# The Routh Array

The Routh Array (dating to 1874) provides a way to analytically determine the stability of a system when the characteristic equation is of higher order. It also gives insight into the range of parameters for which a system is stable.

Given a characteristic equation written in the form

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n = 0$$

the Routh Array is given by

$$\begin{array}{llll}
\text{Row } n: & 1 & a_2 & a_4 \ \dots \\
n-1: & a_1 & a_3 & a_5 \ \dots \\
n-2: & b_1 & b_2 & b_3 \ \dots \\
n-3: & c_1 & c_2 & c_3 \ \dots \\
 & \vdots & & \\
2: & * & * & \\
1: & * & & \\
0: & * & &
\end{array}$$

Where:

$$b_1 = \frac{-\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} \qquad b_2 = \frac{-\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1}$$

$$c_1 = \frac{-\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} \qquad c_2 = \frac{-\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1}$$

and so on

All roots of the characteristic equation are negative iff all terms in the first column are greater than zero. The characteristic equation has as many right half plane roots as there are sign changes in the first column (for example, + + − − is one sign change; + + − + is two sign changes). Slight modifications to this method are necessary when the first column contains a zero (and therefore is neither strictly positive or negative).

With the PI Controller in the car following example, the system type was correct but the characteristic equation became third order. We can check stability using the Routh Array:

$$ms^3 + bs^2 + K_p s + K_I = 0$$

$$\Rightarrow \quad s^3 + b/m\, s^2 + K_p/m\, s + K_I/m = 0$$

Form the Routh array

$$\begin{array}{llll} \text{Row } 3: & 1 & K_p/m & 0 \\ 2: & b/m & K_I/m & 0 \\ 1: & b_1 & & \\ 0: & c_1 & & \end{array}$$

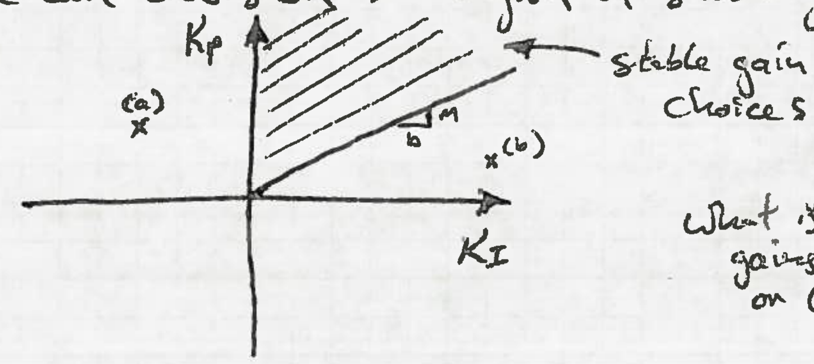
$$b_1 = \frac{-\det\begin{bmatrix} 1 & K_p/m \\ b/m & K_I/m \end{bmatrix}}{b/m} = -\frac{\left(K_I/m - bK_p/m^2\right)}{b/m} = K_p/m - K_I/b$$

$$c_1 = \frac{-\det\begin{bmatrix} b/m & K_I/m \\ (K_p/m - K_I/b) & 0 \end{bmatrix}}{(K_p/m - K_I/b)} = \frac{(K_p/m - K_I/b)(K_I/m)}{(K_p/m - K_I/b)} = K_I/m$$

The first column therefore looks like:

$$\begin{array}{lll} \text{Row } 3: & 1 & \text{Always} > 0 \\ 2: & b/m & \text{Always} > 0 \text{ since } b > 0 \\ 1: & K_p/m - K_I/b & \Rightarrow K_p > K_I(m/b) \\ 0: & K_I/m & \Rightarrow K_I > 0 \end{array}$$

We can envision the region of stable gains graphically:




What if we chose gains at point (a) or (b)?

First column of Routh Array

| | Gains at (a) | Gains at (b) |
|---|---|---|
| Row 3: | $1$ | $1$ |
| 2: | $b/m$ | $b/m$ |
| 1: | $K_P/m - K_I/b > 0$ | $K_P/m - K_I/b < 0$ |
| 0: | $K_I/m < 0$ | $K_I/m > 0$ |
| | $\Downarrow$ | $\Downarrow$ |
| | 1 sign change<br>1 unstable root | 2 sign changes<br>2 unstable roots |

If the first element in a row of the Routh Array is zero, the array requires modification to avoid dividing by zero. The zero can be replaced by a small positive number $\varepsilon$ and the results interpreted in the limit as $\varepsilon \to 0$.

For example, if our system has no damping, $b = 0$.

| | | | |
|---|---|---|---|
| Row 3: | $1$ | $K_P/m$ | $0$ |
| 2: | $0$ replace with $\varepsilon$ | $K_I/m$ | $0$ |
| 1: | $-K_I/m\varepsilon + K_P/m$ | | |
| 0: | $K_I/m$ | | |

For stability, $K_I/m > 0$

$$\lim_{\varepsilon \to 0} -K_I/m\varepsilon + K_P/m > 0$$

$$\lim_{\varepsilon \to 0} K_P > K_I/\varepsilon \quad \text{not possible!}$$

Without damping, the system will have two unstable roots for any positive $K_P$ and $K_I$.

Further modification is required in the special case when an entire row of the Routh Array is zero (see the text for this process).

The proof of the Routh Array is rather involved (though a simpler proof appeared about 125 years after the original). It shouldn't be too surprising, however, that the coefficients of the characteristic equation give information about the sign of the roots.

Back to the example, with PI control, the transfer function becomes

$$\frac{Y(s)}{R(s)} = \frac{X_f(s)}{R(s)} = \frac{K_P s + K_I}{ms^3 + bs^2 + K_P s + K_I}$$

$$= \frac{K_P/m\, s + K_I/m}{s^3 + \underbrace{b/m}_{\text{set by system}}\, s^2 + \underbrace{K_P/m\, s + K_I/m}_{\text{we can choose}}}$$

We can guarantee stability if $b > 0$ but we cannot place all three poles where we want them with only two control gains.

## PID Controller

For the third control knob we can add the derivative of the error to the control signal

$$U(s) = K_P E(s) + K_D s E(s) + \frac{K_I}{s} E(s)$$

$$D(s) = \frac{U(s)}{E(s)} = K_P + K_D s + \frac{K_I}{s}$$

$$u(t) = \underbrace{K_P e(t)}_{\text{"spring"}} + \underbrace{K_D \dot{e}(t)}_{\text{"damper"}} + K_I \int_0^t e(t)\,dt$$

This is easy enough to implement when there is a direct measurement of $\dot{e}(t)$ available. In this example, radar usually gives both range and range rate so this measurement exists. It can be tricky to get $\dot{e}(t)$ from measurements of $e(t)$ alone and this will be discussed in more detail later.

Let's look at our example with the complete PID controller:

$$G(s) = \frac{1}{s(ms+b)} \qquad D(s) = \frac{K_P s + K_D s^2 + K_I}{s}$$

$$\frac{Y(s)}{R(s)} = \frac{DG}{1+DG} = \frac{K_D s^2 + K_P s + K_I}{ms^3 + \underbrace{(b+K_D)}_{\substack{\text{derivative gain}\\ \text{adds to damping}\\ \text{and adds another}\\ \text{zero}}} s^2 + K_P s + K_I}$$

Checking stability with the Routh Array

$$s^3 + (b+K_D)/m \, s^2 + K_P/m \, s + K_I/m = 0$$

$$\begin{array}{lll} \text{Row } 3: & 1 & K_P/m \quad 0 \\ 2: & (b+K_D)/m & K_I/m \\ 1: & K_P/m - K_I/(b+K_D) & \\ 0: & K_I/m & \end{array}$$

So the conditions are:

$$b + K_D > 0$$

$$K_I > 0$$

$$K_P > \frac{m}{b+K_D} K_I$$

Checking system type:

$$\frac{E(s)}{R(s)} = \frac{1}{1+DG} = \frac{s^2(ms+b)}{ms^3+(b+K_D)s^2+K_Ps+K_I} \qquad \text{Type 2}$$

$$\frac{E(s)}{W(s)} = \frac{-G}{1+DG} = \frac{s}{ms^3+(b+K_D)s^2+K_Ps+K_I} \qquad \text{Type 1}$$

The addition of derivative control did not change the system type at all.

PID is often used because the control gains have (at least roughly) intuitive meanings and can often be tuned by hand:

Speed of response → $K_P$

Damping or overshoot → $K_D$

Steady-state error rejection → $K_I$