# MAE6254: Homework 2

Due date: February 22, 2016

**Problem 1** Consider the following equation of motion for a pendulum with friction:

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1 - x_2.$$

Using the following Lyapunov function, show that the equilibrium $x^* = [0,0]^T$ is asymptotically stable.

$$V = \frac{1}{2}x_2^2 + \frac{1}{2}(x_1 + x_2)^2 + c\frac{g}{l}(1 - \cos x_1),$$

where $c$ is a constant that you have to specify to show asymptotic stability.

**Problem 2** Consider the following dynamic system:

$$\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2),$$
$$\dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2).$$

(a) Show that $x^* = [0,0]^T$ is an equilibrium of this system.

(b) Find the linearized equation about $x^* = [0,0]^T$, and determine the type (center, saddle, etc) of the equilibrium using eigenvalues.

(c) Using the following Lyapunov function,

$$V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2,$$

show that $x^*$ is (locally) asymptotically stable (Hint: define a domain by restricting $x_1^2 + x_2^2$).

(d) Find an estimate of the region of attraction from your results at (c).

(e) Show that $x^*$ is not globally asymptotically stable: give any initial condition $x_0$ such that the solution of the state equation $x(t)$ with $x(0) = x_0$ does not asymptotically converge to the origin.

**Problem 3** Consider the following dynamic system:

$$\dot{x}_1 = -\frac{6x_1}{(1 + x_1^2)^2} + 2x_2,$$
$$\dot{x}_2 = -\frac{2x_1 + 2x_2}{(1 + x_1^2)^2}.$$

(a) Show that $x^* = [0,0]^T$ is an equilibrium of this system.

(b) Using the following Lyapunov function, show that $x^*$ is (locally) asymptotically stable:

$$V = \frac{x_1^2}{1 + x_1^2} + x_2^2.$$

(c) Can you claim that $x^*$ is globally asymptotically stable from your results at (b). Why?

(d) Draw the phase portrait of this system (plot the solutions in the $x_1 - x_2$ plane for varying initial conditions using the Matlab `ode45` function), and show that the origin is not globally asymptotically stable.