HOMEWORK 3 SOLUTIONS

1. Taking the Laplace transform of the definition of f_c yields

$$F_c(s) = (-K_V s - K_P)X(s)$$

and so

$$|F_c(i\omega)| = \sqrt{(K_V\omega)^2 + K_P^2}\,|X(i\omega)| = \sqrt{(K_V\omega)^2 + K_P^2}\,\frac{|F(i\omega)|}{\sqrt{(k_{eff} - m\omega^2)^2 + (c_{eff}\omega)^2}}$$

Note that $k_{eff} = m\omega_n^2 = (20kg)(15\text{ rad/s})^2 = 4500\text{ N/m}$,

and $c_{eff} = 2\zeta m\omega_n = 2(0.2)(20kg)(15\text{ rad/s}) = 120\text{ N/(m/s)}$, so these values are fixed. Also, we have no control over ω or $|F(i\omega)|$. Therefore, in order to minimize $|f_c|$, we must minimize $\sqrt{(K_V\omega)^2 + K_P^2}$.

We must first check to see if K_V and K_P are dependent:

$$K_P = m\omega_n^2 - k \text{ and } K_V = 2\zeta m\omega_n - c$$

which shows that these parameters are independent. Therefore, we should try to minimize each one as much as possible while satisfying the constraints of the problem.

Since k_{eff} is greater than any of the available springs, we should choose the stiffest spring so that K_P is the smallest. In this case,

$$K_P = (20kg)(15\text{ rad/s})^2 - 4000\text{ N/m} = 500\text{ N/m}$$

Since c_{eff} is greater than any of the available dampers, we should choose the most viscous damper so that K_V is the smallest. In this case,

$$K_V = 2(0.2)(20kg)(15\text{ rad/s}) - 100\text{ N/(m/s)} = 20\text{ N/(m/s)}$$

2. This system is underdamped since it exhibits overshoot.

From the overshoot: $\zeta = \dfrac{-\ln(0.839)}{\sqrt{\pi^2 + \ln^2(0.839)}} = 0.0558$.

From the settling time: $\zeta\omega_n = \dfrac{4}{77.7\text{ s}} = 0.0515\text{ rad/s}$, and so $\omega_n = \dfrac{0.0515\text{ rad/s}}{0.0558} = 0.923\text{ rad/s}$.

The Laplace transform of the unit step response is

$$X(s) = \frac{1}{ms^2 + cs + k}\frac{1}{s}$$

Using the final value theorem,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0}\frac{1}{ms^2 + cs + k} = \frac{1}{k} = 0.125$$

and so $k = 8\text{ N/m}$. Consequently,

$$m = \frac{k}{\omega_n^2} = \frac{8\text{ N/m}}{(0.923\text{ rad/s})} = 9.39\text{ kg}\text{, and}$$
$$c = 2\zeta m\omega_n = 2(0.0558)(9.39kg)(0.923\text{ rad/s}) = 0.967\text{ N/(m/s)}$$

Now, we can repeat this process for the desired step response.

From the overshoot: $\zeta = \dfrac{-\ln(0.583)}{\sqrt{\pi^2 + \ln^2(0.583)}} = 0.169$.

From the settling time: $\zeta\omega_n = \dfrac{4}{22.9\text{ s}} = 0.175\text{ rad/s}$, and so $\omega_n = \dfrac{0.175\text{ rad/s}}{0.169} = 1.03\text{ rad/s}$.

Using the final value theorem,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) = \lim_{s\to 0}\frac{1}{ms^2 + cs + k} = \frac{1}{k} = 0.0833$$

and so $k = 12\text{ N/m}$. Consequently,

$$m = \frac{k}{\omega_n^2} = \frac{12\text{ N/m}}{(1.03\text{ rad/s})} = 11.65\text{ kg}\text{, and}$$

$$c = 2\zeta m\omega_n = 2(0.169)(11.65kg)(1.03\text{ rad/s}) = 4.056\text{ N/(m/s)}$$

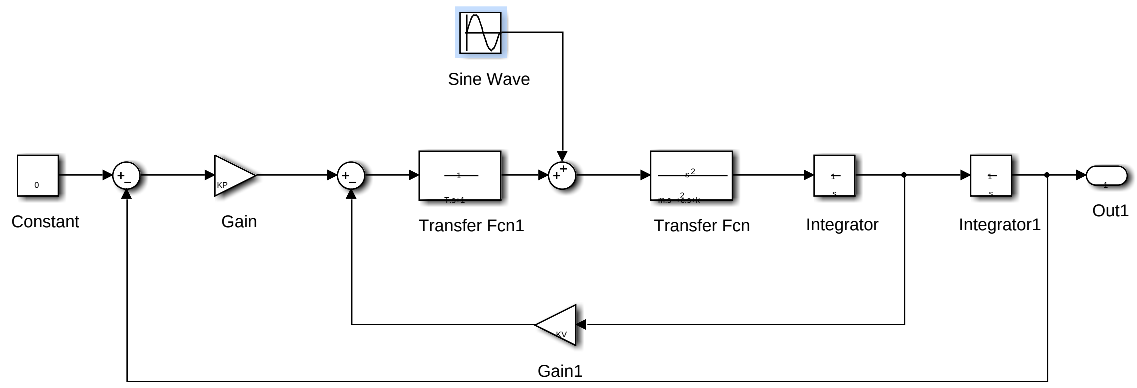
The difference in these values gives the feedback gains:

$$K_A = 11.65\text{ kg} - 9.39\text{ kg} = 2.26\text{ kg}$$

$$K_V = 4.056\text{ N/(m/s)} - 0.967\text{ N/(m/s)} = 3.089\text{ N/(m/s)}$$

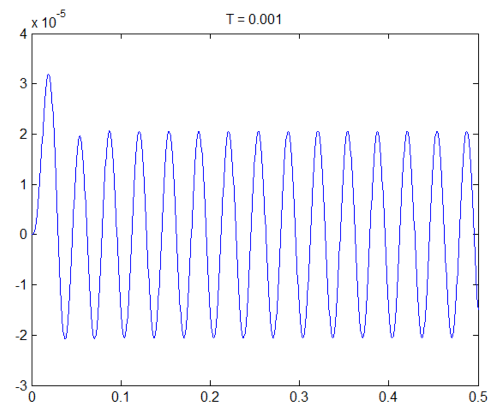
$$K_P = 12\text{ N/m} - 8\text{ N/m} = 4\text{ N/m}$$

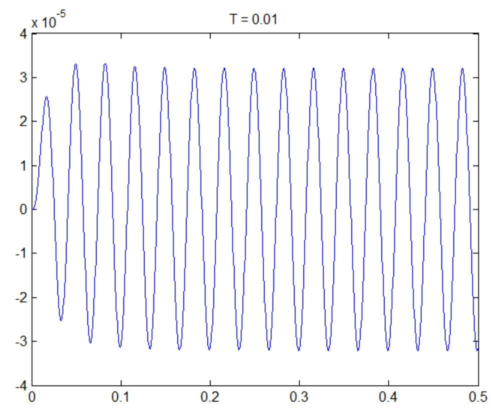
3. a. The Simulink model with actuator dynamics appears as

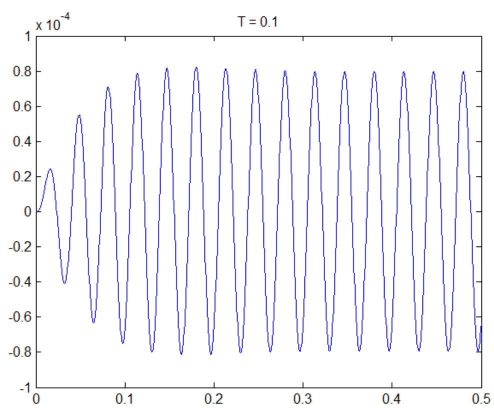


Note that the scope has been replaced by an output port in order to plot the response in the script (below).

Responses for various values of T are plotted:







For small values of T the response looks like the closed-loop response plotted in class because, in this case, $H(s) \approx 1$. This means the actuator is perfect: it can respond instantly and accurately to any command signal. Large values of T mean the actuator is very slow to respond, so at high frequencies it produces almost no output (it acts as a low pass filter with a very low cutoff frequency). Thus, the system behaves as if there is no feedback loop, and the response resembles the case in which there is no feedback control.

b. The EOM is $m\ddot{x} + c\dot{x} + kx = f_c + f$. We wish to draw root loci for the transfer function $\dfrac{X(s)}{R(s)}$, so we will set $f = 0$.

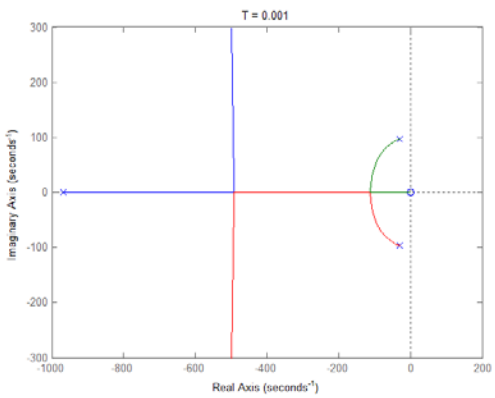
Then taking the Laplace transform yields

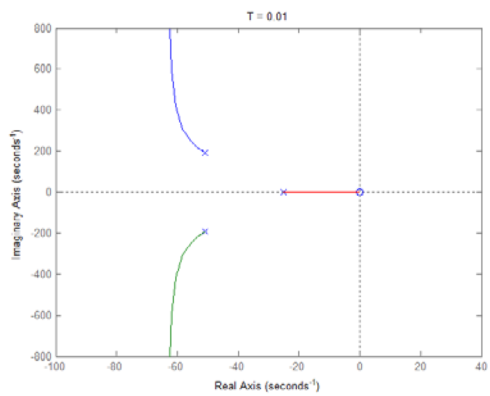
$$ms^2 X(s) + csX(s) + kX(s) = F_c(s) = \frac{-K_V sX(s) - K_P[X(s) - R(s)]}{Ts+1}$$

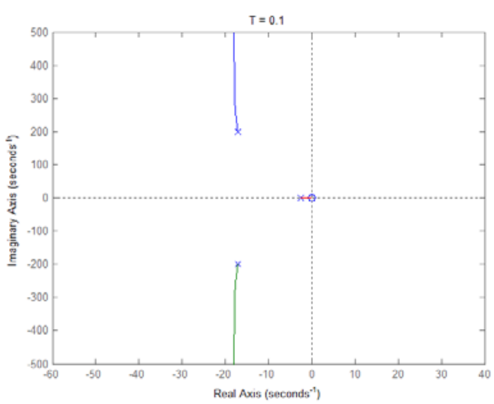
Rearranging this equation results in

$$\frac{X(s)}{R(s)} = \frac{K_P}{\underbrace{mTs^3 + (m + cT)s^2 + (c + kT)s + (k + K_P)}_{d(s)} + K_V \underbrace{s}_{n(s)}}$$

Here are the root locus plots for the same three values of T :







Adding the actuator dynamics adds a third pole to the closed-loop system. For small values of T, this extra pole is far to the left of the plot, so it does not affect the dynamics significantly (it represents a very fast mode). For large values of T, it adds a pole near the origin. This slow pole dominates the plant's two poles, slowing down the whole system. Thus, the system cannot respond effectively to the external force, so it acts similar to the open-loop case.

The following MATLAB code produces the plots shown above:

```
m = 150;
c = 4000;
k = 6e6;
F = 100;
w = 30*2*pi;
KP = -4.5e6;
KV = 11e3;

for T = [1e-3 1e-2 1e-1]
    simout = sim('hw3_3model.slx','SaveTime','on','SaveOutput','on');
    figure;
    plot(get(simout,'tout'),get(simout,'yout'));
    title(sprintf('T = %g',T));
    G = tf([1 0],[m*T m+c*T c+k*T k+KP]);
    figure;
    h = rlocusplot(G);
    p = getoptions(h);
    p.Title.String = sprintf('T = %g',T);
    setoptions(h,p);
end
```