4

Transfer-Function Approach to Modeling Dynamic Systems

4–1 INTRODUCTION

In this chapter, we present the transfer-function approach to modeling and analyzing dynamic systems. We first define the transfer function and then introduce block diagrams. Since MATLAB plays an important role in obtaining computational solutions of transient response problems, we present a detailed introduction to writing MATLAB programs to obtain response curves for time-domain inputs such as the step, impulse, ramp, and others.

In the field of system dynamics, transfer functions are frequently used to characterize the input–output relationships of components or systems that can be described by linear, time-invariant differential equations. We begin this section by defining the transfer function and deriving the transfer function of a mechanical system. Then we discuss the impulse response function, or the weighting function, of the system.

Transfer Function. The *transfer function* of a linear, time-invariant differential-equation system is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the differential equation

$$a_0 \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y$$

$$= b_0 \overset{(m)}{x} + b_1 \overset{(m-1)}{x} + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

where y is the output of the system and x is the input. The transfer function of this system is the ratio of the Laplace-transformed output to the Laplace-transformed input when all initial conditions are zero, or

$$\text{Transfer function} = G(s) = \left.\frac{\mathscr{L}[\text{output}]}{\mathscr{L}[\text{input}]}\right|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n} \qquad (4\text{–}1)$$

By using the concept of a transfer function, it is possible to represent system dynamics by algebraic equations in s. If the highest power of s in the denominator of the transfer function is equal to n, the system is called an *nth-order system.*

Comments on the Transfer Function. The applicability of the concept of the transfer function is limited to linear, time-invariant differential-equation systems. Still, the transfer-function approach is used extensively in the analysis and design of such systems. The following list gives some important comments concerning the transfer function of a system described by a linear, time-invariant differential equation:

1. The transfer function of a system is a mathematical model of that system, in that it is an operational method of expressing the differential equation that relates the output variable to the input variable.
2. The transfer function is a property of a system itself, unrelated to the magnitude and nature of the input or driving function.
3. The transfer function includes the units necessary to relate the input to the output; however, it does not provide any information concerning the physical structure of the system. (The transfer functions of many physically different systems can be identical.)
4. If the transfer function of a system is known, the output or response can be studied for various forms of inputs with a view toward understanding the nature of the system.
5. If the transfer function of a system is unknown, it may be established experimentally by introducing known inputs and studying the output of the system. Once established, a transfer function gives a full description of the dynamic characteristics of the system, as distinct from its physical description.

Example 4–1

Consider the mechanical system shown in Figure 4–1. The displacement x of the mass m is measured from the equilibrium position. In this system, the external force $f(t)$ is the input and x is the output.

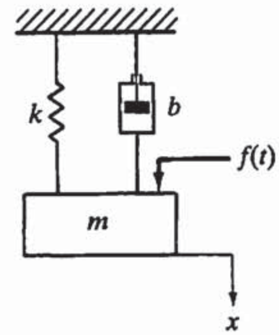

Figure 4–1 Mechanical system.

The equation of motion for the system is

$$m\ddot{x} + b\dot{x} + kx = f(t)$$

Taking the Laplace transform of both sides of this equation and assuming that all initial conditions are zero yields

$$(ms^2 + bs + k)X(s) = F(s)$$

where $X(s) = \mathscr{L}[x(t)]$ and $F(s) = \mathscr{L}[f(t)]$. From Equation (4–1), the transfer function for the system is

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + bs + k}$$

Impulse-Response Function. The transfer function of a linear, time-invariant system is

$$G(s) = \frac{Y(s)}{X(s)}$$

where $X(s)$ is the Laplace transform of the input and $Y(s)$ is the Laplace transform of the output and where we assume that all initial conditions involved are zero. It follows that the output $Y(s)$ can be written as the product of $G(s)$ and $X(s)$, or

$$Y(s) = G(s)X(s) \tag{4–2}$$

Now, consider the output (response) of the system to a unit-impulse input when the initial conditions are zero. Since the Laplace transform of the unit-impulse function is unity, or $X(s) = 1$, the Laplace transform of the output of the system is

$$Y(s) = G(s) \tag{4–3}$$

The inverse Laplace transform of the output given by Equation (4–3) yields the impulse response of the system. The inverse Laplace transform of $G(s)$, or

$$\mathscr{L}^{-1}[G(s)] = g(t)$$

is called the *impulse-response function*, or the *weighting function*, of the system.

The impulse-response function $g(t)$ is thus the response of a linear system to a unit-impulse input when the initial conditions are zero. The Laplace transform of $g(t)$ gives the transfer function. Therefore, the transfer function and impulse-response function of a linear, time-invariant system contain the same information about the system dynamics. It is hence possible to obtain complete information about the dynamic characteristics of a system by exciting it with an impulse input and measuring the response. (In practice, a large pulse input with a very short duration compared with the significant time constants of the system may be considered an impulse.)

Outline of the Chapter. Section 4–1 has presented the concept of the transfer function and impulse-response function. Section 4–2 discusses the block diagram. Section 4–3 sets forth the MATLAB approach to the partial-fraction expansion of a ratio of two polynomials, $B(s)/A(s)$. Section 4–4 details the MATLAB approach to the transient response analysis of transfer-function systems.

4–2 BLOCK DIAGRAMS

Block diagrams of dynamic systems. A block diagram of a dynamic system is a pictorial representation of the functions performed by each component of the system and of the flow of signals within the system. Such a diagram depicts the interrelationships that exist among the various components. Differing from a purely abstract mathematical representation, a block diagram has the advantage of indicating the signal flows of the actual system more realistically.

In a block diagram, all system variables are linked to each other through functional blocks. The *functional block*, or simply *block*, is a symbol for the mathematical operation on the input signal to the block that produces the output. The transfer functions of the components are usually entered in the corresponding blocks, which are connected by arrows to indicate the direction of the flow of signals. Note that a signal can pass only in the direction of the arrows. Thus, a block diagram of a dynamic system explicitly shows a unilateral property.

Figure 4–2 shows an element of a block diagram. The arrowhead pointing toward the block indicates the input to the block, and the arrowhead leading away from the block represents the output of the block. As mentioned, such arrows represent signals.

Note that the dimension of the output signal from a block is the dimension of the input signal multiplied by the dimension of the transfer function in the block.

The advantages of the block diagram representation of a system lie in the fact that it is easy to form the overall block diagram for the entire system merely by connecting the blocks of the components according to the signal flow and that it is possible to evaluate the contribution of each component to the overall performance of the system.

In general, the functional operation of a system can be visualized more readily by examining a block diagram of the system than by examining the physical system itself. A block diagram contains information concerning dynamic behavior, but it

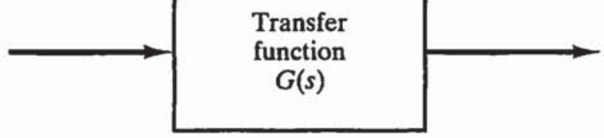

Figure 4–2 Element of a block diagram.

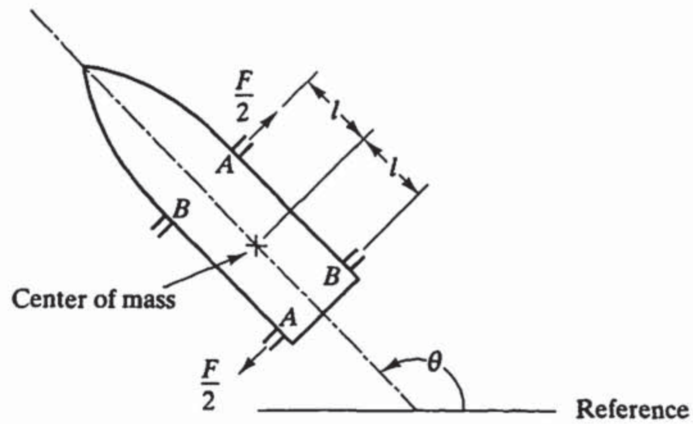

Figure 4–21 Schematic diagram of a satellite attitude control system.

Obtain the transfer function of this system by assuming that the torque $T(t)$ is the input, and the angular displacement $\theta(t)$ of the satellite is the output. (We consider the motion only in the plane of the page.)

Solution Applying Newton's second law to this system and noting that there is no friction in the environment of the satellite, we have

$$J\frac{d^2\theta}{dt^2} = T$$

Taking the Laplace transform of both sides of this last equation and assuming that all initial conditions are zero yields

$$Js^2\Theta(s) = T(s)$$

where $\Theta(s) = \mathcal{L}[\theta(t)]$ and $T(s) = \mathcal{L}[T(t)]$. The transfer function of the system is thus

$$\text{Transfer function} = \frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}$$

Problem A–4–2

Consider the mechanical system shown in Figure 4–22. Displacements x_i and x_o are measured from their respective equilibrium positions. Derive the transfer function of the system wherein x_i is the input and x_o is the output. Then obtain the response $x_o(t)$ when input $x_i(t)$ is a step displacement of magnitude X_i occurring at $t = 0$. Assume that $x_o(0-) = 0$.

Solution The equation of motion for the system is

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2\dot{x}_o$$

Taking the $\mathcal{L}_-$ transform of this equation and noting that $x_i(0-) = 0$ and $x_o(0-) = 0$, we have

$$(b_1 s + k_1)X_i(s) = (b_1 s + k_1 + b_2 s)X_o(s)$$

The transfer function $X_o(s)/X_i(s)$ is

$$\frac{X_o(s)}{X_i(s)} = \frac{b_1 s + k_1}{(b_1 + b_2)s + k_1} \tag{4–28}$$

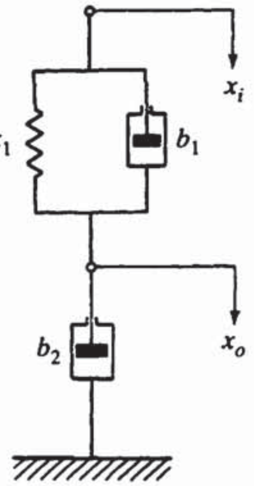

Figure 4–22 Mechanical system.

The response $x_o(t)$ when the input $x_i(t)$ is a step displacement of magnitude X_i occurring at $t = 0$ can be obtained from Equation (4–28). First we have

$$X_o(s) = \frac{b_1 s + k_1}{(b_1 + b_2)s + k_1}\frac{X_i}{s} = \left\{\frac{1}{s} - \frac{b_2}{b_1 + b_2}\frac{1}{s + [k_1/(b_1 + b_2)]}\right\}X_i$$

Then the inverse Laplace transform of $X_o(s)$ gives

$$x_o(t) = \left[1 - \frac{b_2}{b_1 + b_2}e^{-k_1 t/(b_1+b_2)}\right]X_i$$

Notice that $x_o(0+) = [b_1/(b_1 + b_2)]X_i$.

Problem A–4–3

The mechanical system shown in Figure 4–23 is initially at rest. At $t = 0$, a unit-step displacement input is applied to point A. Assume that the system remains linear throughout the response period. The displacement x is measured from the equilibrium position. If $m = 1$ kg, $b = 10$ N-s/m, and $k = 50$ N/m, find the response $x(t)$ as well as the values of $x(0+)$, $\dot{x}(0+)$, and $x(\infty)$.

Solution The equation of motion for the system is

$$m\ddot{x} + b(\dot{x} - \dot{y}) + kx = 0$$

or

$$m\ddot{x} + b\dot{x} + kx = b\dot{y}$$

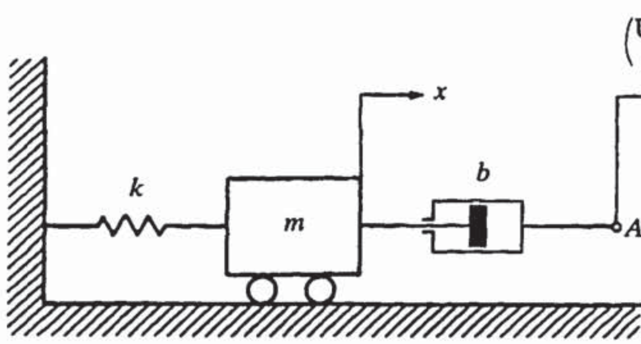

Figure 4–23 Mechanical system.

Noting that $x(0-) = 0$, $\dot{x}(0-) = 0$, and $y(0-) = 0$, we take the $\mathscr{L}_-$ transform of this last equation and obtain

$$(ms^2 + bs + k)X(s) = bsY(s)$$

Thus,

$$\frac{X(s)}{Y(s)} = \frac{bs}{ms^2 + bs + k}$$

Since the input y is a unit step, $Y(s) = 1/s$. Consequently,

$$X(s) = \frac{bs}{ms^2 + bs + k}\frac{1}{s} = \frac{b}{ms^2 + bs + k}$$

Substituting the given numerical values for m, b, and k into this last equation, we get

$$X(s) = \frac{10}{s^2 + 10s + 50} = \frac{10}{(s+5)^2 + 5^2}$$

The inverse Laplace transform of $X(s)$ is

$$x(t) = 2e^{-5t}\sin 5t$$

The values of $x(0+)$, $\dot{x}(0+)$, and $x(\infty)$ are found from the preceding equation and are

$$x(0+) = 0, \qquad \dot{x}(0+) = 10, \qquad x(\infty) = 0$$

Thus, the mass m returns to the original position as time elapses.

Problem A–4–4

Find the transfer function $X_o(s)/X_i(s)$ of the mechanical system shown in Figure 4–24. Obtain the response $x_o(t)$ when the input $x_i(t)$ is a step displacement of magnitude X_i occurring at $t = 0$. Assume that the system is initially at rest $[x_o(0-) = 0$ and $y(0-) = 0]$. Assume also that x_i and x_o are measured from their respective equilibrium positions. The numerical values of b_1, b_2, k_1, and k_2 are as follows:

$$b_1 = 5 \text{ N-s/m}, \qquad b_2 = 20 \text{ N-s/m}, \qquad k_1 = 5 \text{ N/m}, \qquad k_2 = 10 \text{ N/m}$$

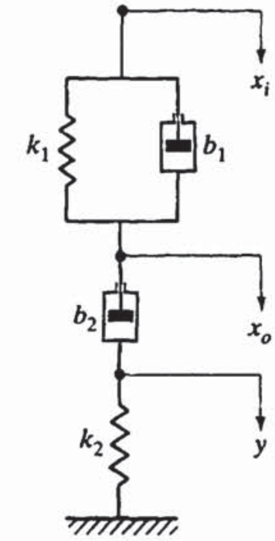

Figure 4–24 Mechanical system.

Solution The equations of motion for the mechanical system are

$$b_1(\dot{x}_i - \dot{x}_o) + k_1(x_i - x_o) = b_2(\dot{x}_o - \dot{y})$$
$$b_2(\dot{x}_o - \dot{y}) = k_2 y$$

Taking the $\mathscr{L}_-$ transform of these two equations, with the initial conditions $x_i(0-) = 0$, $x_o(0-) = 0$ and $y(0-) = 0$, we get

$$b_1[sX_i(s) - sX_o(s)] + k_1[X_i(s) - X_o(s)] = b_2[sX_o(s) - sY(s)]$$
$$b_2[sX_o(s) - sY(s)] = k_2 Y(s)$$

If we eliminate $Y(s)$ from the last two equations, the transfer function $X_o(s)/X_i(s)$ becomes

$$\frac{X_o(s)}{X_i(s)} = \frac{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right)}{\left(\frac{b_1}{k_1}s + 1\right)\left(\frac{b_2}{k_2}s + 1\right) + \frac{b_2}{k_1}s}$$

Substitution of the given numerical values into the transfer function yields

$$\frac{X_o(s)}{X_i(s)} = \frac{(s + 1)(2s + 1)}{(s + 1)(2s + 1) + 4s} = \frac{s^2 + 1.5s + 0.5}{s^2 + 3.5s + 0.5}$$

For an input $x_i(t) = X_i \cdot 1(t)$, the response $x_o(t)$ can be obtained as follows: Since

$$X_o(s) = \frac{s^2 + 1.5s + 0.5}{s^2 + 3.5s + 0.5}\frac{X_i}{s}$$
$$= \left(\frac{0.6247}{s + 3.3508} - \frac{0.6247}{s + 0.1492} + \frac{1}{s}\right)X_i$$

we find that

$$x_o(t) = (0.6247e^{-3.3508t} - 0.6247e^{-0.1492t} + 1)X_i$$

Notice that $x_o(0+) = X_i$.

Problem A–4–5

Obtain the transfer function $X(s)/U(s)$ of the system shown in Figure 4–25, where u is the force input. The displacement x is measured from the equilibrium position.

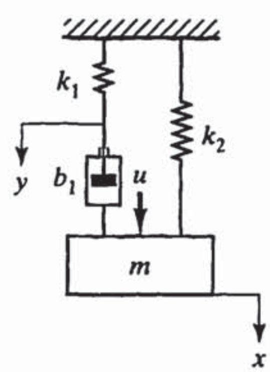

Figure 4–25 Mechanical system.

Solution The equations of motion for the system are

$$m\ddot{x} = -k_2 x - b_1(\dot{x} - \dot{y}) + u$$
$$b_1(\dot{x} - \dot{y}) = k_1 y$$

Laplace transforming these two equations and assuming initial conditions equal to zero, we obtain

$$ms^2 X(s) = -k_2 X(s) - b_1 s X(s) + b_1 s Y(s) + U(s)$$
$$b_1 s X(s) - b_1 s Y(s) = k_1 Y(s)$$

Eliminating $Y(s)$ from the last two equations yields

$$(ms^2 + b_1 s + k_2)X(s) = b_1 s \frac{b_1 s}{b_1 s + k_1} X(s) + U(s)$$

Simplifying, we obtain

$$[(ms^2 + b_1 s + k_2)(b_1 s + k_1) - b_1^2 s^2]X(s) = (b_1 s + k_1)U(s)$$

from which we get the transfer function $X(s)/U(s)$ as

$$\frac{X(s)}{U(s)} = \frac{b_1 s + k_1}{mb_1 s^3 + mk_1 s^2 + b_1(k_1 + k_2)s + k_1 k_2}$$

Problem A–4–6

Figure 4–26(a) shows a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires excite the automobile suspension system, whose motion consists of a translational motion of the center of mass and a rotational motion about the center of mass. Mathematical modeling of the complete system is quite complicated.

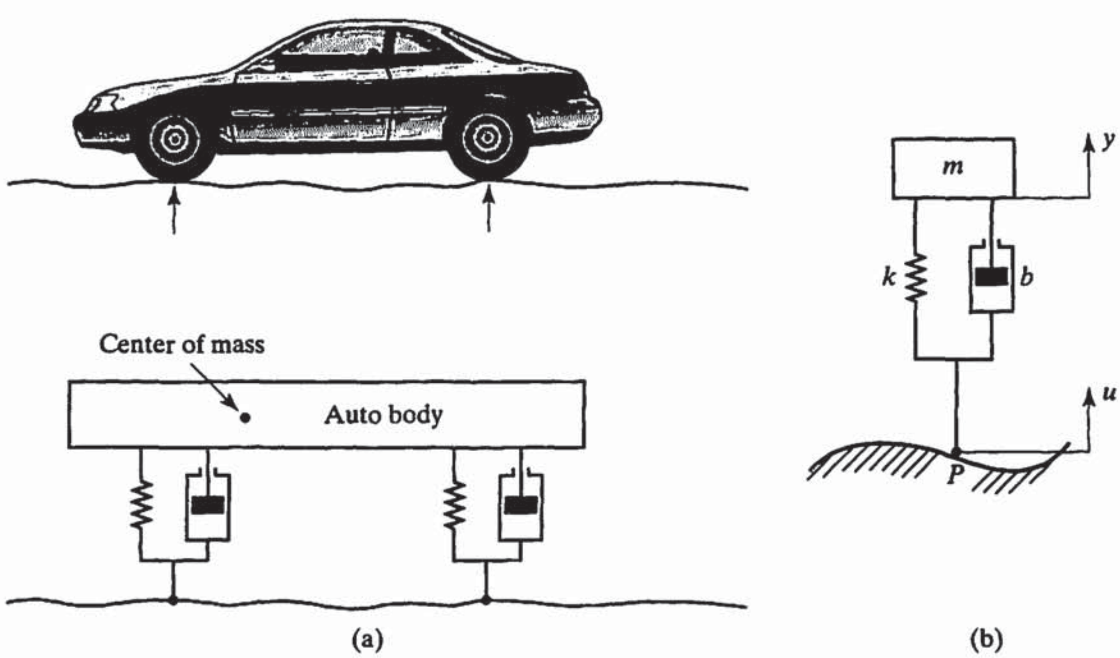

Figure 4–26 (a) Automobile suspension system; (b) simplified suspension system.

A highly simplified version of the suspension system is shown in Figure 4–26(b). Assuming that the motion u at point P is the input to the system and the vertical motion y of the body is the output, obtain the transfer function $Y(s)/U(s)$. (Consider the motion of the body only in the vertical direction.) The displacement y is measured from the equilibrium position in the absence of the input u.

Solution The equation of motion for the system shown in Figure 4–26(b) is

$$m\ddot{y} + b(\dot{y} - \dot{u}) + k(y - u) = 0$$

or

$$m\ddot{y} + b\dot{y} + ky = b\dot{u} + ku$$

Taking the Laplace transform of this last equation, assuming zero initial conditions, we obtain

$$(ms^2 + bs + k)Y(s) = (bs + k)U(s)$$

Hence, the transfer function $Y(s)/U(s)$ is

$$\frac{Y(s)}{U(s)} = \frac{bs + k}{ms^2 + bs + k}$$

Problem A–4–7

Obtain the transfer function $Y(s)/U(s)$ of the system shown in Figure 4–27. The vertical motion u at point P is the input. (Similar to the system of **Problem A–4–6**, this system is also a simplified version of an automobile or motorcycle suspension system. In Figure 4–27, m_1 and k_1 represent the wheel mass and tire stiffness, respectively.) Assume that the displacements x and y are measured from their respective equilibrium positions in the absence of the input u.

Solution Applying Newton's second law to the system, we get

$$m_1\ddot{x} = k_2(y - x) + b(\dot{y} - \dot{x}) + k_1(u - x)$$
$$m_2\ddot{y} = -k_2(y - x) - b(\dot{y} - \dot{x})$$

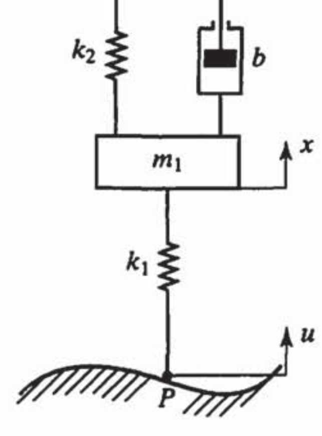

Figure 4–27 Suspension system.

Hence, we have

$$m_1\ddot{x} + b\dot{x} + (k_1 + k_2)x = b\dot{y} + k_2 y + k_1 u$$
$$m_2\ddot{y} + b\dot{y} + k_2 y = b\dot{x} + k_2 x$$

Taking the Laplace transforms of these two equations, assuming zero initial conditions, we obtain

$$[m_1 s^2 + bs + (k_1 + k_2)]X(s) = (bs + k_2)Y(s) + k_1 U(s)$$
$$[m_2 s^2 + bs + k_2]Y(s) = (bs + k_2)X(s)$$

Eliminating $X(s)$ from the last two equations, we have

$$(m_1 s^2 + bs + k_1 + k_2)\frac{m_2 s^2 + bs + k_2}{bs + k_2}Y(s) = (bs + k_2)Y(s) + k_1 U(s)$$

which yields

$$\frac{Y(s)}{U(s)} = \frac{k_1(bs + k_2)}{m_1 m_2 s^4 + (m_1 + m_2)bs^3 + [k_1 m_2 + (m_1 + m_2)k_2]s^2 + k_1 bs + k_1 k_2}$$

Problem A–4–8

Expand the function

$$\frac{B(s)}{A(s)} = \frac{3s^3 + 5s^2 + 10s + 40}{s^4 + 16s^3 + 69s^2 + 94s + 40}$$

into partial fractions with MATLAB.

Solution A MATLAB program for obtaining the partial-fraction expansion is given in MATLAB Program 4–13.

MATLAB Program 4–13

```
>> num = [3  5  10  40];
>> den = [1  16  69  94  40];
>> [r, p, k] = residue(num,den)

r =

    5.2675
   -2.0741
   -0.1934
    1.1852

p =

  -10.0000
   -4.0000
   -1.0000
   -1.0000
k =
       []
```

From the results of the program, we get the following expression:

$$\frac{B(s)}{A(s)} = \frac{5.2675}{s + 10} + \frac{-2.0741}{s + 4} + \frac{-0.1934}{s + 1} + \frac{1.1852}{(s + 1)^2}$$