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THE FORCED VIBRATION OF SINGLY MODIFIED DAMPED ELASTIC SURFACE SYSTEMS

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A technique is developed to predict the forced vibration of membranes, beams, plates or shells when they have attached to them at a single point a linear lumped parameter element or assembly of elements. The distributed parameter element is treated as viscously damped and the lumped parameter assembly may also contain viscous dampers. Solution is obtained in terms of generalized Fourier series in the unmodified eigenfunctions for the distributed portion of the system and the principle of superposition is used to handle the imposed forces and those generated at the attachment. The method is illustrated by investigating a uniformly forced simply supported rectangular plate with a lumped mass at its center and that of a point forced simple beam with a rigid pin support imposed at some arbitrary point.

1. INTRODUCTION

The forced vibration of distributed parameter vibratory systems is a relatively straightforward problem given a spatially uniform system with reasonable boundary conditions. The solutions to these problems are outlined in the excellent text by Meirovitch [1]. The problem of exact analysis of systems which are a combination of a distributed system and lumped parameter elements has been accomplished by classical boundary value techniques for finding the natural frequencies and the associated mode shapes of such systems when damping is not present [2–7]. An alternative approach is that known as eigenvalue modification whereby the dynamic structure of the modified system is evaluated by using the known structure of the system prior to the modification [8–12]. Another paper explores the effect of rigid point constraints on the natural frequencies and modes of elastic surface systems [13]. These methods have been primarily applied to finding eigenvalues and eigenvectors (eigenfunctions) for the modified system.

In a recent paper Howell [14] extends the ideas of Hallquist and Snyder [12] to the forced random vibration of linearly constrained damped linear structures.

The work reported here is based on an extension of the work of Jacquot and Soedel [9] who were concerned with the prediction of natural frequencies and mode shapes associated with modified elastic surface systems. Here a general method will be given to evaluate the forced vibration of a surface system which has a linear lumped parameter element or assembly of these elements attached to the surface system at a single point. These modifications can also be of a nature such as to impose the condition of zero motion at a point on the surface. The technique given relies on eigenfunction expansions in terms of the eigenfunctions of the unmodified surface system. Use of these exact eigenfunctions instead of assumed modes of polynomials as might be done in an assumed mode approach is very useful when evaluating dynamic stresses due to the need to find spatial derivatives of the mode shapes.

The applications given are that of forced vibration of a damped simply supported plate carrying a concentrated mass at its center and the forced vibration of a simply supported beam with a rigid pin support at some arbitrary point.

2. THEORY

2.1. THE UNMODIFIED SYSTEM

In this work the problem considered is one of a damped uniform elastic structure on domain D described by

$$L[w] + c\,\partial w/\partial t + m\,\partial^2 w/\partial t^2 = g(t)\,h(r), \tag{1}$$

where $L[\cdot]$ is a linear spatial elastic operator in one or two dimensions, $w(\mathbf{r}, t)$ is the deflection of the elastic surface normal to the undeformed surface, m is the mass distribution parameter, c is the damping coefficient, and $h(\mathbf{r})$ and $g(t)$ represent, respectively, the spatial and temporal portions of the forcing function. The operator $L[\cdot]$ is normally composed of second and/or fourth spatial derivatives of deflection.

It will be assumed the unforced, undamped problem can initially be described by the equation

$$L[w] + m\,\partial^2 w/\partial t^2 = 0, \tag{2}$$

with a sufficient number of boundary conditions to cause the problem to be well posed,

$$B_i[w(\mathbf{r}_0, t)] = 0, \qquad i = 1, 2, \ldots. \tag{3}$$

The eigenvalues ω_{ij}^2 and eigenfunctions $\phi_{ij}(\mathbf{r})$ satisfy equations (2) and (3) such that

$$L[\phi_{ij}(\mathbf{r})] = m\omega_{ij}\,\phi_{ij}(\mathbf{r}). \tag{4}$$

These eigenfunctions can be normalized with respect to the weighting function in $L[\cdot]$ such that

$$\int_D \phi_{ij}(\mathbf{r})\,\phi_{kl}(\mathbf{r})\,p(\mathbf{r})\mathrm{d}D = \delta_{ij}^{kl}, \tag{5}$$

where $p(\mathbf{r})$ is the weighting function and δ_{ij}^{kl} is the two-dimensional Kronecker delta. Assume that the spatial distribution of the forcing function $h(\mathbf{r})$ is known so it may be expanded in generalized Fourier series of the eigenfunctions of

$$h(\mathbf{r}) = \sum_{i,\,j=1}^{\infty} h_{ij}\,\phi_{ij}(\mathbf{r}), \tag{6}$$

and since the eigenfunctions are orthogonal the Fourier coefficients are given by

$$h_{ij} = \langle h(\mathbf{r}), \phi_{ij}(\mathbf{r})\rangle, \tag{7}$$

where the brackets denote inner product in the sense that the integrand is weighted with $p(\mathbf{r})$. Expression (6) will not be convergent for some forms of $h(\mathbf{r})$, for example delta and doublet functions. This, however, is no serious limitation on what follows. If now the Fourier transform of equation (1) is taken with respect to time the result is

$$L[W(\mathbf{r}, \omega)] + \mathrm{j}\omega c W(\mathbf{r}, \omega) - m\omega^2\,W(\mathbf{r}, \omega) = G(\omega) \sum_{i=1}^{\infty} h_{ij}\,\phi_{ij}(\mathbf{r}). \tag{8}$$

At this point expand the frequency domain motion $W(\mathbf{r}, \omega)$ as a series in the eigenfunctions

$$W(\mathbf{r}, \omega) = \sum_{i,\,j=1}^{\infty} Q_{ij}(\omega)\,\phi_{ij}(\mathbf{r}). \tag{9}$$

It is also known what the effect of the operator $L[\cdot]$ is on the eigenfunctions from equation (4) so

$$L[W(\mathbf{r}, \omega)] = m \sum_{i,\,j=1}^{\infty} Q_{ij}(\omega)\,\omega_{ij}^2\,\phi_{ij}(\mathbf{r}). \tag{10}$$

Substitution of equations (9) and (10) into equation (8) yields

$$\sum_{i,\,j=1}^{\infty} [m\omega_{ij}^2 + \mathrm{j}\omega c - m\omega^2]\, Q_{ij}(\omega)\, \phi_{ij}(\mathbf{r}) = G(\omega) \sum_{i,\,j=1}^{\infty} h_{ij}\, \phi_{ij}(\mathbf{r}). \tag{11}$$

So equating term by term yields

$$Q_{ij}(\omega) = G(\omega)\, h_{ij}/\{m(\omega_{ij}^2 - \omega^2) + \mathrm{j}\omega c\}$$

and the frequency domain motion is, by substitution into equation (9),

$$W(\mathbf{r}, \omega) = G(\omega) \sum_{i,\,j=1}^{\infty} h_{ij}\, \phi_{ij}(\mathbf{r})/\{m(\omega_{ij}^2 - \omega^2) + \mathrm{j}\omega c\}. \tag{12}$$

2.2. THE FLEXIBLY MODIFIED SYSTEM

Now consider the identical system as in equation (1) except let the system be acted on at some point $\mathbf{r}_1$ by a force $f(t)$ such that $f(t)$ is a linear function of the motion $w(\mathbf{r}_1, t)$. In the frequency domain this takes the form

$$F(\omega) = -Z(\omega)\, W(\mathbf{r}_1, \omega), \tag{13}$$

where $Z(\omega)$ is the driving point impedance of the modification. The equation of motion when subjected to this additional force becomes

$$L[w] + c\, \partial w/\partial t + m\, \partial^2 w/\partial t^2 = g(t)\, h(\mathbf{r}) + f(t)\, \delta(\mathbf{r} - \mathbf{r}_1). \tag{14}$$

Expand the delta function in a Fourier series in the eigenfunctions

$$\delta(\mathbf{r} - \mathbf{r}_1) = \sum_{i,\,j=1}^{\infty} a_{ij}\, \phi_{ij}(\mathbf{r}), \tag{15}$$

where

$$a_{ij} = \langle \delta(\mathbf{r} - \mathbf{r}_1), \phi_{ij}(\mathbf{r}) \rangle. \tag{16}$$

Upon employing the sampling property of the delta function the expansion becomes

$$\delta(\mathbf{r} - \mathbf{r}_1) = \sum_{i,\,j=1}^{\infty} p(\mathbf{r}_1)\, \phi_{ij}(\mathbf{r}_1)\, \phi_{ij}(\mathbf{r}). \tag{17}$$

Upon Fourier transforming equation (14) and employing a solution of the form of expression (9) the result is

$$\sum_{i,\,j=1}^{\infty} [m(\omega_{ij}^2 - \omega^2) + \mathrm{j}\omega c]\, Q_{ij}(\omega)\, \phi_{ij}(\mathbf{r}) = G(\omega) \sum_{i,\,j=1}^{\infty} h_{ij}\, \phi_{ij}(\mathbf{r}) + F(\omega)\, p(\mathbf{r}_1) \sum_{i,\,j=1}^{\infty} \phi_{ij}(\mathbf{r}_1)\, \phi_{ij}(\mathbf{r}). \tag{18}$$

Define the quantity

$$Y_{ij}(\omega) = [m(\omega_{ij}^2 - \omega^2) + \mathrm{j}\omega c]^{-1} \tag{19}$$

and solve equation (18) for $Q_{ij}(\omega)$:

$$Q_{ij}(\omega) = Y_{ij}(\omega)[G(\omega)\, h_{ij} + F(\omega)\, p(\mathbf{r}_1)\, \phi_{ij}(\mathbf{r}_1)]. \tag{20}$$

Recall that the force $F(\omega)$ is linearly related to $W(\mathbf{r}_1, \omega)$ as in expression (13) so

$$Q_{ij}(\omega) = Y_{ij}(\omega)[G(\omega)\, h_{ij} - Z(\omega)\, W(\mathbf{r}_1, \omega)\, p(\mathbf{r}_1)\, \phi_{ij}(\mathbf{r}_1)]. \tag{21}$$

Then the frequency domain solution is

$$W(\mathbf{r}, \omega) = G(\omega) \sum_{i,\,j=1}^{\infty} Y_{ij}(\omega)\, h_{ij}\, \phi_{ij}(\mathbf{r}) - Z(\omega)\, W(\mathbf{r}_1, \omega)\, p(\mathbf{r}_1) \sum_{i,\,j=1}^{\infty} Y_{ij}(\omega)\, \phi_{ij}(\mathbf{r})\, \phi_{ij}(\mathbf{r}_1). \tag{22}$$

At this point it appears that an impasse has been reached but if equation (22) is to hold for all $\mathbf{r}$ then it must hold for $\mathbf{r} = \mathbf{r}_1$ and thus equation (22) may be solved for $W(\mathbf{r}_1, \omega)$ yielding

$$W(\mathbf{r}_1, \omega) = \frac{G(\omega) \sum_{i,j=1}^{\infty} Y_{ij}(\omega) h_{ij} \phi_{ij}(\mathbf{r}_1)}{1 + Z(\omega) p(\mathbf{r}_1) \sum_{i,j=1}^{\infty} Y_{ij}(\omega) \phi_{ij}^2(\mathbf{r}_1)}. \tag{23}$$

With expression (23) as a solution for $W(\mathbf{r}_1, \omega)$ it may be substituted back into equation (22) to give the frequency domain solution for any point $\mathbf{r}$ on the elastic surface.

2.3. THE INFLEXIBLY MODIFIED SYSTEM

In the previous section the solution for the motion at point $\mathbf{r}$ in terms of $G(\omega)$ and $F(\omega)$ could be shown to be given by

$$W(\mathbf{r}, \omega) = G(\omega) \sum_{i,j=1}^{\infty} Y_{ij}(\omega) h_{ij} \phi_{ij}(\mathbf{r}) + F(\omega) p(\mathbf{r}_1) \sum_{i,j=1}^{\infty} Y_{ij}(\omega) \phi_{ij}(\mathbf{r}_1) \phi_{ij}(\mathbf{r}). \tag{24}$$

If total rigidity is imposed on the structure at $\mathbf{r} = \mathbf{r}_1$ then the force at that point must be such that the motion at that point is zero. By evaluating expression (24) and equating the result to zero $F(\omega)$ can be found to be

$$F(\omega) = \frac{-G(\omega) \sum_{i,j=1}^{\infty} Y_{ij}(\omega) h_{ij} \phi_{ij}(\mathbf{r}_1)}{p(\mathbf{r}_1) \sum_{i,j=1}^{\infty} Y_{ij}(\omega) \phi_{ij}^2(\mathbf{r}_1)}. \tag{25}$$

Substitution of equation (25) into equation (24) for the motion gives

$$W(\mathbf{r}, \omega) = G(\omega) \sum_{i,j=1}^{\infty} Y_{ij}(\omega) \phi_{ij}(\mathbf{r}) \left[h_{ij} - \phi_{ij}(\mathbf{r}_1) \frac{\sum_{k,l=1}^{\infty} Y_{kl}(\omega) h_{kl} \phi_{kl}(\mathbf{r}_1)}{\sum_{k,l=1}^{\infty} Y_{kl}(\omega) \phi_{kl}^2(\mathbf{r}_1)} \right]. \tag{26}$$

In the case where lumped parameter elements are attached at more than one point then expression (22) will have the corresponding number of terms on the right side each involving the displacement of the corresponding point of attachment and the driving point impedance of the elements driven at that point. The equation could then be evaluated at each point of attachment yielding an inhomogeneous set of linear equations in the frequency domain motions of the points of attachment. This set of linear equations could be solved for all the unknown frequency domain motions which in turn could be substituted back into the equation for the motion at a general location.

What has been developed here is a technique for predicting frequency domain modified system response. This technique could be easily applied to the prediction of the power spectral density function for the structural motion given the power spectral density of the forcing function $g(t)$ and the information that the spatial dependence of the forcing function is deterministic and not random. If the spatial dependence of the forcing function is random then the h_{ij} coefficients will be random variables and the problem should be handled by using mean square calculus [15].

3. APPLICATION I

Consider the forced vibration of a simply supported rectangular plate carrying a mass at its center as illustrated in Figure 1. The forcing function is assumed to be uniform over the surface of the plate and the elastic operator is $L[\cdot] = D\nabla^4[\cdot]$, where D is the flexural rigidity

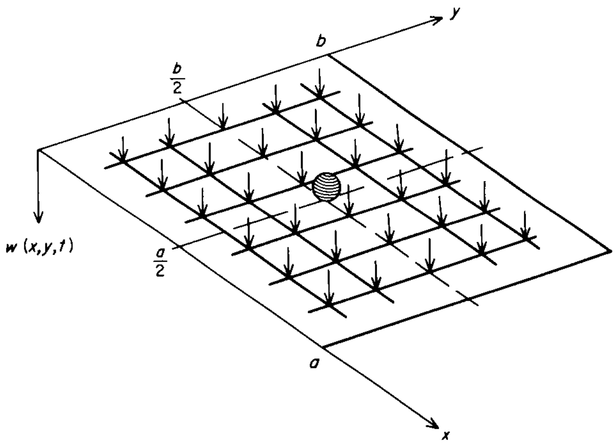

Figure 1. Simply supported rectangular plate carrying a mass subject to a uniform forcing function.

of the plate defined by $D = Eh^3/12(1 - \nu^2)$. The orthonormal eigenfunctions are

$$\phi_{ij}(x, y) = \frac{2}{\sqrt{ab}} \sin\frac{i\pi x}{a} \sin\frac{j\pi y}{b}$$

and the associated natural frequencies are

$$\omega_{ij} = [(i\pi/a)^2 + (j\pi/b)^2]\sqrt{D/m},$$

while the associated weighting function is unity. The forcing function is uniform so $h(x,y) = 1$ and the associated Fourier coefficients are

$$h_{ij} = \begin{cases} ab/\pi^2 ij, & i,j \text{ odd}, \\ 0, & i,j \text{ even}. \end{cases}$$

The driving point impedance of the mass is $Z(\omega) = -M\omega^2$. The motion of the point of attachment is given by application of expression (23) for a location of the mass at the center of the plate,

$$W\left(\frac{a}{2},\frac{b}{2};\omega\right) = \frac{G(\omega) \sum\limits_{i,\,j=1,\,3,\,\ldots}^{\infty} \frac{2\sqrt{ab}}{ij\pi^2} Y_{ij}(\omega) \cos\left[(i-j)\frac{\pi}{2}\right]}{1 - M\omega^2 \sum\limits_{i,\,j=1,\,3,\,\ldots}^{\infty} \frac{4}{ab} Y_{ij}(\omega) \cos\left[(i-j)\frac{\pi}{2}\right]},$$

and the motion of any other point on the plate is given by

$$W(x, y; \omega) = G(\omega) \sum_{i,\,j=1,\,3,\,5,\,\ldots}^{\infty} Y_{ij}(\omega) \frac{ab}{\pi^2 ij} \frac{2}{\sqrt{ab}} \sin\frac{i\pi x}{a} \sin\frac{j\pi x}{b} +$$
$$+ M\omega^2 W\left(\frac{a}{2},\frac{b}{2};\omega\right) \sum_{i,\,j=1,\,3,\,5,\,\ldots}^{\infty} Y_{ij}(\omega) \frac{4}{ab} \cos\left[(i-j)\frac{\pi}{2}\right] \sin\frac{i\pi x}{a} \sin\frac{j\pi x}{b},$$

where the $Y_{ij}(\omega)$ are defined by $Y_{ij}(\omega) = [m(\omega_{ij}^2 - \omega^2) + j\omega c]^{-1}$.

4. APPLICATION II

Consider the point forced vibration of the simply supported beam of Figure 2 with an additional rigid support imposed at some point as indicated. The elastic operator is $L[w] =$

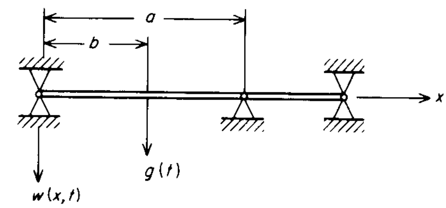

Figure 2. Simply supported beam with intermediate support forced at a point.

$EI\,\partial^4 w/\partial x^4$, where EI is the beam bending stiffness and the associated eigenfunctions are

$$\phi_i(x) = \sqrt{\frac{2}{L}}\sin\frac{i\pi x}{L}\quad i = 1, 2, \ldots.$$

The natural frequencies are

$$\omega_i^2 = (i\pi/L)^4\, EI/m, \qquad i = 1, 2, \ldots,$$

and the weighting function is unity. The spatial distribution of the forcing function is a delta function so the expansion coefficient is $h_i = \sqrt{2/L}\sin i\pi b/L$. Application of expression (26) gives

$$W(x,\omega) = \frac{2G(\omega)}{L}\sum_{i=1}^{\infty} Y_i(\omega)\left[\sin\frac{i\pi b}{L} - \sin\frac{i\pi a}{L}\,\frac{\sum_{k=1}^{\infty} Y_k(\omega)\sin\frac{k\pi b}{L}\sin\frac{k\pi a}{L}}{\sum_{k=1}^{\infty} Y_k(\omega)\sin\frac{2k\pi a}{L}}\right],$$

where $Y_k(\omega)$ is defined by $Y_k(\omega) = [m(\omega_k^2 - \omega^2) + j\omega c]^{-1}$.

5. CONCLUSIONS

A general method to handle the forced vibration of singly modified elastic surface systems for quite general force distributions and general types of linear lumped parameter system modifications or those which are rigid in nature has been developed and applied to the vibration of a simply supported plate carrying a central concentrated mass and the vibration of a simply supported beam with an imposed intermediate pin. This technique could be used in the prediction of second order statistical response properties in the case of stationary random vibration. This technique could also be extended to the case of multiple system modifications for either deterministic or random vibration.

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