

MAE6292: Homework 1

Due date: January 27, 2015

Problem 1 Consider the following linear equation,

$$Ax = y,$$

where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$, and $y \in \mathbb{R}^m$.

- (a) Suppose $m \geq n$, and $\text{rank}(A) = n$. In this case, the above equation is overdetermined, i.e., in general, there is no x satisfying the above equation for a given y . Find the best approximate solution x that minimizes the error, $\|Ax - y\|^2 = (y - Ax)^T(y - Ax)$. Answer in terms of y and A .
- (b) Suppose $n \geq m$, and $\text{rank}(A) = m$. In this case, the above equation is underdetermined, i.e., there are infinite number of x satisfying the above equation for a given y . Find the particular solution x that has the smallest value of $\|x\|^2 = x^T x$. Answer in terms of y and A .

Problem 2

Consider a quadratic cost function,

$$J(x, u) = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru,$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $Q \in \mathbb{R}^{n \times n}$, and $R \in \mathbb{R}^{m \times m}$. Assume that Q is symmetric and positive-semidefinite, and R is symmetric and positive-definite. There is an equality constraint given by

$$f(x, u) = x + Bu + c = 0,$$

where $B \in \mathbb{R}^{n \times m}$ and $c \in \mathbb{R}^n$.

- (a) Derive the conditions for a stationary point, i.e., necessary condition for local minimum.
- (b) Show that the optimal control, state and multiplier values are given by

$$\begin{aligned} u &= -(R + B^T Q B)^{-1} B^T Q c, \\ x &= -(I - B(R + B^T Q B)^{-1} B^T Q) c, \\ \lambda &= (Q^{-1} + B R^{-1} B^T)^{-1} c. \end{aligned}$$

(You may have to use the matrix inversion lemma, http://en.wikipedia.org/wiki/Woodbury_matrix_identity).

- (c) Suppose

$$Q = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad R = I_{2 \times 2}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Write a Matlab code to find the optimal x , u , and λ using the steepest gradient method iteratively.

- (d) Numerically compare your solution of (c) with (b).

Problem 3 Consider the following optimization problem,

$$\text{minimize } J(x) = - \sum_{i=1}^n \log(\alpha_i + x_i), \quad (1)$$

$$\text{subject to } \sum_{i=1}^n x_i = 1, \text{ and } x_i \geq 0, \text{ for all } 1 \leq i \leq n, \quad (2)$$

where the constants $\alpha_i > 0$ are given. It arises in information theory, to allocate the transmitter power x_i with the fixed sum to n channels, in an optimal fashion to maximize the total communication rate.

- (a) Derive the KKT conditions for optimality.
- (b) Solve the condition for stationary point for the multiplier μ_i of inequality constraints, and substitute it to the KKT conditions to rewrite them in terms of x_i, λ, α_i only.
- (c) Show that the optimal state is given by

$$x_i = \max\{0, \frac{1}{\lambda} - \alpha_i\}.$$

- (d) Suppose $n = 3, \alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3$. Find the values of the optimal states x_i and multipliers λ, μ_i .