HOMEWORK 8 SOLUTIONS

1. a. The position coordinates of the pendulum bob, in terms of x and θ, are $(x + l\cos(\theta), l\sin(\theta))$.

The total kinetic energy is then

$$
\begin{aligned}
T &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left\{\left[\frac{d}{dt}(x + l\cos(\theta))\right]^2 + \left[\frac{d}{dt}(l\sin(\theta))\right]^2\right\} \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left\{\left[\dot{x} - l\sin(\theta)\dot{\theta}\right]^2 + \left[l\cos(\theta)\dot{\theta}\right]^2\right\} \\
&= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\left[\dot{x}^2 - 2\dot{x}l\sin(\theta)\dot{\theta} + l^2\sin^2(\theta)\dot{\theta}^2 + l^2\cos^2(\theta)\dot{\theta}^2\right] \\
&= m\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - m\dot{x}l\sin(\theta)\dot{\theta}
\end{aligned}
$$

The total potential energy is

$$
U = \frac{1}{2}kx^2 + mgl[1 - \cos(\theta)]
$$

Note that the gravitational potential energy of the block is not counted because the spring's potential energy is measured with respect to the equilibrium position of the block, where the extra stretch in the spring accounts for the gravitational potential.

There are no external or non-conservative forces in the system, so $Q_1 = Q_2 = 0$. Thus, we can apply Lagrange's equations as follows:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} &= 0 \\
\frac{d}{dt}\left[2m\dot{x} - ml\sin(\theta)\dot{\theta}\right] + kx &= 0 \\
2m\ddot{x} - ml\cos(\theta)\dot{\theta}^2 - ml\sin(\theta)\ddot{\theta} + kx &= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial U}{\partial \theta} &= 0 \\
\frac{d}{dt}\left[ml^2\dot{\theta} - m\dot{x}l\sin(\theta)\right] + m\dot{x}l\cos(\theta)\dot{\theta} + mgl\sin(\theta) &= 0 \\
ml^2\ddot{\theta} - m\ddot{x}l\sin(\theta) - m\dot{x}l\cos(\theta)\dot{\theta} + m\dot{x}l\cos(\theta)\dot{\theta} + mgl\sin(\theta) &= 0 \\
ml^2\ddot{\theta} - m\ddot{x}l\sin(\theta) + mgl\sin(\theta) &= 0
\end{aligned}
$$

b. Linearizing the first equation gives

$$2m\ddot{x} + kx = 0$$

where all terms with products of x and θ (or its derivatives) are considered negligible.

Linearizing the second equation gives

$$ml^2\ddot{\theta} + mgl\theta = 0$$

where all terms with products of x and θ (or its derivatives) are considered negligible, and we use $\sin(\theta) \approx \theta$.

It is interesting that, assuming small deflections, the motion of the two masses becomes decoupled, i.e. neither one affects the other.

Since these are both SDOF equations, the natural frequencies are just

$$\omega_1 = \sqrt{\frac{k}{2m}} \text{ and } \omega_2 = \sqrt{\frac{g}{l}}$$

2. It's easier to write the equations in terms of the translation x_G and rotation θ about point G (the center of gravity of the beam) rather than the endpoints x_1 and x_3.

Then $x_G = \dfrac{x_1 + x_3}{2}$ and $\theta = \dfrac{x_1 - x_3}{5l}$. (Note that we can't write θ in terms of just x_1 or x_3 because the rotation angle is based on the <u>difference</u> between the absolute deflections of the endpoints.)

Similarly, we can write $x_A = x_1 - \dfrac{2}{5}(x_1 - x_3) = \dfrac{3x_1 + 2x_3}{5}$ using similar triangles.

The moment of inertia about G is given by $J_G = \dfrac{1}{12}(2m)(5l)^2 = \dfrac{25}{6}ml^2$.

The total kinetic energy is then

$$\begin{aligned} T &= \frac{1}{2}J_G\dot{\theta}^2 + \frac{1}{2}(2m)\dot{x}_G^2 + \frac{1}{2}(5m)\dot{x}_2^2 \\ &= \frac{1}{2}\left(\frac{25}{6}ml^2\right)\left(\frac{\dot{x}_1 - \dot{x}_3}{5l}\right)^2 + \frac{1}{2}(2m)\left(\frac{\dot{x}_1 + \dot{x}_3}{2}\right)^2 + \frac{1}{2}(5m)\dot{x}_2^2 \end{aligned}$$

The total potential energy is

$$U=\frac{1}{2}kx_1^2+\frac{1}{2}kx_3^2+\frac{1}{2}k\left(x_A-x_2\right)^2=\frac{1}{2}kx_1^2+\frac{1}{2}kx_3^2+\frac{1}{2}k\left(\frac{3x_1+2x_3}{5}-x_2\right)^2$$

The Rayleigh dissipation function is

$$F_{Ray}=\frac{1}{2}c\left(\dot{x}_A-\dot{x}_2\right)^2=\frac{1}{2}c\left(\frac{3\dot{x}_1+2\dot{x}_3}{5}-\dot{x}_2\right)^2$$

Since each force is applied in line with the respective coordinate, the work done is

$$W=F_1x_1+F_2x_2+F_3x_3$$

Thus, we can apply Lagrange's equations as follows for x_1:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right)-\frac{\partial T}{\partial x_1}+\frac{\partial U}{\partial x_1}+\frac{\partial F_{Ray}}{\partial \dot{x}_1}=\frac{\partial W}{\partial x_1}$$

$$\frac{d}{dt}\left[\left(\frac{25}{6}ml^2\right)\left(\frac{1}{5l}\right)\left(\frac{\dot{x}_1-\dot{x}_3}{5l}\right)+(2m)\left(\frac{1}{2}\right)\left(\frac{\dot{x}_1+\dot{x}_3}{2}\right)\right]$$

$$+kx_1+k\left(\frac{3}{5}\right)\left(\frac{3x_1+2x_3}{5}-x_2\right)+c\left(\frac{3}{5}\right)\left(\frac{3\dot{x}_1+2\dot{x}_3}{5}-\dot{x}_2\right)=F_1$$

$$\frac{2}{3}m\ddot{x}_1+\frac{1}{3}m\ddot{x}_3+\frac{9}{25}c\dot{x}_1-\frac{3}{5}c\dot{x}_2+\frac{6}{25}c\dot{x}_3+\frac{34}{25}kx_1-\frac{3}{5}kx_2+\frac{6}{25}kx_3=F_1$$

For x_2:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right)-\frac{\partial T}{\partial x_2}+\frac{\partial U}{\partial x_2}+\frac{\partial F_{Ray}}{\partial \dot{x}_2}=\frac{\partial W}{\partial x_2}$$

$$\frac{d}{dt}\left(5m\dot{x}_2\right)-k\left(\frac{3x_1+2x_3}{5}-x_2\right)-c\left(\frac{3\dot{x}_1+2\dot{x}_3}{5}-\dot{x}_2\right)=F_2$$

$$5m\ddot{x}_2-\frac{3}{5}c\dot{x}_1+c\dot{x}_2-\frac{2}{5}c\dot{x}_3-\frac{3}{5}kx_1+kx_2-\frac{2}{5}kx_3=F_2$$

For x_3:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right)-\frac{\partial T}{\partial x_3}+\frac{\partial U}{\partial x_3}+\frac{\partial F_{Ray}}{\partial \dot{x}_3}=\frac{\partial W}{\partial x_3}$$

$$\frac{d}{dt}\left[-\left(\frac{25}{6}ml^2\right)\left(\frac{1}{5l}\right)\left(\frac{\dot{x}_1-\dot{x}_3}{5l}\right)+(2m)\left(\frac{1}{2}\right)\left(\frac{\dot{x}_1+\dot{x}_3}{2}\right)\right]$$

$$+kx_3+k\left(\frac{2}{5}\right)\left(\frac{3x_1+2x_3}{5}-x_2\right)+c\left(\frac{2}{5}\right)\left(\frac{3\dot{x}_1+2\dot{x}_3}{5}-\dot{x}_2\right)=F_3$$

$$\frac{1}{3}m\ddot{x}_1+\frac{2}{3}m\ddot{x}_3+\frac{6}{25}c\dot{x}_1-\frac{2}{5}c\dot{x}_2+\frac{4}{25}c\dot{x}_3+\frac{6}{25}kx_1-\frac{2}{5}kx_2+\frac{29}{25}kx_3=F_3$$

3. $\mathbf{M}^T=\left(\mathbf{L}\mathbf{L}^T\right)^T=\left(\mathbf{L}^T\right)^T\mathbf{L}^T=\mathbf{L}\mathbf{L}^T=\mathbf{M}$

$$\mathbf{C}^T=\left(\mathbf{L}\tilde{\mathbf{C}}\mathbf{L}^T\right)^T=\left(\mathbf{L}^T\right)^T\tilde{\mathbf{C}}^T\mathbf{L}^T=\mathbf{L}\left(\mathbf{P}\boldsymbol{\Lambda}_1\mathbf{P}^T\right)^T\mathbf{L}^T=\mathbf{L}\left(\mathbf{P}^T\right)^T\boldsymbol{\Lambda}_1^T\mathbf{P}^T\mathbf{L}^T=\mathbf{L}\mathbf{P}\boldsymbol{\Lambda}_1\mathbf{P}^T\mathbf{L}^T=\mathbf{L}\tilde{\mathbf{C}}\mathbf{L}^T=\mathbf{C}$$

$$\mathbf{K}^T=\left(\mathbf{L}\tilde{\mathbf{K}}\mathbf{L}^T\right)^T=\left(\mathbf{L}^T\right)^T\tilde{\mathbf{K}}^T\mathbf{L}^T=\mathbf{L}\left(\mathbf{P}\boldsymbol{\Lambda}_2\mathbf{P}^T\right)^T\mathbf{L}^T=\mathbf{L}\left(\mathbf{P}^T\right)^T\boldsymbol{\Lambda}_2^T\mathbf{P}^T\mathbf{L}^T=\mathbf{L}\mathbf{P}\boldsymbol{\Lambda}_2\mathbf{P}^T\mathbf{L}^T=\mathbf{L}\tilde{\mathbf{K}}\mathbf{L}^T=\mathbf{K}$$

where we have used the fact that $\boldsymbol{\Lambda}_1$ and $\boldsymbol{\Lambda}_2$ are diagonal and, hence, symmetric.