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Tannaka–Kreĭn duality for Roelcke-precompact non-archimedean Polish groups

by

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Abstract. Let G be a Roelcke-precompact non-archimedean Polish group, $\mathcal{A}_G$ the algebra generated by indicator maps of cosets of open subgroups in G. Then $\mathcal{A}_G$ is dense in the algebra of matrix coefficients of G. We prove that multiplicative linear functionals on $\mathcal{A}_G$ are automatically continuous, an analogue of a result of Kreĭn for finite-dimensional representations of topological groups. We deduce two abstract realizations of the Hilbert compactification $\mathbf{H}(G)$ of G. One is the space $\mathbf{P}(\mathcal{M}_G)$ of partial elementary maps with algebraically closed domain on $\mathcal{M}_G$, the countable set of open cosets of G seen as a homogeneous first-order logical structure. This can be seen as a reformulation of a similar identification by Ben Yaacov, Ibarlucía and Tsankov for $\aleph_0$-categorical structures. The other is $\mathbf{T}(G)$, *the Tannaka monoid* of G. The group can be recovered from these constructions, generalizing Tannaka's and Kreĭn's duality theories to this context. Finally, we show that the natural functor that sends G to the category of its representations is full and faithful.

Introduction. A fundamental question in abstract harmonic analysis is: How much information about a topological group can one recover from its representation theory? In this paper, we are interested in cases where the group can be fully reconstructed.

More precisely, a *unitary representation* of a topological group G is a continuous group morphism π from G to the unitary group $\mathcal{U}(\mathcal{H}_\pi)$ of a complex Hilbert space $\mathcal{H}_\pi$. Continuity means here that all the maps of the form $f^\pi_{\xi,\eta}\colon g \mapsto \langle \pi(g)\eta, \xi\rangle$ for $\xi, \eta \in \mathcal{H}_\pi$, called the *matrix coefficients* of π, are continuous. The representation π is said to be *irreducible* if $\mathcal{H}_\pi$ admits no non-trivial G-invariant closed subspace. In the well-behaved cases, every unitary representation of G decomposes uniquely into an aggregate of irre-

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ducible subrepresentations. Harmonic analysis on such a group G reduces to the study of its *unitary dual* $\widehat{G}$, the set of isomorphism classes of irreducible unitary representations of G. In various subcases, duality theories have been established, allowing for the abstract reconstruction of G from $\widehat{G}$.

A fundamental instance of this is the case of locally compact abelian groups with the celebrated Pontryagin–van Kampen duality theory [Po34, vK35]. Indeed, if G is locally compact and abelian, its irreducible representations all have dimension 1 and $\widehat{G}$ can be identified with $\mathrm{Hom}(G, \mathbb{S}^1)$, the continuous group morphisms from G to the unit circle. In particular, $\widehat{G}$ is also an abelian group that is moreover locally compact when endowed with the compact-open topology. Finally, G is canonically isomorphic to its bi-dual.

Another context where representation theory is very tame is the compact case. Indeed, recall the Peter–Weyl Theorem [PW27] which states, in particular, that every representation of a compact group splits in an essentially unique way as a sum of finite dimensional irreducible subrepresentations. This ideal situation allowed Tannaka and Kreĭn to develop, independently and with different approaches, duality theories for compact groups.

Recall that the *Gel'fand spectrum* of an involutive algebra $\mathcal{B}$ is the set of non-zero multiplicative linear functionals $\mathcal{B} \to \mathbb{C}$ that commute with the involution. It is well known that continuity is automatic when $\mathcal{B}$ is a C^*-algebra. Kreĭn considered the algebra $\mathcal{B}_0(G)$ generated by the matrix coefficients of a general topological group G arising from its finite-dimensional representations, which is not complete for the supremum norm in general, and proved the following technical but crucial result: positive linear functionals on $\mathcal{B}_0(G)$ are automatically continuous [Kr14]. As a consequence, the Gel'fand spectrum of $\mathcal{B}_0(G)$ and that of its completion coincide (the involution considered here is the one induced by complex conjugation). With some more data from the representation theory of G, Kreĭn gives this compact space a group structure. It becomes a compact topological group, sometimes called the *Bohr compactification* bG of G, and if G is compact then G and bG are canonically isomorphic.

Tannaka obtained similar duality results from a different perspective. He associated to a compact group G a monoid $\mathbf{T}(G)$ of *operations* on the class of representations of G. More explicitly, an element of $\mathbf{T}(G)$ is a family of operators $(u_\pi)_\pi$ where π ranges over all the finite-dimensional representations of G and u_π is an operator on the same Hilbert space as π. Moreover, the family must commute with representation morphisms and preserve the common operations on representations: sum, tensor product and conjugation. The monoid law is pointwise composition. This structure can be endowed with a natural topology and is in a fact a compact group canonically isomorphic to G [Ta39].

The wider class of *Roelcke-precompact* groups seems to retain a lot of the geometrical properties of compact groups. A topological group G is *Roelcke-precompact* if it is precompact in the Roelcke uniformity, in other words if for every open neighborhood U of the identity, there exists a finite subset F of G such that $G = UFU$. Roelcke-precompact Polish groups are receiving an increasing amount of interest. Cameron started the investigation by carrying out an extensive study of the dynamical properties of Roelcke-precompact permutation groups [Ca90]. Uspenskij [Us98, Us01] and Glasner [Gl12] showed that well-known groups, such as the unitary group of the separable Hilbert space or $\mathrm{Aut}(\mu)$ for an atomless standard Borel probability measure, are Roelcke-precompact and deduced strong properties such as minimality. [Ib16] and the pair of papers [BT16, BIT18] studied compactifications of such groups. More recently, Ibarlucìa showed that Roelcke-precompact Polish groups have Kazhdan's Property (T) [Ib21], like compact groups. Moreover, the Roelcke-precompact Polish groups that are *non-archimedean*, i.e. admit a basis of identity neighborhoods consisting of open subgroups, have seen their unitary representations fully classified [Ts12] in a way that resembles the Peter–Weyl Theorem. Indeed, let $\mathcal{M}_G$ be the set of left translates of open subgroups of G, called the *open cosets*. Then G acts continuously on $\mathcal{M}_G$ seen as a countable (see Remark 1.6(1)) discrete space and it gives rise to a unitary representation:

$$\Lambda_G \colon G \curvearrowright \ell^2(\mathcal{M}_G).$$

This canonical construction captures all the representation theory of G and is reminiscent of the left-regular representation of a compact group. More precisely, the following holds:

Fact 1. *Let G be a Roelcke-precompact non-archimedean Polish group.*

(1) *Every unitary representation of G splits into a sum of irreducible subrepresentations.*
(2) *Every irreducible unitary representation of G is isomorphic to a subrepresentation of Λ_G.*
(3) *The irreducible unitary representations of G separate points. Equivalently, Λ_G is faithful.*
(4) *$\widehat{G}$ is countable.*

The first item is contained in [Ts12, Th. 1.3]. The second item is a consequence of the same result and is proved as Corollary 1.9 below. The third is a straightforward consequence of (1) and (2), and is an analogue of the Gel'fand–Raĭkov Theorem for locally compact groups. The last one is a consequence of (2) and the fact that, $\mathcal{M}_G$ being countable, $\ell^2(\mathcal{M}_G)$ is separable.

The main source of examples for such groups is model theory. Indeed, by the classical Ryll-Nardzewski Theorem (see e.g. [Ho93, Th. 7.3.1]) the automorphism group of any $\aleph_0$-categorical model-theoretical structure is part of this class. This includes the group S_∞ of all permutations of the countable set, the group $\mathrm{Aut}(\mathbb{Q},<)$ of order preserving bijections of $\mathbb{Q}$, $\mathrm{Homeo}(2^{\mathbb{N}})$, $\mathrm{Aut}(R)$ the automorphism group of the random graph, or the group $\mathrm{GL}(\infty,q)$ of linear automorphisms of the countably infinite vector space over the finite field $\mathbb{F}_q$.

There are however fundamental properties that are lost in the Roelcke-precompact case: their irreducible representations can have infinite dimension and there is no Haar measure available. Using dynamical and model-theoretical properties of these groups, we are still able to carry out constructions similar to Tannaka's and Kreĭn's. In particular, we obtain an analogue of Kreĭn's technical result, which is the key to both dualities in both contexts.

To that end, we will consider several algebras of functions on G. They all live in $C_b(G)$, the normed algebra of bounded continuous complex-valued maps on G. $\mathcal{B}(G)$ will denote the set of matrix coefficients of G arising from all the unitary representation of G while $\mathcal{B}_0(G)$ will only contain those arising from finite sums of irreducible representations. Considering sums, tensor products and conjugates of representations, it is easily seen that $\mathcal{B}(G)$ is a subalgebra of $C_b(G)$, closed under complex conjugation. So is $\mathcal{B}_0(G)$ by Lemma 1.15. Let $\mathcal{A}_G$ be the linear span of the indicator maps of open cosets in G. This is in turn a subalgebra of $\mathcal{B}_0(G)$ closed under complex conjugation per Proposition 1.16 below. We insist on the following inclusions:

$$\mathcal{A}_G \subseteq \mathcal{B}_0(G) \subseteq \mathcal{B}(G) \subseteq C_b(G),$$

and on the fact that $\mathcal{A}_G$ is dense in $\mathcal{B}(G)$ (see Proposition 1.16). Note that in the compact case, this definition of $\mathcal{B}_0(G)$ coincides with the previous one. It seems Kreĭn's intent was to exhibit a somewhat *minimal* dual object, whose definition requires as little data as possible (in his case, the algebra $\mathcal{B}_0(G)$ linearly generated by a countable basis, with no need to remember the topology). Here, we will work with the smaller and more intrinsic algebra $\mathcal{A}_G$, which does not even need any harmonic analysis to be defined. Moreover, at the intersection of the compact case and ours, i.e. for G profinite and separable, $\mathcal{A}_G = \mathcal{B}_0(G)$. For all these reasons, and for the useful connections between $\mathcal{A}_G$ and model theory (see e.g. Corollary 2.3), we believe this is the right algebra to consider in this context. One of the main results of this paper is the following analogue of Kreĭn's technical result (restated and proved as Theorem 2.4):

THEOREM 1. *Let G be Roelcke-precompact non-archimedean Polish group. Multiplicative linear functionals $\mathcal{A}_G \to \mathbb{C}$ are automatically positive and continuous.*

Again, it implies that the Gel'fand spectra of $\mathcal{A}_G$ and its completion (i.e. the completion of $\mathcal{B}(G)$) coincide. Since this spectrum is compact, it cannot be isomorphic to G in full generality. However, it can be endowed with a structure of *semitopological* **-monoid*. This structure is known as the *Hilbert compactification* $\mathbf{H}(G)$ of G and has received quite some attention [GM14, BT16]. Along the way, we obtain an identification of the Hilbert compactification of G as the monoid $\mathbf{P}(\mathcal{M}_G)$ of partial elementary maps with algebraically closed domain on $\mathcal{M}_G$ seen as a homogeneous model-theoretical structure (complete definitions are given in Section 1). When G is the automorphism group of some $\aleph_0$-categorical structure $\mathcal{M}$, then $\mathcal{M}_G$ is very similar to $\mathcal{M}^{\mathrm{eq}}$, hence this identification can be seen as a reformulation of [BIT18, Th. 02].

Tannaka's approach also bears fruit in this context, providing another realization of $\mathbf{H}(G)$. Indeed, consider the category $\mathbf{Rep}(G)$ of unitary representations of G that are isomorphic to a finite sum of irreducible representations. Let $\mathbf{T}(G)$ be the monoid of families of operators $(u_\pi)_{\pi\in\mathbf{Rep}(G)}$ that commute with sums, tensor products and representation morphisms (complete definitions are given in Definition 3.5). We obtain the following (restated and proved as Theorem 3.6):

THEOREM 2. *Let G be a Roelcke-precompact non-archimedean Polish group. Then $\mathbf{T}(G)$ is a compact semitopological* **-monoid canonically isomorphic to the Hilbert compactification of G.*

Finally, we establish that the Hilbert compactification fully remembers the original group. Indeed, a Roelcke-precompact non-archimedean Polish group G is homeomorphic to the set of invertible elements of $\mathbf{H}(G)$ via the canonical map that sends g in G to the *evaluation map at* g. Moreover, we can form the category whose objects are the $\mathbf{Rep}(G)$ for every Roelcke-precompact non-archimedean Polish group G. With the right arrows (the *admissible* functors, see Definition 4.2), we have the following (corresponding to Theorems 4.1 and 4.3 respectively):

THEOREM 3.

(1) *Let G be a Roelcke-precompact non-archimedean Polish group. The canonical map $G \to \mathbf{H}(G)$ is a homeomorphic embedding that respects the algebraic structures. Its image is the set of invertibles of $\mathbf{H}(G)$.*
(2) *The contravariant functor* $\mathbf{Rep}$ *that sends a Roelcke-precompact non-archimedean Polish group to the category of its unitary representations is full and faithful, i.e. a duality.*

Numerous treatments, variants and generalizations of Tannaka–Kreĭn duality can be found in the literature. An extensive exposition of the theory, recounting both Kreĭn's and Tannaka's approaches successively, appears in [HR70]. A more modern take on Tannaka's point of view, phrased in the language of category theory, which also lists developments and applications, appears in [JS91].

Tannaka–Kreĭn duality has sprouted roots in different domains of study. One is category theory, with the works of Grothendieck, Deligne and Saavedra-Rivano among others, who introduced *Tannakian formalism* to reverse the process of the classical duality. The idea is that to a *Tannakian category* $\mathcal{C}$ can be associated an algebraic group G such that $\mathcal{C}$ is equivalent to $\mathbf{Rep}(G)$ (see for instance [Sa72]). More recently, Tannaka–Kreĭn duality theory has been reappropriated in a prolific way for the study of *quantum groups* and *knot theory*. It also appears in mathematical physics, related to the *superselection principle* in quantum field theory. See the survey [Va15].

This article is organized as follows. In the first section, we introduce notations and some necessary preliminary results. The second section is inspired from Kreĭn's approach to the duality and contains a proof of Theorem 1. In Section 3, where the formalism follows that of [JS91], we switch to Tannaka's point of view and use the automatic continuity property to prove Theorem 2. In the last section, we prove Theorem 3.

1. Preliminaries. In this section, we set the global framework by fixing notations and stating useful general facts. We start with dynamical and model-theoretical notions.

Let G be a *Polish group*, i.e. a separable and completely metrizable topological group. For subgroups U, V of G, we will write

$$U\backslash G/V = \{UgV : g \in G\}.$$

This is a partition of G. Note that if G is non-archimedean, then it is Roelcke-precompact if and only if $U\backslash G/U$ is finite for every open subgroup U of G.

We will need the dynamical notion of algebraicity, which coincides with the model-theoretical one in the $\aleph_0$-categorical case but not in general. It is only assumed that the definition of first-order structure, substructure and $\aleph_0$-categoricity, in the sense of [Ho93], are known. Anyway, the only structure that is actually involved here, $\mathcal{M}_G$, is relational, so a substructure of $\mathcal{M}_G$ is nothing more than a subset. All the structures considered will be assumed countable. In particular, by an $\aleph_0$-categorical structure, we always mean a countable one.

Definition 1.1. Let G be the automorphism group of some first-order structure $\mathcal{M}$ and $A \subseteq \mathcal{M}$.

(1) We denote by G_A the pointwise stabilizer of A and by $G_{(A)}$ the setwise stabilizer of A.

(2) If A is finite, an element $a \in \mathcal{M}$ is said to be *algebraic over A* if the orbit $G_A \cdot a$ is finite. We denote by $\operatorname{acl}(A)$ the *algebraic closure* of A, i.e. the set of all elements of $\mathcal{M}$ that are algebraic over A. If A is infinite, the algebraic closure of A is

$$\operatorname{acl}(A) = \bigcup \{\operatorname{acl}(B) : B \subseteq A \text{ finite}\}.$$

(3) A is said to be *algebraically closed* if $\operatorname{acl}(A) = A$.

Recall that acl is a closure operator, in particular it is non-decreasing and satisfies $\operatorname{acl}^2 = \operatorname{acl}$.

Definition 1.2. Let $\mathcal{M}$ be a first-order structure. A *partial elementary map* φ on $\mathcal{M}$ is an elementary map between substructures of $\mathcal{M}$, where elementary means a bijection that preserves all formulae in the language of $\mathcal{M}$ with parameters in the domain of φ.

We will see these maps as subsets of $\mathcal{M} \times \mathcal{M}$. Thus, for partial elementary maps s, s' on $\mathcal{M}$, $s \subseteq s'$ means s' extends s. Similarly, $s \cup s'$ denotes the unique common extension of s and s' to the substructure of $\mathcal{M}$ generated by $\operatorname{dom}(s) \cup \operatorname{dom}(s')$, if it exists.

We will denote by $\mathbf{P}(\mathcal{M})$ the set of partial elementary maps on $\mathcal{M}$ with algebraically closed domain and by $\mathrm{F}(\mathcal{M})$ the set of partial elementary maps on $\mathcal{M}$ with finite domain. Note that those sets are stable under *composition where defined* and inversion.

Finally, $\mathcal{M}$ is *homogeneous* if every element of $\mathrm{F}(\mathcal{M})$ extends to a *total* automorphism, i.e. an element of $\operatorname{Aut}(\mathcal{M})$.

We will be needing the following combinatorial result, a variant of B. H. Neumann's Lemma appearing as [BB$^+$76, Th. 1]:

Theorem 1.3 (B. H. Neumann's Lemma). *Let G be a group acting on a set X such that all orbits are infinite. For all finite subsets $A, B \subseteq X$, there exist $g \in G$ such that $g(A) \cap B = \emptyset$.*

More precisely, the following translation of the above theorem will be key in establishing the main result of Section 2.

Lemma 1.4. *Let G be the automorphism group of some homogeneous countable structure $\mathcal{M}$ and assume it is Roelcke-precompact. Let $s, s_1, \ldots, s_n$ be partial elementary map on $\mathcal{M}$ with finite domain such that*

$$\forall i \leqslant n, \qquad \operatorname{dom}(s_i) \not\subseteq \operatorname{acl}(\operatorname{dom}(s))$$

Then there exists $g \in G$ that extends s but none of the s_i's.

Proof. By homogeneity of $\mathcal{M}$, we can pick $g_0 \in G$ extending s. Fix also $x_i \in \operatorname{dom}(s_i) \setminus \operatorname{acl}(\operatorname{dom}(s))$ for every $i \leqslant n$. Using Roelcke-precompactness of G, Lemma 2.4 in [ET16] tells us that the action

$$G_{\operatorname{acl}(\operatorname{dom}(s))} \curvearrowright \mathcal{M} \setminus \operatorname{acl}(\operatorname{dom}(s))$$

only has infinite orbits. Now B. H. Neumann's Lemma above gives u in $G_{\mathrm{acl}(\mathrm{dom}(s))}$ such that

$$u(\{x_1,\dots,x_n\}) \cap \{g_0^{-1}(s_1(x_1)),\dots,g_0^{-1}(s_n(x_n))\} = \emptyset.$$

Then $g = g_0 u$ extends s but none of the s_i's. ■

REMARK 1.5. Let $\mathcal{M}$ be a homogeneous structure. $\mathbf{P}(\mathcal{M})$ is a *semitopological *-monoid compactification* of $G := \mathrm{Aut}(\mathcal{M})$, i.e.

- it is stable under *composition where defined* with a neutral element: $\mathrm{Id}_{\mathcal{M}}$,
- it has an involutive anti-automorphism
$$u \mapsto u^* = \{(y,x) : (x,y) \in u\},$$
- it can be endowed with a compact topology that makes composition separately continuous and the involution continuous,
- there is a continuous action $G \curvearrowright \mathbf{P}(\mathcal{M})$, namely by composition on the left,
- there is a homeomorphic embedding $G \hookrightarrow \mathbf{P}(\mathcal{M})$ with dense image, namely natural inclusion, such that on the image of G the involution restricts to inversion and the monoid law restricts to the product.

An extensive survey regarding such objects can be found in [GM14] and this precise construction is inspired from [BIT18, Prop. 3.8]. While additional assumptions appear in [BIT18], the same proofs apply. We reproduce them for completeness.

To define the topology, consider $K := \mathcal{M} \cup \{\infty\}$ the one-point compactification of the discrete structure $\mathcal{M}$. Elements $p \in K^K$ can be seen as partial maps $\mathcal{M} \to \mathcal{M}$, undefined wherever $p(x) = \infty$. The domain of p corresponds to the set $p^{-1}(\mathcal{M})$.

K^K is endowed with the product topology, which is compact and Hausdorff. Subbasic neighborhoods of an element $u \in \mathbf{P}(\mathcal{M})$ are of two kinds:

$$O^u_A = \{v \in \mathbf{P}(\mathcal{M}) : \forall x \in A,\, v(x) = u(x)\},$$

where $A \subseteq \mathrm{dom}(u)$ is finite, and

$$U^u_{B,C} = \{v \in \mathbf{P}(\mathcal{M}) : \forall x \in B,\, [x \in \mathrm{dom}(v) \Rightarrow v(x) \notin C]\},$$

where $B \subseteq \mathcal{M} \setminus \mathrm{dom}(u)$ and $C \subseteq \mathcal{M}$, both finite.

Clearly, $G \subseteq \mathbf{P}(\mathcal{M})$. Moreover, $\mathrm{Aut}(\mathcal{M})$ does not see the open sets of the second kind in the above description, hence the inclusion is indeed a homeomorphic embedding. In fact, $\mathbf{P}(\mathcal{M})$ is exactly the closure of G, hence is compact.

Indeed, fix $u \in \mathbf{P}(\mathcal{M})$. Let U be a neighborhood of u. By the above description of the topology, we can assume that there exist $A, B, C \subseteq \mathcal{M}$ finite such that $A \subseteq \mathrm{dom}(u)$, $B \subseteq \mathcal{M} \setminus \mathrm{dom}(u)$ and

$$U = O^u_A \cap U^u_{B,C}.$$

Since u is elementary and $\mathcal{M}$ is homogeneous, there exists $g \in G$ such that $u_{|A} = g_{|A}$. Note that $\operatorname{acl}(A) \subseteq \operatorname{acl}(\operatorname{dom}(u)) = \operatorname{dom}(u)$, hence $B \subseteq \mathcal{M} \setminus \operatorname{acl}(A)$. By definition, the (well-defined) action $G_A \curvearrowright \mathcal{M} \setminus \operatorname{acl}(A)$ only has infinite orbits. Using B. H. Neumann's Lemma 1.3, there exists $h \in G_A$ such that $h(B) \cap g^{-1}(C) = \emptyset$. Then $gh \in U \cap G$, proving that $\mathbf{P}(\mathcal{M}) \subseteq \overline{G}$.

For the reverse inclusion, let $u \in \overline{G}$. Considering open sets of the first kind above, finite restrictions of u all extend to automorphism of $\mathcal{M}$: u is thus an elementary map. Toward a contradiction, suppose there exists $a \in \operatorname{acl}(\operatorname{dom}(u)) \setminus \operatorname{dom}(u)$. Fix $A \subseteq \operatorname{dom}(u)$ finite such that $a \in \operatorname{acl}(A)$ and let $g_0 \in G$ be any extension of $u_{|A}$. Set $C := g_0 G_A \cdot a$, which is a finite subset of $\mathcal{M}$. Then $u \in O^u_A \cap U^u_{\{a\},C}$ while $G \cap O^u_A \cap U^u_{\{a\},C} = g_0 G_A \cap U^u_{\{a\},C} = \emptyset$, a contradiction to $u \in \overline{G}$.

Remark 1.6. Let G be a Roelcke-precompact non-archimedean Polish group. We recall well-known facts about the structure $\mathcal{M}_G$ and give sketches of proofs.

(1) $\mathcal{M}_G$ is countable. Indeed, let (U_n) be a countable basis of open neighborhoods of 1_G consisting of open subgroups and let $D \subseteq G$ be a countable dense subset. Let U be an arbitrary open subgroup of G. There exists $n \in \mathbb{N}$ such that $U_n \subseteq U$. Since G is Roelcke-precompact, there is $F \subseteq D$ finite such that $U = U_n F U_n$. This only allows for countably many distinct open subgroups. Moreover, G/U is a collection of disjoint non-empty open sets. Since G is separable, G/U must be countable.

(2) G naturally acts on the left of $\mathcal{M}_G$. This action is continuous and the associated morphism $G \to \operatorname{Sym}(\mathcal{M}_G)$ is a homeomorphic embedding. Because G and $\operatorname{Sym}(\mathcal{M}_G)$ are both Polish, G must be closed in $\operatorname{Sym}(\mathcal{M}_G)$. Thus, after naming the orbits of the diagonal actions $G \curvearrowright \mathcal{M}_G^n$ for every $n \in \mathbb{N}$, we can view $\mathcal{M}_G$ as a discrete first-order homogeneous structure such that
$$G = \operatorname{Aut}(\mathcal{M}_G).$$

(3) Let gU, hV be cosets of G. Then $gU \cap hV \neq \emptyset$ if and only if there is $f \in G$ such that $fU = gU$ and $fV = hV$. In particular, if $gU \cap hV$ is non-empty, then it is a left translate of $U \cap V$.

(4) Suppose that $G = \operatorname{Aut}(\mathcal{M})$ for some countable and homogeneous structure $\mathcal{M}$. There is a natural map $\mathrm{F}(\mathcal{M}) \to \mathcal{M}_G$. Indeed, for s in $\mathrm{F}(\mathcal{M})$, fix an extension $g_0 \in G$. Then a given automorphism g of $\mathcal{M}$ extends s if and only if $g \in g_0 G_{\operatorname{dom}(s)}$ and this coset does not depend on the choice of g_0. Similarly, if $\operatorname{dom}(s) = A \subseteq B$ where A and B are finite, there is a bijection between G_A/G_B and $\{s' \in \mathrm{F}(\mathcal{M}),\ s \subseteq s' \text{ and } \operatorname{dom}(s') = B\}$, namely $s' \mapsto g_0^{-1} g G_B$ where g is any extension of s' (this map does depend on the choice of g_0).

Switching to representation theory, we will write $\mathrm{Hilb}(G)$ for the closure of $\mathcal{B}(G)$ in $C_b(G)$ with respect to the norm topology. This is called the *Hilbert algebra* of G. The *Hilbert compactification* $\mathbf{H}(G)$ of G is the Gel'fand spectrum of $\mathrm{Hilb}(G)$, i.e. the weak*-compact space of non-zero multiplicative linear functionals on $\mathrm{Hilb}(G)$ that commute with complex conjugation. Recall that these are automatically continuous since $\mathrm{Hilb}(G)$ is a C^*-algebra.

Given two representations π, π' of G, a morphism of representations between π and π', or *intertwining operator*, is a bounded operator

$$h\colon \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$$

that commutes with the action of G:

$$\forall g \in G, \qquad h \circ \pi(g) = \pi'(g) \circ h.$$

The set of such operators will be denoted by $\mathrm{Hom}(\pi, \pi')$ or $\mathrm{Hom}_G(\mathcal{H}_\pi, \mathcal{H}_{\pi'})$. An *isomorphism* between π and π' is an element $h \in \mathrm{Hom}(\pi, \pi')$ that is also a surjective isometry. $\widehat{G}$ will denote the set of isomorphism classes of irreducible representations of G.

We now state the following classical theorem for later references; see for instance [Fo95, Th. 3.5].

Theorem 1.7 (Schur's Lemma). *Let G be a topological group and π a continuous unitary representation of G. Then π is irreducible if and only if* $\mathrm{Hom}(\pi, \pi) = \mathbb{C} \cdot 1$.

The following are consequences of the results in [Ts12]. For the definition and properties of *induced representations*, we refer the reader to [Fo95, Section 6]. They are usually defined in the locally compact setting using invariant measures but the basic results still hold in this context with the same proofs, and the formalism is even simpler. Indeed, our quotient spaces are all discrete, hence can be endowed with the counting measure. The distinction between isomorphic representations is often omitted.

Lemma 1.8. *Let G be a Roelcke-precompact non-archimedean Polish group and let π be an irreducible representation of G. There exists an open subgroup V of G such that π is isomorphic to a subrepresentation of the associated quasi-regular representation $\lambda_{G/V}\colon G \curvearrowright \ell^2(G/V)$.*

Proof. By [Ts12, Th. 4.2], π is isomorphic to an induced representation of the form $\mathrm{Ind}_H^G(\sigma)$, where H is an open subgroup of G and σ is an irreducible representation of H that factors through a finite quotient of H. More explicitly, there exists an open normal subgroup V of H of finite index such that σ is the pullback of an irreducible representation of the finite group H/V.

The quasi-regular representations $\lambda_{G/V}\colon G \curvearrowright \ell^2(G/V)$ and $\lambda_{H/V} : H \curvearrowright \ell^2(H/V)$ are linked in the following way:

$$\lambda_{G/V} = \mathrm{Ind}_V^G(\mathbf{1}_V) = \mathrm{Ind}_H^G(\mathrm{Ind}_V^H(\mathbf{1}_V)) = \mathrm{Ind}_H^G(\lambda_{H/V}), \tag{1}$$

where $\mathbf{1}_V$ denotes the trivial one-dimensional representation of V. The first and last equalities are basic properties of induction and the second one is given by the theorem on induction by stages (see [Fo95, Th. 6.14]). Now, applying the Peter–Weyl Theorem to the finite group H/V, we find that σ is a subrepresentation of $\lambda_{H/V}$. Since induction preserves subrepresentations, we deduce from (1) that $\pi = \mathrm{Ind}_H^G(\sigma)$ is a subrepresentation of $\lambda_{G/V} = \mathrm{Ind}_H^G(\lambda_{H/V})$. ∎

COROLLARY 1.9. *Let G be a Roelcke-precompact non-archimedean Polish group. Every irreducible representation of G is a subrepresentation of*

$$\Lambda_G \colon G \curvearrowright \ell^2(\mathcal{M}_G),$$

which is faithful.

Proof. Clearly, $\lambda_{G/V}$ is a subrepresentation of Λ_G for every $V \leqslant G$ open. The faithfulness of Λ_G is easily seen directly and is also a consequence of the fact that closed permutation groups satisfy the Gel'fand–Raĭkov Theorem [Ts12, Th. 1.1]. ∎

The following lemmas are probably well known but we sketch a proof for lack of a reference in such generality.

LEMMA 1.10. *Let G be a topological group, let $\pi_1, \dots, \pi_n$ be irreducible representations of G and let $\pi := \pi_1 \oplus^\perp \cdots \oplus^\perp \pi_n$. If σ is a non-zero subrepresentation of π, then σ contains an irreducible representation isomorphic to π_i for some i.*

Proof. Let $\mathcal{H}, \mathcal{H}_1, \dots, \mathcal{H}_n$ be the underlying spaces of σ and $\pi_1, \dots, \pi_n$ respectively. Let $p, p_1, \dots p_n$ be the orthogonal projections on $\mathcal{H}, \mathcal{H}_1, \dots, \mathcal{H}_n$ respectively. Then $p_1 + \cdots + p_n = 1$ hence $p = pp_1 + \cdots + pp_n$. There must exist i such that $pp_i \neq 0$. Then pp_i is a non-zero intertwining operator between π_i and σ. Applying [BH20, Lemma 1.A.5], we get the existence of a subrepresentation of σ (the closure of $\mathrm{im}\, pp_i$) which is isomorphic to a subrepresentation of π_i (the orthogonal of $\ker pp_i$). This suffices to conclude the proof since p_i is irreducible. ∎

LEMMA 1.11. *Let G be a topological group and let $\pi \colon G \curvearrowright \mathcal{H}$ be a representation of G that splits as a finite sum of irreducible subrepresentations. There is a unique decomposition*

$$\mathcal{H} = \bigoplus_{\lambda \in \widehat{G}} \mathcal{H}_\lambda$$

into closed invariant subspaces where for every $\lambda \in \widehat{G}$, all the irreducible subrepresentations of $\mathcal{H}_\lambda$ have isomorphism type λ.

This is called the isotypical decomposition *of π.*

Proof. For existence, write $\pi = \pi_1 \oplus^\perp \cdots \oplus^\perp \pi_n$ for some irreducible representations $\pi_i \colon G \curvearrowright \mathcal{H}_i$ and simply let

$$\mathcal{H}_\lambda := \bigoplus_{i \in I,\, \pi_i \in \lambda} \mathcal{H}_i.$$

It has the desired property by Lemma 1.10.

As for uniqueness, suppose another such decomposition is given:

$$\mathcal{H} = \bigoplus_{\lambda \in \widehat{G}} \mathcal{H}_\lambda = \bigoplus_{\lambda \in \widehat{G}} \mathcal{H}'_\lambda.$$

Let $\lambda \in \widehat{G}$. It suffices to show that $\mathcal{H}_\lambda \subseteq \mathcal{H}'_\lambda$, which in turn reduces to showing that if $\pi_i \in \lambda$ then $\mathcal{H}_i \subseteq \mathcal{H}'_\lambda$. So let $i \leqslant n$ be such that $\pi_i \in \lambda$. Let p be the orthogonal projection on $\mathcal{H}_i$ and, for every $\lambda \in \widehat{G}$, let p_λ be the projection on $\mathcal{H}'_\lambda$. Then $\sum_{\lambda \in \widehat{G}} p_\lambda = 1$ and

$$p = \sum_{\lambda \in \widehat{G}} p p_\lambda. \tag{2}$$

But if $\mu \neq \lambda$, then $pp_\mu = 0$. Otherwise, by [BH20, Lemma 1.A.5], it implies the existence of a non-zero subrepresentation of π_i (the closure of $\operatorname{im} pp_\lambda$) which is isomorphic to a subrepresentation of $G \curvearrowright \mathcal{H}'_\mu$ (the orthogonal of $\ker pp_\lambda$). By choice of $\mathcal{H}'_\mu$ and irreducibility of π_i, we get $\pi_i \in \mu$, a contradiction. Equation (2) becomes $p = pp_\lambda$, i.e. $\mathcal{H}_i \subseteq \mathcal{H}'_\lambda$. ∎

Lemma 1.12. *Let G be a topological group, let π be an irreducible representation of G and let $n \in \mathbb{N}$. Every subrepresentation of $n \cdot \pi := \bigoplus_{1 \leqslant i \leqslant n} \pi$ is isomorphic to $k \cdot \pi$ for some $k \leqslant n$.*

Proof. Let σ be a subrepresentation of $n \cdot \pi$. Using Lemma 1.10 and Zorn's Lemma, we see that σ is also isomorphic to a multiple of π, say $\sigma \simeq k \cdot \pi$ for some possibly infinite cardinal k. However, $\operatorname{Hom}(\sigma, \sigma) = k^2 \cdot \operatorname{Hom}(\pi, \pi) \simeq k^2 \cdot \mathbb{C}$ embeds into $\operatorname{Hom}(n \cdot \pi, n \cdot \pi) \simeq n^2 \cdot \operatorname{Hom}(\pi, \pi) \simeq n^2 \cdot \mathbb{C}$. Thus $k \leqslant n$. The converse is trivial. ∎

Lemma 1.13. *Let G be a topological group and let $\pi \colon G \curvearrowright \mathcal{H}$ be a representation of G that splits as finite sum of irreducibles. Every subrepresentation of π also splits as a finite sum of irreducibles.*

Proof. Let $\sigma \subseteq \pi$ and consider the isotypical decomposition of π given by Lemma 1.11: $\mathcal{H} = \bigoplus_{\lambda \in \widehat{G}} \mathcal{H}_\lambda$. Let p be the orthogonal projection on the space $\mathcal{K} \subseteq \mathcal{H}$ underlying σ and, for every $\lambda \in \widehat{G}$, let p_λ be the orthogonal projection on $\mathcal{H}_\lambda$. Then $1 = \sum_{\lambda \in \widehat{G}} p_\lambda$, hence

$$p = \sum_{\lambda, \mu \in \widehat{G}} p_\mu p p_\lambda. \tag{3}$$

If $\mu, \lambda \in \widehat{G}$ and $p_\mu p p_\lambda \neq 0$, then $\mathrm{Hom}_G(\mathcal{H}_\lambda, \mathcal{H}_\mu) \neq 0$. In particular, by [BH20, Lemma 1.A.5], there is a non-zero subrepresentation of $\mathcal{H}_\lambda$ isomorphic to a subrepresentation of $\mathcal{H}_\mu$. Using Lemma 1.10 and the properties of the isotypical decomposition, we deduce that $\lambda = \mu$. Thus, (3) becomes $p = \sum_{\lambda \in \widehat{G}} p_\lambda p p_\lambda$ and we get $\mathcal{K} = \bigoplus_{\lambda \in \widehat{G}} \mathcal{K} \cap \mathcal{H}_\lambda$. We conclude the proof using Lemma 1.12. ∎

LEMMA 1.14. *Let G be a Roelcke-precompact non-archimedean Polish group and let V be an open subgroup of G. The associated quasi-regular representation $G \curvearrowright \ell^2(G/V)$ splits as a finite sum of irreducible subrepresentations.*

Proof. Let H be the *commensurator* of V,

$$H = \{g \in G : [V, V \cap gVg^{-1}] < \infty \ \&\ [gVg^{-1}, V \cap gVg^{-1}] < \infty\}.$$

The hypotheses on G imply that V has finite index in H [Ts12, Lem. 2.7]. Hence,

$$V' := \bigcap_{h \in H} hVh^{-1}$$

is in fact a finite intersection of conjugates of V. Thus, it is open and has finite index in V by definition of H.

We have obtained $V' \subseteq V$ open and, as a subgroup of H, normal with finite index. In particular, H/V' is a finite group, hence the quasi-regular representation $\lambda_{H/V'}$ of H is finite-dimensional. It thus splits as a finite sum of irreducible subrepresentations:

$$\lambda_{H/V'} = \bigoplus_{1 \leqslant i \leqslant n} \sigma_i.$$

Then, by the basic properties of induction,

$$\lambda_{G/V'} = \mathrm{Ind}_H^G(\lambda_{H/V}) = \bigoplus_{1 \leqslant i \leqslant n} \mathrm{Ind}_H^G(\sigma_i).$$

Finally, each of the $\mathrm{Ind}_H^G(\sigma_i)$ is irreducible by [Ts12, Prop. 4.1]. Moreover, since $V' \subseteq V$ with finite index, $\lambda_{G/V}$ is a subrepresentation of $\lambda_{G/V'}$, hence also splits as a finite sum of irreducibles by Lemma 1.13. ∎

COROLLARY 1.15. *Let G be a Roelcke-precompact non-archimedean Polish group and let π_1, π_2 be two irreducible representations of G. Then $\pi_1 \otimes \pi_2$ decomposes as a finite sum of irreducible subrepresentations.*

Proof. Using Lemma 1.8, we can find two open subgroups V_1 and V_2 of G such that π_i is a subrepresentation of λ_{G/V_i}, $i = 1, 2$.

Since there is a natural surjective map

$$(V_1 \cap V_2)\backslash G/(V_1 \cap V_2) \to V_1\backslash G/V_2$$

and G is Roelcke-precompact, $V_1\backslash G/V_2$ is finite.

Write

$$V_1\backslash G/V_2 = \{V_1f_1V_2, \dots, V_1f_nV_2\}$$

for some $f_1, \dots, f_n \in G$ (in an injective way). Then

$$\begin{aligned}\ell^2(G/V_1) \otimes \ell^2(G/V_2) &\simeq \ell^2(G/V_1 \times G/V_2)\\ &\simeq \bigoplus_{1\leqslant i\leqslant n} \ell^2(G \cdot (V_1, f_iV_2))\\ &\simeq \bigoplus_{1\leqslant i\leqslant n} \ell^2(G/(V_1 \cap f_iV_2f_i^{-1})).\end{aligned}$$

In particular, we have shown that $\pi_1 \otimes \pi_2$ is a subrepresentation of a finite sum of quasi-regular representations. We conclude the proof using Lemmas 1.14 and 1.13. ■

Recall that $\mathcal{A}_G$ is the linear span of the indicator maps of open cosets in G. From the above observations follow the claims made about $\mathcal{A}_G$ in the introduction:

PROPOSITION 1.16. *Let G be a Roelcke-precompact non-archimedean Polish group. Then $\mathcal{A}_G$ is a subalgebra of $\mathcal{B}_0(G)$ which is closed under the involutions induced by inversion in G and complex conjugation. Hence we have the following chain of algebras:*

$$\mathcal{A}_G \subseteq \mathcal{B}_0(G) \subseteq \mathcal{B}(G)$$

with $\mathcal{A}_G$ dense in $\mathcal{B}(G)$.

Proof. Let gV be an open coset in G and let $\mathbb{1}_{gV}\colon G \to \{0,1\}$ be the indicator map of gV. Denote by $\delta_V, \delta_{gV} \in \ell^2(G/V)$ the Dirac functions at V and gV respectively, and by $\lambda_{G/V}\colon G \curvearrowright \ell^2(G/V)$ the quasi-regular representation of G. Then, for every $h \in G$,

$$\mathbb{1}_{gV}(h) = \langle \lambda_{G/V}(h)\delta_V, \delta_{gV}\rangle.$$

This, in combination with Lemma 1.14, shows that $\mathbb{1}_{gV}$ belongs to $\mathcal{B}_0(G)$. Thus $\mathcal{A}_G \subseteq \mathcal{B}_0(G)$.

By Remark 1.6(3), $\mathcal{A}_G$ is indeed a subalgebra of $\mathcal{B}_0(G)$. It is easily seen on the generators that $\mathcal{A}_G$ is stable under both of the involutions mentioned.

As for density, by Fact 1(1), every matrix coefficient of G can be approximated by linear combinations of matrix coefficients arising from irreducible representations. In turn, by Lemma 1.8, every matrix coefficient arising from an irreducible representation of G also comes from a representation of the form $\ell^2(G/V)$ for some open subgroup $V \leqslant G$. Finally, given an open subgroup V of G, the family $(\delta_{gV})_{gV\in G/V}$ is the canonical Hilbert basis of $\ell^2(G/V)$. Thus, linear combinations of the matrix coefficients associated to these vectors, which all appear in the list of generators of $\mathcal{A}_G$, can approximate all matrix coefficients coming from $G \curvearrowright \ell^2(G/V)$. This shows the density of $\mathcal{A}_G$ in $\mathcal{B}(G)$. ∎

In the next sections, we give two different characterizations of $\mathbf{H}(G)$ for a Roelcke-precompact non-archimedean Polish group G. For a detailed study of the Hilbert compactification of such groups, see [BIT18].

2. Application to Kreĭn's duality. In [Kr14], Kreĭn associated to a compact group G a dual object consisting of the algebra $\mathcal{B}_0(G)$ of finite-dimensional matrix coefficients of G together with a specific *structured* linear basis. More precisely, fix a compact group G and choose a family $(\pi_\lambda)_{\lambda\in\widehat{G}}$ of representatives of the isomorphism classes for the irreducible representations of G. Recalling that these are finite-dimensional, fix an orthonormal basis $(e_1^\lambda, \dots, e_{d_\lambda}^\lambda)$ of the underlying Hilbert space of each representative. For every $\lambda \in \widehat{G}$ and $g \in G$, the matrix of the operator $\pi_\lambda(g)$ in the basis (e_i^λ) is given by

$$[\pi_\lambda(g)]_{(e_i^\lambda)} = (f_{i,j}^\lambda(g))_{1\leqslant i,j\leqslant d_\lambda},$$

where $f_{i,j}^\lambda(g) = \langle \pi_\lambda(g)e_j^\lambda, e_i^\lambda\rangle$ for every $\lambda \in \widehat{G}$ and all $i, j \leqslant d_\lambda$. It is a consequence of the Peter–Weyl Theorem that the following is a linear basis of $\mathcal{B}_0(G)$:

$$B = \big((f_{i,j}^\lambda(g))_{1\leqslant i,j\leqslant d_\lambda}\big)_{\lambda\in\widehat{G}}.$$

Kreĭn's dual object is the pair $(\mathcal{B}_0(G), B)$. The properties of such a pair can be abstracted and such objects are now called *Kreĭn algebras*. Note that the arrangement of the basis in definite squares is part of the data. In particular, an isomorphism of Kreĭn algebras is an isomorphism of algebras that moreover exchanges the bases while preserving the coordinates. Of course, different choices for (π_λ) and the bases yield isomorphic Kreĭn algebras.

To recover G from this object, one of the key steps in Kreĭn's duality is that positive linear functionals on $\mathcal{B}_0(G)$ are automatically continuous. As a consequence, the Gel'fand spectrum $\mathbf{S}_{\mathcal{B}_0(G)}$ of $\mathcal{B}_0(G)$ coincides with that of its completion. Since this completion is precisely $C(G)$ [PW27], one can recover the compact topological space G through Gel'fand duality. Furthermore, the composition law is recovered from the structured basis. Indeed, it can be redefined by setting, for all $\phi, \psi \in \mathbf{S}_{\mathcal{B}_0(G)}$ and every element $f_{i,j}^\lambda$ of the basis,

$$\phi\cdot\psi(f_{i,j}^\lambda) = \sum_{1\leqslant k\leqslant d_\lambda} \phi(f_{i,k}^\lambda)\psi(f_{k,j}^\lambda).$$

In this context, the natural map $G \to \mathbf{S}_{\mathcal{B}_0(G)}$ is a topological group isomorphism.

We describe a similar construction in the Roelcke-precompact non-archimedean and Polish case where we emphasize the following major differences:

- irreducible representations are often *infinite*-dimensional,
- $\mathcal{B}(G)$ is not dense in $C(G)$ in general.

The key result of this section is the automatic continuity of multiplicative linear functionals on $\mathcal{A}_G$, an analogue of the central technical lemma in the establishment of the original duality. We also obtain a model-theoretic description of $\mathbf{H}(G)$ as $\mathbf{P}(\mathcal{M}_G)$.

Recalling Fact 1, we consider the "universal" representation Λ_G given by the action $G \curvearrowright \ell^2(\mathcal{M}_G)$ and form the matrix coefficients obtained from the canonical Hilbert basis $(\delta_x)_{x\in\mathcal{M}_G}$ of $\ell^2(\mathcal{M}_G)$. These are all the maps of the form

$$f_{x,y}\colon G \to \mathbb{C}, \qquad g \mapsto \langle g\cdot\delta_y, \delta_x\rangle = \begin{cases} 1 & \text{if } g(y) = x, \\ 0 & \text{otherwise,} \end{cases}$$

for $x, y \in \mathcal{M}_G$. Note that these are exactly the indicator maps of open cosets in G together with the zero function, hence generate $\mathcal{A}_G$. Indeed, if $x = hU$ and $y = kV$, then $g(y) = x$ exactly when $U = V$ and $g \in hk^{-1}(kUk^{-1})$.

Alternatively, $\mathcal{A}_G$ can be described in terms of finite partial elementary maps on $\mathcal{M}_G$. Indeed, define for every $s \in \mathrm{F}(\mathcal{M}_G)$ a map

$$e_s\colon G \to \mathbb{C}, \qquad g \mapsto \begin{cases} 1 & \text{if } g \text{ extends } s, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from Remark 1.6(4) that these are also exactly the indicator maps of open cosets in G. Note that for $s, s' \in \mathrm{F}(\mathcal{M}_G)$, we have

$$e_s \cdot e_{s'} = \begin{cases} e_{s\cup s'} & \text{if } s \cup s' \in \mathrm{F}(\mathcal{M}_G), \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

A special case of interest is when G can be presented as the automorphism group of some $\aleph_0$-categorical structure $\mathcal{M}$ that admits *weak elimination of imaginaries*. For the formal definition, we refer the reader to [Ho93, Section 4.2]. Informally, an $\aleph_0$-categorical structure $\mathcal{M}$ weakly eliminates imaginaries if we can recover all open subgroups of $G = \mathrm{Aut}(\mathcal{M})$ from the action $G \curvearrowright \mathcal{M}$ up to finite index. More precisely, if $\mathcal{M}$ is an $\aleph_0$-categorical structure, the following property can be taken as a definition of *weakly eliminating imaginaries* (see [Ts12, Lem. 5.1]): For every open subgroup U of G, there exists a unique algebraically closed finite substructure A of $\mathcal{M}$ such that

$$G_A \subseteq U \subseteq G_{(A)}.$$

Note that $G_{(A)}/G_A$ is isomorphic to $\mathrm{Aut}(A)$, hence finite. In particular, G_A has finite index in U. It is a classical fact that all the examples from the introduction weakly eliminate imaginaries (see for example [Ho93, Section 4.2]).

In that case, we can extract a canonical linear basis from the previous generating family of $\mathcal{A}_G$. Namely, defining $e'_s\colon G \to \{0,1\}$ as above for every $s \in \mathrm{F}(\mathcal{M})$, we have the following:

PROPOSITION 2.1. *Let $\mathcal{M}$ be a countable $\aleph_0$-categorical structure that admits weak elimination of imaginaries and let $G = \mathrm{Aut}(\mathcal{M})$. Then*

$$B := \big(e'_s : s \in \mathrm{F}(\mathcal{M}),\ \mathrm{dom}(s) = \mathrm{acl}(\mathrm{dom}(s))\big)$$

is a linear basis of $\mathcal{A}_G$ (and a subfamily of the previous generating family).

Proof. First, recall that $\aleph_0$-categorical structures are homogeneous and have finite algebraicity: algebraic closures of finite sets are finite. Hence, using Remark 1.6(4), the elements of B are indeed indicator functions of open cosets, namely of all cosets associated to subgroups of the form $G_{\mathrm{acl}(A)}$ for $A \subseteq \mathcal{M}$ finite.

Now if U is an open subgroup of G, we can find by weak elimination of imaginaries a finite algebraically closed substructure $A \subset \mathcal{M}$ such that $V := G_{\mathrm{acl}(A)} \subseteq U$ with finite index. Then, writing

$$U = u_1 V \sqcup \cdots \sqcup u_n V$$

for some $u_1, \dots, u_n \in U$, we have, for every $g \in G$,

$$\mathbb{1}_{gU} = \mathbb{1}_{gu_1V} + \cdots + \mathbb{1}_{gu_nV},$$

where $\mathbb{1}_X$ denotes the indicator function of the set $X \subseteq G$. In particular, B generates $\mathcal{A}_G$.

It remains to see that B is free. Assume we have an equation of the form

$$\lambda_1 e'_{s_1} + \cdots + \lambda_n e'_{s_n} = 0 \tag{5}$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and distinct $s_1, \dots, s_n \in \mathrm{F}(\mathcal{M})$ with algebraically closed domains. Let i be such that $\mathrm{dom}(s_i)$ is minimal for inclusion. Using Lemma 1.4, there exists $g \in G$ that extends s_i without extending s_j for any $j \neq i$ such that $\mathrm{dom}(s_j) \nsubseteq \mathrm{dom}(s_i)$. Note that if $j \neq i$ is such that $\mathrm{dom}(s_j) \subseteq \mathrm{dom}(s_i)$ then the domains are equal by choice of i and there exists $a \in \mathrm{dom}(s_i)$ such that $s_j(a) \neq s_i(a) = g(a)$. Thus g only extends s_i. Evaluating the above linking equation (5) on g yields $\lambda_i = 0$. We conclude the proof by induction on n. ∎

The main object of study in this section is the Gel'fand spectrum of $\mathcal{A}_G$, denoted $\mathbf{S}_{\mathcal{A}_G}$, that is, the non-zero multiplicative linear functionals on $\mathcal{A}_G$ that commute with complex conjugation. In the following lemma, we do not need to assume that G is Roelcke-precompact.

LEMMA 2.2. *Let G be a non-archimedean Polish group. Let $\phi\colon \mathcal{A}_G \to \mathbb{C}$ be a non-zero multiplicative linear functional. There exists a unique partial* *elementary map u_ϕ on $\mathcal{M}_G$ with algebraically closed domain such that*

$$\forall s \in \mathrm{F}(\mathcal{M}_G), \qquad \phi(e_s) = \begin{cases} 1 & \text{if } u_\phi \text{ extends } s, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

Proof. First note that for every $s \in \mathrm{F}(\mathcal{M}_G)$, $\phi(e_s) \in \{0,1\}$ since $e_s^2 = e_s$. Then, let

$$u_\phi = \bigcup_{\substack{s \in \mathrm{F}(\mathcal{M}_G) \\ \phi(e_s)=1}} s.$$

We will show that u_ϕ satisfies the claim.

• u_ϕ *is a function*: Let $x \in \mathcal{M}_G$ be such that there exist $s, s' \in \mathrm{F}(\mathcal{M}_G)$ with $x \in \mathrm{dom}(s) \cap \mathrm{dom}(s')$ and $\phi(e_s) = 1 = \phi(e_{s'})$. By multiplicativity of ϕ,

$$\phi(e_s e_{s'}) = \phi(e_s)\phi(e_{s'}) = 1.$$

In particular, $e_s e_{s'} \neq 0$, which implies the existence of $g \in G$ that extends both s and s'. Thus $s(x) = s'(x)$ and u_ϕ is a partial function $\mathcal{M}_G \to \mathcal{M}_G$.

• u_ϕ *is a partial elementary map*: Since $\mathcal{M}_G$ is a relational structure, every subset of $\mathcal{M}_G$ is a substructure. Thus, it is enough to show that each restriction of u_ϕ to a finite set extends to an automorphism of $\mathcal{M}_G$. To that end, let $A = \{a_1, \ldots, a_n\} \subseteq \mathrm{dom}(u_\phi)$ be finite. By definition of u_ϕ, there exist $s_1, \ldots, s_n$ in $\mathrm{F}(\mathcal{M}_G)$ such that for all $i \leqslant n$, $a_i \in \mathrm{dom}(s_i)$ and $\phi(e_{s_i}) = 1$. Up to replacing s_i with $s_{i|A}$, we can assume $\mathrm{dom}(s_i) \subseteq A$ for every $i \leqslant n$. Using the multiplicativity of ϕ, we get

$$\phi(e_{s_1} \cdots e_{s_n}) = \phi(e_{s_1}) \cdots \phi(e_{s_n}) = 1.$$

In particular, $e_{s_1} \cdots e_{s_n} \neq 0$ and there exists $g \in G$ a common extension of $s_1, \ldots, s_n$ hence of $u_{\phi|A}$. Note also that, as

$$e_{s_1} \cdots e_{s_n} = e_{s_1 \cup \cdots \cup s_n} = e_{u_{\phi|A}},$$

we have $\phi(e_{u_{\phi|A}}) = 1$.

• $\mathrm{dom}(u_\phi)$ *is algebraically closed*: Let $A \subseteq \mathrm{dom}(u_\phi)$ be finite and $b \in \mathrm{acl}(A)$. As noted above, $\phi(e_{u_{\phi|A}}) = 1$ and there exists $g_0 \in G$ that extends $u_{\phi|A}$. An element $g \in G$ that coincides with u_ϕ on A lies in $g_0 G_A$, hence must send b into the finite set $g_0 G_A \cdot b$. In other terms, writing

$$g_0 G_A \cdot b = \{g_0(b), \ldots, g_n(b)\}$$

for some $g_1, \ldots, g_n \in G$ such that the $g_i(b)$ are pairwise distinct, we have

$$e_{u_{\phi|A}} = e_{g_{0|A\cup\{b\}}} + \cdots + e_{g_{n|A\cup\{b\}}}.$$

Applying ϕ to the above equation gives, by linearity,

$$1 = \phi(e_{g_{0|A\cup\{b\}}}) + \cdots + \phi(e_{g_{n|A\cup\{b\}}}).$$

Since ϕ is $\{0,1\}$-valued at idempotents, there exists $i \leqslant n$ such that $\phi(e_{g_{i|A\cup\{b\}}}) = 1$. This shows that b lies in $\mathrm{dom}(u_\phi)$, which is thus algebraically closed.

Now that we have built u_ϕ, recalling that $\phi(e_{u_{\phi|A}}) = 1$ for every finite subset A of $\mathcal{M}_G$, the lemma is proved. ∎

In the Roelcke-precompact and Polish case, this construction yields an identification of the Gel'fand spectrum $\mathbf{S}_{\mathcal{A}_G}$ of $\mathcal{A}_G$:

Corollary 2.3. *Let G be a Roelcke-precompact non-archimedean Polish group. The map*

$$\mathbf{S}_{\mathcal{A}_G} \to \mathbf{P}(\mathcal{M}_G), \qquad \phi \mapsto u_\phi,$$

is a homeomorphism.

Proof. It is straightforward from the previous result that the map is injective. We now show surjectivity, i.e. if $u \in \mathbf{P}(\mathcal{M}_G)$ is fixed, the conditions

$$\forall s \in \mathrm{F}(\mathcal{M}_G), \qquad \phi(e_s) = \begin{cases} 1 & \text{if } u \text{ extends } s, \\ 0 & \text{otherwise,} \end{cases}$$

always define a multiplicative linear functional $\phi_u\colon \mathcal{A}_G \to \mathbb{C}$. To that end, assume we have an equation of the form

$$\sum_{i\leqslant n} \lambda_i e_{s_i} = 0 \tag{7}$$

for some $s_1, \dots, s_n \in \mathrm{F}(\mathcal{M}_G)$ and $\lambda_i \in \mathbb{C}$. Recalling the identification of G as $\mathrm{Aut}(\mathcal{M}_G)$ from Remark 1.6(2), we set up for the application of Lemma 1.4.

We define

$$A = \bigcup \{\mathrm{dom}(s_i) : i \leqslant n,\ \mathrm{dom}(s_i) \subseteq \mathrm{dom}(u)\}$$

and

$$I = \{i \leqslant n : \mathrm{dom}(s_i) \not\subseteq \mathrm{acl}(A)\}.$$

By Lemma 1.4, there exists $g_0 \in G$ that extends $u_{|A}$ but none of the s_i for $i \in I$. We claim that

$$\forall i \leqslant n \quad [s_i \subseteq g_0 \Leftrightarrow s_i \subseteq u]. \tag{8}$$

Indeed, assume that $s_i \not\subseteq u$. Then either $\mathrm{dom}(s_i)$ is included in $\mathrm{dom}(u)$ or not. In the first case, there must exist $a \in \mathrm{dom}(s_i)$ such that

$$s_i(a) \neq u(a) = g_0(a),$$

hence $s_i \not\subseteq g_0$. In the second case, we also have $\mathrm{dom}(s_i) \not\subseteq \mathrm{acl}(A)$ otherwise $\mathrm{dom}(s_i) \subseteq \mathrm{acl}(\mathrm{dom}(u)) = \mathrm{dom}(u)$. Hence $i \in I$ and $s_i \not\subseteq g_0$. The converse is clear by construction.

Now, evaluating (7) on g_0 yields

$$0 = \sum_{i \leqslant n} \lambda_i e_{s_i}(g_0) = \sum_{s_i \subseteq g_0} \lambda_i = \sum_{s_i \subseteq u} \lambda_i.$$

Thus ϕ_u is well defined and linear. Complex conjugation is then automatically preserved because $\mathcal{A}_G$ is spanned by real-valued maps. We check the multiplicativity of ϕ_u on the spanning subset $\{e_s : s \in \mathrm{F}(\mathcal{M}_G)\}$. Recalling (4), just observe that, for all $s, s' \in \mathrm{F}(\mathcal{M}_G)$, we have

$$\begin{aligned} \phi_u(e_s e_{s'}) = 1 \iff & [s \cup s' \in \mathrm{F}(\mathcal{M}_G) \text{ and } s \cup s' \subseteq u] \\ \iff & s, s' \subseteq u \iff \phi_u(e_s) = 1 = \phi_u(e_{s'}). \end{aligned}$$

This shows multiplicativity. Finally, it is non-zero since the empty map $\emptyset$ belongs to $\mathrm{F}(\mathcal{M}_G)$ and we always have $\emptyset \subseteq u$.

It is easily checked that $u \mapsto \phi_u$ and $\phi \mapsto u_\phi$ are inverse to each other. To finish the proof, recall that $\mathbf{P}(\mathcal{M}_G)$ is compact and Hausdorff. Since $\mathbf{S}_{\mathcal{A}_G}$ is Hausdorff too, we can use Poincaré's Theorem and it suffices to show that $\mathbf{P}(\mathcal{M}_G) \to \mathbf{S}_{\mathcal{A}_G}$ is continuous.

Recall that $\{e_s : s \in \mathrm{F}(\mathcal{M}_G)\}$ are idempotents and linearly span $\mathcal{A}_G$. In particular, $\phi(e_s) \in \{0, 1\}$ for all $\phi \in \mathbf{S}_{\mathcal{A}_G}$ and $s \in \mathrm{F}(\mathcal{M}_G)$, hence the following sets form a subbasis for the topology on $\mathbf{S}_{\mathcal{A}_G}$:

$$O_{s,\varepsilon} = \{\phi \in \mathbf{S}_{\mathcal{A}_G} : \phi(e_s) = \varepsilon\} \quad \text{for } s \in \mathrm{F}(\mathcal{M}_G) \text{ and } \varepsilon \in \{0, 1\}.$$

Now, let $s \in \mathrm{F}(\mathcal{M}_G)$ and $\varepsilon \in \{0, 1\}$. We treat the case $\varepsilon = 0$; the other is similar. We have

$$\begin{aligned} \phi \in O_{s,0} \iff & \phi(e_s) = 0 \\ \iff & s \nsubseteq u_\phi \\ \iff & \exists x \in \mathrm{dom}(s),\ [x \notin \mathrm{dom}(u_\phi) \text{ or } u_\phi(x) \neq s(x)] \\ \iff & u_\phi \in \bigcup_{x \in \mathrm{dom}(s)} \{v \in \mathbf{P}(\mathcal{M}_G) : x \in \mathrm{dom}(v) \Rightarrow v(x) \notin \{s(x)\}\}. \end{aligned}$$

Recalling the definition of the topology on $\mathbf{P}(\mathcal{M}_G)$ from Remark 1.5, we see that the desired continuity holds. ■

We can now turn to the key result of this section. Recall that a *positive linear functional* on an involutive algebra $\mathcal{B}$ is a linear map $\phi\colon G \to \mathbb{C}$ such that for every $a \in \mathcal{B}$, $\phi(a^* a) \geqslant 0$, where $*$ denotes the involution of $\mathcal{B}$. As mentioned in the introduction, the next theorem is an adaptation of a technical but crucial lemma from the compact duality: Kreĭn [Kr14] proved that for a compact group K, positive linear functionals on $\mathcal{B}_0(K)$ are automatically continuous. Bochner [Bo42] gave an alternative and more elementary proof, based on generalized Fourier analysis and uniform approximation. This also appears in [HR70, Th. 30.2] with Bochner's proof and more context. We prove here that multiplicative linear functionals on $\mathcal{A}_G$ are automatically continuous. Note that we make a stronger assumption on the functionals but work in a smaller algebra.

The key tools from the compact case, namely the Haar measure and the finite dimension, are not available here. Instead, we rely on dynamical properties of Roelcke-precompact groups, in the form of Lemma 1.4.

THEOREM 2.4. *Let G be a Roelcke-precompact non-archimedean Polish group and let $\phi\colon \mathcal{A}_G \to \mathbb{C}$ be multiplicative and linear. Then ϕ is continuous and has operator norm at most* 1.

Proof. As in both [Kr14] and [Bo42], we will show a strong positivity property for ϕ:

$$\forall f \in \mathcal{A}_G \quad [f \geqslant 0 \Rightarrow \phi(f) \geqslant 0],$$

where $f \geqslant 0$ means $f(g) \geqslant 0$ for every $g \in G$.

To that end, fix $f \in \mathcal{A}_G$. The exist $s_1, \ldots, s_n \in \mathrm{F}(\mathcal{M}_G)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ such that $f = \sum_{i \leqslant n} \lambda_i e_{s_i}$. Recalling the identification $G = \mathrm{Aut}(\mathcal{M}_G)$ from Remark 1.6(2), we set up for the application of Lemma 1.4.

Let $A = \bigcup \{\mathrm{dom}(s_i) : i \leqslant n,\ \mathrm{dom}(s_i) \subseteq \mathrm{dom}(u_\phi)\}$, which is finite, and let $I = \{i \leqslant n : \mathrm{dom}(s_i) \nsubseteq \mathrm{acl}(A)\}$. By Lemma 1.4, there exists $g_0 \in G$ that extends $u_{\phi|A}$ but none of the s_i for $i \in I$. Exactly as in the proof of Corollary 2.3, g_0 in fact satisfies

$$\forall i \leqslant n \quad [s_i \subseteq g_0 \Leftrightarrow s_i \subseteq u_\phi].$$

From this and the properties of u_ϕ, it follows that

$$\phi(f) = \sum_{i \leqslant n} \lambda_i \phi(e_{s_i}) = \sum_{s_i \subseteq u_\phi} \lambda_i = \sum_{s_i \subseteq g_0} \lambda_i = f(g_0).$$

In particular, if $f \geqslant 0$, then $\phi(f) \geqslant 0$. Hence ϕ is positive in a strong sense, which is well known to imply continuity. Indeed, for every $f \in \mathcal{A}_G$, the functions $\|f\|_\infty \cdot 1 \pm f$ are in $\mathcal{A}_G$ and only take positive values. Hence $\phi(\|f\|_\infty \cdot 1 \pm f) \geqslant 0$, i.e.

$$|\phi(f)| \leqslant \phi(\|f\|_\infty \cdot 1) = \|f\|_\infty. \ \blacksquare$$

A direct consequence of this result is that the map from Corollary 2.3 lifts to a homeomorphism $\mathbf{H}(G) \to \mathbf{P}(\mathcal{M}_G)$. This homeomorphism gives $\mathbf{H}(G)$ a semitopological *-monoid structure, transported from $\mathbf{P}(\mathcal{M}_G)$.

QUESTION 2.5. Let G be a Roelcke-precompact non-archimedean Polish group. Are multiplicative and linear functionals on $\mathcal{B}_0(G)$ (or $\mathcal{B}(G)$) automatically continuous?

The Hilbert compactification comes with a semitopological monoid structure. The composition law is a form of convolution. Sometimes seen as a variation of the *Arens product*, it also appears in [Bo42]. One way to build it is as follows:

For $\phi \in \mathbf{H}(G)$, one can define a map $\underline{\phi}\colon \mathcal{A}_G \to \mathcal{A}_G$ by setting

$$\forall f \in \mathcal{A}_G, \forall g \in G, \qquad \underline{\phi}(f)(g) = \phi(g^{-1} \cdot f)$$

To see that it is well defined, note that for $g \in G$ and $s \in \mathrm{F}(\mathcal{M}_G)$, we have $\phi(g^{-1} \cdot e_s) = \phi(e_{g^{-1} \circ s}) = 1$ if u_ϕ extends $g^{-1} \circ s$ and 0 otherwise. In other terms,

$$\underline{\phi}(e_s) = \begin{cases} e_{s \circ (u_\phi^{-1})} & \text{if } \operatorname{dom}(s) \subseteq \operatorname{dom}(u_\phi), \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Thus, in both cases, $\underline{\phi}(e_s)$ is an element of $\mathcal{A}_G$. Next, given ϕ and ψ in $\mathbf{H}(G)$, we can define a map $\phi \cdot \psi\colon \mathcal{A}_G \to \mathbb{C}$ by setting $\phi \cdot \psi = \phi \circ \underline{\psi}$.

Then, given $s \in \mathrm{F}(\mathcal{M}_G)$, using (9) we obtain

$$\phi \cdot \psi(e_s) = \phi \circ \underline{\psi}(e_s)) = \begin{cases} 1 & \text{if } s \subseteq u_\phi \circ u_\psi, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 2.3, $\phi \cdot \psi$ is an element of $\mathbf{S}_{\mathcal{A}_G}$ which, by Theorem 2.4, uniquely extends to an element of $\mathbf{H}(G)$. Moreover, we found along the way that the map $\phi \mapsto u_\phi$ sends convolution in $\mathbf{H}(G)$ to composition where defined in $\mathbf{P}(\mathcal{M}_G)$.

The Hilbert compactification also has an involution $\phi \mapsto \phi^*$, which is given by

$$\forall f \in \operatorname{Hilb}(G), \qquad \phi^*(f) = \overline{\phi(f^*)},$$

where $f^*(g) = \overline{f(g^{-1})}$ for all $f \in \operatorname{Hilb}(G)$ and $g \in G$. It is preserved by the map $\mathbf{P}(\mathcal{M}_G) \to \mathbf{H}(G)$. Indeed, if $s \in \mathrm{F}(\mathcal{M}_G)$, we have $e_s^* = e_{s^*}$ and

$$\phi^*(e_s) = \overline{\phi(e_s^*)} = \phi(e_{s^*}) = \begin{cases} 1 & \text{if } s \subseteq u_\phi^*, \\ 0 & \text{otherwise,} \end{cases}$$

hence $u_{\phi^*} = u_\phi^*$.

Consequently, we can identify the Hilbert compactification of G with $\mathbf{P}(\mathcal{M}_G)$. We have in fact just proved the following. Because of the similarity between $\mathcal{M}_G$ and $\mathcal{M}^{\mathrm{eq}}$ when $\mathcal{M}$ is $\aleph_0$-categorical, it can be seen as a reformulation of [BIT18, Th. 0.2], obtained with a different approach.

Theorem 2.6. *The map $\mathbf{H}(G) \to \mathbf{P}(\mathcal{M}_G)$ is an isomorphism of semi-topological monoidal compactifications of G.*

Since G can be fully reconstructed from its Hilbert compactification (see e.g. Theorem 4.1 below), the pair $(\mathcal{A}_G, B)$, where $B = (\mathbb{1}_{gU})_{gU}$ with gU ranging over the open cosets of G, is a dual object to G much like a Kreĭn algebra in the compact case. Again, the topology on $\mathcal{A}_G$ need not be included in the data.

It is interesting to note that it is possible to define this structure on $\mathcal{B}_0(G)$, even more similarly to what Kreĭn did.

To that end, fix a family $(\pi_\lambda)_{\lambda\in\widehat{G}}$ of representatives of the isomorphism classes of irreducible representations of G. Using [Ts12, Th. 1.3], every π_λ is of the form $\mathrm{Ind}_{H_\lambda}^G(\sigma_\lambda)$ where H_λ is an open subgroup of G and σ_λ is a finite-dimensional irreducible representation of H_λ. Then let $(e_1,\dots,e_{d_\lambda})$ be an orthonormal basis of $\mathcal{H}_{\sigma_\lambda}$. We can lift this basis to a Hilbert basis of $\mathcal{H}_{\pi_\lambda}$. Indeed, choose a family $(g_c)_{c\in G/H_\lambda}$ of representatives of the cosets of H_λ and define

$$\xi^\lambda_{i,c}\colon G\to\mathcal{H}_{\sigma_\lambda},\qquad g\mapsto\begin{cases}\sigma_\lambda(h^{-1})e_i & \text{if } g=g_ch \text{ with } h\in H_\lambda,\\ 0 & \text{otherwise,}\end{cases}$$

where $1\leqslant i\leqslant d_\lambda$ and $c\in G/H_\lambda$.

These maps form a Hilbert basis of $\mathcal{H}_{\pi_\lambda}$ and can be used to define the product law on $\mathbf{S}_{\mathcal{B}_0(G)}$. Indeed, the family of matrix coefficients that arises using these vectors, $f^\lambda_{i,c,j,c'}\colon g\mapsto\langle\pi_\lambda(g)\xi_{j,c'},\xi_{i,c}\rangle$, are naturally arranged in block matrices:

$$\big((f^\lambda_{i,c,j,c'})_{\substack{1\leqslant i,j\leqslant d_\lambda\\ c,c'\in G/H_\lambda}}\big)_{\lambda\in\widehat{G}}.$$

In analogy with the compact case, these span a dense subalgebra of $\mathcal{B}_0(G)$.

Given *continuous* $\phi,\psi\in\mathbf{S}_{\mathcal{B}_0(G)}$, we claim that $\phi\cdot\psi$ behaves on the generating family as follows:

$$\phi\cdot\psi(f^\lambda_{i,c,j,c'})=\sum_{\substack{1\leqslant k\leqslant d_\lambda\\ c''\in G/H_\lambda}}\phi(f^\lambda_{i,c,k,c''})\psi(f^\lambda_{k,c'',j,c'}).$$

To see that the sum converges, it suffices, by the Cauchy–Schwarz inequality, to show that for fixed $i\leqslant d_\lambda$ and $c\in G/H_\lambda$,

$$\sum_{\substack{1\leqslant k\leqslant d_\lambda\\ c''\in G/H_\lambda}}|\phi(f^\lambda_{i,c,k,c''})|^2<+\infty.$$

The key is that the matrix coefficients at play here are all bounded by 1 and the collection of their supports have locally finite intersection: $f^\lambda_{i,c,k,c''}(g)=0$ as long as $g\notin g_{c''}H_\lambda g_c^{-1}$. Thus, for any finite subcollection of indices, since ϕ is multiplicative and preserves complex conjugation,

$$\sum_{k,c''}|\phi(f^\lambda_{i,c,k,c''})|^2=\phi\Big(\sum_{k,c''}|f^\lambda_{i,c,k,c''}|^2\Big)\leqslant\|\phi\|\cdot\Big\|\sum_{k,c''}|f^\lambda_{i,c,k,c''}|^2\Big\|_\infty\leqslant d_\lambda.$$

This construction yields the same operation as the above variants. Indeed, for all λ,i,j,c,c' and all $g,h\in G$, we have

$$\pi(h)\xi^\lambda_{j,c'}=\sum_{k,c''}\langle\pi(h)\xi^\lambda_{j,c'},\xi^\lambda_{k,c''}\rangle\xi^\lambda_{k,c''}=\sum_{k,c''}f^\lambda_{k,c'',j,c'}(h)\xi^\lambda_{k,c''},$$

hence

$$\begin{aligned}
f^{\lambda}_{i,c,j,c'}(gh) &= \langle \pi_\lambda(g)\pi_\lambda(h)\xi^{\lambda}_{j,c'}, \xi^{\lambda}_{i,c}\rangle \\
&= \sum_{k,c''} f^{\lambda}_{k,c'',j,c'}(h)\langle \pi_\lambda(g)\xi^{\lambda}_{k,c''}, \xi^{\lambda}_{i,c}\rangle \\
&= \sum_{k,c''} f^{\lambda}_{i,c,k,c''}(g) f^{\lambda}_{k,c'',j,c'}(h).
\end{aligned}$$

Thus, for $\phi, \psi \in \mathbf{S}_{\mathcal{B}_0(G)}$ and λ, i, j, c, c',

$$\begin{aligned}
\psi \cdot \psi(f^{\lambda}_{i,c,j,c'}) &= \phi \circ \underline{\psi}(f^{\lambda}_{i,c,j,c'}) = \phi(g \mapsto \psi(g^{-1} \cdot f^{\lambda}_{i,c,j,c'})) \\
&= \phi\Big(g \mapsto \sum_{k,c''} f^{\lambda}_{i,c,k,c''}(g)\psi(f^{\lambda}_{k,c'',j,c'})\Big) \\
&= \sum_{k,c''} \phi(f^{\lambda}_{i,c,k,c''})\psi(f^{\lambda}_{k,c'',j,c'}).
\end{aligned}$$

The involution can also be defined in terms of the $(f^{\lambda}_{i,c,j,c'})$. Indeed, for all λ, i, j, c, c', we have

$$(f^{\lambda}_{i,c,j,c'})^* = f^{\lambda}_{j,c',i,c}.$$

Thus, a positive answer to Question 2.5 would make the pair $(\mathcal{B}_0(G), B)$, where B is the countable structured family $(f^{\lambda}_{i,c,j,c'})$ of matrix coefficients, a dual object that contains enough data for the full reconstruction of G, even without the topology on $\mathcal{B}_0(G)$.

3. The Tannaka monoid: Another description of $\mathbf{H}(G)$. In [Ta39], Tannaka associated to a compact group G a monoid $\mathbf{T}(G)$ of *operations* on the class of representations of G. More explicitly, an element of $\mathbf{T}(G)$ is a family $(u_\pi)_\pi$ of operators where π ranges over all the finite-dimensional representations of G and u_π is an operator on the same Hilbert space as π. Moreover, the family must commute with intertwining operators and preserve the common operations on representations: sum, tensor product and conjugation. The details can be found in [Ch99] or [HR70]. A more recent treatment of this approach using the terminology from category theory appears in [JS91]. Inspired by this take on Tannaka's duality, we carry out a similar construction for Roelcke-precompact non-archimedean Polish groups. While Tannaka showed in the compact case that $\mathbf{T}(G)$ is a compact group canonically isomorphic to G, we will obtain a compact semitopological *-monoid canonically isomorphic to the Hilbert compactification.

Given a topological group G, we will denote by $\mathbf{Rep}(G)$ the category whose objects are representations of G that split as a finite sum of irreducibles and whose morphisms are the intertwining operators. Then $\mathcal{B}_0(G)$ is exactly the algebra of matrix coefficients of G arising from representations in $\mathbf{Rep}(G)$. Recall that $\widehat{G}$ denotes the unitary dual of G, i.e. the set of equivalence classes of irreducible unitary representations of G.

PROPOSITION 3.1. *Let G be a Roelcke-precompact non-archimedean Polish group. The following properties hold:*

(1) $\mathbf{Rep}(G)$ *is stable under tensor product of representations.*
(2) *Only countably many equivalence classes of representations appear in* $\mathbf{Rep}(G)$.

Proof. (1) This is Lemma 1.15.

(2) This follows from the fact that $\widehat{G}$ is countable, stated as Fact 1(4). ∎

Let $\mathbf{U}_G$ be the forgetful functor from $\mathbf{Rep}(G)$ to the category of Hilbert spaces and let $\mathcal{N}at(\mathbf{U}_G)$ denote the class of natural transformations of $\mathbf{U}_G$. Explicitly, an element of $\mathcal{N}at(\mathbf{U}_G)$ is a family $u = (u_\pi)_{\pi\in\mathbf{Rep}(G)}$ where for every representation $\pi \in \mathbf{Rep}(G)$, u_π is a bounded operator on $\mathcal{H}_\pi$ that commutes with intertwining operators: for all representations $\pi_1, \pi_2 \in \mathbf{Rep}(G)$ and every intertwining operator $h \in \operatorname{Hom}(\pi_1, \pi_2)$, the following diagram is commutative:

$$\begin{tikzcd} \mathcal{H}_{\pi_1} \arrow[r, "u_{\pi_1}"] \arrow[d, "h"'] & \mathcal{H}_{\pi_1} \arrow[d, "h"] \\ \mathcal{H}_{\pi_2} \arrow[r, "u_{\pi_2}"] & \mathcal{H}_{\pi_2} \end{tikzcd}$$

Note that every element g of G induces a natural transformation of $\mathbf{U}_G$, namely $(\pi(g))_{\pi\in\mathbf{Rep}(G)}$.

It is easily seen that $\mathcal{N}at(\mathbf{U}_G)$ is a complex algebra under coordinatewise sum and composition. Moreover, elements of $\mathcal{N}at(\mathbf{U}_G)$ behave well under subrepresentations:

LEMMA 3.2. *Let G be a Roelcke-precompact non-archimedean Polish group and let $u \in \mathcal{N}at(\mathbf{U}_G)$.*

(1) *Let $\pi \in \mathbf{Rep}(G)$ and let F be a closed subspace of $\mathcal{H}_\pi$ that is stable under the action of G. Then F is also stable under u_π.*
(2) *Let $\pi_1, \pi_2 \in \mathbf{Rep}(G)$. Then*

$$u_{\pi_1\oplus\pi_2} = u_{\pi_1} \oplus u_{\pi_2}.$$

Proof. (1) Let $p_F\colon \mathcal{H}_\pi \to \mathcal{H}_\pi$ be the orthogonal projection on F. Then F is stable under the action of G if and only if p_F commutes with the action of G. In particular, if F is G-stable then u_π must also commute with p_F, i.e. $u_\pi(F) \subseteq F$.

(2) Let $\pi = \pi_1 \oplus \pi_2$. By definition, $\mathcal{H}_\pi = \mathcal{H}_{\pi_1} \oplus \mathcal{H}_{\pi_2}$. Moreover, the inclusion maps $\mathcal{H}_{\pi_1} \to \mathcal{H}_\pi$ are intertwining operators. Thus, the following

diagram must be commutative, which proves the claim:

$$\begin{array}{ccc}
\mathcal{H}_{\pi_1} & \xrightarrow{u_{\pi_1}} & \mathcal{H}_{\pi_1} \\
\downarrow & & \downarrow \\
\mathcal{H}_{\pi} & \xrightarrow{u_{\pi}} & \mathcal{H}_{\pi} \\
\uparrow & & \uparrow \\
\mathcal{H}_{\pi_2} & \xrightarrow{u_{\pi_2}} & \mathcal{H}_{\pi_2}
\end{array} \qquad \blacksquare$$

The following is essentially [JS91, Proposition 4]. The proof is the same but we reproduce it for completeness. If $\mathcal{H}$ is a Hilbert space, $\mathcal{L}(\mathcal{H})$ will denote the space of bounded operators $\mathcal{H} \to \mathcal{H}$ and $\mathcal{L}_1(\mathcal{H})$ the subset of those operators with norm at most 1.

Proposition 3.3. *Let G be a Roelcke-precompact non-archimedean Polish group. For every $\lambda \in \widehat{G}$, fix $\pi_\lambda \colon G \to \mathcal{U}(\mathcal{K}_\lambda)$ a representative of λ. Then the restriction map*

$$q\colon \mathcal{N}at(\mathbf{U}_G) \to \prod_{\lambda \in \widehat{G}} \mathcal{L}(\mathcal{K}_\lambda), \qquad u \mapsto (u_{\pi_\lambda})_{\lambda \in \widehat{G}},$$

is a bijective correspondence that respects the algebra structures. In particular, $\mathcal{N}at(\mathbf{U}_G)$ is a set.

Proof. Lemma 3.2 together with the fact that our representations are finite sums of irreducibles show that an element u is determined by its image under q. In particular, q is injective. For surjectivity, fix $(t_\lambda) \in \prod_{\lambda \in \widehat{G}} \mathcal{L}(\mathcal{K}_\lambda)$. We will define a pre-image u for (t_λ).

Let $\pi\colon G \to \mathcal{U}(\mathcal{H})$ be an element of $\mathbf{Rep}(G)$. Recalling Lemma 1.11, there is a unique *isotypical* decomposition $\mathcal{H} = \bigoplus_\lambda \mathcal{H}_\lambda$ into closed invariant subspaces where for every $\lambda \in \widehat{G}$, all the irreducible subrepresentations of $\mathcal{H}_\lambda$ are isomorphic to π_λ.

Claim. *There is a canonical G-equivariant isomorphism*

$$\Psi_\lambda\colon \mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda) \to \mathcal{H}_\lambda,$$

namely, the one induced by the bilinear map

$$\mathcal{K}_\lambda \times \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda) \to \mathcal{H}_\lambda, \qquad (\xi, h) \mapsto h(\xi).$$

Proof of the claim. Because $\mathrm{Hom}(\mathcal{K}_\lambda, \mathcal{H}_\lambda)$ is finite-dimensional (this is an application of Schur's Lemma, detailed below), the subtleties of working with *Hilbert tensor products* rather than *algebraic* ones vanish and the verifications are routine. Still, we give all the details for completeness.

Since $\pi \in \mathbf{Rep}(G)$, $\mathcal{H}_\lambda$ splits as a finite sum $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n$ of subrepresentations where for every $i \leqslant n$, $G \curvearrowright \mathcal{H}_i$ is isomorphic to π_λ, witnessed by an isomorphism of representations $h_i \colon \mathcal{K}_\lambda \to \mathcal{H}_i$.

• $\varPsi_\lambda$ *is well defined*: Actually, $\varPsi_\lambda$ is well defined on the *algebraic* tensor product of $\mathcal{K}_\lambda$ and $\mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda)$. Let $(\xi_i)_{i\in I}$ be a Hilbert basis of $\mathcal{K}_\lambda$. Also,

$$\mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda) = \bigoplus_{1\leqslant i\leqslant n} \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_i)$$

where, by Schur's Lemma, $\mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_i) = \mathbb{C} \cdot h_i$ for every $i \leqslant n$. Thus, $(h_i)_{1\leqslant i\leqslant n}$ is an orthonormal basis of the finite-dimensional Hilbert space $\mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda)$ and

$$(\xi_i \otimes h_j)_{\substack{i\in I\\ 1\leqslant j\leqslant n}}$$

is a Hilbert basis for $\mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda)$ that is moreover contained in the algebraic tensor product. Finally, for every $i \in I$ and $j \leqslant n$, we have

$$\|\varPsi_\lambda(\xi_i \otimes h_j)\| = \|h_j(\xi_i)\| = \|\xi_j\| = 1.$$

Thus, $\varPsi_\lambda$ is bounded (and even isometric), hence extends to the Hilbert tensor product.

• $\varPsi_\lambda$ *is surjective*: For every $j \leqslant n$, $\mathcal{H}_j = h_j(\mathcal{K}_\lambda) = \varPsi_\lambda(\mathcal{K}_\lambda \otimes h_j) \subseteq \operatorname{im} \varPsi_\lambda$, hence $\mathcal{H}_\lambda = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_n \subseteq \operatorname{im} \varPsi_\lambda$.

• $\varPsi_\lambda$ *is injective*: First, note that

$$\begin{aligned}\mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda) &= \mathcal{K}_\lambda \otimes \Big(\bigoplus_{1\leqslant i\leqslant n} \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_i)\Big)\\ &= \bigoplus_{1\leqslant i\leqslant n} \mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_i).\end{aligned}$$

Now, suppose $x \in \ker \varPsi_\lambda$. Using the above decomposition, write $x = x_1 + \cdots + x_n$ where $x_i \in \mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_i)$ for every $i \leqslant n$. Then

$$0 = \varPsi_\lambda(x) = \varPsi_\lambda(x_1) + \cdots + \varPsi_\lambda(x_n).$$

Since $\varPsi_\lambda(x_i) \in \mathcal{H}_i$ for every $i \leqslant n$, and because the $\mathcal{H}_i$'s are pairwise orthogonal, we have $\varPsi_\lambda(x_i) = 0$ for every $i \leqslant n$. We thus reduce to the case $x = x_1$. By Schur's Lemma, $\mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_1) = \mathbb{C} \cdot h_1$, hence x is of the form $\xi \otimes h_1$ for some $\xi \in \mathcal{K}_\lambda$. Then

$$0 = \varPsi_\lambda(x) = h_1(\xi).$$

Since h_1 is isometric, $\xi = 0$ and $x = 0$.

• $\varPsi_\lambda$ *is* G*-equivariant*: By continuity, it suffices to check it on simple tensors. For all $g \in G$, $\xi \in \mathcal{K}_\lambda$ and $h \in \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda)$, we have

$$g \cdot (\xi \otimes h) = g \cdot \xi \otimes g \cdot h = \pi_\lambda(g)\xi \otimes h,$$

hence

$$\Psi_\lambda(g\cdot(\xi\otimes h)) = h(\pi_\lambda(g)\xi) = \pi(g)h(\xi) = \pi(g)\Psi_\lambda(\xi\otimes h). \ \blacksquare_{\text{claim}}$$

Using Ψ_λ, we can define u_π as follows:

$$u_\pi = \bigoplus_\lambda [\Psi_\lambda \circ (t_\lambda \otimes 1) \circ \Psi_\lambda^{-1}].$$

To see that this defines a natural transformation of $\mathbf{U}_G$, fix $\pi, \pi' \in \mathbf{Rep}(G)$ and write $\mathcal{H} = \mathcal{H}_\pi$ and $\mathcal{H}' = \mathcal{H}_{\pi'}$. Let $h \in \mathrm{Hom}(\pi, \pi')$. Recalling the classical general fact that $\mathrm{Hom}(\pi_\lambda, \pi_\mu) = 0$ if $\lambda \neq \mu \in \widehat{G}$, we see that h preserves the isotypical decompositions of π and π':

$$\forall \lambda \in \widehat{G}, \quad h(\mathcal{H}_\lambda) \subseteq \mathcal{H}'_\lambda$$

with the above notations. Moreover, these decompositions are also preserved by u_π and $u_{\pi'}$ by construction. Hence, we can reduce to the case $\mathcal{H} = \mathcal{H}_\lambda$ and $\mathcal{H}' = \mathcal{H}'_\lambda$ for a single $\lambda \in \widehat{G}$, which leaves us with the following diagram:

$$\begin{array}{ccccccc}
\mathcal{H}_\lambda & \xrightarrow{\Psi_\lambda^{-1}} & \mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda) & \xrightarrow{t_\lambda \otimes 1} & \mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}_\lambda) & \xrightarrow{\Psi_\lambda} & \mathcal{H}_\lambda \\
\downarrow h & & \downarrow 1\otimes(h\circ\cdot) & & \downarrow 1\otimes(h\circ\cdot) & & \downarrow h \\
\mathcal{H}'_\lambda & \xrightarrow{\Psi_\lambda'^{-1}} & \mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}'_\lambda) & \xrightarrow{t_\lambda \otimes 1} & \mathcal{K}_\lambda \otimes \mathrm{Hom}_G(\mathcal{K}_\lambda, \mathcal{H}'_\lambda) & \xrightarrow{\Psi'_\lambda} & \mathcal{H}'_\lambda
\end{array}$$

To conclude, we need to prove that the outer square is commutative. But the three small squares are easily seen to be commutative, which is enough. ∎

Consider the weak operator topology on $\mathcal{L}(\mathcal{H}_\pi)$ for every representation π of G. Then we can endow $\prod_{\lambda \in \widehat{G}} \mathcal{L}(\mathcal{H}_\lambda)$ with the product topology. $\mathcal{N}at(\mathbf{U}_G)$ can also be endowed with the coarsest topology for which the projection maps $u \mapsto u_\pi$ are continuous for every representation π of G. Then q is also a homeomorphism for those topologies.

Lemma 3.4. *Let G be a Roelcke-precompact non-archimedean Polish group, let $u \in \mathcal{N}at(\mathbf{U}_G)$ and let $\pi, \sigma \in \mathbf{Rep}(G)$.*

(1) *If σ and π are isomorphic, then $\|u_\sigma\| = \|u_\pi\|$.*
(2) *If σ is a subrepresentation of π, then $\|u_\sigma\| \leqslant \|u_\pi\|$.*

Proof. (1) In this case, there exists a surjective isometry $h \in \mathrm{Hom}(\sigma, \pi)$. Then

$$\|u_\pi\| = \|h \circ u_\sigma \circ h^{-1}\| = \|u_\sigma\|.$$

(2) follows directly from the previous point and Lemma 3.2(2). ∎

Tensor product is the last ingredient we need to add in order to define our version of the Tannaka monoid:

Definition 3.5. The *Tannaka monoid* $\mathbf{T}(G)$ of a Roelcke-precompact non-archimedean Polish group G is the set of elements $u \in \mathcal{N}at(\mathbf{U}_G)$ that are non-zero and commute with tensor product, i.e.

$$u_{\mathbf{1}} = \mathrm{Id}_{\mathbb{C}} \quad \text{and} \quad \forall \pi, \pi' \in \mathbf{Rep}(G),\ u_{\pi \otimes \pi'} = u_\pi \otimes u_{\pi'},$$

where $\mathbf{1}$ denotes the trivial representation of G on $\mathbb{C}$. The operation is coordinatewise composition: for all $u, v \in \mathbf{T}(G)$, $(u \circ v)_\pi = u_\pi \circ v_\pi$. It also admits an involution $u \mapsto u^*$ given by coordinatewise adjunction. It is endowed with the induced topology as a subspace of $\mathcal{N}at(\mathbf{U}_G)$.

Theorem 3.6. *Let G be a Roelcke-precompact non-archimedean Polish group. Then $\mathbf{T}(G)$ is a compact semitopological *-monoid isomorphic to the Hilbert compactification of G.*

Proof. First, it is easily seen that composition is separately continuous in $\mathbf{T}(G)$ as it is in $\mathcal{L}(\mathcal{H})$ with the weak operator topology for every Hilbert space $\mathcal{H}$. Similarly, the involution $*$ of $\mathbf{T}(G)$ is also continuous.

Next, we will define a map $\Phi\colon \mathbf{T}(G) \to \mathbf{H}(G)$. Fix $u \in \mathbf{T}(G)$ in order to define a continuous linear multiplicative functional $\phi\colon \mathrm{Hilb}(G) \to \mathbb{C}$. For every $\pi \in \mathbf{Rep}(G)$ and all $\xi, \eta \in \mathcal{H}_\pi$, denote by $f^\pi_{\xi,\eta}\colon g \mapsto \langle \pi(g)\eta, \xi\rangle$ the associated matrix coefficient. We first define ϕ on $\mathcal{B}_0(G)$ by setting

$$\phi(f^\pi_{\xi,\eta}) = \langle u_\pi(\eta), \xi\rangle. \tag{10}$$

To see that it correctly defines a linear map $\mathcal{B}_0(G) \to \mathbb{C}$, fix representations $\pi_1, \ldots, \pi_n$ in $\mathbf{Rep}(G)$ with respective underlying spaces $\mathcal{H}_1, \ldots, \mathcal{H}_n$ and let $\xi_1, \eta_1 \in \mathcal{H}_1, \ldots, \xi_n, \eta_n \in \mathcal{H}_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be such that

$$\lambda_1 f^{\pi_1}_{\xi_1,\eta_1} + \cdots + \lambda_n f^{\pi_n}_{\xi_n,\eta_n} = 0. \tag{11}$$

Then, let $\pi = \bigoplus_{i \leqslant n} \pi_i$, $\eta = \eta_1 + \cdots + \eta_n$ and $\xi = \lambda_1\xi_1 + \cdots + \lambda_n\xi_n$. Equation (11) becomes

$$\forall g \in G, \qquad 0 = \langle \pi(g)\eta, \xi\rangle = \langle \eta, \pi(g^{-1})\xi\rangle,$$

i.e. η lies in $(\pi(G)\xi)^\perp$, a closed subspace of $\mathcal{H}$ stable under the action of G. Using Lemma 3.2, we also have $u_\pi(\eta) \in (\pi(G)\xi)^\perp$ and even

$$0 = \langle u_\pi(\eta), \xi\rangle_{\mathcal{H}} = \lambda_1\langle u_{\pi_1}(\eta_1), \xi_1\rangle_{\mathcal{H}_1} + \cdots + \lambda_n\langle u_{\pi_n}(\eta_n), \xi_n\rangle_{\mathcal{H}_n}.$$

This shows that ϕ is well defined on $\mathcal{B}_0(G)$. It is clearly non-zero since $u \neq 0$.

To see that ϕ is multiplicative, fix $f^\pi_{\xi,\eta}, f^{\pi'}_{\xi',\eta'}$ in $\mathcal{B}_0(G)$. We have

$$\begin{aligned}
\phi(f^\pi_{\xi,\eta})\phi(f^{\pi'}_{\xi',\eta'}) &= \langle u_\pi(\eta), \xi\rangle\langle u_{\pi'}(\eta'), \xi'\rangle = \langle u_\pi(\eta) \otimes u_{\pi'}(\eta'), \xi \otimes \xi'\rangle \\
&= \langle u_\pi \otimes u_{\pi'}(\eta \otimes \eta'), \xi \otimes \xi'\rangle \ = \langle u_{\pi\otimes\pi'}(\eta \otimes \eta'), \xi \otimes \xi'\rangle \\
&= \phi(f^{\pi\otimes\pi'}_{\xi\otimes\xi',\eta\otimes\eta'}) = \phi(f^\pi_{\xi,\eta} f^{\pi'}_{\xi',\eta'}),
\end{aligned}$$

where we have used the fact that $u \in \mathbf{T}(G)$ to obtain the fourth equality.

Next, we claim that ϕ is bounded. By Theorem 2.4, ϕ is bounded by 1 on $\mathcal{A}_G$. Thus, for every open subgroup $V \leqslant G$ and every $g, h \in G$,

$$|\langle u_{\lambda_{G/V}} \delta_{gV}, \delta_{hV} \rangle| = |\phi(\mathbb{1}_{hVg^{-1}})| \leqslant \|\mathbb{1}_{hVg^{-1}}\| = 1.$$

Since $u_{\lambda_{G/V}}$ is continuous by definition and $(\delta_{gV})_{gV \in G/V}$ is a Hilbert basis of $\ell^2(G/V)$, the above calculation shows that $\|u_{\lambda_{G/V}}\| \leqslant 1$. Now, if $\pi \in \mathbf{Rep}(G)$, one can use Lemma 1.8 to find open subgroups $V_1, \dots, V_n \leqslant G$ such that π is isomorphic to a subrepresentation of $\lambda_{G/V_1} \oplus \cdots \oplus \lambda_{G/V_n}$. Then, by Lemma 3.4, we have

$$\|u_\pi\| \leqslant \|u_{\lambda_{G/V_1} \oplus \cdots \oplus \lambda_{G/V_n}}\| = \|u_{\lambda_{G/V_1}} \oplus \cdots \oplus u_{\lambda_{G/V_n}}\| \leqslant 1.$$

Thus, for every $f^\pi_{\xi,\eta}$ in $\mathcal{B}_0(G)$, we have

$$|\phi(f^\pi_{\xi,\eta})| = |\langle u_\pi \eta, \xi \rangle| \leqslant \|u_\pi\| \cdot \|\xi\| \cdot \|\eta\| \leqslant 1 \cdot \|f^\pi_{\xi,\eta}\|_\infty,$$

and ϕ is bounded by 1.

The map ϕ being multiplicative, linear and bounded on $\mathcal{B}_0(G)$, it uniquely extends to an element of $\mathbf{H}(G)$ and hence $\Phi \colon \mathbf{T}(G) \to \mathbf{H}(G)$ is well defined. The identity (10) above also shows that Φ is continuous as $\mathbf{H}(G)$ is endowed with the pointwise convergence (= weak*) topology.

To define the inverse $\Psi \colon \mathbf{H}(G) \to \mathbf{T}(G)$, fix $\phi \in \mathbf{H}(G)$ and $\pi \in \mathbf{Rep}(G)$. Recall that ϕ is automatically bounded since $\mathrm{Hilb}(G)$ is a C^*-algebra. Then the map

$$\mathcal{H}_\pi \times \mathcal{H}_\pi \to \mathbb{C}, \qquad (\xi, \eta) \mapsto \phi(f^\pi_{\xi,\eta}),$$

is a bounded sesquilinear form. Indeed, for all $\xi, \eta \in \mathcal{H}$,

$$|\phi(f^\pi_{\xi,\eta})| \leqslant \|\phi\| \cdot \|f^\pi_{\xi,\eta}\|_\infty = \|\phi\| \cdot \|\eta\| \cdot \|\xi\|.$$

By the Riesz Representation Theorem, there exists a unique bounded operator $u_\pi \in \mathcal{L}(\mathcal{H}_\pi)$ such that

$$\forall \xi, \eta \in \mathcal{H}_\pi, \qquad \phi(f^\pi_{\xi,\eta}) = \langle u_\pi(\eta), \xi \rangle. \tag{12}$$

It is straightforward to show that $(u_\pi)_{\pi \in \mathbf{Rep}(G)}$ lies in $\mathbf{T}(G)$ and that (12) implies the continuity of the map $\Psi \colon \mathbf{H}(G) \to \mathbf{T}(G)$. Equations (10) and (12) together imply that those maps are inverse to each other, hence homeomorphisms.

To finish the proof, it only remains to show that the *-monoid structures are preserved. To see this, fix $u, v \in \mathbf{T}(G)$, and let $\pi \in \mathbf{Rep}(G)$. For all $\xi, \eta \in \mathcal{H}_\pi$ the following holds:

$$\begin{aligned}
\Phi(u \circ v)(f^\pi_{\xi,\eta}) &= \langle u_\pi \circ v_\pi(\eta), \xi \rangle = \Phi(u)(f^\pi_{\xi, v_\pi(\eta)}) \\
&= \Phi(u)[g \mapsto \langle v_\pi(\eta), \pi(g^{-1})\xi \rangle] = \Phi(u)[g \mapsto \Phi(v)(f^\pi_{\xi, \pi(g^{-1})\eta})] \\
&= \Phi(u)[g \mapsto \Phi(v)(g^{-1} \cdot f^\pi_{\xi,\eta})] = \Phi(u) \cdot \Phi(v)(f^\pi_{\xi,\eta}).
\end{aligned}$$

Finally, since multiplicative linear functionals on $\mathcal{A}_G$ automatically preserve conjugation, Φ preserves the involution. ∎

4. Additional properties of $\mathbf{H}(G)$. We end this article by recovering previously known basic properties of $\mathbf{H}(G)$ and by turning the previous results into a proper duality, i.e. a contravariant functor that is full and faithful.

First, recall that there is a natural map $\iota\colon G \to \mathbf{H}(G)$ that sends $g \in G$ to the *evaluation map at* g. Viewing $\mathbf{H}(G)$ as $\mathbf{T}(G)$, $\iota(g)$ is given by

$$\forall \pi \in \mathbf{Rep}(G), \qquad \iota(g)_\pi = \pi(g),$$

Finally, viewing $\mathbf{H}(G)$ as $\mathbf{P}(\mathcal{M}_G)$ and identifying G with $\mathrm{Aut}(\mathcal{M}_G)$ by Remark 1.6(2), ι becomes the canonical inclusion $G \hookrightarrow \mathbf{P}(\mathcal{M}_G)$. It allows for an abstract reconstruction of G from $\mathbf{H}(G)$:

THEOREM 4.1. *Let G be a non-archimedean Roelcke-precompact Polish group. The natural map*

$$\iota\colon G \to \mathbf{H}(G)$$

is a homeomorphic embedding such that

$$\forall g, h \in G, \qquad \iota(gh) = \iota(g)\cdot\iota(h) \;\&\; \iota(g^{-1}) = \iota(g)^*.$$

Moreover, $\iota(G)$ is exactly the set of invertible elements of $\mathbf{H}(G)$. In other words, G is canonically isomorphic, as a topological group, to the set of invertible elements of $\mathbf{H}(G)$.

Proof. This is contained in Remark 1.5 after identifying G with $\mathrm{Aut}(\mathcal{M}_G)$ by Remark 1.6(2). ∎

Finally, we can reformulate the previous results as an equivalence of categories, or more precisely a duality. Let **RPnA** be the category whose objects are the Roelcke-precompact non-archimedean Polish groups and arrows the continuous group morphisms. Note that if $p\colon G \to H$ is a continuous group morphism between topological groups, it induces a functor $\widetilde{p}\colon \mathbf{Rep}(H) \to \mathbf{Rep}(G)$ by setting $\widetilde{p}(\pi) = \pi \circ p$ for every $\pi \in \mathbf{Rep}(H)$ and $\widetilde{p}(h) = h$ for every morphism h between representations of H. The functor $\widetilde{p}$ has additional properties, it is *admissible* in the following sense:

DEFINITION 4.2. Let G, H be topological groups. A functor

$$F\colon \mathbf{Rep}(H) \to \mathbf{Rep}(G)$$

is said to be *admissible* if it satisfies the following:

(i) F sends the trivial one-dimensional representation of H to the trivial one-dimensional representation of G.
(ii) For every $\pi \in \mathbf{Rep}(H)$, $F(\pi)$ has the same underlying Hilbert space as π.

(iii) For all $\pi, \pi' \in \mathbf{Rep}(H)$ and $h \in \mathrm{Hom}(\pi, \pi')$, $F(h) = h$.
(iv) F commutes with tensor product of representations, i.e. for all $\pi, \pi' \in \mathbf{Rep}(G)$,

$$F(\pi \otimes \pi') = F(\pi) \otimes F(\pi').$$

The collection of such functors is stable under composition and contains the identity maps. We can thus form the category $\mathbf{Rep}(\mathbf{RPnA})$ whose class of objects is $(\mathbf{Rep}(G))_{G\in\mathbf{RPnA}}$ and whose arrows are the admissible functors. We obtain the following, which is also true for compact (resp. locally compact abelian) groups by the usual duality theories of Pontryagin–van Kampen and Tannaka–Kreĭn.

THEOREM 4.3. *The canonical contravariant functor*

$$\mathbf{Rep}\colon \mathbf{RPnA} \to \mathbf{Rep}(\mathbf{RPnA}), \qquad G \mapsto \mathbf{Rep}(G),\ p \mapsto \widetilde{p},$$

is a duality (*i.e. surjective and fully faithful*).

Proof. It is clear by construction that $\mathbf{Rep}$ is surjective with regard to objects. To prove that it is faithful, let $G, H \in \mathbf{RPnA}$ and assume that $p, q\colon G \to H$ are such that $\widetilde{p} = \widetilde{q}$. Then, for every $\pi \in \mathbf{Rep}(G)$, we have $\pi \circ p = \widetilde{p}(\pi) = \widetilde{q}(\pi) = \pi \circ q$. Since non-archimedean Polish groups satisfy the Gel'fand–Raĭkov Theorem (Fact 1(3)), this implies that $p = q$.

Finally, we prove that $\mathbf{Rep}$ is full. Let $G, H \in \mathbf{RPnA}$ and fix an admissible functor $F\colon \mathbf{Rep}(H) \to \mathbf{Rep}(G)$. Let us prove that there exists $p\colon G \to H$ such that $F = \widetilde{p}$. First, we show that F induces a map $p_F\colon \mathbf{T}(G) \to \mathbf{T}(H)$. Indeed, let $u \in \mathbf{T}(G)$ and $\pi \in \mathbf{Rep}(H)$. Since F is admissible, we can define $p_F(u)$ at π by setting $p_F(u)_\pi = u_{F(\pi)}$. Now, let $\pi, \pi' \in \mathbf{Rep}(H)$ and let $h \in \mathrm{Hom}(\pi, \pi')$. Since F is a functor, $F(h)$ lies in $\mathrm{Hom}(F(\pi), F(\pi'))$. Hence, by definition of $\mathbf{T}(G)$, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{H}_{F(\pi)} \arrow[r, "u_{F(\pi)}"] \arrow[d, "F(h)"'] & \mathcal{H}_{F(\pi)} \arrow[d, "F(h)"] \\
\mathcal{H}_{F(\pi')} \arrow[r, "u_{F(\pi')}"] & \mathcal{H}_{F(\pi')}
\end{tikzcd}$$

Since F is admissible, the above diagram is *equal* to the following:

$$\begin{tikzcd}
\mathcal{H}_{\pi} \arrow[r, "p_F(u)_\pi"] \arrow[d, "h"'] & \mathcal{H}_{\pi} \arrow[d, "h"] \\
\mathcal{H}_{\pi'} \arrow[r, "p_F(u)_{\pi'}"] & \mathcal{H}_{\pi'}
\end{tikzcd}$$

This shows that $p_F(u) \in \mathcal{N}at(\mathbf{U}_H)$.

By admissibility of F and since $u \in \mathbf{T}(G)$, we have

$$p_F(u)_{\mathbf{1}} = u_{F(\mathbf{1})} = u_{\mathbf{1}} = \mathrm{Id}_{\mathbb{C}},$$

Moreover, for all $\pi, \pi' \in \mathbf{Rep}(H)$, since u is in $\mathbf{T}(G)$ again,

$$p_F(u)_{\pi\otimes\pi'} = u_{F(\pi\otimes\pi')} = u_{F(\pi)\otimes F(\pi')} = u_{F(\pi)} \otimes u_{F(\pi')} = p_F(u)_\pi \otimes p_F(u)_{\pi'}.$$

Hence $p_F(u)$ is an element of $\mathbf{T}(H)$. Clearly, p_F is continuous and respects the *-monoid structures. Recalling the last statement of Theorem 4.1, we see that p_F restricts to a continuous group morphism $p\colon G \to H$ that induces F. ■

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