2

The Laplace Transform

2–1 INTRODUCTION

The Laplace transform is one of the most important mathematical tools available for modeling and analyzing linear systems. Since the Laplace transform method must be studied in any system dynamics course, we present the subject at the beginning of this text so that the student can use the method throughout his or her study of system dynamics.

The remaining sections of this chapter are outlined as follows: Section 2–2 reviews complex numbers, complex variables, and complex functions. Section 2–3 defines the Laplace transformation and gives Laplace transforms of several common functions of time. Also examined are some of the most important Laplace transform theorems that apply to linear systems analysis. Section 2–4 deals with the inverse Laplace transformation. Finally, Section 2–5 presents the Laplace transform approach to the solution of the linear, time-invariant differential equation.

2–2 COMPLEX NUMBERS, COMPLEX VARIABLES, AND COMPLEX FUNCTIONS

This section reviews complex numbers, complex algebra, complex variables, and complex functions. Since most of the material covered is generally included in the basic mathematics courses required of engineering students, the section can be omitted entirely or used simply for personal reference.

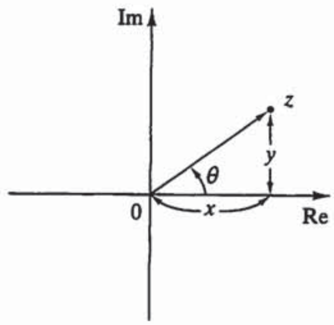

Figure 2–1 Complex plane representation of a complex number z.

Complex numbers. Using the notation $j = \sqrt{-1}$, we can express all numbers in engineering calculations as

$$z = x + jy$$

where z is called a *complex number* and x and jy are its *real* and *imaginary parts*, respectively. Note that both x and y are real and that j is the only imaginary quantity in the expression. The complex plane representation of z is shown in Figure 2–1. (Note also that the real axis and the imaginary axis define the complex plane and that the combination of a real number and an imaginary number defines a point in that plane.) A complex number z can be considered a point in the complex plane or a directed line segment to the point; both interpretations are useful.

The magnitude, or absolute value, of z is defined as the length of the directed line segment shown in Figure 2–1. The angle of z is the angle that the directed line segment makes with the positive real axis. A counterclockwise rotation is defined as the positive direction for the measurement of angles. Mathematically,

$$\text{magnitude of } z = |z| = \sqrt{x^2 + y^2}, \qquad \text{angle of } z = \theta = \tan^{-1}\frac{y}{x}$$

A complex number can be written in rectangular form or in polar form as follows:

$$\left.\begin{aligned} z &= x + jy \\ z &= |z|(\cos\theta + j\sin\theta) \end{aligned}\right\}\text{rectangular forms}$$

$$\left.\begin{aligned} z &= |z| \angle\theta \\ z &= |z|\, e^{j\theta} \end{aligned}\right\}\text{polar forms}$$

In converting complex numbers to polar form from rectangular, we use

$$|z| = \sqrt{x^2 + y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

To convert complex numbers to rectangular form from polar, we employ

$$x = |z|\cos\theta, \qquad y = |z|\sin\theta$$

Complex conjugate. The *complex conjugate* of $z = x + jy$ is defined as

$$\bar{z} = x - jy$$

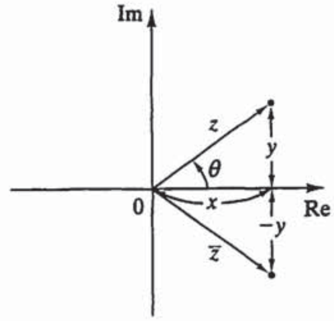

Figure 2–2 Complex number z and its complex conjugate $\bar{z}$.

The complex conjugate of z thus has the same real part as z and an imaginary part that is the negative of the imaginary part of z. Figure 2–2 shows both z and $\bar{z}$. Note that

$$z = x + jy = |z| \angle \theta = |z| (\cos\theta + j\sin\theta)$$
$$\bar{z} = x - jy = |z| \angle -\theta = |z| (\cos\theta - j\sin\theta)$$

Euler's theorem. The power series expansions of $\cos\theta$ and $\sin\theta$ are, respectively,

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

Thus,

$$\cos\theta + j\sin\theta = 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \cdots$$

Since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

it follows that

$$\cos\theta + j\sin\theta = e^{j\theta}$$

This is known as *Euler's theorem.*

Using Euler's theorem, we can express the sine and cosine in complex form. Noting that $e^{-j\theta}$ is the complex conjugate of $e^{j\theta}$ and that

$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$

we find that

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Complex algebra. If the complex numbers are written in a suitable form, operations like addition, subtraction, multiplication, and division can be performed easily.

Equality of complex numbers. Two complex numbers z and w are said to be equal if and only if their real parts are equal and their imaginary parts are equal. So if two complex numbers are written

$$z = x + jy, \qquad w = u + jv$$

then $z = w$ if and only if $x = u$ and $y = v$.

Addition. Two complex numbers in rectangular form can be added by adding the real parts and the imaginary parts separately:

$$z + w = (x + jy) + (u + jv) = (x + u) + j(y + v)$$

Subtraction. Subtracting one complex number from another can be considered as adding the negative of the former:

$$z - w = (x + jy) - (u + jv) = (x - u) + j(y - v)$$

Note that addition and subtraction can be done easily on the rectangular plane.

Multiplication. If a complex number is multiplied by a real number, the result is a complex number whose real and imaginary parts are multiplied by that real number:

$$az = a(x + jy) = ax + jay \qquad (a = \text{real number})$$

If two complex numbers appear in rectangular form and we want the product in rectangular form, multiplication is accomplished by using the fact that $j^2 = -1$. Thus, if two complex numbers are written

$$z = x + jy, \qquad w = u + jv$$

then

$$\begin{aligned} zw &= (x + jy)(u + jv) = xu + jyu + jxv + j^2yv \\ &= (xu - yv) + j(xv + yu) \end{aligned}$$

In polar form, multiplication of two complex numbers can be done easily. The magnitude of the product is the product of the two magnitudes, and the angle of the product is the sum of the two angles. So if two complex numbers are written

$$z = |z| \underline{/\theta}, \qquad w = |w| \underline{/\phi}$$

then

$$zw = |z||w| \underline{/\theta + \phi}$$

Multiplication by j. It is important to note that multiplication by j is equivalent to counterclockwise rotation by 90°. For example, if

$$z = x + jy$$

then

$$jz = j(x + jy) = jx + j^2y = -y + jx$$

or, noting that $j = 1 \underline{/90^\circ}$, if

$$z = |z| \underline{/\theta}$$

then

$$jz = 1 \underline{/90^\circ}\, |z| \underline{/\theta} = |z| \underline{/\theta + 90^\circ}$$

Figure 2–3 illustrates the multiplication of a complex number z by j.

Division. If a complex number $z = |z| \underline{/\theta}$ is divided by another complex number $w = |w| \underline{/\phi}$, then

$$\frac{z}{w} = \frac{|z| \underline{/\theta}}{|w| \underline{/\phi}} = \frac{|z|}{|w|} \underline{/\theta - \phi}$$

That is, the result consists of the quotient of the magnitudes and the difference of the angles.

Division in rectangular form is inconvenient, but can be done by multiplying the denominator and numerator by the complex conjugate of the denominator. This procedure converts the denominator to a real number and thus simplifies division. For instance,

$$\frac{z}{w} = \frac{x + jy}{u + jv} = \frac{(x + jy)(u - jv)}{(u + jv)(u - jv)} = \frac{(xu + yv) + j(yu - xv)}{u^2 + v^2}$$
$$= \frac{xu + yv}{u^2 + v^2} + j\frac{yu - xv}{u^2 + v^2}$$

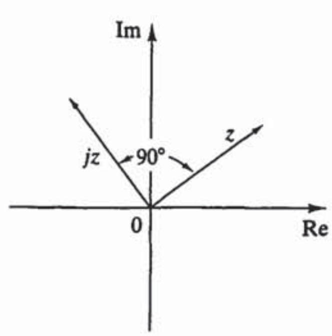

Figure 2–3 Multiplication of a complex number z by j.

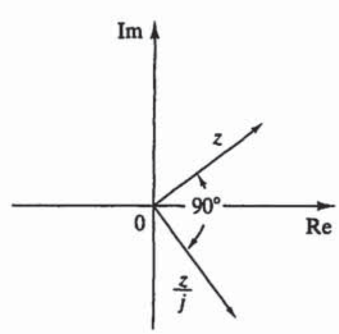

Figure 2–4 Division of a complex number z by j.

Division by j. Note that division by j is equivalent to clockwise rotation by 90°. For example, if $z = x + jy$, then

$$\frac{z}{j} = \frac{x + jy}{j} = \frac{(x + jy)j}{jj} = \frac{jx - y}{-1} = y - jx$$

or

$$\frac{z}{j} = \frac{|z| \angle\theta}{1 \angle 90^\circ} = |z| \angle\theta - 90^\circ$$

Figure 2–4 illustrates the division of a complex number z by j.

Powers and roots. Multiplying z by itself n times, we obtain

$$z^n = (|z| \angle\theta)^n = |z|^n \angle n\theta$$

Extracting the nth root of a complex number is equivalent to raising the number to the $1/n$th power:

$$z^{1/n} = (|z| \angle\theta)^{1/n} = |z|^{1/n} \angle\frac{\theta}{n}$$

For instance,

$$(8.66 - j5)^3 = (10 \angle -30^\circ)^3 = 1000 \angle -90^\circ = 0 - j\,1000 = -j\,1000$$

$$(2.12 - j2.12)^{1/2} = (9 \angle -45^\circ)^{1/2} = 3 \angle -22.5^\circ$$

Comments. It is important to note that

$$|zw| = |z||w|$$

and

$$|z + w| \neq |z| + |w|$$

Complex variable. A complex number has a real part and an imaginary part, both of which are constant. If the real part or the imaginary part (or both) are variables, the complex number is called a *complex variable*. In the Laplace transformation, we use the notation s to denote a complex variable; that is,

$$s = \sigma + j\omega$$

where σ is the real part and $j\omega$ is the imaginary part. (Note that both σ and ω are real.)

Complex function. A complex function $F(s)$, a function of s, has a real part and an imaginary part, or

$$F(s) = F_x + jF_y$$

where F_x and F_y are real quantities. The magnitude of $F(s)$ is $\sqrt{F_x^2 + F_y^2}$, and the angle θ of $F(s)$ is $\tan^{-1}(F_y/F_x)$. The angle is measured counterclockwise from the positive real axis. The complex conjugate of $F(s)$ is $\overline{F}(s) = F_x - jF_y$.

Complex functions commonly encountered in linear systems analysis are single-valued functions of s and are uniquely determined for a given value of s. Typically, such functions have the form

$$F(s) = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

Points at which $F(s)$ equals zero are called *zeros*. That is, $s = -z_1, s = -z_2, \ldots, s = -z_m$ are zeros of $F(s)$. [Note that $F(s)$ may have additional zeros at infinity; see the illustrative example that follows.] Points at which $F(s)$ equals infinity are called *poles*. That is, $s = -p_1, s = -p_2, \ldots, s = -p_n$ are poles of $F(s)$. If the denominator of $F(s)$ involves k-multiple factors $(s + p)^k$, then $s = -p$ is called a *multiple pole* of order k or *repeated pole* of order k. If $k = 1$, the pole is called a *simple pole.*

As an illustrative example, consider the complex function

$$G(s) = \frac{K(s + 2)(s + 10)}{s(s + 1)(s + 5)(s + 15)^2}$$

$G(s)$ has zeros at $s = -2$ and $s = -10$, simple poles at $s = 0, s = -1$, and $s = -5$, and a double pole (multiple pole of order 2) at $s = -15$. Note that $G(s)$ becomes zero at $s = \infty$. Since, for large values of s,

$$G(s) \doteqdot \frac{K}{s^3}$$

it follows that $G(s)$ possesses a triple zero (multiple zero of order 3) at $s = \infty$. If points at infinity are included, $G(s)$ has the same number of poles as zeros. To summarize, $G(s)$ has five zeros $(s = -2, s = -10, s = \infty, s = \infty, s = \infty)$ and five poles $(s = 0, s = -1, s = -5, s = -15, s = -15)$.

2-3 LAPLACE TRANSFORMATION

The Laplace transform method is an operational method that can be used advantageously in solving linear, time-invariant differential equations. Its main advantage is that differentiation of the time function corresponds to multiplication of the transform by a complex variable s, and thus the differential equations in time become algebraic equations in s. The solution of the differential equation can then be found by using a Laplace transform table or the partial-fraction expansion technique. Another advantage of the Laplace transform method is that, in solving the differential equation, the initial conditions are automatically taken care of, and both the particular solution and the complementary solution can be obtained simultaneously.

Laplace transformation. Let us define

$f(t)$ = a time function such that $f(t) = 0$ for $t < 0$

s = a complex variable

$\mathcal{L}$ = an operational symbol indicating that the quantity upon which it operates is to be transformed by the Laplace integral $\int_0^\infty e^{-st}\,dt$

$F(s)$ = Laplace transform of $f(t)$

Then the Laplace transform of $f(t)$ is given by

$$\mathscr{L}[f(t)] = F(s) = \int_0^{\infty} e^{-st}\, dt[f(t)] = \int_0^{\infty} f(t)e^{-st}\, dt$$

The reverse process of finding the time function $f(t)$ from the Laplace transform $F(s)$ is called *inverse Laplace transformation*. The notation for inverse Laplace transformation is $\mathscr{L}^{-1}$. Thus,

$$\mathscr{L}^{-1}[F(s)] = f(t)$$

Existence of Laplace transform. The Laplace transform of a function $f(t)$ exists if the Laplace integral converges. The integral will converge if $f(t)$ is piecewise continuous in every finite interval in the range $t > 0$ and if $f(t)$ is of exponential order as t approaches infinity. A function $f(t)$ is said to be of exponential order if a real, positive constant σ exists such that the function

$$e^{-\sigma t}|f(t)|$$

approaches zero as t approaches infinity. If the limit of the function $e^{-\sigma t}|f(t)|$ approaches zero for σ greater than σ_c and the limit approaches infinity for σ less than σ_c, the value σ_c is called the *abscissa of convergence.*

It can be seen that, for such functions as t, $\sin \omega t$, and $t \sin \omega t$, the abscissa of convergence is equal to zero. For functions like e^{-ct}, te^{-ct}, and $e^{-ct} \sin \omega t$, the abscissa of convergence is equal to $-c$. In the case of functions that increase faster than the exponential function, it is impossible to find suitable values of the abscissa of convergence. Consequently, such functions as e^{t^2} and te^{t^2} do not possess Laplace transforms.

Nevertheless, it should be noted that, although e^{t^2} for $0 \le t \le \infty$ does not possess a Laplace transform, the time function defined by

$$\begin{aligned} f(t) &= e^{t^2} && \text{for } 0 \le t \le T < \infty \\ &= 0 && \text{for } t < 0, T < t \end{aligned}$$

does, since $f(t) = e^{t^2}$ for only a limited time interval $0 \le t \le T$ and not for $0 \le t \le \infty$. Such a signal can be physically generated. Note that the signals that can be physically generated always have corresponding Laplace transforms.

If functions $f_1(t)$ and $f_2(t)$ are both Laplace transformable, then the Laplace transform of $f_1(t) + f_2(t)$ is given by

$$\mathscr{L}[f_1(t) + f_2(t)] = \mathscr{L}[f_1(t)] + \mathscr{L}[f_2(t)]$$

Exponential function. Consider the exponential function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= Ae^{-\alpha t} && \text{for } t \ge 0 \end{aligned}$$

where A and α are constants. The Laplace transform of this exponential function can be obtained as follows:

$$\mathscr{L}[Ae^{-\alpha t}] = \int_0^{\infty} Ae^{-\alpha t}e^{-st}\, dt = A\int_0^{\infty} e^{-(\alpha+s)t}\, dt = \frac{A}{s+\alpha}$$

In performing this integration, we assume that the real part of s is greater than $-\alpha$ (the abscissa of convergence), so that the integral converges. The Laplace transform $F(s)$ of any Laplace transformable function $f(t)$ obtained in this way is valid throughout the entire s plane, except at the poles of $F(s)$. (Although we do not present a proof of this statement, it can be proved by use of the theory of complex variables.)

Step function. Consider the step function

$$\begin{aligned} f(t) &= 0 \qquad \text{for } t < 0 \\ &= A \qquad \text{for } t > 0 \end{aligned}$$

where A is a constant. Note that this is a special case of the exponential function $Ae^{-\alpha t}$, where $\alpha = 0$. The step function is undefined at $t = 0$. Its Laplace transform is given by

$$\mathcal{L}[A] = \int_0^\infty Ae^{-st}\,dt = \frac{A}{s}$$

The step function whose height is unity is called a *unit-step function*. The unit-step function that occurs at $t = t_0$ is frequently written $1(t - t_0)$, a notation that will be used in this book. The preceding step function whose height is A can thus be written $A1(t)$.

The Laplace transform of the unit-step function that is defined by

$$\begin{aligned} 1(t) &= 0 \qquad \text{for } t < 0 \\ &= 1 \qquad \text{for } t > 0 \end{aligned}$$

is $1/s$, or

$$\mathcal{L}[1(t)] = \frac{1}{s}$$

Physically, a step function occurring at $t = t_0$ corresponds to a constant signal suddenly applied to the system at time t equals t_0.

Ramp function. Consider the ramp function

$$\begin{aligned} f(t) &= 0 \qquad \text{for } t < 0 \\ &= At \qquad \text{for } t \geq 0 \end{aligned}$$

where A is a constant. The Laplace transform of this ramp function is

$$\mathcal{L}[At] = A\int_0^\infty te^{-st}\,dt$$

To evaluate the integral, we use the formula for integration by parts:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

In this case, $u = t$ and $dv = e^{-st}\,dt$. [Note that $v = e^{-st}/(-s)$.] Hence,

$$\mathcal{L}[At] = A\int_0^\infty te^{-st}\,dt = A\left(t\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}dt\right)$$
$$= \frac{A}{s}\int_0^\infty e^{-st}\,dt = \frac{A}{s^2}$$

Sinusoidal function. The Laplace transform of the sinusoidal function

$$\begin{aligned} f(t) &= 0 && \text{for } t < 0 \\ &= A\sin\omega t && \text{for } t \geq 0 \end{aligned}$$

where A and ω are constants, is obtained as follows: Noting that

$$e^{j\omega t} = \cos\omega t + j\sin\omega t$$

and

$$e^{-j\omega t} = \cos\omega t - j\sin\omega t$$

we can write

$$\sin\omega t = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

Hence,

$$\mathcal{L}[A\sin\omega t] = \frac{A}{2j}\int_0^\infty (e^{j\omega t} - e^{-j\omega t})e^{-st}\,dt$$
$$= \frac{A}{2j}\frac{1}{s - j\omega} - \frac{A}{2j}\frac{1}{s + j\omega} = \frac{A\omega}{s^2 + \omega^2}$$

Similarly, the Laplace transform of $A\cos\omega t$ can be derived as follows:

$$\mathcal{L}[A\cos\omega t] = \frac{As}{s^2 + \omega^2}$$

Comments. The Laplace transform of any Laplace transformable function $f(t)$ can be found by multiplying $f(t)$ by e^{-st} and then integrating the product from $t = 0$ to $t = \infty$. Once we know the method of obtaining the Laplace transform, however, it is not necessary to derive the Laplace transform of $f(t)$ each time. Laplace transform tables can conveniently be used to find the transform of a given function $f(t)$. Table 2–1 shows Laplace transforms of time functions that will frequently appear in linear systems analysis. In Table 2–2, the properties of Laplace transforms are given.

Translated function. Let us obtain the Laplace transform of the translated function $f(t - \alpha)1(t - \alpha)$, where $\alpha \geq 0$. This function is zero for $t < \alpha$. The functions $f(t)1(t)$ and $f(t - \alpha)1(t - \alpha)$ are shown in Figure 2–5.

By definition, the Laplace transform of $f(t - \alpha)1(t - \alpha)$ is

$$\mathcal{L}[f(t - \alpha)1(t - \alpha)] = \int_0^\infty f(t - \alpha)1(t - \alpha)e^{-st}\,dt$$

TABLE 2–1 Laplace Transform Pairs

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $1(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{s^n}$
5	$t^n \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{1}{(n-1)!}t^{n-1}e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at} \quad (n = 1, 2, 3, \ldots)$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
12	$\sinh \omega t$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh \omega t$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left[1 + \frac{1}{a-b}(be^{-at} - ae^{-bt})\right]$	$\frac{1}{s(s+a)(s+b)}$

TABLE 2–1 *(continued)*

	$f(t)$	$F(s)$
18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$	$\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
23	$-\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t - \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
24	$1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \phi)$ $\phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$	$\frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$
25	$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
26	$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
27	$\sin \omega t - \omega t \cos \omega t$	$\frac{2\omega^3}{(s^2 + \omega^2)^2}$
28	$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2 + \omega^2)^2}$
29	$t \cos \omega t$	$\frac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$
30	$\frac{1}{\omega_2^2 - \omega_1^2}(\cos \omega_1 t - \cos \omega_2 t) \qquad (\omega_1^2 \neq \omega_2^2)$	$\frac{s}{(s^2 + \omega_1^2)(s^2 + \omega_2^2)}$
31	$\frac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$	$\frac{s^2}{(s^2 + \omega^2)^2}$

TABLE 2–2 Properties of Laplace Transforms

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_\pm\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_\pm\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_\pm\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k} \overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \frac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s}$
7	$\mathscr{L}_\pm\left[\iint f(t)\,dt\,dt\right] = \frac{F(s)}{s^2} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s^2} + \frac{\left[\iint f(t)\,dt\,dt\right]_{t=0\pm}}{s}$
8	$\mathscr{L}_\pm\left[\int \cdots \int f(t)(dt)^n\right] = \frac{F(s)}{s^n} + \sum_{k=1}^{n} \frac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$
9	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$
10	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s) \qquad \text{if } \int_0^\infty f(t)\,dt \text{ exists}$
11	$\mathscr{L}[e^{-at}f(t)] = F(s + a)$
12	$\mathscr{L}[f(t - \alpha)1(t - \alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$
13	$\mathscr{L}[tf(t)] = -\frac{dF(s)}{ds}$
14	$\mathscr{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$
15	$\mathscr{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$
16	$\mathscr{L}\left[\frac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds \qquad \text{if } \lim_{t\to 0}\frac{1}{t}f(t) \text{ exists}$
17	$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as)$

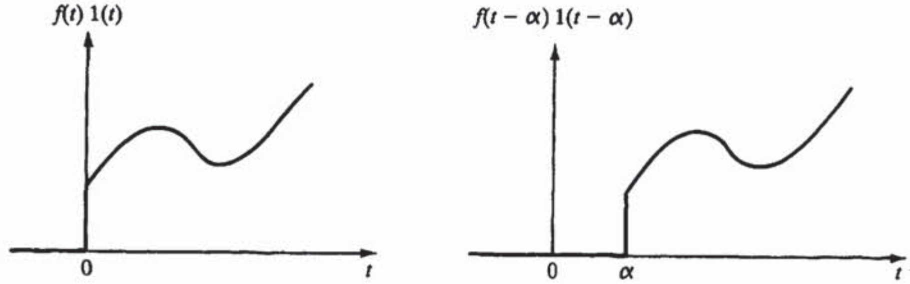

Figure 2–5 Function $f(t)1(t)$ and translated function $f(t-\alpha)1(t-\alpha)$.

By changing the independent variable from t to τ, where $\tau = t - \alpha$, we obtain

$$\int_0^\infty f(t-\alpha)1(t-\alpha)e^{-st}\,dt = \int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau$$

Noting that $f(\tau)1(\tau) = 0$ for $\tau < 0$, we can change the lower limit of integration from $-\alpha$ to 0. Thus,

$$\begin{aligned}\int_{-\alpha}^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau &= \int_0^\infty f(\tau)1(\tau)e^{-s(\tau+\alpha)}\,d\tau \\ &= \int_0^\infty f(\tau)e^{-s\tau}e^{-\alpha s}\,d\tau \\ &= e^{-\alpha s}\int_0^\infty f(\tau)e^{-s\tau}\,d\tau = e^{-\alpha s}F(s)\end{aligned}$$

where

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\,dt$$

Hence,

$$\mathcal{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \qquad \alpha \geq 0$$

This last equation states that the translation of the time function $f(t)1(t)$ by α (where $\alpha \geq 0$) corresponds to the multiplication of the transform $F(s)$ by $e^{-\alpha s}$.

Pulse function. Consider the pulse function shown in Figure 2–6, namely,

$$\begin{aligned} f(t) &= \frac{A}{t_0} \qquad && \text{for } 0 < t < t_0 \\ &= 0 && \text{for } t < 0,\ t_0 < t \end{aligned}$$

where A and t_0 are constants.

The pulse function here may be considered a step function of height A/t_0 that begins at $t = 0$ and that is superimposed by a negative step function of height A/t_0

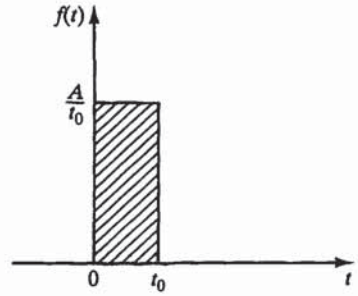

Figure 2–6 Pulse function.

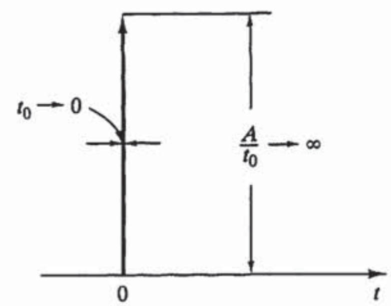

Figure 2–7 Impulse function.

beginning at $t = t_0$; that is,

$$f(t) = \frac{A}{t_0}1(t) - \frac{A}{t_0}1(t - t_0)$$

Then the Laplace transform of $f(t)$ is obtained as

$$\begin{aligned}\mathscr{L}[f(t)] &= \mathscr{L}\left[\frac{A}{t_0}1(t)\right] - \mathscr{L}\left[\frac{A}{t_0}1(t - t_0)\right] \\ &= \frac{A}{t_0 s} - \frac{A}{t_0 s}e^{-st_0} \\ &= \frac{A}{t_0 s}(1 - e^{-st_0}) \qquad (2\text{–}1)\end{aligned}$$

Impulse function. The impulse function is a special limiting case of the pulse function. Consider the impulse function

$$\begin{aligned}f(t) &= \lim_{t_0\to 0}\frac{A}{t_0} \qquad &&\text{for } 0 < t < t_0 \\ &= 0 &&\text{for } t < 0, t_0 < t\end{aligned}$$

Figure 2–7 depicts the impulse function defined here. It is a limiting case of the pulse function shown in Figure 2–6 as t_0 approaches zero. Since the height of the impulse function is A/t_0 and the duration is t_0, the area under the impulse is equal to A. As the duration t_0 approaches zero, the height A/t_0 approaches infinity, but the area under the impulse remains equal to A. Note that the magnitude of the impulse is measured by its area.

From Equation (2–1), the Laplace transform of this impulse function is shown to be

$$\begin{aligned}\mathscr{L}[f(t)] &= \lim_{t_0\to 0}\left[\frac{A}{t_0 s}(1 - e^{-st_0})\right] \\ &= \lim_{t_0\to 0}\frac{\dfrac{d}{dt_0}[A(1 - e^{-st_0})]}{\dfrac{d}{dt_0}(t_0 s)} = \frac{As}{s} = A\end{aligned}$$

Thus, the Laplace transform of the impulse function is equal to the area under the impulse.

The impulse function whose area is equal to unity is called the *unit-impulse function* or the *Dirac delta function*. The unit-impulse function occurring at $t = t_0$ is usually denoted by $\delta(t - t_0)$, which satisfies the following conditions:

$$\delta(t - t_0) = 0 \qquad \text{for } t \neq t_0$$
$$\delta(t - t_0) = \infty \qquad \text{for } t = t_0$$
$$\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$$

An impulse that has an infinite magnitude and zero duration is mathematical fiction and does not occur in physical systems. If, however, the magnitude of a pulse input to a system is very large and its duration very short compared with the system time constants, then we can approximate the pulse input by an impulse function. For instance, if a force or torque input $f(t)$ is applied to a system for a very short time duration $0 < t < t_0$, where the magnitude of $f(t)$ is sufficiently large so that $\int_0^{t_0} f(t)\, dt$ is not negligible, then this input can be considered an impulse input. (Note that, when we describe the impulse input, the area or magnitude of the impulse is most important, but the exact shape of the impulse is usually immaterial.) The impulse input supplies energy to the system in an infinitesimal time.

The concept of the impulse function is highly useful in differentiating discontinuous-time functions. The unit-impulse function $\delta(t - t_0)$ can be considered the derivative of the unit-step function $1(t - t_0)$ at the point of discontinuity $t = t_0$, or

$$\delta(t - t_0) = \frac{d}{dt} 1(t - t_0)$$

Conversely, if the unit-impulse function $\delta(t - t_0)$ is integrated, the result is the unit-step function $1(t - t_0)$. With the concept of the impulse function, we can differentiate a function containing discontinuities, giving impulses, the magnitudes of which are equal to the magnitude of each corresponding discontinuity.

Multiplication of $f(t)$ by $e^{-\alpha t}$. If $f(t)$ is Laplace transformable and its Laplace transform is $F(s)$, then the Laplace transform of $e^{-\alpha t} f(t)$ is obtained as

$$\mathscr{L}[e^{-\alpha t} f(t)] = \int_0^{\infty} e^{-\alpha t} f(t) e^{-st}\, dt = F(s + \alpha) \tag{2–2}$$

We see that the multiplication of $f(t)$ by $e^{-\alpha t}$ has the effect of replacing s by $(s + \alpha)$ in the Laplace transform. Conversely, changing s to $(s + \alpha)$ is equivalent to multiplying $f(t)$ by $e^{-\alpha t}$. (Note that α may be real or complex.)

The relationship given by Equation (2–2) is useful in finding the Laplace transforms of such functions as $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$. For instance, since

$$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2} = F(s) \qquad \text{and} \qquad \mathscr{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2} = G(s)$$

it follows from Equation (2–2) that the Laplace transforms of $e^{-\alpha t}\sin\omega t$ and $e^{-\alpha t}\cos\omega t$ are given, respectively, by

$$\mathscr{L}[e^{-\alpha t}\sin\omega t] = F(s+\alpha) = \frac{\omega}{(s+\alpha)^2+\omega^2}$$

and

$$\mathscr{L}[e^{-\alpha t}\cos\omega t] = G(s+\alpha) = \frac{s+\alpha}{(s+\alpha)^2+\omega^2}$$

Comments on the lower limit of the Laplace integral. In some cases, $f(t)$ possesses an impulse function at $t = 0$. Then the lower limit of the Laplace integral must be clearly specified as to whether it is $0-$ or $0+$, since the Laplace transforms of $f(t)$ differ for these two lower limits. If such a distinction of the lower limit of the Laplace integral is necessary, we use the notations

$$\mathscr{L}_+[f(t)] = \int_{0+}^{\infty} f(t)e^{-st}\,dt$$

and

$$\mathscr{L}_-[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}\,dt = \mathscr{L}_+[f(t)] + \int_{0-}^{0+} f(t)e^{-st}\,dt$$

If $f(t)$ involves an impulse function at $t = 0$, then

$$\mathscr{L}_+[f(t)] \neq \mathscr{L}_-[f(t)]$$

since

$$\int_{0-}^{0+} f(t)e^{-st}\,dt \neq 0$$

for such a case. Obviously, if $f(t)$ does not possess an impulse function at $t = 0$ (i.e., if the function to be transformed is finite between $t = 0-$ and $t = 0+$), then

$$\mathscr{L}_+[f(t)] = \mathscr{L}_-[f(t)]$$

Differentiation theorem. The Laplace transform of the derivative of a function $f(t)$ is given by

$$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0) \tag{2–3}$$

where $f(0)$ is the initial value of $f(t)$, evaluated at $t = 0$. Equation (2–3) is called the differentiation theorem.

For a given function $f(t)$, the values of $f(0+)$ and $f(0-)$ may be the same or different, as illustrated in Figure 2–8. The distinction between $f(0+)$ and $f(0-)$ is important when $f(t)$ has a discontinuity at $t = 0$, because, in such a case, $df(t)/dt$ will

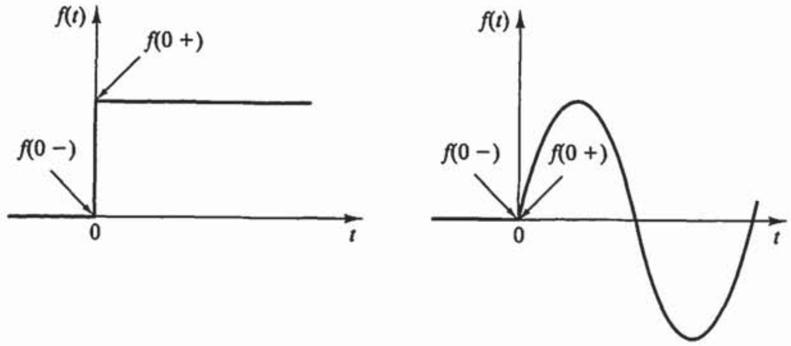

Figure 2–8 Step function and sine function indicating initial values at $t = 0-$ and $t = 0+$.

involve an impulse function at $t = 0$. If $f(0+) \neq f(0-)$, Equation (2–3) must be modified to

$$\mathscr{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+)$$

$$\mathscr{L}_-\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0-)$$

To prove the differentiation theorem, we proceed as follows: Integrating the Laplace integral by parts gives

$$\int_0^\infty f(t)e^{-st}\,dt = f(t)\frac{e^{-st}}{-s}\bigg|_0^\infty - \int_0^\infty \left[\frac{d}{dt}f(t)\right]\frac{e^{-st}}{-s}\,dt$$

Hence,

$$F(s) = \frac{f(0)}{s} + \frac{1}{s}\mathscr{L}\left[\frac{d}{dt}f(t)\right]$$

It follows that

$$\mathscr{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)$$

Similarly, for the second derivative of $f(t)$, we obtain the relationship

$$\mathscr{L}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

where $\dot{f}(0)$ is the value of $df(t)/dt$ evaluated at $t = 0$. To derive this equation, define

$$\frac{d}{dt}f(t) = g(t)$$

Then

$$\begin{aligned}\mathscr{L}\left[\frac{d^2}{dt^2}f(t)\right] &= \mathscr{L}\left[\frac{d}{dt}g(t)\right] = s\mathscr{L}[g(t)] - g(0)\\ &= s\mathscr{L}\left[\frac{d}{dt}f(t)\right] - \dot{f}(0)\\ &= s^2F(s) - sf(0) - \dot{f}(0)\end{aligned}$$

Similarly, for the nth derivative of $f(t)$, we obtain

$$\mathscr{L}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - \overset{(n-1)}{f}(0)$$

where $f(0), \dot{f}(0), \ldots, \overset{(n-1)}{f}(0)$ represent the values of $f(t), df(t)/dt, \ldots, d^{n-1}f(t)/dt^{n-1}$, respectively, evaluated at $t = 0$. If the distinction between $\mathscr{L}_+$ and $\mathscr{L}_-$ is necessary, we substitute $t = 0+$ or $t = 0-$ into $f(t), df(t)/dt, \ldots, d^{n-1}f(t)/dt^{n-1}$, depending on whether we take $\mathscr{L}_+$ or $\mathscr{L}_-$.

Note that, for Laplace transforms of derivatives of $f(t)$ to exist, $d^nf(t)/dt^n$ $(n = 1, 2, 3, \ldots)$ must be Laplace transformable.

Note also that, if all the initial values of $f(t)$ and its derivatives are equal to zero, then the Laplace transform of the nth derivative of $f(t)$ is given by $s^nF(s)$.

Final-value theorem. The final-value theorem relates the steady-state behavior of $f(t)$ to the behavior of $sF(s)$ in the neighborhood of $s = 0$. The theorem, however, applies if and only if $\lim_{t\to\infty} f(t)$ exists [which means that $f(t)$ settles down to a definite value as $t \to \infty$]. If all poles of $sF(s)$ lie in the left half s plane, then $\lim_{t\to\infty} f(t)$ exists, but if $sF(s)$ has poles on the imaginary axis or in the right half s plane, $f(t)$ will contain oscillating or exponentially increasing time functions, respectively, and $\lim_{t\to\infty} f(t)$ will not exist. The final-value theorem does not apply to such cases. For instance, if $f(t)$ is a sinusoidal function $\sin \omega t$, then $sF(s)$ has poles at $s = \pm j\omega$, and $\lim_{t\to\infty} f(t)$ does not exist. Therefore, the theorem is not applicable to such a function.

The final-value theorem may be stated as follows: If $f(t)$ and $df(t)/dt$ are Laplace transformable, if $F(s)$ is the Laplace transform of $f(t)$, and if $\lim_{t\to\infty} f(t)$ exists, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

To prove the theorem, we let s approach zero in the equation for the Laplace transform of the derivative of $f(t)$, or

$$\lim_{s\to 0}\int_0^\infty \left[\frac{d}{dt}f(t)\right]e^{-st}\,dt = \lim_{s\to 0}[sF(s) - f(0)]$$

Since $\lim_{s\to 0} e^{-st} = 1$, if $\lim_{t\to\infty} f(t)$ exists, then we obtain

$$\begin{aligned}\int_0^\infty \left[\frac{d}{dt}f(t)\right]dt &= f(t)\Big|_0^\infty = f(\infty) - f(0)\\ &= \lim_{s\to 0} sF(s) - f(0)\end{aligned}$$

from which it follows that

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Initial-value theorem. The initial-value theorem is the counterpart of the final-value theorem. Using the initial-value theorem, we are able to find the value of $f(t)$ at $t = 0+$ directly from the Laplace transform of $f(t)$. The theorem does not give the value of $f(t)$ at exactly $t = 0$, but rather gives it at a time slightly greater than zero.

The initial-value theorem may be stated as follows: If $f(t)$ and $df(t)/dt$ are both Laplace transformable and if $\lim_{s\to\infty} sF(s)$ exists, then

$$f(0+) = \lim_{s\to\infty} sF(s)$$

To prove this theorem, we use the equation for the $\mathscr{L}_+$ transform of $df(t)/dt$:

$$\mathscr{L}_+\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0+)$$

For the time interval $0+ \le t \le \infty$, as s approaches infinity, e^{-st} approaches zero. (Note that we must use $\mathscr{L}_+$ rather than $\mathscr{L}_-$ for this condition.) Hence,

$$\lim_{s\to\infty} \int_{0+}^{\infty} \left[\frac{d}{dt}f(t)\right] e^{-st}\, dt = \lim_{s\to\infty} [sF(s) - f(0+)] = 0$$

or

$$f(0+) = \lim_{s\to\infty} sF(s)$$

In applying the initial-value theorem, we are not limited as to the locations of the poles of $sF(s)$. Thus, the theorem is valid for the sinusoidal function.

Note that the initial-value theorem and the final-value theorem provide a convenient check on the solution, since they enable us to predict the system behavior in the time domain without actually transforming functions in s back to time functions.

Integration theorem. If $f(t)$ is of exponential order, then the Laplace transform of $\int f(t)\, dt$ exists and is given by

$$\mathscr{L}\left[\int f(t)\, dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} \tag{2–4}$$

where $F(s) = \mathscr{L}[f(t)]$ and $f^{-1}(0) = \int f(t)\, dt$, evaluated at $t = 0$. Equation (2–4) is called the integration theorem.

The integration theorem can be proven as follows: Integration by parts yields

$$\begin{aligned}\mathscr{L}\left[\int f(t)\, dt\right] &= \int_0^{\infty} \left[\int f(t)\, dt\right] e^{-st}\, dt \\ &= \left[\int f(t)\, dt\right] \frac{e^{-st}}{-s}\bigg|_0^{\infty} - \int_0^{\infty} f(t) \frac{e^{-st}}{-s}\, dt\end{aligned}$$

$$= \frac{1}{s}\int f(t)\,dt\bigg|_{t=0} + \frac{1}{s}\int_0^{\infty} f(t)e^{-st}\,dt$$
$$= \frac{f^{-1}(0)}{s} + \frac{F(s)}{s}$$

and the theorem is proven.

Note that, if $f(t)$ involves an impulse function at $t = 0$, then $f^{-1}(0+) \neq f^{-1}(0-)$. So if $f(t)$ involves an impulse function at $t = 0$, we must modify Equation (2–4) as follows:

$$\mathscr{L}_+\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0+)}{s}$$
$$\mathscr{L}_-\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0-)}{s}$$

We see that integration in the time domain is converted into division in the s domain. If the initial value of the integral is zero, the Laplace transform of the integral of $f(t)$ is given by $F(s)/s$.

The integration theorem can be modified slightly to deal with the definite integral of $f(t)$. If $f(t)$ is of exponential order, the Laplace transform of the definite integral $\int_0^t f(t)\,dt$ can be given by

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} \tag{2–5}$$

To prove Equation (2–5), first note that

$$\int_0^t f(t)\,dt = \int f(t)\,dt - f^{-1}(0)$$

where $f^{-1}(0)$ is equal to $\int f(t)\,dt$, evaluated at $t = 0$, and is a constant. Hence,

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \mathscr{L}\left[\int f(t)\,dt - f^{-1}(0)\right]$$
$$= \mathscr{L}\left[\int f(t)\,dt\right] - \mathscr{L}[f^{-1}(0)]$$

Referring to Equation (2–4) and noting that $f^{-1}(0)$ is a constant, so that

$$\mathscr{L}[f^{-1}(0)] = \frac{f^{-1}(0)}{s}$$

we obtain

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s} + \frac{f^{-1}(0)}{s} - \frac{f^{-1}(0)}{s} = \frac{F(s)}{s}$$

Note that, if $f(t)$ involves an impulse function at $t = 0$, then $\int_{0+}^{t} f(t)\,dt \neq \int_{0-}^{t} f(t)\,dt$, and the following distinction must be observed:

$$\mathcal{L}_+\left[\int_{0+}^{t} f(t)\,dt\right] = \frac{\mathcal{L}_+[f(t)]}{s}$$

$$\mathcal{L}_-\left[\int_{0-}^{t} f(t)\,dt\right] = \frac{\mathcal{L}_-[f(t)]}{s}$$

2–4 INVERSE LAPLACE TRANSFORMATION

The inverse Laplace transformation refers to the process of finding the time function $f(t)$ from the corresponding Laplace transform $F(s)$. Several methods are available for finding inverse Laplace transforms. The simplest of these methods are (1) to use tables of Laplace transforms to find the time function $f(t)$ corresponding to a given Laplace transform $F(s)$ and (2) to use the partial-fraction expansion method. In this section, we present the latter technique. [Note that MATLAB is quite useful in obtaining the partial-fraction expansion of the ratio of two polynomials, $B(s)/A(s)$. We shall discuss the MATLAB approach to the partial-fraction expansion in Chapter 4.]

Partial-fraction expansion method for finding inverse Laplace transforms. If $F(s)$, the Laplace transform of $f(t)$, is broken up into components, or

$$F(s) = F_1(s) + F_2(s) + \cdots + F_n(s)$$

and if the inverse Laplace transforms of $F_1(s), F_2(s), \ldots, F_n(s)$ are readily available, then

$$\begin{aligned}\mathcal{L}^{-1}[F(s)] &= \mathcal{L}^{-1}[F_1(s)] + \mathcal{L}^{-1}[F_2(s)] + \cdots + \mathcal{L}^{-1}[F_n(s)] \\ &= f_1(t) + f_2(t) + \cdots + f_n(t)\end{aligned}$$

where $f_1(t), f_2(t), \ldots, f_n(t)$ are the inverse Laplace transforms of $F_1(s), F_2(s), \cdots, F_n(s)$, respectively. The inverse Laplace transform of $F(s)$ thus obtained is unique, except possibly at points where the time function is discontinuous. Whenever the time function is continuous, the time function $f(t)$ and its Laplace transform $F(s)$ have a one-to-one correspondence.

For problems in systems analysis, $F(s)$ frequently occurs in the form

$$F(s) = \frac{B(s)}{A(s)}$$

where $A(s)$ and $B(s)$ are polynomials in s and the degree of $B(s)$ is not higher than that of $A(s)$.

The advantage of the partial-fraction expansion approach is that the individual terms of $F(s)$ resulting from the expansion into partial-fraction form are very simple functions of s; consequently, it is not necessary to refer to a Laplace transform table if we memorize several simple Laplace transform pairs. Note, however, that in applying the partial-fraction expansion technique in the search for the inverse Laplace transform of $F(s) = B(s)/A(s)$, the roots of the denominator polynomial $A(s)$ must be known in advance. That is, this method does not apply until the denominator polynomial has been factored.

Consider $F(s)$ written in the factored form

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2)\cdots(s + z_m)}{(s + p_1)(s + p_2)\cdots(s + p_n)}$$

where $p_1, p_2, \ldots, p_n$ and $z_1, z_2, \ldots, z_m$ are either real or complex quantities, but for each complex p_i or z_i, there will occur the complex conjugate of p_i or z_i, respectively. Here, the highest power of s in $A(s)$ is assumed to be higher than that in $B(s)$.

In the expansion of $B(s)/A(s)$ into partial-fraction form, it is important that the highest power of s in $A(s)$ be greater than the highest power of s in $B(s)$ because if that is not the case, then the numerator $B(s)$ must be divided by the denominator $A(s)$ in order to produce a polynomial in s plus a remainder (a ratio of polynomials in s whose numerator is of lower degree than the denominator). (For details, see **Example 2–2.**)

Partial-fraction expansion when $F(s)$ involves distinct poles only. In this case, $F(s)$ can always be expanded into a sum of simple partial fractions; that is,

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n} \tag{2–6}$$

where $a_k (k = 1, 2, \ldots, n)$ are constants. The coefficient a_k is called the *residue* at the pole at $s = -p_k$. The value of a_k can be found by multiplying both sides of Equation (2–6) by $(s + p_k)$ and letting $s = -p_k$, giving

$$\begin{aligned}\left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} &= \left[\frac{a_1}{s + p_1}(s + p_k) + \frac{a_2}{s + p_2}(s + p_k) + \cdots \right. \\ &\quad \left. + \frac{a_k}{s + p_k}(s + p_k) + \cdots + \frac{a_n}{s + p_n}(s + p_k)\right]_{s=-p_k} \\ &= a_k\end{aligned}$$

We see that all the expanded terms drop out, with the exception of a_k. Thus, the residue a_k is found from

$$a_k = \left[(s + p_k)\frac{B(s)}{A(s)}\right]_{s=-p_k} \tag{2–7}$$

Note that since $f(t)$ is a real function of time, if p_1 and p_2 are complex conjugates, then the residues a_1 and a_2 are also complex conjugates. Only one of the conjugates, a_1 or a_2, need be evaluated, because the other is known automatically.

Since

$$\mathscr{L}^{-1}\left[\frac{a_k}{s + p_k}\right] = a_k e^{-p_k t}$$

$f(t)$ is obtained as

$$f(t) = \mathscr{L}^{-1}[F(s)] = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t} \qquad t \geq 0$$

Example 2–1

Find the inverse Laplace transform of

$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

where a_1 and a_2 are found by using Equation (2–7):

$$a_1 = \left[(s+1)\frac{s+3}{(s+1)(s+2)}\right]_{s=-1} = \left[\frac{s+3}{s+2}\right]_{s=-1} = 2$$

$$a_2 = \left[(s+2)\frac{s+3}{(s+1)(s+2)}\right]_{s=-2} = \left[\frac{s+3}{s+1}\right]_{s=-2} = -1$$

Thus,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{-1}{s+2}\right] \\ &= 2e^{-t} - e^{-2t} \qquad t \geq 0 \end{aligned}$$

Example 2–2

Obtain the inverse Laplace transform of

$$G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s+1)(s+2)}$$

Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator:

$$G(s) = s + 2 + \frac{s+3}{(s+1)(s+2)}$$

Note that the Laplace transform of the unit-impulse function $\delta(t)$ is unity and that the Laplace transform of $d\delta(t)/dt$ is s. The third term on the right-hand side of this last equation is $F(s)$ in **Example 2–1**. So the inverse Laplace transform of $G(s)$ is given as

$$g(t) = \frac{d}{dt}\delta(t) + 2\delta(t) + 2e^{-t} - e^{-2t} \qquad t \geq 0-$$

Comment. Consider a function $F(s)$ that involves a quadratic factor $s^2 + as + b$ in the denominator. If this quadratic expression has a pair of complex-conjugate roots, then it is better not to factor the quadratic, in order to avoid complex numbers. For example, if $F(s)$ is given as

$$F(s) = \frac{p(s)}{s(s^2 + as + b)}$$

where $a \geq 0$ and $b > 0$, and if $s^2 + as + b = 0$ has a pair of complex-conjugate roots, then expand $F(s)$ into the following partial-fraction expansion form:

$$F(s) = \frac{c}{s} + \frac{ds + e}{s^2 + as + b}$$

(See **Example 2–3** and **Problems A–2–15, A–2–16,** and **A–2–19.**)

Example 2–3

Find the inverse Laplace transform of

$$F(s) = \frac{2s + 12}{s^2 + 2s + 5}$$

Notice that the denominator polynomial can be factored as

$$s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2)$$

The two roots of the denominator are complex conjugates. Hence, we expand $F(s)$ into the sum of a damped sine and a damped cosine function.

Noting that $s^2 + 2s + 5 = (s + 1)^2 + 2^2$ and referring to the Laplace transforms of $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$, rewritten as

$$\mathscr{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2}$$

and

$$\mathscr{L}[e^{-\alpha t} \cos \omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}$$

we can write the given $F(s)$ as a sum of a damped sine and a damped cosine function:

$$\begin{aligned} F(s) &= \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s + 1)}{(s + 1)^2 + 2^2} \\ &= 5\frac{2}{(s + 1)^2 + 2^2} + 2\frac{s + 1}{(s + 1)^2 + 2^2} \end{aligned}$$

It follows that

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}[F(s)] \\ &= 5\mathscr{L}^{-1}\left[\frac{2}{(s + 1)^2 + 2^2}\right] + 2\mathscr{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right] \\ &= 5e^{-t} \sin 2t + 2e^{-t} \cos 2t \qquad t \geq 0 \end{aligned}$$

Partial-fraction expansion when $F(s)$ involves multiple poles. Instead of discussing the general case, we shall use an example to show how to obtain the partial-fraction expansion of $F(s)$. (See also **Problems A–2–17** and **A–2–19.**)

Consider

$$F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3}$$

The partial-fraction expansion of this $F(s)$ involves three terms:

$$F(s) = \frac{B(s)}{A(s)} = \frac{b_3}{(s+1)^3} + \frac{b_2}{(s+1)^2} + \frac{b_1}{s+1}$$

where b_3, b_2, and b_1 are determined as follows: Multiplying both sides of this last equation by $(s+1)^3$, we have

$$(s+1)^3 \frac{B(s)}{A(s)} = b_3 + b_2(s+1) + b_1(s+1)^2 \tag{2–8}$$

Then, letting $s = -1$, we find that Equation (2–8) gives

$$\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_3$$

Also, differentiating both sides of Equation (2–8) with respect to s yields

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = b_2 + 2b_1(s+1) \tag{2–9}$$

If we let $s = -1$ in Equation (2–9), then

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_2$$

Differentiating both sides of Equation (2–9) with respect to s, we obtain

$$\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = 2b_1$$

From the preceding analysis, it can be seen that the values of b_3, b_2, and b_1 are found systematically as follows:

$$\begin{aligned}
b_3 &= \left[(s+1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} \\
&= (s^2 + 2s + 3)_{s=-1} \\
&= 2 \\
b_2 &= \left\{\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]\right\}_{s=-1} \\
&= \left[\frac{d}{ds}(s^2 + 2s + 3)\right]_{s=-1} \\
&= (2s + 2)_{s=-1} \\
&= 0 \\
b_1 &= \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right]\right\}_{s=-1} \\
&= \frac{1}{2!}\left[\frac{d^2}{ds^2}(s^2 + 2s + 3)\right]_{s=-1} \\
&= \frac{1}{2}(2) = 1
\end{aligned}$$

We thus obtain

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= \mathcal{L}^{-1}\left[\frac{2}{(s+1)^3}\right] + \mathcal{L}^{-1}\left[\frac{0}{(s+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] \\ &= t^2e^{-t} + 0 + e^{-t} \\ &= (t^2+1)e^{-t} \qquad t \geq 0 \end{aligned}$$

2–5 SOLVING LINEAR, TIME-INVARIANT DIFFERENTIAL EQUATIONS

In this section, we are concerned with the use of the Laplace transform method in solving linear, time-invariant differential equations.

The Laplace transform method yields the complete solution (complementary solution and particular solution) of linear, time-invariant differential equations. Classical methods for finding the complete solution of a differential equation require the evaluation of the integration constants from the initial conditions. In the case of the Laplace transform method, however, this requirement is unnecessary because the initial conditions are automatically included in the Laplace transform of the differential equation.

If all initial conditions are zero, then the Laplace transform of the differential equation is obtained simply by replacing d/dt with s, d^2/dt^2 with s^2, and so on.

In solving linear, time-invariant differential equations by the Laplace transform method, two steps are followed:

1. By taking the Laplace transform of each term in the given differential equation, convert the differential equation into an algebraic equation in s and obtain the expression for the Laplace transform of the dependent variable by rearranging the algebraic equation.
2. The time solution of the differential equation is obtained by finding the inverse Laplace transform of the dependent variable.

In the discussion that follows, two examples are used to demonstrate the solution of linear, time-invariant differential equations by the Laplace transform method.

Example 2–4

Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0, \qquad x(0) = a, \qquad \dot{x}(0) = b$$

where a and b are constants.

Writing the Laplace transform of $x(t)$ as $X(s)$, or

$$\mathcal{L}[x(t)] = X(s)$$

we obtain

$$\begin{aligned} \mathcal{L}[\dot{x}] &= sX(s) - x(0) \\ \mathcal{L}[\ddot{x}] &= s^2X(s) - sx(0) - \dot{x}(0) \end{aligned}$$

The Laplace transform of the given differential equation becomes

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 3[sX(s) - x(0)] + 2X(s) = 0$$

Substituting the given initial conditions into the preceding equation yields

$$[s^2X(s) - as - b] + 3[sX(s) - a] + 2X(s) = 0$$

or

$$(s^2 + 3s + 2)X(s) = as + b + 3a$$

Solving this last equation for $X(s)$, we have

$$X(s) = \frac{as + b + 3a}{s^2 + 3s + 2} = \frac{as + b + 3a}{(s + 1)(s + 2)} = \frac{2a + b}{s + 1} - \frac{a + b}{s + 2}$$

The inverse Laplace transform of $X(s)$ produces

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[\frac{2a + b}{s + 1}\right] - \mathcal{L}^{-1}\left[\frac{a + b}{s + 2}\right] \\ &= (2a + b)e^{-t} - (a + b)e^{-2t} \qquad t \geq 0 \end{aligned}$$

which is the solution of the given differential equation. Notice that the initial conditions a and b appear in the solution. Thus, $x(t)$ has no undetermined constants.

Example 2–5

Find the solution $x(t)$ of the differential equation

$$\ddot{x} + 2\dot{x} + 5x = 3, \qquad x(0) = 0, \qquad \dot{x}(0) = 0$$

Noting that $\mathcal{L}[3] = 3/s$, $x(0) = 0$, and $\dot{x}(0) = 0$, we see that the Laplace transform of the differential equation becomes

$$s^2X(s) + 2sX(s) + 5X(s) = \frac{3}{s}$$

Solving this equation for $X(s)$, we obtain

$$\begin{aligned} X(s) &= \frac{3}{s(s^2 + 2s + 5)} \\ &= \frac{3}{5}\frac{1}{s} - \frac{3}{5}\frac{s + 2}{s^2 + 2s + 5} \\ &= \frac{3}{5}\frac{1}{s} - \frac{3}{10}\frac{2}{(s + 1)^2 + 2^2} - \frac{3}{5}\frac{s + 1}{(s + 1)^2 + 2^2} \end{aligned}$$

Hence, the inverse Laplace transform becomes

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X(s)] \\ &= \frac{3}{5}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{3}{10}\mathcal{L}^{-1}\left[\frac{2}{(s + 1)^2 + 2^2}\right] - \frac{3}{5}\mathcal{L}^{-1}\left[\frac{s + 1}{(s + 1)^2 + 2^2}\right] \\ &= \frac{3}{5} - \frac{3}{10}e^{-t}\sin 2t - \frac{3}{5}e^{-t}\cos 2t \qquad t \geq 0 \end{aligned}$$

which is the solution of the given differential equation.

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–2–1

Obtain the real and imaginary parts of

$$\frac{2 + j1}{3 + j4}$$

Also, obtain the magnitude and angle of this complex quantity.

Solution

$$\frac{2 + j1}{3 + j4} = \frac{(2 + j1)(3 - j4)}{(3 + j4)(3 - j4)} = \frac{6 + j3 - j8 + 4}{9 + 16} = \frac{10 - j5}{25}$$
$$= \frac{2}{5} - j\frac{1}{5}$$

Hence,

$$\text{real part} = \frac{2}{5}, \qquad \text{imaginary part} = -j\frac{1}{5}$$

The magnitude and angle of this complex quantity are obtained as follows:

$$\text{magnitude} = \sqrt{\left(\frac{2}{5}\right)^2 + \left(\frac{-1}{5}\right)^2} = \sqrt{\frac{5}{25}} = \frac{1}{\sqrt{5}} = 0.447$$
$$\text{angle} = \tan^{-1}\frac{-1/5}{2/5} = \tan^{-1}\frac{-1}{2} = -26.565°$$

Problem A–2–2

Find the Laplace transform of

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= te^{-3t} && t \geq 0 \end{aligned}$$

Solution Since

$$\mathscr{L}[t] = G(s) = \frac{1}{s^2}$$

referring to Equation (2–2), we obtain

$$F(s) = \mathscr{L}[te^{-3t}] = G(s + 3) = \frac{1}{(s + 3)^2}$$

Problem A–2–3

What is the Laplace transform of

$$\begin{aligned} f(t) &= 0 && t < 0 \\ &= \sin(\omega t + \theta) && t \geq 0 \end{aligned}$$

where θ is a constant?

Solution Noting that

$$\sin(\omega t + \theta) = \sin \omega t \cos \theta + \cos \omega t \sin \theta$$

we have

$$\begin{aligned}\mathcal{L}[\sin(\omega t + \theta)] &= \cos\theta\,\mathcal{L}[\sin \omega t] + \sin\theta\,\mathcal{L}[\cos \omega t] \\ &= \cos\theta \frac{\omega}{s^2 + \omega^2} + \sin\theta \frac{s}{s^2 + \omega^2} \\ &= \frac{\omega\cos\theta + s\sin\theta}{s^2 + \omega^2}\end{aligned}$$

Problem A–2–4

Find the Laplace transform $F(s)$ of the function $f(t)$ shown in Figure 2–9. Also, find the limiting value of $F(s)$ as a approaches zero.

Solution The function $f(t)$ can be written

$$f(t) = \frac{1}{a^2}1(t) - \frac{2}{a^2}1(t-a) + \frac{1}{a^2}1(t-2a)$$

Then

$$\begin{aligned}F(s) &= \mathcal{L}[f(t)] \\ &= \frac{1}{a^2}\mathcal{L}[1(t)] - \frac{2}{a^2}\mathcal{L}[1(t-a)] + \frac{1}{a^2}\mathcal{L}[1(t-2a)] \\ &= \frac{1}{a^2}\frac{1}{s} - \frac{2}{a^2}\frac{1}{s}e^{-as} + \frac{1}{a^2}\frac{1}{s}e^{-2as} \\ &= \frac{1}{a^2 s}(1 - 2e^{-as} + e^{-2as})\end{aligned}$$

As a approaches zero, we have

$$\begin{aligned}\lim_{a\to 0} F(s) &= \lim_{a\to 0}\frac{1 - 2e^{-as} + e^{-2as}}{a^2 s} = \lim_{a\to 0}\frac{\dfrac{d}{da}(1 - 2e^{-as} + e^{-2as})}{\dfrac{d}{da}(a^2 s)} \\ &= \lim_{a\to 0}\frac{2se^{-as} - 2se^{-2as}}{2as} = \lim_{a\to 0}\frac{e^{-as} - e^{-2as}}{a}\end{aligned}$$

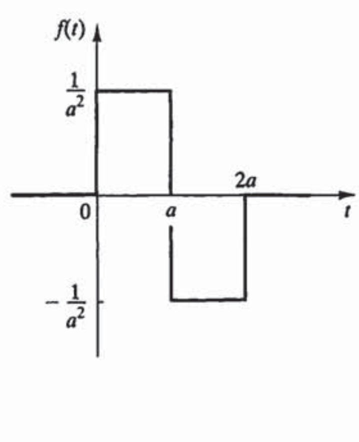

Figure 2–9 Function $f(t)$.

$$= \lim_{a\to 0} \frac{\frac{d}{da}(e^{-as} - e^{-2as})}{\frac{d}{da}(a)} = \lim_{a\to 0} \frac{-se^{-as} + 2se^{-2as}}{1}$$
$$= -s + 2s = s$$

Problem A-2-5

Obtain the Laplace transform of the function $f(t)$ shown in Figure 2-10.

Solution The given function $f(t)$ can be defined as follows:

$$\begin{aligned} f(t) &= 0 && t \le 0 \\ &= \frac{b}{a}t && 0 < t \le a \\ &= 0 && a < t \end{aligned}$$

Notice that $f(t)$ can be considered a sum of the three functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ shown in Figure 2-11. Hence, $f(t)$ can be written as

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) + f_3(t) \\ &= \frac{b}{a}t \cdot 1(t) - \frac{b}{a}(t-a)\cdot 1(t-a) - b\cdot 1(t-a) \end{aligned}$$

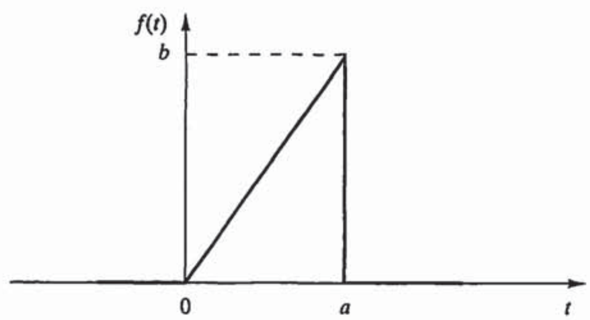

Figure 2-10 Function $f(t)$.

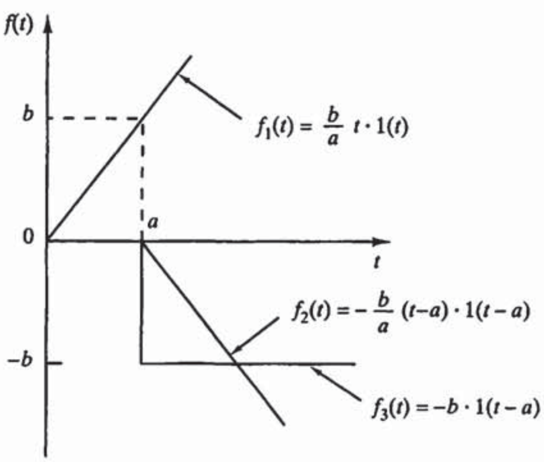

Figure 2-11 Functions $f_1(t)$, $f_2(t)$, and $f_3(t)$.

Then the Laplace transform of $f(t)$ becomes

$$F(s) = \frac{b}{a}\frac{1}{s^2} - \frac{b}{a}\frac{1}{s^2}e^{-as} - b\frac{1}{s}e^{-as}$$
$$= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as}$$

The same $F(s)$ can, of course, be obtained by performing the following Laplace integration:

$$\mathcal{L}[f(t)] = \int_0^a \frac{b}{a}te^{-st}\,dt + \int_a^\infty 0\,e^{-st}\,dt$$
$$= \frac{b}{a}t\frac{e^{-st}}{-s}\bigg|_0^a - \int_0^a \frac{b}{a}\frac{e^{-st}}{-s}\,dt$$
$$= b\frac{e^{-as}}{-s} + \frac{b}{as}\frac{e^{-st}}{-s}\bigg|_0^a$$
$$= b\frac{e^{-as}}{-s} - \frac{b}{as^2}(e^{-as} - 1)$$
$$= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as}$$

Problem A–2–6

Prove that if the Laplace transform of $f(t)$ is $F(s)$, then, except at poles of $F(s)$,

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$
$$\mathcal{L}[t^2f(t)] = \frac{d^2}{ds^2}F(s)$$

and in general,

$$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$$

Solution

$$\mathcal{L}[tf(t)] = \int_0^\infty tf(t)e^{-st}\,dt = -\int_0^\infty f(t)\frac{d}{ds}(e^{-st})\,dt$$
$$= -\frac{d}{ds}\int_0^\infty f(t)e^{-st}\,dt = -\frac{d}{ds}F(s)$$

Similarly, by defining $tf(t) = g(t)$, the result is

$$\mathcal{L}[t^2f(t)] = \mathcal{L}[tg(t)] = -\frac{d}{ds}G(s) = -\frac{d}{ds}\left[-\frac{d}{ds}F(s)\right]$$
$$= (-1)^2\frac{d^2}{ds^2}F(s) = \frac{d^2}{ds^2}F(s)$$

Repeating the same process, we obtain

$$\mathcal{L}[t^nf(t)] = (-1)^n\frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3, \ldots$$

Problem A-2-7

Find the Laplace transform of

$$f(t) = 0 \qquad t < 0$$
$$= t^2 \sin \omega t \qquad t \ge 0$$

Solution Since

$$\mathscr{L}[\sin \omega t] = \frac{\omega}{s^2 + \omega^2}$$

referring to **Problem A-2-6**, we have

$$\mathscr{L}[f(t)] = \mathscr{L}[t^2 \sin \omega t] = \frac{d^2}{ds^2}\left[\frac{\omega}{s^2 + \omega^2}\right] = \frac{-2\omega^3 + 6\omega s^2}{(s^2 + \omega^2)^3}$$

Problem A-2-8

Prove that if the Laplace transform of $f(t)$ is $F(s)$, then

$$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = aF(as) \qquad a > 0$$

Solution If we define $t/a = \tau$ and $as = s_1$, then

$$\mathscr{L}\left[f\left(\frac{t}{a}\right)\right] = \int_0^\infty f\left(\frac{t}{a}\right)e^{-st}\,dt = \int_0^\infty f(\tau)e^{-as\tau}a\,d\tau$$
$$= a\int_0^\infty f(\tau)e^{-s_1\tau}\,d\tau = aF(s_1) = aF(as)$$

Problem A-2-9

Prove that if $f(t)$ is of exponential order and if $\int_0^\infty f(t)\,dt$ exists [which means that $\int_0^\infty f(t)\,dt$ assumes a definite value], then

$$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$$

where $F(s) = \mathscr{L}[f(t)]$.

Solution Note that

$$\int_0^\infty f(t)\,dt = \lim_{t\to\infty}\int_0^t f(t)\,dt$$

Referring to Equation (2–5), we have

$$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$$

Since $\int_0^\infty f(t)\,dt$ exists, by applying the final-value theorem to this case, we obtain

$$\lim_{t\to\infty}\int_0^t f(t)\,dt = \lim_{s\to 0} s\frac{F(s)}{s}$$

or

$$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$$

Problem A–2–10

The convolution of two time functions is defined by

$$\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau$$

A commonly used notation for the convolution is $f_1(t)*f_2(t)$, which is defined as

$$f_1(t)*f_2(t) = \int_0^t f_1(\tau)f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau)f_2(\tau)\,d\tau$$

Show that if $f_1(t)$ and $f_2(t)$ are both Laplace transformable, then

$$\mathscr{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] = F_1(s)F_2(s)$$

where $F_1(s) = \mathscr{L}[f_1(t)]$ and $F_2(s) = \mathscr{L}[f_2(t)]$.

Solution Noting that $1(t-\tau) = 0$ for $t < \tau$, we have

$$\begin{aligned}\mathscr{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] &= \mathscr{L}\left[\int_0^\infty f_1(\tau)f_2(t-\tau)1(t-\tau)\,d\tau\right]\\ &= \int_0^\infty e^{-st}\left[\int_0^\infty f_1(\tau)f_2(t-\tau)1(t-\tau)\,d\tau\right]dt\\ &= \int_0^\infty f_1(\tau)\,d\tau\int_0^\infty f_2(t-\tau)1(t-\tau)e^{-st}\,dt\end{aligned}$$

Changing the order of integration is valid here, since $f_1(t)$ and $f_2(t)$ are both Laplace transformable, giving convergent integrals. If we substitute $\lambda = t - \tau$ into this last equation, the result is

$$\begin{aligned}\mathscr{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)\,d\tau\right] &= \int_0^\infty f_1(\tau)e^{-s\tau}\,d\tau\int_0^\infty f_2(\lambda)e^{-s\lambda}\,d\lambda\\ &= F_1(s)F_2(s)\end{aligned}$$

or

$$\mathscr{L}[f_1(t)*f_2(t)] = F_1(s)F_2(s)$$

Thus, the Laplace transform of the convolution of two time functions is the product of their Laplace transforms.

Problem A–2–11

Determine the Laplace transform of $f_1(t)*f_2(t)$, where

$$\begin{aligned}f_1(t) &= f_2(t) = 0 && \text{for } t < 0\\ f_1(t) &= t && \text{for } t \geq 0\\ f_2(t) &= 1 - e^{-t} && \text{for } t \geq 0\end{aligned}$$

Solution Note that

$$\mathscr{L}[t] = F_1(s) = \frac{1}{s^2}$$

$$\mathscr{L}[1 - e^{-t}] = F_2(s) = \frac{1}{s} - \frac{1}{s+1}$$

The Laplace transform of the convolution integral is given by

$$\mathcal{L}[f_1(t)*f_2(t)] = F_1(s)F_2(s) = \frac{1}{s^2}\left(\frac{1}{s} - \frac{1}{s+1}\right)$$
$$= \frac{1}{s^3} - \frac{1}{s^2(s+1)} = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1}$$

To verify that the expression after the rightmost equal sign is indeed the Laplace transform of the convolution integral, let us first integrate the convolution integral and then take the Laplace transform of the result. We have

$$f_1(t)*f_2(t) = \int_0^t \tau[1 - e^{-(t-\tau)}]\, d\tau$$
$$= \int_0^t (t-\tau)(1 - e^{-\tau})\, d\tau$$
$$= \int_0^t (t - \tau - te^{-\tau} + \tau e^{-\tau})\, d\tau$$

Noting that

$$\int_0^t (t-\tau)\, d\tau = \frac{t^2}{2}$$
$$\int_0^t te^{-\tau}\, d\tau = -te^{-t} + t$$
$$\int_0^t \tau e^{-\tau}\, d\tau = -\tau e^{-\tau}\Big|_0^t + \int_0^t e^{-\tau}\, d\tau = -te^{-t} - e^{-t} + 1$$

we have

$$f_1(t)*f_2(t) = \frac{t^2}{2} - t + 1 - e^{-t}$$

Thus,

$$\mathcal{L}\left[\frac{t^2}{2} - t + 1 - e^{-t}\right] = \frac{1}{s^3} - \frac{1}{s^2} + \frac{1}{s} - \frac{1}{s+1}$$

Problem A–2–12

Prove that if $f(t)$ is a periodic function with period T, then

$$\mathcal{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\, dt}{1 - e^{-Ts}}$$

Solution

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}\, dt = \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} f(t)e^{-st}\, dt$$

By changing the independent variable from t to $\tau = t - nT$, we obtain

$$\mathcal{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs} \int_0^T f(\tau + nT)e^{-s\tau}\, d\tau$$

Since $f(t)$ is a periodic function with period $T, f(\tau + nT) = f(\tau)$. Hence,

$$\mathscr{L}[f(t)] = \sum_{n=0}^{\infty} e^{-nTs} \int_0^T f(\tau)e^{-s\tau}\, d\tau$$

Noting that

$$\begin{aligned}\sum_{n=0}^{\infty} e^{-nTs} &= 1 + e^{-Ts} + e^{-2Ts} + \cdots \\ &= 1 + e^{-Ts}(1 + e^{-Ts} + e^{-2Ts} + \cdots) \\ &= 1 + e^{-Ts}\left(\sum_{n=0}^{\infty} e^{-nTs}\right)\end{aligned}$$

we obtain

$$\sum_{n=0}^{\infty} e^{-nTs} = \frac{1}{1 - e^{-Ts}}$$

It follows that

$$\mathscr{L}[f(t)] = \frac{\int_0^T f(t)e^{-st}\, dt}{1 - e^{-Ts}}$$

Problem A–2–13

What is the Laplace transform of the periodic function shown in Figure 2–12?

Solution Note that

$$\begin{aligned}\int_0^T f(t)e^{-st}\, dt &= \int_0^{T/2} e^{-st}\, dt + \int_{T/2}^T (-1)e^{-st}\, dt \\ &= \frac{e^{-st}}{-s}\bigg|_0^{T/2} - \frac{e^{-st}}{-s}\bigg|_{T/2}^T \\ &= \frac{e^{-(1/2)Ts} - 1}{-s} + \frac{e^{-Ts} - e^{-(1/2)Ts}}{s} \\ &= \frac{1}{s}[e^{-Ts} - 2e^{-(1/2)Ts} + 1] \\ &= \frac{1}{s}[1 - e^{-(1/2)Ts}]^2\end{aligned}$$

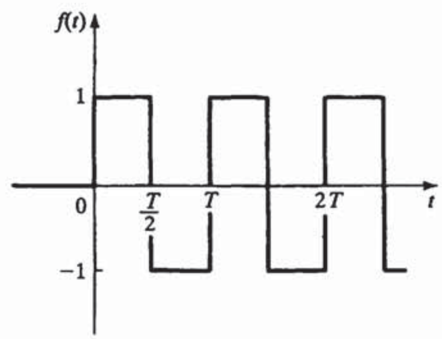

Figure 2–12 Periodic function (square wave).

Consequently,

$$F(s) = \frac{\int_0^T f(t)e^{-st}\,dt}{1 - e^{-Ts}} = \frac{(1/s)[1 - e^{-(1/2)Ts}]^2}{1 - e^{-Ts}}$$

$$= \frac{1 - e^{-(1/2)Ts}}{s[1 + e^{-(1/2)Ts}]} = \frac{1}{s}\tanh\frac{Ts}{4}$$

Problem A–2–14

Find the initial value of $df(t)/dt$, where the Laplace transform of $f(t)$ is given by

$$F(s) = \mathscr{L}[f(t)] = \frac{2s + 1}{s^2 + s + 1}$$

Solution Using the initial-value theorem, we obtain

$$\lim_{t\to 0+} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{s(2s + 1)}{s^2 + s + 1} = 2$$

Since the $\mathscr{L}_+$ transform of $df(t)/dt = g(t)$ is given by

$$\mathscr{L}_+[g(t)] = sF(s) - f(0+)$$

$$= \frac{s(2s + 1)}{s^2 + s + 1} - 2 = \frac{-s - 2}{s^2 + s + 1}$$

the initial value of $df(t)/dt$ is obtained as

$$\lim_{t\to 0+}\frac{df(t)}{dt} = g(0+) = \lim_{s\to\infty} s[sF(s) - f(0+)]$$

$$= \lim_{s\to\infty}\frac{-s^2 - 2s}{s^2 + s + 1} = -1$$

To verify this result, notice that

$$F(s) = \frac{2(s + 0.5)}{(s + 0.5)^2 + (0.866)^2} = \mathscr{L}[2e^{-0.5t}\cos 0.866t]$$

Hence,

$$f(t) = 2e^{-0.5t}\cos 0.866t$$

and

$$\dot f(t) = -e^{-0.5t}\cos 0.866t + 2e^{-0.5t}0.866\sin 0.866t$$

Thus,

$$\dot f(0) = -1 + 0 = -1$$

Problem A–2–15

Obtain the inverse Laplace transform of

$$F(s) = \frac{cs + d}{(s^2 + 2as + a^2) + b^2}$$

where a, b, c, and d are real and a is positive.

Solution Since $F(s)$ can be written as

$$F(s) = \frac{c(s+a)+d-ca}{(s+a)^2+b^2}$$
$$= \frac{c(s+a)}{(s+a)^2+b^2} + \frac{d-ca}{b}\frac{b}{(s+a)^2+b^2}$$

we obtain

$$f(t) = ce^{-at}\cos bt + \frac{d-ca}{b}e^{-at}\sin bt$$

Problem A-2-16

Find the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2+2s+2)}$$

Solution Since

$$s^2+2s+2 = (s+1+j1)(s+1-j1)$$

it follows that $F(s)$ involves a pair of complex-conjugate poles, so we expand $F(s)$ into the form

$$F(s) = \frac{1}{s(s^2+2s+2)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2+2s+2}$$

where a_1, a_2, and a_3 are determined from

$$1 = a_1(s^2+2s+2) + (a_2 s + a_3)s$$

By comparing corresponding coefficients of the s^2, s, and s^0 terms on both sides of this last equation respectively, we obtain

$$a_1 + a_2 = 0, \qquad 2a_1 + a_3 = 0, \qquad 2a_1 = 1$$

from which it follows that

$$a_1 = \frac{1}{2}, \qquad a_2 = -\frac{1}{2}, \qquad a_3 = -1$$

Therefore,

$$F(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s+2}{s^2+2s+2}$$
$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{(s+1)^2+1^2} - \frac{1}{2}\frac{s+1}{(s+1)^2+1^2}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{1}{2} - \frac{1}{2}e^{-t}\sin t - \frac{1}{2}e^{-t}\cos t \qquad t \geq 0$$

Problem A–2–17

Derive the inverse Laplace transform of

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)}$$

Solution

$$F(s) = \frac{5(s+2)}{s^2(s+1)(s+3)} = \frac{b_2}{s^2} + \frac{b_1}{s} + \frac{a_1}{s+1} + \frac{a_2}{s+3}$$

where

$$a_1 = \frac{5(s+2)}{s^2(s+3)}\bigg|_{s=-1} = \frac{5}{2}$$

$$a_2 = \frac{5(s+2)}{s^2(s+1)}\bigg|_{s=-3} = \frac{5}{18}$$

$$b_2 = \frac{5(s+2)}{(s+1)(s+3)}\bigg|_{s=0} = \frac{10}{3}$$

$$b_1 = \frac{d}{ds}\left[\frac{5(s+2)}{(s+1)(s+3)}\right]_{s=0}$$

$$= \frac{5(s+1)(s+3) - 5(s+2)(2s+4)}{(s+1)^2(s+3)^2}\bigg|_{s=0} = -\frac{25}{9}$$

Thus,

$$F(s) = \frac{10}{3}\frac{1}{s^2} - \frac{25}{9}\frac{1}{s} + \frac{5}{2}\frac{1}{s+1} + \frac{5}{18}\frac{1}{s+3}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \frac{10}{3}t - \frac{25}{9} + \frac{5}{2}e^{-t} + \frac{5}{18}e^{-3t} \qquad t \geq 0$$

Problem A–2–18

Find the inverse Laplace transform of

$$F(s) = \frac{s^4 + 2s^3 + 3s^2 + 4s + 5}{s(s+1)}$$

Solution Since the numerator polynomial is of higher degree than the denominator polynomial, by dividing the numerator by the denominator until the remainder is a fraction, we obtain

$$F(s) = s^2 + s + 2 + \frac{2s+5}{s(s+1)} = s^2 + s + 2 + \frac{a_1}{s} + \frac{a_2}{s+1}$$

where

$$a_1 = \frac{2s+5}{s+1}\bigg|_{s=0} = 5$$

$$a_2 = \frac{2s+5}{s}\bigg|_{s=-1} = -3$$

It follows that

$$F(s) = s^2 + s + 2 + \frac{5}{s} - \frac{3}{s+1}$$

The inverse Laplace transform of $F(s)$ is

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{d^2}{dt^2}\delta(t) + \frac{d}{dt}\delta(t) + 2\delta(t) + 5 - 3e^{-t} \qquad t \geq 0-$$

Problem A–2–19

Obtain the inverse Laplace transform of

$$F(s) = \frac{2s^2 + 4s + 6}{s^2(s^2 + 2s + 10)} \tag{2–10}$$

Solution Since the quadratic term in the denominator involves a pair of complex-conjugate roots, we expand $F(s)$ into the following partial-fraction form:

$$F(s) = \frac{a_1}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 10}$$

The coefficient a_1 can be obtained as

$$a_1 = \left.\frac{2s^2 + 4s + 6}{s^2 + 2s + 10}\right|_{s=0} = 0.6$$

Hence, we obtain

$$\begin{aligned} F(s) &= \frac{0.6}{s^2} + \frac{a_2}{s} + \frac{bs + c}{s^2 + 2s + 10} \\ &= \frac{(a_2 + b)s^3 + (0.6 + 2a_2 + c)s^2 + (1.2 + 10a_2)s + 6}{s^2(s^2 + 2s + 10)} \end{aligned} \tag{2–11}$$

By equating corresponding coefficients in the numerators of Equations (2–10) and (2–11), respectively, we obtain

$$\begin{aligned} a_2 + b &= 0 \\ 0.6 + 2a_2 + c &= 2 \\ 1.2 + 10a_2 &= 4 \end{aligned}$$

from which we get

$$a_2 = 0.28, \qquad b = -0.28, \qquad c = 0.84$$

Hence,

$$\begin{aligned} F(s) &= \frac{0.6}{s^2} + \frac{0.28}{s} + \frac{-0.28s + 0.84}{s^2 + 2s + 10} \\ &= \frac{0.6}{s^2} + \frac{0.28}{s} + \frac{-0.28(s+1) + (1.12/3) \times 3}{(s+1)^2 + 3^2} \end{aligned}$$

The inverse Laplace transform of $F(s)$ gives

$$f(t) = 0.6t + 0.28 - 0.28e^{-t}\cos 3t + \frac{1.12}{3}e^{-t}\sin 3t$$

Problem A–2–20

Derive the inverse Laplace transform of

$$F(s) = \frac{1}{s(s^2 + \omega^2)}$$

Solution

$$\begin{aligned} F(s) &= \frac{1}{s(s^2 + \omega^2)} = \frac{1}{\omega^2}\left(\frac{1}{s} - \frac{s}{s^2 + \omega^2}\right) \\ &= \frac{1}{\omega^2}\frac{1}{s} - \frac{1}{\omega^2}\frac{s}{s^2 + \omega^2} \end{aligned}$$

Thus, the inverse Laplace transform of $F(s)$ is obtained as

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{\omega^2}(1 - \cos \omega t) \qquad t \geq 0$$

Problem A–2–21

Obtain the solution of the differential equation

$$\dot{x} + ax = A \sin \omega t, \qquad x(0) = b$$

Solution Laplace transforming both sides of this differential equation, we have

$$[sX(s) - x(0)] + aX(s) = A\frac{\omega}{s^2 + \omega^2}$$

or

$$(s + a)X(s) = \frac{A\omega}{s^2 + \omega^2} + b$$

Solving this last equation for $X(s)$, we obtain

$$\begin{aligned} X(s) &= \frac{A\omega}{(s + a)(s^2 + \omega^2)} + \frac{b}{s + a} \\ &= \frac{A\omega}{a^2 + \omega^2}\left(\frac{1}{s + a} - \frac{s - a}{s^2 + \omega^2}\right) + \frac{b}{s + a} \\ &= \left(b + \frac{A\omega}{a^2 + \omega^2}\right)\frac{1}{s + a} + \frac{Aa}{a^2 + \omega^2}\frac{\omega}{s^2 + \omega^2} - \frac{A\omega}{a^2 + \omega^2}\frac{s}{s^2 + \omega^2} \end{aligned}$$

The inverse Laplace transform of $X(s)$ then gives

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}[X(s)] \\ &= \left(b + \frac{A\omega}{a^2 + \omega^2}\right)e^{-at} + \frac{Aa}{a^2 + \omega^2}\sin \omega t - \frac{A\omega}{a^2 + \omega^2}\cos \omega t \qquad t \geq 0 \end{aligned}$$

5

State-Space Approach to Modeling Dynamic Systems

5–1 INTRODUCTION

The modern trend in dynamic systems is toward greater complexity, due mainly to the twin requirements of complex tasks and high accuracy. Complex systems may have multiple inputs and multiple outputs. Such systems may be linear or nonlinear and may be time invariant or time varying. A very powerful approach to treating such systems is the state-space approach, based on the concept of state. This concept, by itself, is not new; it has been in existence for a long time in the field of classical dynamics and in other fields. What is new is the combination of the concept of state and the capability of high-speed solution of differential equations with the use of the digital computer.

This chapter presents an introductory account of modeling dynamic systems in state space and analyzing simple dynamic systems with MATLAB. (More on the state-space analysis of dynamic systems is given in Chapter 8.) If the dynamic system is formulated in the state space, it is very easy to simulate it on the computer and find the computer solution of the system's differential equations, because the state-space formulation is developed precisely with such computer solution in mind. Although we treat only linear, time-invariant systems in this chapter, the state-space approach can be applied to both linear and nonlinear systems and to both time-invariant and time-varying systems.

In what follows, we shall first give definitions of state, state variables, state vector, and state space. Then we shall present the outline of the chapter.

State. The *state* of a dynamic system is the smallest set of variables (called *state variables*) such that knowledge of these variables at $t = t_0$, together with knowledge of the input for $t \geq t_0$, completely determines the behavior of the system for any time $t \geq t_0$.

Thus, the state of a dynamic system at time t is uniquely determined by the state at time t_0 and the input $t \geq t_0$ and is independent of the state and input before t_0. In dealing with linear time-invariant systems, we usually choose the reference time t_0 to be zero.

State variables. The *state variables* of a dynamic system are the variables making up the smallest set of variables that determines the state of the dynamic system. If at least n variables $x_1, x_2, \ldots, x_n$ are needed to completely describe the behavior of a dynamic system (so that, once the input is given for $t \geq t_0$ and the initial state at $t = t_0$ is specified, the future state of the system is completely determined), then those n variables are a set of state variables. It is important to note that variables that do not represent physical quantities can be chosen as state variables.

State vector. If n state variables are needed to completely describe the behavior of a given system, then those state variables can be considered the n components of a vector $\mathbf{x}$ called a *state vector*. A state vector is thus a vector that uniquely determines the system state $\mathbf{x}(t)$ for any time $t \geq t_0$, once the state at $t = t_0$ is given and the input $\mathbf{u}(t)$ for $t \geq t_0$ is specified.

State space. The n-dimensional space whose coordinate axes consist of the x_1-axis, x_2-axis, $\ldots$, x_n-axis is called a *state space*. Any state can be represented by a point in the state space.

State-space equations. In state-space analysis, we are concerned with three types of variables that are involved in the modeling of dynamic systems: input variables, output variables, and state variables. As we shall see later, the state-space representation for a given system is not unique, except that the number of state variables is the same for any of the different state-space representations of the same system.

If a system is linear and time invariant and if it is described by n state variables, r input variables, and m output variables, then the state equation will have the form

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1r}u_r \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2r}u_r \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nr}u_r
\end{aligned}$$

and the output equation will have the form

$$\begin{aligned}
y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1r}u_r \\
y_2 &= c_{21}x_1 + c_{22}x_2 + \cdots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2r}u_r \\
&\vdots \\
y_m &= c_{m1}x_1 + c_{m2}x_2 + \cdots + c_{mn}x_n + d_{m1}u_1 + d_{m2}u_2 + \cdots + d_{mr}u_r
\end{aligned}$$

where the coefficients a_{ij}, b_{ij}, c_{ij}, and d_{ij} are constants, some of which may be zero. If we use vector–matrix expressions, these equations can be written as

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{5–1}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \tag{5–2}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix}, \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}, \mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1r} \\ d_{21} & d_{22} & \cdots & d_{2r} \\ \vdots & \vdots & & \vdots \\ d_{m1} & d_{m2} & \cdots & d_{mr} \end{bmatrix}$$

Matrices **A**, **B**, **C**, and **D** are called the *state matrix*, *input matrix*, *output matrix*, and *direct transmission matrix*, respectively. Vectors **x**, **u**, and **y** are the *state vector*, *input vector*, and *output vector*, respectively. (In control systems analysis and design, the input matrix **B** and input vector **u** are called the control matrix and control vector, respectively.) The elements of the state vector are the state variables. The elements of the input vector **u** are the input variables. (If the system involves only one input variable, then u is a scalar.) The elements of the output vector **y** are the output variables. (The system may involve one or more output variables.) Equation (5–1) is called the *state equation*, and Equation (5–2) is called the *output equation*. [In this book, whenever we discuss state-space equations, they are described by Equations (5–1) and (5–2).]

A block diagram representation of Equations (5–1) and (5–2) is shown in Figure 5–1. (In the figure, double-line arrows are used to indicate that the signals are vector quantities.)

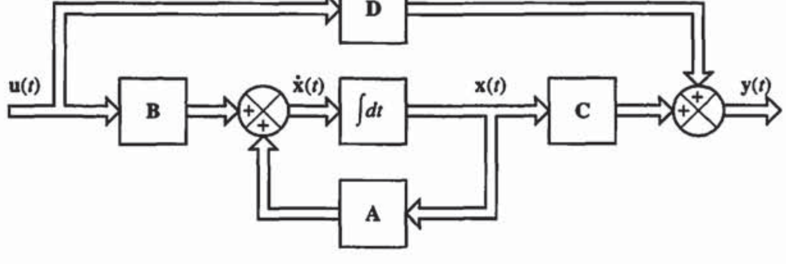

Figure 5–1 Block diagram of the linear, continuous-time system represented in state space.

Example 5–1

Consider the mechanical system shown in Figure 5–2. The displacement y of the mass is the output of the system, and the external force u is the input to the system. The displacement y is measured from the equilibrium position in the absence of the external force. Obtain a state-space representation of the system.

From the diagram, the system equation is

$$m\ddot{y} + b\dot{y} + ky = u \tag{5–3}$$

This system is of second order. (This means that the system involves two integrators.) Thus, we need two state variables to describe the system dynamics. Since $y(0)$, $\dot{y}(0)$, and $u(t) \geq 0$ completely determine the system behavior for $t \geq 0$, we choose $y(t)$ and $\dot{y}(t)$ as state variables, or define

$$x_1 = y$$
$$x_2 = \dot{y}$$

Then we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \ddot{y} = \frac{1}{m}(-ky - b\dot{y}) + \frac{1}{m}u$$

or

$$\dot{x}_1 = x_2 \tag{5–4}$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u \tag{5–5}$$

The output equation is

$$y = x_1 \tag{5–6}$$

In vector–matrix form, Equations (5–4) and (5–5) can be written as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \tag{5–7}$$

The output equation, Equation (5–6), can be written as

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5–8}$$

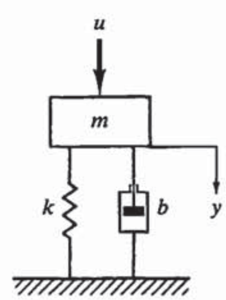

Figure 5–2 Mechanical system.

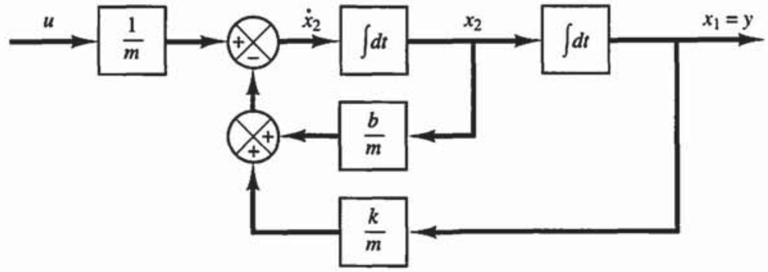

Figure 5–3 Block diagram of the mechanical system shown in Figure 5–2.

Equation (5–7) is a state equation, and Equation (5–8) is an output equation for the system. Equations (5–7) and (5–8) are in the standard form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$
$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0], \qquad D = 0$$

Note that Equation (5–3) can be modified to

$$\frac{u}{m} - \frac{k}{m}y - \frac{b}{m}\dot{y} = \ddot{y}$$

or

$$\frac{1}{m}u - \frac{k}{m}x_1 - \frac{b}{m}x_2 = \dot{x}_2$$

On the basis of this last equation, we can draw the block diagram shown in Figure 5–3. Notice that the outputs of the integrators are state variables.

In a state-space representation, a system is represented by a state equation and an output equation. In this representation, the internal structure of the system is described by a first-order vector–matrix differential equation. This fact indicates that the state-space representation is fundamentally different from the transfer-function representation, in which the dynamics of the system are described by the input and the output, but the internal structure is put in a black box.

Outline of the chapter. Section 5–1 has defined some terms that are necessary for the modeling of dynamic systems in state space and has derived a state-space model of a simple dynamic system. Section 5–2 gives a transient-response analysis of systems in state-space form with MATLAB. Section 5–3 discusses the state-space modeling of systems wherein derivative terms of the input function do not appear in the system differential equations. Numerical response analysis is done with MATLAB. Section 5–4 presents two methods for obtaining state-space models of systems in which derivative terms of the input function appear explicitly in the system differential equations. Section 5–5 treats the transformation of system models from transfer-function representation to state-space representation and vice versa. The section also examines the transformation of one state-space representation to another.

5–2 TRANSIENT-RESPONSE ANALYSIS OF SYSTEMS IN STATE-SPACE FORM WITH MATLAB

This section presents the MATLAB approach to obtaining transient-response curves of systems that are written in state-space form.

Step response. We first define the system with

```
sys = ss(A,B,C,D)
```

For a unit-step input, the MATLAB command

```
step(sys)    or    step(A,B,C,D)
```

will generate plots of unit-step responses. The time vector is automatically determined when t is not explicitly included in the step commands.

Note that when step commands have left-hand arguments, such as

```
y = step(sys,t),        [y,t,x] = step(sys,t),
[y,x,t] = step(A,B,C,D,iu),        [y,x,t] = step(A,B,C,D,iu,t)
```

no plot is shown on the screen. Hence, it is necessary to use a plot command to see the response curves. The matrices y and x contain the output and state response of the system, respectively, evaluated at the computation time points t. (Matrix y has as many columns as outputs and one row for each element in t. Matrix x has as many columns as states and one row for each element in t.)

Note also that the scalar iu is an index into the inputs of the system and specifies which input is to be used for the response; t is the user-specified time. If the system involves multiple inputs and multiple outputs, the step commands produces a series of step response plots, one for each input and output combination of

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

(For details, see **Example 5–2.**)

Transfer matrix. Next, consider a multiple-input–multiple-output system. Assume that there are r inputs $u_1, u_2, \ldots, u_r$, and m outputs $y_1, y_2, \ldots, y_m$. Define

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

The transfer matrix $\mathbf{G}(s)$ relates the output $\mathbf{Y}(s)$ to the input $\mathbf{U}(s)$, or

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s)$$

where

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} \tag{5–9}$$

[The derivation of Equation (5–9) is given in **Example 5–2**, to follow.] Since the input vector $\mathbf{u}$ is r dimensional and the output vector $\mathbf{y}$ is m dimensional, the transfer matrix $\mathbf{G}(s)$ is an $m \times r$ matrix.

Example 5–2

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Obtain the unit-step response curves.

Although it is not necessary to obtain the transfer-function expression for the system in order to obtain the unit-step response curves with MATLAB, we shall derive such an expression for reference purposes. For the system defined by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$$

the transfer matrix $\mathbf{G}(s)$ is a matrix that relates $\mathbf{Y}(s)$ and $\mathbf{U}(s)$ through the formula

$$\mathbf{Y}(s) = \mathbf{G}(s)\mathbf{U}(s) \tag{5–10}$$

Taking Laplace transforms of the state-space equations, we obtain

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \tag{5–11}$$

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) \tag{5–12}$$

In deriving the transfer matrix, we assume that $\mathbf{x}(0) = \mathbf{0}$. Then, from Equation (5–11), we get

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

Substituting this equation into Equation (5–12) yields

$$\mathbf{Y}(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]\mathbf{U}(s)$$

Upon comparing this last equation with Equation (5–10), we see that

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

The transfer matrix $\mathbf{G}(s)$ for the given system becomes

$$\begin{aligned} \mathbf{G}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -6.5 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

$$= \frac{1}{s^2 + s + 6.5}\begin{bmatrix} s & -1 \\ 6.5 & s+1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
$$= \frac{1}{s^2 + s + 6.5}\begin{bmatrix} s-1 & s \\ s+7.5 & 6.5 \end{bmatrix}$$

Hence,

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{s-1}{s^2+s+6.5} & \dfrac{s}{s^2+s+6.5} \\ \dfrac{s+7.5}{s^2+s+6.5} & \dfrac{6.5}{s^2+s+6.5} \end{bmatrix}\begin{bmatrix} U_1(s) \\ U_2(s) \end{bmatrix}$$

Since this system involves two inputs and two outputs, four transfer functions can be defined, depending on which signals are considered as input and output. Note that, when considering the signal u_1 as the input, we assume that signal u_2 is zero, and vice versa. The four transfer functions are

$$\frac{Y_1(s)}{U_1(s)} = \frac{s-1}{s^2+s+6.5}, \qquad \frac{Y_1(s)}{U_2(s)} = \frac{s}{s^2+s+6.5}$$
$$\frac{Y_2(s)}{U_1(s)} = \frac{s+7.5}{s^2+s+6.5}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{6.5}{s^2+s+6.5}$$

The four individual step-response curves can be plotted with the use of the command

```
step(A,B,C,D)
```

or

```
sys = ss(A,B,C,D);        step(sys)
```

MATLAB Program 5–1 produces four individual unit-step response curves, shown in Figure 5–4.

MATLAB Program 5–1

```
>> A = [-1  -1;6.5  0];
>> B = [1  1;1  0];
>> C = [1  0;0  1];
>> D = [0  0;0  0];
>> sys = ss(A,B,C,D);
>> step(sys)
>> grid
>> title('Unit-Step Responses')
>> xlabel('t')
>> ylabel('Outputs')
```

To plot two step-response curves for the input u_1 in one diagram and two step-response curves for the input u_2 in another diagram, we may use the commands

```
step(A,B,C,D,1)
```

and

```
step(A,B,C,D,2)
```

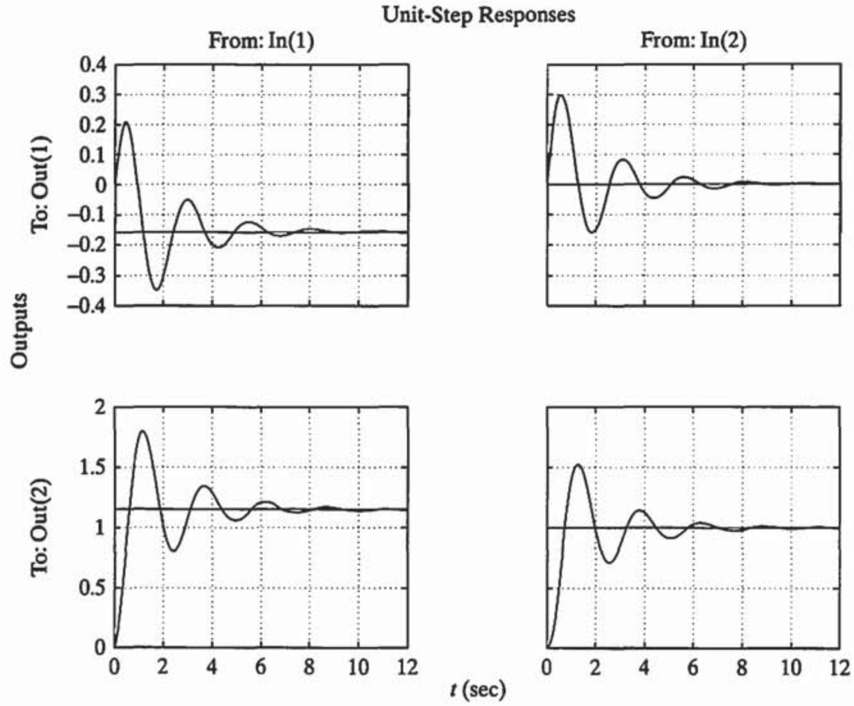

Figure 5–4 Unit-step response curves.

MATLAB Program 5–2

```
>> % ----- In this program, we first plot step-response curves
>> % when the input is u1. Then we plot response curves when
>> % the input is u2. -----
>>
>> A = [-1  -1;6.5  0];
>> B = [1  1;1  0];
>> C = [1  0;0  1];
>> D = [0  0;0  0];
>>
>> step(A,B,C,D,1)
>> grid
>> title('Step-Response Plots (u_1 = Unit-Step Input, u_2 = 0)')
>> xlabel('t'); ylabel('Outputs')
>>
>> step(A,B,C,D,2)
>> grid
>> title('Step-Response Plots (u_1 = 0, u_2 = Unit-Step Input)')
>> xlabel('t'); ylabel('Outputs')
```

respectively. MATLAB Program 5–2 does just that. Figures 5–5 and 5–6 show the two diagrams produced, each consisting of two unit-step response curves.

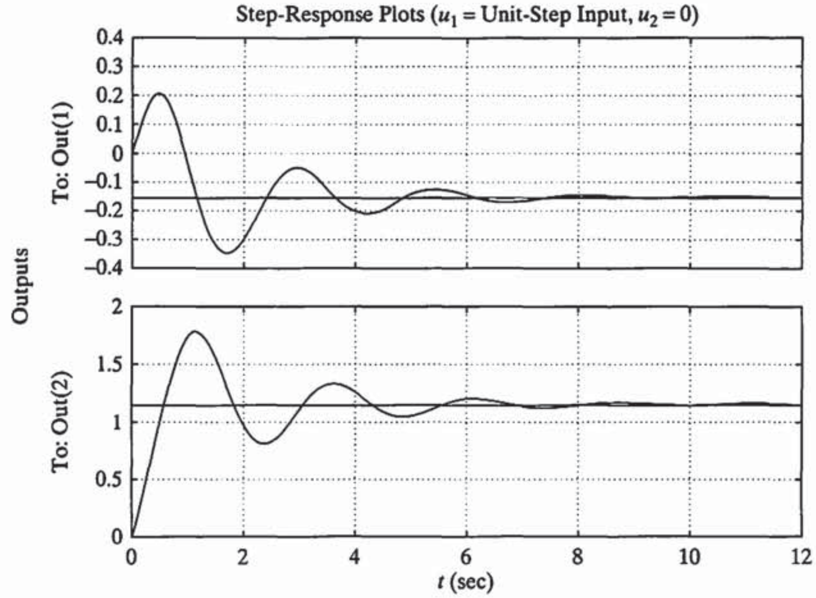

Figure 5–5 Unit-step response curves when u_1 is the input and $u_2 = 0$.

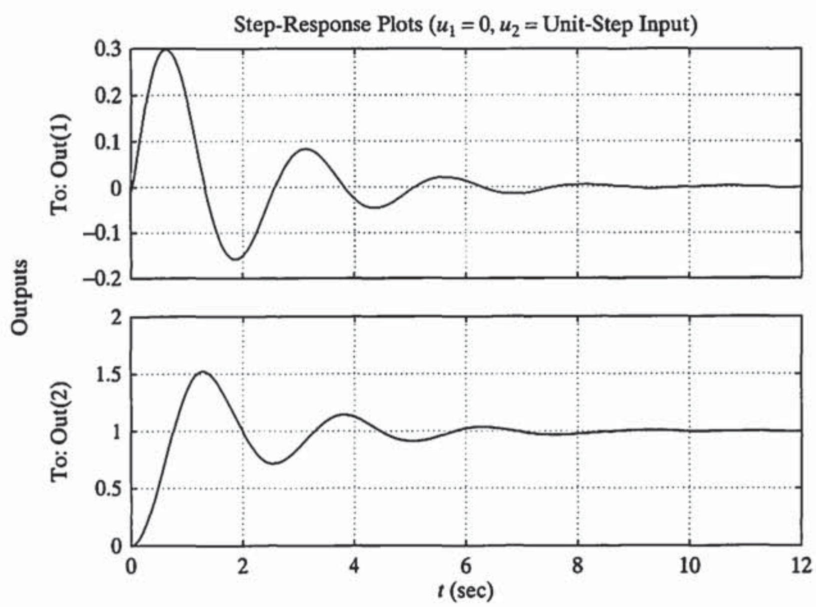

Figure 5–6 Unit-step response curves when u_2 is the input and $u_1 = 0$.

Impulse response. The unit-impulse response of a dynamic system defined in a state space may be obtained with the use of one of the following MATLAB commands:

```
sys = ss(A,B,C,D);        impulse(sys),        y = impulse(sys, t),
[y,t,x] = impulse(sys),        [y,t,x] = impulse(sys,t),
impulse(A,B,C,D),        [y,x,t] = impulse(A,B,C,D),
[y,x,t] = impulse(A,B,C,D,iu),        [y,x,t] = impulse(A,B,C,D,iu,t)
```

The command impulse(sys) or impulse(A,B,C,D) produces a series of unit-impulse response plots, one for each input–output combination of the system

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$
$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

with the time vector automatically determined. If the right-hand side of a command includes the scalar iu (an index into the inputs of the system), then that scalar specifies which input to use for the impulse response.

Note that if a command includes t, it is the user-supplied time vector, which specifies the times at which the impulse response is to be computed.

If MATLAB is invoked with the left-hand argument [y,t,x], as in the case of [y,t,x] = impulse(sys,t), the command returns the output and state responses of the system and the time vector t. No plot is drawn on the screen. The matrices y and x contain the output and state responses of the system, evaluated at the time points t. (Matrix y has as many columns as outputs and one row for each element in t. Matrix x has as many columns as state variables and one row for each element in t.)

Response to arbitrary input. The command lsim produces the response of linear time-invariant systems to arbitrary inputs. If the initial conditions of the system in state-space form are zero, then

```
lsim(sys,u,t)
```

produces the response of the system to an arbitrary input u with user-specified time t.

If the initial conditions are nonzero in a state-space model, the command

```
lsim(sys,u,t,x0)
```

where x_0 is the initial state, produces the response of the system, subject to the input u and the initial condition x_0.

The command

```
[y,t] = lsim(sys,u,t,x0)
```

returns the output response y. No plot is drawn. To plot the response curves, it is necessary to use the command plot(t,y).

Response to initial condition. To find the response to the initial condition x_0 given to a system in a state-space form, the following command may be used:

```
[y,t] = lsim(sys,u,t,x0)
```

Here, u is a vector consisting of zeros having length size(t). Alternatively, if we choose $\mathbf{B} = \mathbf{0}$ and $\mathbf{D} = \mathbf{0}$, then u can be *any* input having length size(t).

Another way to obtain the response to the initial condition given to a system in a state-space form is to use the command

initial(A,B,C,D,x_0,t)

Example 5–3 is illustrative.

Example 5–3

Consider the system shown in Figure 5–7. The system is at rest for $t < 0$. At $t = 0$, the mass is pulled downward by 0.1 m and is released with an initial velocity of 0.05 m/s. That is, $x(0) = 0.1$ m and $\dot{x}(0) = 0.05$ m/s. The displacement x is measured from the equilibrium position. There is no external input to this system.

Assuming that $m = 1$ kg, $b = 3$ N-s/m, and $k = 2$ N/m, obtain the response curves $x(t)$ versus t and $\dot{x}(t)$ versus t with MATLAB. Use the command initial.

The system equation is

$$m\ddot{x} + b\dot{x} + kx = 0$$

Substituting the given numerical values for m, b, and k yields

$$\ddot{x} + 3\dot{x} + 2x = 0$$

If we define the state variables as

$$x_1 = x$$
$$x_2 = \dot{x}$$

and the output variables as

$$y_1 = x_1$$
$$y_2 = x_2$$

then the state equation becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

The output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

Thus,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}_0 = \begin{bmatrix} 0.1 \\ 0.05 \end{bmatrix}$$

Using the command

initial(A,B,C,D,x_0,t)

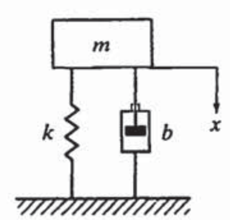

Figure 5–7 Mechanical system.

we can obtain the responses $x(t) = y_1(t)$ versus t and $\dot{x}(t) = y_2(t)$ versus t. MATLAB Program 5–3 produces the response curves, which are shown in Figure 5–8.

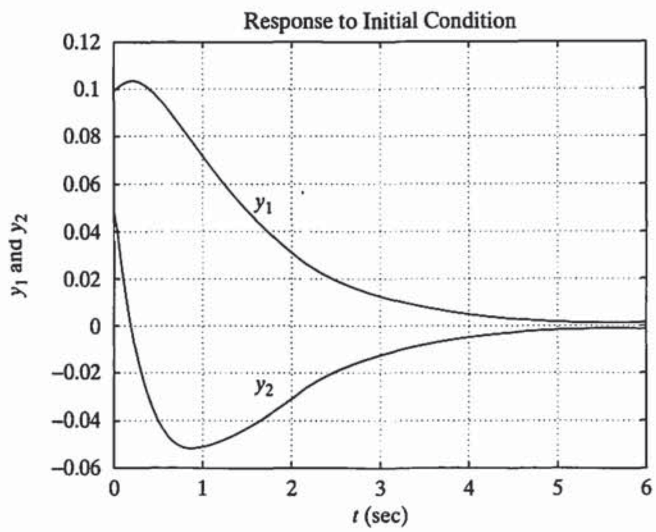

Figure 5–8 Response curves to initial condition.

MATLAB Program 5–3

```
>> t = 0:0.01:6;
>> A = [0  1;−2  −3];
>> B = [0;0];
>> C = [1  0;0  1];
>> D = [0;0];
>> [y, x, t] = initial(A,B,C,D,[0.1; 0.05],t);
>> y1 = [1     0] *y';
>> y2 = [0     1] *y';
>> plot(t,y1,t,y2)
>> grid
>> title('Response to Initial Condition')
>> xlabel('t (sec)')
>> ylabel('y_1 and y_2')
>> text(1.6, 0.05,'y_1')
>> text(1.6, −0.026,'y_2')
```

5–3 STATE-SPACE MODELING OF SYSTEMS WITH NO INPUT DERIVATIVES

In this section, we present two examples of the modeling of dynamic systems in state-space form. The systems used are limited to the case where derivatives of the input functions do not appear explicitly in the equations of motion. In each example, we first derive state-space models and then find the response curves with MATLAB, given the numerical values of all of the variables and the details of the input functions.

Example 5–4

Consider the mechanical system shown in Figure 5–9. The system is at rest for $t < 0$. At $t = 0$, a unit-impulse force, which is the input to the system, is applied to the mass. The displacement x is measured from the equilibrium position before the mass m is hit by the unit-impulse force.

Assuming that $m = 5$ kg, $b = 20$ N-s/m, and $k = 100$ N/m, obtain the response curves $x(t)$ versus t and $\dot{x}(t)$ versus t with MATLAB.

The system equation is

$$m\ddot{x} + b\dot{x} + kx = u$$

The response of such a system depends on the initial conditions and the forcing function u. The variables that provide the initial conditions qualify as state variables. Hence, we choose the variables that specify the initial conditions as state variables x_1 and x_2. Thus,

$$x_1 = x$$
$$x_2 = \dot{x}$$

The state equation then becomes

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m}(u - kx - b\dot{x}) = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

For the output variables, we choose

$$y_1 = x$$
$$y_2 = \dot{x}$$

Rewriting the state equation and output equation, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} u$$

and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

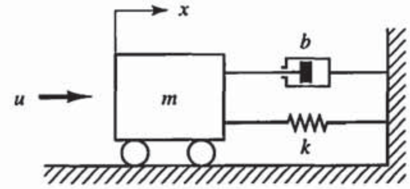

Figure 5–9 Mechanical system.

Substituting the given numerical values for m, b, and k into the state space equations yields

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -20 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

MATLAB Program 5–4 produces the impulse-response curves $x(t)$ versus t and $\dot{x}(t)$ versus t, shown in Figure 5–10.

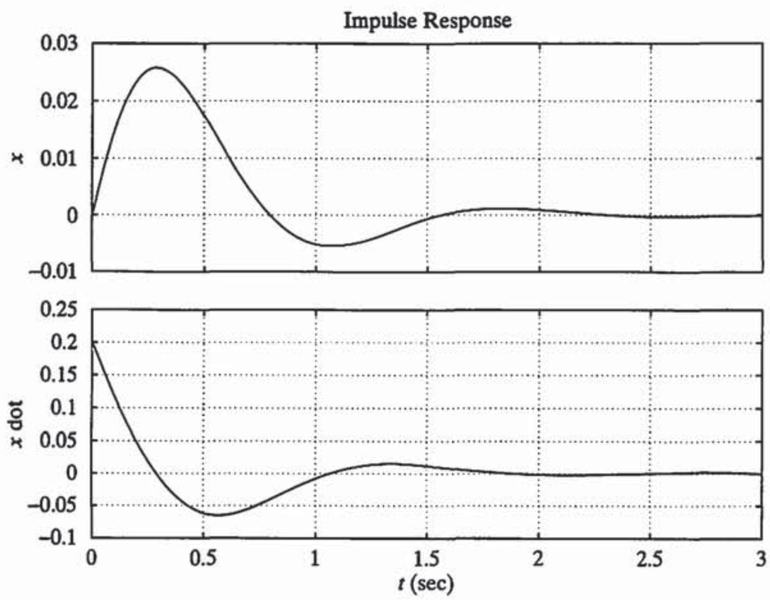

Figure 5–10 Impulse-response curves.

MATLAB Program 5–4

```
>> t = 0:0.01:3;
>> A = [0  1; -20  -4];
>> B = [0;0.2];
>> C = [1  0;0  1];
>> D = [0;0];
>> sys = ss(A,B,C,D);
>> [y, t] = impulse(sys,t);
>> y1 = [1  0] *y';
>> y2 = [0  1] *y';
>> subplot(211); plot(t,y1); grid
>> title('Impulse Response')
>> ylabel('x')
>> subplot(212); plot(t,y2); grid
>> xlabel('t (sec)'); ylabel('x dot')
```

Example 5–5

Consider the mechanical system shown in Figure 5–11. The system is at rest for $t < 0$. At $t = 0$, a step force f of α newtons is applied to mass m_2. [The force $f = \alpha u$, where u is a step force of 1 newton.] The displacements z_1 and z_2 are measured from the respective equilibrium positions of the carts before f is applied. Derive a state-space representation of the system. Assuming that $m_1 = 10$ kg, $m_2 = 20$ kg, $b = 20$ N-s/m, $k_1 = 30$ N/m, $k_2 = 60$ N/m, and $\alpha = 10$, obtain the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, and $z_1(t) - z_2(t)$ versus t with MATLAB. Also, obtain $z_1(\infty)$ and $z_2(\infty)$.

The equations of motion for this system are

$$m_1\ddot{z}_1 = -k_1 z_1 - k_2(z_1 - z_2) - b(\dot{z}_1 - \dot{z}_2) \tag{5–13}$$

$$m_2\ddot{z}_2 = -k_2(z_2 - z_1) - b(\dot{z}_2 - \dot{z}_1) + \alpha u \tag{5–14}$$

In the absence of a forcing function, the initial conditions of any system determine the response of the system. The initial conditions for this system are $z_1(0)$, $\dot{z}_1(0)$, $z_2(0)$, and $\dot{z}_2(0)$. Hence, we choose z_1, $\dot{z}_1$, z_2, and $\dot{z}_2$ as state variables for the system and thus define

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= \dot{z}_1 \\ x_3 &= z_2 \\ x_4 &= \dot{z}_2 \end{aligned}$$

Then Equation (5–13) can be rewritten as

$$\dot{x}_2 = -\frac{k_1 + k_2}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k_2}{m_1}x_3 + \frac{b}{m_1}x_4$$

and Equation (5–14) can be rewritten as

$$\dot{x}_4 = \frac{k_2}{m_2}x_1 + \frac{b}{m_2}x_2 - \frac{k_2}{m_2}x_3 - \frac{b}{m_2}x_4 + \frac{1}{m_2}\alpha u$$

The state equation now becomes

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k_1 + k_2}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k_2}{m_1}x_3 + \frac{b}{m_1}x_4 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k_2}{m_2}x_1 + \frac{b}{m_2}x_2 - \frac{k_2}{m_2}x_3 - \frac{b}{m_2}x_4 + \frac{1}{m_2}\alpha u \end{aligned}$$

Note that z_1 and z_2 are the outputs of the system; hence, the output equations are

$$\begin{aligned} y_1 &= z_1 \\ y_2 &= z_2 \end{aligned}$$

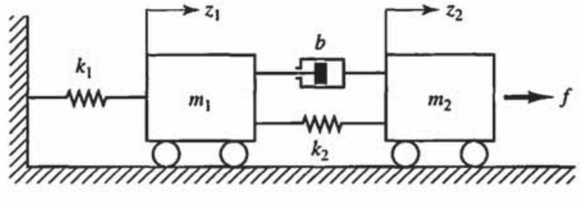

Figure 5–11 Mechanical system.

In terms of vector-matrix equations, we have

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1 + k_2}{m_1} & -\frac{b}{m_1} & \frac{k_2}{m_1} & \frac{b}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & \frac{b}{m_2} & -\frac{k_2}{m_2} & -\frac{b}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\alpha}{m_2} \end{bmatrix} u \tag{5–15}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u \tag{5–16}$$

Equations (5–15) and (5–16) represent the system in state-space form.

Next, we substitute the given numerical values for m_1, m_2, b, k_1, and k_2 into Equation (5–15). The result is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9 & -2 & 6 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u \tag{5–17}$$

From Equations (5–17) and (5–16), we have

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -9 & -2 & 6 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & -3 & -1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

MATLAB Program 5–5 produces the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, and $z_1(t) - z_2(t)$ versus t. The curves are shown in Figure 5–12.

MATLAB Program 5–5

```
>> t = 0:0.1:200;
>> A = [0  1  0  0;-9  -2  6  2;0  0  0  1;3  1  -3  -1];
>> B = [0;0;0;0.5];
>> C = [1  0  0  0;0  0  1  0];
>> D = [0;0];
>> sys = ss(A,B,C,D);
>> [y,t] = step(sys,t);
>> y1 = [1 0]*y';
>> y2 = [0 1]*y';
>> z1 = y1; subplot(311); plot(t,z1); grid
>> title('Responses z_1 Versus t, z_2 Versus t, and z_1 - z_2 Versus t')
>> ylabel('Output z_1')
>> z2 = y2; subplot(312); plot(t,z2); grid
>> ylabel('Output z_2')
>> subplot(313); plot(t,z1-z2); grid
>> xlabel('t (sec)'); ylabel('z_1 - z_2')
```

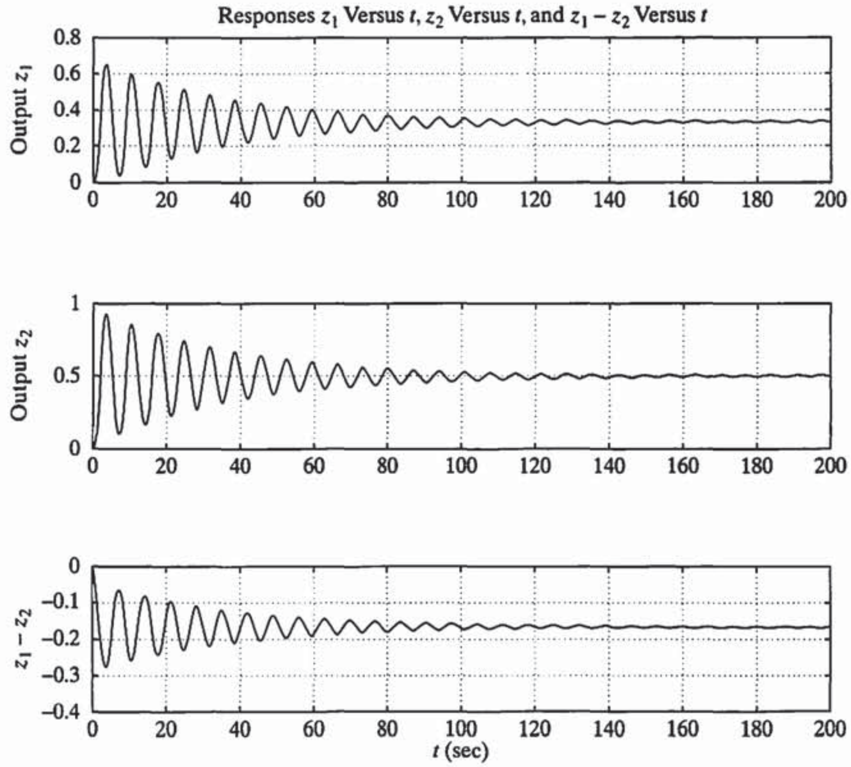

Figure 5–12 Step-response curves.

To obtain $z_1(\infty)$ and $z_2(\infty)$, we set all derivatives of z_1 and z_2 in Equations (5–13) and (5–14) equal to zero, because all derivative terms must approach zero at steady state in the system. Then, from Equation (5–14), we get

$$k_2[z_2(\infty) - z_1(\infty)] = \alpha u$$

from which it follows that

$$z_2(\infty) - z_1(\infty) = \frac{\alpha u}{k_2} = \frac{10}{60} = \frac{1}{6}$$

From Equation (5–13), we have

$$k_1 z_1(\infty) = k_2[z_2(\infty) - z_1(\infty)]$$

Hence,

$$z_1(\infty) = \frac{k_2}{k_1}[z_2(\infty) - z_1(\infty)] = \frac{60}{30}\frac{1}{6} = \frac{1}{3}$$

and

$$z_2(\infty) = \frac{1}{6} + z_1(\infty) = \frac{1}{2}$$

Thus,

$$z_1(\infty) = \frac{1}{3}\text{m}, \qquad z_2(\infty) = \frac{1}{2}\text{m}$$

The final values of $z_1(t)$ and $z_2(t)$ obtained with MATLAB (see the response curves in Figure 5–12) agree, of course, with the result obtained here.

5–4 STATE-SPACE MODELING OF SYSTEMS WITH INPUT DERIVATIVES

In this section, we take up the case where the equations of motion of a system involve one or more derivatives of the input function. In such a case, the variables that specify the initial conditions do not qualify as state variables. The main problem in defining the state variables is that they must be chosen such that they will eliminate the derivatives of the input function u in the state equation.

For example, consider the mechanical system shown in Figure 5–13. The displacements y and u are measured from their respective equilibrium positions. The equation of motion for this system is

$$m\ddot{y} = -ky - b(\dot{y} - \dot{u})$$

or

$$\ddot{y} = -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u}$$

If we choose the state variables

$$x_1 = y$$
$$x_2 = \dot{y}$$

then we get

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{b}{m}\dot{u} \tag{5–18}$$

The right-hand side of Equation (5–18) involves the derivative term $\dot{u}$. Note that, in formulating state-space representations of dynamic systems, we constrain the input function to be any function of time of order up to the impulse function, but not any higher order impulse functions, such as $d\delta(t)/dt$, $d^2\delta(t)/dt^2$, etc.

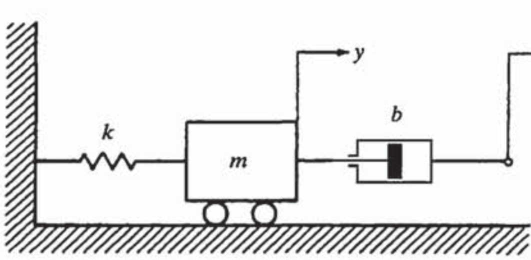

Figure 5–13 Mechanical system.

To explain why the right-hand side of the state equation should not involve the derivative of the input function u, suppose that u is the unit-impulse function $\delta(t)$. Then the integral of Equation (5–18) becomes

$$x_2 = -\frac{k}{m}\int y\,dt - \frac{b}{m}y + \frac{k}{m}\delta(t)$$

Notice that x_2 includes the term $(k/m)\,\delta(t)$. This means that $x_2(0) = \infty$, which is not acceptable as a state variable. We should choose the state variables such that the state equation will not include the derivative of u.

Suppose that we try to eliminate the term involving $\dot{u}$ from Equation (5–18). One possible way to accomplish this is to define

$$x_1 = y$$
$$x_2 = \dot{y} - \frac{b}{m}u$$

Then

$$\begin{aligned}\dot{x}_2 &= \ddot{y} - \frac{b}{m}\dot{u}\\ &= -\frac{k}{m}y - \frac{b}{m}\dot{y} + \frac{b}{m}\dot{u} - \frac{b}{m}\dot{u}\\ &= -\frac{k}{m}x_1 - \frac{b}{m}\left(x_2 + \frac{b}{m}u\right)\\ &= -\frac{k}{m}x_1 - \frac{b}{m}x_2 - \left(\frac{b}{m}\right)^2 u\end{aligned}$$

Thus, we have eliminated the term that involves $\dot{u}$. The acceptable state equation can now be given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\dfrac{k}{m} & -\dfrac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\dfrac{b}{m}\\ -\left(\dfrac{b}{m}\right)^2\end{bmatrix}u$$

If equations of motion involve u, $\dot{u}$, $\ddot{u}$, etc., the choice of state variables becomes more complicated. Fortunately, there are systematic methods for choosing state variables for a general case of equations of motion that involve derivatives of the input function u. In what follows we shall present two systematic methods for eliminating derivatives of the input function from the state equations. Note that MATLAB can also be used to obtain state-space representations of systems involving derivatives of the input function u. (See Section 5–5.)

State-space representation of dynamic systems in which derivatives of the input function appear in the system differential equations. We consider the case where the input function u is a scalar. (That is, only one input function u is involved in the system.)

The differential equation of a system that involves derivatives of the input function has the general form

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u \tag{5–19}$$

To apply the methods presented in this section, it is necessary that the system be written as a differential equation in the form of Equation (5–19) or its equivalent transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

We examine two methods when $n = 2$; for an arbitrary $n = 1, 2, 3, \ldots$, see **Problems A–5–12 and A–5–13.**

Method 1. Consider the second-order system

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u \tag{5–20}$$

As a set of state variables, suppose that we choose

$$x_1 = y - \beta_0 u \tag{5–21}$$

$$x_2 = \dot{x}_1 - \beta_1 u \tag{5–22}$$

where

$$\beta_0 = b_0 \tag{5–23}$$

$$\beta_1 = b_1 - a_1\beta_0 \tag{5–24}$$

Then, from Equation (5–21), we have

$$y = x_1 + \beta_0 u \tag{5–25}$$

Substituting this last equation into Equation (5–20), we obtain

$$\ddot{x}_1 + \beta_0\ddot{u} + a_1(\dot{x}_1 + \beta_0\dot{u}) + a_2(x_1 + \beta_0 u) = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

Noting that $\beta_0 = b_0$ and $\beta_1 = b_1 - a_1\beta_0$, we can simplify the preceding equation to

$$\ddot{x}_1 + a_1\dot{x}_1 + a_2 x_1 = \beta_1\dot{u} + (b_2 - a_2\beta_0)u \tag{5–26}$$

From Equation (5–22), we have

$$\dot{x}_1 = x_2 + \beta_1 u \tag{5–27}$$

Substituting Equation (5–27) into Equation (5–26), we obtain

$$\dot{x}_2 + \beta_1\dot{u} + a_1(x_2 + \beta_1 u) + a_2 x_1 = \beta_1\dot{u} + (b_2 - a_2\beta_0)u$$

which can be simplified to

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + \beta_2 u \tag{5–28}$$

where

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 \tag{5–29}$$

From Equations (5–27) and (5–28), we obtain the state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} u \tag{5–30}$$

From Equation (5–25), we get the output equation:

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \beta_0 u \tag{5–31}$$

Equations (5–30) and (5–31) represent the system in a state space.

Note that if $\beta_0 = b_0 = 0$, then the state variable x_1 is the output signal y, which can be measured, and, in this case, the state variable x_2 is the output velocity $\dot{y}$ minus $b_1 u$.

Note that, for the case of the nth-order differential-equation system

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

the state equation and output equation can be given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix} u$$

and

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u$$

where $\beta_0, \beta_1, \beta_2, \ldots, \beta_n$ are determined from

$$\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 \\
&\ \vdots \\
\beta_n &= b_n - a_1\beta_{n-1} - \cdots - a_{n-1}\beta_1 - a_n\beta_0
\end{aligned}$$

Method 2. Consider the second-order system

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

or its equivalent transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^2 + b_1 s + b_2}{s^2 + a_1 s + a_2} \tag{5–32}$$

Equation (5–32) can be split into two equations as follows:

$$\frac{Z(s)}{U(s)} = \frac{1}{s^2 + a_1 s + a_2}, \qquad \frac{Y(s)}{Z(s)} = b_0 s^2 + b_1 s + b_2$$

We then have

$$\ddot{z} + a_1\dot{z} + a_2 z = u \tag{5–33}$$

$$b_0\ddot{z} + b_1\dot{z} + b_2 z = y \tag{5–34}$$

If we define

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{z} \end{aligned} \tag{5–35}$$

then Equation (5–33) can be written as

$$\dot{x}_2 = -a_2 x_1 - a_1 x_2 + u \tag{5–36}$$

and Equation (5–34) can be written as

$$b_0\dot{x}_2 + b_1 x_2 + b_2 x_1 = y$$

Substituting Equation (5–36) into this last equation, we obtain

$$b_0(-a_2 x_1 - a_1 x_2 + u) + b_1 x_2 + b_2 x_1 = y$$

which can be rewritten as

$$y = (b_2 - a_2 b_0)x_1 + (b_1 - a_1 b_0)x_2 + b_0 u \tag{5–37}$$

From Equations (5–35) and (5–36), we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_2 x_1 - a_1 x_2 + u \end{aligned}$$

These two equations can be combined into the vector–matrix differential equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{5–38}$$

Equation (5–37) can be rewritten as

$$y = [b_2 - a_2 b_0 \;\; \vdots \;\; b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u \tag{5–39}$$

Equations (5–38) and (5–39) are the state equation and output equation, respectively. Note that the state variables x_1 and x_2 in this case may not correspond to any physical signals that can be measured.

If the system equation is given by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

or its equivalent transfer function

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1}s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

then the state equation and the output equation obtained with the use of Method 2 are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} u \quad (5\text{–}40)$$

and

$$y = [b_n - a_n b_0 \;\vdots\; b_{n-1} - a_{n-1} b_0 \;\vdots\; \cdots \;\vdots\; b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + b_0 u \quad (5\text{–}41)$$

Examples 5–6 and **5–7** illustrate the use of the preceding two analytical methods for obtaining state-space representations of a differential-equation system involving derivatives of the input signal.

Example 5–6

Consider the spring-mass-dashpot system mounted on a cart as shown in Figure 5–14. Assume that the cart is standing still for $t < 0$. In this system, $u(t)$ is the displacement of the cart and is the input to the system. At $t = 0$, the cart is moved at a constant speed, or $\dot{u}$ = constant. The displacement y of the mass is the output. (y is measured

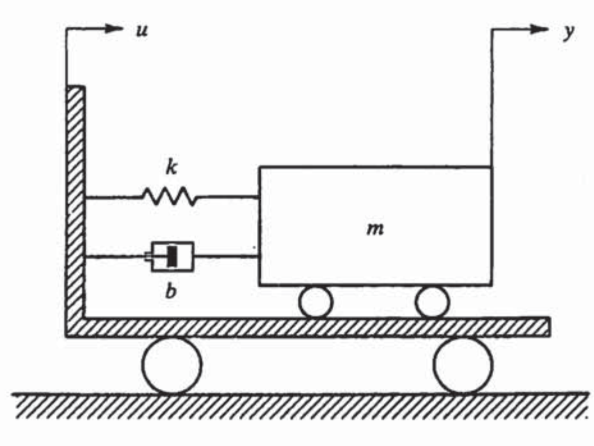

Figure 5–14 Spring-mass-dashpot system mounted on a cart.

from the rest position and is relative to the ground.) In this system, m denotes the mass of the small cart on the large cart (assume that the large cart is massless), b denotes the viscous-friction coefficient, and k is the spring constant. We assume that the entire system is a linear system.

Obtain a state-space representations of the system based on methods 1 and 2 just presented. Assuming that $m = 10$ kg, $b = 20$ N-s/m, $k = 100$ N/m, and the input is a ramp function such that $\dot{u} = 1$ m/s, obtain the response curve $y(t)$ versus t with MATLAB.

First, we shall obtain the system equation. Applying Newton's second law, we obtain

$$m\frac{d^2y}{dt^2} = -b\left(\frac{dy}{dt} - \frac{du}{dt}\right) - k(y - u)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = b\frac{du}{dt} + ku \tag{5–42}$$

Equation (5–42) is the differential equation (mathematical model) of the system. The transfer function is

$$\frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Method 1. We shall obtain a state-space model of this system based on Method 1. We first compare the differential equation of the system,

$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k}{m}y = \frac{b}{m}\dot{u} + \frac{k}{m}u$$

with the standard form

$$\ddot{y} + a_1\dot{y} + a_2y = b_0\ddot{u} + b_1\dot{u} + b_2u$$

and identify

$$a_1 = \frac{b}{m}, \qquad a_2 = \frac{k}{m}, \qquad b_0 = 0, \qquad b_1 = \frac{b}{m}, \qquad b_2 = \frac{k}{m}$$

From Equations (5–23), (5–24), and (5–29), we have

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1\beta_0 = \frac{b}{m}$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k}{m} - \left(\frac{b}{m}\right)^2$$

From Equations (5–21) and (5–22), we define

$$x_1 = y - \beta_0 u = y$$
$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 - \frac{b}{m}u \tag{5–43}$$

From Equations (5–43) and (5–28), we obtain

$$\dot{x}_1 = x_2 + \beta_1 u = x_2 + \frac{b}{m}u \tag{5–44}$$

$$\dot{x}_2 = -a_2x_1 - a_1x_2 + \beta_2 u = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \left[\frac{k}{m} - \left(\frac{b}{m}\right)^2\right]u \tag{5–45}$$

and the output equation is

$$y = x_1 \tag{5–46}$$

Combining Equations (5–44) and (5–45) yields the state equation, and from Equation (5–46), we get the output equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \dfrac{b}{m} \\ \dfrac{k}{m} - \left(\dfrac{b}{m}\right)^2 \end{bmatrix} u$$

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These two equations give a state-space representation of the system.

Next, we shall obtain the response curve $y(t)$ versus t for the unit-ramp input $\dot{u} = 1$ m/s. Substituting the given numerical values for m, b, and k into the state equation, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \end{bmatrix} u$$

and the output equation is

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

MATLAB Program 5–6 produces the response $y(t)$ of the system to the ramp input $\dot{u} = 1$ m/s. The response curve $y(t)$ versus t and the unit-ramp input are shown in Figure 5–15.

MATLAB Program 5–6

```
>> % ----- The response y(t) is obtained by use of the
>> % state-space equation obtained by Method 1. -----
>>
>> t = 0:0.01:4;
>> A = [0  1;-10  -2];
>> B = [2;6];
>> C = [1  0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u = t;
>> lsim(sys,u,t)
>> grid
>> title('Unit-Ramp Response (Method 1)')
>> xlabel('t')
>> ylabel('Output y and Unit-Ramp Input u')
>> text(0.85, 0.25,'y')
>> text(0.15,0.8,'u')
```

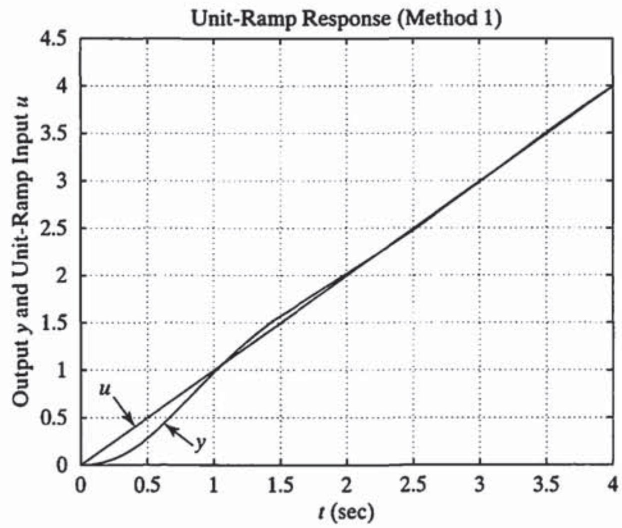

Figure 5–15 Unit-ramp response obtained with the use of Method 1.

Method 2. Since

$$b_0 = 0$$
$$b_2 - a_2 b_0 = \frac{k}{m} - \frac{k}{m} \times 0 = \frac{k}{m}$$
$$b_1 - a_1 b_0 = \frac{b}{m} - \frac{b}{m} \times 0 = \frac{b}{m}$$

from Equations (5–38) and (5–39), we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} \dfrac{k}{m} & \dfrac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The last two equations give another state-space representation of the same system.

Substituting the given numerical values for m, b, and k into the state equation, we get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -10 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

and the output equation is

$$y = [10 \quad 2] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

MATLAB Program 5–7 produces the response $y(t)$ to the unit-ramp input $\dot{u} = 1$ m/s. The resulting response curve $y(t)$ versus t and the unit-ramp input are shown in Figure 5–16. Notice that the response curve here is identical to that shown in Figure 5–15.

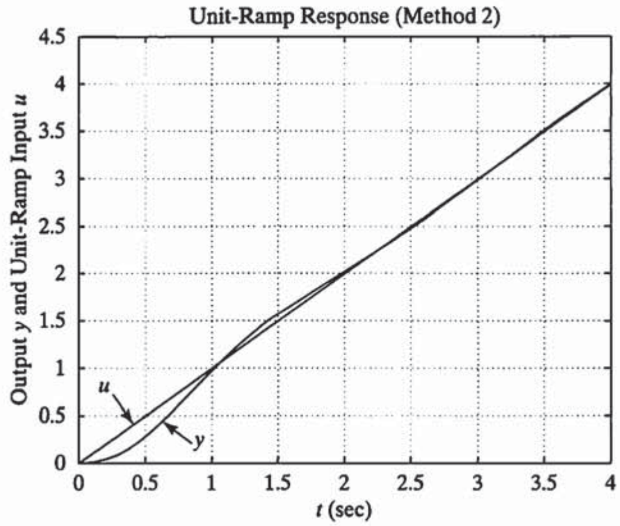

Figure 5–16 Unit-ramp response obtained with the use of Method 2.

MATLAB Program 5–7

```
>> % ----- The response y(t) is obtained by use of the
>> % state-space equation obtained by Method 2. -----
>>
>> t = 0:0.01:4;
>> A = [0  1;-10  -2];
>> B = [0;1];
>> C = [10  2];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u = t;
>> lsim(sys,u,t)
>> grid
>> title('Unit-Ramp Response (Method 2)')
>> xlabel('t')
>> ylabel('Output y and Unit-Ramp Input u')
>> text(0.85,0.25,'y')
>> text(0.15,0.8,'u')
```

Example 5–7

Consider the front suspension system of a motorcycle. A simplified version is shown in Figure 5–17(a). Point P is the contact point with the ground. The vertical displacement u of point P is the input to the system. The displacements x and y are measured from their respective equilibrium positions before the input u is given to the system. Assume that

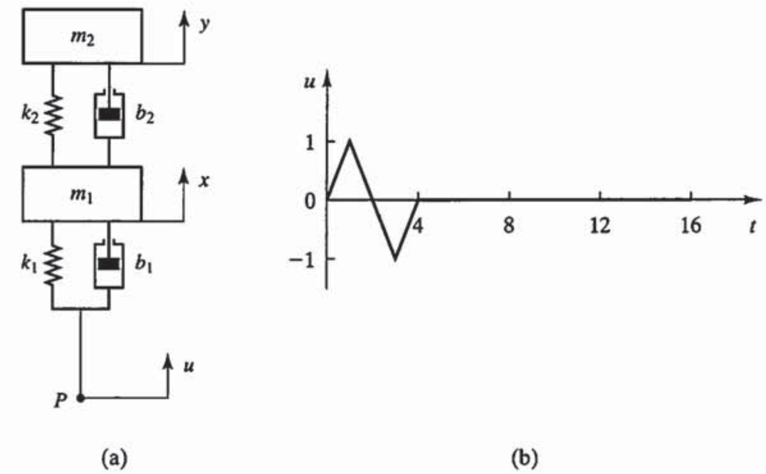

Figure 5–17 (a) Mechanical system; (b) triangular bump input u.

m_1, b_1, and k_1 represent the front tire and shock absorber assembly and m_2, b_2, and k_2 represent half of the body of the vehicle. Assume also that the system is at rest for $t < 0$. At $t = 0$, P is given a triangular bump input as shown in Figure 5–17(b). Point P moves only in the vertical direction. Assume that $m_1 = 10$ kg, $m_2 = 100$ kg, $b_1 = 50$ N-s/m, $b_2 = 100$ N-s/m, $k_1 = 50$ N/m, and $k_2 = 200$ N/m. (These numerical values are chosen to simplify the computations involved.) Obtain a state-space representation of the system. Plot the response curve $y(t)$ versus t with MATLAB.

Method 1. Applying Newton's second law to the system, we obtain

$$m_1\ddot{x} = -k_1(x - u) - b_1(\dot{x} - \dot{u})$$
$$m_2\ddot{y} = -k_2(y - x) - b_2(\dot{y} - \dot{x})$$

which can be rewritten as

$$m_1\ddot{x} + b_1\dot{x} + k_1x = b_1\dot{u} + k_1u$$
$$m_2\ddot{y} + b_2\dot{y} + k_2y = b_2\dot{x} + k_2x$$

If we substitute the given numerical values for m_1, m_2, b_1, b_2, k_1, and k_2, the equations of motion become

$$10\ddot{x} + 50\dot{x} + 50x = 50\dot{u} + 50u$$
$$100\ddot{y} + 100\dot{y} + 200y = 100\dot{x} + 200x$$

which can be simplified to

$$\ddot{x} + 5\dot{x} + 5x = 5\dot{u} + 5u \tag{5–47}$$
$$\ddot{y} + \dot{y} + 2y = \dot{x} + 2x \tag{5–48}$$

Laplace transforming Equations (5–47) and (5–48), assuming the zero initial conditions, we obtain

$$(s^2 + 5s + 5)X(s) = (5s + 5)U(s)$$
$$(s^2 + s + 2)Y(s) = (s + 2)X(s)$$

Eliminating $X(s)$ from these two equations, we get

$$(s^2 + 5s + 5)(s^2 + s + 2)Y(s) = 5(s + 1)(s + 2)U(s)$$

or

$$(s^4 + 6s^3 + 12s^2 + 15s + 10)Y(s) = (5s^2 + 15s + 10)U(s) \tag{5–49}$$

Equation (5–49) corresponds to the differential equation

$$\ddot{\ddot{y}} + 6\dddot{y} + 12\ddot{y} + 15\dot{y} + 10y = 5\ddot{u} + 15\dot{u} + 10u$$

Comparing this last equation with the standard fourth-order differential equation

$$\ddot{\ddot{y}} + a_1\dddot{y} + a_2\ddot{y} + a_3\dot{y} + a_4 y = b_0\ddot{\ddot{u}} + b_1\dddot{u} + b_2\ddot{u} + b_3\dot{u} + b_4 u$$

we find that

$$\begin{aligned} &a_1 = 6, \quad a_2 = 12, \quad a_3 = 15, \quad a_4 = 10 \\ &b_0 = 0, \quad b_1 = 0, \quad b_2 = 5, \quad b_3 = 15, \quad b_4 = 10 \end{aligned}$$

Next, we define the state variables as follows:

$$\begin{aligned} x_1 &= y - \beta_0 u \\ x_2 &= \dot{x}_1 - \beta_1 u \\ x_3 &= \dot{x}_2 - \beta_2 u \\ x_4 &= \dot{x}_3 - \beta_3 u \end{aligned}$$

where

$$\begin{aligned} \beta_0 &= b_0 = 0 \\ \beta_1 &= b_1 - a_1\beta_0 = 0 \\ \beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 = 5 \\ \beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = 15 - 6 \times 5 = -15 \end{aligned}$$

Hence,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + 5u \\ \dot{x}_3 &= x_4 - 15u \\ \dot{x}_4 &= -a_4x_1 - a_3x_2 - a_2x_3 - a_1x_4 + \beta_4 u \\ &= -10x_1 - 15x_2 - 12x_3 - 6x_4 + \beta_4 u \end{aligned}$$

where

$$\begin{aligned} \beta_4 &= b_4 - a_1\beta_3 - a_2\beta_2 - a_3\beta_1 - a_4\beta_0 \\ &= 10 + 6 \times 15 - 12 \times 5 - 15 \times 0 - 10 \times 0 = 40 \end{aligned}$$

Thus,

$$\dot{x}_4 = -10x_1 - 15x_2 - 12x_3 - 6x_4 + 40u$$

and the state equation and output equation become

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -15 & -12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ -15 \\ 40 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0u$$

MATLAB Program 5–8 produces the response $y(t)$ to the triangular bump input shown in Figure 5–17(b). The resulting response curve $y(t)$ versus t, as well as the input $u(t)$ versus t, is shown in Figure 5–18.

MATLAB Program 5–8

```
>> t = 0:0.01:16;
>> A = [0  1  0  0;0  0  1  0;0  0  0  1;-10  -15  -12  -6];
>> B = [0;5;-15;40];
>> C = [1  0  0  0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u1 = [0:0.01:1];
>> u2 = [0.99:-0.01:-1];
>> u3 = [-0.99:0.01:0];
>> u4 = 0*[4.01:0.01:16];
>> u = [u1  u2  u3  u4];
>> y = lsim(sys,u,t);
>> plot(t,y,t,u)
>> v = [0  16  -1.5  1.5]; axis(v)
>> grid
>> title('Response to Triangular Bump (Method 1)')
>> xlabel('t (sec)')
>> ylabel('Triangular Bump and Response')
```

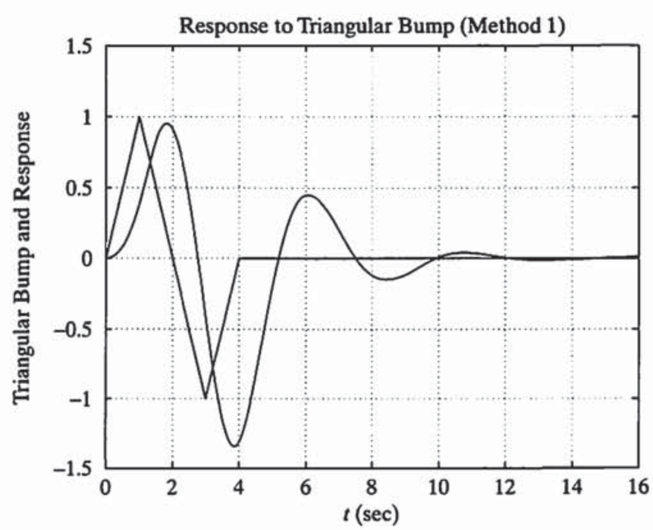

Figure 5–18 Response curve $y(t)$ and triangular bump input $u(t)$.

Method 2. From Equation (5–49), the transfer function of the system is given by

$$\frac{Y(s)}{U(s)} = \frac{5s^2 + 15s + 10}{s^4 + 6s^3 + 12s^2 + 15s + 10}$$

Figure 5–19 shows a block diagram in which the transfer function is split into two parts. If we define the output of the first block as $Z(s)$, then

$$\begin{aligned}\frac{Z(s)}{U(s)} &= \frac{1}{s^4 + 6s^3 + 12s^2 + 15s + 10} \\ &= \frac{1}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4}\end{aligned}$$

and

$$\frac{Y(s)}{Z(s)} = 5s^2 + 15s + 10 = b_0 s^4 + b_1 s^3 + b_2 s^2 + b_3 s + b_4$$

from which we get

$$\begin{matrix} a_1 = 6, & a_2 = 12, & a_3 = 15, & a_4 = 10, & \\ b_0 = 0, & b_1 = 0, & b_2 = 5, & b_3 = 15, & b_4 = 10 \end{matrix}$$

Next, we define the state variables as follows:

$$\begin{aligned} x_1 &= z \\ x_2 &= \dot{x}_1 \\ x_3 &= \dot{x}_2 \\ x_4 &= \dot{x}_3 \end{aligned}$$

From Equation (5–40), noting that $a_1 = 6, a_2 = 12, a_3 = 15$, and $a_4 = 10$, we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -15 & -12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$

Similarly, from the output equation given by Equation (5–41), we have

$$y = [b_4 - a_4 b_0 \,\vdots\, b_3 - a_3 b_0 \,\vdots\, b_2 - a_2 b_0 \,\vdots\, b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + b_0 u$$

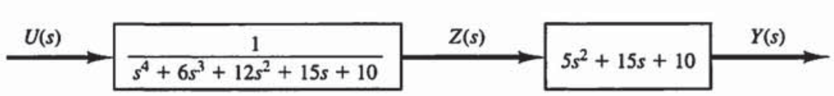

Figure 5–19 Block diagram of $Y(s)/U(s)$.

or

$$y = [10 \quad 15 \quad 5 \quad 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0u$$

MATLAB Program 5–9 produces the response $y(t)$ to the triangular bump input. The response curve is shown in Figure 5–20. (This response curve is identical to that shown in Figure 5–19.)

MATLAB Program 5–9

```
>> t = 0:0.01:16;
>> A = [0  1  0  0;0  0  1  0;0  0  0  1;-10  -15  -12  -6];
>> B = [0;0;0;1];
>> C = [10   15   5   0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u1 = [0:0.01:1];
>> u2 = [0.99:-0.01:-1];
>> u3 = [-0.99:0.01:0];
>> u4 = 0*[4.01:0.01:16];
>> u = [u1  u2  u3  u4];
>> y = lsim(sys,u,t);
>> plot(t,y,t,u)
>> v = [0   16   -1.5   1.5]; axis(v)
>> grid
>> title('Response to Triangular Bump (Method 2)')
>> xlabel('t (sec)')
>> ylabel('Triangular Bump and Response')
```

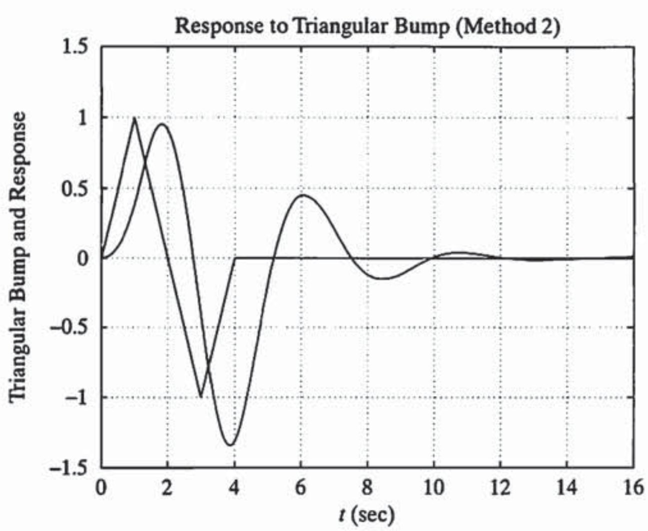

Figure 5–20 Response $y(t)$ to the triangular bump input $u(t)$.

5–5 TRANSFORMATION OF MATHEMATICAL MODELS WITH MATLAB

MATLAB is quite useful in transforming a system model from transfer function to state space and vice versa. We shall begin our discussion with the transformation from transfer function to state space.

Let us write the transfer function $Y(s)/U(s)$ as

$$\frac{Y(s)}{U(s)} = \frac{\text{numerator polynomial in } s}{\text{denominator polynomial in } s} = \frac{\text{num}}{\text{den}}$$

Once we have this transfer-function expression, the MATLAB command

```
[A, B, C, D] = tf2ss(num,den)
```

will give a state-space representation. Note that the command can be used when the system equation involves one or more derivatives of the input function. (In such a case, the transfer function of the system involves a numerator polynomial in s.)

It is important to note that the state-space representation of any system is not unique. There are many (indeed, infinitely many) state-space representations of the same system. The MATLAB command gives one possible such representation.

Transformation from transfer function to state space. Consider the transfer function system

$$\frac{Y(s)}{U(s)} = \frac{s}{s^3 + 14s^2 + 56s + 160} \tag{5–50}$$

Of the infinitely many possible state-space representations of this system, one is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -160 & -56 & -14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -14 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u$$

Another is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -14 & -56 & -160 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \tag{5–51}$$

$$y = [0 \quad 1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0]u \tag{5–52}$$

MATLAB transforms the transfer function given by Equation (5–50) into the state-space representation given by Equations (5–51) and (5–52). For the system considered here, MATLAB Program 5–10 will produce matrices **A**, **B**, **C**, and D.

MATLAB Program 5–10

```
>> % ----- Transforming transfer-function model to
>> %       state-space model -----
>>
>> num = [0  0  1  0];
>> den = [1  14  56  160];
>>
>> % ----- Enter the following transformation command -----
>>
>> [A, B, C, D] = tf2ss(num,den)
A =

   -14    -56    160
     1      0      0
     0      1      0
B =

     1
     0
     0
C =

     0      1      0
D =

     0
```

Transformation from state space to transfer function. To obtain the transfer function from state-space equations, use the command

```
[num,den] = ss2tf(A,B,C,D,iu)
```

Note that iu must be specified for systems with more than one input. For example, if the system has three inputs ($u1$, $u2$, $u3$), then iu must be either 1, 2, or 3, where 1 implies $u1$, 2 implies $u2$, and 3 implies $u3$.

If the system has only one input, then either

```
[num,den] = ss2tf(A,B,C,D)
```

or

```
[num,den] = ss2tf(A,B,C,D,1)
```

may be used. (For the case where the system has multiple inputs and multiple outputs, see **Example 5–9**.)

Example 5–8

Obtain the transfer function of the system defined by the following state-space equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & -15 & -12 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ -15 \\ 40 \end{bmatrix} u$$

$$Y = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0u$$

MATLAB Program 5–11 produces the transfer function of the system, namely,

$$\frac{Y(s)}{U(s)} = \frac{5s^2 + 15s + 10}{s^4 + 6s^3 + 12s^2 + 15s + 10}$$

MATLAB Program 5–11

```
>> % ----- Transforming state-space model to
>> %       transfer function model -----
>>
>> A = [0  1  0  0;0  0  1  0;0  0  0  1;-10  -15  -12  -6];
>> B = [0;5;-15;40];
>> C = [1  0  0  0];
>> D = 0;
>>
>> % ----- Enter the following transformation command ----
>>
>> [num,den] = ss2tf(A,B,C,D)
num =
          0          0     5.0000    15.0000    10.0000

den =
     1.0000     6.0000    12.0000    15.0000    10.0000
```

Example 5–9

Consider a system with multiple inputs and multiple outputs. When the system has more than one output, the command

```
[NUM,den] = ss2tf(A,B,C,D,iu)
```

produces transfer functions for all outputs to each input. (The numerator coefficients are returned to matrix NUM with as many rows as there are outputs.)

Let the system be defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

This system involves two inputs and two outputs. Four transfer functions are involved: $Y_1(s)/U_1(s)$, $Y_2(s)/U_1(s)$, $Y_1(s)/U_2(s)$, and $Y_2(s)/U_2(s)$. (When considering input u_1, we assume that input u_2 is zero, and vice versa.)

MATLAB Program 5–12 produces representations of the following four transfer functions:

$$\frac{Y_1(s)}{U_1(s)} = \frac{s+4}{s^2+4s+25}, \qquad \frac{Y_1(s)}{U_2(s)} = \frac{s+5}{s^2+4s+25}$$
$$\frac{Y_2(s)}{U_1(s)} = \frac{-25}{s^2+4s+25}, \qquad \frac{Y_2(s)}{U_2(s)} = \frac{s-25}{s^2+4s+25}$$

MATLAB Program 5–12

```
>> A = [0   1;-25   -4];
>> B = [1   1;0   1];
>> C = [1   0;0   1];
>> D = [0   0;0   0];
>> [NUM,den] = ss2tf(A,B,C,D,1)

NUM =

          0    1.0000     4.0000
          0         0   -25.0000

den =

     1.0000    4.0000    25.0000
>> [NUM,den] = ss2tf(A,B,C,D,2)

NUM =

          0    1.0000     5.0000
          0    1.0000   -25.0000

den =

     1.0000    4.0000    25.0000
```

Nonuniqueness of a set of state variables. A set of state variables is not unique for a given system. Suppose that $x_1, x_2, \ldots, x_n$ are a set of state variables. Then we may take as another set of state variables any set of functions

$$\begin{aligned} \hat{x}_1 &= X_1(x_1, x_2, \ldots, x_n) \\ \hat{x}_2 &= X_2(x_1, x_2, \ldots, x_n) \\ &\vdots \\ \hat{x}_n &= X_n(x_1, x_2, \ldots, x_n) \end{aligned}$$

provided that, for every set of values $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n$, there corresponds a unique set of values $x_1, x_2, \ldots, x_n$, and vice versa. Thus, if $\mathbf{x}$ is a state vector, then

$$\hat{\mathbf{x}} = \mathbf{P}\mathbf{x}$$

is also a state vector, provided that the matrix $\mathbf{P}$ is nonsingular. (Note that a square matrix $\mathbf{P}$ is nonsingular if the determinant $|\mathbf{P}|$ is nonzero.) Different state vectors convey the same information about the system behavior.

Transformation of a state-space model into another state-space model. A state-space model

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{5–53}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \tag{5–54}$$

can be transformed into another state-space model by transforming the state vector $\mathbf{x}$ into state vector $\hat{\mathbf{x}}$ by means of the transformation

$$\mathbf{x} = \mathbf{P}\hat{\mathbf{x}}$$

where $\mathbf{P}$ is nonsingular. Then Equations (5–53) and (5–54) can be written as

$$\mathbf{P}\dot{\hat{\mathbf{x}}} = \mathbf{A}\mathbf{P}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\hat{\mathbf{x}} + \mathbf{D}\mathbf{u}$$

or

$$\dot{\hat{\mathbf{x}}} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}\hat{\mathbf{x}} + \mathbf{P}^{-1}\mathbf{B}\mathbf{u} \tag{5–55}$$

$$\mathbf{y} = \mathbf{C}\mathbf{P}\hat{\mathbf{x}} + \mathbf{D}\mathbf{u} \tag{5–56}$$

Equations (5–55) and (5–56) represent another state-space model of the same system. Since infinitely many $n \times n$ nonsingular matrices can be used as a transformation matrix $\mathbf{P}$, there are infinitely many state-space models for a given system.

Eigenvalues of an $n \times n$ matrix A. The *eigenvalues* of an $n \times n$ matrix $\mathbf{A}$ are the roots of the characteristic equation

$$|\lambda\mathbf{I} - \mathbf{A}| = 0 \tag{5–57}$$

The eigenvalues are also called the *characteristic roots*.

Consider, for example, the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

The characteristic equation is

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{A}| &= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix} \\ &= \lambda^3 + 6\lambda^2 + 11\lambda + 6 \\ &= (\lambda + 1)(\lambda + 2)(\lambda + 3) = 0 \end{aligned}$$

The eigenvalues of $\mathbf{A}$ are the roots of the characteristic equation, or -1, -2, and -3.

It is sometimes desirable to transform the state matrix into a diagonal matrix. This may be done by choosing an appropriate transformation matrix **P**. In what follows, we shall discuss the diagonalization of a state matrix.

Diagonalization of state matrix A. Consider an $n \times n$ state matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \tag{5–58}$$

We first consider the case where matrix **A** has distinct eigenvalues only. If the state vector **x** is transformed into another state vector **z** with the use of a transformation matrix **P**, or

$$\mathbf{x} = \mathbf{P}\mathbf{z}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \tag{5–59}$$

in which $\lambda_1, \lambda_2, \ldots,$ and λ_n are n distinct eigenvalues of **A**, then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ becomes a diagonal matrix, or

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \lambda_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & \lambda_n \end{bmatrix} \tag{5–60}$$

Note that each column of the transformation matrix **P** in Equation (5–59) is an eigenvector of the matrix **A** given by Equation (5–58). (See **Problem A–5–18** for details.)

Next, consider the case where matrix **A** involves multiple eigenvalues. In this case, diagonalization is not possible, but matrix **A** can be transformed into a Jordan canonical form. For example, consider the 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Assume that $\mathbf{A}$ has eigenvalues λ_1, λ_1, and λ_3, where $\lambda_1 \neq \lambda_3$. In this case, the transformation $\mathbf{x} = \mathbf{S}\mathbf{z}$, where

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 1 \\ \lambda_1 & 1 & \lambda_3 \\ \lambda_1^2 & 2\lambda_1 & \lambda_3^2 \end{bmatrix} \tag{5–61}$$

will yield

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{5–62}$$

This matrix is in Jordan canonical form.

Example 5–10

Consider a system with the state-space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

or

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + Du \end{aligned} \tag{5–63}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}, \qquad \mathbf{C} = [1 \quad 0 \quad 0], \qquad D = 0$$

The eigenvalues of the state matrix $\mathbf{A}$ are -1, -2, and -3, or

$$\lambda_1 = -1, \qquad \lambda_2 = -2, \qquad \lambda_3 = -3$$

We shall show that Equation (5–63) is not the only possible state equation for the system. Suppose we define a set of new state variables z_1, z_2, and z_3 by the transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

or

$$\mathbf{x} = \mathbf{P}\mathbf{z} \tag{5–64}$$

where

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \tag{5–65}$$

Then, substituting Equation (5–64) into Equation (5–63), we obtain

$$\mathbf{P}\dot{\mathbf{z}} = \mathbf{APz} + \mathbf{B}u$$

Premultiplying both sides of this last equation by $\mathbf{P}^{-1}$, we get

$$\dot{\mathbf{z}} = \mathbf{P}^{-1}\mathbf{APz} + \mathbf{P}^{-1}\mathbf{B}u \tag{5–66}$$

or

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

Simplifying gives

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u \tag{5–67}$$

Equation (5–67) is a state equation that describes the system defined by Equation (5–63).

The output equation is modified to

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$$

$$= [1 \quad 1 \quad 1] \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \tag{5–68}$$

Notice that the transformation matrix $\mathbf{P}$ defined by Equation (5–65) changes the coefficient matrix of $\mathbf{z}$ into the diagonal matrix. As is clearly seen from Equation (5–67), the three separate state equations are uncoupled. Notice also that the diagonal elements of the matrix $\mathbf{P}^{-1}\mathbf{AP}$ in Equation (5–66) are identical to the three eigenvalues of $\mathbf{A}$. (For a proof, see **Problem A–5–20.**)

EXAMPLE PROBLEMS AND SOLUTIONS

Problem A–5–1

Consider the pendulum system shown in Figure 5–21. Assuming angle θ to be the output of the system, obtain a state-space representation of the system.

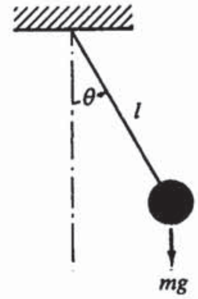

Figure 5–21 Pendulum system.

Solution The equation for the pendulum system is

$$ml^2\ddot{\theta} = -mgl\sin\theta$$

or

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

This is a second-order system; accordingly, we need two state variables, x_1 and x_2, to completely describe the system dynamics. If we define

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$

then we get

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin x_1$$

(There is no input u to this system.) The output y is angle θ. Thus,

$$y = \theta = x_1$$

A state-space representation of the system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{g}{l}\dfrac{\sin x_1}{x_1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Note that the state equation just obtained is a nonlinear differential equation.

If the angle θ is limited to be small, then the system can be linearized. For small angle θ, we have $\sin\theta = \sin x_1 \doteqdot x_1$ and $(\sin x_1)/x_1 \doteqdot 1$. A state-space representation of the linearized model is then given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Problem A–5–2

Obtain a state-space representation of the mechanical system shown in Figure 5–22. The external force $u(t)$ applied to mass m_2 is the input to the system. The displacements y and z are measured from their respective equilibrium positions and are the outputs of the system.

Solution Applying Newton's second law to this system, we obtain

$$m_2\ddot{y} + b_1(\dot{y} - \dot{z}) + k_1(y - z) + k_2 y = u \tag{5–69}$$

$$m_1\ddot{z} + b_1(\dot{z} - \dot{y}) + k_1(z - y) = 0 \tag{5–70}$$

If we define the state variables

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = z$$
$$x_4 = \dot{z}$$

then, from Equation (5–69), we get

$$m_2\dot{x}_2 = -(k_1 + k_2)x_1 - b_1x_2 + k_1x_3 + b_1x_4 + u$$

Also, from Equation (5–70), we obtain

$$m_1\dot{x}_4 = k_1x_1 + b_1x_2 - k_1x_3 - b_1x_4$$

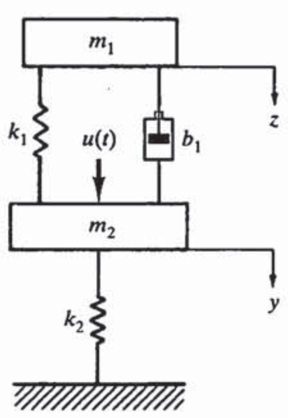

Figure 5–22 Mechanical system.

Hence, the state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{k_1 + k_2}{m_2} & -\dfrac{b_1}{m_2} & \dfrac{k_1}{m_2} & \dfrac{b_1}{m_2} \\ 0 & 0 & 0 & 1 \\ \dfrac{k_1}{m_1} & \dfrac{b_1}{m_1} & -\dfrac{k_1}{m_1} & -\dfrac{b_1}{m_1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m_2} \\ 0 \\ 0 \end{bmatrix} u \tag{5–71}$$

The outputs of the system are y and z. Consequently, if we define the output variables as

$$y_1 = y$$
$$y_2 = z$$

then we have

$$y_1 = x_1$$
$$y_2 = x_3$$

The output equation can now be put in the form

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{5–72}$$

Equations (5–71) and (5–72) give a state-space representation of the mechanical system shown in Figure 5–22.

Problem A–5–3

Obtain a state-space representation of the system defined by

$$\overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u \tag{5–73}$$

where u is the input and y is the output of the system.

Solution Since the initial conditions $y(0), \dot{y}(0), \ldots, \overset{(n-1)}{y}(0)$, together with the input $u(t)$ for $t \geq 0$, determines completely the future behavior of the system, we may take $y(t), \dot{y}(t), \ldots, \overset{(n-1)}{y}(t)$ as a set of n state variables. (Mathematically, such a choice of state variables is quite convenient. Practically, however, because higher order derivative terms are inaccurate due to the noise effects that are inherent in any practical system, this choice of state variables may not be desirable.)

Let us define

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$\vdots$$
$$x_n = \overset{(n-1)}{y}$$

Then Equation (5–73) can be written as

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\vdots \\
\dot{x}_{n-1} &= x_n \\
\dot{x}_n &= -a_n x_1 - \cdots - a_1 x_n + u
\end{aligned}
$$

or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \tag{5–74}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

The output can be given by

$$y = [1 \quad 0 \quad \cdots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or

$$y = \mathbf{C}\mathbf{x} \tag{5–75}$$

where

$$\mathbf{C} = [1 \quad 0 \quad \cdots \quad 0]$$

Equation (5–74) is the state equation and Equation (5–75) is the output equation.

Note that the state-space representation of the transfer function of the system,

$$\frac{Y(s)}{U(s)} = \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

is also given by Equations (5–74) and (5–75).

Problem A–5–4

Consider a system described by the state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

and output equation

$$y = \mathbf{C}\mathbf{x} + Du$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.125 & -1.375 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -0.25 \\ 0.34375 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0], \quad D = 1$$

Obtain the transfer function of this system.

Solution From Equation (5–9), the transfer function $G(s)$ can be given in terms of matrices **A**, **B**, **C**, and D as

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

Since

$$s\mathbf{I} - \mathbf{A} = \begin{bmatrix} s & -1 \\ 0.125 & s + 1.375 \end{bmatrix}$$

we have

$$(s\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{s^2 + 1.375s + 0.125}\begin{bmatrix} s + 1.375 & 1 \\ -0.125 & s \end{bmatrix}$$

Therefore, the transfer function of the system is

$$\begin{aligned} G(s) &= [1 \quad 0]\frac{1}{s^2 + 1.375s + 0.125}\begin{bmatrix} s + 1.375 & 1 \\ -0.125 & s \end{bmatrix}\begin{bmatrix} -0.25 \\ 0.34375 \end{bmatrix} + 1 \\ &= \frac{-0.25(s + 1.375) + 0.34375}{s^2 + 1.375s + 0.125} + 1 \\ &= \frac{s^2 + 1.125s + 0.125}{s^2 + 1.375s + 0.125} \\ &= \frac{8s^2 + 9s + 1}{8s^2 + 11s + 1} \end{aligned}$$

Problem A–5–5

Consider the following state equation and output equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 6.5 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

The system involves two inputs and two outputs, so there are four input–output combinations. Obtain the impulse-response curves of the four combinations. (When u_1 is a unit-impulse input, we assume that $u_2 = 0$, and vice versa.)

Next, find the outputs y_1 and y_2 when both inputs, u_1 and u_2, are given at the same time (i.e., $u_1 = u_2 =$ unit-impulse function occurring at the same time $t = 0$).

Solution The command

```
sys = ss(A,B,C,D1),        impulse(sys,t)
```

produces the impulse-response curves for the four input–output combinations. (See MATLAB Program 5–13; when u_1 is a unit-impulse function, we assume that $u_2 = 0$, and vice versa.) The resulting curves are shown in Figure 5–23.

When both unit-impulse inputs $u_1(t)$ and $u_2(t)$ are given at the same time $t = 0$, the responses are

$$y_1(t) = y_{11}(t) + y_{21}(t)$$
$$y_2(t) = y_{12}(t) + y_{22}(t)$$

MATLAB Program 5–13

```
>> t = 0:0.01:10;
>> A = [-1   -1;6.5   0];
>> B = [1   1;1   0];
>> C = [1   0;0   1];
>> D = [0   0;0   0];
>> sys = ss(A,B,C,D);
>> impulse(sys,t)
>> grid
>> title('Impulse-Response Curves')
>> xlabel('t'); ylabel('Outputs')
```

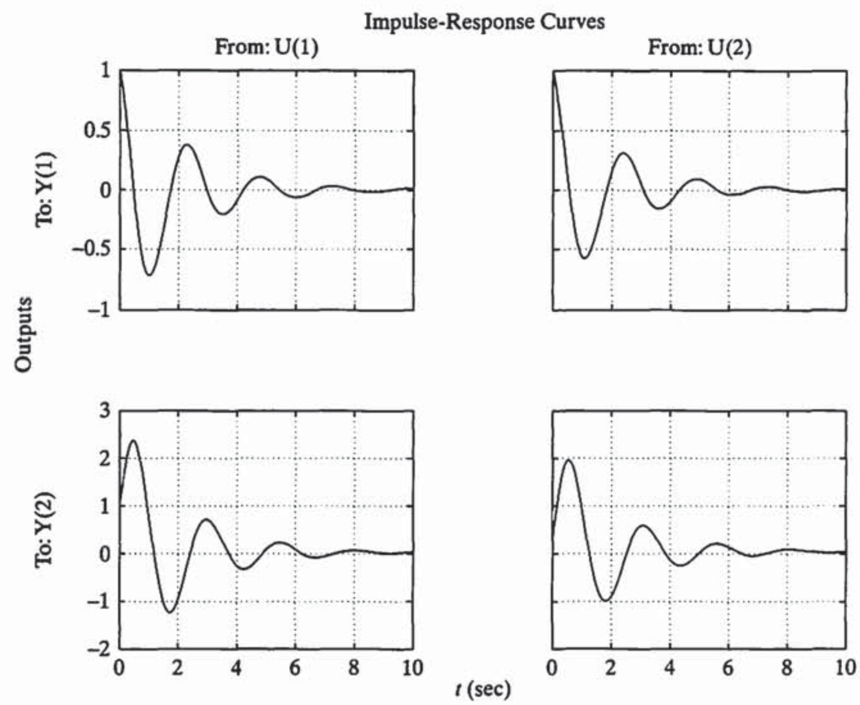

Figure 5–23 Unit-impulse response curves. (The left column corresponds to u_1 = unit-impulse input and $u_2 = 0$. The right column corresponds to $u_1 = 0$ and u_2 = unit-impulse input.)

where

$$
\begin{aligned}
y_{11} &= y_1 && \text{when } u_1 = \delta(t),\ u_2 = 0 \\
y_{12} &= y_2 && \text{when } u_1 = \delta(t),\ u_2 = 0 \\
y_{21} &= y_1 && \text{when } u_1 = 0,\ u_2 = \delta(t) \\
y_{22} &= y_2 && \text{when } u_1 = 0,\ u_2 = \delta(t)
\end{aligned}
$$

MATLAB Program 5–14 produces the responses $y_1(t) = y_{11}(t) + y_{21}(t)$ and $y_2(t) = y_{12}(t) + y_{22}(t)$. The resulting response curves are shown in Figure 5–24.

MATLAB Program 5–14

```
>> t = 0:0.01:10;
>> A = [-1   -1;6.5   0];
>> B = [1   1;1   0];
>> C = [1   0;0   1];
>> D = [0   0;0   0];
>> sys = ss(A,B,C,D);
>> [y,t,x] = impulse(sys,t);
>> y11 = [1   0]*y(:,:,1)';
>> y12 = [0   1]*y(:,:,1)';
>> y21 = [1   0]*y(:,:,2)';
>> y22 = [0   1]*y(:,:,2)';
>> subplot(211); plot(t,y11+y21); grid
>> title('Impulse Response when Both u_1 and u_2 are given at t = 0')
>> ylabel('y_1')
>> subplot(212); plot(t,y12+y22); grid
>> xlabel('t (sec)'); ylabel('y_2')
```

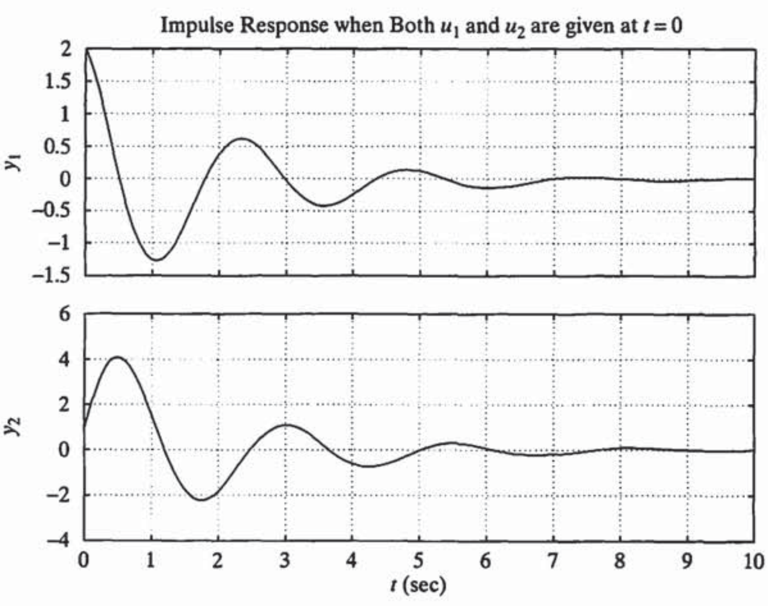

Figure 5–24 Response curves $y_1(t)$ versus t and $y_2(t)$ versus t when $u_1(t)$ and $u_2(t)$ are given at the same time. [Both $u_1(t)$ and $u_2(t)$ are unit-impulse inputs occurring at $t = 0$.]

Problem A–5–6

Obtain the unit-step response and unit-impulse response of the following system with MATLAB:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -0.01 & -0.1 & -0.5 & -1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.04 \\ -0.012 \\ 0.008 \end{bmatrix} u$$

$$y = [1 \quad 0 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The initial conditions are zeros.

Solution To obtain the unit-step response of this system, the following command may be used:

```
[y, x, t] = step(A, B, C, D)
```

Since the unit-impulse response is the derivative of the unit-step response, the derivative of the output (y = x1) will give the unit-impulse response. From the state equation, we see that the derivative of y is

```
x2 = [0  1  0  0]*x'
```

Hence, x2 versus t will give the unit-impulse response.

MATLAB Program 5–15 produces both the unit-step and unit-impulse responses. The resulting unit-step response curve and unit-impulse curve are shown in Figure 5–25.

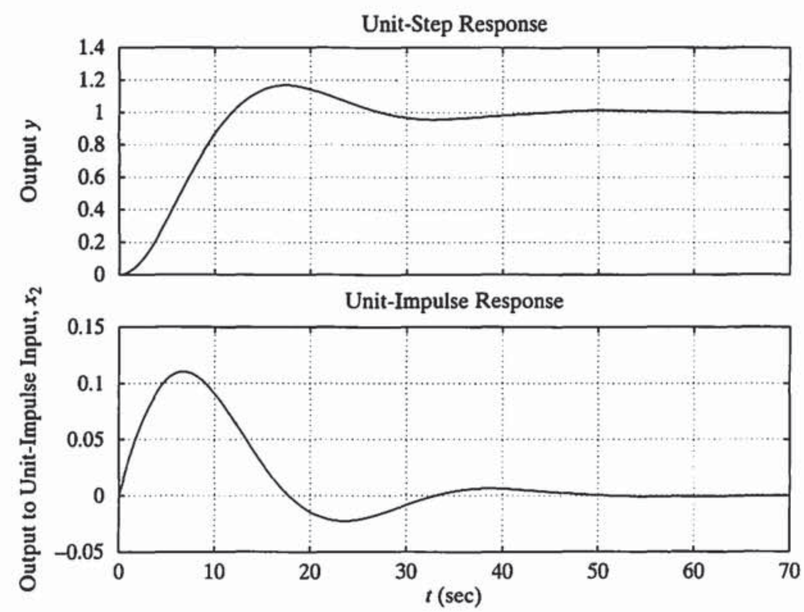

Figure 5–25 Unit-step response curve and unit-impulse response curve.

MATLAB Program 5–15

```
>> A = [0  1  0  0;0  0  1  0;0  0  0  1;-0.01  -0.1  -0.5  -1.5];
>> B = [0;0.04;-0.012;0.008];
>> C = [1  0  0  0];
>> D = 0;
>>
>> % To get the step response, enter, for example, the following
>> % command:
>>
>> [y,x,t] = step(A,B,C,D);
>> subplot(211); plot(t,y); grid
>> title('Unit-Step Response')
>> ylabel('Output y')
>>
>> % The unit-impulse response of the system is the same as the
>> % derivative of the unit-step response. (Note that x_1dot
>> % = x_2 in this system.) Hence, the unit-impulse response
>> % of this system is given by ydot = x_2. To plot the unit-
>> % impulse response curve, enter the following command:
>>
>> x2 = [0  1  0  0]*x'; subplot(212); plot(t,x2); grid
>> title('Unit-Impulse Response')
>> xlabel('t (sec)'); ylabel('Output to Unit-Impulse Input, x_2')
```

Problem A–5–7

Two masses m_1 and m_2 are connected by a spring with spring constant k, as shown in Figure 5–26. Assuming no friction, derive a state-space representation of the system, which is at rest for $t < 0$. The displacements y_1 and y_2 are the outputs of the system and are measured from their rest positions relative to the ground.

Assuming that $m_1 = 40$ kg, $m_2 = 100$ kg, $k = 40$ N/m, and f is a step force input of magnitude of 10 N, obtain the response curves $y_1(t)$ versus t and $y_2(t)$ versus t with MATLAB. Also, obtain the relative motion between m_1 and m_2. Define $y_2 - y_1 = x$ and plot the curve $x(t)$ versus t. Assume that we are interested in the period $0 \leq t \leq 20$.

Solution Let us define a step force input of magnitude 1 N as u. Then the equations of motion for the system are

$$m_1\ddot{y}_1 + k(y_1 - y_2) = 0$$
$$m_2\ddot{y}_2 + k(y_2 - y_1) = f$$

We choose the state variables for the system as follows:

$$x_1 = y_1$$
$$x_2 = \dot{y}_1$$

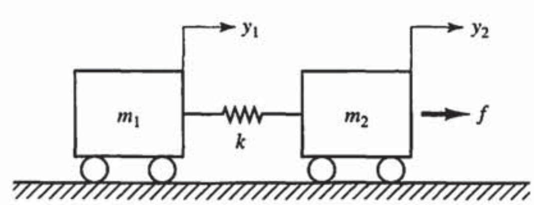

Figure 5–26 Mechanical system.

$$x_3 = y_2$$
$$x_4 = \dot{y}_2$$

Then we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{m_1}x_1 + \frac{k}{m_1}x_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{k}{m_2}x_1 - \frac{k}{m_2}x_3 + \frac{1}{m_2}f$$

Noting that $f = 10u$ and substituting the given numerical values for m_1, m_2, and k, we obtain the state equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.4 & 0 & -0.4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1 \end{bmatrix} u$$

The output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + 0u$$

MATLAB Program 5–16 produces the outputs y_1 and y_2 and the relative motion $x(= y_2 - y_1 = x_3 - x_1)$. The resulting response curves $y_1(t)$ versus t, $y_2(t)$ versus t, and $x(t)$ versus t are shown in Figure 5–27. Notice that the vibration between m_1 and m_2 continues forever.

MATLAB Program 5–16

```
>> t = 0:0.02:20;
>> A = [0  1  0  0;-1  0  1  0;0  0  0  1;0.4  0  -0.4  0];
>> B = [0;0;0;0.1];
>> C = [1  0  0  0;0  0  1  0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> [y,t,x] = step(sys,t);
>> y1 = [1  0]*y';
>> y2 = [0  1]*y';
>> subplot(311); plot(t,y1), grid
>> title('Step Response')
>> ylabel('Output y_1')
>> subplot(312); plot(t,y2), grid
>> ylabel('Output y_2')
>> subplot(313); plot(t,y2 - y1), grid
>> xlabel('t (sec)'); ylabel('x = y_2 - y_1')
```

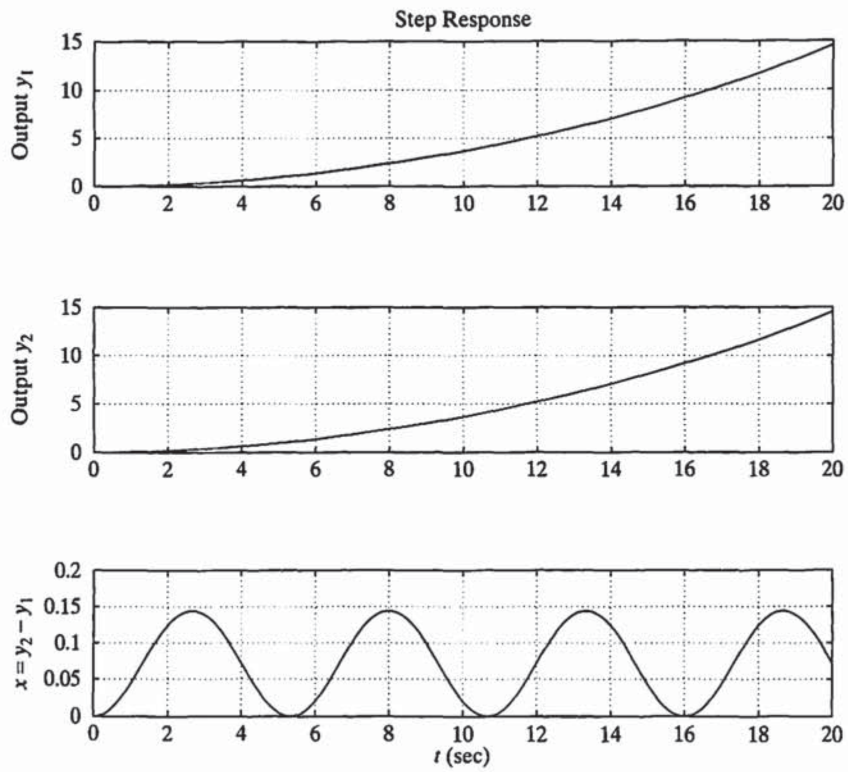

Figure 5–27 Response curves $y_1(t)$ versus t, $y_2(t)$ versus t, and $x(t)$ versus t.

Problem A–5–8

Obtain the unit-ramp response of the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + [0]u$$

The system is initially at rest.

Solution Noting that the unit-ramp input is defined by

$$u = t \qquad (0 \leq t)$$

we may use the command

lsim(sys, u, t)

as shown in MATLAB Program 5–17. The unit-ramp response curve and the unit-ramp input are shown in Figure 5–28.

MATLAB Program 5–17

```
>> t = 0:0.01:18;
>> A = [0   1;-1   -0.4];
>> B = [0;1];
>> C = [1   0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> u = t;
>> lsim(sys,u,t)
>> grid
>> title('Unit-Ramp Response')
>> xlabel('t')
>> ylabel('Output y')
>> text(3.5,0.6,'y')
>> text(0.5,3.2,'u')
```

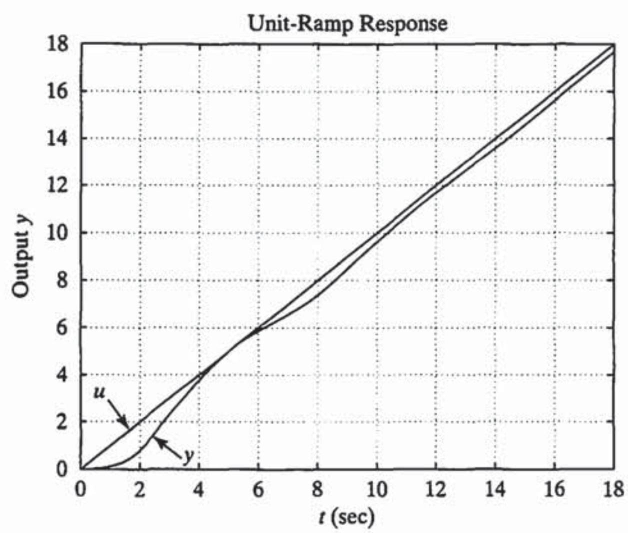

Figure 5–28 Plot of unit-ramp response curve, together with unit-ramp input.

Problem A–5–9

A mass M (where $M = 8$ kg) is supported by a spring (where $k = 400$ N/m) and a damper (where $b = 40$ N-s/m), as shown in Figure 5–29. At $t = 0$, a mass $m = 2$ kg is gently placed on the top of mass M, causing the system to exhibit vibrations. Assuming that the displacement x of the combined mass is measured from the equilibrium position before m is placed on M, obtain a state-space representation of the system. Then plot the response curve $x(t)$ versus t. (For an analytical solution, see **Problem A–3–16**.)

Solution The equation of motion for the system is

$$(M + m)\ddot{x} + b\dot{x} + kx = mg \qquad (0 < t)$$

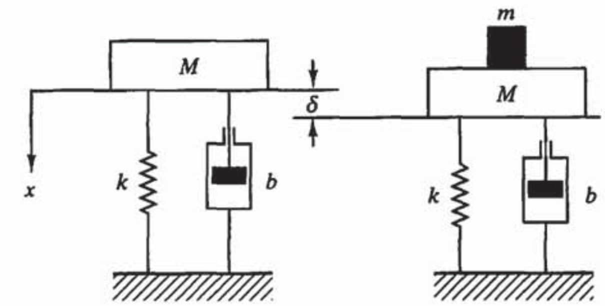

Figure 5–29 Mechanical system.

Substituting the given numerical values for M, m, b, k, and $g = 9.807\ \text{m/s}^2$ into this last equation, we obtain

$$10\ddot{x} + 40\dot{x} + 400x = 2 \times 9.807$$

or

$$\ddot{x} + 4\dot{x} + 40x = 1.9614$$

The input here is a step force of magnitude 1.9614 N.

Let us define a step force input of magnitude 1 N as u. Then we have

$$\ddot{x} + 4\dot{x} + 40x = 1.9614u$$

If we now choose state variables

$$x_1 = x$$
$$x_2 = \dot{x}$$

then we obtain

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -40x_1 - 4x_2 + 1.9614u$$

The state equation is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -40 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1.9614 \end{bmatrix} u$$

and the output equation is

$$y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0u$$

MATLAB Program 5–18 produces the response curve $y(t)$ [$= x(t)$] versus t, shown in Figure 5–30. Notice that the static deflection $x(\infty) = y(\infty) \doteqdot y(600)$ is 0.049035 m.

Problem A–5–10

Consider the system shown in Figure 5–31. The system is at rest for $t < 0$. The displacements z_1 and z_2 are measured from their respective equilibrium positions relative to the ground. Choosing z_1, $\dot{z}_1$, z_2, and $\dot{z}_2$ as state variables, derive a state-space representation of the system. Assuming that $m_1 = 10$ kg, $m_2 = 20$ kg, $b = 20$ N-s/m, $k = 60$ N/m, and f is a step force input of magnitude 10 N, plot the response curves $z_1(t)$ versus t, $z_2(t)$ versus t, $z_2(t) - z_1(t)$ versus t, and $\dot{z}_2(t) - \dot{z}_1(t)$ versus t. Also, obtain the steady-state values of $\ddot{z}_1$, $\ddot{z}_2$, and $z_2 - z_1$.

MATLAB Program 5–18

```
>> t = 0:0.01:6;
>> A = [0   1;-40   -4];
>> B = [0;1.9614];
>> C = [1   0];
>> D = 0;
>> sys = ss(A,B,C,D);
>> [y,t] = step(sys,t);
>> plot(t,y)
>> grid
>> title('Step Response')
>> xlabel('t (sec)'); ylabel('Output y')
>>
>> format long;
>> y(600)

ans =

   0.04903515818520
```

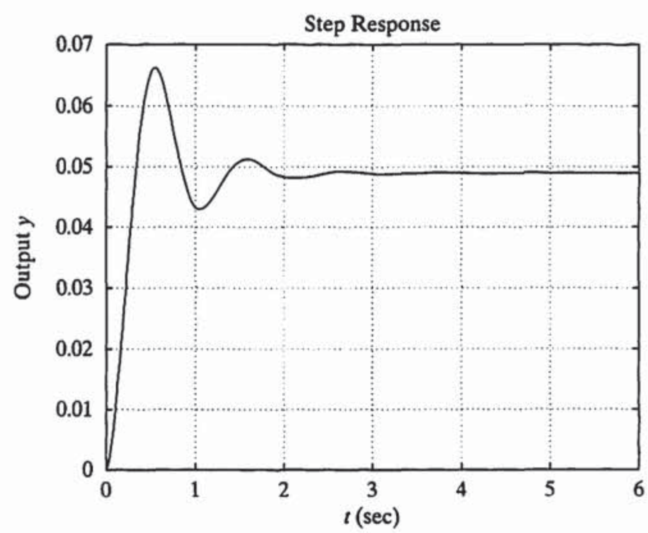

Figure 5–30 Step-response curve.

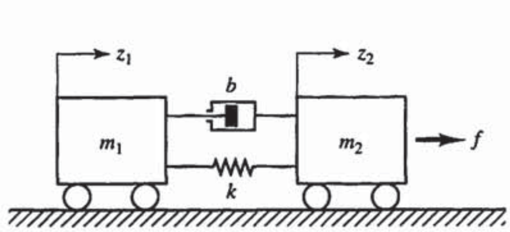

Figure 5–31 Mechanical system.

Solution The equations of motion for the system are

$$m_1\ddot{z}_1 = k(z_2 - z_1) + b(\dot{z}_2 - \dot{z}_1) \tag{5–76}$$

$$m_2\ddot{z}_2 = -k(z_2 - z_1) - b(\dot{z}_2 - \dot{z}_1) + f \tag{5–77}$$

Since we chose state variables as

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= \dot{z}_1 \\ x_3 &= z_2 \\ x_4 &= \dot{z}_2 \end{aligned}$$

Equations (5–76) and (5–77) can be written as

$$\begin{aligned} m_1\dot{x}_2 &= k(x_3 - x_1) + b(x_4 - x_2) \\ m_2\dot{x}_4 &= -k(x_3 - x_1) - b(x_4 - x_2) + f \end{aligned}$$

We thus have

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{k}{m_1}x_1 - \frac{b}{m_1}x_2 + \frac{k}{m_1}x_3 + \frac{b}{m_1}x_4 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \frac{k}{m_2}x_1 + \frac{b}{m_2}x_2 - \frac{k}{m_2}x_3 - \frac{b}{m_2}x_4 + \frac{1}{m_2}f \end{aligned}$$

Let us define z_1 and z_2 as the system outputs. Then

$$\begin{aligned} y_1 &= z_1 = x_1 \\ y_2 &= z_2 = x_3 \end{aligned}$$

After substitution of the given numerical values and $f = 10u$ (where u is a step force input of magnitude 1 N occurring at $t = 0$), the state equation becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & -2 & 6 & 2 \\ 0 & 0 & 0 & 1 \\ 3 & 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix} u$$

The output equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

MATLAB Program 5–19 produces the response curves z_1 versus t, z_2 versus t, $z_2 - z_1$ versus t, and $\dot{z}_2 - \dot{z}_1$ versus t. The resulting curves are shown in Figure 5–32.

Note that at steady state $\ddot{z}_1(t)$ and $\ddot{z}_2(t)$ approach a constant value, or

$$\ddot{z}_1(\infty) = \ddot{z}_2(\infty) = \alpha$$

Also, at steady state the value of $z_2(t) - z_1(t)$ approaches a constant value, or

$$z_2(\infty) - z_1(\infty) = \beta$$

MATLAB Program 5–19

```
>> t = 0:0.01:15;
>> A = [0  1  0  0;-6  -2  6  2;0  0  0  1;3  1  -3  -1];
>> B = [0;0;0;0.5];
>> C = [1  0  0  0;0  0  1  0];
>> D = [0;0];
>> sys = ss(A,B,C,D);
>> [y,t,x] = step(sys, t);
>> x1 = [1  0  0  0]*x';
>> x2 = [0  1  0  0]*x';
>> x3 = [0  0  1  0]*x';
>> x4 = [0  0  0  1]*x';
>> subplot(221); plot(t,x1); grid
>> xlabel('t (sec)'); ylabel('Output z_1')
>> subplot(222); plot(t,x3); grid
>> xlabel('t (sec)'); ylabel('Output z_2')
>> subplot(223); plot(t,x3 - x1); grid
>> xlabel('t (sec)'); ylabel('Output z_2 - z_1')
>> subplot(224); plot(t,x4 - x2); grid
>> xlabel('t (sec)'); ylabel('z_2dot - z_1dot')
```

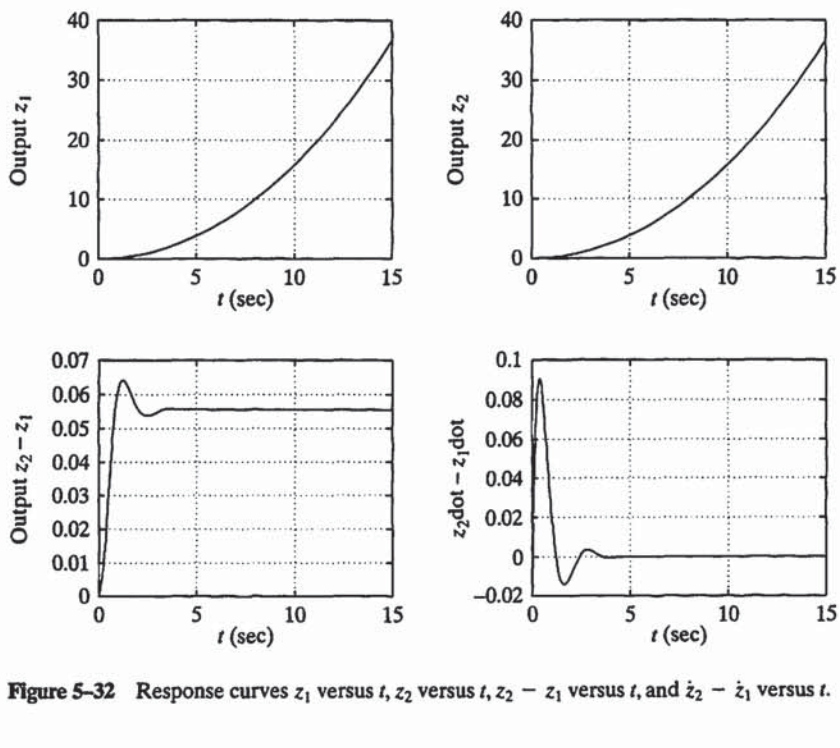

Figure 5–32 Response curves z_1 versus t, z_2 versus t, $z_2 - z_1$ versus t, and $\dot{z}_2 - \dot{z}_1$ versus t.

The steady-state value of $\dot{z}_2(t) - \dot{z}_1(t)$ is zero, or

$$\dot{z}_2(\infty) - \dot{z}_1(\infty) = 0$$

For $t = \infty$, Equation (5–76) becomes

$$m_1\ddot{z}_1(\infty) = k[z_2(\infty) - z_1(\infty)] + b[\dot{z}_2(\infty) - \dot{z}_1(\infty)]$$

or

$$10\alpha = k\beta + b \times 0$$

Also, Equation (5–77) becomes

$$m_2\ddot{z}_2(\infty) = -k[z_2(\infty) - z_1(\infty)] - b[\dot{z}_2(\infty) - \dot{z}_1(\infty)] + f$$

or

$$20\alpha = -k\beta - b \times 0 + f$$

Hence,

$$10\alpha = 60\beta$$
$$20\alpha = -60\beta + f$$

from which we get

$$\alpha = \frac{f}{30} = \frac{10}{30} = \frac{1}{3}$$

and

$$\beta = \frac{10\alpha}{60} = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18}$$

Thus,

$$\ddot{z}_1(\infty) = \ddot{z}_2(\infty) = \alpha = \frac{1}{3}\,\text{m/s}^2$$
$$z_2(\infty) - z_1(\infty) = \beta = \frac{1}{18}\,\text{m}$$

Problem A–5–11

Obtain two state-space representations of the mechanical system shown in Figure 5–33 where u is the input displacement and y is the output displacement. The system is initially at rest. The displacement y is measured from the rest position before the input u is given.

Solution The equation of motion for the mechanical system shown in Figure 5–33 is

$$f_1(\dot{u} - \dot{y}) + k_1(u - y) = f_2\dot{y}$$

Rewriting, we obtain

$$(f_1 + f_2)\dot{y} + k_1 y = f_1\dot{u} + k_1 u$$

or

$$\dot{y} + \frac{k_1}{f_1 + f_2}y = \frac{f_1}{f_1 + f_2}\dot{u} + \frac{k_1}{f_1 + f_2}u \tag{5–78}$$

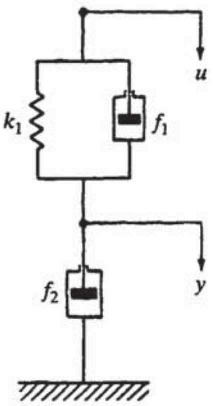

Figure 5–33 Mechanical system.

Comparing this last equation with

$$\dot{y} + a_1 y = b_0 \dot{u} + b_1 u \tag{5–79}$$

we get

$$a_1 = \frac{k_1}{f_1 + f_2}, \qquad b_0 = \frac{f_1}{f_1 + f_2}, \qquad b_1 = \frac{k_1}{f_1 + f_2}$$

We shall obtain two state-space representations of the system, based on Methods 1 and 2 presented in Section 5–4.

Method 1. First calculate β_0 and β_1:

$$\beta_0 = b_0 = \frac{f_1}{f_1 + f_2}$$

$$\beta_1 = b_1 - a_1\beta_0 = \frac{k_1 f_2}{(f_1 + f_2)^2}$$

Define the state variable x by

$$x = y - \beta_0 u = y - \frac{f_1}{f_1 + f_2}u$$

Then the state equation can be obtained from Equation (5–78) as follows:

$$\dot{x} = -\frac{k_1}{f_1 + f_2}x + \frac{k_1 f_2}{(f_1 + f_2)^2}u \tag{5–80}$$

The output equation is

$$y = x + \frac{f_1}{f_1 + f_2}u \tag{5–81}$$

Equations (5–80) and (5–81) give a state-space representation of the system.

Method 2. From Equation (5–79), we have

$$\frac{Y(s)}{U(s)} = \frac{b_0 s + b_1}{s + a_1}$$

If we define

$$\frac{Z(s)}{U(s)} = \frac{1}{s + a_1}, \qquad \frac{Y(s)}{Z(s)} = b_0 s + b_1$$

then we get

$$\dot{z} + a_1 z = u \tag{5–82}$$
$$b_0 \dot{z} + b_1 z = y \tag{5–83}$$

Next, we define the state variable x by

$$x = z$$

Then Equation (5–82) can be written as

$$\dot{x} = -a_1 x + u$$

or

$$\dot{x} = -\frac{k_1}{f_1 + f_2} x + u \tag{5–84}$$

and Equation (5–83) becomes

$$b_0 \dot{x} + b_1 x = y$$

or

$$y = \frac{k_1}{f_1 + f_2} x + \frac{f_1}{f_1 + f_2} \dot{x} \tag{5–85}$$

Substituting Equation (5–84) into Equation (5–85), we get

$$y = \frac{k_1 f_2}{(f_1 + f_2)^2} x + \frac{f_1}{f_1 + f_2} u \tag{5–86}$$

Equations (5–84) and (5–86) give a state-space representation of the system.

Problem A–5–12

Show that, for the differential-equation system

$$\dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y = b_0 \dddot{u} + b_1 \ddot{u} + b_2 \dot{u} + b_3 u \tag{5–87}$$

state and output equations can be given, respectively, by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u \tag{5–88}$$

and

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \beta_0 u \tag{5–89}$$

where the state variables are defined by

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$
$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$

The constants, β_0, β_1, β_2, and β_3 are defined by

$$\begin{aligned}
\beta_0 &= b_0 \\
\beta_1 &= b_1 - a_1\beta_0 \\
\beta_2 &= b_2 - a_1\beta_1 - a_2\beta_0 \\
\beta_3 &= b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0
\end{aligned}$$

Solution From the definition of the state variables x_2 and x_3, we have

$$\dot{x}_1 = x_2 + \beta_1 u \tag{5–90}$$

$$\dot{x}_2 = x_3 + \beta_2 u \tag{5–91}$$

To derive the equation for $\dot{x}_3$, we note that

$$\dddot{y} = -a_1\ddot{y} - a_2\dot{y} - a_3 y + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

Since

$$x_3 = \ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u$$

we have

$$\begin{aligned}
\dot{x}_3 &= \dddot{y} - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\
&= (-a_1\ddot{y} - a_2\dot{y} - a_3 y) + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\
&= -a_1(\ddot{y} - \beta_0\ddot{u} - \beta_1\dot{u} - \beta_2 u) - a_1\beta_0\ddot{u} - a_1\beta_1\dot{u} - a_1\beta_2 u \\
&\quad - a_2(\dot{y} - \beta_0\dot{u} - \beta_1 u) - a_2\beta_0\dot{u} - a_2\beta_1 u - a_3(y - \beta_0 u) - a_3\beta_0 u \\
&\quad + b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u - \beta_0\dddot{u} - \beta_1\ddot{u} - \beta_2\dot{u} \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_0 - \beta_0)\dddot{u} + (b_1 - \beta_1 - a_1\beta_0)\ddot{u} \\
&\quad + (b_2 - \beta_2 - a_1\beta_1 - a_2\beta_0)\dot{u} + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + (b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0)u \\
&= -a_1 x_3 - a_2 x_2 - a_3 x_1 + \beta_3 u
\end{aligned}$$

Hence, we get

$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + \beta_3 u \tag{5–92}$$

Combining Equations (5–90), (5–91), and (5–92) into a vector-matrix differential equation, we obtain Equation (5–88). Also, from the definition of state variable x_1, we get the output equation given by Equation (5–89).

Note that the derivation presented here can be easily extended to the general case of an nth-order system.

Problem A–5–13

Show that, for the system

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

or

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

state and output equations may be given, respectively, by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

and

$$y = [b_3 - a_3 b_0 \vdots b_2 - a_2 b_0 \vdots b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$

Solution Let us define

$$\frac{Z(s)}{U(s)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}, \qquad \frac{Y(s)}{Z(s)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$$

Then we obtain

$$\dddot{z} + a_1 \ddot{z} + a_2 \dot{z} + a_3 z = u$$
$$b_0 \dddot{z} + b_1 \ddot{z} + b_2 \dot{z} + b_3 z = y$$

Now we define

$$x_1 = z$$
$$x_2 = \dot{x}_1 \tag{5–93}$$
$$x_3 = \dot{x}_2 \tag{5–94}$$

Then, noting that $\dot{x}_3 = \ddot{x}_2 = \dddot{x}_1 = \dddot{z}$, we obtain

$$\dot{x}_3 = -a_3 z - a_2 \dot{z} - a_1 \ddot{z} + u$$

or

$$\dot{x}_3 = -a_3 x_1 - a_2 x_2 - a_1 x_3 + u \tag{5–95}$$

Also,

$$\begin{aligned} y &= b_0 \dddot{z} + b_1 \ddot{z} + b_2 \dot{z} + b_3 z \\ &= b_0 \dot{x}_3 + b_1 x_3 + b_2 x_2 + b_3 x_1 \\ &= b_0[(-a_3 x_1 - a_2 x_2 - a_1 x_3) + u] + b_1 x_3 + b_2 x_2 + b_3 x_1 \\ &= (b_3 - a_3 b_0) x_1 + (b_2 - a_2 b_0) x_2 + (b_1 - a_1 b_0) x_3 + b_0 u \end{aligned} \tag{5–96}$$

From Equations (5–93), (5–94), and (5–95), we obtain

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

which is the state equation. From Equation (5–96), we get

$$y = [b_3 - a_3 b_0 \vdots b_2 - a_2 b_0 \vdots b_1 - a_1 b_0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$

which is the output equation.

Note that the derivation presented here can be easily extended to the general case of an nth-order system.

Problem A–5–14

Consider the mechanical system shown in Figure 5–34. The system is initially at rest. The displacements u, y, and z are measured from their respective rest positions.

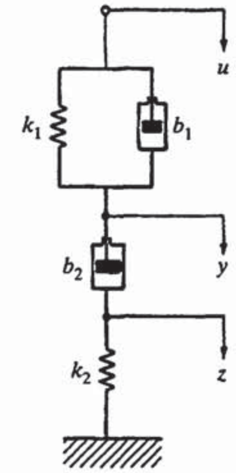

Figure 5–34 Mechanical system.

Assuming that u is the input and y is the output, obtain the transfer function $Y(s)/U(s)$ of the system. Then obtain a state-space representation of the system.

Solution The equations of motion for the system are

$$b_1(\dot{u} - \dot{y}) + k_1(u - y) = b_2(\dot{y} - \dot{z})$$
$$b_2(\dot{y} - \dot{z}) = k_2 z$$

Laplace transforming these two equations, assuming zero initial conditions, we obtain

$$b_1[sU(s) - sY(s)] + k_1[U(s) - Y(s)] = b_2[sY(s) - sZ(s)]$$
$$b_2[sY(s) - sZ(s)] = k_2 Z(s)$$

Eliminating $Z(s)$ from the last two equations yields

$$(b_1 s + k_1)U(s) = \left(b_1 s + k_1 + b_2 s - \frac{b_2{}^2 s^2}{b_2 s + k_2}\right)Y(s)$$

Multiplying both sides of this last equation by $(b_2 s + k_2)$, we get

$$(b_1 s + k_1)(b_2 s + k_2)U(s) = [(b_1 s + k_1)(b_2 s + k_2) + b_2 k_2 s]Y(s)$$

The transfer function of the system then becomes

$$\frac{Y(s)}{U(s)} = \frac{(b_1 s + k_1)(b_2 s + k_2)}{(b_1 s + k_1)(b_2 s + k_2) + b_2 k_2 s}$$
$$= \frac{s^2 + \left(\frac{k_1}{b_1} + \frac{k_2}{b_2}\right)s + \frac{k_1 k_2}{b_1 b_2}}{s^2 + \left(\frac{k_1}{b_1} + \frac{k_2}{b_2} + \frac{k_2}{b_1}\right)s + \frac{k_1 k_2}{b_1 b_2}} \qquad (5\text{–}97)$$

Next, we shall obtain a state-space representation of the system. The differential equation corresponding to Equation (5–97) is

$$\ddot{y} + \left(\frac{k_1}{b_1} + \frac{k_2}{b_2} + \frac{k_2}{b_1}\right)\dot{y} + \frac{k_1 k_2}{b_1 b_2}y = \ddot{u} + \left(\frac{k_1}{b_1} + \frac{k_2}{b_2}\right)\dot{u} + \frac{k_1 k_2}{b_1 b_2}u$$

Comparing this equation with the standard second-order differential equation given by Equation (5–20), namely,

$$\ddot{y} + a_1\dot{y} + a_2 y = b_0\ddot{u} + b_1\dot{u} + b_2 u$$

we find that

$$a_1 = \frac{k_1}{b_1} + \frac{k_2}{b_2} + \frac{k_2}{b_1}, \qquad a_2 = \frac{k_1 k_2}{b_1 b_2}$$

$$b_0 = 1, \qquad b_1 = \frac{k_1}{b_1} + \frac{k_2}{b_2}, \qquad b_2 = \frac{k_1 k_2}{b_1 b_2}$$

From Equations (5–23), (5–24), and (5–29), we have

$$\beta_0 = b_0 = 1$$

$$\beta_1 = b_1 - a_1\beta_0 = -\frac{k_2}{b_1}$$

$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = \frac{k_1 k_2}{b_1^2} + \frac{k_2^2}{b_1 b_2} + \frac{k_2^2}{b_1^2}$$

From Equations (5–21) and (5–22), we define the state variables x_1 and x_2 as

$$x_1 = y - \beta_0 u = y - u$$

$$x_2 = \dot{x}_1 - \beta_1 u = \dot{x}_1 + \frac{k_2}{b_1}u$$

The state equation is given by Equation (5–30) as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_2 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\beta_1\\ \beta_2\end{bmatrix}u$$

or

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\dfrac{k_1 k_2}{b_1 b_2} & -\left(\dfrac{k_1}{b_1} + \dfrac{k_2}{b_2} + \dfrac{k_2}{b_1}\right)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}-\dfrac{k_2}{b_1}\\ \dfrac{k_1 k_2}{b_1^2} + \dfrac{k_2^2}{b_1 b_2} + \dfrac{k_2^2}{b_1^2}\end{bmatrix}u \qquad (5\text{–}98)$$

The output equation is given by Equation (5–31) as

$$y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \beta_0 u$$

or

$$y = [1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix} + u \qquad (5\text{–}99)$$

Equations (5–98) and (5–99) constitute a state-space representation of the system.

Problem A–5–15

Consider the mechanical system shown in Figure 5–35, in which $m = 0.1$ kg, $b = 0.4$ N-s/m, $k_1 = 6$ N/m, and $k_2 = 4$ N/m. The displacements y and z are measured from their respective equilibrium positions. Assume that force u is the input to the system. Considering that displacement y is the output, obtain the transfer function $Y(s)/U(s)$. Also, obtain a state-space representation of the system.

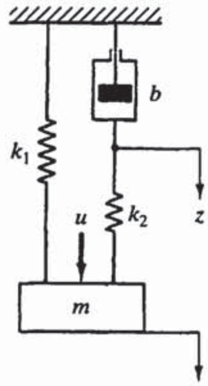

Figure 5–35 Mechanical system.

Solution The equations of motion for the system are

$$m\ddot{y} + k_1 y + k_2(y - z) = u \tag{5–100}$$

$$k_2(y - z) = b\dot{z} \tag{5–101}$$

Taking the Laplace transforms of Equations (5–100) and (5–101), assuming zero initial conditions, we obtain

$$[ms^2 + (k_1 + k_2)]Y(s) = k_2 Z(s) + U(s)$$

$$k_2 Y(s) = (k_2 + bs)Z(s)$$

Eliminating $Z(s)$ from these two equations yields

$$\frac{Y(s)}{U(s)} = \frac{k_2 + bs}{mbs^3 + mk_2 s^2 + (k_1 + k_2)bs + k_1 k_2}$$

$$= \frac{\frac{1}{m}s + \frac{k_2}{mb}}{s^3 + \frac{k_2}{b}s^2 + \frac{k_1 + k_2}{m}s + \frac{k_1 k_2}{mb}}$$

Substituting numerical values for m, b, k_1, and k_2 into this last equation results in

$$\frac{Y(s)}{U(s)} = \frac{10s + 100}{s^3 + 10s^2 + 100s + 600} \tag{5–102}$$

This is the transfer function of the system.

Next, we shall obtain a state-space representation of the system using Method 1 presented in Section 5–4. From Equation (5–102), we obtain

$$\dddot{y} + 10\ddot{y} + 100\dot{y} + 600y = 10\dot{u} + 100u$$

Comparing this equation with the standard third-order differential equation, namely,

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

we find that

$$a_1 = 10, \qquad a_2 = 100, \qquad a_3 = 600$$

$$b_0 = 0, \qquad b_1 = 0, \qquad b_2 = 10, \qquad b_3 = 100$$

Referring to **Problem A-5-12**, define

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{x}_1 - \beta_1 u$$
$$x_3 = \dot{x}_2 - \beta_2 u$$

where

$$\beta_0 = b_0 = 0$$
$$\beta_1 = b_1 - a_1\beta_0 = 0$$
$$\beta_2 = b_2 - a_1\beta_1 - a_2\beta_0 = 10$$

Also, note that

$$\beta_3 = b_3 - a_1\beta_2 - a_2\beta_1 - a_3\beta_0 = 100 - 10 \times 10 = 0$$

Then the state equation for the system becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -600 & -100 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} u \tag{5-103}$$

and the output equation becomes

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5-104}$$

Equations (5-103) and (5-104) give a state-space representation of the system.

Problem A-5-16

Consider the system defined by

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u \tag{5-105}$$

Obtain a state-space representation of the system by the partial-fraction expansion technique.

Solution First, rewrite Equation (5-105) in the form of a transfer function:

$$\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6} = \frac{6}{(s+1)(s+2)(s+3)}$$

Next, expanding this transfer function into partial fractions, we get

$$\frac{Y(s)}{U(s)} = \frac{3}{s+1} + \frac{-6}{s+2} + \frac{3}{s+3}$$

from which we obtain

$$Y(s) = \frac{3}{s+1}U(s) + \frac{-6}{s+2}U(s) + \frac{3}{s+3}U(s) \tag{5-106}$$

Let us define

$$X_1(s) = \frac{3}{s+1}U(s)$$
$$X_2(s) = \frac{-6}{s+2}U(s)$$
$$X_3(s) = \frac{3}{s+3}U(s)$$

Then, rewriting these three equations, we have

$$sX_1(s) = -X_1(s) + 3U(s)$$
$$sX_2(s) = -2X_2(s) - 6U(s)$$
$$sX_3(s) = -3X_3(s) + 3U(s)$$

The inverse Laplace transforms of the last three equations give

$$\dot{x}_1 = -x_1 + 3u \tag{5–107}$$
$$\dot{x}_2 = -2x_2 - 6u \tag{5–108}$$
$$\dot{x}_3 = -3x_3 + 3u \tag{5–109}$$

Since Equation (5–106) can be written as

$$Y(s) = X_1(s) + X_2(s) + X_3(s)$$

we obtain

$$y = x_1 + x_2 + x_3 \tag{5–110}$$

Combining Equations (5–107), (5–108), and (5–109) into a vector–matrix differential equation yields the following state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix} u \tag{5–111}$$

From Equation (5–110), we get the following output equation:

$$y = [1 \quad 1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{5–112}$$

Equations (5–111) and (5–112) constitute a state-space representation of the system given by Equation (5–105). (Note that this representation is the same as that obtained in **Example 5–10.**)

Problem A–5–17

Show that the 2×2 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

has two distinct eigenvalues and that the eigenvectors are linearly independent of each other.

Solution The eigenvalues, obtained from

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)(\lambda - 2) = 0$$

are

$$\lambda_1 = 1 \qquad \text{and} \qquad \lambda_2 = 2$$

Thus, matrix $\mathbf{A}$ has two distinct eigenvalues.

There are two eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ associated with λ_1 and λ_2, respectively. If we define

$$\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}, \qquad \mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix}$$

then the eigenvector $\mathbf{x}_1$ can be found from

$$\mathbf{A}\mathbf{x}_1 = \lambda_1\mathbf{x}_1$$

or

$$(\lambda_1\mathbf{I} - \mathbf{A})\mathbf{x}_1 = \mathbf{0}$$

Noting that $\lambda_1 = 1$, we have

$$\begin{bmatrix} 1-1 & -1 \\ 0 & 1-2 \end{bmatrix}\begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which gives

$$x_{11} = \text{arbitrary constant} \qquad \text{and} \qquad x_{21} = 0$$

Hence, eigenvector $\mathbf{x}_1$ may be written as

$$\mathbf{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix}$$

where $c_1 \neq 0$ is an arbitrary constant.

Similarly, for the eigenvector $\mathbf{x}_2$, we have

$$\mathbf{A}\mathbf{x}_2 = \lambda_2\mathbf{x}_2$$

or

$$(\lambda_2\mathbf{I} - \mathbf{A})\mathbf{x}_2 = \mathbf{0}$$

Noting that $\lambda_2 = 2$, we obtain

$$\begin{bmatrix} 2-1 & -1 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

from which we get

$$x_{12} - x_{22} = 0$$

Thus, the eigenvector associated with $\lambda_2 = 2$ may be selected as

$$\mathbf{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} = \begin{bmatrix} c_2 \\ c_2 \end{bmatrix}$$

where $c_2 \neq 0$ is an arbitrary constant.

The two eigenvectors are therefore given by

$$\mathbf{x}_1 = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x}_2 = \begin{bmatrix} c_2 \\ c_2 \end{bmatrix}$$

That eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent can be seen from the fact that the determinant of the matrix $[\mathbf{x}_1\,\mathbf{x}_2]$ is nonzero:

$$\begin{vmatrix} c_1 & c_2 \\ 0 & c_2 \end{vmatrix} \neq 0$$

Problem A–5–18

Obtain the eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Assume that the eigenvalues are λ_1, λ_2, and λ_3; that is,

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ a_3 & a_2 & \lambda + a_1 \end{vmatrix}$$
$$= \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$$
$$= (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)$$

Assume also that λ_1, λ_2, and λ_3 are distinct.

Solution The eigenvector $\mathbf{x}_i$ associated with an eigenvalue λ_i is a vector that satisfies the equation

$$\mathbf{A}\mathbf{x}_i = \lambda_i\mathbf{x}_i \tag{5–113}$$

which can be written as

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} = \lambda_i\begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix}$$

Simplifying this last equation, we obtain

$$x_{i2} = \lambda_i x_{i1}$$
$$x_{i3} = \lambda_i x_{i2}$$
$$-a_3x_{i1} - a_2x_{i2} - a_1x_{i3} = \lambda_i x_{i3}$$

Thus,

$$\begin{bmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \end{bmatrix} = \begin{bmatrix} x_{i1} \\ \lambda_i x_{i1} \\ \lambda_i^2 x_{i1} \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_i \\ \lambda_i^2 \end{bmatrix} x_{i1}$$

Hence, the eigenvectors are

$$\begin{bmatrix} x_{11} \\ \lambda_1 x_{11} \\ \lambda_1^2 x_{11} \end{bmatrix}, \quad \begin{bmatrix} x_{21} \\ \lambda_2 x_{21} \\ \lambda_2^2 x_{21} \end{bmatrix}, \quad \begin{bmatrix} x_{31} \\ \lambda_3 x_{31} \\ \lambda_3^2 x_{31} \end{bmatrix} \tag{5–114}$$

Note that if $\mathbf{x}_i$ is an eigenvector, then $a\mathbf{x}_i$ (where a = scalar $\neq 0$) is also an eigenvector, because Equation (5–113) can be written as

$$a(\mathbf{A}\mathbf{x}_i) = a(\lambda_i\mathbf{x}_i)$$

or

$$\mathbf{A}(a\mathbf{x}_i) = \lambda_i(a\mathbf{x}_i)$$

Thus, by dividing the eigenvectors given by (5–114) by x_{11}, x_{21}, and x_{31}, respectively, we obtain

$$\begin{bmatrix} 1 \\ \lambda_1 \\ \lambda_1^2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \lambda_2 \\ \lambda_2^2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \lambda_3 \\ \lambda_3^2 \end{bmatrix}$$

These are also a set of eigenvectors.

Problem A–5–19

Consider a matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Assume that λ_1, λ_2, and λ_3 are distinct eigenvalues of matrix **A**.

Show that if a transformation matrix **P** is defined by

$$\mathbf{P} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Solution First note that

$$\mathbf{AP} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ -a_3 - a_2\lambda_1 - a_1\lambda_1^2 & -a_3 - a_2\lambda_2 - a_1\lambda_2^2 & -a_3 - a_2\lambda_3 - a_1\lambda_3^2 \end{bmatrix} \tag{5–115}$$

Since λ_1, λ_2, and λ_3 are eigenvalues, they satisfy the characteristic equation, or

$$\lambda_i^3 + a_1\lambda_i^2 + a_2\lambda_i + a_3 = 0$$

Thus,

$$\lambda_i^3 = -a_3 - a_2\lambda_i - a_1\lambda_i^2$$

Hence,

$$\begin{aligned} -a_3 - a_2\lambda_1 - a_1\lambda_1^2 &= \lambda_1^3 \\ -a_3 - a_2\lambda_2 - a_1\lambda_2^2 &= \lambda_2^3 \\ -a_3 - a_2\lambda_3 - a_1\lambda_3^2 &= \lambda_3^3 \end{aligned}$$

Consequently, Equation (5–115) can be written as

$$\mathbf{AP} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{bmatrix} \tag{5–116}$$

Next, define

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Then

$$\mathbf{PD} = \begin{bmatrix} 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \\ \lambda_1^3 & \lambda_2^3 & \lambda_3^3 \end{bmatrix} \tag{5–117}$$

Comparing Equations (5–116) and (5–117), we have

$$\mathbf{AP} = \mathbf{PD}$$

Thus, we have shown that

$$\mathbf{P}^{-1}\mathbf{AP} = \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Problem A–5–20

Prove that the eigenvalues of a square matrix $\mathbf{A}$ are invariant under a linear transformation.

Solution To prove the invariance of the eigenvalues under a linear transformation, we must show that the characteristic polynomials $|\lambda\mathbf{I} - \mathbf{A}|$ and $|\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}|$ are identical.

Since the determinant of a product is the product of the determinants, we obtain

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| &= |\lambda\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{AP}| \\ &= |\mathbf{P}^{-1}(\lambda\mathbf{I} - \mathbf{A})\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| \\ &= |\mathbf{P}^{-1}||\mathbf{P}||\lambda\mathbf{I} - \mathbf{A}| \end{aligned}$$

Noting that the product of the determinants $|\mathbf{P}^{-1}|$ and $|\mathbf{P}|$ is the determinant of the product $|\mathbf{P}^{-1}\mathbf{P}|$, we obtain

$$\begin{aligned} |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| &= |\mathbf{P}^{-1}\mathbf{P}||\lambda\mathbf{I} - \mathbf{A}| \\ &= |\lambda\mathbf{I} - \mathbf{A}| \end{aligned}$$

Thus, we have proven that the eigenvalues of $\mathbf{A}$ are invariant under a linear transformation.

PROBLEMS

Problem B–5–1

Obtain state-space representations of the mechanical systems shown in Figures 5–36(a) and (b).

Problem B–5–2

For the spring–mass–pulley system of Figure 5–37, the moment of inertia of the pulley about the axis of rotation is J and the radius is R. Assume that the system is initially in equilibrium. The gravitational force of mass m causes a static deflection of the spring such that $k\delta = mg$. Assuming that the displacement y of mass m is measured from the equilibrium position, obtain a state-space representation of the system. The external force u applied to mass m is the input and the displacement y is the output of the system.