HOMEWORK 2 SOLUTIONS

1. a. The rate of oscillation of the road is given by

$$\omega = 2\pi\left(\frac{55\text{ mph}}{12\text{ ft.}}\right)\left(\frac{5280\text{ ft.}}{\text{mi.}}\right)\left(\frac{\text{hr.}}{3600\text{ s}}\right) = 42.24\text{ rad/s}$$

empty car: $m = \dfrac{1000\text{ lbs.}}{32.2\text{ ft./s}^2} = 31.1\text{ slugs}$

$$\omega_n = \sqrt{\frac{30{,}000\text{ lbs./ft.}}{31.1\text{ slugs}}} = 31.1\text{ rad/s}$$

$$r = \frac{42.24\text{ rad/s}}{31.1\text{ rad/s}} = 1.36$$

$$\frac{X}{Y} = \sqrt{\frac{1+(2\cdot 0.2\cdot 1.36)^2}{\left(1-1.36^2\right)^2+(2\cdot 0.2\cdot 1.36)^2}} = 1.13$$

$$X = 1.13\cdot 4\text{ in.} = 4.52\text{ in.}$$

fully loaded car: $m = \dfrac{3000\text{ lbs.}}{32.2\text{ ft./s}^2} = 93.2\text{ slugs}$

$$\omega_n = \sqrt{\frac{30{,}000\text{ lbs./ft.}}{93.2\text{ slugs}}} = 17.9\text{ rad/s}$$

$$r = \frac{42.24\text{ rad/s}}{17.9\text{ rad/s}} = 2.36$$

$$\frac{X}{Y} = \sqrt{\frac{1+(2\cdot 0.2\cdot 2.36)^2}{\left(1-2.36^2\right)^2+(2\cdot 0.2\cdot 2.36)^2}} = 0.295$$

$$X = 0.295\cdot 4\text{ in.} = 1.18\text{ in.}$$

b. To find the maximum speed, we first differentiate the transmissibility equation with respect to r and set it equal to 0.

$$\frac{\partial}{\partial r}\sqrt{\frac{1+(2\zeta r)^2}{\left(1-r^2\right)^2+(2\zeta r)^2}}$$

$$=\frac{1}{2}\left[\frac{1+(2\zeta r)^2}{\left(1-r^2\right)^2+(2\zeta r)^2}\right]^{-1/2}\frac{\frac{\partial}{\partial r}\left[\left(1-r^2\right)^2+(2\zeta r)^2\right]\left[1+(2\zeta r)^2\right]-\left[\left(1-r^2\right)^2+(2\zeta r)^2\right]\frac{\partial}{\partial r}\left[1+(2\zeta r)^2\right]}{\left[\left(1-r^2\right)^2+(2\zeta r)^2\right]^2}$$

If we assume that $\zeta \neq 0$, then the denominator will never be 0. Thus, for the whole equation to be 0, the numerator must be 0:

$$\begin{aligned}0&=\frac{\partial}{\partial r}\left[\left(1-r^2\right)^2+(2\zeta r)^2\right]\left[1+(2\zeta r)^2\right]-\left[\left(1-r^2\right)^2+(2\zeta r)^2\right]\frac{\partial}{\partial r}\left[1+(2\zeta r)^2\right]\\&=\left(-4r+4r^3+8\zeta^2 r\right)\left(1+4\zeta^2 r^2\right)-\left(1-2r^2+r^4+4\zeta^2 r^2\right)\left(8\zeta^2 r\right)\\&=-4r+4r^3+8\zeta^2 r^5\end{aligned}$$

This gives the solution $r=0$ or $r=\frac{1}{2\zeta}\sqrt{\sqrt{1+8\zeta^2}-1}$. (The other solutions are negative or complex.) The first solution corresponds to the car not moving, so this is also not correct.

Plugging in the values for the problem yields

$$r=\frac{1}{2\cdot 0.2}\sqrt{\sqrt{1+8\cdot 0.2^2}-1}=0.965$$

$$\omega=0.965\cdot 17.9\text{ rad/s}=17.3\text{ rad/s}$$

$$V=(17.3\text{ rad/s})\left(\frac{3600\text{ s}}{\text{hr.}}\right)\left(\frac{\text{mi.}}{5280\text{ ft.}}\right)\left(\frac{12\text{ ft.}}{2\pi}\right)=22.49\text{ mph}$$

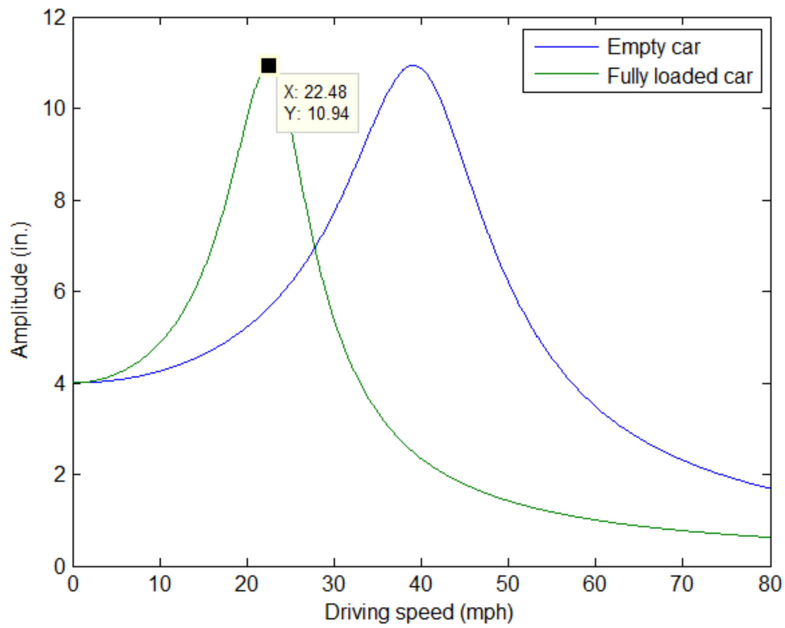
c. The following MATLAB code plots amplitude vs. driving speed for both scenarios:

```
v = linspace(0,80,1001);          % driving speed (mph)
w = (2*pi/12*5280/3600)*v;        % road frequency (rad/s)
zeta = 0.2;                       % damping ratio

% empty car
w_n = 31.1;                       % natural frequency (rad/s)
r = w/w_n;
X_empty = 4*sqrt((1+(2*zeta*r).^2)./((1-r.^2).^2+(2*zeta*r).^2));

% fully loaded car
w_n = 17.9;                       % natural frequency (rad/s)
r = w/w_n;
X_full = 4*sqrt((1+(2*zeta*r).^2)./((1-r.^2).^2+(2*zeta*r).^2));
```

```
% plot responses
figure;
plot(v,X_empty,v,X_full);
ylabel('Amplitude (in.)');
xlabel('Driving speed (mph)');
legend('Empty car','Fully loaded car');
```



Note that for the fully loaded car, the peak response occurs at the driving speed found in part (b).

2. The transfer function for this system is

$$\frac{X(s)}{F(s)} = \frac{1}{5s^2 + cs + 10} = \left(\frac{1}{5}\right)\frac{1}{s^2 + 0.2cs + 2} = \left(\frac{1}{10}\right)\frac{2}{s^2 + 0.2cs + 2}$$

where we have converted this into the standard form

$$\frac{X(s)}{F(s)} = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

The peak response of this transfer function can be found by solving the following:

$$0 = \frac{\partial}{\partial\omega}\left|\frac{X(i\omega)}{F(i\omega)}\right| = \frac{\partial}{\partial\omega}K\frac{\omega_n^2}{\sqrt{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2}} = K\omega_n^2\frac{-2\omega\left(\omega_n^2 - \omega^2\right) + 4\zeta^2\omega_n^2\omega}{\left[\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega_n\omega\right)^2\right]^{3/2}}$$

Since the denominator is always positive, this equation is solved when the numerator is 0:

$$\omega = 0 \text{ or } \omega = \omega_n \sqrt{1 - 2\zeta^2}$$

The first solution turns out to be a local minimum (this can be verified by computing the second derivative). The second solution is the peak frequency ω_n. Plugging this into the frequency response amplitude yields the peak magnitude

$$M_p = \left| \frac{X(i\omega_p)}{F(i\omega_p)} \right| = K \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega_p^2)^2 + (2\zeta\omega_n\omega_p)^2}} = \frac{K}{2\zeta\sqrt{1 - \zeta^2}}$$

Setting $M_p = \frac{3}{22}$ and $K = \frac{1}{10}$ and solving for ζ yields $\zeta = 0.92$ or $\zeta = 0.40$.

The $\zeta = 0.92$ is invalid because the preceding formula only holds for $\zeta < \frac{1}{\sqrt{2}} = 0.707$.

We can see that $\omega_n = \sqrt{2}$ rad/s, and, furthermore, $2\zeta\omega_n = 0.2c$.

Solving for c yields $c = 5.66$ N/(m/s).

3. a. Displacement transmissibility for a spring-mass-damper system has the form

$$\left| \frac{X(i\omega)}{F(i\omega)} \right| = \frac{1}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}$$

For a rotating imbalance, the input force can be written as

$$f(t) = mR\omega^2 \sin(\omega t)$$

Hence, the magnitude X of the vibration is given by

$$X = \frac{mR\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \text{ or } \frac{MX}{mR} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

where $\omega_n = \sqrt{\frac{k}{M}}$, $r = \frac{\omega}{\omega_n}$, and $\zeta = \frac{c}{2\sqrt{Mk}}$. This frequency response thus has a double zero at $\omega = 0$ and a pair of complex conjugate poles with corner frequency ω_n.

Since the peak in the response is significant, we can assume light damping, and so $\omega_n \approx \omega_p = 3\text{ Hz} = 18.8$ rad/s.

This yields $k = M\omega_n^2 = 35$ kN/m.

At high speeds, we have

$$X \to \frac{mR}{M}, \text{ and so } mR = MX = (100\ \text{kg})(0.005\ \text{m}) = 0.5\ \text{kg - m}$$

By setting $\frac{d}{dr}\left(\frac{MX}{mR}\right) = 0$, the peak frequency is found to be $r = \frac{1}{\sqrt{1-2\zeta^2}}$, with a value of

$\left(\frac{MX}{mR}\right)_{\max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$.

Hence, $\frac{(100\ \text{kg})(0.025\ \text{m})}{0.5\ \text{kg - m}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$, giving $\zeta = 1$ or $\zeta = 0.1$.

The $\zeta = 1$ is invalid because the preceding formula only holds for $\zeta < \frac{1}{\sqrt{2}} = 0.707$.

Therefore, $c = 2\zeta\sqrt{Mk} = 374\ \text{N/(m/s)}$.

b. Force transmissibility for a spring-mass-damper system has the form

$$\left|\frac{F_t(i\omega)}{F(i\omega)}\right| = \frac{\sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}$$

Plugging in $\omega = 6\ \text{Hz} = 37.7\ \text{rad/s}$, $M = 100\ \text{kg}$, $c = 374\ \text{N/(m/s)}$, and $k = 35\ \text{kN/m}$ yields

$$\left|\frac{F_t(i\omega)}{F(i\omega)}\right| = 0.35 = 35\%$$

c. The force transmissibility is always greater than 1 when $\omega < \sqrt{2}\omega_n$. Since the frequency range includes resonance (and lower frequencies), this task is impossible.