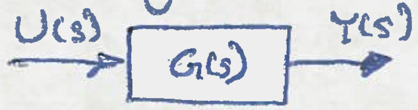
Closed-Loop System Response

From our study of Laplace transforms, it is clear that the poles of a transfer function are a key characteristic of the system. The poles tell us whether or not the system is stable and provide the basic "building blocks" that combine to give the time response.

This leads us to the key idea of E105...

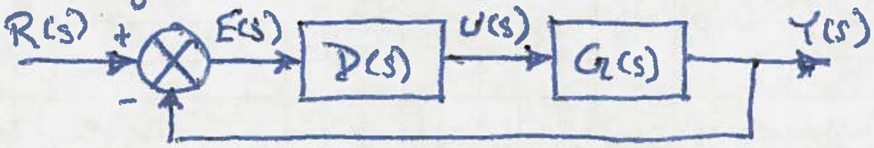
Closing a feedback loop around a system changes the location of the poles!

Let's say we start with a system $G(s) = \dfrac{B(s)}{A(s)}$



The poles satisfy $A(s) = 0$
(these are "open loop" poles)

Now let's close a feedback loop around $G(s)$ together with a controller, $D(s)$...



$$\frac{Y(s)}{R(s)} = \frac{D(s)\,G(s)}{1 + D(s)\,G(s)}$$

The poles now satisfy:

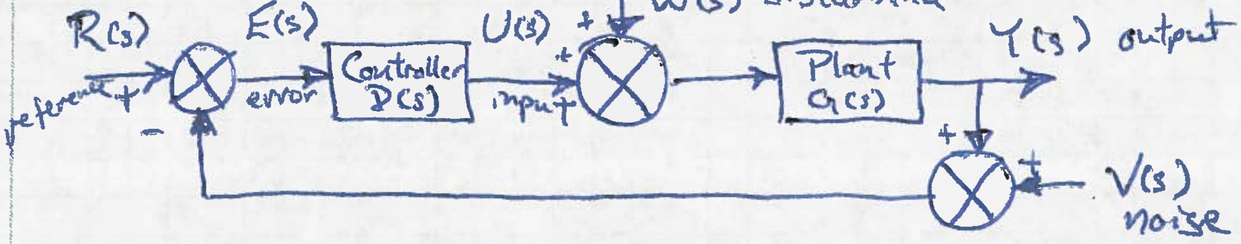
$$1 + D(s)\,G(s) = 0$$

-or-

$$A(s) + D(s)\,B(s) = 0$$

These are the "closed-loop" poles of our system with feedback (we have closed the feedback loop and changed the behavior)

By changing the poles, we can change the system's stability, speed of response and oscillation. We may also be interested in things like disturbance rejection or noise sensitivity, requiring a slightly expanded system form:



We now have three signals coming in from outside our system (the reference, a disturbance and noise).

To determine the output resulting from these signals we just work our way through the diagram:

$$Y(s) = G(s)\left[W(s) + U(s)\right]$$
$$= G(s)\left[W(s) + D(s)E(s)\right]$$
$$= G(s)\left[W(s) + D(s)\left(R(s) - Y(s) - V(s)\right)\right]$$

$$Y(s) + D(s)G(s)Y(s) = D(s)G(s)R(s) + G(s)W(s) - D(s)G(s)V(s)$$

$$\Rightarrow Y(s) = \frac{DG}{1+DG}R(s) + \frac{G}{1+DG}W(s) - \frac{DG}{1+DG}V(s)$$

Similarly, for our error and input signals we have

$$E(s) = \frac{1}{1+DG}R(s) - \frac{G}{1+DG}W(s) + \frac{DG}{1+DG}V(s)$$

$$U(s) = \frac{D}{1+DG}R(s) - \frac{DG}{1+DG}W(s) + \frac{D}{1+DG}V(s)$$

We have nine transfer functions we can use to determine the response of output, error or input to reference, disturbance or noise. All of these transfer functions share a characteristic...

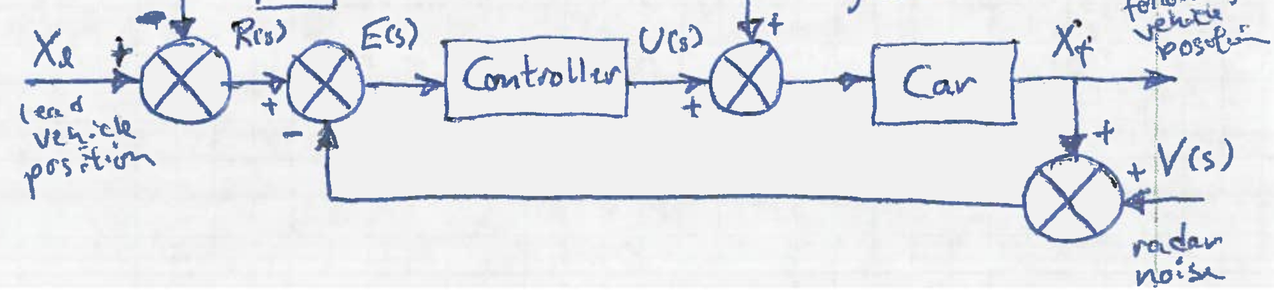
The denominator of each transfer function is $1+DG$

$\Rightarrow$ They all have the same poles!

Solutions to $1 + D(s)G(s) = 0$

So poles are a characteristic of the system. Taking our definition of stability to be having all of the poles in the left half plane, stability is the same for any of these responses.

Let's work through concepts of closed-loop system response using the example of car following:



We can model the vehicle simply as:

$$m\ddot{x} = u - b\dot{x}$$

so $$ms^2 X(s) = U(s) - bsX(s)$$

$$\Rightarrow \frac{X_f(s)}{U(s)} = \frac{1}{ms^2+bs} = G(s)$$

The open loop system poles are at $s=0$ and $s=-b/m$.
Does this make sense? What does the pole at $s=0$ imply?

Once we specify a controller, we can look at the system poles and characteristics such as stability.

Let's start with the simplest possible controller...

<u>Proportional Control</u> - $D(s) = K$ a constant

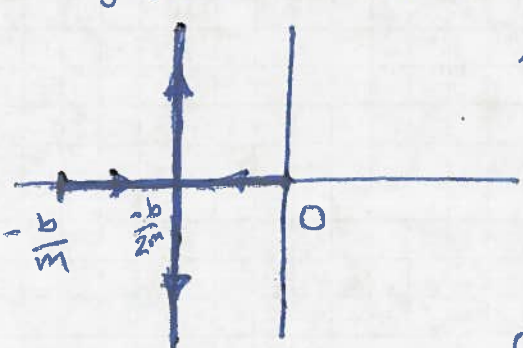
$$1 + D(s)G(s) = 1 + \frac{K}{ms^2+bs} = 0$$

$$\Rightarrow ms^2 + bs + K = 0$$

open loop poles are at $s=0$, $s=-b/m$

closed loop poles are at $s = -\frac{b}{2m} \pm \frac{\sqrt{b^2-4mK}}{2m}$

As K increases, the closed loop pole locations move away from the open loop locations.



At low values of K, the poles are real.

As K increases, the poles become complex but with a constant real part at $-b/2m$.

(What does this mean for the system response?)

So we have some ability to locate the poles of the closed-loop system. What is a good location? One approach to answering this is to look at the step response of the system.

Performance Specifications

Usually when putting performance specifications on the system, we look at the transfer function from the reference to the output:

$$\frac{Y(s)}{R(s)} = \frac{DG}{1+DG}$$

The most common input used to specify performance is the unit step. It is much easier, in practice to produce something resembling a step change in the reference than it is to produce an impulse in most cases.

For the car example, we have:

$$\frac{X_f}{R} = \frac{\frac{K}{ms^2+bs}}{1+\frac{K}{ms^2+bs}} = \frac{K}{ms^2+bs+K}$$

Does a step change in $r(t)$ make any sense? This would be a step change in the position of the lead car which is not physically possible. However,

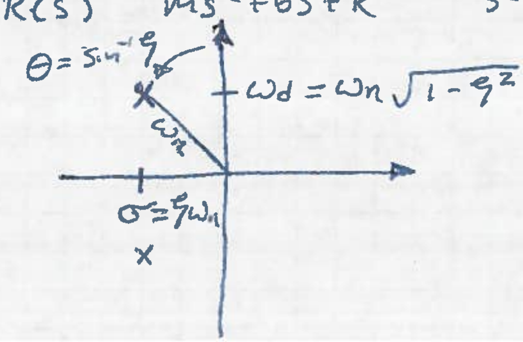
$$\frac{X_f(s)}{R(s)} = \frac{sX_f}{sR} = \frac{\dot{X}_f}{\dot{R}} = \frac{K}{ms^2+bs+K}$$

So we can think of the step response as also describing the change in speed of the following vehicle if the lead vehicle speed suddenly changes.

This transfer function has a particular form that makes it easy to specify the step response. More complicated systems with a dominant set of closed-loop poles (a single pair close to the imaginary axis) will often resemble this response even if they don't fit it exactly.

We can consider the response in terms of a natural frequency, ω_n, and a damping ratio, ζ.

$$\frac{X_f(s)}{R(s)} = \frac{K}{ms^2+bs+K} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$



$$\omega_n^2 = K/m$$

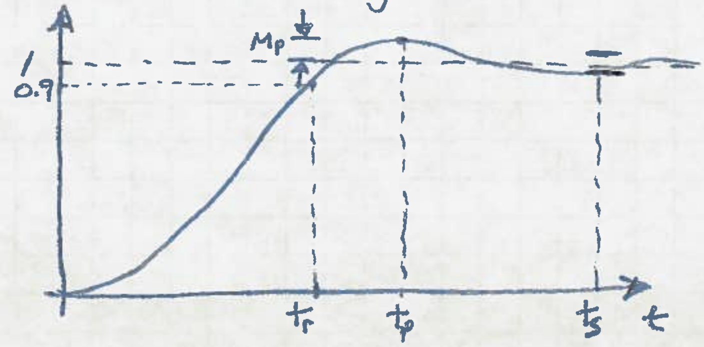
$$\zeta = \frac{b}{2\sqrt{mK}}$$

$$\sigma = b/2m$$

Step response: $\quad Y(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \cdot \dfrac{1}{s}$

$$\Rightarrow y(t) = 1 - e^{-\sigma t}\left[\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right]$$

This looks something like:



See Section 3.3 for more examples with different damping ratios.

We can specify a number of things:

<u>Rise time</u> $\quad t_r \approx \dfrac{1.8}{\omega_n}$ $\quad$ (This is based on observation)

At the peak, $\quad \dot{y}(t) = \sigma e^{-\sigma t}\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)$

$$\qquad - e^{-\sigma t}\left(-\omega_d \sin\omega_d t + \sigma\cos\omega_d t\right)$$

$$= e^{-\sigma t}\left(\frac{\sigma^2}{\omega_d}\sin\omega_d t + \omega_d \sin\omega_d t\right) = 0$$

$$\Rightarrow \sin\omega_d t = 0 \text{ at the peak}$$

<u>Peak time</u> $\quad t_p = \dfrac{\pi}{\omega_d}$ $\quad$ (This is exact)

$$y(t_p) = 1 - e^{-\sigma\pi/\omega_d}\left(\underbrace{\cos\pi}_{-1} + \frac{\sigma}{\omega_d}\underbrace{\sin\pi}_{0}\right)$$

$$= 1 + e^{-\zeta\omega_n\pi/\omega_n\sqrt{1-\zeta^2}}$$

$$= 1 + \underbrace{e^{-\pi\zeta/\sqrt{1-\zeta^2}}}_{M_p}$$

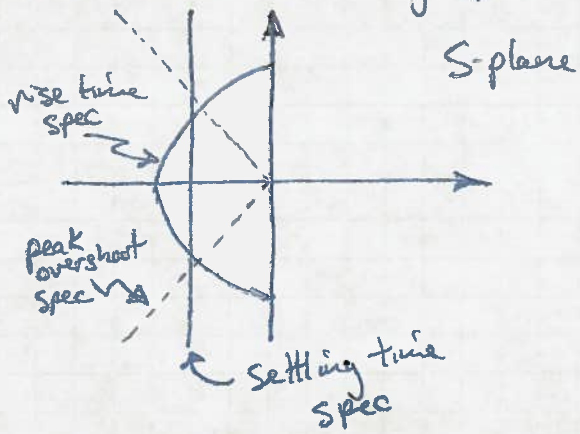
<u>Peak overshoot</u> $\quad M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}}$ $\qquad$ If $\zeta = 0.5$, $M_p = 0.16$
$\zeta = 0.7$, $M_p = 0.05$

$$y(t) = 1 - \underbrace{e^{-\sigma t}}_{\text{decay}}\underbrace{\left(\cos\omega_d t + \frac{\sigma}{\omega_d}\sin\omega_d t\right)}_{\text{oscillate}}$$

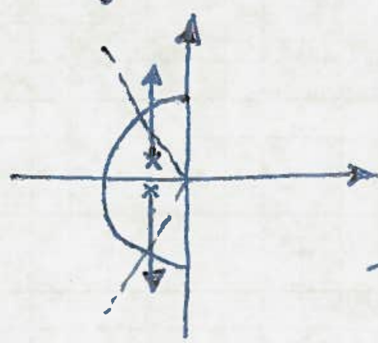
When $e^{-\sigma t} = 0.01$, the response must be within 1% of the final value of 1 $\Rightarrow \sigma t_s = 4.6$

<u>Settling time (1%)</u> $\quad t_s = \dfrac{4.6}{\sigma}$ $\quad$ (other settling times can be defined similarly)

These have nice graphical representations in the s-plane



What can we do by changing the proportional gain in the car following example?



We can make a damping spec with a long rise time

-or-

We can make a rise time spec with low damping

The gain does not effect the settling time.