MAE 6254 Midterm Exam

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1 Problem 1

For the following system:

$$\dot{x_1} = -x_1^3 + x_2$$
$$\dot{x_2} = x_1 - x_2^3$$

a) find three equilibria

Equilibria are at x^* where $\dot{x}^* = 0$. Therefore

$$0 = -x_1^3 + x_2$$
$$0 = x_1 - x_2^3$$

This is true at:

$$x_1^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T \tag{1.1}$$
$$x_2^* = \begin{bmatrix} 1 & 1 \end{bmatrix}^T \tag{1.2}$$
$$x_3^* = \begin{bmatrix} -1 & -1 \end{bmatrix}^T \tag{1.3}$$

b) Find the type of each equilibrium

$$x = x^* + \delta x$$
$$\dot{x} = \dot{x}^* + \delta\dot{x} = \frac{\partial f}{\partial x}\Big|_{x^*}$$
$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}$$

By evaluating matrix A at each equilibrium and finding it's eigenvalues, we can determine the type of equilibrium.
Equilibrium 1:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.4}$$

$$\lambda = -1,\ 1 \Rightarrow \ \textit{saddle point} \tag{1.5}$$

Equilibrium 2:

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \tag{1.6}$$

$$\lambda = -4,\ -2 \Rightarrow \ \textit{stable node} \tag{1.7}$$

Equilibrium 3:

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \tag{1.8}$$

$$\lambda = -4,\ -2 \Rightarrow \ \textit{stable node} \tag{1.9}$$

2 Problem 2

a) Find the equilibrium of the system: The equilibrium is at $x^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. This makes

$$\dot{x_1} = (1+0)(0-0) = 0$$
$$\dot{x_2} = 0(1+0) = 0$$

b) Make the strongest possible statement about the stability of the system using the given Lyapunov equation:

$$V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2} \tag{2.1}$$

V is positive definite becauase $V = 0$ only if $x = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} && (2.2)\\
&= \frac{(1+x_1^2)2x_1 - x_1^2(2x_1}{(1+x_1^2)^2}(1+x_1^2)^2(-x_1 - x_2) + \frac{(1+x_2^2)2x_2 - x_2^2(2x_2}{(1+x_2^2)^2}x_1(1+x_1^2)^2 && (2.3)\\
&= (2x_1 + 2x_1^3 - 2x_1^3)(-x_1 - x_2) + (2x_2 + 2x_2^3 - 2x_2^3)x_1 && (2.4)\\
&= -2x_1^2 && (2.5)
\end{aligned}$$

Therefore $\dot{V}$ is negative semi-definite, and the equilibrium is stable. We can use LaSalle's theorem to show that the equilibrium of this time-invariant system is asymptotically stable.
Let $S = \{x \in D | x_1 = 0\}$. Let x_1, x_2 be solutions staying in S. $V = \dot{V} = 0$ implies that $x_1 = 0$, and therefore $\dot{x_1} = 0$. This leaves the equation for V as:

$$0 = \frac{x_2^2}{1+x_2^2} \tag{2.6}$$

The only solution for which this is true is $x_2 = 0$. By LaSalle's theorem, the equilibrium is asymptotically stable.
The above is true for $x \in D = \mathbb{R}^2$, and additionally V is radially unbounded. Therefore, the equilibrium is globally asymptotically stable.

3 Problem 3

a) Show that the given Lyapunov equation is positive definite (p.d.).

$$
\begin{aligned}
V(x_1, x_2) &= \frac{3}{2}x_1^2 - x_1x_2 + x_2^2 && (3.1)\\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \mathbf{P} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T && (3.2)\\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3/2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T && (3.3)
\end{aligned}
$$

V is p.d. if $\mathbf{P}$ is p.d. Matrix $\mathbf{P}$ is p.d. if the eigenvalues of $[\mathbf{P}+\mathbf{P}^T]/2 > 0$, or equivalently if the determinant of each leading principle minor is positive.

$$
[P + P^T]/2 = Q = \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \tag{3.4}
$$

Both leading principle minors of $\mathbf{Q}$ are positive, and therefore V is positive definite.

b) Show that the equilibrium is asymptotically stable:

$$
\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x}_1 + \frac{\partial V}{\partial x_2}\dot{x}_2 && (3.5)\\
&= (3x_1 - x_2)(-x_2) + (2x_2 - x_1)((x_1^2 - 1)x_2 + x_1) && (3.6)\\
&= -x_1^2 - x_2^2 + 2x_1^2x_2^2 - x_1^3x_2 && (3.7)\\
&= -x_1^2(1 + x_1x_2) - x_2^2(1 - 2x_1^2) && (3.8)
\end{aligned}
$$

In the domain $D = \{x_1, x_2 \in \mathbb{R} \mid 1 + x_1x_2 > 0,\ x_1^2 < \frac{1}{2}\}$, $\dot{V}$ is negative definite, and therefore the equilibrium is asymptotically stable.

c) For a constant c, the sublevel set Ω_c of V is described by an ellipse. This ellipse can be found from the equation of V. The equation of an ellipse is given by

$$
0 = Ax^2 + Bxy + Cy^2 + Dx + Ey + F \tag{3.9}
$$

where the coefficients are calculated as functions of the semi-major (a) and semi-minor (b) axes, and angle (θ) of the semi-major axis. In this case

$$
\begin{aligned}
A &= \frac{3}{2} && = a^2 sin^2\theta + b^2 cos^2\theta && (3.10)\\
B &= -1 && = 2(b^2 - a^2) sin\theta cos\theta && (3.11)\\
C &= 1 && = a^2 cos^2\theta + b^2 sin^2\theta && (3.12)\\
F &= c && = -a^2b^2 && (3.13)
\end{aligned}
$$

This system of equations can be solved for a, b, θ. For a constant c:

$$a^2 = \frac{-c}{b^2} \tag{3.14}$$

$$b^2 = \frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4} + 4cos^2\theta sin^2\theta}}{2cos^2\theta} \tag{3.15}$$

$$sin(2\theta) = \frac{-1}{b(\theta)^2 + \frac{c}{b(\theta)}} \tag{3.16}$$

Once this system of equations is solved, the semi-major axis is given by a, the semi-minor axis is given by b, the angle of the semi-major axis is given by θ, and the angle of the semi-minor axis is given by $\theta + \frac{\pi}{2}$. The ellipse described above can be seen in the contour plot of V.

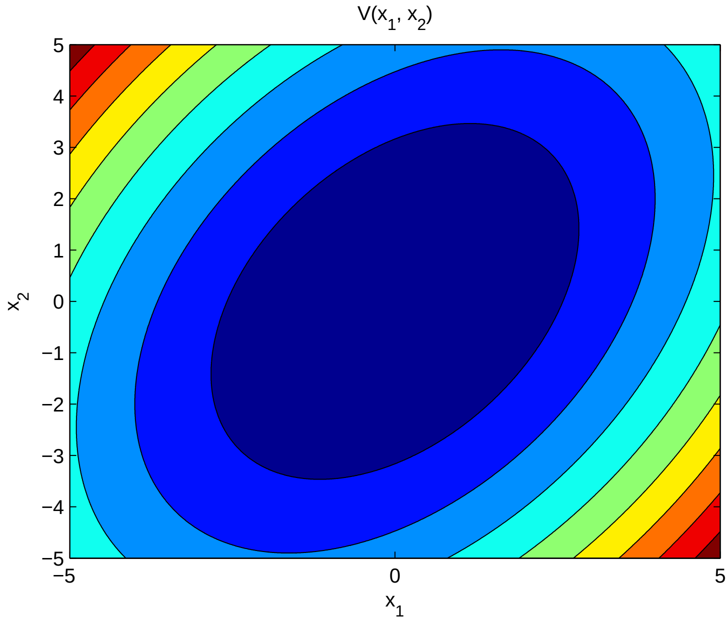


Figure 3.1: Contour plot of V. The sublevel set of V is anything inside a given contour corresponding to the constant c.

(d) The region of attraction (ROA) for V is bounded by trajectories. This region is described by the largest sublevel set in which V always remains. A conservative estimate of this region of attraction is given by the ellipse found when $a = 4, b = 1/4, \theta = 0$. This region can be expanded out to the trajectories given as the limits of the domain D. A larger estimate of the ROA is given by the largest subset of V for which $\dot{V}$ remains negative. This is true for the ellipse at $\theta = 1.02, a = 5/3, b = 1$. This ellipse is shown in the image below.

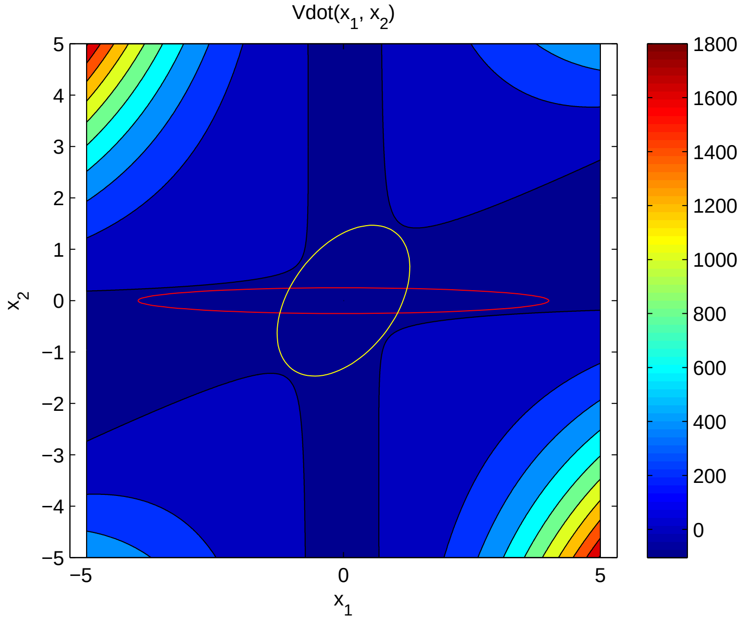


Figure 3.2: Estimates of the region of attraction plotted over a contour plot of $\dot{V}$

4 Problem 4

a) Show that V is positive definite and decrescent for the given time-varying system:

$$V(t, x_1, x_2) = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(1 + b(t))x_1^2 \tag{4.1}$$

$$\geq \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = W_1(x) \tag{4.2}$$

Since $W_1(x)$ is p.d., and $V \geq W_1(x) \forall t$, V must be p.d. Also,

$$V(t, x_1, x_2) = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(1 + b(t))x_1^2 \tag{4.3}$$

$$\leq \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T = W_2(x) \tag{4.4}$$

Since $W_2(x)$ is p.d. and $V \leq W_2(x) \forall t$, V must be decrescent.

b) Show that the origin is globally asymptotically stable, and find the constants of the exponential bound:
First, we have already found that V is positive definite and decrescent. Additionally,

because $V \to \infty \Rightarrow x \to \infty$, V is radially unbounded. These properties hold for $D = \mathbb{R}^2$.

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} && (4.5)\\
&= (4x_1 + x_2)\dot{x_1} + (x_1 + x_2)\dot{x_2} && (4.6)\\
&= -2x_1^2 + (2 - b(t))x_1x_2 - (b(t) - 1)x_2^2 && (4.7)\\
&\leq -2x_1^2 - 2x_1x_2 - 3x_2^2 && (4.8)\\
&\leq -3||x|| && (4.9)
\end{aligned}$$

Therefore, V is negative definite. This holds for $D = \mathbb{R}^2$, so the equilibrium is globally exponentially stable and bounded by $||x|| \leq 3||x_0||exp[-(t - t_0)]$. The constants for the exponential bound are found by

$$\begin{aligned}
W_1(x) &= \frac{1}{2}||x|| && (4.10)\\
W_2(x) &= \frac{3}{2}||x|| && (4.11)\\
W_3(x) &= -3||x|| && (4.12)\\
k &= \frac{k_2}{k_1} = 3 && (4.13)\\
\gamma &= \frac{-k_3}{2k_2} = 1 && (4.14)
\end{aligned}$$

5 Problem 5

a) The equation of motion can be written as

$$\dot{x} = \begin{bmatrix} \dot{x_1}^T & \dot{x_2}^T \end{bmatrix}^T \tag{5.1}$$
$$= \begin{bmatrix} x_2{}^T & (ge_3 + \frac{u}{m} - \ddot{p}_d(t))^T \end{bmatrix}^T \tag{5.2}$$

b) Substituting the proposed control input into the equations of motion, we should get $\dot{x} = 0$ for an equilibrium position at $x = 0$.

$$u = -k_p x_1 - k_v x_2 + m\ddot{p}_d(t) - mge_3 \tag{5.3}$$
$$\dot{x} = \begin{bmatrix} \vec{0}^T & (ge_3 - 0 - 0 + \frac{m\ddot{p}_d(t) - mge_3}{m} - \ddot{p}_d(t))^T \end{bmatrix}^T \tag{5.4}$$
$$= \begin{bmatrix} \vec{0}^T & \vec{0}^T \end{bmatrix}^T \tag{5.5}$$

c) $V_0 = \frac{1}{2} m x_2^T x_2 + \frac{1}{2} k_p x_1^T x_1$ must be positive definite, because $m, k_p > 0$ and $x^T x > 0$ by definition.

$$\dot{V_0} = \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + \frac{\partial V}{\partial t} \tag{5.6}$$
$$= 2(k_p x_1 x_2 + m x_2(\ddot{p} - \ddot{p}_d)) \tag{5.7}$$
$$= 2(k_p x_1 x_2 + m x_2(ge_3 + \frac{1}{m}(-k_p x_1 - k_v x_2 + m\ddot{p}_d(t) - mge_3) - \ddot{p}_d(t)) \tag{5.8}$$
$$= 2k_p x_1 x_2 - 2k_p x_1 x_2 - 2mk_v x_2^2 \tag{5.9}$$
$$= -2mk_v x_2^2 \tag{5.10}$$

$\dot{V_0}$ is negative semi-definite, and therefore the equilibrium is stable. We can also show that it is universally stable, since this solution is not a direct function of time. Next, we use the LaSalle-Yoshizawa theorem to check if the origin is asymptotically stable.
Let $S = \{x \in D \; x_2 = 0\}$. $x_2 = 0$ is implied by $\dot{V} = 0$, and therefore $\dot{x_2} = 0$. $x_1(t), x_2(t)$ be solutions staying in S. This leaves

$$V_0 = 0 = \frac{1}{2} k_p x_1^T x_1 \tag{5.11}$$

Therefore, the only solution that stays in $Sast \to \infty$ is $x_1 = 0$. This means that the equilibrium of the system is u.a.s, and since the region of attraction is $\mathbb{R}^n$, the equilibrium is g.u.a.s.

d)

$$V_0 = \frac{1}{2} k_p x_1^T x_1 + c x_1^T x_2 + \frac{1}{2} m x_2^T x_2 \tag{5.12}$$
$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} \frac{k_p}{2} & \frac{c}{2} \\ \frac{c}{2} & \frac{m}{2} \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \tag{5.13}$$

This becomes positive definite when $\frac{mk_p}{4} > \frac{c}{4}$, therefore it is p.d. when $mk_p > c$. Let $\alpha = max(kp, c, m)$.

$$V \leq \alpha(x_1 + x_2)^T (x_1 + x_2) = W(x) \tag{5.14}$$

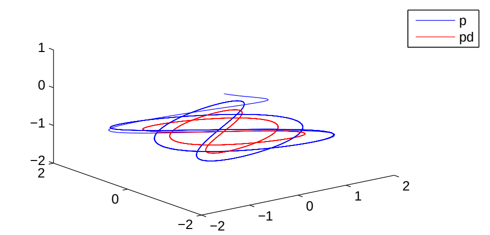
The function $W(x)$ is positive definite, so V is decrescent.

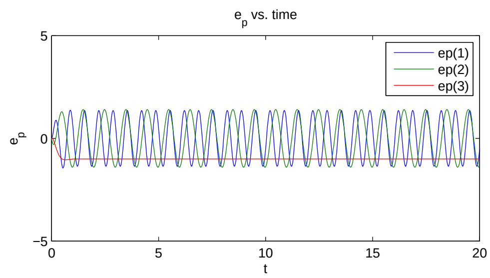
e)

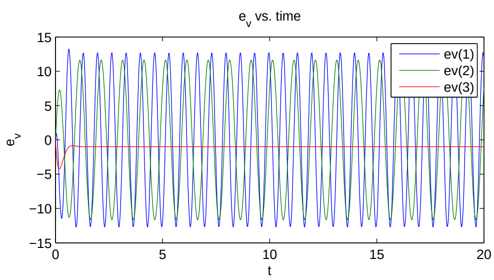
$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} + \frac{\partial V}{\partial t} && (5.15)\\
&= 2(k_p x_1 x_2 + m x_2(\ddot{p} - \ddot{p}_d)) + c\dot{x_1}^T x_2 + c x_1^T \dot{x_2} && (5.16)\\
&= -2(k_v x_2^T x_2 + c x_2^T x_2 - c k_p x_1^T x_1 - c k_v x_1^T x_2 && (5.17)\\
&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} (c - k_p) & \frac{-ck_v}{2} \\ \frac{-ck_v}{2} & (c - k_v) \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T && (5.18)\\
& && (5.19)
\end{aligned}$$

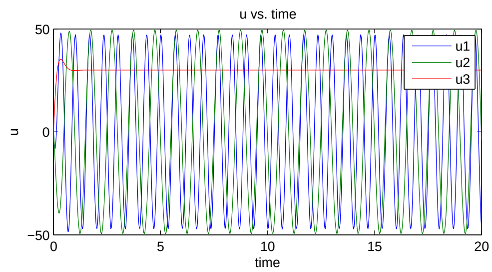
The equilibrium is globally exponentially stable when $\dot{V} < 0$, which is true for any c satisfying $c^2 - (k_p k_v)c - k_p k_v > \frac{c^2 k_v^2}{4}$.

f) For the gains $kp = 25, k_v = 5$, the following plots were generated by numerically integrating the equations of motion.









6 Problem 6

a) The equilibrium points at $\dot{x} = 0$ are found by:

$$\dot{x} = \begin{bmatrix} \dot{\Omega} \\ \dot{g} \end{bmatrix} \tag{6.1}$$

$$= \begin{bmatrix} J^{-1}u - J^{-1}\Omega \times J\Omega \\ -\Omega \times g \end{bmatrix} \tag{6.2}$$

$$= \begin{bmatrix} J^{-1}(-k\Omega + g \times s) - J^{-1}\Omega \times J\Omega \\ -\Omega \times g \end{bmatrix} \tag{6.3}$$

$x_2 = 0$ only when $\Omega = 0$, and then for $x_1 = 0, g = \pm s$. So the two equilibria are at $x^* = \begin{bmatrix} 0 & s \end{bmatrix}^T, \begin{bmatrix} 0 & -s \end{bmatrix}^T$.

b) V can be rewritten as $V = 1/2\Omega^T J\Omega + 1/2(g+s)^T(g+s)$. Since J is positive by definition, V is positive definite. $V = 0$ implies we are looking at the second equilibrium, $s^E = \begin{bmatrix} 0 & -s \end{bmatrix}^T$, and if V is p.d. and $\dot{V}$ is n.d., then this implies that g is aligned with s.

$$\dot{V} = \dot{\Omega}J + (\dot{g} + s) \tag{6.4}$$

$$= [J^{-1}u - J^{-1}\Omega \times J\Omega]J + [\Omega \times g - s] \tag{6.5}$$

The sign indefinite terms are:

$$J^{-1}\Omega \times J\Omega J = -\Omega J \times J^{-1}\Omega J = 0.g \times s \qquad = 0 \tag{6.6}$$

Therefore, $\dot{V}$ is n.d. Breaking V into parts,

$$\frac{1}{2}\gamma_{min}(J) \leq \frac{1}{2}\Omega^T J\Omega \tag{6.7}$$

$$\frac{1}{2} \leq \frac{1}{2}(g+s)^T(g+s) \tag{6.8}$$

Therefore, the ROA can be estimated as a ball with radius $\frac{1}{2}(min(J) + 1)$.