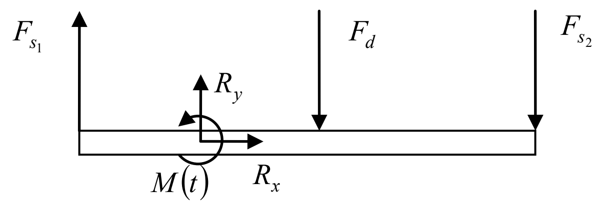
HOMEWORK 1 SOLUTIONS

1.

a.



$$J_0\ddot{\theta} = \sum M_0 = -F_{s_1}\frac{l}{4} - F_d\frac{l}{4} - F_{s_2}\frac{3l}{4} + M(t)$$

$$\left[\frac{1}{12}ml^2 + m\left(\frac{l}{4}\right)^2\right]\ddot{\theta} = -k\left(\frac{l}{4}\right)^2\theta - c\left(\frac{l}{4}\right)^2\dot{\theta} - k\left(\frac{3l}{4}\right)^2\theta + M(t)$$

$$\frac{7}{48}ml^2\ddot{\theta} + \frac{1}{16}cl^2\dot{\theta} + \frac{5}{8}kl^2\theta = M(t)$$

$$1.46\ddot{\theta} + 62.5\dot{\theta} + 3125\theta = M(t)$$

$$1.46s^2\Theta(s) + 62.5s\Theta(s) + 3125\Theta(s) = M(s)$$

$$G(s) = \frac{\Theta(s)}{M(s)} = \frac{1}{1.46s^2 + 62.5s + 3125}$$

b. $\omega = 1000\text{ rpm} = 105\text{ rad/s}$, so $M(t) = 100\cos(105t)$. Thus, $M(s) = \dfrac{100s}{s^2 + 11{,}025}$.

Since the ICs are both 0, we can use the transfer function, which assumes 0 ICs:

$$\Theta(s) = G(s)M(s) = \frac{1}{1.46s^2 + 62.5s + 3125}\frac{100s}{s^2 + 11{,}025}$$

Taking the inverse Laplace transform yields

$$\theta(t) = 6.1\times10^{-3}e^{-21.4t}\cos(41t) - 4.7\times10^{-3}e^{-21.4t}\sin(41t)$$
$$-6.1\times10^{-3}\cos(105t) + 3.1\times10^{-3}\sin(105t)$$

Note that the last two terms can be combined using the formula

$a\cos(\omega t) + b\sin(\omega t) = \sqrt{a^2 + b^2}\cos\left[\omega t - \tan^{-1}\left(\dfrac{b}{a}\right)\right]$, which yields

$$\theta(t) = 6.1\times10^{-3}e^{-21.4t}\cos(41t) - 4.7\times10^{-3}e^{-21.4t}\sin(41t) = 6.9\times10^{-3}\cos(105t - 2.67)$$

c. Since this part only asks for the steady-state response, we can use the frequency transfer function approach:

$$G(i\omega)=\frac{1}{1.46(105i)^2+62.5(105i)+3125(105i)}=\frac{1}{-13{,}000+6560i}$$

$$|G(i\omega)|=\frac{1}{\sqrt{(-13{,}000)^2+(6560)^2}}=6.9\times10^{-5}$$

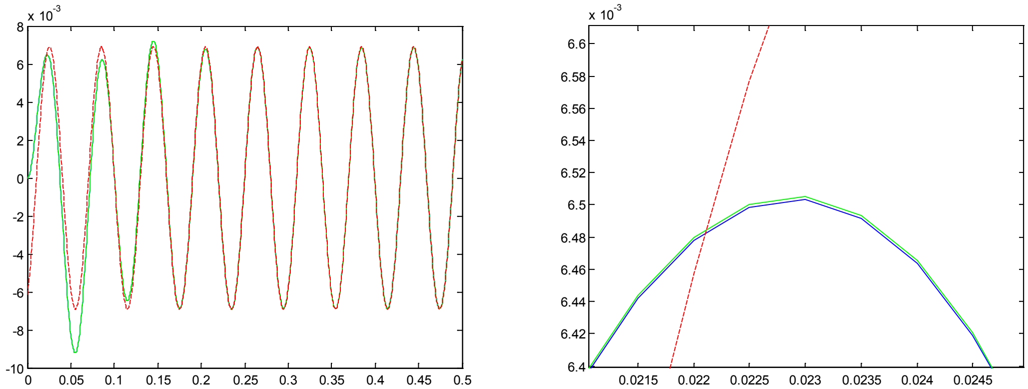
$$\angle G(i\omega)=\tan^{-1}\left(\frac{0}{1}\right)-\tan^{-1}\left(\frac{6560}{13{,}000}\right)=-2.67\text{ rad}$$

$$\theta(t)=6.9\times10^{-5}\cdot100\cos(105t-2.67)=6.9\times10^{-3}\cos(105t-2.67)$$

Note that the steady-state solution found in part (c) matches the solution of part (b) as $t\to\infty$.

d. The following Matlab code simulates the system using `lsim` and `ode45`, starting from rest, and compares the result to the steady-state solution calculated in part (c).

```
function hw9_1()

clc;
clear all;
close all;

% simulate the system using lsim
G = tf(1,[1.46 62.5 3125]);
t = linspace(0,0.5,1001);
u = 100*cos(105*t);
y_lsim = lsim(G,u,t);

% simulate the system using ode45
y0 = [0 0]';
[t,y_ode45] = ode45(@loc_eom,t,y0);

    function xdot = loc_eom(this_t,this_x)

        M = 100*cos(105*this_t);
        xdot = [this_x(2); (-62.5*this_x(2)-3125*this_x(1)+M)/1.46];

    end

% steady-state response
y_ss = 6.9e-3*cos(105*t-2.67);
figure;
plot(t,y_lsim,'b',t,y_ode45(:,1),'g',t,y_ss,'r--');

end
```



The blue (`lsim`) and green (`ode45`) curves are almost identical, as expected. They are both seen to converge to the steady-state solution (red, dashed).

2. a. Let $G(s)=\dfrac{Y(s)}{U(s)}$ be the transfer function of a system.

Let $y_{step}(t)$ be the step response of the system, and denote its Laplace Transform as $Y_{step}(s)$.

Note that the Laplace Transform of $\int_0^t y_{step}(\tau)$ is $\dfrac{Y_{step}(s)}{s}$, assuming 0 initial conditions.

Also note that the Laplace Transform of a step is $\dfrac{1}{s}$, and the Laplace Transform of a ramp is $\dfrac{1}{s^2}$.

Hence, $y_{ramp}(t)=\mathcal{L}^{-1}\left[G(s)\dfrac{1}{s^2}\right]=\mathcal{L}^{-1}\left[\dfrac{G(s)\frac{1}{s}}{s}\right]=\mathcal{L}^{-1}\left[\dfrac{Y_{step}(s)}{s}\right]$, which shows that the ramp response is the integral of the step response.

b. Suppose we have the state space model

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{Ax}+\mathbf{B}u\\ y_{step} &= \mathbf{Cx}+\mathbf{D}u\end{aligned}$$

where $u(t)$ is a unit step and $y_{step}(t)$ is the step response of the system.

Suppose this system has n states. Let's add a new state $x_{n+1} = \int_0^t y_{step}(\tau)$, which, from part (a), is the ramp response of the system.

Hence, $\dot{x}_{n+1} = y_{step} = \mathbf{C}\mathbf{x} + \mathbf{D}u$.

Thus, if we augment the original system matrices in the following way:

$\tilde{\mathbf{x}} = \left[\begin{array}{c}\mathbf{x} \\ \hline x_{n+1}\end{array}\right]$, $\tilde{\mathbf{A}} = \left[\begin{array}{c|c}\mathbf{A} & \mathbf{0} \\ \hline \mathbf{C} & 0\end{array}\right]$, and $\tilde{\mathbf{B}} = \left[\begin{array}{c}\mathbf{B} \\ \hline \mathbf{D}\end{array}\right]$, then

$$\begin{aligned} \dot{\tilde{\mathbf{x}}} &= \tilde{\mathbf{A}}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}u \\ y_{ramp} &= \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}\tilde{\mathbf{x}} + [0]u \end{aligned}$$

3. $\ddot{x} + 4\dot{x} + 40x = 10$, $x_0 = \dot{x}_0 = 0$

Taking the Laplace transform of this ODE gives

$$s^2 X(s) + 4sX(s) + 40X(s) = \frac{10}{s}$$
$$X(s) = \frac{10}{(s^2 + 4s + 40)s}$$

Check discriminant: $4^2 - 4 \cdot 40 = -144 < 0$, so we cannot factor into linear terms

$$X(s) = \frac{a_1 s + a_2}{s^2 + 4s + 40} + \frac{a_3}{s}$$
$$\Rightarrow 10 = s(a_1 s + a_2) + (s^2 + 4s + 40)a_3 = (a_1 + a_3)s^2 + (a_2 + 4a_3)s + 40a_3$$
$$\begin{cases} a_1 + a_3 = 0 \Rightarrow a_1 = -\dfrac{1}{4} \\ a_2 + 4a_3 = 0 \Rightarrow a_2 = -1 \\ 40a_3 = 10 \Rightarrow a_3 = \dfrac{1}{4} \end{cases}$$
$$X(s) = \frac{-\frac{1}{4}s - 1}{s^2 + 4s + 40} + \frac{\frac{1}{4}}{s}$$

Complete the squares: $s^2 + 4s + 40 = (s+2)^2 + 36$

$$X(s)=\frac{-\frac{1}{4}s-1}{(s+2)^2+36}+\frac{\frac{1}{4}}{s}$$
$$=\left(-\frac{1}{4}\right)\frac{s+4}{(s+2)^2+36}+\left(\frac{1}{4}\right)\frac{1}{s}$$
$$=\left(-\frac{1}{4}\right)\frac{s+2}{(s+2)^2+36}+\left(-\frac{1}{4}\right)\frac{2}{(s+2)^2+36}+\left(\frac{1}{4}\right)\frac{1}{s}$$
$$=\left(-\frac{1}{4}\right)\frac{s+2}{(s+2)^2+36}+\left(-\frac{1}{12}\right)\frac{6}{(s+2)^2+36}+\left(\frac{1}{4}\right)\frac{1}{s}$$
$$x(t)=-\frac{1}{4}e^{-2t}\cos(6t)-\frac{1}{12}e^{-2t}\sin(6t)+\frac{1}{4}$$

4. a. This system is underdamped since it exhibits overshoot. Using the overshoot and settling time, the position of the system's complex conjugate poles can be found.

From the overshoot: $\zeta=\dfrac{-\ln(0.527)}{\sqrt{\pi^2+\ln^2(0.527)}}=0.2$.

From the settling time: $\zeta\omega_n=\dfrac{4}{9.36\text{ s}}=0.427\text{ rad/s}$, which is the negative of the real part of the poles.

Thus, $\omega_n=\dfrac{0.427\text{ rad/s}}{0.2}=2.14\text{ rad/s}$, and

$\omega_d=2.14\text{ rad/s}\sqrt{1-0.2^2}=2.09\text{ rad/s}$, which is the imaginary part of the poles.

Therefore, the poles are located at $s=-0.427\pm 2.09$.

b. Since we assume the system is 2nd order, we can write the transfer function in the form

$$G(s)=\frac{A}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

The step response of this system is $X(s)=G(s)\dfrac{1}{s}$.

Using the Final Value Theorem, the steady-state value of this response is

$\lim_{t\to\infty}x(t)=\lim_{s\to 0}sX(s)=G(0)=\dfrac{A}{\omega_n^2}=0.684$ according to the plot.

Hence, $A = 0.684(2.14\ \text{rad/s})^2 = 3.13$.

Therefore, $G(s) = \dfrac{3.13}{s^2 + 0.856s + 4.58}$.

c. To eliminate overshoot, we would want the system to be critically damped or overdamped. Hence, the minimum required damping is $\zeta = 1$. Therefore, we would increase damping by a factor of $1/0.2 = 5$.

d. In this case, the denominator of the transfer function would be $(s + \omega_n)^2$, which would place the poles at $s = -\omega_n = -2.14\ \text{rad/s}$ with multiplicity 2.