Sinusoidal Inputs

The transfer function of a system is the Laplace transform of the impulse response so it is easy to obtain the impulse response. Using partial fraction expansion, it is also straightforward to solve for the response to any general input. Sinusoidal inputs give another interesting interpretation of the transfer function.

We know that poles at $s = \pm j\omega$ are associated with sinusoids. In particular,

$$\mathcal{L}[\sin \omega t] = \int_0^\infty (\sin \omega t)\, e^{-st}\, dt$$

$$= \int_0^\infty \frac{e^{j\omega t} - e^{-j\omega t}}{2j}\, e^{-st}\, dt$$

$$= \frac{1}{2j} \int_0^\infty e^{(j\omega - s)t} - e^{-(j\omega - s)t}\, dt$$

$$= \frac{1}{2j}\left[\frac{-1}{j\omega - s} + \frac{1}{-j\omega - s}\right]$$

$$= \frac{1}{2j}\left[\frac{2j\omega}{(j\omega - s)(-j\omega - s)}\right]$$

$$= \frac{\omega}{s^2 + \omega^2}$$

Similarly, $\mathcal{L}[\cos \omega t] = \dfrac{s}{s^2+\omega^2}$

Consider a stable system $H(s)$ and sinusoidal $U(s) = \dfrac{\omega A}{s^2+\omega^2}$

$$U(s) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$$

$$Y(s) = \underbrace{\frac{a_1}{s-p_1} + \frac{a_2}{s-p_2} + \ldots + \frac{a_n}{s-p_n}}_{\text{poles of } H(s) \text{ are stable so response dies out as } t \to \infty} + \underbrace{\frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}}_{\text{poles from the input oscillate without decaying}}$$

So after the response from the stable poles dies out,

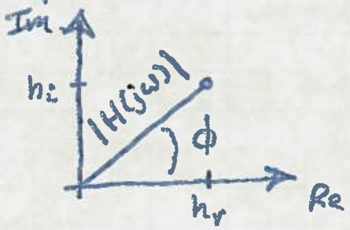
$$y(t) = a e^{-j\omega t} + \bar{a} e^{j\omega t}$$

The output of the system therefore becomes some sort of sinusoid. To be more specific, we can just evaluate the residues using:

$$a = (s+j\omega)\, Y(s) \big|_{s=-j\omega}$$

$$a = (s+j\omega)\,H(s)\,\frac{\omega A}{s^2+\omega^2}\Big|_{s=-j\omega}$$
$$= H(s)\,\frac{\omega A}{s-j\omega}\Big|_{s=-j\omega}$$
$$= -\frac{1}{2j}\,H(-j\omega)A$$
$$\bar{a} = (s-j\omega)\,H(s)\,\frac{\omega A}{s^2+\omega^2}\Big|_{s=j\omega}$$
$$= \frac{1}{2j}\,H(j\omega)A$$

$H(j\omega)$ is just a complex number for any value of ω.
It can be considered in terms of real and imaginary components or a magnitude and phase



$$\{H(j\omega)\} = |H(j\omega)|\,(\cos\phi + j\sin\phi)$$
$$= |H(j\omega)|\,e^{j\phi}$$
$$H(-j\omega) = \overline{H}(j\omega) = |H(j\omega)|\,e^{-j\phi}$$

Putting this together gives the time response:

$$y(t) = -\frac{1}{2j}\,|H(j\omega)|\,Ae^{-j\phi}e^{-j\omega t} + \frac{1}{2j}\,|H(j\omega)|\,Ae^{j\phi}e^{j\omega t}$$
$$= A\,|H(j\omega)|\,\frac{1}{2j}\left[e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}\right]$$
$$= \underbrace{A}_{\text{input amplitude}}\;\underbrace{|H(j\omega)|}_{\text{change in amplitude}}\;\sin(\underbrace{\omega}_{\text{input frequency}} t + \underbrace{\phi}_{\text{phase shift}})$$

So $H(j\omega)$ is a complex number representing the change in magnitude and phase experienced by a sinusoidal input of frequency ω.

$$u = A\sin\omega t \longrightarrow \boxed{H(j\omega)} \longrightarrow y = A\,|H(j\omega)|\,\sin(\omega t+\phi)$$

Another way to get the same result is to use convolution:

$$y(t) = \int_0^\infty h(\tau)\,u(t-\tau)\,d\tau$$
$$= \int_0^\infty h(\tau)\,e^{q(t-\tau)}\,d\tau \qquad \text{if } u(t) = e^{qt}$$
$$= e^{qt}\int_0^\infty h(\tau)\,e^{-q\tau}\,d\tau$$
$$= e^{qt}\,H(q)$$

If $u(t)$ is a sinusoid, it can be written as

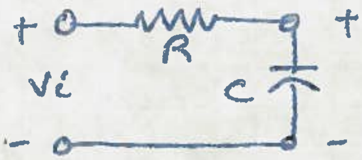
$$u(t) = \frac{A}{2j} e^{j\omega t} - \frac{A}{2j} e^{-j\omega t}$$

which gives

$$y(t) = \frac{A}{2j} H(j\omega) e^{j\omega t} - \frac{A}{2j} H(-j\omega) e^{-j\omega t}$$

which can be simplified as above.

Example



What is the frequency response of this RC circuit?

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{RCs+1}$$

$$H(j\omega) = \frac{1}{RCj\omega+1} = \frac{1-j\omega RC}{1+\omega^2R^2C^2}$$

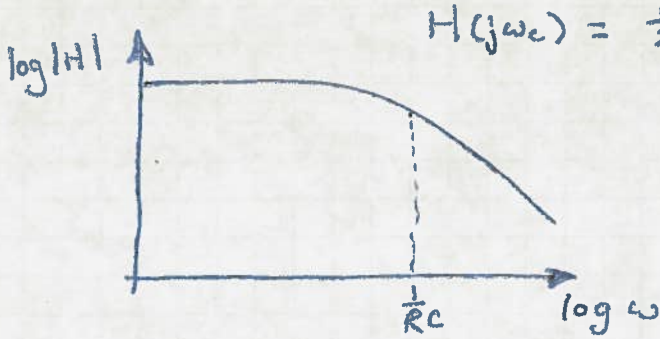
$$|H(j\omega)| = \sqrt{\left(\frac{1}{1+\omega^2R^2C^2}\right)^2 + \left(\frac{\omega RC}{1+\omega^2R^2C^2}\right)^2}$$

$|H(j\omega)| = \phi$ at $\omega = 0$ => DC gain of 1

$|H(j\omega)| \to 0$ as $\omega \to \infty$ => A low pass filter

when $\omega_c = \frac{1}{RC}$ $\quad |H(j\omega_c)| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{2}$

$H(j\omega_c) = \frac{1}{2} - \frac{1}{2}j$ so $\phi = -45°$



A sinusoid at frequency $\omega_c = \frac{1}{RC}$ has been shifted in phase by 45° and reduced in magnitude by $\frac{\sqrt{2}}{2}$ relative to the input.

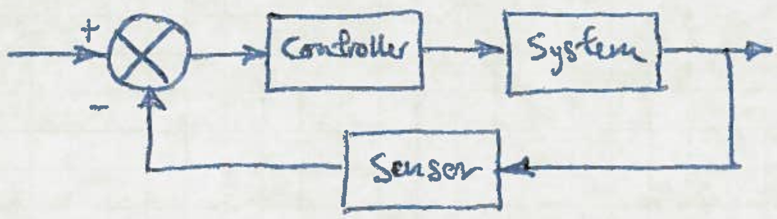
Block Diagrams

The transfer function can be used to tell us a lot about the system, including:

* The impulse response
* Stability (through pole locations)
* The response to general inputs
* Steady-state value (if poles are stable)
* The frequency response

The goal of feedback control is to shape the response of the system by closing a feedback loop which changes the open loop transfer function into the closed-loop transfer function.

To do this, we need to be comfortable working with systems in a form that looks like:



This is a Block Diagram

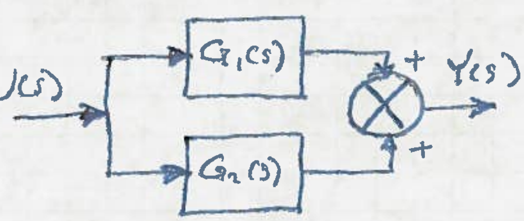
There are some simple rules of block diagram algebra that make it easy to manipulate systems into a form we want.

Basic Connections

Cascade

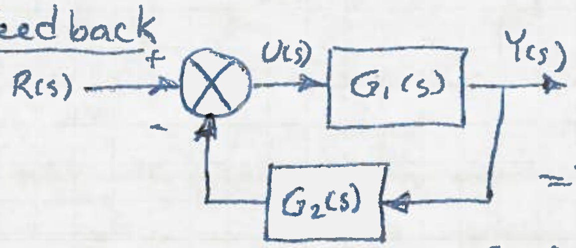
$$Y_2(s) = G_2(s)\,Y_1(s) = G_1(s)\,G_2(s)\,U(s)$$

Sum



$$Y(s) = \left[G_1(s) + G_2(s)\right]U(s)$$

Feedback

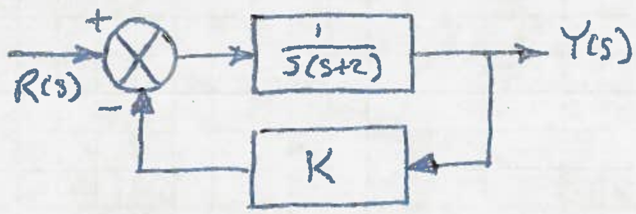


$$Y(s) = G_1(s)\,U(s)$$
$$= G_1(s)\left[R(s) - G_2(s)\,Y(s)\right]$$

$$\Rightarrow Y(s)\left[1 + G_1(s)G_2(s)\right] = G_1(s)\,R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

Example



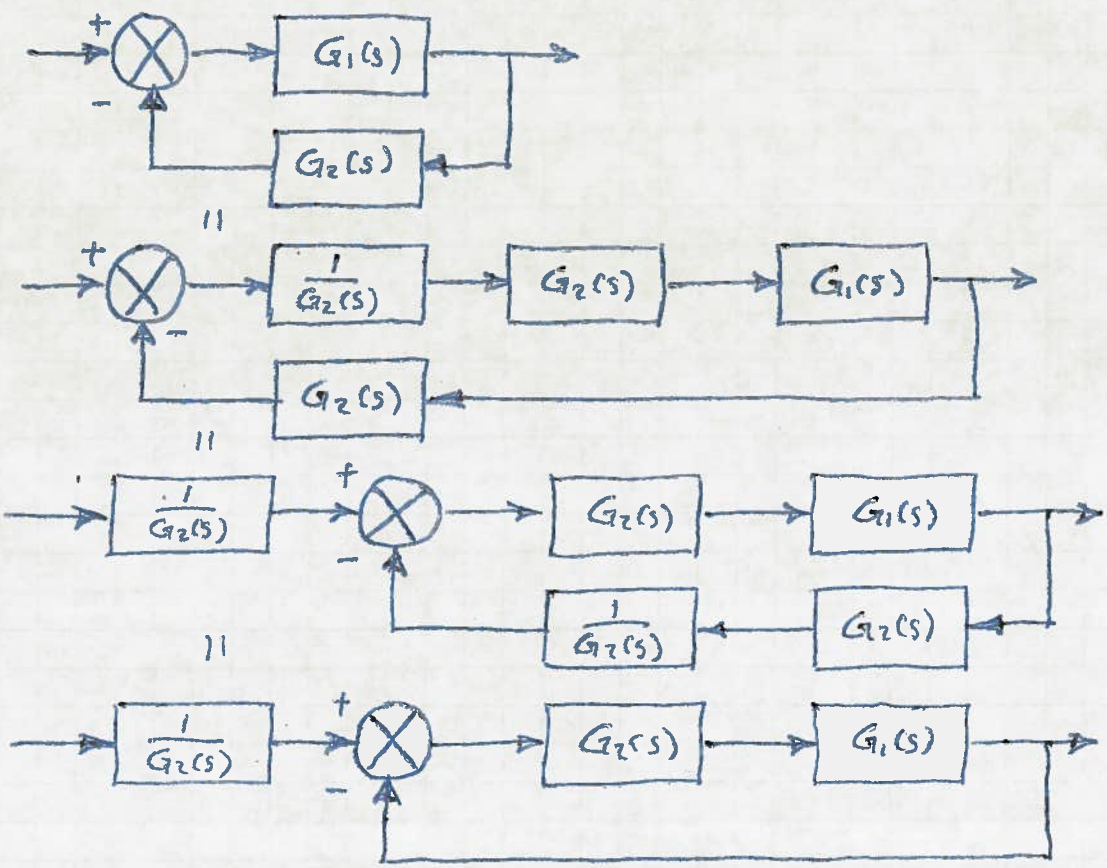
This could be a DC motor only now with position feedback.

$$\frac{Y(s)}{R(s)} = \frac{\frac{1}{s(s+2)}}{1+\frac{K}{s(s+2)}} = \frac{\frac{1}{s(s+2)}}{\frac{s(s+2)}{s(s+2)}+\frac{K}{s(s+2)}} = \frac{1}{s^2+2s+K}$$

The response to a change in reference input is stable instead of continuing to integrate as in open loop.

Unity feedback

It is often helpful to rearrange the system to have a unity feedback loop (no blocks in the feedback path). This can be accomplished with some manipulation of the diagram.



We can move blocks into the feedforward or feedback paths to simplify analysis.