MAE6254: Homework 1

Due date: February 8, 2015

Problem 1 Consider the following time-invariant state equation:

$$\frac{dx}{dt} = f(x), \tag{1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$. Here, we consider a time-scaling property.

(a) Suppose that $x(t)$ is a solution of (1) for $-\infty < t < \infty$ with an initial condition given by $x(0) = x_\circ$. Define $\tau = at$ for a non-zero constant a, and let $y(\tau) = x(\frac{\tau}{a})$. Find the differential equation satisfied by $y(\tau)$, i.e. find the expression for $\frac{dy(\tau)}{d\tau}$ in terms of f, y, and a.

(b) *(Backward integration)* Specialize your answer to (a) when $a = -1$.

Problem 2 Consider a planar pendulum with a friction. The equation of motion is given by

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - c\dot{\theta},$$

where $g = 9.81\,\mathrm{m/s^2}$, $l = 5\,\mathrm{m}$, and $c = 0.8\,/\mathrm{s}$.

(a) Define the state vector be $x = [\theta,\, \dot{\theta}]^T \in \mathbb{R}^2$. Rewrite the equation of motion as $\dot{x} = f(x)$.

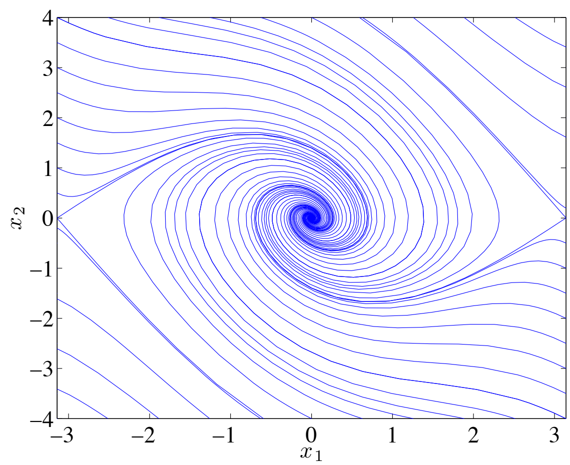
(b) Show that there are two equilibria at $x_1^\star = [0, 0]^T$, and $x_2^\star = [\pi, 0]^T$.

(c) Determine the type of each equilibrium (e.g., node or focus) using the linearized equation of motion.

(d) Write a Matlab code to draw the following phase portrait, according to the following steps:

- Select a bounding box for the phase portrait: $|x_1| \leq \pi$ and $|x_2| \leq 4$.
- Choose initial conditions on the bounding box (this should be done iteratively so that all essential qualitative behaviors are illustrated).
- Numerically integrate the state equation for each of the above initial condition, both forward in time and backward in time (using the results of Problem 1). You may use Matlab function, `ode45`.
- Plot them in the selected bounding box. You may use the Matlab command, `axis`.

It is also required to put arrowheads on top of several curves (by hand), to illustrate the direction of flows.



(e) Shortly discuss whether this phase portrait is consistent with your result of (c).

(f) On your phase portrait, mark the trajectories that asymptotically converge to the inverted equilibrium $x_2^\star = [\pi, 0]^T$.

Problem 3 Consider the following linear system:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

for constants α and β.

(a) Find the eigenvalues.

(b) Define the polar coordinates (r, θ) as

$$r = \sqrt{z_1^2 + z_2^2}, \quad \theta = \tan^{-1} \frac{z_2}{z_1}.$$

Show that the state equation of (r, θ) is given by

$$\dot{r} = \alpha r, \quad \dot{\theta} = \beta.$$

(Hint: $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$, $\frac{d}{dx}\tan^{-1} f(x) = \frac{f'(x)}{1+f^2(x)}$)

Problem 4 Determine whether each of the following functions $f : \mathbb{R} \to \mathbb{R}$ is (i) locally Lipschitz; (ii) globally Lipschitz.

(a) $f(x) = -x + a$ for a constant a

(b) $f(x) = |x|$

(c) $f(x) = \tan x$

Problem 5 Suppose that $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are globally Lipschitz, i.e.,

$$|f_1(x) - f_1(y)| \leq L_1|x - y|, \quad |f_2(x) - f_2(y)| \leq L_2|x - y|,$$

for some Lipschitz constants L_1 and L_2. Show that each of the following functions are globally Lipschitz, and find the corresponding Lipschitz constant in terms of L_1 and L_2.

(a) $f = f_1 + f_2$

(b) $f = f_1 \circ f_2$, i.e., $f(x) = f_1(f_2(x))$