HOMEWORK 3 SOLUTIONS

1. Since the collision is perfectly elastic, we can use conservation of momentum and energy to calculate the velocity of the ball and the block after the collision:

$$m_0 v_0(0-) = Mv(0+) + m_0 v_0(0+) \text{ and } \frac{1}{2} m_0 v_0^2(0-) = \frac{1}{2} M v^2(0+) + \frac{1}{2} m_0 v_0^2(0+)$$

The velocity of the ball before impact is $v_0(0-) = \sqrt{2gh} = 6.26\text{ m/s}$ (positive downward). Solving this system of equations gives

$$v(0+) = \frac{2m_0 v_0(0-)}{M + m_0} = 0.60\text{ m/s} \text{ and } v_0(0+) = -\frac{(M - m_0)v_0(0-)}{M + m_0} = -5.66\text{ m/s}$$

Since the ball is removed from the situation after the impact, we can now consider the spring-mass-damper system as a free vibration with initial position of 0 and initial velocity of 0.60 m/s.

The equation of motion is $M\ddot{x} + c\dot{x} + kx = 0$. Taking the Laplace transform yields

$$M\left(s^2 X(s) - v(0+)\right) + csX(s) + kX(s) = 0$$

Solving for $X(s)$ and plugging in the values gives

$$X(s) = \frac{Mv(0+)}{Ms^2 + cs + k} = \frac{1.2}{2s^2 + 5s + 100} = \frac{0.6}{s^2 + 2.5s + 50}$$

Checking the discriminant: $2.5^2 - 4 \cdot 50 = -193.75 < 0$

Completing the squares: $s^2 + 2.5s + 50 = (s + 1.25)^2 + 48.4$

Hence, $X(s) = \dfrac{0.6}{(s+1.25)^2 + 48.4} = \left(\dfrac{0.6}{7}\right)\dfrac{7}{(s+1.25)^2 + 48.4}$

The inverse Laplace transform is then

$$x(t) = 0.086e^{-1.25t}\sin(7t)$$

2. The area of the piston is $A = \frac{\pi}{4} d^2 = 0.0078\,\mathrm{m}$. Thus, the applied force on the piston is

$$f(t) = Ap(t) = 390\left(1 - e^{-3t}\right)$$

The EOM is $m\ddot{x} + kx = f(t)$. Taking the Laplace transform and assuming 0 ICs gives $ms^2 X(s) + kX(s) = F(s)$.

At this point, there are two methods for solving this problem:

1. <u>Inverse Laplace transform method</u>

Taking the Laplace transform of the input force gives

$$F(s) = \frac{390}{s} - \frac{390}{s+3}$$

Solving for $X(s)$ and substituting in $F(s)$ gives

$$X(s) = \frac{F(s)}{ms^2 + k} = \frac{390}{s\left(10s^2 + 1000\right)} - \frac{390}{(s+3)\left(10s^2 + 1000\right)}$$

Taking the inverse Laplace transform yields

$$x(t) = 0.39 - 0.36e^{-3t} - 0.11\sin(10t) - 0.032\cos(10t)$$

2. <u>Convolution integral method</u>

First, we must compute the transfer function, which is derived from the EOM assuming 0 ICs:

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + k} = \frac{1}{10s^2 + 1000}$$

This inverse Laplace transform of $G(s)$ is the unit impulse response:

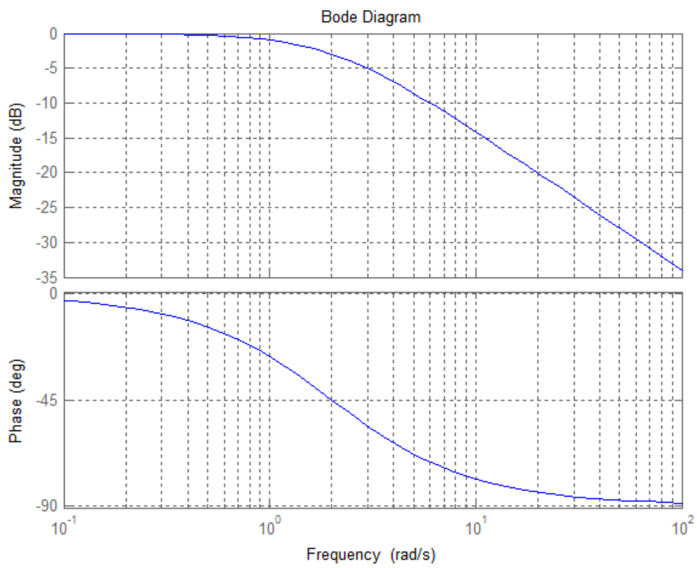
$$g(t) = 0.01\sin(10t)$$

Now, we can use the convolution integral to get $x(t)$ as follows:

$$\begin{aligned} x(t) &= \int_0^t f(\tau) g(t-\tau)\, d\tau = \int_0^t 390\left(1 - e^{-3\tau}\right) 0.01\sin[10(t-\tau)]\, d\tau \\ &= 0.39 - 0.36e^{-3t} - 0.11\sin(10t) - 0.032\cos(10t) \end{aligned}$$

3. a. The Bode plot of $\dfrac{1}{0.5s+1}$ has a corner frequency at $\omega = 2\text{ rad/s}$, a slope of 0 dB/decade to the left of this and a slope of -20 dB/decade to the right. The gain as $\omega \to 0$ is 0 dB. The phase varies from 0 deg. to -90 deg.

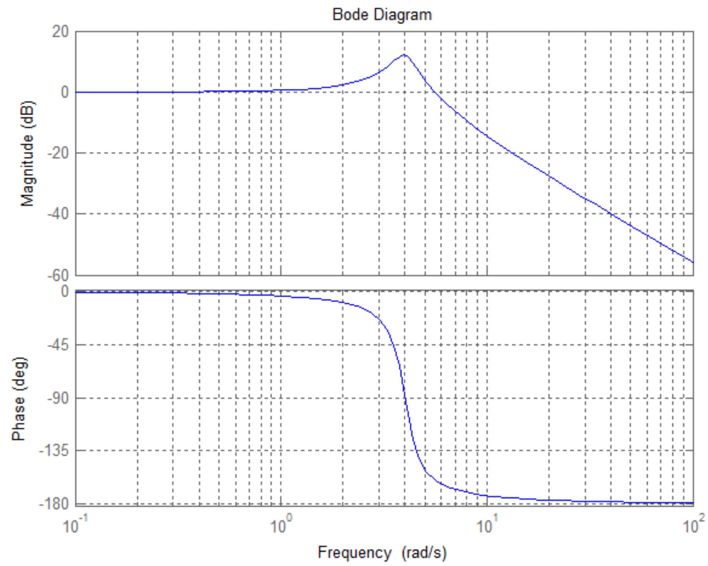
Hence, the Bode plot looks like



b. The term $\dfrac{16}{s^2+s+16}$ has complex conjugate roots and so cannot be factored into first-order terms. Writing the denominator as $s^2 + 2\zeta\omega_n s + \omega_n^2$, we can see that $\omega_n = 4\text{ rad/s}$, which is the corner frequency for this term, and $\zeta = 0.125$, which indicates that there will be a peak near the corner frequency. Since it's a second-order term, it will have a slope of 0 dB/decade to the left of this and a slope of -40 dB/decade to the right. The gain as $\omega \to 0$ is 0 dB. The phase varies from 0 deg. to -180 deg.

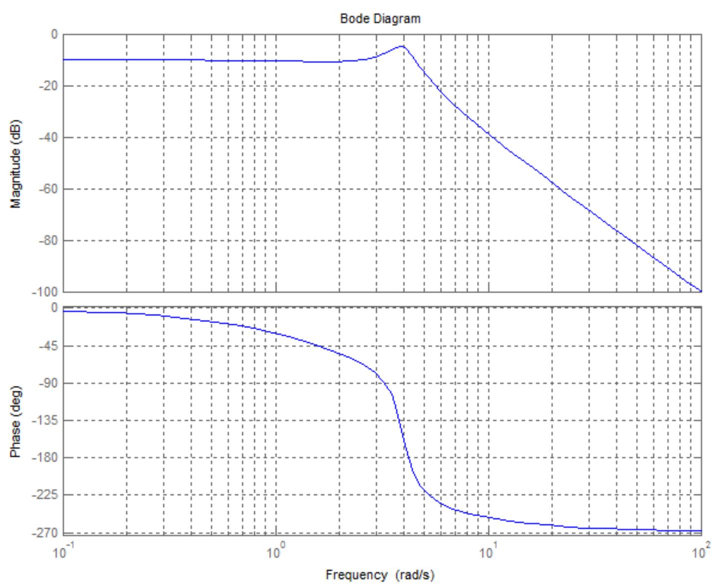
The magnitude of the peak is given by $M_p = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} = 4.03 = 12.1\,\text{dB}$.

Hence, the Bode plot looks like



c. The term $\frac{5}{16}$ is a straight line with magnitude $20\log\left(\frac{5}{16}\right) = -10.1\,\text{dB}$ and 0 phase angle.

d. Combining these terms yields



4. a. $G(s) = \frac{1}{0.1s+1}$ b. $G(s) = \frac{100s+1}{0.1s+1}$

c. $G(s) = 10(10s+1)$ d. $G(s) = \frac{1}{(s+1)^2}$