1. (10 points) Consider a sequence $s_n = 1^2 + 2^2 + \cdots + n^2$ for all positive integers n.
   (a) (2 points) Build a linear non-homogeneous recurrence relation for this sequence.
   (b) (3 points) Write an algorithm to compute s_n for any n in pseudo-code.
   (c) (5 points) Prove that $s_n = \frac{1}{6}n(n+1)(2n+1)$ with any method you know.

(a) $S_n = S_{n-1} + n^2$, $S_1 = 1$.

(b)

```
procedure sum-sq(n: integer)
S := 0
for i := 1 to n
    S := S + i^2
return S
```

```
procedure recursive sum_sq(n: integer)
if n = 1 then return 1
else return recursive sum_sq(n-1) + n^2
```

(c) Either use math. induction or solve (a) with initial value $S_1 = 1$.
I will provide another method below:

$$+\begin{cases} (n+1)^3 = \cancel{n^3} + 3n^2 + 3n + 1 \\ \cancel{n^3} = \cancel{(n-1)^3} + 3(n-1)^2 + 3(n-1) + 1 \\ \quad \vdots \\ \cancel{3^3} = \cancel{2^3} + 3\cdot 2^2 + 3\cdot 2 + 1 \\ \cancel{2^3} = 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1 \end{cases}$$

$$(n+1)^3 = 1^3 + 3\cdot(1^2+2^2+\dots+n^2) + 3(1+2+\dots+n) + n$$

$$\Rightarrow S_n = \frac{1}{3}\left((n+1)^3 - 1 - 3\cdot\frac{n+1}{2}\cdot n - n\right) = \frac{1}{3}\left(n^3 + 3n^2 + 3n - \frac{3n^2+3n}{2} - n\right)$$

$$= \frac{1}{6}\left(2n^3 + 6n^2 + 6n - 3n^2 - 3n - 2n\right) = \frac{1}{6}\left(2n^3 + 3n^2 + n\right) = \frac{n}{6}\left(2n^2 + 3n + 1\right)$$

$$= \frac{n}{6}\left(2n^2 + n + 2n + 1\right) = \frac{n(n+1)(2n+1)}{6}.$$

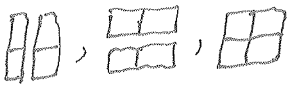
2. (10 points) Let a_n be the number of all possible ways to tile a rectangle of size $2 \times n$ by the tiles of sizes 1×2, 2×2 and 1×4. When we refer to tiling a board, we imply that tiles can be rotated, but they are not allowed to be flipped, overlapped or cut. They must precisely cover the entire board without extending beyond its edges.

(a) (1 point) Find a_1 and a_2. (b) (2 points) Find a_3 and a_4.

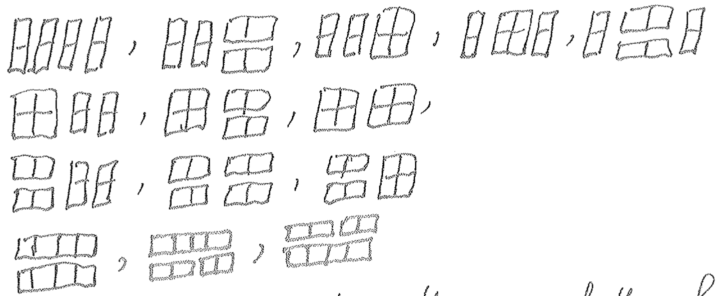
(c) (3 points) Describe clearly with words a **recursive** algorithm to generate all such tiling for any n.

(d) (2 points) Build a recurrence relation to compute a_n.

(e) (2 points) Find the characteristic equation for the recurrence relation to compute a_n. Do not solve it. How many initial conditions are required to solve the recurrence relation to compute a_n?

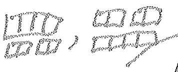
(a) $n=1$  $a_1 = 1$, $n=2$  $a_2 = 3$

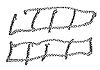
(b) $n=3$  $a_3 = 5$

$n=4$  $a_4 = 12 + 2 = 14$

(c) For $n \geq 4$ we can start with one of the following:

- put a block , then there are a_{n-1} ways to continue;
- put blocks , then to continue we tile $2\times(n-2)$ rectangle in a_{n-2} ways;
- put a block , =//= a_{n-2} ways;

same for 

- put blocks , the rest $2 \times (n-4)$ is tiled in a_{n-4} ways

Clearly, we continue recursively and the first cases are in (a), (b).

(d) $a_n = a_{n-1} + 2a_{n-2} + 3a_{n-4}$

(e) $\lambda^n = \lambda^{n-1} + 2\lambda^{n-2} + 3\lambda^{n-4} \Rightarrow \lambda^4 - \lambda^3 - 2\lambda^2 - 3 = 0$

Four initial values are needed to find the value of a_n, $\forall n \geq 4$.

3. (10 points) Relations

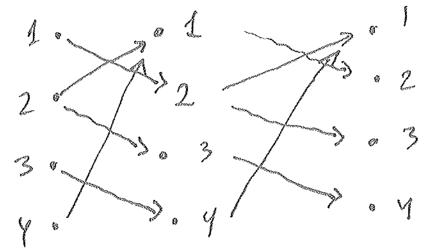
(a) (2 points) Let A be of n elements and B be of m elements. How many different relations are there from set A to set B?

(b) (2 points) Find $R \circ R$ for $R = \{(1,2),(2,1),(2,3),(3,4),(4,1)\}$.

(c) (6 points) Find the transitive closure of the relation R from (b).

(a) $|A| = n$, $|B| = m$ $\Rightarrow$ $|A \times B| = nm$ and $R \subset A \times B$. Since there are $2^{|A\times B|}$ subsets of the set $A \times B$, we get 2^{nm} different relations.

(b)



$\Rightarrow R \circ R = \{(1,1),(1,3),(2,2),(2,4),(3,1),(4,2)\}$ or in the matrix form $M_{R\circ R} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(c) $M_{TRAN.CL.} = M_R \vee M_{R\circ R} \vee M_{R\circ R\circ R} \vee M_{R\circ R\circ R\circ R}$

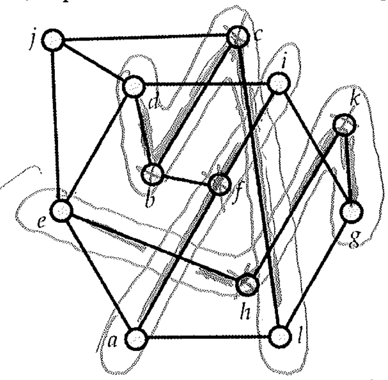
$M_{R\circ R\circ R} = M_{R\circ R} \odot M_R = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

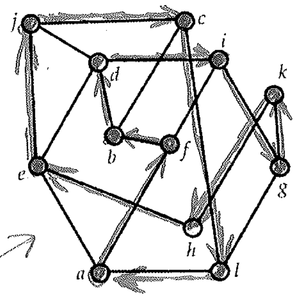
$M_{R\circ R\circ R\circ R} = M_{R\circ R\circ R} \odot M_R = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

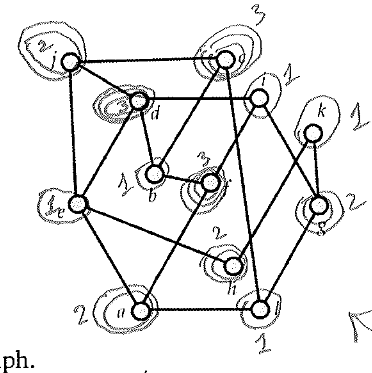
$M_{TRAN.CL} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

4. (10 points) Consider the following graph (2 more copies are provided here for you to use as a draft):







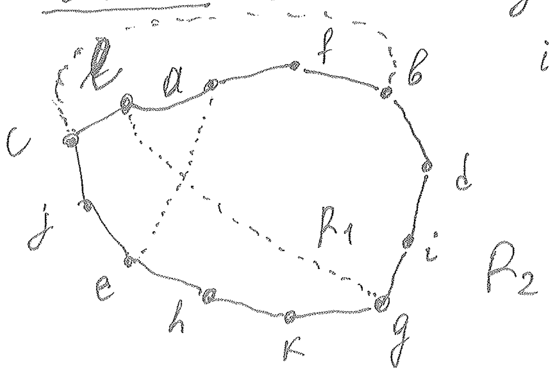
(a) (2 points) Describe a cycle (closed path) of the maximal length in this graph.

(a) (2 points) Is this a bipartite graph? NO, because j, d, e is a triangle

(b) (2 points) Find the chromatic number of this graph. $\chi(G)=3$ since $\exists\Delta$ and 3 is enough

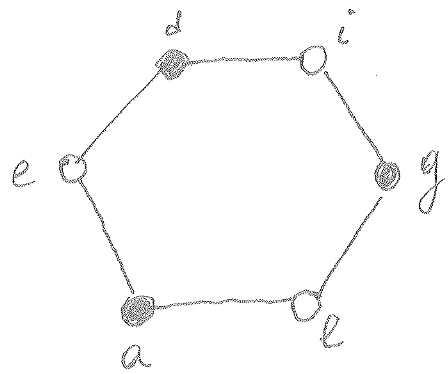
(c) (4 points) Is this a planar graph? NO

(c) Solution 1: Consider the cycle of the max length:



it splits the plane into 2 regions R_1 and R_2. 6 edges left to be drawn, but the edges ae, bc, gl are already intersecting no matter how you try to redraw the graph.

Solution 2: let's find a homeomorphic copy of $K_{3,3}$ in it.



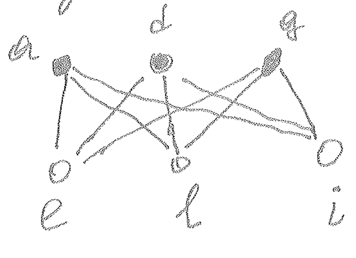
→ each has degree 2, now we need only to "deform" edges

ai (through f)

ld (through ~~c~~ c, b)

eg (through h, k)

and we get $K_{3,3}$:



5. (10 points) Circle your answers. Use the margins and the rest of the page as a draft.

5.1 A recursive algorithm solves a problem of size n by dividing it into 4 sub-problems of size $n/2$, solving each sub-problem and combining the solution with additional 3 operations. Find Big-Theta estimate of this algorithm.

(a) n^2 (b) $n^2 \log n$ (c) n (d) $n \log n$ (e) no correct answer

$f(n) = 4f(\frac{n}{2}) + 3$ $\quad n^{\log_2 4} = n^2$

5.2 A recursive algorithm solves a problem of size n by dividing it into 7 sub-problems of size $n/4$, solving each sub-problem, and then combining the solutions in $n^3 + 1$ additional operations. Find Big-Theta estimate of this algorithm.

(a) n^2 (b) n^3 (c) $n^{\log_4 7}$ (d) $n^2 \log n$ (e) no correct answer

$f(n) = 7f(\frac{n}{4}) + n^3 + 1$ $\quad \log_4 7 < 3$ $\quad n^3$

5.3 A recursive algorithm solves a problem of size n by solving two sub-problems of size $n-1$, then combining the solution in $n \log n$ operations. Which of the following functions is the Big-Theta estimate for its complexity:

(a) n^2 (b) $n^2 \log n$ (c) 2^n (d) $2^n \log n$ (e) no correct answer

$a_n = 2a_{n-1} + n \log n$ $\quad (a_n)_{hom} = \alpha 2^n$

5.4 The general form of the homogeneous part of a recurrence relation given by $a_n = 2a_{n-1} - a_{n-2} + n$ is of the form $\alpha + \beta n$. In which form a particular solution should be searched for?

$\alpha \cdot 1^n + \beta n \cdot 1^n$

(a) $An + B$ (b) $n(An + B)$ (c) $n^2(An + B)$ (d) $n^3(An + B)$ (e) no correct answer

$n \cdot 1^n$, 1 is used in hom. part

5.5 In which form a particular solution for a recurrence relation $a_n = 2a_{n-1} - a_{n-2} + n2^n$ should be searched for?

2 is NOT a root

(a) $(An + B)2^n$ (b) $n(An + B)2^n$ (c) $n^2(An + B)2^n$ (d) $n^3(An + B)2^n$ (e) no correct answer

5.6 A set of bitstrings S is generated by the following rule: empty string $\lambda \in S$, and whenever $s \in S$, then $s000 \in S$ and $111s \in S$. How many bitstrings of length 9 are in S?

(a) 2^9 (b) 2^3 (c) 4 (d) 5 (e) no correct answer

λ
000 111
000000 111000 111111
000000000 111000000 111111000 111111111

5.7 Given $f(1) = 2$ and $f(n) = f(\frac{n}{3}) + n$ for any n divisible by 3, compute $f(27)$. $= f(9) + 27 = f(3) + 9 + 27 = f(1) + 3 + 9 + 27 = 2 + 30 + 9 = 41$

(a) 41 (b) 37 (c) 29 (d) 81 (e) no correct answer

5.8 There are 3 groups in a certain course. If 17 students are failing the course, which of the following is definitely true? $\lceil 17/3 \rceil = 6$

(a) Group 1 has at least 11 students failing (b) Group 2 has at least 6 students failing (c) Group 3 has at least 4 students failing (d) There is a group with at least 6 students failing (e) no correct answer

5.9 How many bitstrings of length 5 are there that do not contain 0000?

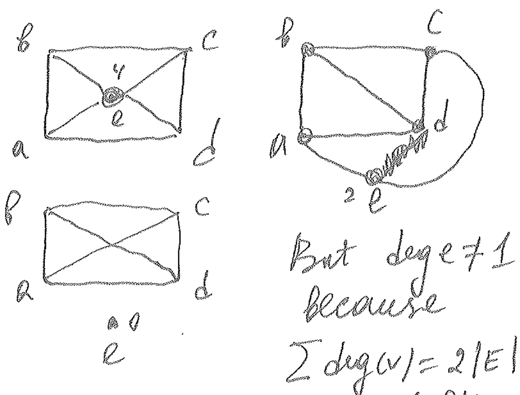
(a) 25 (b) 29 (c) 31 (d) 28 (e) no correct answer

5.10 A graph with vertices $\{a, b, c, d, e\}$ satisfies $\deg a = \deg b = \deg c = \deg d = 3$. Which of the following cannot be the value of $\deg e$?

(a) 4 (b) 2 (c) 1 (d) 0 (e) no correct answer

10000
00001
00000 } the only bitstrings that contain 0000

$\Rightarrow 2^5 - 3 = 32 - 2 = 29$ do not contain 0000



But $\deg e \neq 1$ because $\sum \deg(v) = 2|E|$

ODD $= 3+3+3+3+1 \neq$ even

End of Exam