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Transient Response of a Dynamic System Under Random Excitation

This paper analyzes the transient response of a simple harmonic oscillator to a stationary random input having an arbitrary power spectrum. The application of the results of this analysis to the response of structures to strong-motion earthquakes is discussed.

A NUMBER of recent papers [1, 2, 3, 4][1] have considered the response of dynamic systems to random excitation. With the exception of Bogdanoff's paper [3], the analysis has been confined to the stationary aspects of the motion. The study of the transient motion of a system under random excitation has a history extending over the past 30 years. The first solution to the problem was given by Uhlenbeck and Ornstein [5] in 1930, and in a later paper Uhlenbeck and Wang [6] solved the same problem using the Fokker-Planck equation for the system. Housner [7] was the first to treat the earthquake problem as an example of random excitation; unfortunately, he did not publish his results. The most recent paper on the subject is that of Bogdanoff and Goldberg [3] who solved exactly the same problem as Uhlenbeck and Ornstein [5]. With the exception of Housner [7], all these authors assumed the input process to be white; this assumption radically simplifies the mathematical problems of analysis.

The purpose of this paper is to analyze the transient motion of a single-degree-of-freedom oscillator subjected to a stationary random input having an arbitrary power spectrum. An approximate solution is presented for the case of small damping and a smooth power spectrum having no sharp peaks. The application of the results of this analysis to determining the response of structures to strong motion earthquakes is discussed.

Analysis

Consider a simple harmonic oscillator acted upon by a random force. The equation of motion is

$$m\ddot{x} + \beta\dot{x} + kx = m\alpha(t) \tag{1}$$

where

$$\left.\begin{aligned} m &= \text{mass of oscillator} \\ \beta &= \text{constant of viscous damper} \\ k &= \text{linear spring constant} \end{aligned}\right\} \tag{2}$$

$$m\alpha(t) = \text{external force acting on mass}$$

Divide both sides of (1) by m:

Let

$$\begin{aligned} \beta/m &= 2\zeta\omega_0 \\ k/m &= \omega_0^2 \end{aligned} \tag{3}$$

[1] Numbers in brackets designate References at the end of the paper.

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Equation (1) becomes

$$\ddot{x} + 2\omega_0\zeta\dot{x} + \omega_0^2 x = \alpha(t) \tag{4}$$

In the analysis which follows, it will be assumed that:

$$\left.\begin{aligned} &(a)\ \alpha(t) \text{ is stationary} \\ &(b)\ \alpha(t) \text{ is Gaussian} \\ &(c)\ \alpha(t) \text{ has mean zero} \\ &(d)\ \alpha(t) \text{ has power spectrum } \phi(\omega) \end{aligned}\right\} \tag{5}$$

Let $h(t)$ be the impulse response of the system:

$$\begin{aligned} h(t) &= \frac{e^{-\zeta\omega_0 t}}{\omega_1}\sin\omega_1 t, \quad t \geq 0 \\ &= 0 \qquad\qquad\qquad\quad t < 0 \end{aligned} \tag{6}$$

where

$$\omega_1 = \omega_0(1 - \zeta^2)^{1/2}$$

Only the case where $\zeta < 1$ will be considered, since it is the case of most interest.

If the initial conditions are

$$x(0) = a; \quad \dot{x}(0) = b \tag{7}$$

Then, if $\alpha(t)$ is assumed to be mean square continuous, a valid solution of (4) is

$$x(t) = ae^{-\zeta\omega_0 t}\left[\cos\omega_1 t + \frac{\zeta\omega_0}{\omega_1}\sin\omega_1 t\right] + \frac{b}{\omega_1}e^{-\zeta\omega_0 t}\sin\omega_1 t + \int_0^t h(t-\tau)\alpha(\tau)d\tau \tag{8}$$

Since it was assumed that $\alpha(t)$ was Gaussian, and since the system is linear, it may be shown that $x(t)$ is Gaussian also [6, 9]. For a Gaussian process, only the mean and the variance are required to characterize the complete process. It is therefore necessary to compute the stochastic, or ensemble average, $\langle x(t)\rangle$, and the variance $\sigma^2(t)$ to characterize the $x(t)$ process. Once these two functions of time are known, the probabilistic description of $x(t)$ is known.

Stochastic Average of $x(t)$

The stochastic average of $x(t)$ is obtained by taking the stochastic average of equation (8).

$$\therefore\ \mu(t) = \langle x(t\ \rangle = ae^{-\omega_0\zeta t}\left[\cos\omega_1 t + \frac{\omega_0\zeta}{\omega_1}\sin\omega_1 t\right] + \frac{b}{\omega_1}e^{-\omega_0\zeta t}\sin\omega_1 t + \int_0^t h(t-\tau)\langle\alpha(\tau)\rangle d\tau \tag{9}$$

But by assumption (5c), $\langle \alpha(\tau) \rangle = 0$. Hence,

$$\mu(t) = \langle x(t) \rangle = ae^{-\omega_0\zeta t}\left[\cos\omega_1 t + \frac{\omega_0\zeta}{\omega_1}\sin\omega_1 t\right] + \frac{b}{\omega_1}e^{-\omega_0\zeta t}\sin\omega_1 t \quad (10)$$

It is seen from the foregoing that the stochastic mean of the motion depends only on the initial conditions.

Variance of $x(t)$

The variance of $x(t)$ is defined to be

$$\sigma^2(t) = \langle [x(t) - \mu(t)]^2 \rangle \quad (11)$$

Using equations (8) and (10) and substituting in (11),

$$\sigma^2(t) = \int_0^t \int_0^t h(t-\tau)h(t-\tau')\langle \alpha(\tau)\alpha(\tau')\rangle d\tau\, d\tau' \quad (12)$$

The quantity $\langle \alpha(\tau)\ \alpha(\tau') \rangle$ is by definition, $R_\alpha(\tau,\tau')$, the autocorrelation function for $\alpha(\tau)$. Since it was assumed in (5a) that $\alpha(t)$ was stationary, then

$$\langle \alpha(\tau)\ \alpha(\tau') \rangle = R_\alpha(\tau,\tau') = R_\alpha(\tau - \tau') \quad (13)$$

The autocorrelation function for a stationary process depends only on the time difference $(\tau - \tau')$, and not on τ and τ' individually. The simplest problem to treat is that of a completely random function, for which $R(\tau - \tau')$ is given simply by $2D\ \delta(\tau - \tau')$. Such a process has a white power spectrum with a spectral density $4D$/cycle. Although such a process allows considerable mathematical simplification, it is a process which can never be realized physically since it would involve infinite mean-squared power. All physically realizable processes involve power spectra which go to zero for sufficiently high frequencies. If $\phi(\omega)$ is the power spectrum under consideration, the requirement of physical realizability is that

$$\int_0^\infty \phi(\omega)d\omega < \infty \quad (14)$$

For a stationary random process, the autocorrelation function may be obtained from the power spectrum or vice versa, since they form a Fourier cosine pair (6).

The autocorrelation function $R_\alpha(\tau - \tau')$ is given by

$$R_\alpha(\tau - \tau') = \int_0^\infty \phi(\omega)\cos\omega(\tau - \tau')d\omega \quad (15)$$

where $\phi(\omega)$ is the power spectrum of $\alpha(t)$. Substituting (15) into (12), the variance is given by

$$\sigma^2(t) = \int_0^t \int_0^t \int_0^\infty \phi(\omega)\cos\omega(\tau-\tau')h(t-\tau)\, h(t-\tau')d\omega d\tau d\tau' \quad (16)$$

Since the integrals involved in (16) are convergent, the order of integration may be reversed. Using the expression for $h(t)$ given in equation (6),

$$\sigma^2(t) = \int_0^\infty \frac{\phi(\omega)}{\omega_1^2}\int_0^t \int_0^t e^{-\zeta\omega_0(2t-\tau-\tau')}\sin\omega_1(t-\tau)\, \sin\omega_1(t-\tau')\cos\omega(\tau-\tau')d\omega d\tau\, d\tau' \quad (17)$$

The double integral on t occurring in (17) may be evaluated after some tedious algebra, giving

$$\sigma^2(t) = \int_0^\infty \frac{\phi(\omega)}{|z(\omega)|^2}\Bigg[1 + e^{-2\omega_0\zeta t}\Bigg\{1 + \frac{2\omega_0}{\omega_1}\zeta\sin\omega_1 t\cos\omega_1 t - e^{\omega_0\zeta t}\left(2\cos\omega_1 t + \frac{2\omega_0\zeta}{\omega_1}\sin\omega_1 t\right)\cos\omega t - e^{\omega_0\zeta t}\frac{2\omega}{\omega_1}\sin\omega_1 t\sin\omega t + \frac{(\omega_0\zeta)^2 - \omega_1^2 + \omega^2}{\omega_1^2}\sin^2\omega_1 t\Bigg\}\Bigg]d\omega \quad (18)$$

where

$$|z(\omega)|^2 = (\omega_0^2 - \omega^2)^2 + (2\omega\omega_0\zeta)^2 \quad (19)$$

Equation (18) exhibits some interesting properties. They are:
(1) As $t \to 0$

$\sigma^2(t) \to 0$, as would be expected;

(2) As $t \to \infty$

$\sigma^2(t) \to \int_0^\infty \frac{\phi(\omega)d\omega}{|z(\omega)|^2}$, the result which would be predicted from generalized harmonic analysis for the stationary problem;

(3) If $\phi(\omega)$ is set equal to $(2D/\pi)$ and the integral (18) evaluated by contour integration, then

$$\sigma_x^2(t) = \frac{D}{2\zeta\omega_0^3}\left[1 - \frac{e^{-2\omega_0\zeta t}}{\omega_1^2}\left\{\omega_1^2 + \omega_0\omega_1\zeta\sin 2\omega_1 t + \frac{(2\omega_0\zeta)^2}{2}\sin^2\omega_1 t\right\}\right] \quad (20)$$

As would be expected, this result is identical with the result obtained by Uhlenbeck [5] for a white process.

Approximate Evaluation of Equation (18)

For any analytic function $\phi(\omega)$, equation (18) can be evaluated by contour integration. If $\phi(\omega)$ is given numerically, then equation (18) can be evaluated numerically. However, if $\phi(\omega)$ is a smooth function of ω, having no sharp peaks, and ζ is small, then a very good approximation may be obtained in the following way: If ζ is small, the function $1/|z(\omega)|^2$ is sharply peaked at $\omega = \omega_0$; therefore, the main contribution to the integral comes from the region around $\omega = \omega_0$. By analogy with Laplace's method of evaluating integrals, equation (18) may be approximated by

$$\sigma_x^2(t) \simeq \phi(\omega_0)\Bigg[\Bigg[\int_0^\infty \frac{1}{|z(\omega)|^2}\Bigg[1 + e^{-2\omega_0\zeta t}\Bigg\{1 + \frac{2\omega_0\zeta}{\omega_1}\sin\omega_1 t\cos\omega_1 t - e^{\omega_0\zeta t}\left(2\cos\omega_1 t + \frac{2\omega_0\zeta}{\omega_1}\sin\omega_1 t\right)\cos\omega t - e^{\omega_0\zeta t}(\sin\omega_1 t\sin\omega t)\left(\frac{2\omega}{\omega_1}\right) + \frac{(\omega_0\zeta)^2 - \omega_1^2 + \omega^2}{\omega_1^2}\sin^2\omega_1 t\Bigg\}\Bigg]d\omega\Bigg]\Bigg] \quad (21)$$

Evaluating the integral in (21) by contour integration,

$$\sigma_x^2(t) \simeq \frac{\pi\phi(\omega_0)}{4\zeta\omega_0^3}\left[1 - \frac{e^{-2\omega_0\zeta t}}{\omega_1^2}\left\{\omega_1^2 + \frac{(2\omega_0\zeta)^2}{2}\sin^2\omega_1 t + \omega_0\omega_1\zeta\sin 2\omega_1 t\right\}\right] \quad (22)$$

Zero Damped Oscillator

Of special interest is the case when ζ is zero. The results for this case can be obtained from equation (22) by a limiting process:

$$\sigma_x^2(t)_{\zeta=0} = \lim_{\zeta\to 0} \frac{\pi}{4}\,\frac{\phi(\omega_0)}{\zeta\omega_0^3}\left[1 - \frac{e^{-2\omega_0\zeta t}}{\omega_1^2}\left\{\omega_1^2 + \frac{(2\omega_0\zeta)^2}{2}\sin^2\omega_1 t + \omega_0\omega_1\zeta\sin 2\,\omega_1 t\right\}\right]$$

$$= \frac{\pi}{4}\,\frac{\phi(\omega_0)}{\omega_0^3}\lim_{\zeta\to 0}\left[\{1 - (1 - 2\omega_0\zeta t)(1 + \zeta\sin 2\,\omega_0 t) + 0[(\zeta t)^2]\}\zeta^{-1}\right] \quad (23)$$

$$\therefore\ \sigma_x^2(t) \simeq \frac{\pi}{4}\,\frac{\phi(\omega_0)}{\omega_0^3}\,[2\omega_0 t - \sin 2\omega_0 t] \quad (24)$$

Results

For convenience, equation (22) may be written

$$\frac{2\sigma_x^2}{\pi}\,\frac{\omega_0^3}{\phi(\omega_0)} \simeq \frac{1}{2\zeta}\left[1 - e^{-2\zeta\theta}\left\{1 + \frac{2\zeta^2}{1-\zeta^2}\sin^2(1-\zeta^2)^{1/2}\theta + \frac{\zeta}{(1-\zeta^2)^{1/2}}\sin 2\,(1-\zeta^2)^{1/2}\theta\right\}\right] \quad (25)$$

where $\theta = \omega_0 t$.

Plots of equation (25) are shown in Fig. 1 for $\zeta = 0$, 0.025, 0.05, and 0.10. It will be observed that for $\zeta = 0.1$, the system approaches stationarity in roughly three cycles. Hence, even though the output process is nonstationary, only a slight error is made by treating it as a stationary process, provided the stationary input is applied for a sufficiently large number of cycles.

Distribution Functions for $x(t)$

As was pointed out previously, $x(t)$ is Gaussian if $\alpha(t)$ is Gaussian. Hence the probability that x lies in the interval x to $x + dx$ is given by

$$p(x)dx = \frac{1}{(2\pi)^{1/2}\sigma(t)}\exp\left\{-\frac{(x-\mu(t))^2}{2\sigma^2(t)}\right\}dx \quad (26)$$

where

$$\mu(t) = \langle x(t)\rangle \qquad \sigma^2(t) = \langle[x(t) - \mu(t)]^2\rangle \quad (27)$$

The stochastic average $\mu(t)$ is given by equation (9), and depends only on the initial velocity and displacement. The variance $\sigma^2(t)$ is given by equation (18), and is independent of the initial conditions.

Probability of Exceeding a Given Value

The probability that $x(t)$ exceeds a given value, $k\sigma$, is obtained by integration of equation (26):

$$P(x > k\sigma) = \int_{k\sigma}^{\infty}\frac{1}{(2\pi)^{1/2}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx \quad (28)$$

$$= \tfrac{1}{2}\left[1 - \operatorname{erf}\left\{\frac{k\sigma - \mu}{2^{1/2}\sigma}\right\}\right] \quad (29)$$

where erf is the error function.

For some physical processes, for example brittle failure, the sign of $x(t)$ is unimportant. The probability that the modulus of $x(t)$, $|x|$, exceeds a given value, $k\sigma$, is obtained by integration of equation (26):

$$\therefore\ P(|x| > k\sigma) = \int_{k\sigma}^{\infty}\frac{1}{(2\pi)^{1/2}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx + \int_{-\infty}^{-k\sigma}\frac{1}{(2\pi)^{1/2}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}dx \quad (30)$$

$$\therefore\ P(|x| > k\sigma) = 1 - \tfrac{1}{2}\left\{\operatorname{erf}\frac{(k\sigma - \mu)}{2^{1/2}\sigma} + \operatorname{erf}\frac{(k\sigma + \mu)}{2^{1/2}\sigma}\right\} \quad (31)$$

In the special case of a system starting from rest, the probability that x exceeds $k\sigma$ is given by

$$P(|x| > k\sigma) = \left(1 - \operatorname{erf}\frac{k}{2^{1/2}}\right) \quad (32)$$

It is interesting to note that the right-hand side of equation (32) does not involve time, even though x and σ are both functions of time. This is in contrast to the situation in equations (29) and (31), where the right-hand side of the equations is also a function of time.

Application of Results to Strong Motion Earthquakes

An intensive study of strong motion earthquakes has been

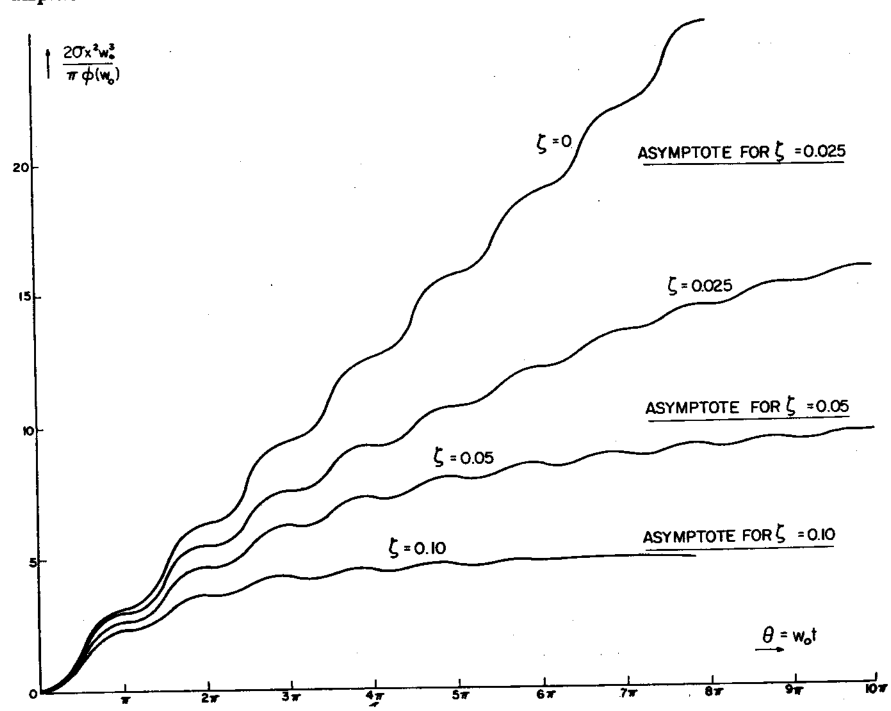

Fig. 1 Transient response of a dynamic system under random excitation

made at the California Institute of Technology [8] over the past 25 years. As a result of these studies, the following statements may be made:

1 The duration of strong motion earthquakes is in direct relation to their intensity. The stronger the earthquake, the longer it lasts.

2 The strong earthquakes tend to be quite random in nature. The weak ones tend to be more deterministic.

3 While no earthquake is completely stationary, since it must first build up, and must eventually die out, the very strong earthquakes exhibit long portions of quasi-stationary behavior.

4 No earthquake exhibits a white-power spectrum. All earthquakes so far analyzed show a peaked power spectrum falling off sharply at high frequencies.

5 Unfortunately, the amount of data on strong motion earthquakes is quite limited, so it has not been possible to obtain anything like a complete statistical description of earthquakes. However, the available data show that, at least out to the 3σ point, the stronger earthquakes exhibit Gaussian statistics with mean zero.

Applying these statements to the foregoing analysis we see that:

(a) The results of the analysis may be applied with some confidence to the very strong, long-duration earthquakes, since the analysis did not assume a white process.

(b) For the shorter earthquakes, some attempt should be made to account for the nonstationarity of the input process. This has been done for a particular class of inputs by Stumpf [9] in his Doctoral thesis.

(c) Some caution must be exercised in applying the distribution function for the displacement in the case of earthquakes. It is well known that the probability of exceeding a specified value is quite sensitive to the tail of the distribution function, and this is the very area where our ignorance is greatest in the earthquake problem.

Example

As an illustration of the application of the foregoing theory to earthquakes, consider the response of a single-story building to a strong motion earthquake. For the purposes of analysis, the building will be treated as a roof of mass m, supported by columns of shear stiffness k. The variable x in equation (1) will then represent the displacement of the roof relative to the base of the building.

Let ω_0 and ζ in equation (4) be given by

$$\left.\begin{aligned} \omega_0 &= 5 \text{ rad/sec} \\ \zeta &= 0.025 \end{aligned}\right. \qquad (33)$$

Kanai [10] has analyzed a large number of earthquakes and has suggested that the spectral density, $\phi(\omega)$, of the ground motion of earthquakes may be expressed by

$$\phi(\omega) = \frac{1 + 4h_g^2 \dfrac{\omega^2}{\nu_g^2}}{\left(1 - \dfrac{\omega^2}{\nu_g^2}\right)^2 + 4h_g^2 \dfrac{\omega^2}{\nu_g^2}} B \qquad (34)$$

$$B = \text{const}$$

where

$$B = \text{spectral density at bedrock}$$

$$\nu_g, h_g = \text{parameters depending on local geology}$$

For the purposes of illustration let

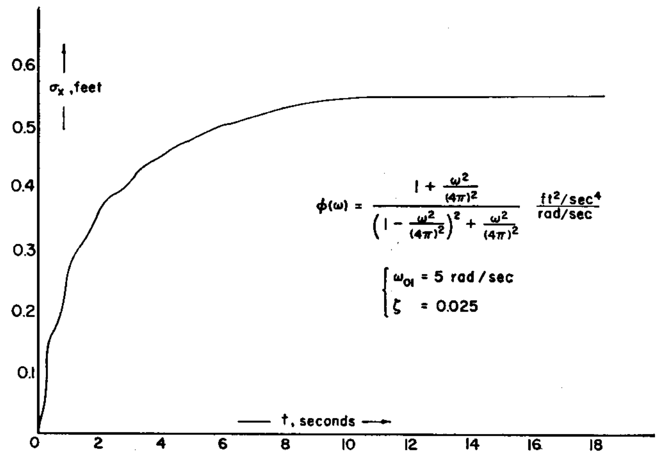

Fig. 2 Transient response of building to random earthquake

$$\left.\begin{aligned} h_g &= 0.5 \\ \nu_g &= 4\pi \\ B &= 1 \text{ ft}^2/\text{sec}^4/\text{rad/sec} \end{aligned}\right\} \qquad (35)$$

These values were so chosen that $\phi(\omega)$ approximates that for the El Centro, Calif., earthquake of May 1940 [8], which lasted for 25 sec. Using equation (22), the root-mean-squared displacement, $\sigma_x(t)$, is given by

$$\sigma_x(t) \simeq \left[\left[\frac{\pi\phi(\omega_0)}{4\zeta\omega_0^3}\left\{1 - \frac{e^{-2\omega_0\zeta t}}{\omega_1^2}\left[\omega_1^2 + \frac{(2\omega_0\zeta)^2}{2}\sin^2\omega_1 t + \omega_0\omega_1\zeta\sin 2\omega_1 t\right]\right\}\right]\right]^{1/2} \qquad (36)$$

Fig. 2 shows the numerical results of substituting (33), (34), and (35) into (36).

It will be seen that the root-mean-squared displacement σ_x reaches a stationary value of 0.57 ft in roughly 9 sec. Since the earthquake is assumed to last for 25 sec, little error would result in treating the problem as a stationary one.

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