MAE 6254 Midterm Exam

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1. Problem 1

For the following system:

$$\dot{x_1} = -x_1^3 + x_2$$
$$\dot{x_2} = x_1 - x_2^3$$

a) find three equilibria
b) Find the type of each equilibrium

1.a.

Equilibria are at x^* where $\dot{x}^* = 0$. Therefore

$$0 = -x_1^3 + x_2$$
$$0 = x_1 - x_2^3$$

This is true at:

$$x^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T x^* = \begin{bmatrix} 1 & 1 \end{bmatrix}^T x^* = \begin{bmatrix} -1 & -1 \end{bmatrix}^T \tag{1.1}$$

1.b.

$$x = x^* + \delta x$$
$$\dot{x} = \dot{x}^* + \delta\dot{x} = \frac{\partial f}{\partial x}\Big|_{x^*}$$
$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix}$$

By evaluating matrix A at each equilibrium and finding it's eigenvalues, we can determine the type of equilibrium.
Equilibrium 1:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.2}$$

$$\lambda = -1,\ 1 \Rightarrow\ \textit{saddle point} \tag{1.3}$$

Equilibrium 2:

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \tag{1.4}$$

$$\lambda = -4,\ -2 \Rightarrow\ \textit{stable node} \tag{1.5}$$

Equilibrium 3:

$$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \tag{1.6}$$

$$\lambda = -4,\ -2 \Rightarrow\ \textit{stable node} \tag{1.7}$$

2. Problem 2

a) Find the equilibrium of the system: The equilibrium is at $x^* = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$. This makes

$$\dot{x_1} = (1+0)(0-0) = 0$$
$$\dot{x_2} = 0(1+0) = 0$$

b) Make the strongest possible statement about the stability of the system using the given Lyapunov equation:

$$V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + \frac{x_2^2}{1+x_2^2} \tag{2.1}$$

V is positive definite becauase $V = 0$ only if $x\!= \begin{bmatrix} 0 & 0 \end{bmatrix}^T$.

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x_1}\dot{x_1} + \frac{\partial V}{\partial x_2}\dot{x_2} && (2.2)\\
&= \frac{(1+x_1^2)2x_1 - x_1^2(2x_1}{(1+x_1^2)^2}(1+x_1^2)^2(-x_1-x_2) + \frac{(1+x_2^2)2x_2 - x_2^2(2x_2}{(1+x_2^2)^2}x_1(1+x_1^2)^2 && (2.3)\\
&= (2x_1 + 2x_1^3 - 2x_1^3)(-x_1-x_2) + (2x_2 + 2x_2^3 - 2x_2^3)x_1 && (2.4)\\
&= -2x_1^2 && (2.5)
\end{aligned}$$

Therefore $\dot{V}$ is negative semi-definite, and the equilibrium is stable. We can use LaSalle's theorem to show that the equilibrium of this time-invariant system is asymptotically stable.
Let $S = \{x \in D | x_1 = 0\}$. Let x_1, x_2 be solutions staying in S. $V = \dot{V} = 0$ implies that $x_1 = 0$, and therefore $\dot{x_1} = 0$. This leaves the equation for V as:

$$0 = \frac{x_2^2}{1+x_2^2} \tag{2.6}$$

The only solution for which this is true is $x_2 = 0$. By LaSalle's theorem, the equilibrium is asymptotically stable.
The above is true for $x \in D = \mathbb{R}^2$, and additionally V is radially unbounded. Therefore, the equilibrium is globally asymptotically stable.

3. Problem 3

a) Show that the given Lyapunov equation is positive definite (p.d.).

$$V(x_1, x_2) = \frac{3}{2}x_1^2 - x_1 x_2 + x_2^2 \tag{3.1}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \mathbf{P} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \tag{3.2}$$

$$= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3/2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \tag{3.3}$$

V is p.d. if $\mathbf{P}$ is p.d. Matrix $\mathbf{P}$ is p.d. if the eigenvalues of $\mathbf{P} + \mathbf{P}^T/2 > 0$, or equivalently if the determinant of each leading principle minor is positive.

$$[P + P^T]/2 = Q = \begin{bmatrix} 3/2 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix} \tag{3.4}$$

Both leading principle minors of $\mathbf{Q}$ are positive, and therefore V is positive definite.

4. Lists

4.a. Example of list (3*itemize)

- First item in a list
  - First item in a list
    - First item in a list
    - Second item in a list
  - Second item in a list
- Second item in a list

4.b. Example of list (enumerate)

1. First item in a list
2. Second item in a list
3. Third item in a list