REVIEW OF LINEAR SINGLE-DEGREE-OF-FREEDOM SYSTEMS

All vibrating systems are made up of the following three components:

1. A means of storing potential energy (e.g. spring, elasticity, gravity field)
2. A means of storing kinetic energy (linear or angular inertia)
3. A means by which energy is removed (e.g. damper, friction)

Some energy is lost every cycle to surroundings in the form of heat or sound (i.e. there is always friction), so an external source of energy is required to sustain a vibration.

Degrees of freedom (DOF) – the number of *independent* variables required to determine the position of all parts of a system

For each DOF, we can write an ordinary differential equation (ODE) in the form

$$m_{eq}\ddot{x} + c_{eq}\dot{x} + k_{eq}x = u \tag{1}$$

m_{eq} – the equivalent mass

c_{eq} – the equivalent damping

k_{eq} – the equivalent stiffness

u – the input (usually an external force or torque)

States – the number of *independent* variables, along with the external inputs, required to determine the future behavior of the system

Since Eq. (1) is a second-order ODE (the highest derivative is the second), two initial conditions (ICs) are required. Thus, for mechanical systems, usually # of states = 2 × # of DOFs.

1. Classifying Vibrating Systems

Free response – response of a system to non-zero initial conditions, no inputs are applied for $t > 0$
Forced response – response of a system to a non-zero input, assuming zero initial conditions

Transient response – initial aperiodic motion of a system in response to change in state or input, usually decays to 0 as $t \to \infty$ (if the system is stable), behavior determined by characteristic roots of the system
Steady-state response – asymptotic behavior as $t \to \infty$, behavior determined by form of input (e.g. step, ramp, sinusoid)

(Asymptotically) stable – transient response decays to 0 as $t \to \infty$, all characteristic roots have negative real part
Unstable – transient response is unbounded as $t \to \infty$, at least one characteristic root has positive real part
(Neutrally) stable – transient response is bounded as $t \to \infty$, one or more simple characteristic roots have 0 real part

Undamped – no energy is lost to surroundings, amplitude of motion remains constant (this is a special case of neutral stability)
Damped – energy is dissipated each cycle (stable)

Single degree of freedom (SDOF) – only one natural frequency of the system, which is the frequency at which it vibrates freely
Multi-degree of freedom (MDOF) - # of DOFs = # of natural frequencies, each frequency corresponds to a mode of vibrations in which all masses pass through their equilibria simultaneously

Discrete or lumped parameter – system has a finite # of components, equivalently a finite # of DOFs
Continuous or distributed parameter – system is approximated as a continuum, equivalently an infinite # of DOFs, can be approximated as a discrete system using FEA or modal analysis

Deterministic – ICs and system parameters are given constants, and inputs can be evaluated for any $t \geq 0$
Stochastic (random) – ICs, system parameters, and inputs are not knowable ahead of time, but their statistics, e.g. mean, standard deviation, are given functions (may be stationary or time-varying)

Linear – all components behave linearly, all EOMs are linear, Principle of Superposition applies, analytical solutions exist
Nonlinear – at least one component behaves nonlinearly, at least one EOM is nonlinear, superposition of solutions is not possible, usually there is no analytical solution
Note: any differentiable function can be linearized about a point in its domain, but the linearization generally becomes less accurate with distance from the point of linearization

2. Solving for the response of a system using Laplace Transform and State Space methods

First, the following equation relating the Laplace Transform and the matrix exponential to the solution to a system of linear homogeneous ODEs will be proven:

> The solution to the linear system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ with initial condition $\mathbf{x}(0) = \mathbf{x}_0$ is given by $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0 = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}_0$.

Proof: Assume the solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is given by a power series $\mathbf{x}(t) = \mathbf{x}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \mathbf{b}_3 t^3 + \cdots$

Plugging this into the ODE gives

$$\mathbf{b}_1 + 2\mathbf{b}_2 t + 3\mathbf{b}_3 t^2 + \cdots = \mathbf{A}\left(\mathbf{x}_0 + \mathbf{b}_1 t + \mathbf{b}_2 t^2 + \mathbf{b}_3 t^3 + \cdots\right)$$

Equating like powers of t gives:

$$
\begin{aligned}
\mathbf{b}_1 &= \mathbf{A}\mathbf{x}_0 \\
\mathbf{b}_2 &= \frac{1}{2}\mathbf{A}\mathbf{b}_1 = \frac{1}{2}\mathbf{A}^2\mathbf{x}_0 \\
\mathbf{b}_3 &= \frac{1}{3}\mathbf{A}\mathbf{b}_2 = \frac{1}{3\cdot 2}\mathbf{A}^3\mathbf{x}_0 \\
&\vdots \\
\mathbf{b}_k &= \frac{1}{k!}\mathbf{A}^k\mathbf{x}_0
\end{aligned}
$$

Thus, the power series resembles a matrix version of the scalar exponential function, which justifies the use of the matrix exponential notation, as follows:

$$\mathbf{x}(t) = \sum_{k=0}^{\infty}\frac{1}{k!}\mathbf{A}^k t^k \mathbf{x}_0 \equiv e^{\mathbf{A}t}\mathbf{x}_0$$

Taking the Laplace Transform of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ and rearranging yields:

$$
\begin{aligned}
s\mathbf{X}(s) - \mathbf{x}(0) &= \mathbf{A}\mathbf{X}(s) \\
(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) &= \mathbf{x}_0 \\
\mathbf{X}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}_0
\end{aligned}
$$

Taking the inverse Laplace Transform of the latter gives

$$\mathbf{x}(t) = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]\mathbf{x}_0$$

Since the solution to the ODE is unique, it must be the case that

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I} - \mathbf{A})^{-1}\right]$$ □

Example 1: Consider the system shown in Figure 1 with the following values: $m = 2\text{ kg}$, $c = 40\text{ N/(m/s)}$, $k = 108\text{ N/m}$, $l = 0.3\text{ m}$, $l_1 = 0.1\text{ m}$, $l_2 = 0.15\text{ m}$, $f(t) = \text{unit step}$, $x_0 = 0.1\text{ m}$, and $\dot{x}_0 = 0\text{ m/s}$. Find $x(t)$.

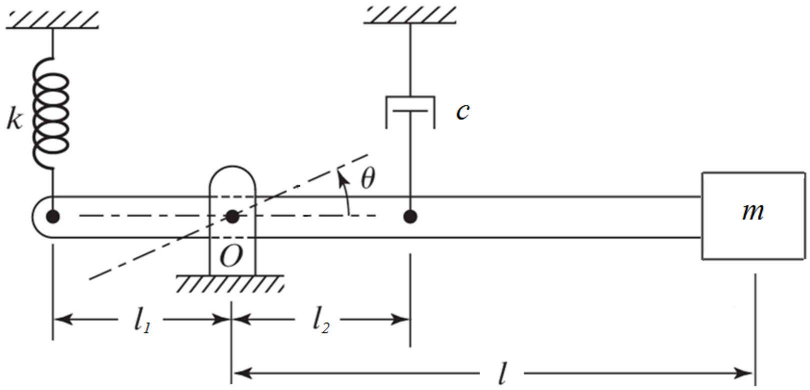


Figure 1

This system has one DOF, rotation about the pivot point. Taking a sum of moments about this point yields:

$$J_0\ddot{\theta} = -F_{spring}l_1 - F_{damper}l_2 - fl_1$$

$$ml^2\frac{\ddot{x}}{l} = -\left(kx\frac{l_1}{l}\right)l_1 - \left(c\dot{x}\frac{l_2}{l}\right)l_2 - fl_1$$

$$\underbrace{m}_{m_{eq}}\ddot{x} + \underbrace{c\left(\frac{l_2}{l}\right)^2}_{c_{eq}}\dot{x} + \underbrace{k\left(\frac{l_1}{l}\right)^2}_{k_{eq}}x = -\frac{l_1}{l}f$$

$$2\ddot{x} + 10\dot{x} + 12x = -0.33f$$

<u>Solution using Laplace Transform:</u>

Taking the Laplace Transform of the above ODE and rearranging gives

$$2\left(s^2X(s) - sx_0 - \dot{x}_0\right) + 10\left(sX(s) - x_0\right) + 12X(s) = -0.33F(s)$$

$$\left(2s^2 + 10s + 12\right)X(s) = 0.2s + 1 - \frac{0.33}{s}$$

$$X(s) = \underbrace{\frac{0.2s+1}{2s^2+10s+12}}_{\text{free response}} - \underbrace{\frac{0.33}{2s^2+10s+12}\frac{1}{s}}_{\text{forced response}}$$

At this point, the terms should be expanded into partial fractions. This can be accomplished using MATLAB's residue function. Alternatively, if the Symbolic Toolbox is installed, the command ilaplace can be used to get the symbolic inverse Laplace Transform.

The partial fraction expansion is

$$X(s)=\frac{0.3}{s+2}+\frac{-0.2}{s+3}+\frac{0.08}{s+2}+\frac{-0.06}{s+3}+\frac{-0.03}{s}$$

Taking the inverse Laplace Transform yields

$$x(t)=0.38e^{-2t}-0.26e^{-3t}-0.03$$

Note that the time constants of the response are 0.5 and 0.33 seconds. A system's transient response can be considered negligible after 5 dominant (i.e. slowest) time constants, in this case after 2.5 seconds.

Solution using state space:

To write the equation in state space, first rearrange the EOM(s) such that the highest derivative appears by itself on the left side:

$$\ddot{x}=-6x-5\dot{x}-0.17f$$

This equation must be written as a system of first-order ODEs:

Let $u_1 = f$.

Let $x_1 = x$. Then $\dot{x}_1 = \dot{x} = x_2$.

Let $x_2 = \dot{x}$. Then $\dot{x}_2 = \ddot{x} = -6x_1 - 5x_2 - 0.17u_1$.

In state-space form this becomes

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}}_{\dot{\mathbf{x}}}=\underbrace{\begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}}_{\mathbf{A}}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{\mathbf{x}}+\underbrace{\begin{bmatrix}0\\ -0.17\end{bmatrix}}_{\mathbf{B}}\underbrace{\begin{bmatrix}u_1\end{bmatrix}}_{\mathbf{u}}$$

The output of the system is merely the position x :

Let $y = x = x_1$

In state-space form this becomes

$$\underbrace{\begin{bmatrix}y\end{bmatrix}}_{\mathbf{y}}=\underbrace{\begin{bmatrix}1 & 0\end{bmatrix}}_{\mathbf{C}}\underbrace{\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{\mathbf{x}}+\underbrace{\begin{bmatrix}0\end{bmatrix}}_{\mathbf{D}}\underbrace{\begin{bmatrix}u_1\end{bmatrix}}_{\mathbf{u}}$$

The solution is then

$$\mathbf{x}(t) = \underbrace{e^{\mathbf{A}t}\mathbf{x}_0}_{\text{free response}} + \underbrace{\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau}_{\text{forced response}}$$

Two ways to compute the matrix exponential:

1. $e^{\mathbf{A}t} = \mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{A})^{-1}\right]$

$$(s\mathbf{I}-\mathbf{A})^{-1} = \frac{1}{s(s+5)+6}\begin{bmatrix} s+5 & 1 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} \dfrac{3}{s+2}+\dfrac{-2}{s+3} & \dfrac{1}{s+2}+\dfrac{-1}{s+3} \\ \dfrac{-6}{s+2}+\dfrac{6}{s+3} & \dfrac{-2}{s+2}+\dfrac{3}{s+3} \end{bmatrix}$$

Taking the inverse Laplace Transform yields the matrix exponential:

$$e^{\mathbf{A}t} = \begin{bmatrix} 3e^{-2t}-2e^{-3t} & e^{-2t}-e^{-3t} \\ -6e^{-2t}+6e^{-3t} & -2e^{-2t}+3e^{-3t} \end{bmatrix}$$

2. $e^{\mathbf{A}t} = \mathbf{V}\begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix}\mathbf{V}^{-1}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$, and $\mathbf{V}$ is a matrix of corresponding eigenvectors. Then

$$\begin{aligned} e^{\mathbf{A}t} &= \begin{bmatrix} 0.447 & -0.316 \\ -0.894 & 0.949 \end{bmatrix}\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix}\begin{bmatrix} 6.71 & 2.24 \\ 6.32 & 03.16 \end{bmatrix} \\ &= \begin{bmatrix} 3e^{-2t}-2e^{-3t} & e^{-2t}-e^{-3t} \\ -6e^{-2t}+6e^{-3t} & -2e^{-2t}+3e^{-3t} \end{bmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} 3e^{-2t}-2e^{-3t} & e^{-2t}-e^{-3t} \\ -6e^{-2t}+6e^{-3t} & -2e^{-2t}+3e^{-3t} \end{bmatrix}\begin{bmatrix} 0.1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 3e^{-2(t-\tau)}-2e^{-3(t-\tau)} & e^{-2(t-\tau)}-e^{-3(t-\tau)} \\ -6e^{-2(t-\tau)}+6e^{-3(t-\tau)} & -2e^{-2(t-\tau)}+3e^{-3(t-\tau)} \end{bmatrix}\begin{bmatrix} 0 \\ -0.17 \end{bmatrix}d\tau \\ &= \begin{bmatrix} 0.3e^{-2t}-0.2e^{-3t} \\ -0.6e^{-2t}+0.6e^{-3t} \end{bmatrix} + \begin{bmatrix} 0.08e^{-2t}-0.06e^{-3t}-0.03 \\ -0.17e^{-2t}+0.17e^{-3t} \end{bmatrix} \\ &= \begin{bmatrix} 0.38e^{-2t}-0.26e^{-3t}-0.03 \\ -0.77e^{-2t}+0.77e^{-3t} \end{bmatrix} \end{aligned}$$

The output $y = x = x_1$ is just the first row of $\mathbf{x}(t)$.

3. Transient Specifications of Second-order Systems

Given a generic second-order ODE of the form $m\ddot{x}+c\dot{x}+kx=u$, the transfer function $G(s)$ can be found by taking the Laplace Transform of this equation and assuming 0 ICs:

$$ms^2X(s)+csX(s)+kX(s)=U(s)$$
$$G(s)=\frac{\mathcal{L}[\text{output}]}{\mathcal{L}[\text{input}]}=\frac{X(s)}{U(s)}=\frac{1}{ms^2+cs+k}$$

The <u>characteristic roots</u> are the roots of the denominator polynomial:

$$ms^2+cs+k=0$$
$$s_{1,2}=\frac{-c\pm\sqrt{c^2-4mk}}{2m}$$

The <u>discriminant</u> c^2-4mk determines the nature of the roots, i.e. real vs. complex.

Define <u>critical damping</u>, c_c, as the value of damping at which the nature of the roots changes:

$$c_c^2-4mk=0 \text{, which implies } c_c=2\sqrt{mk}$$

Define the <u>damping ratio</u> by $\zeta=\frac{c}{c_c}=\frac{c}{2\sqrt{mk}}$.

Define the <u>natural frequency</u> by $\omega_n=\sqrt{\frac{k}{m}}$. This is the frequency of oscillation when $c=0$.

The transfer function can be rewritten in terms of these two quantities:

$$G(s)=\frac{1}{ms^2+cs+k}=\frac{1}{m}\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}=\frac{1}{k}\frac{\omega_n^2}{\underbrace{s^2+2\zeta\omega_n s+\omega_n^2}_{\text{standard form}}}$$

classification	damping ratio	characteristic roots	behavior	Laplace Transform
undamped	$\zeta=0$	$s_{1,2}=\pm\omega_n$	oscillatory, constant amplitude $x(t)=A_1\cos(\omega_n t)+A_2\sin(\omega_n t)$	$\frac{*}{(s^2+\omega_n^2)*}$
underdamped	$0<\zeta<1$	$s_{1,2}=-\zeta\omega_n\pm i\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d}$	oscillatory, exponentially decaying amplitude $x(t)=A_1e^{-\zeta\omega_n t}\cos(\omega_d t)+A_2e^{-\zeta\omega_n t}\sin(\omega_d t)$	$\frac{*}{(s^2+2\zeta\omega_n s+\omega_n^2)*}$ $\frac{*}{\underbrace{[(s+\zeta\omega_n)^2+\omega_d^2]}_{\text{complete the squares}}*}$

critically damped	$\zeta = 1$	$s_{1,2} = -\omega_n \equiv -a$	no oscillations, fast decay $x(t) = A_1 e^{-at} + A_2 e^{-at}$	$\dfrac{*}{(s+a)^2 *}$
overdamped	$\zeta > 1$	$s_{1,2} = -\left(\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \equiv -a_1, -a_2$	no oscillations, slower decay than critically damped case $x(t) = A_1 e^{-a_1 t} + A_2 e^{-a_2 t}$	$\dfrac{*}{(s+a_1)(s+a_2)*}$

4. Impulse Response

From the Laplace Transform: $\underbrace{X(s)}_{\text{response}} = \underbrace{G(s)}_{\substack{\text{transfer}\\ \text{function}}}\underbrace{F(s)}_{\text{input}}$, so $x(t) = \mathcal{L}^{-1}[G(s)F(s)]$.

What if $f(t)$ is complicated or doesn't have an exact formula, i.e. it is only given at discrete times $f(t_1), f(t_2), \ldots, f(t_n)$? Using the following formula,

$$\mathcal{L}\left[\int_0^t f_1(\tau) f_2(t-\tau)d\tau\right] = \int_0^\infty \int_0^t f_1(\tau) f_2(t-\tau)d\tau e^{-st}dt = \int_0^\infty \int_0^\infty f_1(\tau) f_2(t-\tau)e^{-s(t-\tau)}e^{-s\tau}d\tau dt$$

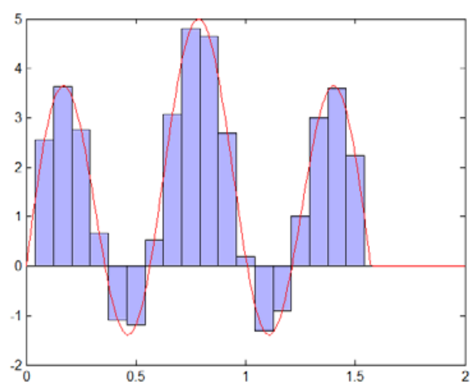
$$= \int_0^\infty \int_0^\infty f_1(\tau) f_2(\lambda)e^{-s\lambda}e^{-s\tau}d\tau d\lambda = \left(\int_0^\infty f_1(\tau)e^{-s\tau}d\tau\right)\left(\int_0^\infty f_2(\lambda)e^{-s\lambda}d\lambda\right) = F_1(s)F_2(s)$$

the response can be written directly in the time domain:

$$x(t) = \mathcal{L}^{-1}[G(s)F(s)] = \int_0^t g(\tau) f(t-\tau)d\tau = \int_0^t f(\tau) g(t-\tau)d\tau$$

Mathematically, this is a nice, compact formula. But what is its physical interpretation?

Any arbitrary (bounded, piecewise continuous) input can be approximated by a sequence of finite duration impulses. Like the Riemann integral, the approximation gets better as the duration gets smaller.



Input force (red) approximated by a sequence of finite duration pulses.

Consider a single pulse with a constant force of A/t_f with a duration of t_f. Then the strength of this impulse is A N-s (newtons × seconds). The Laplace Transform of this pulse is

$$\int_0^{\infty} \frac{A}{t_f} e^{-st} dt = \int_0^{t_f} \frac{A}{t_f} e^{-st} dt = \frac{A}{t_f s}\left(1 - e^{-st_f}\right)$$

If $t_f \to 0$, then the pulse will become an instantaneously applied infinite force; however, the strength of the impulse, and hence the change in momentum after the force, remains a constant A N-s. Although this is a mathematical abstraction, it is a good approximation to a rapid impact. The Laplace Transform of this instantaneous impulse is thus

$$\lim_{t_f \to 0} \frac{A}{t_f s}\left(1 - e^{-st_f}\right) = \frac{A}{s} \lim_{t_f \to 0} \frac{\frac{d}{dt_f}\left(1 - e^{-st_f}\right)}{\frac{d}{dt_f}\left(t_f\right)} = \frac{A}{s} \lim_{t_f \to 0} \frac{se^{-st_f}}{1} = A$$

Define $\delta(t)$ to be the unit impulse (or impulse with unit strength). Then $\mathcal{L}[\delta(t)] = 1$.

The unit impulse response of a system (or its response to a unit impulse input) is

$$x(t) = \mathcal{L}^{-1}[G(s)F(s)] = \mathcal{L}^{-1}[G(s) \cdot 1] \equiv g(t)$$

Therefore, the inverse Laplace Transform of a system's transfer function, $G(s)$, is the response of that system to a unit impulse input.

Suppose there are two impulses, one at $t = t_1$ with strength A_1, and one at $t = t_2$ with strength A_2. Then $f(t) = A_1\delta(t - t_1) + A_2\delta(t - t_2)$. The response is then

$$x(t) = \begin{cases} 0, & 0 < t < t_1 \\ A_1 g(t - t_1), & t_1 < t < t_2 \\ A_1 g(t - t_1) + A_2 g(t - t_2), & t \geq t_2 \end{cases}$$

This can also just be written $x(t) = A_1 g(t - t_1) + A_2 g(t - t_2)$ if it is understood that $g(t) = 0$ for all $t < 0$.

Returning to the approximation of a general force $f(t)$ by a sequence of impulses, the response is then approximately

$$\begin{aligned} x(t) &\approx \sum \text{all impulse responses up to time } t \\ &= \sum_{\{k | t_k < t\}} \underbrace{f(t_k)\Delta t_k}_{A_k} g(t - t_k) \end{aligned}$$

Now, let $\max_k(t_k) \to 0$. Then $x(t) \to \int_0^t f(\tau) g(t - \tau) d\tau$, assuming the functions are bounded and piecewise continuous (i.e. Riemann integrable).

5. Frequency response

These are the main reasons for designing in the frequency domain, where input frequency ω is the independent variable:

1. You don't need a mathematical model of the system (e.g. transfer function, state space). Instead, feedback control or vibration suppression can be accomplished using experimental data from frequency sweeps or broadband excitation.

2. You can address performance requirements over a wide range of input signals, especially those that don't have a simple Laplace Transform.

3. Frequency domain techniques extend readily to nonlinear or uncertain systems.

Example 2: Consider once more the system from Example 1. This time, suppose the input is sinusoidal, given by $f(t) = 2\cos(5t)$.

From previously:

$$2\left(s^2X(s) - sx_0 - \dot{x}_0\right) + 10\left(sX(s) - x_0\right) + 12X(s) = -0.33F(s)$$

$$\left(2s^2 + 10s + 12\right)X(s) = 0.2s + 1 - 0.33\left(2\frac{s}{s^2 + 5^2}\right)$$

$$X(s) = \underbrace{\frac{0.2s + 1}{2s^2 + 10s + 12}}_{\text{free response}} - \underbrace{\frac{0.67}{2s^2 + 10s + 12}\frac{1}{s^2 + 25}}_{\text{forced response}}$$

Partial fraction expansion yields

$$X(s) = \frac{0.3}{s+2} + \frac{-0.2}{s+3} + \frac{0.023}{s+2} + \frac{-0.029}{s+3} + \frac{0.0032 + 0.0042i}{s - 5i} + \frac{0.00320.0042i}{s + 5i}$$

Because the original transfer function has only real coefficients, any complex terms in the expansion will appear in conjugate pairs. Although an inverse Laplace Transform of the last two terms exists, they are complex exponentials (although, when added together, the imaginary parts do cancel out). It is preferred, however, to deal only in real functions, so the last two terms must be manipulated to yield a sine and a cosine term instead. The following formula is useful for this, which can be verified algebraically:

$$\boxed{\frac{r}{s-p} + \frac{\bar{r}}{s-\bar{p}} = \frac{2\,\mathrm{Re}(r)s - 2\,\mathrm{Re}(r\bar{p})}{s^2 - 2\,\mathrm{Re}(p)s + |p|^2}}$$

where overline denotes complex conjugation. Thus, we get

$$X(s) = \frac{0.323}{s+2} + \frac{-0.229}{s+3} + \frac{0.0064s - 0.0423}{s^2 + 25} = \frac{0.323}{s+2} + \frac{-0.229}{s+3} + 0.0064\frac{s}{s^2 + 25} - \frac{0.0423}{5}\frac{5}{s^2 + 25}$$

Taking the inverse Laplace Transform yields

$$
\begin{aligned}
x(t) &= 0.323e^{-2t} - 0.229e^{-3t} + 0.0064\cos(5t) - 0.0085\sin(5t) \\
&= 0.323e^{-2t} - 0.229e^{-3t} + \sqrt{(0.0064)^2 + (-0.0085)^2}\cos\left[5t - \tan^{-1}\left(\frac{-0.0085}{0.0064}\right)\right] \\
&= \underbrace{0.323e^{-2t} - 0.229e^{-3t}}_{\text{transient response}} + \underbrace{0.0106\cos(5t + 0.921)}_{\text{steady-state response}}
\end{aligned}
$$

If only the steady-state response is required, then the frequency transfer function can be utilized. The following statement is proved in the document "Frequency transfer function derivation":

> Let $G(s) = Y(s)/U(s)$ be the transfer function of a stable system (i.e. all roots of its characteristic equation have negative real parts) subjected to the input $u(t) = U\sin(\omega t)$. Then the steady-state output is given by $y_{ss}(t) = y(t \to \infty) = |G(i\omega)|U\sin(\omega t + \phi)$, where $\phi = \angle G(i\omega) \equiv \tan^{-1}\left[\frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}\right]$.
> Hence $|G(i\omega)|$ is the amplitude ratio of the output sinusoid, $y(t)$, to the input sinusoid, $u(t)$, and $\angle G(i\omega)$ is the phase shift of the output sinusoid with respect to the input sinusoid. $G(i\omega)$ is called the frequency transfer function of the system. Note that sin can be replaced by cos in the above equations.

Returning to the previous example, the frequency transfer function can be derived from the transfer function.

$$G(s) = \frac{-0.33}{2s^2 + 10s + 12}$$

With and input frequency of $\omega = 5$ rad/s, we get

$$G(i\omega) = G(5i) = \frac{-0.33}{2(5i)^2 + 10(5i) + 12} = \frac{-0.33}{-38 + 50i}$$

Thus,

$$|G(5i)| = \frac{-0.33}{\sqrt{(-38)^2 + (50)^2}} = 0.0053 \text{ and } \angle G(5i) = \tan^{-1}\left(\frac{0}{-0.33}\right) - \tan^{-1}\left(\frac{50}{-38}\right) = 0.921$$

Therefore, the steady-state solution is

$$x(t \to \infty) = 0.0053 \cdot 2\cos(5t + 0.921) = 0.0106\cos(5t + 0.921)$$

which matches the full solution given above. Note that the transient solution cannot be found this way; however, this is seldom needed when persistent excitations are being studied.

Note that sometimes the frequency transfer function is written $G(\omega) = X(\omega)/F(\omega)$ instead of $G(i\omega) = X(i\omega)/F(i\omega)$. The latter notation reflects more clearly the connection to the transfer function $G(s)$.

6. Bode plots

Bode Diagrams are an effective way to visualize transfer functions and frequency responses $G(i\omega)$ by plotting its magnitude $|G(i\omega)|$ and phase $\angle G(i\omega)$. (Recall that a complex function is uniquely determined by its magnitude and phase.)

They use a log scale, so powers of ω become straight lines whose slope is proportional to the power of ω. For example, $\log(\omega^2) = 2\log(\omega)$ and $\log(\omega^5) = 5\log(\omega)$.

Magnitudes (y-axis) are plotted in <u>decibels</u>: $20\log|G(i\omega)|$. The following table shows some common values worth memorizing:

$\lvert G(i\omega)\rvert$	$20\log\lvert G(i\omega)\rvert$
0.01	-40 dB
0.1	-20 dB
1	0 dB
10	20 dB
100	40 dB

Frequencies (x-axis) are plotted in <u>decades</u> (powers of ten).

Slopes are stated in terms of dB/decade.

Phase angles $\angle G(i\omega)$ are plotted in <u>degrees</u>.

Using a log scale makes serial transfer functions easy to combine:

$$\longrightarrow \boxed{K(s)} \longrightarrow \boxed{G(s)} \longrightarrow$$

magnitude: $20\log|K(i\omega)G(i\omega)| = 20\log|K(i\omega)| + 20\log|G(i\omega)|$

phase: $\angle[K(i\omega)G(i\omega)] = \angle K(i\omega) + \angle G(i\omega)$

Using a log scale also makes ratios of polynomials of ω (like a frequency transfer function) easy to analyze by separating the numerator and denominator.

If $G(i\omega) = \dfrac{n(i\omega)}{d(i\omega)}$, then $20\log|G(i\omega)| = 20\log|n(i\omega)| - 20\log|d(i\omega)|$, and

$$\angle[G(i\omega)] = \angle n(i\omega) - \angle d(i\omega).$$

Note that real polynomials can always be factored into 1st- and 2nd-order terms. So if we know how to plot 1st- and 2nd-order terms, any function $G(i\omega)$ can be plotted using a sum or difference of its constituent terms.

Basic constituent terms: