Solving for the Impulse Response

Writing the transfer function as a partial fraction expansion:

$$Y(s) = H(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \dots + \frac{c_n}{s-p_n}$$

The impulse response is given by

$$y(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + \dots + c_n e^{p_n t}$$

The c_i are known as residues and can be solved for using

$$c_i = Y(s)(s-p_i)\big|_{s=p_i}$$

To see this, consider that

$$Y(s)(s-p_1)\big|_{s=p_1} = c_1 + \frac{c_2 \cancel{(s-p_1)}^{\,0}}{(s-p_2)} + \dots + \frac{c_n \cancel{(s-p_1)}^{\,0}}{s-p_n}$$

if there are no repeated roots (more on that shortly)

<u>Example</u>

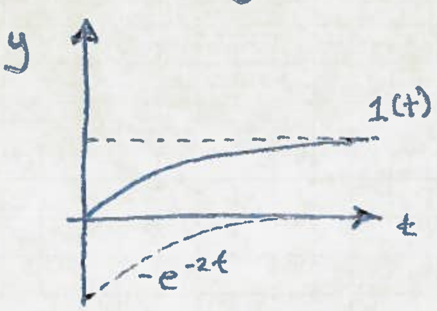
$$H(s) = \frac{2}{s(s+2)} = \frac{c_1}{s} + \frac{c_2}{s+2} = Y(s)$$

$$c_1 = Y(s)\cdot s\big|_{s=0} = \frac{2}{s+2}\Big|_{s=0} = 1$$

$$c_2 = Y(s)\cdot(s+2)\big|_{s=-2} = \frac{2}{s}\Big|_{s=-2} = -1$$

$$\Rightarrow \quad Y(s) = \frac{1}{s} - \frac{1}{s+2}$$

$$y(t) = 1(t) - e^{-2t}$$



Impulse response of the system $H(s) = \frac{2}{s(s+2)}$

What if poles are repeated?

$$Y(s) = \frac{c_1}{s-p_1} + \frac{c_2}{(s-p_1)^2} + \dots + \frac{c_n}{s-p_n}$$

$$c_2 = Y(s)(s-p_1)^2\big|_{s=p_1}$$

$$c_1 = \frac{d}{ds} Y(s)(s-p_1)^2\big|_{s=p_1}$$

$$y(t) = c_1 e^{p_1 t} + c_2 t e^{p_1 t} + \dots + c_n e^{p_n t}$$

Is this OK?

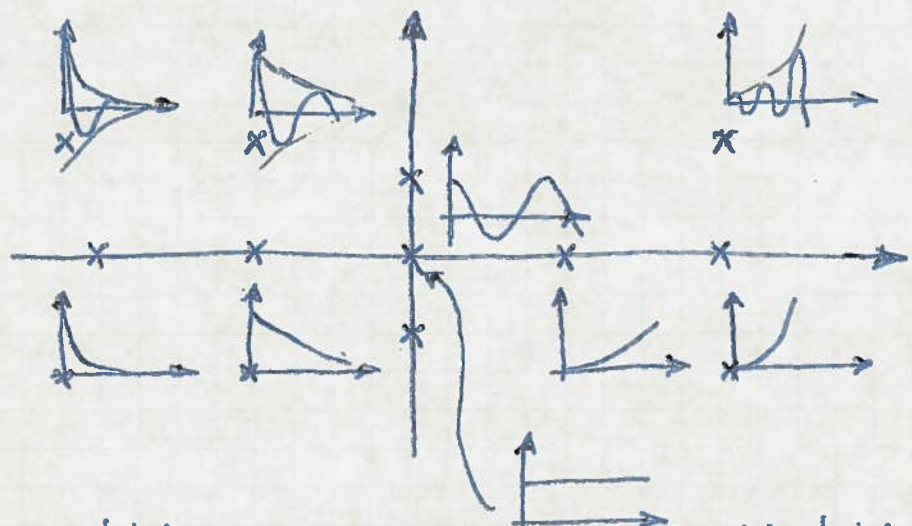
In general, for a repeated pole of multiplicity m

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+a)^m}\right\} = \frac{1}{(m-1)!}\, t^{m-1} e^{-at}$$

Since $\lim_{t \to \infty} \frac{t^{m-1}}{e^{at}} = 0 \quad \forall m$, the response dies out.
$\rightarrow$ repeated stable poles are OK

It may grow quite large before it decays however, suggesting that stability is not our only system requirement.

So if we have the transfer function, we can do a partial fraction expansion, look at the poles and tell both stability and the building blocks of the response.



Stable

Marginally Stable

Unstable

The response gets faster the farther you move from the origin on the real axis. The response gets more oscillatory the farther you move from the origin on the imaginary axis.

Response to General Inputs

The same techniques can be used to solve for the system response to any input

$$Y(s) = H(s)\,U(s) = \frac{K \prod^{m}(s-z_i)}{\prod^{n}(s-p_i)} \cdot K_u \frac{\prod^{m_u}(s-z_i)}{\prod^{n_u}(s-p_i)}$$

$$= \underbrace{\frac{a_1}{s-p_1} + \frac{a_2}{s-p_2} + \dots + \frac{a_n}{s-p_n}}_{\text{poles from system}} + \underbrace{\frac{a_{n+1}}{s-p_{n+1}} + \dots + \frac{a_{n+n_u}}{s-p_{n+n_u}}}_{\text{poles from input}}$$

We can solve for the system response in the same manner by taking the inverse Laplace transform. Keep in mind that the residues for a given input will be different from those calculated for the impulse response, so we have to solve for them again.

Example

$$H(s) = \frac{1}{s(s+2)}$$

This might be the response of a motor position to voltage if inductance is small.

$$\underset{V(s)}{\text{Voltage}} \longrightarrow \boxed{H(s)} \longrightarrow \underset{\Theta(s)}{\text{Motor position}}$$

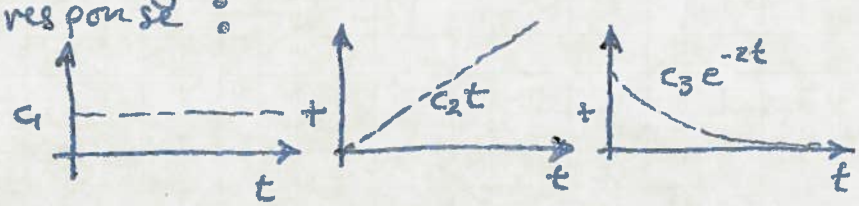
$$\Theta(s) = H(s)\,V(s)$$

What is the response to a step change in voltage? Let's use a step of 4 Volts.

$$V(s) = \frac{4}{s} \quad \Rightarrow \quad \Theta(s) = \frac{4}{s^2(s+2)}$$

$$\Theta(s) = \frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s+2}$$

We already have an idea of the qualitative response:



The response keeps growing in time (as we would expect). Mathematically this happens because the input results in a double pole at $s=0$. Multiple stable poles result in a stable system but the same is not true for the marginally stable pole at the origin.

To get the response quantitatively, we just solve for the residues:

$$c_1 = \frac{d}{ds}\Theta(s)\,s^2\Big|_{s=0} = \frac{d}{ds}\frac{4}{s+2}\Big|_{s=0} = \frac{-4}{(s+2)^2}\Big|_{s=0} = -1$$

$$c_2 = s^2\,\Theta(s)\big|_{s=0} = \frac{4}{s+2}\Big|_{s=0} = 2$$

$$c_3 = (s+2)\Theta(s)\big|_{s=-2} = \frac{4}{s^2}\Big|_{s=-2} = 1$$

$$\Rightarrow \Theta(t) = -1 + 2t + e^{-2t}$$

Poles at $s=0$ act as integrators in the system. This should make sense - multiplication by s represents differentiation and division by s represents integration.

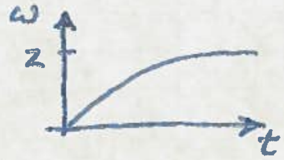
$$\mathcal{L}\left\{ \int_0^t f(\tau)\,d\tau \right\} = \frac{1}{s}F(s)$$

The integrator in this transfer function turns the step into a ramp (so position keeps increasing).

What about motor velocity $\omega = \dot{\theta}$? $\mathcal{L}\{\omega(t)\} = \Omega(s)$

$$\Omega(s) = s\,\Theta(s) = \frac{4}{s(s+2)} = \frac{2}{s} + \frac{-2}{s+2}$$

$$\Rightarrow \omega(t) = 2(1-e^{-2t})$$



If the system is stable, it is very easy to find the steady-state value that an input will produce.

From the derivative relationship in Laplace transforms:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0) = \int_0^\infty e^{-st}\frac{dy}{dt}\,dt$$

Taking the limit as $s \to 0$

$$\lim_{s\to 0}\left[sY(s) - y(0)\right] = \lim_{s\to 0}\int_0^\infty e^{-st}\frac{dy}{dt}\,dt$$

$$\lim_{s\to 0} sY(s) - y(0) = \int_0^\infty \frac{dy}{dt}\,dt$$

$$\lim_{s\to 0} sY(s) - y(0) = y(\infty) - y(0)$$

$$\Rightarrow \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

This is the Final Value Theorem.

In the example above:

$$\lim_{t\to\infty}\omega(t) = \lim_{s\to 0} s\,\Omega(s) = \lim_{s\to 0} s\,\frac{4}{s(s+2)} = 2$$

If we know the transfer function and input then finding the steady-state value (if it exists) is extremely simple.