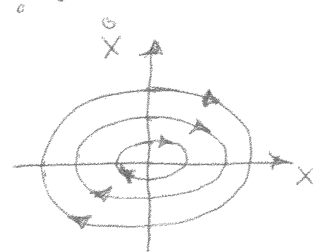




When the system has complex eigenvalues, it also has complex eigenvectors so sketching the eigenvectors in the same way is not possible. It is easy to see the behavior of the second-order system with complex eigenvalues in the phase plane, however, by considering the mass-spring damper and the states x and $\dot{x}$:

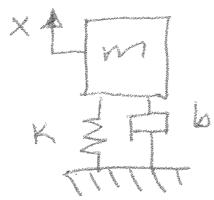


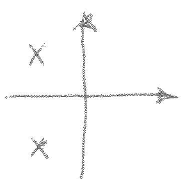


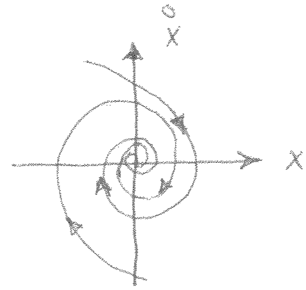


This is a center

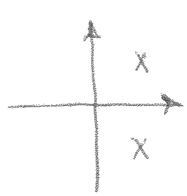
If you add damping, the system becomes a spiral point or a stable focus.

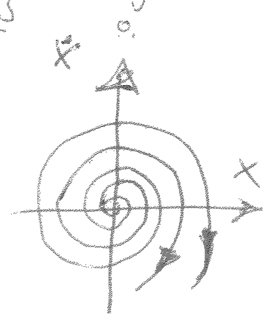






With positive real parts (negative damping or stiffness), the system is an unstable focus





There are other special cases. For instance, the motivating example at the beginning is a proper node, resulting from the repeated eigenvalue at -1:

$$\dot{\underset{\sim}{x}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \underset{\sim}{x} \qquad \lambda^2 + 2\lambda + 1 = 0$$

The two state bicycle model with constant speed will have negative real eigenvalues at low speed regardless of whether it is understeering or oversteering. Hence the vehicle will behave like a stable node.

An understeering vehicle at higher speeds will show a decrease in damping and show characteristics of a stable focus. An oversteering vehicle above the critical speed will have one stable and one unstable pole and thus look like a saddle point.

When things like longitudinal forces or weight transfer are considered, the same vehicle may have different equilibria corresponding to different conditions. A car may look like a stable focus in normal cornering but a saddle point around its drift equilibrium.