HOMEWORK 6 SOLUTIONS

1. a. Starting with the equation $\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$, we assume a solution of the form $\mathbf{q}(t) = Ae^{i\omega t}\mathbf{u}$. Substituting this solution into the ODE and simplifying yields

$$-\omega^2\mathbf{M}\mathbf{u} + \mathbf{K}\mathbf{u} = \mathbf{0} \text{ or } \mathbf{M}^{-1}\mathbf{K}\mathbf{u} = \omega^2\mathbf{u}$$

Thus, the eigenvalues of $\mathbf{M}^{-1}\mathbf{K}$ are the squares of the system's natural frequencies.

The Cholesky decomposition of $\mathbf{M}$ is $\mathbf{M} = \mathbf{L}\mathbf{L}^T$, so $\mathbf{L}^{-1}\mathbf{M}\mathbf{L}^{-T} = \mathbf{I}$. If we make the substitution $\mathbf{q} = \mathbf{L}^{-T}\mathbf{r}$, then

$$\mathbf{M}\mathbf{L}^{-T}\ddot{\mathbf{r}} + \mathbf{K}\mathbf{r} = \mathbf{0} \text{ or } \mathbf{I}\ddot{\mathbf{r}} + \tilde{\mathbf{K}}\mathbf{r} = \mathbf{0}, \text{ where } \tilde{\mathbf{K}} \equiv \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}$$

We assume a solution of the form $\mathbf{r}(t) = Be^{i\omega t}\mathbf{v}$. Substituting this solution into the ODE and simplifying yields

$$-\omega^2\mathbf{I}\mathbf{v} + \tilde{\mathbf{K}}\mathbf{v} = \mathbf{0} \text{ or } \tilde{\mathbf{K}}\mathbf{v} = \omega^2\mathbf{v}$$

Thus, the eigenvalues of $\tilde{\mathbf{K}}$ are also the squares of the system's natural frequencies.

b. Let's see how $\mathbf{M}^{-1}\mathbf{K}$ and $\tilde{\mathbf{K}}$ are related:

$$\begin{aligned}
\tilde{\mathbf{K}} &= \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T} \\
\mathbf{L}^{-T}\tilde{\mathbf{K}}\mathbf{L}^{T} &= \mathbf{L}^{-T}\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}\mathbf{L}^{T} \\
\mathbf{L}^{-T}\tilde{\mathbf{K}}\mathbf{L}^{T} &= \left(\mathbf{L}\mathbf{L}^{T}\right)^{-1}\mathbf{K}\left(\mathbf{L}\mathbf{L}^{-1}\right)^{T} \\
\mathbf{L}^{-T}\tilde{\mathbf{K}}\mathbf{L}^{T} &= \mathbf{M}^{-1}\mathbf{K}
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{L}^{-T}\tilde{\mathbf{K}}\mathbf{L}^{T}\mathbf{u} &= \omega^2\mathbf{u} \\
\tilde{\mathbf{K}}\mathbf{L}^{T}\mathbf{u} &= \omega^2\mathbf{L}^{T}\mathbf{u}
\end{aligned}$$

If we compare this to the equation $\tilde{\mathbf{K}}\mathbf{v} = \omega^2\mathbf{v}$, we can see that $\mathbf{v} = \mathbf{L}^T\mathbf{u}$, which shows how the eigenvectors are related.

In linear algebra terms, $\mathbf{M}^{-1}\mathbf{K}$ and $\tilde{\mathbf{K}}$ are <u>similar matrices</u>. Similar matrices have the same eigenvalues, and their eigenvectors are related by a coordinate transformation, in this case $\mathbf{L}^T$.

c. Suppose $\mathbf{K}$ is symmetric, i.e. $\mathbf{K}=\mathbf{K}^T$. Then

$$\tilde{\mathbf{K}}^T=\left(\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}\right)^T=\left(\mathbf{L}^{-T}\right)^T\mathbf{K}^T\left(\mathbf{L}^{-1}\right)^T=\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}=\tilde{\mathbf{K}}$$

which shows that $\tilde{\mathbf{K}}$ is also symmetric.

d. Suppose $\mathbf{K}$ is positive semi-definite, i.e. $\mathbf{a}^T\mathbf{K}\mathbf{a}\geq 0$ for all $\mathbf{a}\neq 0$. Then

$$\mathbf{a}^T\tilde{\mathbf{K}}\mathbf{a}=\mathbf{a}^T\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}\mathbf{a}=\left(\mathbf{L}^{-T}\mathbf{a}\right)^T\mathbf{K}\left(\mathbf{L}^{-T}\mathbf{a}\right)$$

Let $\mathbf{b}=\mathbf{L}^{-T}\mathbf{a}$. Then, since $\mathbf{L}^{-T}$ is invertible, $\mathbf{a}\neq 0\Leftrightarrow\mathbf{b}\neq 0$. Thus,

$$\mathbf{a}^T\tilde{\mathbf{K}}\mathbf{a}=\mathbf{a}^T\mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T}\mathbf{a}=\mathbf{b}^T\mathbf{K}\mathbf{b}\geq 0 \text{ for all } \mathbf{a}\neq 0$$

which shows that $\tilde{\mathbf{K}}$ is also positive semi-definite.

e. Let $\mathbf{C}=\alpha\mathbf{M}+\beta\mathbf{K}$. Then

$$\begin{aligned}\mathbf{C}\mathbf{M}^{-1}\mathbf{K}&=(\alpha\mathbf{M}+\beta\mathbf{K})\mathbf{M}^{-1}\mathbf{K}=\alpha\mathbf{M}\mathbf{M}^{-1}\mathbf{K}+\beta\mathbf{K}\mathbf{M}^{-1}\mathbf{K}=\alpha\mathbf{K}+\beta\mathbf{K}\mathbf{M}^{-1}\mathbf{K}\\&=\mathbf{K}\left(\alpha\mathbf{I}+\beta\mathbf{M}^{-1}\mathbf{K}\right)=\mathbf{K}\left(\alpha\mathbf{M}^{-1}\mathbf{M}+\beta\mathbf{M}^{-1}\mathbf{K}\right)=\mathbf{K}\mathbf{M}^{-1}(\alpha\mathbf{M}+\beta\mathbf{K})\\&=\mathbf{K}\mathbf{M}^{-1}\mathbf{C}\end{aligned}$$

2. a. The EOMs for each of the masses respectively are

$$\begin{aligned}&m_1\ddot{x}_1=\sum F_x=-k_1x_1-c_1\dot{x}_1+k_2(x_2-x_1)+c_2(\dot{x}_2-\dot{x}_1)\\&m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+(k_1+k_2)x_1-k_2x_2=0\end{aligned}$$

$$\begin{aligned}&m_2\ddot{x}_2=\sum F_x=-k_2(x_2-x_1)-c_2(\dot{x}_2-\dot{x}_1)\\&m_2\ddot{x}_2-c_2\dot{x}_1+c_2\dot{x}_2-k_2x_1+k_2x_2=0\end{aligned}$$

Arranging these into matrix-vector form yields

$$\underbrace{\begin{bmatrix}m_1&0\\0&m_2\end{bmatrix}}_{\mathbf{M}}\underbrace{\begin{bmatrix}\ddot{x}_1\\\ddot{x}_2\end{bmatrix}}_{\ddot{\mathbf{q}}}+\underbrace{\begin{bmatrix}c_1+c_2&-c_2\\-c_2&c_2\end{bmatrix}}_{\mathbf{C}}\underbrace{\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}}_{\dot{\mathbf{q}}}+\underbrace{\begin{bmatrix}k_1+k_2&-k_2\\-k_2&k_2\end{bmatrix}}_{\mathbf{K}}\underbrace{\begin{bmatrix}x_1\\x_2\end{bmatrix}}_{\mathbf{q}}=\begin{bmatrix}0\\0\end{bmatrix}$$

From Matlab (see below), we calculate the natural frequencies $\omega_1=0.40\text{ rad/s}$ and $\omega_2=1.76\text{ rad/s}$.

Using the following relationships

$$\zeta_1 = \frac{\alpha}{2\omega_1} + \frac{\beta\omega_1}{2} \text{ and } \zeta_2 = \frac{\alpha}{2\omega_2} + \frac{\beta\omega_2}{2}$$

we can solve this system of equations for the unknowns

$$\alpha = \frac{2\omega_1\omega_2(\zeta_1\omega_2 - \zeta_2\omega_1)}{\omega_2^2 - \omega_1^2} \text{ and } \beta = \frac{2(\zeta_2\omega_2 - \zeta_1\omega_1)}{\omega_2^2 - \omega_1^2}$$

Plugging in the values gives $\alpha = 0.20$ and $\beta = 0.28$.

b-c. The following Matlab code plots the responses:

```
clc;
clear all;
close all;

% system constants
m1 = 1;
m2 = 4;
k1 = 2;
k2 = 1;

% populate matrices
M = diag([m1 m2])
K = [k1+k2 -k2; -k2 k2]

% compute mass-normalized modes
L = chol(M,'lower')
Ktilde = (L\K)/L'            % same as Ktilde = inv(L)*K*inv(L')
[P,D] = eig(Ktilde)
w = sqrt(diag(D))
N = length(w);

% proportional damping constants
w1 = w(1);
w2 = w(2);
z1 = 0.3;
z2 = 0.3;
alpha = 2*w1*w2*(z1*w2-z2*w1)/(w2^2-w1^2)
beta = 2*(z2*w2-z1*w1)/(w2^2-w1^2)

% initial conditions
q0 = [1 -1]';
qdot0 = [0 2]';

% compute solutions to modal equations
z0 = (L*P)'*q0
zdot0 = (L*P)'*qdot0
Z = zeros(length(w),1);      % modal amplitudes
tht = Z;                     % modal phase angles
zeta = [z1 z2]';             % modal damping ratios
wd = w.*sqrt(1-zeta.^2);     % modal damped natural frequencies
for i = 1:length(Z)
    Z(i) = sqrt(z0(i)^2+((zdot0(i)+zeta(i)*w(i)*z0(i))/wd(i))^2);
    tht(i) = atan2(wd(i)*z0(i),zdot0(i)+zeta(i)*w(i)*z0(i));
end
Z
```

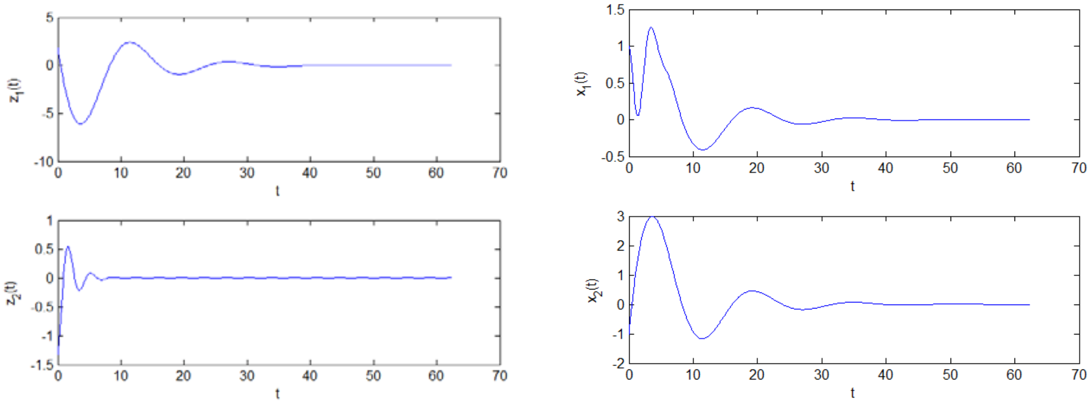
```
tht

t = linspace(0,4*2*pi/w(1),1001);
z = zeros(N,length(t));
for i = 1:N
    z(i,:) = Z(i)*exp(-zeta(i)*w(i)*t).*sin(w(i)*t+tht(i));
end

figure;
for i = 1:N
    subplot(N,1,i);
    plot(t,z(i,:));
    ylabel(sprintf('z_%d(t)',i));
    xlabel('t');
end

% compute response trajectories
q = zeros(N,length(t));
G = L'\P;                         % same as G = inv(L')*P
for i = 1:N
    for j = 1:N
        q(i,:) = q(i,:) + G(i,j)*z(j,:);
    end
end

figure;
for i = 1:N
    subplot(N,1,i);
    plot(t,q(i,:));
    ylabel(sprintf('x_%d(t)',i));
    xlabel('t');
end
```



3. a. The EOMs for each of the masses respectively are

$$m\ddot{x}_1 = \sum F_x = k\left(x_2 - x_1\right)$$
$$m\ddot{x}_1 + kx_1 - kx_2 = 0$$

$$4m\ddot{x}_2 = \sum F_x = -k\left(x_2 - x_1\right) + k\left(x_3 - x_2\right)$$
$$4m\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0$$

$$m\ddot{x}_3 = \sum F_x = -k(x_3 - x_2)$$
$$m\ddot{x}_3 - kx_2 + kx_3 = 0$$

Arranging these into matrix-vector form yields

$$\underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & 4m & 0 \\ 0 & 0 & m \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{q}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b. From Matlab (see below), the natural frequencies and corresponding mode shapes are

$\omega_1 = 0$ rad/s, $\omega_2 = 2.12$ rad/s, and $\omega_3 = 2.59$ rad/s

$$\mathbf{u}_1 = \begin{bmatrix} 0.577 \\ 0.577 \\ 0.577 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}, \text{ and } \mathbf{u}_3 = \begin{bmatrix} 0.667 \\ -0.333 \\ 0.667 \end{bmatrix}$$

The first mode is a rigid body mode, since it has 0 natural frequency. This represents the entire aircraft heaving vertically. The second mode represents the wings flexing antisymmetrically while the fuselage remains fixed. The third mode represents the wings flexing symmetrically while the fuselage vibrates out of phase with the wings.

c. From Matlab (see below), the mass-normalized mode shapes are

$$\mathbf{p}_1 = \begin{bmatrix} 0.408 \\ 0.816 \\ 0.408 \end{bmatrix}, \ \mathbf{p}_2 = \begin{bmatrix} -0.707 \\ 0 \\ 0.707 \end{bmatrix}, \text{ and } \mathbf{p}_3 = \begin{bmatrix} -0.577 \\ 0.577 \\ -0.577 \end{bmatrix}$$

These vectors are similar to the ones found in part (b) except the displacements have been normalized by the square root of the masses of the components.

d-e. The rigid body mode must be dealt with specially. In this case, the modal EOM reduces to $\ddot{z}_1 = 0$. This has the solution

$$z_1(t) = \dot{z}_1(0)t + z_1(0)$$

The rest of the solution proceeds as usual.

```
clc;
clear all;
close all;

% system constants
m = 3000;
E = 6.9e9;
I = 5.2e-6;
l = 2;
k = 3*E*I/l^3;

% populate matrices
M = diag([m 4*m m])
K = [k -k 0; -k 2*k -k; 0 -k k]

% compute natural frequencies and mode shapes
[U,D] = eig(inv(M)*K);
w = sqrt(diag(D));
[w,ind] = sort(w);                    % natural frequencies (rad/s)
w
U = U(:,ind')

% compute mass-normalized modes
L = chol(M,'lower')
Ktilde = (L\K)/L'             % same as Ktilde = inv(L)*K*inv(L')
[P,D] = eig(Ktilde);
w = sqrt(diag(D));
[w,ind] = sort(w);                    % natural frequencies (rad/s)
w
P = P(:,ind')
N = length(w);

% initial conditions
q0 = [0.2 0 0]';
qdot0 = [0 0 0]';

% compute solutions to modal equations
z0 = (L*P)'*q0
zdot0 = (L*P)'*qdot0
t = linspace(0,10*2*pi/w(2),1001);
z = zeros(N,length(t));
z(1,:) = zdot0(1)*t + z0(1);            % rigid body mode
z(2,:) = sqrt(z0(2)^2+(zdot0(2)/w(2))^2)*sin(w(2)*t+atan2(w(2)*z0(2),zdot0(2)));
z(3,:) = sqrt(z0(3)^2+(zdot0(3)/w(3))^2)*sin(w(3)*t+atan2(w(3)*z0(3),zdot0(3)));

figure;
for i = 1:N
    subplot(N,1,i);
    plot(t,z(i,:));
    ylabel(sprintf('z_%d(t)',i));
    xlabel('t');
end

% compute response trajectories
q = zeros(N,length(t));
G = L'\P;                           % same as G = inv(L')*P
for i = 1:N
    for j = 1:N
        q(i,:) = q(i,:) + G(i,j)*z(j,:);
    end
end
```

```
figure;
for i = 1:N
    subplot(N,1,i);
    plot(t,q(i,:));
    ylabel(sprintf('x_%d(t)',i));
    xlabel('t');
end
```