MAE6292: Homework 4

Due date: April 16, 2015

Problem 1 In this problem, we illustrate the use of Bayes' theorem with a simple example. You have an electronic device, such as a cell phone or a notebook. Let F be the events that your device fails. Suppose that it may fail because of one of the following three causes, overheating, wrong components, and being dropped. Those events are denoted by C_1, C_2, and C_3, respectively. Assume that there is no chance that the device fails due to more than one cases, i.e., the causes of failure are mutually exclusive. The probability of each causes, and the probability of failure of each causes are summarized as follows.

$$P[C_1] = 0.05, \quad P[C_2] = 0.01, \quad P[C_3] = 0.1,$$
$$P[F|C_1] = 0.1, \quad P[F|C_2] = 0.5, \quad P[F|C_3] = 0.7.$$

(a) Find the probability of failure from any cause, i.e., $P[F]$.

(b) Suppose your device fails. Find the probability that the failure is from specific causes, i.e., $P[C_1|F]$, $P[C_2|F]$, and $P[C_3|F]$. What is the most probable cause of failure between C_1, C_2, or C_3.

Problem 2 The joint probability density of X and Y are given by

$$p_{XY}(x, y) = \begin{cases} \dfrac{1}{\pi} & \text{if } x^2 + y^2 \leq 1, \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the marginal probability densities $p_X(x)$, $p_Y(y)$.

(b) Find the conditional probability density $p_{X|Y}(x|y)$.

(c) Find the conditional probability density $p_{Y|X}(y|x)$ from Baye's rule and the above results.

(d) Are X and Y independent random variables?

(e) Find the probability of $P[0 \leq X \leq \frac{1}{2}, 0 \leq Y]$.

Problem 3 The random variable X follows the Gaussian distribution. Its probability density is given by

$$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}.$$

(a) Show that $E[X] = \mu$ from the definition of the mean, i.e., show

$$\mu = \int_{-\infty}^{\infty} x p_X(x) dx.$$

(Hint: use the substitution $z = \frac{1}{\sigma}(x - \mu)$.)

(b) Show that $\mathrm{Var}[X] = \sigma^2$ from the definition of the variance, i.e., show

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p_X(x) dx.$$

Problem 4 Consider the determination of the location of a point via multiple cameras, studied at the first midterm exam question. Let $x \in \mathbb{R}^3$ be the location of the point, and let the image of the point on the i-th camera be located at $z_i \in \mathbb{R}^2$ at the image plane of the camera. They are related as

$$z_i = h_i(x) + v_i = \frac{1}{c_i^T x + d_i}(A_i x + b_i) + v_i,$$

where $A_i \in \mathbb{R}^{2\times 3}$, $b_i \in \mathbb{R}^{2\times 1}$, $c_i \in \mathbb{R}^{3\times 1}$, and $d_i \in \mathbb{R}$ are sub-matrices of the *camera matrix*, $C_i \in \mathbb{R}^{3\times 4}$ defining the location, orientation and properties of the i-th camera,

$$C_i = \begin{bmatrix} A_i & b_i \\ c_i^T & d_i \end{bmatrix}.$$

The measurement noise is denoted by $v_i \in \mathbb{R}^2$, which follows $v_i \sim \mathcal{N}(0_{2\times 1}, R_i)$. (Note that the notation has been revised to be compatible with the classroom notation for parameter estimation).

There are four cameras. The combined camera matrix $C \in \mathbb{R}^{3\times 4\times 4}$, the measurement vector $Z = [z_1; z_2; z_3; z_4] \in \mathbb{R}^8$, and the covariance matrix $R = \text{diag}[R_1, R_2, R_3, R_4] \in \mathbb{R}^{8\times 8}$ are stored at the Matlab file `prob4_data.mat`.

(a) Find the optimal estimate x, and the error covariance matrix P.

(b) Determine the direction that the estimate is most uncertain, i.e., the semi-major axis of the Gaussian density ellipsoid.
(Hint: Geometry of an ellipsoid is available at google, such as `http://en.wikipedia.org/wiki/Ellipsoid`)

(c) Determine the direction that the estimate is least uncertain, i.e., the semi-minor axis of the Gaussian density ellipsoid.

(d) A marginal distribution over a subset of multivariate Gaussian random variables is also Gaussian, and it is obtained by dropping the irrelevant variables (the variables that one wants to marginalize out) from the mean vector and the covariance matrix. For example, the covariance matrix of the first two components of the position x is simply the first 2×2 block of P. Using this fact and the Matlab function `plot_gaussian_ellipsoid`, draw the 86% Gaussian ellipsoid for the first two components of x.

(e) Draw the 86% Gaussian ellipsoid for the last two components of x.