

## MAE6292: Homework 3

Due date: March 3, 2015

**Problem 1** In class, we covered an optimal orbital transfer of a spacecraft. Let the location of the spacecraft be defined by  $(r, \alpha)$  in polar coordinates. The state vector is defined as  $x = [r, \alpha, \dot{r}, r\dot{\alpha}]^T \in \mathbb{R}^4$ . The equations of motion are given by

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{x_4}{x_1}, \\ \dot{x}_3 &= \frac{x_4^2}{x_1} - \frac{gR^2}{x_1} + \frac{T}{m} \sin u, \\ \dot{x}_4 &= -\frac{x_3}{x_4}x_1 + \frac{T}{m} \cos u.\end{aligned}$$

The control input corresponds to the angle of the fixed thrust from the local horizontal direction.

An optimal control problem to transfer the spacecraft from a surface of the Earth with the radius  $R$  to a circular orbit with the altitude  $D$  in a minimum time is considered in class, and we showed that the boundary conditions to solve the optimality conditions and to determine the terminal time are given by

$$x_1(t_f) = R + D, \quad \lambda_2(t_f) = 0, \quad x_3(t_f) = 0, \quad x_4(t_f) = \sqrt{\frac{gR^2}{R + D}}, \quad H(t_f) = 0.$$

In this problem, we derive boundary conditions for other optimal spacecraft missions.

- (a) In this mission, the spacecraft is launched from  $x(0) = [R, 0, 0, 0]^T$  and it is to rendezvous with another spacecraft that is on a fixed circular orbit with its altitude  $D$ . The orbital period of the target spacecraft is 2 hours, and both spacecraft are on the reference axis of  $\alpha = 0$  initially when  $t = 0$ . This implies that the terminal state should satisfy

$$\psi(t_f, x_f) = x_f - [R + D, \frac{\pi}{3600}t_f, 0, \frac{\pi}{3600}(R + D)]^T = 0_{4 \times 1}.$$

Find another boundary condition to determine the final time  $t_f$ .

- (b) In this mission, the spacecraft should be transferred to a circle centered at  $(R + E)(\cos \gamma, \sin \gamma)$  with its radius  $C$ , i.e., the terminal state constraint is given by

$$\psi(t_f, x_f) = (x_1 \cos x_2 - (R + E) \cos \gamma)^2 + (x_1 \sin x_2 - (R + E) \sin \gamma)^2 - C^2 = 0.$$

In addition to this, derive five more equations to determine four boundary conditions, one multiplier  $\nu$ , and the terminal time  $t_f$ .

**Problem 2** In class, we studied linear quadratic regulators. Here, we generalize it into linear quadratic *tracking* problems. Consider a linear dynamics given by

$$\dot{x} = Ax + Bu.$$

The cost function is defined as

$$J = \frac{1}{2}(x(t_f) - r(t_f))^T Q_f(x(t_f) - r(t_f)) + \frac{1}{2} \int_{t_0}^{t_f} (x(t) - r(t))^T Q(t)(x(t) - r(t)) + u(t)^T R(t)u(t) dt,$$

where  $r(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  is a reference trajectory to be followed.

(a) Derive necessary conditions for optimality to obtain

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ Qr \end{bmatrix},$$

$$u = -R^{-1}B^T\lambda.$$

(b) Show that the terminal boundary conditions are given by  $\lambda(t_f) = Q_f(x(t_f) - r(t_f))$ .

(c) We can show that the optimal state and the multiplier satisfying the above necessary conditions satisfy

$$\lambda = P(t)x + s(t),$$

for some  $P(t) : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$  and  $s(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ . Substitute it into the optimality condition, and derive differential equations that should be satisfied for  $P(t)$  and  $s(t)$ . Also, find the boundary conditions for  $P(t)$  and  $s(t)$ .

(Hint: the differential equation for  $P(t)$  is identical to the Riccati equation for LQR).

(d) Suppose

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and the cost function is given by

$$J = \frac{1}{2} \int_0^{20} (x_1 - \sin \frac{\pi}{2}t)^2 + 0.01u^2 dt.$$

Numerically simulate the linear quadratic trackers derived above, and plot the corresponding optimal  $x(t)$ ,  $r(t)$ ,  $u(t)$  and  $K(t)$ ,  $s(t)$ .