Vehicle Fixed Coordinates

To apply Newton's laws and generate equations of motion, we need to have an inertial system to use as a reference. We will use a set of axes fixed to the surface of the Earth and using East-North-Up coordinate axes for this purpose. While this isn't a true inertial system (which would have to lie at the center of the Earth), it is close enough for our purposes.

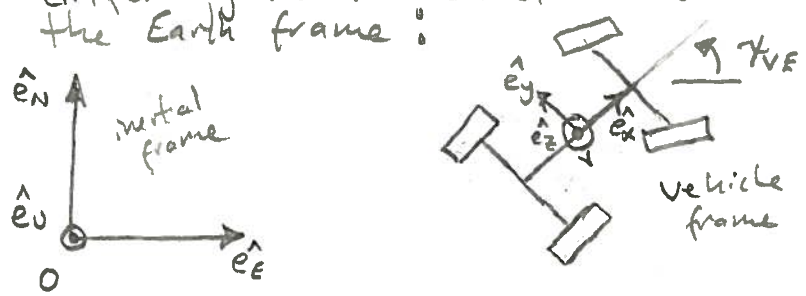
It will often be convenient to express forces, velocities and accelerations in terms of axes fixed to the vehicle instead of those fixed to the Earth. This lets us think in terms of acceleration due to braking or cornering instead of acceleration in the East or North direction. The vehicle, however, is not an inertial system and it is necessary to quickly switch back and forth as needed.

We do this by defining two sets of coordinates:

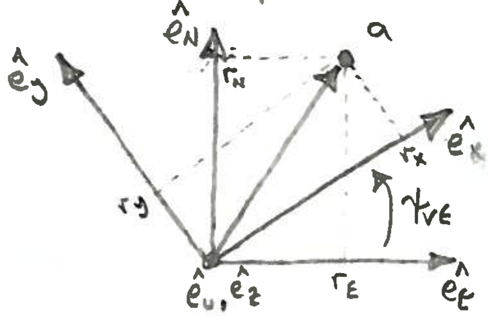
* An inertial system fixed to the earth, denoted by O, with unit vectors $\hat{e}_E$, $\hat{e}_N$ and $\hat{e}_U$

* A vehicle fixed system referenced to a point v on the vehicle with unit vectors $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$

For planar models, the vehicle's z-axis coincides with the up axis of the inertial system. The two axes differ by the rotation of the vehicle relative to the Earth frame:



Vectors, such as the position of a point a relative to the origin, can be expressed in either coordinate system:



$$\underset{\sim}{r_{oa}} = r_E \hat{e}_E + r_N \hat{e}_N + r_U \hat{e}_U$$

$$= r_x \hat{e}_x + r_y \hat{e}_y + r_z \hat{e}_z$$

where the unit vectors are related by:

$$\hat{e}_x = \cos\gamma_{VE}\, \hat{e}_E + \sin\gamma_{VE}\, \hat{e}_N$$
$$\hat{e}_y = -\sin\gamma_{VE}\, \hat{e}_E + \cos\gamma_{VE}\, \hat{e}_N$$

$$\hat{e}_E = \cos\gamma_{VE}\, \hat{e}_x - \sin\gamma_{VE}\, \hat{e}_y$$
$$\hat{e}_N = \sin\gamma_{VE}\, \hat{e}_x + \cos\gamma_{VE}\, \hat{e}_y$$

So $\underset{\sim}{r_{oa}} = r_E \hat{e}_E + r_N \hat{e}_N + r_U \hat{e}_U$

$$= r_E \cos\psi_{VE}\, \hat{e}_x - r_E \sin\psi_{VE}\, \hat{e}_y + r_N \sin\psi_{VE}\, \hat{e}_x + r_N \cos\psi_{VE}\, \hat{e}_y + r_U \hat{e}_z$$

$$= \underbrace{(r_E \cos\psi_{VE} + r_N \sin\psi_{VE})}_{r_x}\hat{e}_x + \underbrace{(-r_E \sin\psi_{VE} + r_N \cos\psi_{VE})}_{r_y}\hat{e}_y + \underbrace{r_U}_{r_z}\hat{e}_z$$

We will use an additional subscript to denote the coordinate frame in order to avoid confusion.

$\underset{\sim}{r_{oa,o}} = \begin{bmatrix} r_E \\ r_N \\ r_U \end{bmatrix}$ vector from point o to point a in inertial coords.

$\underset{\sim}{r_{oa,v}} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$ vector from point o to point a in vehicle fixed coords.

$$\underset{\sim}{r_{oa,v}} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi_{VE} & \sin\psi_{VE} & 0 \\ -\sin\psi_{VE} & \cos\psi_{VE} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_{vo}} \begin{bmatrix} r_E \\ r_N \\ r_U \end{bmatrix}$$

rotation matrix from o to v

$\Rightarrow \underset{\sim}{r_{oa,v}} = A_{vo}\, \underset{\sim}{r_{oa,o}}$

Similarly,
$$\begin{bmatrix} r_E \\ r_N \\ r_U \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi_{VE} & -\sin\psi_{VE} & 0 \\ \sin\psi_{VE} & \cos\psi_{VE} & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A_{ov}} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$$

$\Rightarrow \underset{\sim}{r_{oa,o}} = A_{ov}\, \underset{\sim}{r_{oa,v}}$

These are elementary rotation matrices. It is easy to see that:

(1) $A_{vo} = A_{ov}^T$ (so $A_{ov} = A_{vo}^T$)

(2) $A_{ov} \cdot A_{vo} = \begin{bmatrix} \cos^2\psi_{VE} + \sin^2\psi_{VE} & 0 & 0 \\ 0 & \cos^2\psi_{VE} + \sin^2\psi_{VE} & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

So $A_{ov} = A_{vo}^{-1}$

(that makes sense - if you rotate through an angle and back through the same angle, you return to the original position).

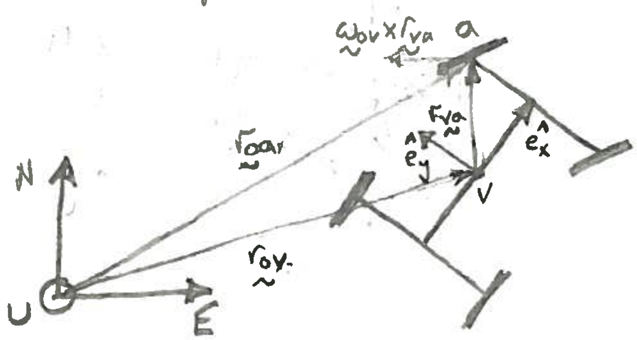
The velocity of a point is its derivative with respect to the inertial system:

$$\underset{\sim}{v}_{oa,o} = \dot{\underset{\sim}{r}}_{oa,o}$$

We can express absolute velocities with respect to the vehicle-fixed coordinates as well:

$$\underset{\sim}{v}_{oa,v} = A_{vo}\,\underset{\sim}{v}_{oa,o}$$

With the rotation matrices, we can calculate the velocity of any point a fixed to the vehicle:



$$\underset{\sim}{r}_{oa,o} = \underset{\sim}{r}_{ov,o} + \underset{\sim}{r}_{va,o}$$

$$= \underset{\sim}{r}_{ov,o} + A_{ov}\,\underset{\sim}{r}_{va,v}$$

constant vector in vehicle frame

$$\underset{\sim}{v}_{oa,o} = \dot{\underset{\sim}{r}}_{oa,o} = \dot{\underset{\sim}{r}}_{ov,o} + \dot{A}_{ov}\,\underset{\sim}{r}_{va,v} + \cancel{A_{ov}\,\dot{\underset{\sim}{r}}_{va,v}} \rightarrow 0$$

$$\Rightarrow \quad \underset{\sim}{v}_{oa,o} = \underset{\sim}{v}_{ov,o} + \dot{A}_{ov}\,\underset{\sim}{r}_{va,v}$$

velocity of point a — velocity of point v — velocity component due to rotating frame

The derivative of the rotation matrix can be written in a more familiar form:

$$\dot{A}_{ov} = \begin{bmatrix} -\sin\psi_{VE}\,\dot{\psi}_{VE} & -\cos\psi_{VE}\,\dot{\psi}_{VE} & 0 \\ \cos\psi_{VE}\,\dot{\psi}_{VE} & -\sin\psi_{VE}\,\dot{\psi}_{VE} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \underset{\sim}{r}_{va,v} = \begin{bmatrix} r_{vax} \\ r_{vay} \\ r_{vaz} \end{bmatrix}$$

$$\Rightarrow \dot{A}_{ov}\,\underset{\sim}{r}_{va,v} = \begin{bmatrix} (-r_{vax}\sin\psi_{VE} - r_{vay}\cos\psi_{VE})\dot{\psi}_{VE} \\ (r_{vax}\cos\psi_{VE} - r_{vay}\sin\psi_{VE})\dot{\psi}_{VE} \\ 0 \end{bmatrix}$$

This is just the cross-product of the angular velocity of the frame and the vector $\underset{\sim}{r}_{va}$:

$$\underset{\sim}{\omega}_{ov,o} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_{VE} \end{bmatrix} \qquad A_{ov}\,\underset{\sim}{r}_{va,v} = \begin{bmatrix} r_{vax}\cos\psi_{VE} - r_{vay}\sin\psi_{VE} \\ r_{vax}\sin\psi_{VE} + r_{vay}\cos\psi_{VE} \\ r_{vaz} \end{bmatrix}$$

$$\underset{\sim}{\omega}_{ov,o} \times A_{ov}\,\underset{\sim}{r}_{va,v} = \begin{bmatrix} -(r_{vax}\sin\psi_{VE} + r_{vay}\cos\psi_{VE})\dot{\psi}_{VE} \\ (r_{vax}\cos\psi_{VE} - r_{vay}\sin\psi_{VE})\dot{\psi}_{VE} \\ 0 \end{bmatrix}$$

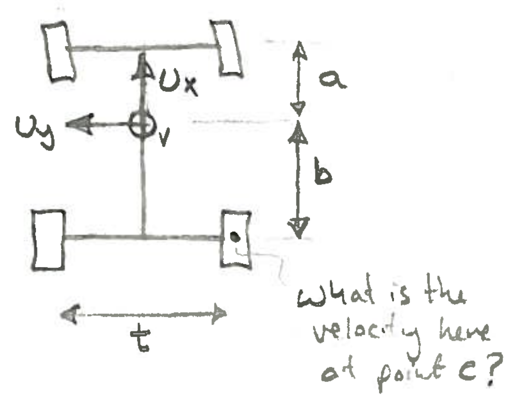
So $\mathring{A}_{ov}\, \underset{\sim}{r}_{va,v} = \underset{\sim}{\omega}_{ov,o} \times A_{ov}\, \underset{\sim}{r}_{va,v}$

$$\Rightarrow \quad \underset{\sim}{V}_{oa,o} = \underset{\sim}{V}_{ov,o} + \underset{\sim}{\omega}_{ov,o} \times A_{ov}\, \underset{\sim}{r}_{va,v}$$

Or if we want to reference this to vehicle-fixed coords

$$\underset{\sim}{V}_{oa,v} = \underset{\sim}{V}_{ov,v} + \underset{\sim}{\omega}_{ov,v} \times \underset{\sim}{r}_{va,v}$$

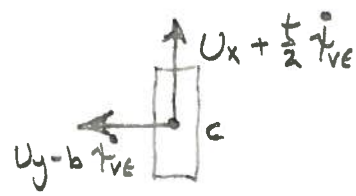
Now we can easily get the velocity at any point on the vehicle:



$$\underset{\sim}{\omega}_{ov,v} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_{VE} \end{bmatrix} \qquad \underset{\sim}{r}_{vc,v} = \begin{bmatrix} -b \\ -\frac{t}{2} \\ 0 \end{bmatrix}$$

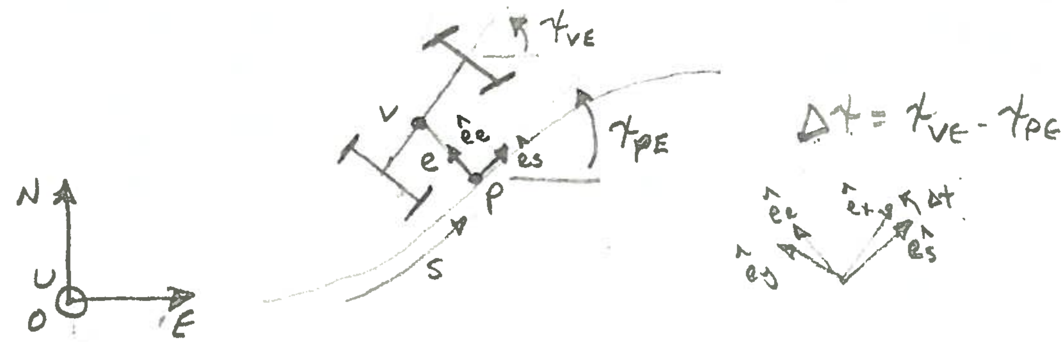
$$\underset{\sim}{V}_{ov,v} = \begin{bmatrix} U_x \\ U_y \\ 0 \end{bmatrix}$$

$$\underset{\sim}{V}_{oc,v} = \begin{bmatrix} U_x \\ U_y \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_{VE} \end{bmatrix} \times \begin{bmatrix} -b \\ -t/2 \\ 0 \end{bmatrix} = \begin{bmatrix} U_x + t/2\, \dot{\psi}_{VE} \\ U_y - b\, \dot{\psi}_{VE} \\ 0 \end{bmatrix}$$



Path Fixed Coordinates

When controlling a vehicle, its motion relative to a desired path (which could be the middle of the lane or a racing line) is often important. Describing this requires writing the equations in terms of coordinates fixed to the path.



Consider a path through space which can be described in terms of the distance along the path, s

At any distance s along the path, we know the

* path heading $\psi_{PE}(s)$
* curvature $K(s)$ where $\dot{\psi}_{PE}(s) = K(s)\,\dot{s}$
* coordinate system: $\hat{e}_s$ - tangent to the path
  $\hat{e}_e$ - normal to the path

Suppose we project our reference point on the vehicle, v, to the closest point on the path. The line between the path and v will fall on the path normal for some point p.

We can use the distance s along the path to p, the distance e along the path normal and the heading error $\Delta\psi$ to locate the vehicle relative to the path. We want to be able to describe how these variables change as the vehicle moves.

$$\underset{\sim}{r}_{ov,o} = \underset{\sim}{r}_{op,o} + \underset{\sim}{r}_{pv,o}$$

$$= \underset{\sim}{r}_{op,o} + A_{op}\,\underset{\sim}{r}_{pv,p}$$

This is no longer zero

$$\underset{\sim}{V}_{ov,o} = \underset{\sim}{V}_{op,o} + \dot{A}_{op}\,\underset{\sim}{r}_{pv,p} + A_{op}\,\dot{\underset{\sim}{r}}_{pv,p}$$

$$\underset{\sim}{V}_{op,o} = \underset{\sim}{V}_{ov,o} - \underset{\sim}{\omega}_{op,o} \times A_{op}\,\underset{\sim}{r}_{pv,p} - A_{op}\,\dot{\underset{\sim}{r}}_{pv,p}$$

In the path coordinates

$$\underset{\sim}{V}_{op,p} = A_{po}\,\underset{\sim}{V}_{ov,o} - \underset{\sim}{\omega}_{op,p} \times \underset{\sim}{r}_{pv,p} - \dot{\underset{\sim}{r}}_{pv,p}$$

Let's look at each term separately.

Since point p can only move along the path:

* $\underset{\sim}{V}_{op,p} = \begin{bmatrix} \dot{s} \\ 0 \\ 0 \end{bmatrix}$

* $A_{po}\,\underset{\sim}{V}_{ov,o} = A_{po}\,A_{ov}\,\underset{\sim}{V}_{ov,v}$ $\qquad \underset{\sim}{V}_{ov,v} \triangleq \begin{bmatrix} U_x \\ U_y \\ 0 \end{bmatrix}$

$= A_{pv}\,\underset{\sim}{V}_{ov,v}$

$= \begin{bmatrix} \cos\Delta\psi & -\sin\Delta\psi & 0 \\ \sin\Delta\psi & \cos\Delta\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} U_x \\ U_y \\ 0 \end{bmatrix}$

$$A_{po}\,\underset{\sim}{V}_{ov,o} = \begin{bmatrix} U_x\cos\Delta\psi - U_y\sin\Delta\psi \\ U_x\sin\Delta\psi + U_y\cos\Delta\psi \\ 0 \end{bmatrix}$$

* $\underset{\sim}{\omega}_{op,p} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi}_{PE} \end{bmatrix}$ $\qquad \underset{\sim}{r}_{pv,p} = \begin{bmatrix} 0 \\ e \\ 0 \end{bmatrix}$

$$\underset{\sim}{\omega}_{op,p} \times \underset{\sim}{r}_{pv,p} = \begin{bmatrix} -e\dot{\psi}_{PE} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -e\kappa\dot{s} \\ 0 \\ 0 \end{bmatrix}$$

Since the curvature defines the rate at which heading changes as we move along the path

* $\underset{\sim}{\dot{r}}_{pv,p} = \begin{bmatrix} 0 \\ \dot{e} \\ 0 \end{bmatrix}$

$$\underset{\sim}{V}_{op,p} = A_{po}\,\underset{\sim}{V}_{ov,o} - \underset{\sim}{\omega}_{op,p} \times \underset{\sim}{r}_{pv,p} - \underset{\sim}{\dot{r}}_{pv,p}$$

velocity of point p in path coords

velocity of the vehicle projected to the path

motion from change in orientation of path axes

motion of vehicle relative to point on path

$$\begin{bmatrix} \dot{s} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} U_x\cos\Delta\psi - U_y\sin\Delta\psi \\ U_x\sin\Delta\psi + U_x\cos\Delta\psi \\ 0 \end{bmatrix} - \begin{bmatrix} -e\kappa\dot{s} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \dot{e} \\ 0 \end{bmatrix}$$

These can be rearranged to give the equations we need for changes in s, e and $\Delta\psi$

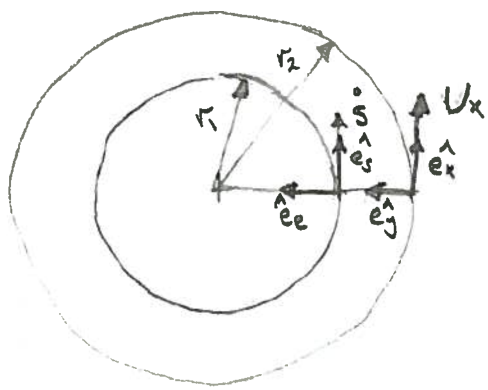
$$(1-eK_1)\dot{s} = U_x \cos\Delta\psi - U_y \sin\Delta\psi$$

$$\Rightarrow \dot{s} = \left(\frac{1}{1-eK_1}\right)(U_x \cos\Delta\psi - U_y \sin\Delta\psi)$$

$$\dot{e} = U_x \sin\Delta\psi + U_y \cos\Delta\psi$$

$$\Delta\dot{\psi} = \dot{\psi}_{VE} - \dot{\psi}_{PE} = \dot{\psi}_{VE} - K_1\dot{s}$$

A simple example that illustrates why the $\underset{\sim}{\omega}_{op,p} \times \underset{\sim}{r}_{pv,p}$ term is needed is to consider a vehicle tracking a circular path at a constant larger radius



Clearly, $\dot{s} \neq U_x$

$$\dot{s} = \frac{r_1}{r_2} U_x$$

Check: $K_1 = \frac{1}{r_1}$

$e = -(r_2 - r_1)$

$$\left(1 + \frac{r_2 - r_1}{r_1}\right)\dot{s} = U_x$$

$$\frac{r_2}{r_1}\dot{s} = U_x$$

$$\Rightarrow \dot{s} = \frac{r_1}{r_2} U_x$$

Now all we need is a vehicle model!