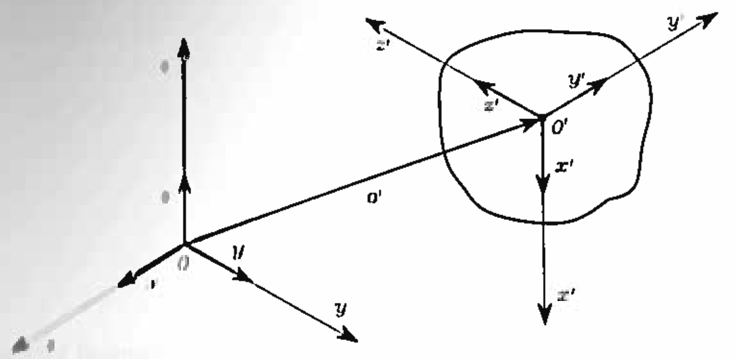

Figure 2.1 Position and orientation of a rigid body.

In order to describe the rigid body orientation, it is convenient to consider an orthonormal frame attached to the body and express its unit vectors with respect to the reference frame. Let then O'–$x'y'z'$ be such frame with origin in O' and $\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{z}'$ be the unit vectors of the frame axes. These vectors are expressed with respect to the reference frame O–xyz by the equations:

$$\begin{aligned} \boldsymbol{x}' &= x'_x\boldsymbol{x} + x'_y\boldsymbol{y} + x'_z\boldsymbol{z} \\ \boldsymbol{y}' &= y'_x\boldsymbol{x} + y'_y\boldsymbol{y} + y'_z\boldsymbol{z} \\ \boldsymbol{z}' &= z'_x\boldsymbol{x} + z'_y\boldsymbol{y} + z'_z\boldsymbol{z}. \end{aligned} \tag{2.2}$$

from $\cos\theta = \frac{A\cdot B}{\|A\cdot B\|}$

The components of each unit vector are the direction cosines of the axes of frame O'–$x'y'z'$ with respect to the reference frame O–xyz.

2.2 Rotation Matrix

By adopting a compact notation, the three unit vectors in (2.2) describing the body orientation with respect to the reference frame can be combined in the (3×3) matrix

$$\boldsymbol{R} = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}'^T\boldsymbol{x} & \boldsymbol{y}'^T\boldsymbol{x} & \boldsymbol{z}'^T\boldsymbol{x} \\ \boldsymbol{x}'^T\boldsymbol{y} & \boldsymbol{y}'^T\boldsymbol{y} & \boldsymbol{z}'^T\boldsymbol{y} \\ \boldsymbol{x}'^T\boldsymbol{z} & \boldsymbol{y}'^T\boldsymbol{z} & \boldsymbol{z}'^T\boldsymbol{z} \end{bmatrix}, \tag{2.3}$$

which is termed *rotation matrix*.

It is worth noting that the column vectors of matrix $\boldsymbol{R}$ are mutually orthogonal since they represent the unit vectors of an orthonormal frame, *i.e.*,

$$\boldsymbol{x}'^T\boldsymbol{y}' = 0 \qquad \boldsymbol{y}'^T\boldsymbol{z}' = 0 \qquad \boldsymbol{z}'^T\boldsymbol{x}' = 0.$$

Also, they have unit norm

$$\boldsymbol{x}'^T\boldsymbol{x}' = 1 \qquad \boldsymbol{y}'^T\boldsymbol{y}' = 1 \qquad \boldsymbol{z}'^T\boldsymbol{z}' = 1.$$

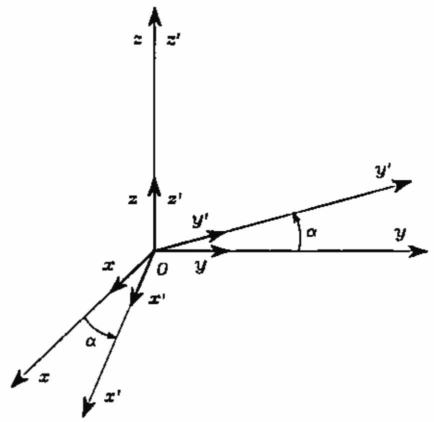

Figure 2.2 Rotation of the frame O–xyz by an angle α about axis z.

As a consequence, $\boldsymbol{R}$ is an *orthogonal* matrix meaning that

$$\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I} \tag{2.4}$$

where $\boldsymbol{I}$ denotes the (3×3) identity matrix.

If both sides of (2.4) are postmultiplied by the inverse matrix $\boldsymbol{R}^{-1}$, the useful result is obtained:

$$\boldsymbol{R}^T = \boldsymbol{R}^{-1}, \tag{2.5}$$

that is, the transpose of the rotation matrix is equal to its inverse. Further, observe that $\det(\boldsymbol{R}) = 1$ if the frame is right-handed, while $\det(\boldsymbol{R}) = -1$ if the frame is left-handed. (?)

2.2.1 Elementary Rotations

Consider the frames that can be obtained via *elementary rotations* of the reference frame about one of the coordinate axes. These rotations are positive if they are made counter-clockwise about the relative axis.

Suppose that the reference frame O–xyz is rotated by an angle α about axis z (Figure 2.2), and let O–$x'y'z'$ be the rotated frame. The unit vectors of the new frame can be described in terms of their components with respect to the reference frame, *i.e.*,

$$\boldsymbol{x}' = \begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{y}' = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \qquad \boldsymbol{z}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Hence, the rotation matrix of frame O–$x'y'z'$ with respect to frame O–xyz is

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.6}$$

In a similar manner, it can be shown that the rotations by an angle β about axis y and by an angle γ about axis x are respectively given by:

$$\boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2.7}$$

$$\boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}. \tag{2.8}$$

These matrices will be useful to describe rotations about an arbitrary axis in space.

It is easy to verify that for the elementary rotation matrices in (2.6)–(2.8) the following property holds:

$$\boldsymbol{R}_k(-\vartheta) = \boldsymbol{R}_k^T(\vartheta) \qquad k = x, y, z. \tag{2.9}$$

In view of (2.6)–(2.8), the rotation matrix can be attributed a geometrical meaning; namely, the matrix $\boldsymbol{R}$ describes the rotation about an axis in space needed to align the axes of the reference frame with the corresponding axes of the body frame.

2.2.2 Representation of a Vector

In order to understand a further geometrical meaning of a rotation matrix, consider the case when the origin of the body frame coincides with the origin of the reference frame (Figure 2.3); it follows that $\boldsymbol{o}' = \boldsymbol{0}$, where $\boldsymbol{0}$ denotes the (3×1) null vector. A point P in space can be represented either as

$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

with respect to frame O–xyz, or as

$$\boldsymbol{p}' = \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix}$$

with respect to frame O–$x'y'z'$.

Since $\boldsymbol{p}$ and $\boldsymbol{p}'$ are representations of the same point P, it is

$$\boldsymbol{p} = p'_x\boldsymbol{x}' + p'_y\boldsymbol{y}' + p'_z\boldsymbol{z}' = \begin{bmatrix} \boldsymbol{x}' & \boldsymbol{y}' & \boldsymbol{z}' \end{bmatrix} \boldsymbol{p}'$$

and, accounting for (2.3), it is

$$\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}'. \tag{2.10}$$

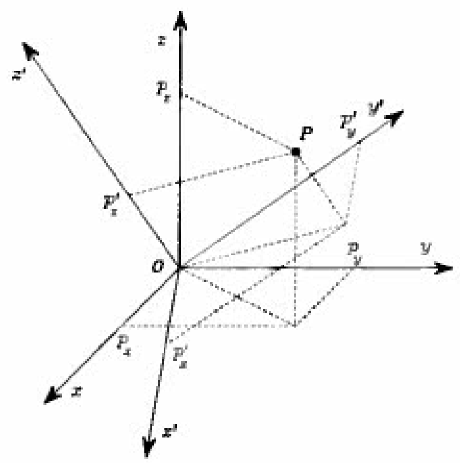

Figure 2.3 Representation of a point P in two different coordinate frames.

The rotation matrix $\boldsymbol{R}$ represents the *transformation matrix* of the vector coordinates in frame O–$x'y'z'$ into the coordinates of the same vector in frame O–xyz. In view of the orthogonality property (2.4), the inverse transformation is simply given by

$$\boldsymbol{p}' = \boldsymbol{R}^T\boldsymbol{p}. \tag{2.11}$$

Example 2.1

Consider two frames with common origin mutually rotated by an angle α about the axis z. Let $\boldsymbol{p}$ and $\boldsymbol{p}'$ be the vectors of the coordinates of a point P, expressed in the frames O–xyz and O–$x'y'z'$, respectively (Figure 2.4). On the basis of simple geometry, the relationship between the coordinates of P in the two frames is:

$$\begin{aligned} p_x &= p'_x \cos\alpha - p'_y \sin\alpha \\ p_y &= p'_x \sin\alpha + p'_y \cos\alpha \\ p_z &= p'_z. \end{aligned}$$

Therefore, the matrix (2.6) represents not only the orientation of a frame with respect to another frame, but it also describes the transformation of a vector from a frame to another frame with the same origin.

2.2.3 Rotation of a Vector

A rotation matrix can be also interpreted as the matrix operator allowing rotation of a vector by a given angle about an arbitrary axis in space. In fact, let $\boldsymbol{p}'$ be a vector in the reference frame O–xyz; in view of orthogonality of the matrix $\boldsymbol{R}$, the product $\boldsymbol{R}\boldsymbol{p}'$ yields a vector $\boldsymbol{p}$ with the same norm as that of $\boldsymbol{p}'$ but rotated with respect to

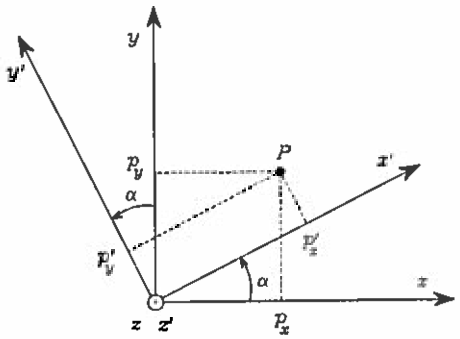

Figure 2.4 Representation of a point P in rotated frames.

$\boldsymbol{p}'$ according to the matrix $\boldsymbol{R}$. The norm equality can be proved by observing that $\boldsymbol{p}^T\boldsymbol{p} = \boldsymbol{p}'^T\boldsymbol{R}^T\boldsymbol{R}\boldsymbol{p}'$ and applying (2.4). This interpretation of the rotation matrix will be revisited later.

Example 2.2

Consider the vector $\boldsymbol{p}$ which is obtained by rotating a vector $\boldsymbol{p}'$ in the plane xy by an angle α about axis z of the reference frame (Figure 2.5). Let (p'_x, p'_y, p'_z) be the coordinates of the vector $\boldsymbol{p}'$. The vector $\boldsymbol{p}$ has components

$$p_x = p'_x \cos\alpha - p'_y \sin\alpha$$
$$p_y = p'_x \sin\alpha + p'_y \cos\alpha$$
$$p_z = p'_z.$$

It is easy to recognize that $\boldsymbol{p}$ can be expressed as

$$\boldsymbol{p} = \boldsymbol{R}_z(\alpha)\boldsymbol{p}',$$

where $\boldsymbol{R}_z(\alpha)$ is the same rotation matrix as in (2.6).

In sum, a rotation matrix attains three *equivalent geometrical meanings*:

- It describes the mutual orientation between two coordinate frames; its column vectors are the direction cosines of the axes of the rotated frame with respect to the original frame.
- It represents the coordinate transformation between the coordinates of a point expressed in two different frames (with common origin).
- It is the operator that allows rotating a vector in the same coordinate frame.

2.3 Composition of Rotation Matrices

In order to derive composition rules of rotation matrices, it is useful to consider the expression of a vector in two different reference frames. Let then O–$x_0y_0z_0$, O–$x_1y_1z_1$, O–$x_2y_2z_2$ be three frames with common origin O. The vector $\boldsymbol{p}$ describing the position of a generic point in space can be expressed in each of the above frames; let $\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{p}^2$ denote the expressions of $\boldsymbol{p}$ in the three frames[1].

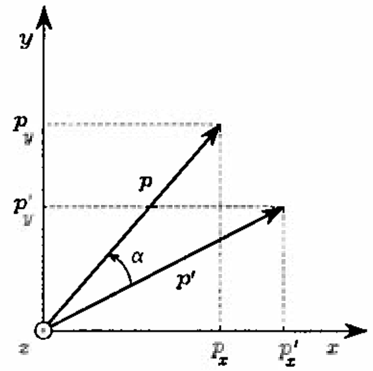

Figure 2.5 Rotation of a vector.

At first, consider the relationship between the expression $\boldsymbol{p}^2$ of the vector $\boldsymbol{p}$ in Frame 2 and the expression $\boldsymbol{p}^1$ of the same vector in Frame 1. If $\boldsymbol{R}_i^j$ denotes the rotation matrix of Frame i with respect to Frame j, it is

$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1\boldsymbol{p}^2. \tag{2.12}$$

Similarly, it turns out that

$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0\boldsymbol{p}^1 \tag{2.13}$$
$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0\boldsymbol{p}^2. \tag{2.14}$$

On the other hand, substituting (2.12) in (2.13) and using (2.14) gives

$$\boldsymbol{R}_2^0 = \boldsymbol{R}_1^0\boldsymbol{R}_2^1. \tag{2.15}$$

The relationship in (2.15) can be interpreted as the composition of successive rotations. Consider a frame initially aligned with the frame O–$x_0y_0z_0$. The rotation expressed by matrix $\boldsymbol{R}_2^0$ can be regarded as obtained in two steps:

- first rotate the given frame according to $\boldsymbol{R}_1^0$, so as to align it with frame O–$x_1y_1z_1$;
- then rotate the frame, now aligned with frame O–$x_1y_1z_1$, according to $\boldsymbol{R}_2^1$, so as to align it with frame O–$x_2y_2z_2$.

[1] Hereafter, the superscript of a vector or a matrix denotes the frame in which its components are expressed.

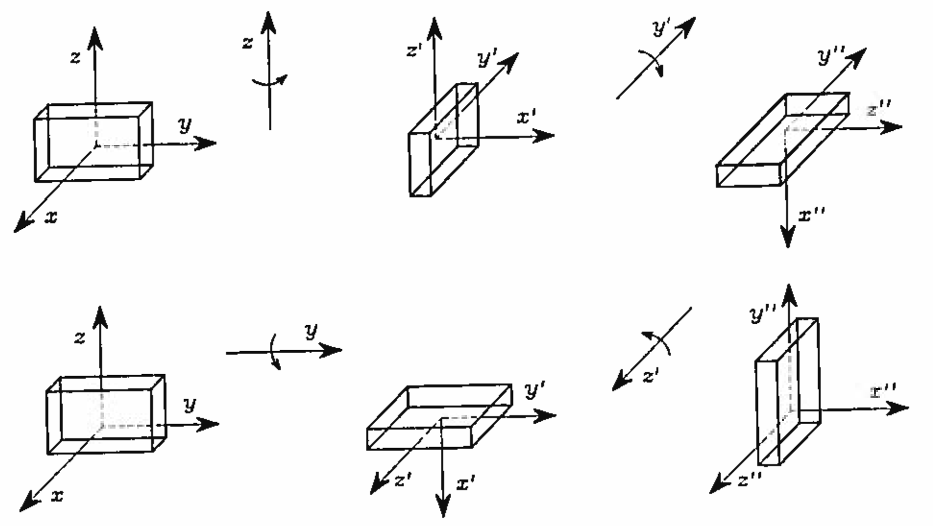

Figure 2.6 Successive rotations of an object about axes of current frame.
(i.e. Euler angle rotation)

Notice that the overall rotation can be expressed as a sequence of partial rotations; each rotation is defined with respect to the preceding one. The frame with respect to which the rotation occurs is termed *current frame*. Composition of successive rotations is then obtained by postmultiplication of the rotation matrices following the given order of rotations, as in (2.15). With the adopted notation, in view of (2.5), it is

$$R^j_i = (R^i_j)^{-1} = (R^i_j)^T. \tag{2.16}$$

Successive rotations can be also specified by constantly referring them to the initial frame; in this case, the rotations are made with respect to a *fixed frame*. Let R^0_1 be the rotation matrix of frame O–$x_1y_1z_1$ with respect to the fixed frame O–$x_0y_0z_0$. Let then $\bar{R}^0_2$ denote the matrix characterizing frame O–$x_2y_2z_2$ with respect to Frame 0, which is obtained as a rotation of Frame 1 according to the matrix $\bar{R}^1_2$. Since (2.15) gives a composition rule of successive rotations about the axes of the current frame, the overall rotation can be regarded as obtained in the following steps:

- first realign Frame 1 with Frame 0 by means of rotation R^1_0;
- then make the rotation expressed by $\bar{R}^1_2$ with respect to the current frame;
- finally compensate for the rotation made for the realignment by means of the inverse rotation R^0_1.

Since the above rotations are described with respect to the current frame, application of the composition rule (2.15) yields

$$\bar{R}^0_2 = R^0_1 R^1_0 \bar{R}^1_2 R^0_1.$$

In view of (2.16), it is

$$\bar{R}^0_2 = \bar{R}^1_2 R^0_1 \tag{2.17}$$

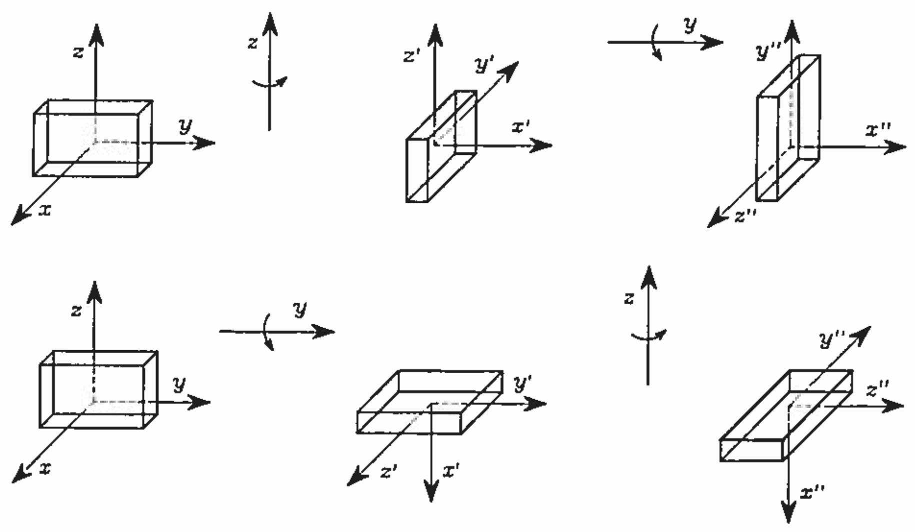

Figure 2.7 Successive rotations of an object about axes of fixed frame.
(i.e. rotations w/respect to global coord system)

where the resulting $\bar{R}^0_2$ is different from the matrix R^0_2 in (2.15). Hence, it can be stated that composition of successive rotations with respect to a fixed frame is obtained by premultiplication of the single rotation matrices in the order of the given sequence of rotations.

By recalling the meaning of a rotation matrix in terms of the orientation of a current frame with respect to a fixed frame, it can be recognized that its columns are the direction cosines of the axes of the current frame with respect to the fixed frame, while its rows (columns of its transpose and inverse) are the direction cosines of the axes of the fixed frame with respect to the current frame.

An important issue of composition of rotations is that the matrix product is not commutative. In view of this, it can be concluded that two rotations in general do not commute and its composition depends on the order of the single rotations.

Example 2.3

Consider an object and a frame attached to it. Figure 2.6 shows the effects of two successive rotations of the object with respect to the current frame by changing the order of rotations. It is evident that the final object orientation is different in the two cases. Also in the case of rotations made with respect to the current frame, the final orientations differ (Figure 2.7). It is interesting to note that the effects of the sequence of rotations with respect to the fixed frame are interchanged with the effects of the sequence of rotations with respect to the current frame. This can be explained by observing that the order of rotations in the fixed frame commutes with respect to the order of rotations in the current frame.

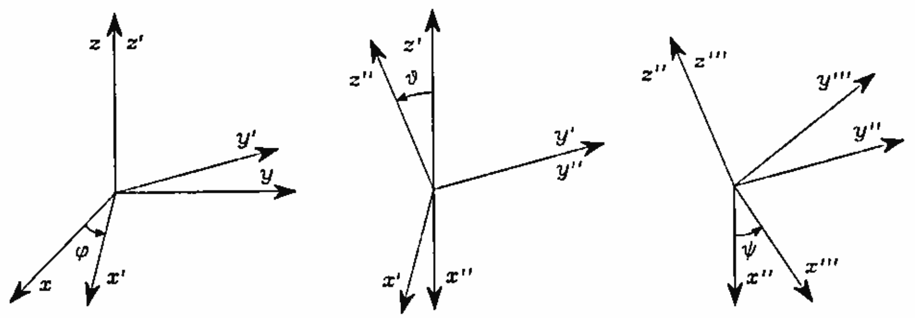

Figure 2.8 Representation of Euler angles ZYZ.

2.4 Euler Angles

– rotation about local coordinate systems

Rotation matrices give a redundant description of frame orientation; in fact, they are characterized by nine elements which are not independent but related by six constraints due to the orthogonality conditions given in (2.4). This implies that *three parameters* are sufficient to describe orientation of a rigid body in space. A representation of orientation in terms of three independent parameters constitutes a *minimal representation*.

A minimal representation of orientation can be obtained by using a set of three angles $\boldsymbol{\phi} = [\varphi \quad \vartheta \quad \psi]^T$. Consider the rotation matrix expressing the elementary rotation about one of the coordinate axes as a function of a single angle. Then, a generic rotation matrix can be obtained by composing a suitable sequence of three elementary rotations while guaranteeing that two successive rotations are not made about parallel axes. This implies that 12 distinct sets of angles are allowed out of all 27 possible combinations; each set represents a triplet of *Euler angles*. In the following, two sets of Euler angles are analyzed; namely, the ZYZ angles and the ZYX (or Roll–Pitch–Yaw) angles.

2.4.1 ZYZ Angles

The rotation described by *ZYZ angles* is obtained as composition of the following elementary rotations (Figure 2.8):

- Rotate the reference frame by the angle φ about axis z; this rotation is described by the matrix $\boldsymbol{R}_z(\varphi)$ which is formally defined in (2.6).
- Rotate the current frame by the angle ϑ about axis y'; this rotation is described by the matrix $\boldsymbol{R}_{y'}(\vartheta)$ which is formally defined in (2.7).
- Rotate the current frame by the angle ψ about axis z''; this rotation is described by the matrix $\boldsymbol{R}_{z''}(\psi)$ which is again formally defined in (2.6).

The resulting frame orientation is obtained by composition of rotations with respect to *current frames*, and then it can be computed via postmultiplication of the matrices of

elementary rotation, *i.e.*,[2]

$$\begin{aligned}\boldsymbol{R}(\boldsymbol{\phi}) &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_{y'}(\vartheta)\boldsymbol{R}_{z''}(\psi)\\ &= \begin{bmatrix} c_\varphi c_\vartheta c_\psi - s_\varphi s_\psi & -c_\varphi c_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta \\ s_\varphi c_\vartheta c_\psi + c_\varphi s_\psi & -s_\varphi c_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta \\ -s_\vartheta c_\psi & s_\vartheta s_\psi & c_\vartheta \end{bmatrix}.\end{aligned} \tag{2.18}$$

It is useful to solve the *inverse problem*, that is to determine the set of Euler angles corresponding to a given rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

Compare this expression with that of $\boldsymbol{R}(\boldsymbol{\phi})$ in (2.18). By considering the elements $[1,3]$ and $[2,3]$, on the assumption that $r_{13} \neq 0$ and $r_{23} \neq 0$, it follows that

$$\varphi = \mathrm{Atan2}(r_{23}, r_{13}),$$

where $\mathrm{Atan2}(y, x)$ is the arctangent function of two arguments[3]. Then, squaring and summing the elements $[1,3]$ and $[2,3]$ and using the element $[3,3]$ yields

$$\vartheta = \mathrm{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right).$$

The choice of the positive sign for the term $\sqrt{r_{13}^2 + r_{23}^2}$ limits the range of feasible values of ϑ to $(0, \pi)$. On this assumption, considering the elements $[3,1]$ and $[3,2]$ gives

$$\psi = \mathrm{Atan2}(r_{32}, -r_{31}).$$

In sum, the requested solution is

$$\begin{aligned}\varphi &= \mathrm{Atan2}(r_{23}, r_{13})\\ \vartheta &= \mathrm{Atan2}\left(\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right)\\ \psi &= \mathrm{Atan2}(r_{32}, -r_{31}).\end{aligned} \tag{2.19}$$

[2] The notations c_ϕ and s_ϕ are the abbreviations for $\cos\phi$ and $\sin\phi$, respectively; short-hand notations of this kind will be adopted often throughout the text.

[3] The function $\mathrm{Atan2}(y, x)$ computes the arctangent of the ratio y/x but utilizes the sign of each argument to determine which quadrant the resulting angle belongs to; this allows the correct determination of an angle in a range of 2π.

It is possible to derive another solution which produces the same effects as solution (2.19). Choosing ϑ in the range $(-\pi, 0)$ leads to

$$\begin{aligned}
\varphi &= \mathrm{Atan2}(-r_{23}, -r_{13}) \\
\vartheta &= \mathrm{Atan2}\left(-\sqrt{r_{13}^2 + r_{23}^2}, r_{33}\right) \\
\psi &= \mathrm{Atan2}(-r_{32}, r_{31}).
\end{aligned} \tag{2.19'}$$

Solutions (2.19) and (2.19′) degenerate when $s_\vartheta = 0$; in this case, it is possible to determine only the sum or difference of φ and ψ. In fact, if $\vartheta = 0, \pi$, the successive rotations of φ and ψ are made about axes of current frames which are parallel, thus giving equivalent contributions to the rotation.[4]

2.4.2 Roll–Pitch–Yaw Angles

Another set of Euler angles originates from a representation of orientation in the (aero)nautical field. These are the ZYX angles, also called *Roll–Pitch–Yaw angles*, to denote the typical motions of an (air)craft. In this case, the angles $\boldsymbol{\phi} = [\varphi \quad \vartheta \quad \psi]^T$ represent rotations defined with respect to a fixed frame attached to the centre of mass of the craft (Figure 2.9).

The rotation resulting from Roll–Pitch–Yaw angles can be obtained as follows:

- Rotate the reference frame by the angle ψ about axis x (yaw); this rotation is described by the matrix $\boldsymbol{R}_x(\psi)$ which is formally defined in (2.8).
- Rotate the reference frame by the angle ϑ about axis y (pitch); this rotation is described by the matrix $\boldsymbol{R}_y(\vartheta)$ which is formally defined in (2.7).
- Rotate the reference frame by the angle φ about axis z (roll); this rotation is described by the matrix $\boldsymbol{R}_z(\varphi)$ which is formally defined in (2.6).

The resulting frame orientation is obtained by composition of rotations with respect to the *fixed frame*, and then it can be computed via premultiplication of the matrices of elementary rotation, *i.e.*,[5]

$$\begin{aligned}
\boldsymbol{R}(\boldsymbol{\phi}) &= \boldsymbol{R}_z(\varphi)\boldsymbol{R}_y(\vartheta)\boldsymbol{R}_x(\psi) \\
&= \begin{bmatrix} c_\varphi c_\vartheta & c_\varphi s_\vartheta s_\psi - s_\varphi c_\psi & c_\varphi s_\vartheta c_\psi + s_\varphi s_\psi \\ s_\varphi c_\vartheta & s_\varphi s_\vartheta s_\psi + c_\varphi c_\psi & s_\varphi s_\vartheta c_\psi - c_\varphi s_\psi \\ -s_\vartheta & c_\vartheta s_\psi & c_\vartheta c_\psi \end{bmatrix}.
\end{aligned} \tag{2.20}$$

[4] In the following chapter, it will be seen that these configurations characterize the so-called representation *singularities* of the Euler angles.

[5] The ordered sequence of rotations XYZ about axes of the fixed frame is equivalent to the sequence ZYX about axes of the current frame.

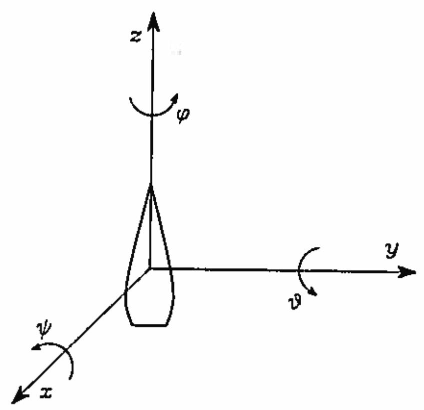

Figure 2.9 Representation of Roll–Pitch–Yaw angles.

As for the Euler angles ZYZ, the *inverse solution* to a given rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

can be obtained by comparing it with the expression of $\boldsymbol{R}(\boldsymbol{\phi})$ in (2.20). The solution for ϑ in the range $(-\pi/2, \pi/2)$ is

$$\begin{aligned}
\varphi &= \mathrm{Atan2}(r_{21}, r_{11}) \\
\vartheta &= \mathrm{Atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right) \\
\psi &= \mathrm{Atan2}(r_{32}, r_{33}),
\end{aligned} \tag{2.21}$$

whereas the other equivalent solution for ϑ in the range $(\pi/2, 3\pi/2)$ is

$$\begin{aligned}
\varphi &= \mathrm{Atan2}(-r_{21}, -r_{11}) \\
\vartheta &= \mathrm{Atan2}\left(-r_{31}, -\sqrt{r_{32}^2 + r_{33}^2}\right) \\
\psi &= \mathrm{Atan2}(-r_{32}, -r_{33}).
\end{aligned} \tag{2.21'}$$

Solutions (2.21) and (2.21′) degenerate when $c_\vartheta = 0$; in this case, it is possible to determine only the sum or difference of φ and ψ.

2.5 Angle and Axis

A nonminimal representation of orientation can be obtained by resorting to *four parameters* expressing a rotation of a given angle about an axis in space. This can be advantageous in the problem of trajectory planning for a manipulator's end-effector orientation.

Let $\boldsymbol{r} = [r_x \quad r_y \quad r_z]^T$ be the unit vector of a rotation axis with respect to the reference frame O–xyz. In order to derive the rotation matrix $\boldsymbol{R}(\vartheta, \boldsymbol{r})$ expressing the rotation of an *angle* ϑ about *axis* $\boldsymbol{r}$, it is convenient to compose elementary rotations about the coordinate axes of the reference frame. The angle is taken to be positive if the rotation is made counter-clockwise about axis $\boldsymbol{r}$.

As shown in Figure 2.10, a possible solution is to rotate first $\boldsymbol{r}$ by the angles necessary to align it with axis z, then to rotate by ϑ about z and finally to rotate by the angles necessary to align the unit vector with the initial direction. In detail, the sequence of rotations, to be made always with respect to axes of fixed frame, is the following:

- align $\boldsymbol{r}$ with z, which is obtained as the sequence of a rotation by $-\alpha$ about z and a rotation by $-\beta$ about y;
- rotate by ϑ about z;
- realign with the initial direction of $\boldsymbol{r}$, which is obtained as the sequence of a rotation by β about y and a rotation by α about z.

In sum, the resulting rotation matrix is

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_z(\vartheta)\boldsymbol{R}_y(-\beta)\boldsymbol{R}_z(-\alpha). \tag{2.22}$$

From the components of the unit vector $\boldsymbol{r}$ it is possible to extract the transcendental functions needed to compute the rotation matrix in (2.22), so as to eliminate the dependence from α and β; in fact, it is

$$\sin\alpha = \frac{r_y}{\sqrt{r_x^2 + r_y^2}} \qquad \cos\alpha = \frac{r_x}{\sqrt{r_x^2 + r_y^2}}$$

$$\sin\beta = \sqrt{r_x^2 + r_y^2} \qquad \cos\beta = r_z.$$

Then, it can be found that the rotation matrix corresponding to a given angle and axis is

$$\boldsymbol{R}(\vartheta, \boldsymbol{r}) = \begin{bmatrix} r_x^2(1-c_\vartheta) + c_\vartheta & r_x r_y(1-c_\vartheta) - r_z s_\vartheta & r_x r_z(1-c_\vartheta) + r_y s_\vartheta \\ r_x r_y(1-c_\vartheta) + r_z s_\vartheta & r_y^2(1-c_\vartheta) + c_\vartheta & r_y r_z(1-c_\vartheta) - r_x s_\vartheta \\ r_x r_z(1-c_\vartheta) - r_y s_\vartheta & r_y r_z(1-c_\vartheta) + r_x s_\vartheta & r_z^2(1-c_\vartheta) + c_\vartheta \end{bmatrix}. \tag{2.23}$$

For this matrix, the following property holds:

$$\boldsymbol{R}(-\vartheta, -\boldsymbol{r}) = \boldsymbol{R}(\vartheta, \boldsymbol{r}), \tag{2.24}$$

i.e., a rotation by $-\vartheta$ about $-\boldsymbol{r}$ cannot be distinguished from a rotation by ϑ about $\boldsymbol{r}$; hence, such representation is not unique.

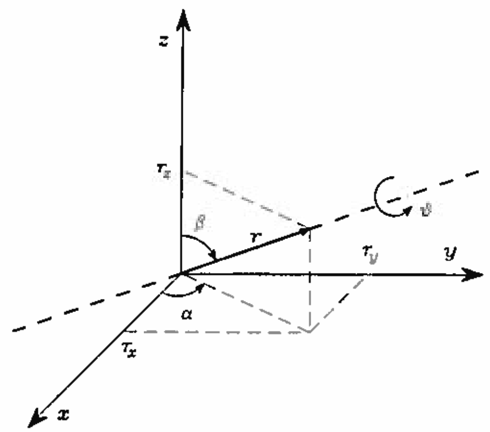

Figure 2.10 Rotation of an angle about an axis.

If it is desired to solve the *inverse problem* to compute the axis and angle corresponding to a given rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

the following result is useful:

$$\begin{aligned} \vartheta &= \cos^{-1}\left(\frac{r_{11} + r_{22} + r_{33} - 1}{2}\right) \\ \boldsymbol{r} &= \frac{1}{2\sin\vartheta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}, \end{aligned} \tag{2.25}$$

for $\sin\vartheta \neq 0$. Notice that (2.25) expresses the rotation in terms of four parameters; namely, the angle and the three components of the axis unit vector. However, it can be observed that the three components of $\boldsymbol{r}$ are not independent but are constrained by the condition

$$r_x^2 + r_y^2 + r_z^2 = 1. \tag{2.26}$$

If $\sin\vartheta = 0$, (2.25) becomes meaningless. To solve the inverse problem, it is necessary to directly refer to the particular expressions attained by the rotation matrix $\boldsymbol{R}$ and find the solving formulæ in the two cases $\vartheta = 0$ and $\vartheta = \pi$. Notice that, when $\vartheta = 0$ (null rotation), the unit vector $\boldsymbol{r}$ is arbitrary (singularity).

2.6 Unit Quaternion

The drawbacks of the angle/axis representation can be overcome by a different four-parameter representation; namely, the unit *quaternion*, viz. Euler parameters, defined as $\mathcal{Q} = \{\eta, \boldsymbol{\epsilon}\}$ where:

$$\eta = \cos\frac{\vartheta}{2}$$
$$\boldsymbol{\epsilon} = \sin\frac{\vartheta}{2}\boldsymbol{r}; \tag{2.27}$$

η is called the scalar part of the quaternion while $\boldsymbol{\epsilon} = [\epsilon_x \quad \epsilon_y \quad \epsilon_z]^T$ is called the vector part of the quaternion. They are constrained by the condition

$$\eta^2 + \epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2 = 1, \tag{2.28}$$

hence, the name *unit* quaternion. It is worth remarking that, differently from the angle/axis representation, a rotation by $-\vartheta$ about $-\boldsymbol{r}$ gives the same quaternion as that associated with a rotation by ϑ about $\boldsymbol{r}$; this solves the above nonuniqueness problem. In view of (2.23), (2.27) and (2.28), the rotation matrix corresponding to a given quaternion takes on the form

$$\boldsymbol{R}(\eta, \boldsymbol{\epsilon}) = \begin{bmatrix} 2(\eta^2 + \epsilon_x^2) - 1 & 2(\epsilon_x\epsilon_y - \eta\epsilon_z) & 2(\epsilon_x\epsilon_z + \eta\epsilon_y) \\ 2(\epsilon_x\epsilon_y + \eta\epsilon_z) & 2(\eta^2 + \epsilon_y^2) - 1 & 2(\epsilon_y\epsilon_z - \eta\epsilon_x) \\ 2(\epsilon_x\epsilon_z - \eta\epsilon_y) & 2(\epsilon_y\epsilon_z + \eta\epsilon_x) & 2(\eta^2 + \epsilon_z^2) - 1 \end{bmatrix}. \tag{2.29}$$

If it is desired to solve the *inverse problem* to compute the quaternion corresponding to a given rotation matrix

$$\boldsymbol{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

the following result is useful:

$$\eta = \frac{1}{2}\sqrt{r_{11} + r_{22} + r_{33} + 1}$$
$$\boldsymbol{\epsilon} = \frac{1}{2}\begin{bmatrix} \operatorname{sgn}(r_{32} - r_{23})\sqrt{r_{11} - r_{22} - r_{33} + 1} \\ \operatorname{sgn}(r_{13} - r_{31})\sqrt{r_{22} - r_{33} - r_{11} + 1} \\ \operatorname{sgn}(r_{21} - r_{12})\sqrt{r_{33} - r_{11} - r_{22} + 1} \end{bmatrix}, \tag{2.30}$$

where conventionally $\operatorname{sgn}(x) = 1$ for $x \geq 0$ and $\operatorname{sgn}(x) = -1$ for $x < 0$. Notice that in (2.30) it has been implicitly assumed $\eta \geq 0$; this corresponds to an angle $\vartheta \in [-\pi, \pi]$, and thus any rotation can be described. Also, compared to the inverse solution in (2.25) for the angle and axis representation, no singularity occurs for (2.30).

The quaternion extracted from $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$ is denoted as $\mathcal{Q}^{-1}$, and can be computed as

$$\mathcal{Q}^{-1} = \{\eta, -\boldsymbol{\epsilon}\}. \tag{2.31}$$

Let $\mathcal{Q}_1 = \{\eta_1, \boldsymbol{\epsilon}_1\}$ and $\mathcal{Q}_2 = \{\eta_2, \boldsymbol{\epsilon}_2\}$ denote the quaternions corresponding to the rotation matrices $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$, respectively. The quaternion corresponding to the product $\boldsymbol{R}_1\boldsymbol{R}_2$ is given by

$$\mathcal{Q}_1 * \mathcal{Q}_2 = \{\eta_1\eta_2 - \boldsymbol{\epsilon}_1^T\boldsymbol{\epsilon}_2, \eta_1\boldsymbol{\epsilon}_2 + \eta_2\boldsymbol{\epsilon}_1 + \boldsymbol{\epsilon}_1 \times \boldsymbol{\epsilon}_2\} \tag{2.32}$$

where the quaternion product operator "$*$" has been formally introduced. It is easy to see that if $\mathcal{Q}_2 = \mathcal{Q}_1^{-1}$ then the quaternion $\{1, \boldsymbol{0}\}$ is obtained from (2.32) which is the identity element for the product.

2.7 Homogeneous Transformations

As illustrated at the beginning of the chapter, the position of a rigid body in space is expressed in terms of the position of a suitable point on the body with respect to a reference frame (translation), while its orientation is expressed in terms of the components of the unit vectors of a frame attached to the body—with origin in the above point—with respect to the same reference frame (rotation).

As shown in Figure 2.11, consider an arbitrary point P in space. Let $\boldsymbol{p}^0$ be the vector of coordinates of P with respect to the reference frame O_0–$x_0y_0z_0$. Consider then another frame in space O_1–$x_1y_1z_1$. Let $\boldsymbol{o}_1^0$ be the vector describing the origin of Frame 1 with respect to Frame 0, and $\boldsymbol{R}_1^0$ be the rotation matrix of Frame 1 with respect to Frame 0. Let also $\boldsymbol{p}^1$ be the vector of coordinates of P with respect to Frame 1. On the basis of simple geometry, the position of point P with respect to the reference frame can be expressed as

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0\boldsymbol{p}^1. \tag{2.33}$$

Hence, (2.33) represents the *coordinate transformation* (*translation + rotation*) of a bound vector between two frames.

The inverse transformation can be obtained by premultiplying both sides of (2.33) by $\boldsymbol{R}_1^{0T}$; in view of (2.4), it follows that

$$\boldsymbol{p}^1 = -\boldsymbol{R}_1^{0T}\boldsymbol{o}_1^0 + \boldsymbol{R}_1^{0T}\boldsymbol{p}^0 \tag{2.34}$$

which, via (2.16), can be written as

$$\boldsymbol{p}^1 = -\boldsymbol{R}_0^1\boldsymbol{o}_1^0 + \boldsymbol{R}_0^1\boldsymbol{p}^0. \tag{2.35}$$

In order to achieve a compact representation of the relationship between the coordinates of the same point in two different frames, the *homogeneous representation* of a generic vector $\boldsymbol{p}$ can be introduced as the vector $\tilde{\boldsymbol{p}}$ formed by adding a fourth unit component, *i.e.*,

$$\tilde{\boldsymbol{p}} = \begin{bmatrix} \boldsymbol{p} \\ 1 \end{bmatrix}. \tag{2.36}$$

By adopting this representation for the vectors $\boldsymbol{p}^0$ and $\boldsymbol{p}^1$ in (2.33), the coordinate transformation can be written in terms of the (4×4) matrix

$$\boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} \tag{2.37}$$