Euler–Bournoulli Beam Theory

Figure 6.10 illustrates a cantilevered beam with the transverse direction of vibration indicated [i.e., the deflection, $w(x, t)$, is in the y direction]. The beam is of rectangular cross section $A(x)$ with width h_y, thickness h_z, and length l. Also associated with the beam is a flexural (bending) stiffness $EI(x)$, where E is Young's elastic modulus for the beam and $I(x)$ is the cross-sectional area moment of inertia about the "z axis." From mechanics of materials, the beam sustains a bending moment $M(x, t)$, which is related to the beam deflection, or bending deformation, $w(x, t)$, by

$$M(x, t) = EI(x)\frac{\partial^2 w(x, t)}{\partial x^2} \tag{6.85}$$

A model of bending vibration may be derived from examining the force diagram of an infinitesimal element of the beam as indicated in Figure 6.10. Assuming the deformation to be small enough such that the shear deformation is much smaller than

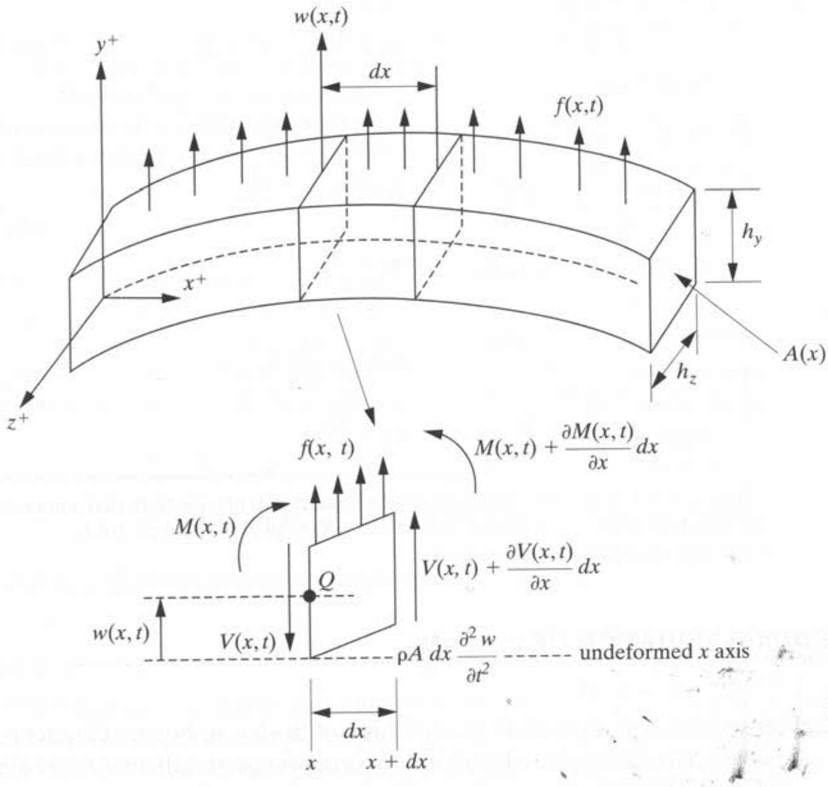


Figure 6.10 Simple Euler–Bernoulli beam of length (l) in transverse vibration and a free-body diagram of a small element of the beam as it is deformed by a distributed force per unit length, denoted $f(x, t)$.

$w(x, t)$ (i.e., so that the sides of the element dx do not bend), a summation of forces in the y direction yields

$$\left(V(x,t) + \frac{\partial V(x,t)}{\partial x}\,dx\right) - V(x,t) + f(x,t)\,dx = \rho A(x)\,dx\,\frac{\partial^2 w(x,t)}{\partial t^2} \tag{6.86}$$

Here $V(x, t)$ is the shear force at the left end of the element dx, $V(x, t) + V_x(x, t)\,dx$ is the shear force at the right end of the element dx, $f(x, t)$ is the total external force applied to the element per unit length, and the term on the right side of the equality is the inertial force of the element. The assumption of small shear deformation used in the force balance of equation (6.86) is true if $l/h_z \geq 10$ and $l/h_y \geq 10$ (i.e., for long slender beams or Euler–Bernoulli beams).

Next the moments acting on the element dx about the z axis through point Q are summed. This yields

$$\left[M(x,t) + \frac{\partial M(x,t)}{\partial x}\,dx\right] - M(x,t) + \left[V(x,t) + \frac{\partial V(x,t)}{\partial x}\,dx\right]dx + [f(x,t)\,dx]\,\frac{dx}{2} = 0 \tag{6.87}$$

Here the left-hand side of the equation is zero since it is also assumed that the rotary inertia of the element dx is negligible. Simplifying this expression yields

$$\left[\frac{\partial M(x,t)}{\partial x} + V(x,t)\right]dx + \left[\frac{\partial V(x,t)}{\partial x} + \frac{f(x,t)}{2}\right](dx)^2 = 0 \tag{6.88}$$

Since dx is assumed to be very small, $(dx)^2$ is assumed to be almost zero, so that this moment expression yields (dx is small, but not zero)

$$V(x,t) = -\frac{\partial M(x,t)}{\partial x} \tag{6.89}$$

This states that the shear force is proportional to the spatial change in the bending moment. Substitution of this expression for the shear force into equation (6.86) yields

$$-\frac{\partial^2}{\partial x^2}[M(x,t)]dx + f(x,t)dx = \rho A(x)dx\,\frac{\partial^2 w(x,t)}{\partial t^2} \tag{6.90}$$

Further substitution of equation (6.85) into (6.90) and dividing by dx yields

$$\rho A(x)\,\frac{\partial^2 w(x,t)}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\,\frac{\partial^2 w(x,t)}{\partial x^2}\right] = f(x,t) \tag{6.91}$$

If no external force is applied so that $f(x, t) = 0$ and if $EI(x)$ and $A(x)$ are assumed to be constant, equation (6.91) simplifies so that free vibration is governed by

$$\frac{\partial^2 w(x,t)}{\partial t^2} + c^2\,\frac{\partial^4 w(x,t)}{\partial x^4} = 0 \quad c = \sqrt{\frac{EI}{\rho A}} \tag{6.92}$$

Note that unlike the previous equations, the free vibration equation (6.92) contains four spatial derivatives and hence requires four (instead of two) boundary conditions in calculating a solution. The presence of the two time derivatives again requires that two initial conditions, one for the displacement and one for the velocity, be specified.

The boundary conditions required to solve the spatial equation in a separation-of-variables solution of equation (6.92) are obtained by examining the deflection $w(x, t)$, the slope of the deflection $\partial w(x, t)/\partial x$, the bending moment $EI\partial^2 w(x, t)/\partial x^2$, and the shear force $\partial[EI\partial^2 w(x, t)/\partial x^2]/\partial x$ at each end of the beam. A common configuration is *clamped–free* or *cantilevered* as illustrated in Figure 6.10. In addition to a boundary being clamped or free, the end of a beam could be resting on a support restrained from bending or deflecting. The situation is called *simply supported* or *pinned*. A *sliding* boundary is one in which displacement is allowed but rotation is not. The shear load at a sliding boundary is zero.

If a beam in transverse vibration is free at one end, the deflection and slope at that end are unrestricted, but the bending moment and shear force must vanish:

$$\begin{aligned} \text{bending moment} &= EI\frac{\partial^2 w}{\partial x^2} = 0 \\ \text{shear force} &= \frac{-\partial}{\partial x}\left[EI\frac{\partial^2 w}{\partial x^2}\right] = 0 \end{aligned} \tag{6.93}$$

If, on the other hand, the end of a beam is clamped (or fixed), the bending moment and shear force are unrestricted, but the deflection and slope must vanish at that end:

$$\begin{aligned} \text{deflection} &= w = 0 \\ \text{slope} &= \frac{\partial w}{\partial x} = 0 \end{aligned} \tag{6.94}$$

At a simply supported or pinned end, the slope and shear force are unrestricted and the deflection and bending moment must vanish:

$$\begin{aligned} \text{deflection} &= w = 0 \\ \text{bending moment} &= EI\frac{\partial^2 w}{\partial x^2} = 0 \end{aligned} \tag{6.95}$$

At a sliding end, the slope or rotation is zero and no shear force is allowed. On the other hand, the deflection and bending moment are unrestricted. Hence, at a sliding boundary,

$$\begin{aligned} \text{slope} &= \frac{\partial w}{\partial x} = 0 \\ \text{shear force} &= \frac{-\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = 0 \end{aligned} \tag{6.96}$$

Other boundary conditions are possible by connecting the ends of a beam to a variety of devices such as lumped masses, springs, and so on. These boundary conditions can be determined by force and moment balances.

In addition to satisfying four boundary conditions, the solution of equation (6.92) for free vibration can be calculated only if two initial conditions (in time) are specified. As in the case of the rod, string, and bar, these initial conditions are specified initial deflection and velocity profiles:

$$w(x, 0) = w_0(x) \quad \text{and} \quad w_t(x, 0) = \dot{w}_0(x)$$

assuming that $t = 0$ is the initial time. Note that w_0 and $\dot{w}_0$ cannot both be zero, or no motion will result.

The solution of equation (6.92) subject to four boundary conditions and two initial conditions proceeds following exactly the same steps used in previous sections. A separation-of-variables solution of the form $w(x,t) = X(x)T(t)$ is assumed. This is substituted into the equation of motion, equation (6.92), to yield (after rearrangement)

$$c^2 \frac{X''''(x)}{X(x)} = -\frac{\ddot{T}(t)}{T(t)} = \omega^2 \tag{6.97}$$

where the partial derivatives have been replaced with total derivatives as before (*note:* $X'''' = d^4X/dx^4, \ddot{T} = d^2T/dt^2$). Here the choice of separation constant, ω^2, is made, based on experience with the systems of Section 6.4, that the natural frequency comes from the temporal equation:

$$\ddot{T}(t) + \omega^2 T(t) = 0 \tag{6.98}$$

which is the right side of equation (6.97). This temporal equation has a solution of the form

$$T(t) = A \sin \omega t + B \cos \omega t \tag{6.99}$$

where the constants A and B will eventually be determined by the specified initial conditions after being combined with the spatial solution.

The spatial equation comes from rearranging equation (6.97), which yields

$$X''''(x) - \left(\frac{\omega}{c}\right)^2 X(x) = 0 \tag{6.100}$$

By defining [recall equation (6.92)]

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI} \tag{6.101}$$

and assuming a solution to equation (6.100) of the form $Ae^{\sigma x}$, the general solution of equation (6.100) can be calculated to be of the form (see Problem 6.42)

$$X(x) = a_1 \sin \beta x + a_2 \cos \beta x + a_3 \sinh \beta x + a_4 \cosh \beta x \tag{6.102}$$

Here the value for β and three of the four constants of integration a_1, a_2, a_3, and a_4 will be determined from the four boundary conditions. The fourth constant becomes combined with the constants A and B from the temporal equation, which are then determined from the initial conditions. The following example illustrates the solution procedure for a beam fixed at one end and simply supported at the other end.

Example 6.5.1

Calculate the natural frequencies and mode shapes for the transverse vibration of a beam of length l that is fixed at one end and pinned at the other end.

Solution The boundary conditions in this case are given by equation (6.94) at the fixed end and equation (6.95) at the pinned end. Substitution of the general solution given by equation (6.102) into equation (6.94) at $x = 0$ yields

$$X(0) = 0 \Rightarrow a_2 + a_4 = 0 \tag{a}$$

$$X'(0) = 0 \Rightarrow \beta(a_1 + a_3) = 0 \tag{b}$$

Similarly, at $x = l$ the boundary conditions result in

$$X(l) = 0 \Rightarrow a_1 \sin\beta l + a_2 \cos\beta l + a_3 \sinh\beta l + a_4 \cosh\beta l = 0 \tag{c}$$

$$EIX''(l) = 0 \Rightarrow \beta^2(-a_1 \sin\beta l - a_2 \cos\beta l + a_3 \sinh\beta l + a_4 \cosh\beta l) = 0 \tag{d}$$

These four boundary conditions thus yield four equations [(a) through (d)] in the four unknown coefficients a_1, a_2, a_3, and a_4. These can be written as the single vector equation

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ \beta & 0 & \beta & 0 \\ \sin\beta l & \cos\beta l & \sinh\beta l & \cosh\beta l \\ -\beta^2 \sin\beta l & -\beta^2 \cos\beta l & \beta^2 \sinh\beta l & \beta^2 \cosh\beta l \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Recall from Chapter 4 that this vector equation can have a nonzero solution for the vector $\mathbf{a} = [a_1 \quad a_2 \quad a_3 \quad a_4]^T$ only if the determinant of the coefficient matrix vanishes (i.e., if the coefficient matrix is singular). Furthermore, recall that since the coefficient matrix is singular, not all of the elements of the vector $\mathbf{a}$ can be calculated.

Setting the determinant above equal to zero yields the characteristic equation

$$\tan\beta l = \tanh\beta l$$

This equality is satisfied for an infinite number of choices for β, denoted β_n. The solution can be visualized by plotting both $\tan\beta l$ and $\tanh\beta l$ versus βl on the same plot. This is similar to the solution technique used in Example 6.4.1 and illustrated in Figure 6.9. The first five solutions are

$$\beta_1 l = 3.926602 \qquad \beta_2 l = 7.068583 \qquad \beta_3 l = 10.210176$$

$$\beta_4 l = 13.351768 \qquad \beta_5 l = 16.49336143$$

For the rest of the modes (i.e., for values of the index $n > 5$), the solutions to the characteristic equation are well approximated by

$$\beta_n l = \frac{(4n + 1)\pi}{4}$$

With these values of the weighted frequencies $\beta_n l$, the individual modes of vibration can be calculated. Solving the preceding matrix equation for the individual coefficients a_i yields $a_1 = -a_3, a_2 = -a_4$, and

$$(\sinh\beta_n l - \sin\beta_n l)a_3 + (\cosh\beta_n l - \cos\beta_n l)a_4 = 0$$

Thus

$$a_3 = -\frac{\cosh \beta_n l - \cos \beta_n l}{\sinh \beta_n l - \sin \beta_n l} a_4$$

for each n. The fourth coefficient a_4 cannot be determined by this set of equations, because the coefficient matrix is singular (otherwise, each a_i would be zero). This remaining coefficient becomes the arbitrary magnitude of the eigenfunction. As this constant depends on n, denote it by $(a_4)_n$. Substitution of these values of a_i in the expression $X(x)$ for the spatial solution yields the result that the eigenfunctions or mode shapes have the form

$$X_n(x) = (a_4)_n \left[\frac{\cosh \beta_n l - \cos \beta_n l}{\sinh \beta_n l - \sin \beta_n l} (\sinh \beta_n x - \sin \beta_n x) - \cosh \beta_n x + \cos \beta_n x \right],$$
$$n = 1, 2, 3, \ldots$$

The first three mode shapes are plotted in Figure 6.11 for $(a_2)_n = 1$ and $n = 1, 2, 3$.

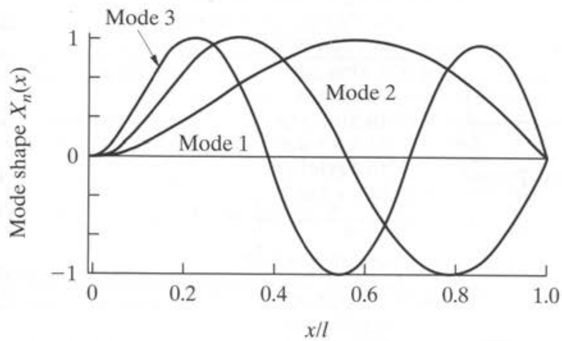


Figure 6.11 Plot of the first three mode shapes of the clamped–pinned beam of Example 6.5.1, arbitrarily normalized to unity.

These mode shapes can be shown to be orthogonal, so that

$$\int_0^l X_n(x) X_m(x) dx = 0$$

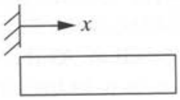
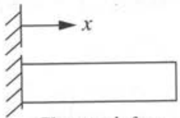
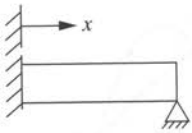
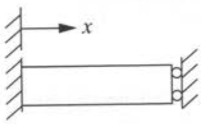
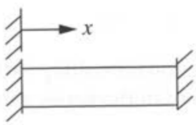
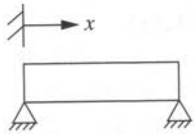
for $n \neq m$ (see Exercise 6.45). As in Example 6.3.2, this orthogonality, along with the initial conditions, can be used to calculate the constants A_n and B_n in the series solution for the displacement

$$w(x, t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) X_n(x)$$

□

Table 6.6 summarizes a number of different boundary configurations for the slender beam. The slender beam model given in equation (6.91) is often referred to

TABLE 6.6 SAMPLE OF VARIOUS BOUNDARY CONFIGURATIONS OF A SLENDER BEAM IN TRANSVERSE VIBRATION OF LENGTH l ILLUSTRATING WEIGHTED NATURAL FREQUENCIES AND MODE SHAPES[a]

Configuration	Weighted frequencies $\beta_n l$ and characteristic equation	Mode shape	σ_n
 Free–free	0 (rigid-body mode) 4.73004074 7.85320462 10.9956078 14.1371655 17.2787597 $\frac{(2n+1)\pi}{2}$ for $n > 5$ $\cos \beta l \cosh \beta l = 1$	$\cosh \beta_n x + \cos \beta_n x$ $-\sigma_n (\sinh \beta_n x + \sin \beta_n x)^b$	 0.9825 1.0008 0.9999 1.0000 0.9999 1 for $n > 5$
 Clamped–free	1.87510407 4.69409113 7.85475744 10.99554073 14.13716839 $\frac{(2n-1)\pi}{2}$ for $n > 5$ $\cos \beta l \cosh \beta l = -1$	$\cosh \beta_n x - \cos \beta_n x$ $-\sigma_n (\sinh \beta_n x - \sin \beta_n x)$	0.7341 1.0185 0.9992 1.0000 1.0000 1 for $n > 5$
 Clamped–pinned	3.92660231 7.06858275 10.21017612 13.35176878 16.49336143 $\frac{(4n+1)\pi}{4}$ for $n > 5$ $\tan \beta l = \tanh \beta l$	$\cosh \beta_n x - \cos \beta_n x$ $-\sigma_n (\sinh \beta_n x - \sin \beta_n x)$	1.0008 1 for $n > 1$
 Clamped–sliding	2.36502037 5.49780392 8.63937983 11.78097245 14.92256510 $\frac{(4n-1)\pi}{4}$ for $n > 5$ $\tan \beta l + \tanh \beta l = 0$	$\cosh \beta_n x - \cos \beta_n x$ $-\sigma_n (\sinh \beta_n x - \sin \beta_n x)$	0.9825 1 for $n > 1$
 Clamped–clamped	4.73004074 7.85320462 10.9956079 14.1371655 17.2787597 $\frac{(2n+1)\pi}{2}$ for $n > 5$ $\cos \beta l \cosh \beta l = 1$	$\cosh \beta_n x - \cos \beta_n x$ $-\sigma_n (\sinh \beta_n x - \sin \beta_n x)$	0.982502 1.00078 0.999966 1.0000 1.0000 1 for $n > 5$
 Pinned–pinned	$n\pi$ $\sin \beta l = 0$	$\sin \frac{n\pi x}{l}$	none

[a]Here the weighted natural frequencies $\beta_n l$ are related to the natural frequencies by equation (6.101) or $\omega_n = \beta_n^2 \sqrt{EI/\rho A}$, as used in Example 6.5.1. The values of σ_i for the mode shapes are computed from the formulas given in Table 6.5.

[b]There are two free–free mode shapes: $X_0 =$ constant and $X_0 = A(x - l/2)$; the first is translational, the second rotational.

as the *Euler–Bernoulli* or Bernoulli–Euler beam equation. The assumptions used in formulating this model are that the beam be

- Uniform along its span, or length, and slender
- Composed of a linear, homogeneous, isotropic elastic material without axial loads
- Such that plane sections remain plane
- Such that the plane of symmetry of the beam is also the plane of vibration so that rotation and translation are decoupled
- Such that rotary inertia and shear deformation can be neglected

The key to solving for the time response of distributed-parameter systems is the orthogonality of the mode shapes. Note from Table 6.7 that the mode shapes are quite complicated in many configurations. This does not mean that orthogonality is necessarily violated, just that evaluating the integrals in the modal analysis procedure becomes more difficult.

TABLE 6.7 EQUATIONS FOR THE MODE SHAPE COEFFICIENTS σ_n FOR USE IN TABLE 6.4[a]

Boundary condition	Formula for σ_n
Free–free	$\sigma_n = \dfrac{\cosh \beta_n l - \cos \beta_n l}{\sinh \beta_n l - \sin \beta_n l}$
Clamped–free	$\sigma_n = \dfrac{\sinh \beta_n l - \sin \beta_n l}{\cosh \beta_n l + \cos \beta_n l}$
Clamped–pinned	$\sigma_n = \dfrac{\cosh \beta_n l - \cos \beta_n l}{\sinh \beta_n l - \sin \beta_n l}$
Clamped–sliding	$\sigma_n = \dfrac{\sinh \beta_n l - \sin \beta_n l}{\cosh \beta_n l + \cos \beta_n l}$
Clamped–clamped	Same as free–free

[a]These coefficients are used in the calculations for the mode shapes, as illustrated in Example 6.5.1.

Timoshenko Beam Theory

The model of the transverse vibration of the beam presented in equation (6.91) ignores the effects of shear deformation and rotary inertia. Beams modeled including the effects of rotary inertia and shear deformation are called *Timoshenko beams*. These effects are considered next. As mentioned previously, it is safe to ignore the shear deformation as long as the h_z and h_y illustrated in Figure 6.10 are small compared with the length of the beam. As the beam becomes shorter, the effect of shear deformation becomes evident. This is illustrated in Figure 6.12, which is a repeat of the element dx of Figure 6.10 with shear deformation included.

Referring to the figure, the line OA is a line through the center of the element dx perpendicular to the face at the right side. The line OB, on the other hand, is the