Transient Response and the Yaw Rate Transfer Function

If we assume that the vehicle:

(1) Travels at a constant longitudinal speed U_x
(2) Operates in the linear region of the tires (which corresponds to lateral acceleration up to about 0.6g on dry roads)
(3) Operates at small steer and slip angles

We can rearrange the lateral and yaw equations of motion into a transfer function that enables all sorts of analysis of the dynamic properties of this model.

Lateral velocity

$$m(\dot{U}_y + rU_x) = F_{yf} + F_{yr}$$
$$= -C_{\alpha f}\alpha_f - C_{\alpha r}\alpha_r$$
$$= -C_{\alpha f}\left(\frac{U_y + ar}{U_x} - \delta\right) - C_{\alpha r}\left(\frac{U_y - br}{U_x}\right)$$
$$m(\dot{U}_y + rU_x) = \frac{-(C_{\alpha f} + C_{\alpha r})}{U_x}U_y + \frac{(bC_{\alpha r} - aC_{\alpha f})}{U_x}r + C_{\alpha f}\delta$$

Yaw rate

$$I_z\dot{r} = aF_{yf} - bF_{yr}$$
$$= -aC_{\alpha f}\left(\frac{U_y + ar}{U_x} - \delta\right) - b\left(-C_{\alpha r}\left(\frac{U_y - br}{U_x}\right)\right)$$
$$I_z\dot{r} = \frac{bC_{\alpha r} - aC_{\alpha f}}{U_x}U_y - \frac{a^2C_{\alpha f} + b^2C_{\alpha r}}{U_x}r + aC_{\alpha f}\delta$$

More compactly, in state space form:

$$\begin{bmatrix}\dot{U}_y \\ \dot{r}\end{bmatrix} = \begin{bmatrix}\frac{-(C_{\alpha f} + C_{\alpha r})}{mU_x} & \frac{bC_{\alpha r} - aC_{\alpha f}}{mU_x} - U_x \\ \frac{bC_{\alpha r} - aC_{\alpha f}}{I_zU_x} & \frac{-(a^2C_{\alpha f} + b^2C_{\alpha r})}{I_zU_x}\end{bmatrix}\begin{bmatrix}U_y \\ r\end{bmatrix} + \begin{bmatrix}\frac{C_{\alpha f}}{m} \\ \frac{aC_{\alpha f}}{I_z}\end{bmatrix}\delta$$

Laplace transforming such that $\mathcal{L}\{U_y(t)\} = U_y(s)$, $\mathcal{L}\{r(t)\} = R(s)$ and $\mathcal{L}\{\delta(t)\} = \Delta(s)$

$$sU_y(s) = \frac{-(C_{\alpha f} + C_{\alpha r})}{mU_x}U_y(s) + \left[\frac{(bC_{\alpha r} - aC_{\alpha f})}{mU_x} - U_x\right]R(s) + \frac{C_{\alpha f}}{m}\Delta(s)$$

$$U_y(s) = \frac{\left[\frac{(bC_{\alpha r} - aC_{\alpha f})}{mU_x} - U_x\right]R(s) + \frac{C_{\alpha f}}{m}\Delta(s)}{s + \frac{(C_{\alpha f} + C_{\alpha r})}{mU_x}}$$

substitute

$$sR(s) = \frac{bC_{\alpha r} - aC_{\alpha f}}{I_zU_x}U_y(s) - \frac{(a^2C_{\alpha f} + b^2C_{\alpha r})}{I_zU_x}R(s) + \frac{aC_{\alpha f}}{I_z}\Delta(s)$$

After rearranging...

$$\frac{R(s)}{\Delta(s)} = \frac{a C_{\alpha f} s + \frac{L C_{\alpha f} C_{\alpha r}}{m U_x}}{I_z s^2 + \left[\frac{I_z (C_{\alpha f} + C_{\alpha r})}{m U_x} + \frac{a^2 C_{\alpha f} + b^2 C_{\alpha r}}{U_x}\right] s + \left[\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + b C_{\alpha r} - a C_{\alpha f}\right]}$$

"mass" always >0

"damper" always >0

"spring" sign uncertain

The transfer function has a second order denominator similar to a mass-spring-damper system. Unlike the mass-spring-damper, however, it represents the relationship between the steer angle input and the angular velocity, not a position variable. We can learn a lot about basic vehicle design from this transfer function.

Stability

One of the most fundamental questions about a dynamic system is its stability. A second order system is stable if and only if each of the coefficients in the characteristic equation (denominator) are positive.

Two of these clearly are. The third can be rearranged in terms of the understeer gradient:

$$K = \left(\frac{W_f}{C_{\alpha f}} - \frac{W_r}{C_{\alpha r}}\right)\left(\frac{1}{g}\right)$$

$$= \left(\frac{b m g}{L C_{\alpha f}} - \frac{a m g}{L C_{\alpha r}}\right)\left(\frac{1}{g}\right)$$

$$= \frac{m}{L}\left(\frac{b}{C_{\alpha f}} - \frac{a}{C_{\alpha r}}\right)$$

$$\Rightarrow \quad K = \frac{m}{L}\left(\frac{b C_{\alpha r} - a C_{\alpha f}}{C_{\alpha f} C_{\alpha r}}\right)$$

If $K \geq 0$, $b C_{\alpha r} - a C_{\alpha f} \geq 0$ and the "spring" term > 0

$\Rightarrow$ An understeer or neutral steering car cannot go unstable!

However, if $K < 0$ $b C_{\alpha r} - a C_{\alpha f} < 0$. The "spring" term < 0 when

$$\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + b C_{\alpha r} - a C_{\alpha f} < 0$$

$$\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + \frac{C_{\alpha f} C_{\alpha r} K L}{m} < 0$$

$$\frac{1}{U_x^2} + \frac{K}{L} < 0$$

$$L + K U_x^2 < 0$$

$$K U_x^2 < -L$$

$$U_x > \sqrt{-L/K} \quad (\text{where } K < 0)$$

This is defined as the critical speed of the oversteer vehicle

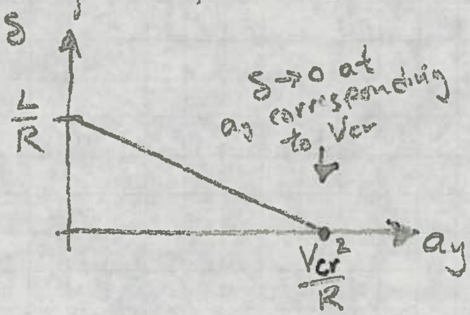
$$V_{cr} = \sqrt{-L/K}$$

When the longitudinal speed exceeds the critical speed, the transfer function becomes unstable

$\Rightarrow$ An oversteer vehicle is unstable above its critical speed!

What does this instability mean? Turning the steering wheel results in a yaw rate that continues to grow and never reaches equilibrium. Eventually this causes one of the axles to reach its peak force capability at which point the vehicle will spin (most likely) or plow (possible in some cases). This is very hard for the driver to control and manufacturers work hard to avoid oversteer in the linear region of handling (and, as we'll see later, at the limits as well).

The critical speed shows up on the handling diagram for a linear oversteer vehicle:



$$\delta = \frac{L}{R} + K a_y = 0$$

$$a_y = \frac{V^2}{R}$$

$$\frac{L}{R} + \frac{K V^2}{R} = 0$$

$$\Rightarrow V = \sqrt{-L/K}$$

Avoiding oversteer in the linear region of handling is largely a matter of avoiding a rear weight bias. In the case of the Porsche 911, it involves having much stiffer rear tires in order to compensate for the weight on the rear.

The Understeer vehicle

We could simply substitute numbers for parameters into the transfer function and examine its behavior for a particular vehicle. There are some general properties that can be derived by looking at the analytical form of the transfer function, however. These are worth diving through a bit of algebra to discover.

Steady-state

The steady-state yaw rate can be found from the final value theorem.

$$r_{ss} = \lim_{s\to 0} sR(s) \qquad \text{when } \Delta(s) = \frac{1}{s}\delta$$

$$= \lim_{s\to 0} s\left(\frac{1}{s}\delta\right)\frac{R(s)}{\Delta(s)}$$

$$= \frac{\delta\left(\frac{L C_{\alpha f} C_{\alpha r}}{m U_x}\right)}{\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + bC_{\alpha r} - aC_{\alpha f}}$$

$$= \frac{\delta\left(\frac{L C_{\alpha f} C_{\alpha r}}{m U_x}\right)}{\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + \frac{K C_{\alpha f} C_{\alpha r} L}{m}}$$

$$= \frac{\delta\left(\frac{L}{U_x}\right)}{\frac{L^2}{U_x^2} + KL}$$

$$r_{ss} = \frac{U_x}{L + KU_x^2}\,\delta$$

In comparison, the kinematic model gave

$$\dot{\psi}_{ve} = r = \frac{V}{L}\tan\delta$$

So the linear bicycle model modifies this according to the understeer gradient.

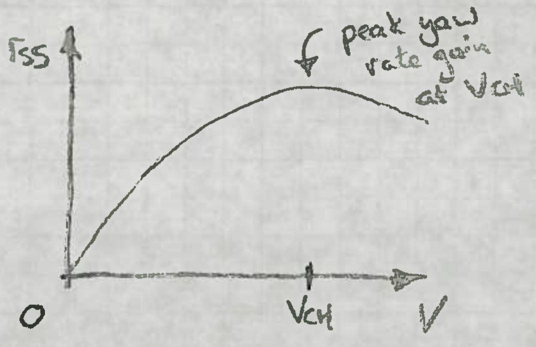
The yaw rate gain or steady-state yaw rate varies as a function of speed. When is it at its maximum?

$$\frac{\partial r_{ss}}{\partial U_x} = \frac{\delta}{L + KU_x^2} - 2KU_x\frac{U_x\delta}{(L+KU_x^2)^2} = 0$$

$$(L + KU_x^2) - 2KU_x^2 = 0$$

$$KU_x^2 = L$$

$$U_x = \sqrt{L/K} \triangleq V_{CH}$$ – the characteristic speed of an understeering vehicle



The gain increases until V_{CH} and then gradually decreases.

The more understeer a vehicle has, the lower its characteristic speed:

K (rad/m/s²)	K (deg/g)	V_{CH}
0.00178	1	37.5 m/s
0.00356	2	26.5 m/s
0.00534	3	21.6 m/s

For this normal range of passenger car understeer gradient, V_{CH} is in the range of highway speeds.

The difference in yaw rate gain between the kinematic model and the dynamic model can be significant. At V_{CH} (26.5 m/s) for a vehicle with 2 deg/g understeer:

$$\frac{U_x}{L} = 10.6 \qquad \frac{U_x}{L + KU_x^2} = 5.3$$

⟹ A factor of 2 difference in the yaw rate produced per radian of steer angle!!

(The difference between these two models will always be a factor of 2 when $U_x = V_{CH}$)

Natural frequency and damping

The denominator of the yaw rate transfer function can be written in the form:

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

with some rearranging...

$$\omega_n^2 = \frac{1}{I_z}\left[\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + b C_{\alpha r} - a C_{\alpha f}\right]$$

Since $\frac{C_{\alpha f} C_{\alpha r} L}{m} = \frac{b C_{\alpha r} - a C_{\alpha f}}{K}$ we can rearrange this in a couple of ways:

$$\omega_n^2 = \frac{1}{I_z}\left[\frac{L(b C_{\alpha r} - a C_{\alpha f})}{K U_x^2} + b C_{\alpha r} - a C_{\alpha f}\right]$$

$$= \left(\frac{b C_{\alpha r} - a C_{\alpha f}}{I_z}\right)\left[\frac{L}{K U_x^2} + 1\right]$$

$$\omega_n^2 = \left(\frac{b C_{\alpha r} - a C_{\alpha f}}{I_z}\right)\left[\left(\frac{V_{CH}^2}{U_x^2}\right) + 1\right] \qquad \text{since } V_{CH}^2 = \frac{L}{K}$$

This is a useful formula for calculating and thinking about the natural frequency. For calculating the damping ratio, a slightly different form is helpful:

$$\omega_n^2 = \frac{1}{I_z}\left[\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + \frac{K C_{\alpha f} C_{\alpha r} L}{m}\right]$$

$$= \frac{C_{\alpha f} C_{\alpha r} L^2}{m I_z}\left[\frac{1}{U_x^2} + \frac{K}{L}\right]$$

$$= \frac{C_{\alpha f} C_{\alpha r} L^2}{m I_z}\left[\frac{1}{U_x^2} + \frac{1}{V_{CH}^2}\right]$$

$$= \frac{C_{\alpha f} C_{\alpha r} L^2}{m I_z U_x^2}\left[1 + \left(\frac{U_x^2}{V_{CH}^2}\right)\right]$$

To examine the damping ratio, two useful bits of algebra prove helpful.

<u>Useful bit of algebra #1</u>

(i) $(C_{\alpha f} + C_{\alpha r})(a^2 C_{\alpha f} + b^2 C_{\alpha r}) = a^2 C_{\alpha f}^2 + b^2 C_{\alpha r}^2 + (a^2 + b^2) C_{\alpha f} C_{\alpha r}$

(ii) $(b C_{\alpha r} - a C_{\alpha f})^2 = b^2 C_{\alpha r}^2 + a^2 C_{\alpha f}^2 - 2ab C_{\alpha f} C_{\alpha r}$

(iii) $(a+b)^2 C_{\alpha f} C_{\alpha r} = (a^2 + b^2) C_{\alpha f} C_{\alpha r} + 2ab C_{\alpha f} C_{\alpha r}$

So $\boxed{L^2 C_{\alpha f} C_{\alpha r} = (a+b)^2 C_{\alpha f} C_{\alpha r} = (C_{\alpha f} + C_{\alpha r})(a^2 C_{\alpha f} + b^2 C_{\alpha r}) - (b C_{\alpha r} - a C_{\alpha f})^2}$

The next useful bit is a more general principle...

Useful bit of algebra #2

The arithmetic mean of two positive numbers is always greater than their geometric mean or

$$\frac{1}{2}(m+n) \geq \sqrt{mn} \qquad \text{for } m,n \geq 0$$

so $$\frac{\frac{1}{2}(m+n)}{\sqrt{mn}} \geq 1$$

Why?
$$(m-n)^2 \geq 0$$
$$m^2+n^2-2mn \geq 0$$
$$m^2+n^2+2mn \geq 4mn$$
$$(m+n)^2 \geq 4mn$$
$$\Longrightarrow \frac{1}{2}(m+n) \geq \sqrt{mn}$$

$$2\zeta\omega_n = \frac{(C_{\alpha f}+C_{\alpha r})}{mU_x} + \frac{a^2C_{\alpha f}+b^2C_{\alpha r}}{I_z U_x}$$

$$\zeta = \frac{I_z(C_{\alpha f}+C_{\alpha r}) + m(a^2C_{\alpha f}+b^2C_{\alpha r})}{2mI_zU_x\omega_n}$$

$$= \frac{I_z(C_{\alpha f}+C_{\alpha r}) + m(a^2C_{\alpha f}+b^2C_{\alpha r})}{2mI_zU_x\sqrt{\frac{C_{\alpha f}C_{\alpha r}L^2}{mI_zU_x^2}\left(1+\left(\frac{U_x}{V_{CH}}\right)^2\right)}}$$

$$= \frac{\frac{1}{2}\left[I_z(C_{\alpha f}+C_{\alpha r}) + m(a^2C_{\alpha f}+b^2C_{\alpha r})\right]}{\sqrt{mI_zC_{\alpha f}C_{\alpha r}L^2}\sqrt{\left(1+\left(\frac{U_x}{V_{CH}}\right)^2\right)}}$$

$$\zeta = \frac{\zeta_0}{\sqrt{1+\left(\frac{U_x}{V_{CH}}\right)^2}}$$

$$\zeta_0 = \frac{\frac{1}{2}\left[I_z(C_{\alpha f}+C_{\alpha r}) + m(a^2C_{\alpha f}+b^2C_{\alpha r})\right]}{\sqrt{mI_zC_{\alpha f}C_{\alpha r}L^2}}$$

$$\zeta_0 = \frac{\frac{1}{2}\left[I_z(C_{\alpha f}+C_{\alpha r}) + m(a^2C_{\alpha f}+b^2C_{\alpha r})\right]}{\sqrt{I_z(C_{\alpha f}+C_{\alpha r})m(a^2C_{\alpha f}+b^2C_{\alpha r}) - mI_z(bC_{\alpha r}-aC_{\alpha f})^2}} \quad \text{by useful fact \#1}$$

$$\zeta_0 \geq \frac{\frac{1}{2}\left[I_z(C_{\alpha f}+C_{\alpha r}) + m(a^2C_{\alpha f}+b^2C_{\alpha r})\right]}{\sqrt{I_z(C_{\alpha f}+C_{\alpha r})m(a^2C_{\alpha f}+b^2C_{\alpha r})}} \geq 1 \quad \text{by useful fact \#2}$$

Since $\zeta = \dfrac{\zeta_o}{\sqrt{1+\left(\frac{U_x}{V_{cr}}\right)^2}}$ and $\zeta_o \geq 1$, the poles of the understeering vehicle are critically damped ($\zeta \geq 1$) at very low speed. Damping decreases as speed increases and a vehicle with a higher understeer gradient experiences a greater decrease in damping at a given speed than a vehicle with a lower understeer gradient.

This fact explains why cars designed to run on Germany's Autobahn tend to have low understeer gradients - too much understeer and the yaw response becomes very oscillatory at high speed! The system is still "stable" but may not feel that way to the driver. A large overshoot in yaw rate may feel like the unstable behavior of an oversteering vehicle if the driver doesn't wait for the oscillation to die out before correcting steering...

So, to summarize, the understeering car cannot go unstable, has real poles at low speed and exhibits a loss of damping (and eventually complex conjugate poles) as speed increases.

The Neutral Steer Vehicle

Neutral steering avoids the stability problems of an oversteer vehicle and the damping issues of the understeer vehicle.

The steady-state yaw rate of a neutral steering vehicle is the same as the kinematic model:

$$r_{ss} = \frac{U_x}{L + K U_x^2}\delta = \frac{U_x \delta}{L} \text{ when } K = 0$$

The natural frequency becomes:

$$\omega_n^2 = \frac{C_{\alpha f} C_{\alpha r} L^2}{m I_z U_x^2}$$

and the damping ratio becomes constant and no longer a function of longitudinal speed:

$$\zeta = \zeta_o = \frac{\frac{1}{2}\left[I_z(C_{\alpha f}+C_{\alpha r}) + m(a^2 C_{\alpha f} + b^2 C_{\alpha r})\right]}{\sqrt{I_z(C_{\alpha f}+C_{\alpha r})\, m\,(a^2 C_{\alpha f} + b^2 C_{\alpha r})}} \geq 1$$

So the poles lie on the real axis at any speed.

The neutral steer vehicle has an even more interesting property than the poles remaining stable and damped.

When $K = 0$, $aC_{\alpha f} - bC_{\alpha r} = 0$ so $C_{\alpha f} = \frac{b}{a} C_{\alpha r}$

this means $C_{\alpha f} + C_{\alpha r} = \frac{b}{a} C_{\alpha r} + C_{\alpha r} = \frac{L}{a} C_{\alpha r}$ and our useful bit of algebra #1 reduces to:

$$L^2 C_{\alpha f} C_{\alpha r} = (C_{\alpha f} + C_{\alpha r})(a^2 C_{\alpha f} + b^2 C_{\alpha r})$$

These can be used to rearrange the transfer function:

$$\frac{R(s)}{\Delta(s)} = \frac{aC_{\alpha f} s + \frac{L C_{\alpha f} C_{\alpha r}}{mU_x}}{I_z s^2 + \left[\frac{I_z (C_{\alpha f} + C_{\alpha r})}{mU_x} + \frac{a^2 C_{\alpha f} + b^2 C_{\alpha r}}{U_x}\right] s + \frac{C_{\alpha f} C_{\alpha r} L^2}{mU_x^2}}$$

$$= \frac{aC_{\alpha f}\left(s + \frac{L C_{\alpha r}}{a m U_x}\right)}{I_z s^2 + \left[\frac{I_z (C_{\alpha f} + C_{\alpha r})}{mU_x} + \frac{a^2 C_{\alpha f} + b^2 C_{\alpha r}}{U_x}\right] s + \frac{(C_{\alpha f} + C_{\alpha r})(a^2 C_{\alpha f} + b^2 C_{\alpha r})}{mU_x^2}}$$

$$= \frac{aC_{\alpha f}\left(s + \frac{C_{\alpha f} + C_{\alpha r}}{mU_x}\right)}{\left(I_z s + \frac{a^2 C_{\alpha f} + b^2 C_{\alpha r}}{U_x}\right)\left(s + \frac{C_{\alpha f} + C_{\alpha r}}{mU_x}\right)}$$

A pole-zero cancellation!

$$\frac{R(s)}{\Delta(s)} = \frac{aC_{\alpha f}}{I_z s + \frac{a^2 C_{\alpha f} + b^2 C_{\alpha r}}{U_x}}$$

So the transfer function of the neutral steer vehicle actually drops to first order! This is why vehicles that are very balanced with respect to mass and tire properties "handle like they are on rails." The dynamic response is simpler.

In practice, perfect neutral steering is impossible to achieve – any shift in the mass due to a different driver, driver position, fuel use or luggage will alter this perfect balance and result in a vehicle that is slightly oversteering or understeering. However vehicles that are close to neutral steering remain damped and stable over a very wide range of speeds.

The Oversteer Vehicle

Below the critical speed, it makes sense to calculate the steady-state yaw rate, natural frequency and yaw rate of the oversteer vehicle.

$$r_{ss} = \frac{U_x}{L + K U_x^2}\delta \qquad U_x < V_{cr}$$

The steady-state yaw rate has the same expression as the understeer vehicle. Since $K < 0$, the yaw rate gain gets increasingly large as the vehicle approaches its critical speed.

The natural frequency can be written as:

$$\omega_n^2 = \frac{bC_{\alpha r} - aC_{\alpha f}}{I_z}\left[\frac{L}{KU_x^2} + 1\right]$$

$$= \frac{aC_{\alpha f} - bC_{\alpha r}}{I_z}\left[\frac{-L}{KU_x^2} - 1\right]$$

$$\omega_n^2 = \frac{aC_{\alpha f} - bC_{\alpha r}}{I_z}\left[\left(\frac{V_{cr}}{U_x}\right)^2 - 1\right]$$

Looking at this equation or the "spring" term in the denominator of the transfer function:

$$\frac{C_{\alpha f} C_{\alpha r} L^2}{m U_x^2} + bC_{\alpha r} - aC_{\alpha f}$$

it is clear that that this term becomes less positive as speed increases up to the critical speed, becomes zero at the critical speed and is negative at speeds above the critical speed.

By Routh's stability criterion, the fact that there is only one sign change in the characteristic equation's coefficients above the critical speed means that the vehicle has one unstable pole. That means above the critical speed, the poles are real with one being positive and one negative.

What do the poles look like below the critical speed? Using the other form of the ω_n relationship:

$$\omega_n^2 = \frac{C_{\alpha f} C_{\alpha r} L^2}{m I_z}\left[\frac{1}{U_x^2} + \frac{K}{L}\right]$$

$$\omega_n^2 = \frac{C_{\alpha f} C_{\alpha r} L^2}{m I_z U_x^2}\left[1 - \left(\frac{U_x}{V_{cr}}\right)^2\right]$$

and the related expression for the damping ratio becomes:

$$\zeta = \frac{\zeta_0}{\sqrt{1-\left(\frac{U_x}{V_{cr}}\right)^2}} \qquad U_x < V_{cr}$$

Since the denominator is always less than or equal to one and $\zeta_0 \geq 1$, the poles of the oversteer vehicle are critically damped - or real - below the critical speed.

From this analysis, we see that the oversteer vehicle has real poles at all speeds. At low speeds, both lie on the negative real axis and one moves towards the origin as speed increases. That pole hits the origin at the critical speed and lies on the positive real axis at higher speeds.

Designing a vehicle with slight oversteer such that V_{cr} is outside the expected speed range (or capability) of a car is in principle not a problem. This may happen in the pursuit of neutral steer, for instance.

The challenge is making sure that this instability does not appear as additional complexity is added to the vehicle model. Weight will transfer from inside to outside tires in a turn and from rear tires to front tires while braking. As the tire forces approach their limits, forces from braking or acceleration compete for friction with lateral forces. Eliminating oversteer under all (or most) of these conditions requires more than simply changing mass distribution and tire stiffness. In fact, this is a primary objective of suspension design.

But even as models get more detailed, these second order system dynamics remain fundamental. So understanding the behavior of the poles in this simple linear system builds the foundation of intuition necessary to understand more complex situations.

Note that while oversteer vehicles are unstable above their critical speed, that doesn't mean they are undriveable at such speeds. Their response can be stabilized by a human driver if that driver has sufficient skill and isn't surprised by the oversteer.