2. In *probability* if for every $\varepsilon > 0$

$$\lim_{N\to\infty} \text{Prob}[|I_N - I| \geq \varepsilon] = 0$$

From the Chebyshev inequality of Equation (3.22), it follows directly that convergence in the mean square sense implies convergence in probability. In practice, most integral expressions involving random variables exist by assuming convergence in the mean square sense.

5.4 DERIVATIVE RANDOM PROCESSES

The derivative of any particular sample function $x(t)$ from an arbitrary random process $\{x(t)\}$ is defined by

$$\dot{x}(t) = \frac{dx(t)}{dt} = \lim_{\varepsilon\to 0}\left[\frac{x(t+\varepsilon) - x(t)}{\varepsilon}\right] \tag{5.139}$$

Existence of this limit may occur in different ways. The derivative $\dot{x}(t)$ is said to exist

1. In the *usual sense* if the limit exists for all functions $x(t)$ in $\{x(t)\}$.
2. In the *mean square sense* if

$$\lim_{\varepsilon\to 0} E\left[\left|\frac{x(t+\varepsilon) - x(t)}{\varepsilon} - \dot{x}(t)\right|^2\right] = 0 \tag{5.140}$$

For a stationary random process, a necessary and sufficient condition for $\dot{x}(t)$ to exist in the mean square sense is that its autocorrelation function $R_{xx}(\tau)$ should have derivatives of order up to 2, that is, $R'_{xx}(\tau)$ and $R''_{xx}(\tau)$ must exist [Reference 5.4].

5.4.1 *Correlation Functions*

Consider the following derivative functions, which are assumed to be well defined:

$$R'_{xx}(\tau) = \frac{dR_{xx}(\tau)}{d\tau} \qquad R''_{xx}(\tau) = \frac{d^2R_{xx}(\tau)}{d\tau^2}$$

$$\dot{x}(t) = \frac{dx(t)}{dt} \qquad \ddot{x}(t) = \frac{d^2x(t)}{dt^2} \tag{5.141}$$

By definition, for stationary random data,

$$R_{xx}(\tau) = E[x(t)x(t+\tau)] = E[x(t-\tau)x(t)]$$

$$R_{x\dot{x}}(\tau) = E[x(t)\dot{x}(t+\tau)] = E[x(t-\tau)\dot{x}(t)] \tag{5.142}$$

$$R_{\dot{x}\dot{x}}(\tau) = E[\dot{x}(t)\dot{x}(t+\tau)] = E[\dot{x}(t-\tau)\dot{x}(t)]$$

Now

$$R'_{xx}(\tau) = \frac{d}{d\tau}E[x(t)x(t+\tau)] = E[x(t)\dot{x}(t+\tau)] = R_{x\dot{x}}(\tau) \tag{5.143}$$

Also

$$R'_{xx}(\tau) = \frac{d}{d\tau}E[x(t-\tau)x(t)] = -E[\dot{x}(t-\tau)x(t)] = -R_{\dot{x}x}(\tau)$$

Hence

$$R'_{xx}(0) = R_{x\dot{x}}(0) = -R_{\dot{x}x}(0) = 0 \tag{5.144}$$

since $R'_{xx}(0)$ equals the positive and negative of the same quantity. The corresponding $R_{xx}(0)$ is a maximum value of $R_{xx}(\tau)$. This proves that for stationary random data

$$E[x(t)\dot{x}(t)] = 0 \tag{5.145}$$

In words, at any t, Equation (5.145) indicates that the derivative $\{\dot{x}(t)\}$ for stationary random data $\{x(t)\}$ is equally likely to be positive or negative. Equation (5.143) states that the derivative $R'_{xx}(\tau)$ of the autocorrelation function $R_{xx}(\tau)$ with respect to τ is the same as the cross-correlation function between $\{x(t)\}$ and $\{\dot{x}(t)\}$. A maximum value for the autocorrelation function $R_{xx}(\tau)$ corresponds to a zero crossing for its derivative $R'_{xx}(\tau)$, which becomes a zero crossing for the cross-correlation function between $\{x(t)\}$ and $\{\dot{x}(t)\}$. This crossing of zero by $R'_{xx}(\tau)$ will be with *negative* slope, that is

$$R'_{xx}(0-) > 0 \quad \text{and} \quad R'_{xx}(0+) < 0 \tag{5.146}$$

as can be seen from the picture in Figure 5.9. In practice, determining the location where zero crossings will occur is usually easier than determining the location of maximum values.

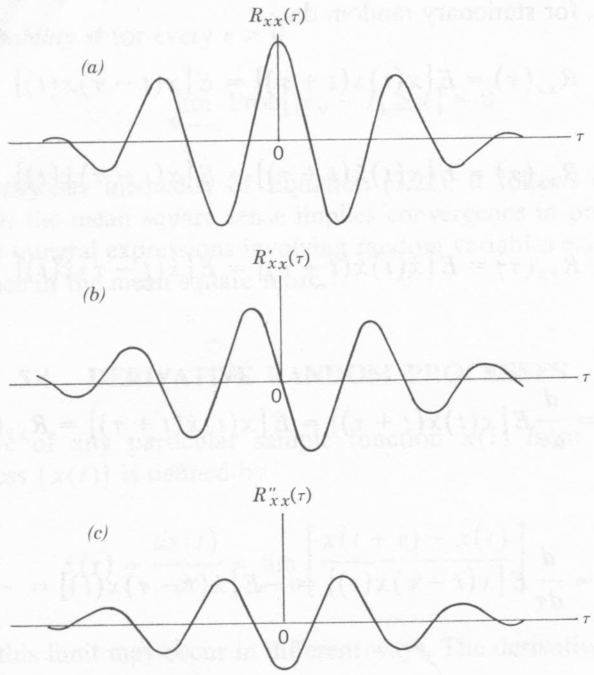

Figure 5.9 Illustration of derivatives of autocorrelation functions. (*a*) Original function. (*b*) First derivative. (*c*) Second derivative.

It will now be shown that $R'_{xx}(\tau)$ is an odd function of τ corresponding to $R_{xx}(\tau)$ being an even function of τ. By definition

$$R_{xx}(-\tau) = E[x(t)x(t-\tau)] = E[x(t+\tau)x(t)] \qquad (5.147)$$

Hence

$$R'_{xx}(-\tau) = \frac{d}{d\tau}E[x(t+\tau)x(t)] = E[\dot{x}(t+\tau)x(t)] = R_{\dot{x}x}(\tau) \qquad (5.148)$$

But, from Equation (5.143), $R_{\dot{x}x}(\tau) = -R'_{xx}(\tau)$. Hence Equation (5.148) becomes

$$R'_{xx}(-\tau) = -R'_{xx}(\tau) \qquad (5.149)$$

This proves that $R'_{xx}(\tau)$ is an odd function of τ.

The second derivative gives

$$R''_{xx}(\tau) = \frac{d}{d\tau} R'_{xx}(\tau) = \frac{d}{d\tau} R_{x\dot{x}}(\tau) = \frac{d}{d\tau} E[x(t-\tau)\dot{x}(t)]$$

$$= -E[\dot{x}(t-\tau)\dot{x}(t)] = -R_{\dot{x}\dot{x}}(\tau) \tag{5.150}$$

Also

$$R''_{xx}(\tau) = \frac{d}{d\tau} R'_{xx}(\tau) = \frac{d}{d\tau} R_{x\dot{x}}(\tau) = \frac{d}{d\tau} E[x(t)\dot{x}(t+\tau)]$$

$$= E[x(t)\ddot{x}(t+\tau)] = R_{x\ddot{x}}(\tau) \tag{5.151}$$

One can also verify directly that $R''_{xx}(\tau)$ is an even function of τ, namely,

$$R''_{xx}(-\tau) = R''_{xx}(\tau) \tag{5.152}$$

At $\tau = 0$, one obtains

$$E[\dot{x}^2(t)] = R_{\dot{x}\dot{x}}(0) = -R_{x\ddot{x}}(0) = -R''_{xx}(0) \tag{5.153}$$

As shown earlier,

$$R_{x\dot{x}}(\tau) = \frac{d}{d\tau} R_{xx}(\tau) = R'_{xx}(\tau) \tag{5.154}$$

Typical plots for $R_{xx}(\tau)$, $R'_{xx}(\tau)$, and $R''_{xx}(\tau)$ are drawn in Figure 5.9, based on a sine wave process where

$$R_{xx}(\tau) = X\cos 2\pi f_0\tau$$

$$R'_{xx}(\tau) = -X(2\pi f_0)\sin 2\pi f_0\tau \tag{5.155}$$

$$R''_{xx}(\tau) = -X(2\pi f_0)^2\cos 2\pi f_0\tau$$

The results given above can be extended to higher-order derivatives. For example,

$$R_{\dot{x}\ddot{x}}(\tau) = \frac{d}{d\tau} R_{\dot{x}\dot{x}}(\tau) = -R'''_{xx}(\tau) \tag{5.156}$$

$$R_{\ddot{x}\ddot{x}}(\tau) = -\frac{d}{d\tau} R_{\dot{x}\ddot{x}}(\tau) = R''''_{xx}(\tau) \tag{5.157}$$

At $\tau = 0$, one obtains

$$E[\ddot{x}^2(t)] = R_{\ddot{x}\ddot{x}}(0) = R''''_{xx}(0) \tag{5.158}$$

Thus knowledge of $R_{xx}(\tau)$ and its successive derivatives can enable one to state properties for autocorrelation and cross-correlation functions between $\{x(t)\}$ and its successive derivatives $\{\dot{x}(t)\}$, $\{\ddot{x}(t)\}$, and so on.

5.4.2 *Spectral Density Functions*

It is easy to derive corresponding properties for autospectral and cross-spectral density functions between $\{x(t)\}$ and its successive derivatives $\{\dot{x}(t)\}$ and $\{\ddot{x}(t)\}$. Let

$$X(f) = \mathscr{F}[x(t)] = \text{Fourier transform}[x(t)] \tag{5.159}$$

Then

$$\mathscr{F}[\dot{x}(t)] = (j2\pi f)X(f) \tag{5.160}$$

$$\mathscr{F}[\ddot{x}(t)] = -(2\pi f)^2 X(f) \tag{5.161}$$

From Equations (5.66) and (5.67), it follows directly that

$$G_{x\dot{x}}(f) = j(2\pi f)G_{xx}(f) \tag{5.162}$$

$$G_{\dot{x}\dot{x}}(f) = (2\pi f)^2 G_{xx}(f) \tag{5.163}$$

$$G_{\dot{x}\ddot{x}}(f) = j(2\pi f)^3 G_{xx}(f) \tag{5.164}$$

$$G_{\ddot{x}\ddot{x}}(f) = (2\pi f)^4 G_{xx}(f) \tag{5.165}$$

and so on. These formulas are the same with one-sided G's replaced by the corresponding two-sided S's.

These results can also be derived from the Wiener–Khinchine relations of Equation (5.28). Start with the basic relation

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f)e^{j2\pi f\tau}\,df \tag{5.166}$$

Then successive derivatives will be

$$R'_{xx}(\tau) = j\int_{-\infty}^{\infty} (2\pi f)S_{xx}(f)e^{j2\pi f\tau}\,df \tag{5.167}$$

$$R''_{xx}(\tau) = -\int_{-\infty}^{\infty} (2\pi f)^2 S_{xx}(f)e^{j2\pi f\tau}\,df \tag{5.168}$$

$$R'''_{xx}(\tau) = -j\int_{-\infty}^{\infty} (2\pi f)^3 S_{xx}(f)e^{j2\pi f\tau}\,df \tag{5.169}$$

$$R''''_{xx}(\tau) = \int_{-\infty}^{\infty} (2\pi f)^4 S_{xx}(f)e^{j2\pi f\tau}\,df \tag{5.170}$$

The Wiener–Khinchine relations, together with previous formulas in Section 5.4.1, show that these four derivative expressions are the same as

$$R'_{xx}(\tau) = R_{x\dot{x}}(\tau) = \int_{-\infty}^{\infty} S_{x\dot{x}}(f)e^{j2\pi f\tau}\,df \tag{5.171}$$

$$R''_{xx}(\tau) = -R_{\dot{x}\dot{x}}(\tau) = -\int_{-\infty}^{\infty} S_{\dot{x}\dot{x}}(f)e^{j2\pi f\tau}\,df \tag{5.172}$$

$$R'''_{xx}(\tau) = \quad R_{\dot{x}\ddot{x}}(\tau) = -\int_{-\infty}^{\infty} S_{\dot{x}\ddot{x}}(f)e^{j2\pi f\tau}\,df \tag{5.173}$$

$$R''''_{xx}(\tau) = R_{\ddot{x}\ddot{x}}(\tau) = \int_{-\infty}^{\infty} S_{\ddot{x}\ddot{x}}(f)e^{j2\pi f\tau}\,df \tag{5.174}$$

Corresponding terms in the last eight formulas yield Equations (5.162)–(5.165).

5.4.3 *Expected Number of Zero Crossings*

Consider a stationary random noise record $x(t)$ that has the time derivative $\dot{x}(t)$. Let $p(\alpha, \beta)$ represent the joint probability density function of $x(t)$ and $\dot{x}(t)$ at $x(t) = \alpha$ and $\dot{x}(t) = \beta$. By definition, for all t,

$$p(\alpha, \beta)\,\Delta\alpha\,\Delta\beta \simeq \text{Prob}[\alpha < x(t) \leq +\alpha + \Delta\alpha \text{ and } \beta < \dot{x}(t) \leq \beta + \Delta\beta] \tag{5.175}$$

In words, $p(\alpha, \beta)\,\Delta\alpha\,\Delta\beta$ estimates the probability over all time that $x(t)$ lies in the interval $[\alpha, \alpha + \Delta\alpha]$ when its derivative $\dot{x}(t)$ is between β and $\beta + \Delta\beta$. For unit total time, this represents the amount of time that $x(t)$ spends in the interval $[\alpha, \alpha + \Delta\beta]$ with a given derivative value between β and $\beta + \Delta\beta$. When $\Delta\beta$ is negligible compared to β, this means that the derivative value is essentially β.

To find the expected number of crossings of $x(t)$ through the interval $[\alpha, \alpha + \Delta\alpha]$, the amount of time that $x(t)$ is inside this interval should be divided by the time required to cross the interval. If t_β is the crossing time for a particular derivative value β, then

$$t_\beta = \frac{\Delta\alpha}{|\beta|} \tag{5.176}$$

where the absolute value of β is used since crossing time must be a positive quantity. Hence, the expected number of passages per unit time of $x(t)$ through the interval $[\alpha, \alpha + \Delta\alpha]$ for a given value of $\dot{x}(t) = \beta$ is

$$\frac{p(\alpha, \beta)\,\Delta\alpha\,\Delta\beta}{t_\beta} = |\beta|\,p(\alpha, \beta)\,\Delta\beta \tag{5.177}$$

In the limit as $\Delta\beta \to 0$, the total expected number of passages per unit time of $x(t)$ through the line $x(t) = \alpha$ for all possible values of β is found by

$$\overline{N}_\alpha = \int_{-\infty}^{\infty} |\beta| p(\alpha, \beta)\, d\beta \tag{5.178}$$

This represents the expected number of crossings of α per unit time with *both* positive and negative slopes. Assuming that $x(t)$ passes the value α half of the time with positive slope and half of the time with negative slope, then $\frac{1}{2}\overline{N}_\alpha$ gives the expected number of times per unit time that $x(t)$ exceeds the value α, that is, crosses the line $x(t) = \alpha$ with positive slope.

The expected number of zeros of $x(t)$ per unit time is found by the number of crossings of the line $x(t) = 0$ with both positive and negative slopes. This is given by $\overline{N}_\alpha$ when $\alpha = 0$, namely,

$$\overline{N}_0 = \int_{-\infty}^{\infty} |\beta| p(0, \beta)\, d\beta \tag{5.179}$$

The value of $\overline{N}_0$ can be interpreted as twice the "apparent frequency" of the noise record. For example, if the record were a pure sine wave of frequency f_0 Hz, then N_0 would be $2f_0$ zeros per sec (e.g., a 60 Hz sine wave has 120 zeros/sec). For noise, the situation is more complicated but, still, knowledge of $\overline{N}_0$ together with other quantities helps to characterize a particular noise.

For an arbitrary record $x(t)$ and its derivative $\dot{x}(t)$ from a zero mean value stationary random process, it follows from Equations (5.145) and (5.150) that

$$\sigma_x^2 = E[x^2(t)] = R_{xx}(0) \tag{5.180}$$

$$\sigma_{\dot{x}}^2 = E[\dot{x}^2(t)] = R_{\dot{x}\dot{x}}(0) = -R''_{xx}(0) \tag{5.181}$$

$$\sigma_{x\dot{x}} = E[x(t)\dot{x}(t)] = 0 \tag{5.182}$$

From Equations (5.166) and (5.168), it also follows that

$$\sigma_x^2 = \int_{-\infty}^{\infty} S_{xx}(f)\, df = \int_0^{\infty} G_{xx}(f)\, df \tag{5.183}$$

$$\sigma_{\dot{x}}^2 = \int_{-\infty}^{\infty} (2\pi f)^2 S_{xx}(f)\, df = \int_0^{\infty} (2\pi f)^2 G_{xx}(f)\, df \tag{5.184}$$

Assume now that $x(t)$ and $\dot{x}(t)$ have zero mean values and form a two-dimensional normal distribution with the above variances and zero covariance. Then

$$p(\alpha, \beta) = p(\alpha) p(\beta) \tag{5.185}$$

with

$$p(\alpha) = \frac{1}{\sigma_x\sqrt{2\pi}} \exp(-\alpha^2/2\sigma_x^2) \tag{5.186}$$

$$p(\beta) = \frac{1}{\sigma_{\dot{x}}\sqrt{2\pi}} \exp(-\beta^2/2\sigma_{\dot{x}}^2) \tag{5.187}$$

Substitution of Equation (5.185) into Equation (5.178) shows that

$$\begin{aligned} \overline{N}_\alpha &= \frac{\exp(-\alpha^2/2\sigma_x^2)}{2\pi\sigma_x\sigma_{\dot{x}}} \int_{-\infty}^{\infty} |\beta| \exp(-\beta^2/2\sigma_{\dot{x}}^2)\, d\beta \\ &= \frac{1}{\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right)\exp(-\alpha^2/2\sigma_x^2) \end{aligned} \tag{5.188}$$

In particular, for $\alpha = 0$, one obtains

$$\begin{aligned} \overline{N}_0 &= \frac{1}{\pi}\left(\frac{\sigma_{\dot{x}}}{\sigma_x}\right) = \frac{1}{\pi}\left[\frac{-R''_{xx}(0)}{R_{xx}(0)}\right]^{1/2} \\ &= \frac{1}{\pi}\left[\frac{\int_0^\infty (2\pi f)^2 G_{xx}(f)\, df}{\int_0^\infty G_{xx}(f)\, df}\right]^{1/2} \end{aligned} \tag{5.189}$$

In terms of $\overline{N}_0$, one can express

$$\overline{N}_\alpha = \overline{N}_0 \exp(-\alpha^2/2\sigma_x^2) \tag{5.190}$$

These results from Reference 5.1 were derived originally by Rice [Reference 5.5], who used a different method of proof.

Example 5.12. **Zero Crossings of Low-Pass White Noise.** To illustrate the above formulas, consider the case of low-pass white noise from 0 to B Hz, where

$$G_{xx}(f) = K \qquad 0 \le f \le B \quad \text{otherwise zero}$$

Now

$$\sigma_x^2 = \int_0^B K\, df = KB$$

$$\sigma_{\dot{x}}^2 = \int_0^B (2\pi f)^2 K\, df = \frac{4\pi^2}{3} KB^3$$

From Equation (5.189),

$$\overline{N}_0 = \frac{2}{\sqrt{3}} B \approx 2(0.58B)$$

A pure sine wave of frequency B Hz would have $\overline{N}_0 = 2B$ zeros/sec. Here, the conclusion is that for low-pass white noise cutting off at B Hz, the apparent frequency of the noise is about 0.58 of the cutoff frequency.

PROBLEMS

5.1 Given data with an autocorrelation function defined by $R_{xx}(\tau) = 25e^{-4|\tau|}\cos 4\pi\tau + 16$, determine
(a) the mean value and variance.
(b) the associated one-sided autospectral density function.

5.2 Which of the following properties are always true of autocorrelation functions of stationary data?
(a) must be an even function.
(b) must be nonnegative.
(c) must be bounded by its value at zero.
(d) can determine the mean value of the data.
(e) can determine the variance of the data.

5.3 Which of the properties in Problem 5.2 are always true of cross-correlation functions of stationary data?

5.4 Given data with a two-sided autospectral density function defined by

$$S_{xx}(f) = \begin{cases} 16\delta(f) + 20\left(1 - \dfrac{|f|}{10}\right) & f \leq 10 \\ 0 & |f| > 10 \end{cases}$$

determine for the data
(a) the mean value and variance.
(b) the associated autocorrelation function.

5.5 Which of the properties in Problem 5.2 are always true of the two-sided
(a) autospectral density functions?
(b) cross-spectral density functions?

5.6 Which of the following properties are always true for two ergodic random processes?
(a) $R_{xy}(\infty) = \mu_x \mu_y$.
(b) $R_{xy}(0) = 0$ when $\mu_x = 0$ or $\mu_y = 0$.
(c) $R_{xy}(\tau) = 0$ when $R_{xx}(\tau) = 0$ or $R_{yy}(\tau) = 0$.

(d) $|R_{xy}(\tau)|^2 \leq R_{xx}(\tau)R_{yy}(\tau)$.
(e) $G_{xy}(0) = 0$ when $\mu_x = 0$ or $\mu_y = 0$.
(f) $|G_{xy}(f)|^2 \leq G_{xx}(0)G_{yy}(0)$.
(g) $G_{xy}(f) = 0$ when $G_{xx}(f) = 0$ or $G_{yy}(f) = 0$.

5.7 Assume data have a one-sided cross-spectral density function given by $G_{xy}(f) = (6/f^2) + j(8/f^3)$. Determine the two-sided cross-spectral density function $S_{xy}(f)$ for all frequencies in terms of
(a) real and imaginary functions.
(b) gain and phase functions.

5.8 If a record $x(t)$ from an ergodic random process has an autocorrelation function given by $R_{xx}(\tau) = e^{-a|\tau|}\cos 2\pi f_0\tau$ with $a > 0$, determine the autocorrelation function for the first time derivative of the data, $\dot{x}(t)$.

5.9 Assume a record $x(t)$ from an ergodic random process has a one-sided autospectral spectral density function given by

$$G_{xx}(f) = \frac{1}{25 + f^2} \qquad 0 \leq f \leq 25 \qquad \text{otherwise zero}$$

Determine the average number of zero crossings per second in the record $x(t)$.

5.10 Prove the result in Equation (5.30) from Equation (5.59).

REFERENCES

5.1 Bendat, J. S., *Principles and Applications of Random Noise Theory*, Wiley, New York, 1958. Reprinted by Krieger, Melbourne, Florida, 1977.

5.2 Bendat, J. S., and Piersol, A. G., *Engineering Applications of Correlation and Spectral Analysis*, Wiley-Interscience, New York, 1980.

5.3 Doob, J. L., *Stochastic Processes*, Wiley, New York, 1953.

5.4 Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, McGraw-Hill, New York, 1965.

5.5 Rice, S. O., "Mathematical Analysis of Random Noise," in *Selected Papers on Noise and Stochastic Processes*, (N. Wax, ed.) Dover, New York, 1954.