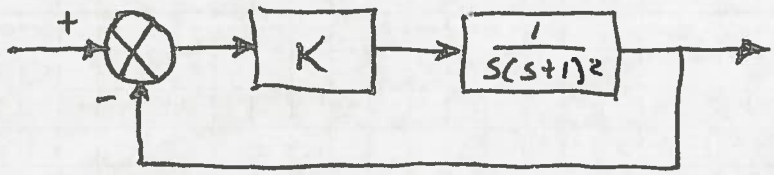
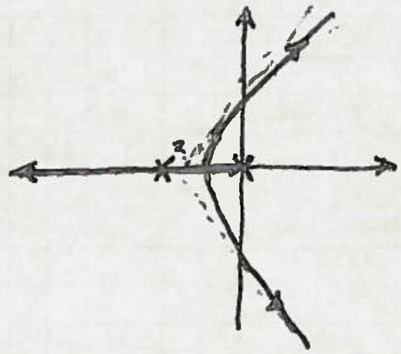
Neutral Stability

Consider the following system:



We can sketch the Root Locus and know that increasing the gain will cause instability...



(This happens at a gain $K=2$ in this system)

We know that roots of the characteristic equation satisfy

$$1+KG(s)=0$$

or, equivalently, that $|KG(s)|=1$ and $\angle G(s)=180°$

When the roots cross over the imaginary axis, we know we have a purely imaginary root. Call this value of $s=j\omega$.

Then we have $|KG(j\omega)|=1$ and $\angle G(j\omega)=180°$.

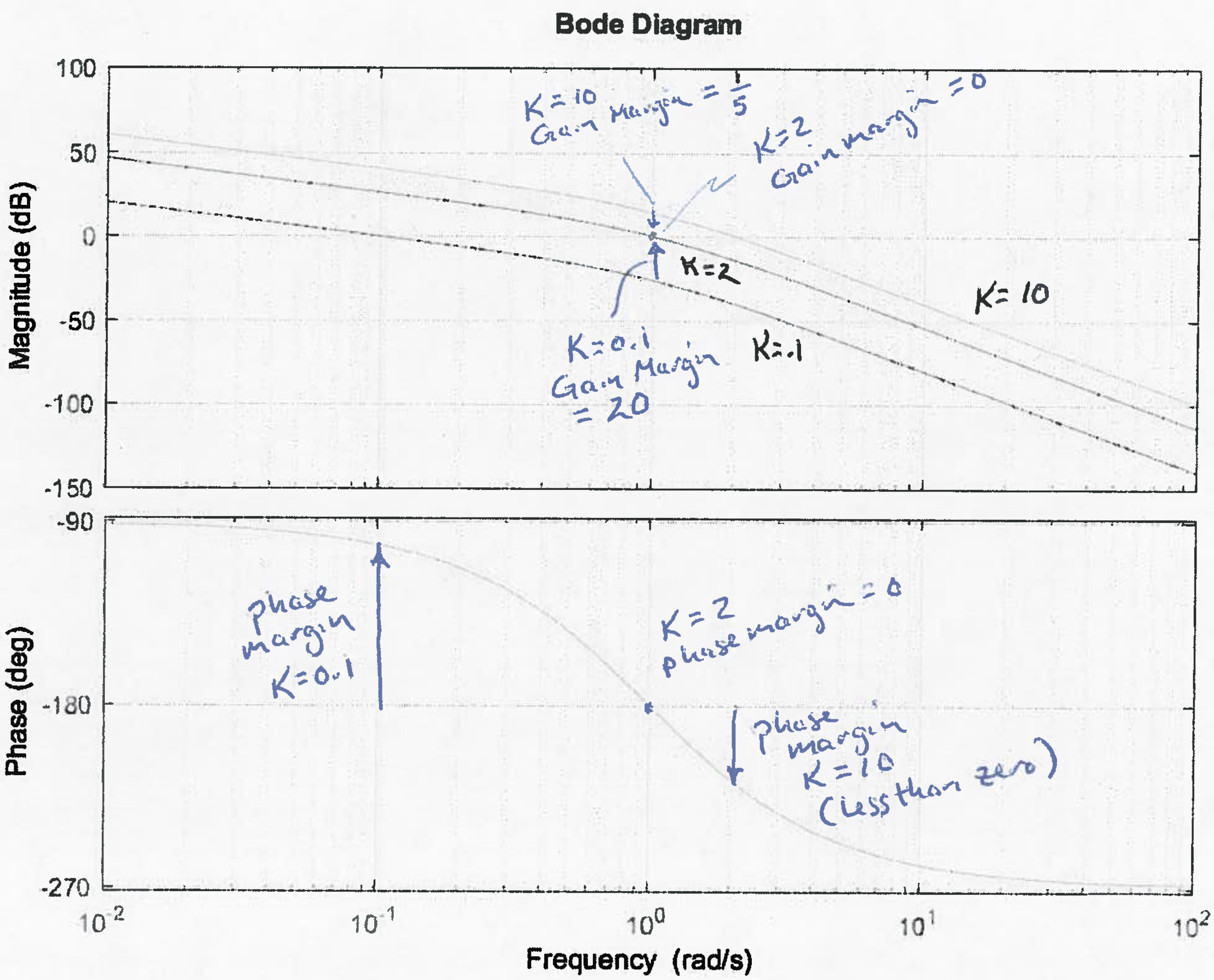
We can see these conditions for neutral stability directly from the Bode Plot. It tells us how much we can increase the gain before neutral stability.

We can define a

Gain margin - How much we can further increase the gain before hitting neutral stability (i.e. making $|KG(j\omega)|=1$ when $\angle G(j\omega)=180°$)

Phase margin - How many degrees we are from neutral stability (i.e. difference in phase from 180° when $|KG(j\omega)|=1$

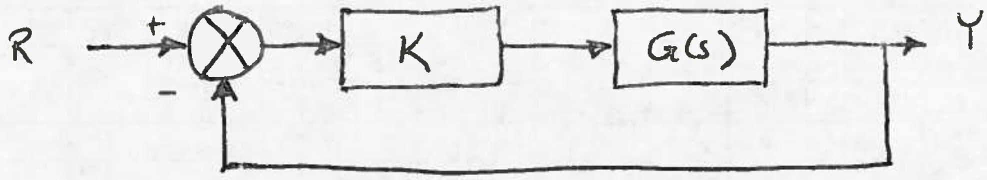
These are very useful design concepts for the simple case that the system crosses 180° at exactly one frequency. Other cases require more complexity (and Nyquist stability analysis)



Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
K=10 Gain Margin = 1/5
K=2 Gain margin = 0
K=2
K= 10
K=0.1 Gain Margin = 20
K=.1
phase margin K=0.1
K=2 phase margin = 0
phase margin K=10 (less than zero)

Frequency Domain Compensator Design

Considering a system in a feedback form:



We can see from the root locus that the system has poles on the imaginary axis when

$$|KG(j\omega)| = 1 \quad \text{and} \quad \angle G(j\omega) = 180° \text{ or } -180°$$

We can also understand the instability by looking at the closed loop transfer function near the crossover frequency of the open loop system. ω_c is the frequency corresponding to a magnitude of 1 on the Bode plot:

$$|KG(j\omega_c)| = 1 \quad \text{at crossover frequency } \omega_c$$

Looking at the closed-loop transfer function from R to Y, which we will call $T(s)$,

$$T(j\omega) = \frac{KG(j\omega)}{1 + KG(j\omega)}$$

When $|KG(j\omega)| = 1$ and $\angle G(j\omega) = -180°$, the denominator is not defined. This makes sense since any small input will set up an oscillation at this neutral stability condition.

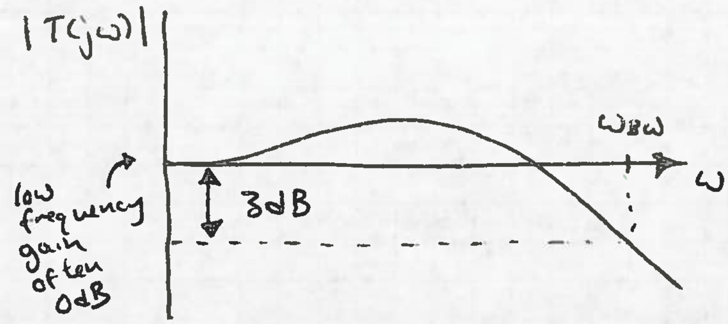
Most systems have the following behavior across their frequency range:

$|K(j\omega)| >> 1$ at low frequencies since we generally want $|T(j\omega)| \approx 1$ for good reference tracking

$|K(j\omega)| << 1$ at high frequencies since we generally have more poles than zeros. This simply reflects the fact that physical systems have limits to how fast they can respond.

Because of this, $|T(j\omega)|$ drops at higher frequencies. We call the <u>bandwidth</u> of the system the highest frequency it can reproduce sufficiently accurately.

Commonly, the bandwidth is defined to be the frequency at which the output of the system is down 3dB from its low frequency behavior.



-3dB is 0.707 so this is the point where the output sinusoid is 70% of the reference sinusoid's amplitude.

If performance is more critical for a particular system, the bandwidth can be defined as something other than a 3dB drop. If no other information is given, 3dB is generally being assumed. Control engineers also use the term "bandwith" more qualitatively when speaking about the frequency range of a system.

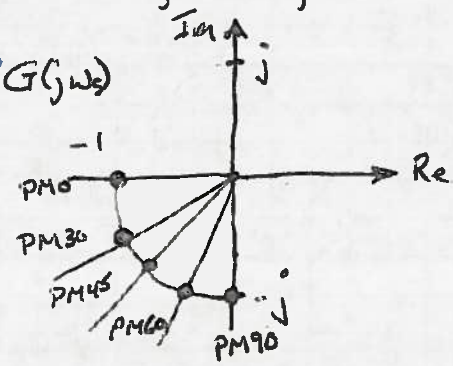
Bandwidth of a closed-loop system generally lies near the crossover frequency of the open loop system.

Usually, $\omega_c \leq \omega_{BW} \leq 2\omega_c$

Bandwith is to the frequency domain what specifications like rise time and peak time are to the time domain. A higher bandwidth means the system responds well to faster reference commands

Just as the crossover frequency parallels response time, the phase margin parallels damping. To see this, we can look at the behavior of the closed-loop system at the crossover frequency and notice that it exhibits peaking. The magnitude of $KG(j\omega_c)$ is always one (that is the definition of crossover) so the only thing that changes is the phase.

$KG(j\omega_c)$



$KG(j\omega_c)$ is just a complex number

A phase margin of 0° corresponds to a phase of 180°

A phase margin of 90° corresponds to a phase of -90°.

$|KG(j\omega_c)|$ is always 1

So the magnitude of the closed-loop system at the crossover frequency depends only upon the phase margin (since the magnitude is, by definition, 1).

We can therefore calculate the closed-loop transfer function's magnitude and phase at crossover for different values of the phase margin. Doing this, we can clearly see that phase margin is related to damping or resonance on the closed-loop Bode plot.

These are calculated by hand here as a reminder that these are simply complex numbers and the closed loop magnitude and phase follow directly from the open loop magnitude and phase at a given frequency.

PM 90° (phase = -90°)

$$KG(j\omega_c) = -j$$

$$T(j\omega_c) = \frac{-j}{1-j} = \frac{-j}{1-j}\cdot\frac{1+j}{1+j} = \frac{-j+1}{2} = \frac{1}{2} - \frac{1}{2}j$$

$|T(j\omega_c)| = \frac{\sqrt{2}}{2}$ or 0.707 (so amplitude is reduced) $\angle T(j\omega_c) = -45°$

PM 60° (phase = -120°)

$$KG(j\omega_c) = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$$

$$T(j\omega_c) = \frac{-\frac{1}{2}-\frac{\sqrt{3}}{2}j}{1-\frac{1}{2}-\frac{\sqrt{3}}{2}j} = \frac{-\frac{1}{2}-\frac{\sqrt{3}}{2}j}{\frac{1}{2}-\frac{\sqrt{3}}{2}j}\cdot\frac{\frac{1}{2}+\frac{\sqrt{3}}{2}j}{\frac{1}{2}+\frac{\sqrt{3}}{2}j} = \frac{-\frac{1}{4}-\frac{\sqrt{3}}{2}j+\frac{3}{4}}{\frac{1}{4}+\frac{3}{4}} = \frac{1}{2}-\frac{\sqrt{3}}{2}j$$

$|T(j\omega_c)| = 1$ $\angle T(j\omega_c) = -60°$

I reproduce the output 1:1 relative to the reference but with a -60° degree phase shift.

PM 45° (phase = -135°)

$$KG(j\omega_c) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}j$$

$$T(j\omega_c) = \frac{1}{2} - \frac{\sqrt{2}}{2(2-\sqrt{2})}j$$

$|T(j\omega_c)| = \underline{1.30}$ $\angle T(j\omega_c) = -67.5°$

Resonance as the closed loop system goes above 1 at ω_c

PM 30° (phase = −150°)

$$KG(j\omega_c) = -\frac{\sqrt{3}}{2} - \frac{1}{2}j$$

$$T(j\omega_c) = \frac{1}{2} - \frac{1}{2(2-\sqrt{3})}j$$

$$|T(j\omega_c)| = \underline{\underline{1.93}} \qquad \angle T(j\omega_c) = -75°$$

↑ Resonance amplitude increasing

As phase margin decreases further, the resonance peak increases further...

$$PM\,10° \Rightarrow |T(j\omega_c)| = 5.74 \quad \angle T(j\omega_c) = -85°$$
$$PM\,5° \Rightarrow |T(j\omega_c)| = 11.5 \quad \angle T(j\omega_c) = -87.5°$$

So this leads to thinking in the frequency domain about closed-loop bandwidth (or crossover frequency) and phase margin as analogous to our system response time and damping.

Lead Compensation

In the frequency domain, we often think of performance in terms of the bandwidth of the closed-loop system. This is closely related to the crossover frequency of the open loop system. Generally:

$$\omega_c \leq \omega_{BW} \leq 2\omega_c$$

We also want some amount of damping in the system which we can think of in terms of phase margin since

$$\zeta \approx \frac{PM}{100}$$

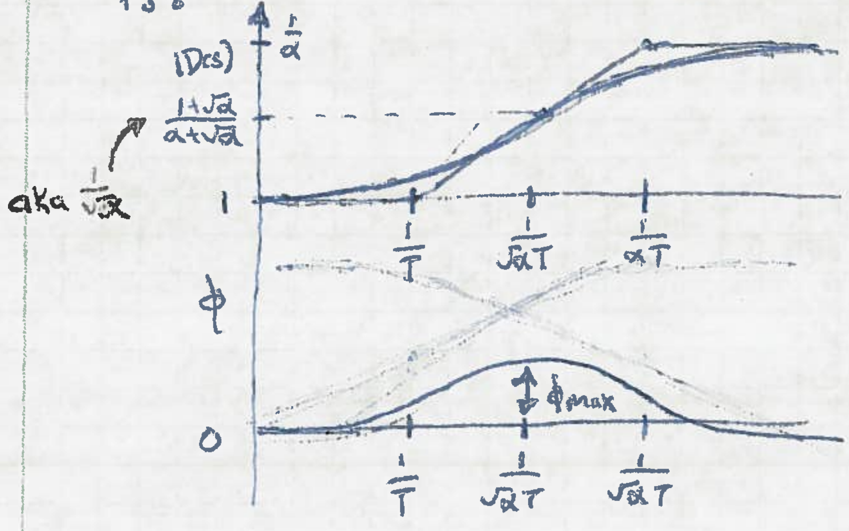
So instead of tuning natural frequency and damping ratio to get a desired time domain response like rise time and overshoot, we tune the crossover frequency and phase margin to get a desired closed-loop bandwidth and an acceptable amount of resonance.

In terms of a design process, we can adjust the gain to get the desired crossover frequency then add a lead compensator to get the desired phase margin.

The lead compensator has two parameters, T and α and a form

$$D(s) = \frac{Ts+1}{\alpha Ts+1} \qquad 0 < \alpha < 1$$

α is a number less than 1 that determines the separation of the zero and pole and T locates the zero in the frequency domain. The Bode plot is:



$\phi = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega)$ from the transfer function,

The maximum phase occurs at $\frac{1}{\sqrt{\alpha}T}$ and is

$$\sin\phi_{max} = \frac{1-\alpha}{1+\alpha}$$

$$\alpha = \frac{1-\sin\phi_{max}}{1+\sin\phi_{max}}$$

If we like the crossover frequency we have with a given gain, we can add phase margin by choosing α and add that margin at the right frequency by choosing T.

This will also have a change in the gain and crossover frequency unless we also change the gain of our compensator.

The lead compensator has a gain of $\frac{1+\sqrt{\alpha}}{\alpha+\sqrt{\alpha}}$ at frequency $\frac{1}{\sqrt{\alpha}T}$. If we divide by this value, we have a magnitude of 1 at $\frac{1}{\sqrt{\alpha}T}$ and can add phase without changing the gain or crossover frequency.

So our final lead compensator looks like:

$$D(s) = \left(\frac{\alpha+\sqrt{\alpha}}{1+\sqrt{\alpha}}\right)\left(\frac{Ts+1}{\alpha Ts+1}\right)$$

Or, even more simply, realizing that $\frac{\alpha+\sqrt{\alpha}}{1+\sqrt{\alpha}} = \sqrt{\alpha}$

$$D(s) = \sqrt{\alpha}\left(\frac{Ts+1}{\alpha Ts+1}\right)$$