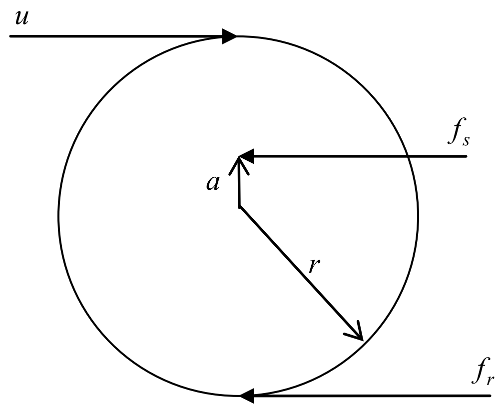
HOMEWORK 1 SOLUTIONS

1. a. The free body diagram of this system looks like



where f_r is the reaction (friction) force of the ground, and f_s is the spring force.

Balance of forces in the horizontal direction and moments about the center of the disc give:

$$\sum F = m\ddot{z} = -f_s - f_r + u$$
$$\sum M = J\ddot{\theta} = -f_s a + f_r r + ur$$

where $z = r\theta$ is the horizontal displacement of the center of the disc.

The compression of the spring is measured by $x = z + a\theta = (r + a)\theta$. Hence,

$$J\ddot{\theta} = -f_s a + f_r r + ur = -f_s a + (u - m\ddot{z} - f_s)r + ur = -f_s(r + a) - m\ddot{z}r + 2ur$$
$$J\ddot{\theta} + m\ddot{z}r + f_s(r + a) = 2ur$$
$$\frac{1}{2}mr^2\ddot{\theta} + mr^2\ddot{\theta} + k(r + a)^2\theta = 2ru$$
$$\frac{\Theta(s)}{U(s)} = \frac{2r}{\frac{3}{2}mr^2s^2 + k(r + a)^2}$$

b. From part (a), $\frac{X(s)}{\Theta(s)} = r + a$. Therefore, $\frac{X(s)}{U(s)} = \frac{X(s)}{\Theta(s)}\frac{\Theta(s)}{U(s)} = \frac{2r(r + a)}{\frac{3}{2}mr^2s^2 + k(r + a)^2}$.

c. The characteristic polynomial is the same, and therefore so is the natural frequency. This has to remain invariant because no matter what states are chosen, they will oscillate freely at the same frequency.

2. $X(s)=\dfrac{3s^2+23s+100}{(s+3)(s^2+2s+26)}$

Check discriminant: $2^2-4\cdot 26=-100<0$, so cannot factor into linear terms

$$
\begin{aligned}
&X(s)=\frac{a_1}{s+3}+\frac{a_2 s+a_3}{s^2+2s+26}\\
&\Rightarrow 3s^2+23s+100=a_1(s^2+2s+26)+(a_2 s+a_3)(s+3)\\
&=(a_1+a_2)s^2+(2a_1+3a_2+a_3)s+(26a_1+3a_3)\\
&\begin{cases} a_1+a_2=3\Rightarrow a_1=2\\ 2a_1+3a_2+a_3=23\Rightarrow a_2=1\\ 26a_1+3a_3=100\Rightarrow a_3=16\end{cases}\\
&X(s)=\frac{2}{s+3}+\frac{s+16}{s^2+2s+26}
\end{aligned}
$$

Complete the squares: $s^2+2s+26=(s+1)^2+25$

$$
\begin{aligned}
X(s)&=\frac{2}{s+3}+\frac{s+1}{(s+1)^2+25}+\frac{15}{(s+1)^2+25}\\
&=\frac{2}{s+3}+\frac{s+1}{(s+1)^2+25}+\frac{15}{5}\frac{5}{(s+1)^2+25}\\
X(s)&=2e^{-3t}+e^{-t}\cos(5t)+3e^{-t}\sin(5t)
\end{aligned}
$$

3. Using the formula for the Laplace Transform of the derivative of a function, we can transform the ODE into

$$\left(s^3X(s)-s^2x_0-s\dot{x}_0-\ddot{x}_0\right)+5\left(s^2X(s)-sx_0-\dot{x}_0\right)+7\left(sX(s)-x_0\right)+3X(s)=0$$

Plugging in the ICs and solving for $X(s)$ yields

$$X(s)=\frac{s^2+5s+7}{s^3+5s^2+7s+13}=\frac{s^2+5s+7}{(s+1)^2(s+3)}$$

This can be expanded by partial fractions into

$$X(s)=\frac{a_1}{(s+1)^2}+\frac{a_2}{s+1}+\frac{a_3}{s+3}$$

Since there are no complex poles, we can use the explicit formulas for the unknown residues:

$$a_1 = (s+1)^2 X(s)\Big|_{s=-1} = \left.\frac{s^2+5s+7}{s+3}\right|_{s=-1} = \frac{3}{2}$$

$$a_2 = \frac{d}{ds}\left[(s+1)^2 X(s)\right]_{s=-1} = \frac{d}{ds}\left[\frac{s^2+5s+7}{s+3}\right]_{s=-1} = \left.\frac{(2s+5)(s+3)-s^2-5s-7}{(s+3)^2}\right|_{s=-1} = \frac{3}{4}$$

$$a_3 = (s+3)X(s)\Big|_{s=-3} = \left.\frac{s^2+5s+7}{(s+1)^2}\right|_{s=-3} = \frac{1}{4}$$

Putting this together, we have

$$X(s) = \frac{3}{2}\frac{1}{(s+1)^2} + \frac{3}{4}\frac{1}{s+1} + \frac{1}{4}\frac{1}{s+3}$$

Taking the inverse Laplace Transform yields

$$x(t) = \frac{3}{2}te^{-t} + \frac{3}{4}e^{-t} + \frac{1}{4}e^{-3t}$$

4. a. A balance of moments about the pivot point 0 gives

$$J_0\ddot{\theta} = -F_{s,left}\left(\frac{l}{4}\right) - F_{s,right}\left(\frac{3l}{4}\right)$$

In terms of θ, the extension of the left spring is $\frac{l}{4}\theta$. In terms of θ and z, the extension of the right spring is $\frac{3l}{4}\theta - z$. Hence,

$$J_0\ddot{\theta} = -k\left(\frac{l}{4}\theta\right)\left(\frac{l}{4}\right) - k\left(\frac{3l}{4}\theta - z\right)\left(\frac{3l}{4}\right)$$

The moment of inertia of the rod about 0 is $J_0 = \frac{1}{12}ml^2 + m\left(\frac{l}{4}\right)^2 = \frac{7}{48}ml^2$, where we have used the Parallel Axis Theorem to shift the axis of rotation from the center of the rod to a point $\frac{l}{4}$ to the left.

Plugging this in and rearranging the EOM into standard form gives

$$\frac{7}{48}ml^2\ddot{\theta} + \frac{5}{8}kl^2\theta = \frac{3}{4}klz$$

Taking the Laplace transform of this equation, with 0 initial conditions, yields

$$\frac{7}{48}ml^2s^2\Theta(s)+\frac{5}{8}kl^2\Theta(s)=\frac{3}{4}klZ(s)$$

Hence, the transfer function is

$$G(s)=\frac{\Theta(s)}{Z(s)}=\frac{\frac{3}{4}kl}{\frac{7}{48}ml^2s^2+\frac{5}{8}kl^2}=\frac{36k}{7mls^2+30kl}=\frac{180{,}000}{70s^2+150{,}000}$$

b. $z(t)=0.01e^{-t}$, and so $Z(s)=\dfrac{0.01}{s+1}$. Therefore,

$$\Theta(s)=G(s)Z(s)=\frac{180{,}000}{70s^2+150{,}000}\frac{0.01}{s+1}=\frac{25.7}{(s^2+2140)(s+1)}$$

The quadratic term clearly has imaginary roots, and so we can write the partial fraction expansion as

$$\Theta(s)=\frac{a_1s+a_2}{s^2+2140}+\frac{a_3}{s+1}$$

Multiplying through by the common denominator yields

$$\begin{aligned}25.7&=(a_1s+a_2)(s+1)+a_3(s^2+2140)\\&=(a_1+a_3)s^2+(a_1+a_2)s+(a_2+2140a_3)\end{aligned}$$

which then gives the system of equations:

$$\begin{cases}25.7=a_2+2140a_3\\ \quad 0=a_1+a_2\\ \quad 0=a_1+a_3\end{cases}$$

The solution to this system is $a_1=-0.012$, $a_2=0.012$, and $a_3=0.012$. Thus,

$$\begin{aligned}\Theta(s)&=\frac{-0.012s+0.012}{s^2+2140}+\frac{0.012}{s+1}\\&=-0.012\frac{s}{s^2+2140}+\left(\frac{0.012}{46.3}\right)\frac{46.3}{s^2+2140}+\frac{0.012}{s+1}\end{aligned}$$

Taking the inverse Laplace transform of this gives

$$\theta(t)=-0.012\cos(46.3t)+0.000259\sin(46.3t)+0.012e^{-t}$$

c. Rearranging the EOM into state-space form gives

$$\frac{7}{48}ml^2\ddot{\theta}+\frac{5}{8}kl^2\theta=\frac{3}{4}klz$$

$$\ddot{\theta}=-\frac{30k}{7m}\theta+\frac{36k}{7ml}z$$

Let's choose $x_1=\theta$ and $x_2=\dot{\theta}$ as the states, $u_1=z$ as the input, and $y_1=\theta$ as the output. Then

$$\dot{x}_1=\dot{\theta}=x_2\ ,\quad \dot{x}_2=\ddot{\theta}=-\frac{30k}{7m}\theta+\frac{36k}{7ml}z=-\frac{30k}{7m}x_1+\frac{36k}{7ml}u_1,\ \text{and}\ y_1=\theta=x_1.$$

Thus, the state space model is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\underbrace{\begin{bmatrix}0 & 1\\ -\frac{30k}{7m} & 0\end{bmatrix}}_{\mathbf{A}}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\underbrace{\begin{bmatrix}0\\ \frac{36k}{7ml}\end{bmatrix}}_{\mathbf{B}}[u_1]=\underbrace{\begin{bmatrix}0 & 1\\ -2140 & 0\end{bmatrix}}_{\mathbf{A}}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\underbrace{\begin{bmatrix}0\\ 2570\end{bmatrix}}_{\mathbf{B}}[u_1]$$

$$[y_1]=\underbrace{[1\quad 0]}_{\mathbf{C}}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\underbrace{[0]}_{\mathbf{D}}[u_1]$$

d. The following code provides a plot comparing the `lsim` results and the analytical solution

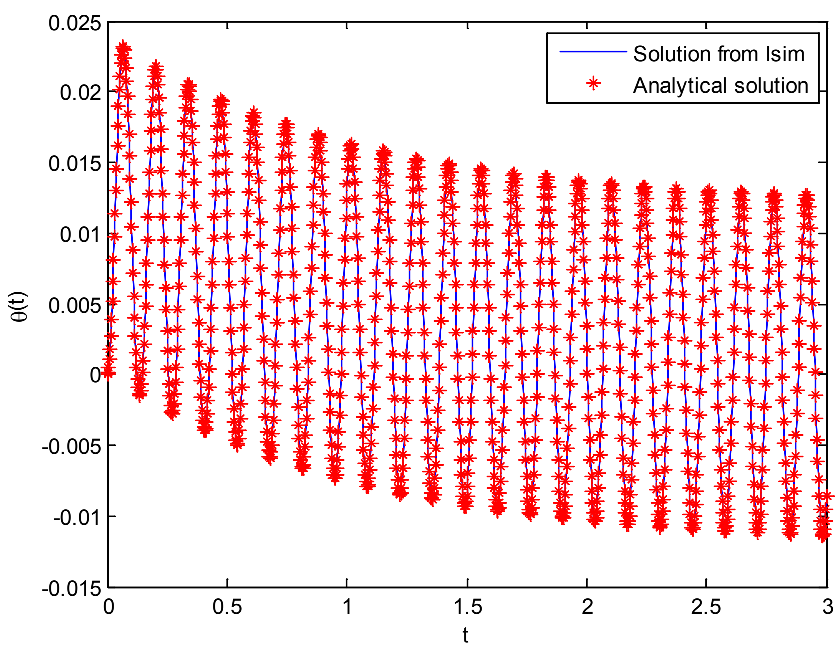
```
% constants
k = 5000;
m = 10;
l = 1;

% set up state space matrices
A = [0 1; -30*k/(7*m) 0];
B = [0 36*k/(7*m*l)]';
C = [1 0];
D = [0];

% simulate system using lsim
sys = ss(A,B,C,D);
t = linspace(0,3,1001);
u = 0.01*exp(-t);
y = lsim(sys,u,t);

% plot solution using lsim
figure;
plot(t,y);

% plot solution using analytical formula
tht = 0.012*exp(-t) + 0.000259*sin(46.3*t) - 0.012*cos(46.3*t);
hold on;
plot(t,tht,'r*');
ylabel('\theta(t)');
xlabel('t');
legend('Solution from lsim','Analytical solution');
```



As you can see, the two curves are the same.