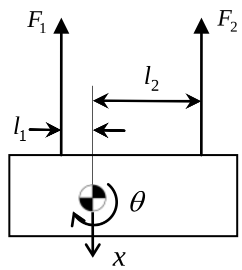
HOMEWORK 5 SOLUTIONS

1. a. We can consider the masses to be one rigid body since there is no relative motion between them. The free body diagram of this combined body is



We can combine the springs on the left into one spring with stiffness $2k$ since they are acting in parallel; same for the springs on the right side.

The distances to the center of gravity are given by

$$l_1 = \frac{m_2}{m_1+m_2} l \text{ and } l_2 = l - \frac{m_2}{m_1+m_2} l = \frac{m_1}{m_1+m_2} l$$

The sum of the forces in the vertical direction is

$$(m_1+m_2)\ddot{x} = \sum F_x = -F_1 - F_2 = -2k(x - l_1\theta) - 2k(x + l_2\theta)$$
$$(m_1+m_2)\ddot{x} + 4kx - 2k(l_1 - l_2)\theta = 0$$
$$(m_1+m_2)\ddot{x} + 4kx - 2k\frac{m_2-m_1}{m_1+m_2} l\theta = 0$$

The sum of the moments about the center of gravity is

$$(m_1 l_1^2 + m_2 l_2^2)\ddot{\theta} = \sum M_0 = F_1 l_1 - F_2 l_2 = 2k(x - l_1\theta)l_1 - 2k(x + l_2\theta)l_2$$
$$\left(m_1 \frac{m_2^2}{(m_1+m_2)^2} l^2 + m_2 \frac{m_1^2}{(m_1+m_2)^2} l^2\right)\ddot{\theta} - 2k(l_1 - l_2)x + 2k(l_1^2 + l_2^2)\theta = 0$$
$$\frac{m_1 m_2}{m_1+m_2} l^2\ddot{\theta} - 2k\frac{m_2-m_1}{m_1+m_2} lx + 2k\frac{m_1^2+m_2^2}{(m_1+m_2)^2} l^2\theta = 0$$

Writing these equations in matrix-vector form yields

$$\underbrace{\begin{bmatrix} m_1+m_2 & 0 \\ 0 & \dfrac{m_1 m_2}{m_1+m_2} l^2 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} 4k & -2k\dfrac{m_2-m_1}{m_1+m_2} l \\ -2k\dfrac{m_2-m_1}{m_1+m_2} l & 2k\dfrac{m_1^2+m_2^2}{(m_1+m_2)^2} l^2 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{\mathbf{q}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b. Let's write these equations as

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}}_{\mathbf{M}}\underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{\mathbf{q}}}+\underbrace{\begin{bmatrix} k_{11} & -k_{12} \\ -k_{12} & k_{22} \end{bmatrix}}_{\mathbf{K}}\underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{\mathbf{q}}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The natural frequencies are solutions to $\det\left(-\omega^2\mathbf{M}+\mathbf{K}\right)=0$:

$$\det\left(-\omega^2\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}+\begin{bmatrix} k_{11} & -k_{12} \\ -k_{12} & k_{22} \end{bmatrix}\right)=0$$

$$\det\left(\begin{bmatrix} k_{11}-m\omega^2 & -k_{12} \\ -k_{12} & k_{22}-J\omega^2 \end{bmatrix}\right)=0$$

$$\left(k_{11}-m\omega^2\right)\left(k_{22}-J\omega^2\right)-\left(-k_{12}\right)\left(-k_{12}\right)=0$$

$$mJ\omega^4-\left(mk_{22}+Jk_{11}\right)\omega^2+\left(k_{11}k_{22}-k_{12}^2\right)=0$$

$$\omega^2=\frac{\left(mk_{22}+Jk_{11}\right)\pm\sqrt{\left(mk_{22}+Jk_{11}\right)^2-4mJ\left(k_{11}k_{22}-k_{12}^2\right)}}{2mJ}$$

$$\omega^2=\frac{\left(mk_{22}+Jk_{11}\right)\pm\sqrt{\left(mk_{22}-Jk_{11}\right)^2+4mJk_{12}^2}}{2mJ}$$

Now we are in a position to start substituting:

$$\omega^2=\frac{\left[(m_1+m_2)2k\dfrac{m_1^2+m_2^2}{(m_1+m_2)^2}l^2+4k\dfrac{m_1m_2}{m_1+m_2}l^2\right]}{2(m_1+m_2)\dfrac{m_1m_2}{m_1+m_2}l^2}$$

$$\pm\frac{1}{2(m_1+m_2)\dfrac{m_1m_2}{m_1+m_2}l^2}\sqrt{\left[(m_1+m_2)2k\frac{m_1^2+m_2^2}{(m_1+m_2)^2}l^2-4k\frac{m_1m_2}{m_1+m_2}l^2\right]^2+4(m_1+m_2)\frac{m_1m_2}{m_1+m_2}l^2\left(2k\frac{m_2-m_1}{m_1+m_2}l\right)^2}$$

$$=\frac{2k\left(\dfrac{m_1^2+m_2^2}{m_1+m_2}+\dfrac{2m_1m_2}{m_1+m_2}\right)l^2\pm\sqrt{\left[2k\left(\dfrac{m_1^2+m_2^2}{m_1+m_2}-\dfrac{2m_1m_2}{m_1+m_2}\right)l^2\right]^2+16k^2\dfrac{m_1m_2(m_2-m_1)^2}{(m_1+m_2)^2}l^4}}{2m_1m_2l^2}$$

$$=\frac{2k(m_1+m_2)l^2\pm\sqrt{\left[2k\dfrac{(m_2-m_1)^2}{m_1+m_2}l^2\right]^2+16k^2\dfrac{m_1m_2(m_2-m_1)^2}{(m_1+m_2)^2}l^4}}{2m_1m_2l^2}$$

$$=\frac{2k(m_1+m_2)l^2\pm\sqrt{4k^2\dfrac{(m_2-m_1)^4}{(m_1+m_2)^2}l^4+16k^2\dfrac{m_1m_2(m_2-m_1)^2}{(m_1+m_2)^2}l^4}}{2m_1m_2l^2}$$

$$=\frac{2k(m_1+m_2)l^2\pm 2k\dfrac{|m_2-m_1|}{m_1+m_2}l^2\sqrt{(m_2-m_1)^2+4m_1m_2}}{2m_1m_2l^2}$$

$$=\frac{2k(m_1+m_2)l^2\pm 2k\dfrac{|m_2-m_1|}{m_1+m_2}l^2(m_1+m_2)}{2m_1m_2l^2}$$

$$=\frac{k(m_1+m_2)\pm k|m_2-m_1|}{m_1m_2}$$

Assume, without loss of generality, that $m_2 > m_1$.

Then $\omega_1 = \sqrt{\frac{k(m_1+m_2)-k(m_2-m_1)}{m_1 m_2}} = \sqrt{\frac{2k}{m_2}}$ and $\omega_2 = \sqrt{\frac{k(m_1+m_2)+k(m_2-m_1)}{m_1 m_2}} = \sqrt{\frac{2k}{m_1}}$

c. The first mode shape is found from $\left(-\omega_1^2\mathbf{M}+\mathbf{K}\right)\mathbf{u}_1 = \mathbf{0}$:

$$\begin{bmatrix} k_{11}-m\omega_1^2 & -k_{12} \\ -k_{12} & k_{22}-J\omega_1^2 \end{bmatrix}\begin{bmatrix} u_{11} \\ u_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left(k_{11}-m\omega_1^2\right)u_{11} - k_{12}u_{21} = 0$$

$$u_{21} = \frac{k_{11}-m\omega_1^2}{k_{12}}u_{11} = \frac{4k-(m_1+m_2)\frac{2k}{m_2}}{2k\frac{m_2-m_1}{m_1+m_2}l}u_{11} = \frac{m_1+m_2}{m_2 l}u_{11}$$

So, $\mathbf{u}_1 = \begin{bmatrix} 1 \\ \frac{m_1+m_2}{m_2 l} \end{bmatrix}$.

Similarly, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ -\frac{m_1+m_2}{m_1 l} \end{bmatrix}$.

d. If $m_1 = m_2$, then $\omega_{1,2} = \sqrt{\frac{4k}{m}}$ and the mode shapes become $\mathbf{u}_{1,2} = \begin{bmatrix} 1 \\ \pm\frac{2}{l} \end{bmatrix}$. Since the natural frequencies are the same and the mode shapes are linearly independent, this states that any initial condition results in simple harmonic motion (only one frequency of motion). Physically, both translation and rotation oscillate at the same frequency, and since these are the only degrees of freedom, any general motion of the system will be at this frequency.

2. Using the first equation for the first mode shape, we can solve for k_{12}:

$$\begin{bmatrix} 12-\omega_1^2 & -k_{12} \\ -k_{12} & k_{22}-4\omega_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ 9.11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} \left(12-\omega_1^2\right) - 9.11k_{12} &= 0 \\ \left(12-1.7^2\right) - 9.11k_{12} &= 0 \\ k_{12} &= 1 \end{aligned}$$

Now, using the second equation, we can solve for k_{22}:

$$\begin{bmatrix} 12-\omega_1^2 & -k_{12} \\ -k_{12} & k_{22}-4\omega_1^2 \end{bmatrix} \begin{bmatrix} 1 \\ 9.11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -k_{12} + 9.11\left(k_{22} - 4\omega_1^2\right) &= 0 \\ -1 + 9.11\left(k_{22} - 4\cdot 1.7^2\right) &= 0 \\ k_{22} &= 11.67 \end{aligned}$$

Finally, we can use the first equation for the second mode shape to solve for ω_2:

$$\begin{bmatrix} 12-\omega_2^2 & -k_{12} \\ -k_{12} & k_{22}-4\omega_2^2 \end{bmatrix} \begin{bmatrix} -9.16 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -9.16\left(12-\omega_2^2\right) - 0.25k_{12} &= 0 \\ -9.16\left(12-\omega_2^2\right) - 0.25 &= 0 \\ \omega_2 &= 3.47 \text{ rad/s} \end{aligned}$$

3. The general form of the response is $\mathbf{q}(t)=\sum_{k=1}^{N} A_k \sin(\omega_k t+\phi_k)\mathbf{u}_k$, where

$$A_i=\frac{\mathbf{u}_i^T\mathbf{M}\mathbf{q}(0)}{\mu_i\sin(\phi_i)} \text{ and } \phi_i=\tan^{-1}\left(\frac{\omega_i\mathbf{u}_i^T\mathbf{M}\mathbf{q}(0)}{\mathbf{u}_i^T\mathbf{M}\dot{\mathbf{q}}(0)}\right)$$

If we want only the j[th] mode to appear in the response, then we must have $A_i=0$ for $i\neq j$.

a. $\dot{\mathbf{q}}(0)=\mathbf{0}$. In this case $\phi_i=\pm\frac{\pi}{2}$, and so $\sin(\phi_i)=\pm1$.

Thus, we must have $\mathbf{u}_i^T\mathbf{M}\mathbf{q}(0)=0$. Since, $\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j=0$, this can be achieved with the initial condition $\mathbf{q}(0)=a\mathbf{u}_j$ for any a.

b. $\mathbf{q}(0)=\mathbf{0}$. In this case $\phi_i=0,\pi$, and so $\sin(\phi_i)=0$. Therefore, the equation $A_i=\frac{\mathbf{u}_i^T\mathbf{M}\mathbf{q}(0)}{\mu_i\sin(\phi_i)}$ is undetermined.

We can use the alternative equation $A_i=\frac{\mathbf{u}_i^T\mathbf{M}\dot{\mathbf{q}}(0)}{\mu_i\cos(\phi_i)}$, which can be derived from the equation $\dot{\mathbf{q}}(t)=\sum_{k=1}^{N} A_k\omega_k \cos(\omega_k t+\phi_k)\mathbf{u}_k$. In this case, $\cos(\phi_i)=\pm1$

Thus, we must have $\mathbf{u}_i^T\mathbf{M}\dot{\mathbf{q}}(0)=0$. Since, $\mathbf{u}_i^T\mathbf{M}\mathbf{u}_j=0$, this can be achieved with the initial condition $\dot{\mathbf{q}}(0)=b\mathbf{u}_j$ for any b.

c. In the most general case, both conditions must be met: $\mathbf{q}(0)=a\mathbf{u}_j$ for any a, and $\dot{\mathbf{q}}(0)=b\mathbf{u}_j$ for any b.

4. a. The EOMs for each of the masses respectively are

$$m_1\ddot{x}_1 = \sum F_x = -kx_1 + k(x_2 - x_1) + 5k(x_3 - x_1)$$
$$m_1\ddot{x}_1 + 7kx_1 - kx_2 - 5kx_3 = 0$$

$$m_2\ddot{x}_2 = \sum F_x = -k(x_2 - x_1) + k(x_3 - x_2)$$
$$m_2\ddot{x}_2 - kx_1 + 2kx_2 - kx_3 = 0$$

$$m_3\ddot{x}_3 = \sum F_x = -k(x_3 - x_2) - 5k(x_3 - x_1) - kx_3$$
$$m_3\ddot{x}_3 - 5kx_1 - kx_2 + 7kx_3 = 0$$

Arranging these into matrix-vector form yields

$$\underbrace{\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} 7k & -k & -5k \\ -k & 2k & -k \\ -5k & -k & 7k \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{q}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

b. The following Matlab code plots the responses:

```
clc;
clear all;
close all;

% system parameters
m1 = 10;
m2 = 2;
m3 = 10;
k = 6;

% system matrices
M = diag([m1 m2 m3]);
K = [7*k -k -5*k; -k 2*k -k; -5*k -k 7*k];

% compute natural frequencies and mode shapes
[V,D] = eig(inv(M)*K);
V = fliplr(V);
w = flipud(sqrt(diag(D)));                  % natural frequencies (rad/s)

% initial conditions
q0 = [0 2 0]';
qdot0 = [0 0 0]';

% compute unknown coefficients using ICs
N = length(w);                              % # of modes
mu = zeros(N,1);                            % modal masses
phi = mu;                                   % phase angles
A = mu;                                     % amplitudes
for i = 1:N
    mu(i) = V(:,i)'*M*V(:,i);
    phi(i) = atan2(w(i)*V(:,i)'*M*q0,V(:,i)'*M*qdot0);
    A(i) = V(:,i)'*M*q0/(mu(i)*sin(phi(i)));
end
```

```
% compute response trajectories
t = linspace(0,10*2*pi/w(1),1001);
q = zeros(N,length(t));
for i = 1:N
    for j = 1:N
        q(i,:) = q(i,:) + A(j)*sin(w(j)*t+phi(j))*V(i,j);
    end
end

figure;
for i = 1:N
    subplot(N,1,i);
    plot(t,q(i,:));
    ylabel(sprintf('x_%d(t)',i));
    xlabel('t');
end
```