5.3 VIBRATION ABSORBERS

Another approach to protecting a device from steady-state harmonic disturbance at a constant frequency is a *vibration absorber*. Unlike the isolator of the previous sections, an absorber consists of a second mass–spring combination added to the primary device to protect it from vibrating. The major effect of adding the second mass–spring system is to change from a single-degree-of-freedom system to a two-degree-of-freedom system. The new system has two natural frequencies (recall Section 4.1). The added spring–mass system is called the absorber. The values of the absorber mass and stiffness are chosen such that the motion of the original mass is a minimum. This is accompanied by substantial motion of the added absorber system, as illustrated in the following.

Absorbers are often used on machines that run at constant speed, such as sanders, compactors, reciprocating tools, and electric razors. Probably the most visible vibration absorbers can be seen on transmission lines and telephone lines. A dumbbell-shaped vibration absorber is often used on such wires to provide vibration suppression against wind blowing, which can cause the wire to oscillate at its natural frequency. The presence of the absorber prevents the wire from vibrating so much at resonance that it breaks (or fatigues). Figure 5.14 illustrates a simple vibration absorber attached to a spring–mass system. The equations of motion [summing forces in the vertical direction (refer to Chapter 4)] are

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{x}_a \end{bmatrix} + \begin{bmatrix} k + k_a & -k_a \\ -k_a & k_a \end{bmatrix}\begin{bmatrix} x \\ x_a \end{bmatrix} = \begin{bmatrix} F_0 \sin \omega t \\ 0 \end{bmatrix} \tag{5.15}$$

where $x = x(t)$ is the displacement of the table modeled as having mass m and stiffness k, x_a is the displacement of the absorber mass (of mass m_a and stiffness k_a), and the harmonic force $F_0 \sin \omega t$ is the disturbance applied to the table mass. It is desired to design the absorber (i.e., choose m_a and k_a) such that the displacement of the primary system is as small as possible in steady state. Here it is desired to reduce the vibration of the table, which is the primary mass.

In contrast to the solution technique of modal analysis used in Chapter 4, here it is desired to obtain a solution in terms of parameters $(m, k, m_a, \text{and } k_a)$ that can

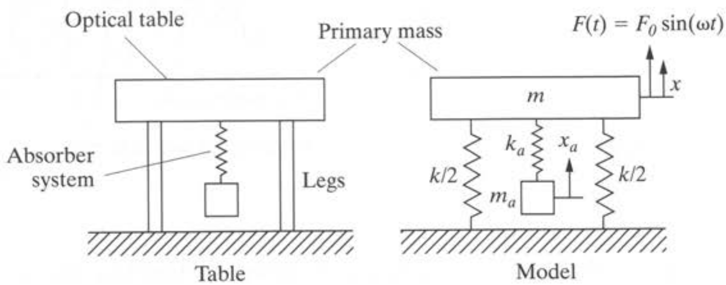

Figure 5.14 An optical table protected by an added vibration absorber. The table and its supporting legs are modeled as a single-degree-of-freedom system with mass m and stiffness k.

then be solved for as part of a design process. To this end, let the steady-state solution of $x(t)$ and $x_a(t)$ be of the form

$$x(t) = X \sin \omega t$$
$$x_a(t) = X_a \sin \omega t$$

Substitution of these steady-state forms into equation (5.15) yields (after some manipulation)

$$\begin{bmatrix} k + k_a - m\omega^2 & -k_a \\ -k_a & k_a - m_a\omega^2 \end{bmatrix}\begin{bmatrix} X \\ X_a \end{bmatrix} \sin \omega t = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t \tag{5.17}$$

which is an equation in the vector $[X \quad X_a]^T$. Dividing by $\sin \omega t$, taking the inverse of the matrix coefficient of $[X \quad X_a]^T$ (see Window 5.3), and multiplying from the left yields

$$\begin{aligned}\begin{bmatrix} X \\ X_a \end{bmatrix} &= \frac{1}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2}\begin{bmatrix} k_a - m_a\omega^2 & k_a \\ k_a & k + k_a - m\omega^2 \end{bmatrix}\begin{bmatrix} F_0 \\ 0 \end{bmatrix} \\ &= \frac{1}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2}\begin{bmatrix} (k_a - m_a\omega^2)F_0 \\ k_aF_0 \end{bmatrix}\end{aligned} \tag{5.18}$$

Window 5.3

> Recall that the inverse of a 2×2 matrix A given by
>
> $$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
>
> is defined to be
>
> $$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$
>
> where
>
> $$\det A = ad - bc$$

Equating elements of the vector equality given by equation (5.18) yields the result that the magnitude of the steady-state vibration of the device (table) becomes

$$X = \frac{(k_a - m_a\omega^2)F_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} \tag{5.19}$$

while the magnitude of vibration of the absorber mass becomes

$$X_a = \frac{k_aF_0}{(k + k_a - m\omega^2)(k_a - m_a\omega^2) - k_a^2} \tag{5.20}$$

Note from equation (5.19) that the absorber parameters k_a and m_a can be chosen such that the magnitude of steady-state vibration, X, is exactly zero. This is accomplished by equating the coefficient of F_0 in equation (5.19) to zero:

$$\omega^2 = \frac{k_a}{m_a} \tag{5.21}$$

Hence if the absorber parameters are chosen to satisfy the tuning condition of equation (5.21), the steady-state motion of the primary mass is zero (i.e., $X = 0$). In this event the steady-state motion of the absorber mass is calculated from equations (5.20) and (5.16) with $k_a = m_a\omega^2$ to be

$$x_a(t) = -\frac{F_0}{k_a}\sin\omega t \tag{5.22}$$

Thus the absorber mass oscillates at the driving frequency with amplitude $X_a = F_0/k_a$.

Note that the magnitude of the force acting on the absorber mass is just $k_a x_a = k_a(-F_0/k_a) = -F_0$. Hence when the absorber system is tuned to the driving frequency and has reached steady state, the force provided by the absorber mass is equal in magnitude and opposite in direction to the disturbance force. With zero net force acting on the primary mass, it does not move and the motion is "absorbed" by motion of the absorber mass. Note that while the applied force is completely absorbed by the motion of the absorber mass, the system is not experiencing resonance because $\sqrt{k_a/m_a}$ is not a natural frequency of the two-mass system.

The success of the vibration absorber discussed previously depends on several factors. First the harmonic excitation must be well known and not deviate much from its constant value. If the driving frequency drifts much, the tuning condition will not be satisfied, and the primary mass will experience some oscillation. There is also some danger that the driving frequency could shift to one of the combined systems' resonant frequencies, in which case one or the other of the system's coordinates would be driven to resonance and potentially fail. The analysis used to design the system assumes that it can be constructed without introducing any appreciable damping. If damping is introduced, the equations cannot necessarily be decoupled and the magnitude of the displacement of the primary mass will not be zero. In fact, damping defeats the purpose of a tuned vibration absorber and is desirable only if the frequency range of the driving force is too wide for effective operation of the absorber system. This is discussed in the next section. Another key factor in absorber design is that the absorber spring stiffness k_a must be capable of withstanding the full force of the excitation and hence must be capable of the corresponding deflections. The issue of spring size and deflection as well as the value of the absorber mass places a geometric limitation on the design of a vibration absorber system.

The issue of avoiding resonance in absorber design in case the driving frequency shifts can be quantified by examining the mass ratio μ, defined as the ratio of the absorber mass to the primary mass:

$$\mu = \frac{m_a}{m}$$

In addition, it is convenient to define the frequencies

$\omega_p = \sqrt{\dfrac{k}{m}}$ original natural frequency of the primary system without the absorber attached

$\omega_a = \sqrt{\dfrac{k_a}{m_a}}$ the natural frequency of the absorber system before it is attached to the primary system

With these definitions, also note that

$$\frac{k_a}{k} = \mu\,\frac{\omega_a^2}{\omega_p^2} = \mu\beta^2 \tag{5.23}$$

where the frequency ratio β is $\beta = \omega_a/\omega_p$. Substitution of the values for μ, ω_p, and ω_a into equation (5.19) for the amplitude of vibration of the primary mass yields (after some manipulation)

$$\frac{Xk}{F_0} = \frac{1 - \omega^2/\omega_a^2}{\left[1 + \mu(\omega_a/\omega_p)^2 - (\omega/\omega_p)^2\right]\left[1 - (\omega/\omega_a)^2\right] - \mu(\omega_a/\omega_p)^2} \tag{5.24}$$

The absolute value of this expression is plotted in Figure 5.15 for the case $\mu = 0.25$. Such plots can be used to illustrate how much drift in driving frequency can be tolerated by the absorber design. Note that if ω should drift to either $0.781\omega_a$ or $1.28\omega_a$, the combined system would experience resonance and fail, since these are the natural frequencies of the combined system. In fact, if the driving frequency shifts such that $|Xk/F_0| > 1$, the force transmitted to the primary system is amplified and the absorber system is not an improvement over the original design of the primary system. The shaded area of Figure 5.15 illustrates the values of ω/ω_a such that $|Xk/F_0| \le 1$. This illustrates the useful operating range of the absorber design (i.e., $0.908\omega_a < \omega < 1.118\omega_a$). Hence if the driving frequency drifts within this range, the absorber design still offers some protection to the primary system by reducing its steady-state vibration magnitude.

The design of an absorber can be further illuminated by examining the mass ratio μ and the frequency ratio β. These two dimensionless quantities indirectly specify both the mass and stiffness of the absorber system. The frequency equation (characteristic equation) for the two-mass system is obtained by setting the determinant of the matrix coefficient in equation (5.17) [i.e., the denominator of equation (5.18)] to zero and interpreting ω as the system natural frequency ω_n. Substitution for the value of β and rearranging yields

$$\beta^2\left(\frac{\omega_n^2}{\omega_a^2}\right)^2 - \left[1 + \beta^2(1 + \mu)\right]\frac{\omega_n^2}{\omega_a^2} + 1 = 0 \tag{5.25}$$

which is a quadratic equation in (ω_n^2/ω_a^2). Solving this yields

$$\left(\frac{\omega_n}{\omega_a}\right)^2 = \frac{1 + \beta^2(1 + \mu)}{2\beta^2} \pm \frac{1}{2\beta^2}\sqrt{\beta^4(1 + \mu)^2 - 2\beta^2(1 - \mu) + 1} \tag{5.26}$$

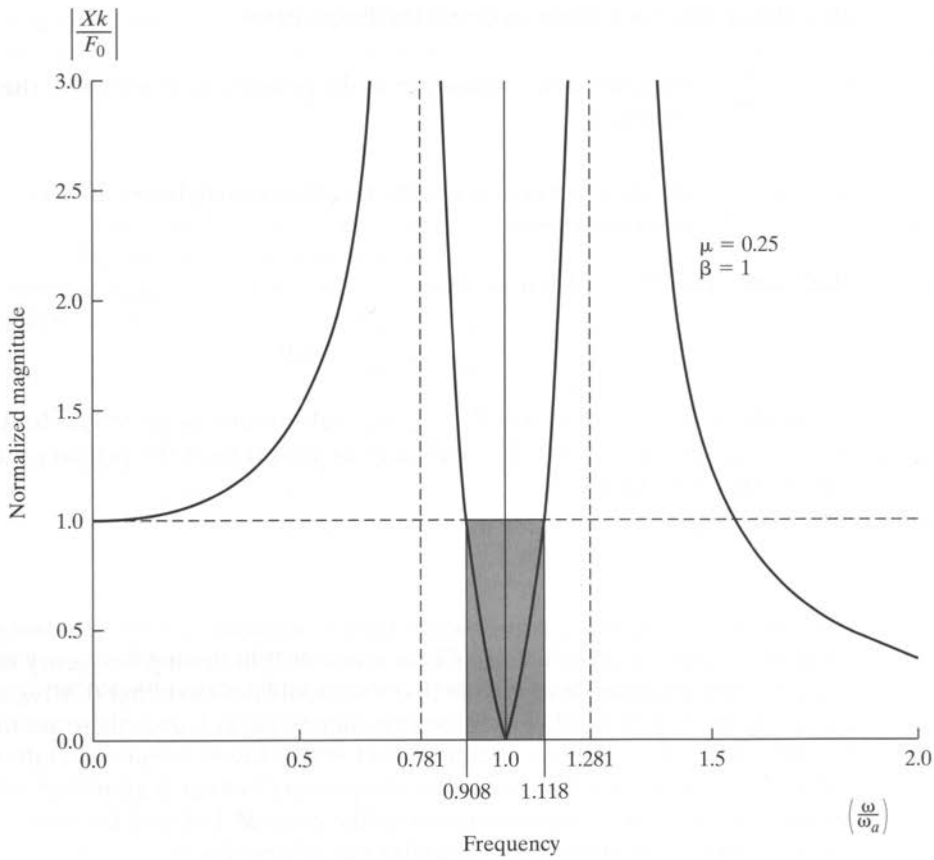

Figure 5.15 Plot of normalized magnitude of the primary mass versus the normalized driving frequency for the case $\mu = 0.25$. The two natural frequencies of the system occur at 0.781 and 1.281.

which illustrates how the system's natural frequencies vary with the mass ratio μ and the frequency ratio β. This is plotted for $\beta = 1$ in Figure 5.16. Note that as μ is increased, the natural frequencies split farther apart, and farther from the operating

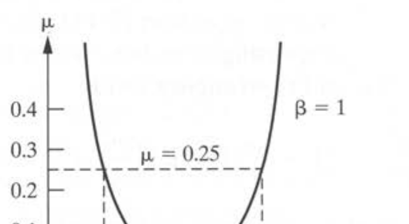

Figure 5.16 Plot of mass ratio versus system natural frequency (normalized to the frequency of the absorber system), illustrating that increasing the mass ratio increases the useful frequency range of a vibration absorber. Here ω_1 and ω_2 indicate the normalized value of the system's natural frequencies.

point $\omega = \omega_a$ of the absorber. Therefore, if μ is too small, the combined system will not tolerate much fluctuation in the driving frequency before it fails. As a rule of thumb, μ is usually taken to be between 0.05 and 0.25 (i.e., $0.05 \le \mu \le 0.25$), as larger values of μ tend to indicate a poor design. Vibration absorbers can also fail because of fatigue if $x_a(t)$ and the stresses associated with this motion of the absorber are large. Hence limits are often placed on the maximum value of X_a by the designer. The following example illustrates an absorber design.

Example 5.3.1

A radial saw base has a mass of 73.16 kg and is driven harmonically by a motor that turns the saw's blade as illustrated in Figure 5.17. The motor runs at constant speed and produces a 13-N force at 180 cycles/min because of a small unbalance in the motor. The resulting forced vibration was not detected until after the saw had been manufactured. The manufacturer wants a vibration absorber designed to drive the table oscillation to zero simply by retrofitting an absorber onto the base. Design the absorber assuming that the effective stiffness provided by the table legs is 2600 N/m. In addition, the absorber must fit inside the table base and hence has a maximum deflection of 0.2 cm.

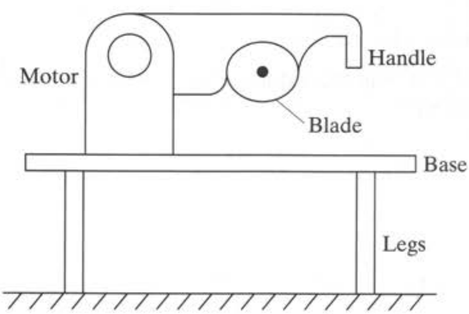

Figure 5.17 Schematic of a radial saw system in need of a vibration absorber.

Solution To meet the deflection requirement, the absorber stiffness is chosen first. This is calculated by assuming that $X = 0$, so that $|X_a k_a| = |F_0|$ [i.e., so that the mass m_a absorbs all of the applied force, i.e., equation (5.20) with $k_a = m_a\omega^2$]. Hence

$$k_a = \frac{F_0}{X_a} = \frac{13\text{ N}}{0.2\text{ cm}} = \frac{13\text{ N}}{0.002\text{ m}} = 6500\text{ N/m}$$

Since the absorber is designed such that $\omega = \omega_a$,

$$m_a = \frac{k_a}{\omega^2} = \frac{6500\text{ N/m}}{[(180/60)2\pi]^2} = 18.29\text{ kg}$$

Note in this case that $\mu = 18.29/73.16 = 0.25$.

□

Example 5.3.2

Calculate the bandwidth of operation of the absorber design of Example 5.3.1. Assume that the useful range of an absorber is defined such that $|Xk/F_0| < 1$. For values of $|Xk/F_0| > 1$, the machine could easily drift into resonance and the amplitude of vibration actually becomes an amplification of the effective driving force amplitude.

Solution From equation (5.24) with $Xk/F_0 = 1$,

$$1 - \left(\frac{\omega}{\omega_a}\right)^2 = \left[1 + \mu\left(\frac{\omega_a}{\omega_p}\right)^2 - \left(\frac{\omega}{\omega_p}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_a}\right)^2\right] - \mu\left(\frac{\omega_a}{\omega_p}\right)^2$$

Solving this for ω/ω_a yields the two solutions

$$\frac{\omega}{\omega_a} = \pm\sqrt{1 + \mu}$$

For the system of Example 5.3.1, $\mu = 0.25$, so that the second solution becomes

$$\frac{\omega}{\omega_a} = 1.1180$$

The condition that $|Xk/F_0| = 1$ is also satisfied for $Xk/F_0 = -1$. Substitution of this into equation (5.24) followed by some manipulation yields

$$\left(\frac{\omega_a}{\omega_p}\right)^2\left(\frac{\omega}{\omega_a}\right)^4 - \left[2 + (\mu + 1)\left(\frac{\omega_a}{\omega_p}\right)^2\right]\left(\frac{\omega}{\omega_a}\right)^2 + 2 = 0$$

which is quadratic in $(\omega/\omega_a)^2$. Using the values of $\omega_a^2 = 6500/18.29$, $\omega_p^2 = 2600/73.16$, and $\mu = 0.25$, this simplifies to

$$10\left(\frac{\omega}{\omega_a}\right)^4 - 14.5\left(\frac{\omega}{\omega_a}\right)^2 + 2 = 0$$

Solving for ω/ω_a yields

$$\left(\frac{\omega}{\omega_a}\right)^2 = 0.1544, 1.2956 \quad \text{or} \quad \frac{\omega}{\omega_a} = 0.3929, 1.1382$$

Hence the three roots satisfying $|Xk/F_0| = 1$ are 0.3929, 1.1180, and 1.1382. Following the example of Figure 5.15 indicates that the driving frequency may vary between $0.3929\omega_a$ and $1.1180\omega_a$, or since $\omega_a = 18.857$,

$$7.4089 < \omega < 21.0821 \text{ (rad/s)}$$

before the response of the primary mass is amplified or the system is in danger of experiencing resonance.

□

The preceding discussion and examples illustrate the concept of *performance robustness;* that is, the examples illustrate how the design holds up as the parameter values (k, k_a, etc.) drift from the values used in the original design. Example 5.3.2 illustrates that the mass ratio greatly affects the robustness of absorber designs. This is stated in the caption of Figure 5.16; up to a certain point, increasing μ increases the robustness of the absorber design. The effects of damping on absorber design are examined in the next section.

5.4 DAMPING IN VIBRATION ABSORPTION

As mentioned in Section 5.3, damping is often present in devices and has the potential for destroying the ability of a vibration absorber to protect the primary system fully by driving X to zero. In addition, damping is sometimes added to vibration absorbers to prevent resonance or to improve the effective bandwidth of operation of a vibration absorber. Also, a damper by itself is often used as a vibration absorber by dissipating the energy supplied by an applied force. Such devices are called *vibration dampers* rather than absorbers.

First consider the effect of modeling damping in the standard vibration absorber problem. A vibration absorber with damping in both the primary and absorber system is illustrated in Figure 5.18. This system is dynamically equal to the system of Figure 4.15 of Section 4.5. The equations of motion are given in matrix form by equation (4.116) as

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \ddot{x}_a(t) \end{bmatrix} + \begin{bmatrix} c + c_a & -c_a \\ -c_a & c_a \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{x}_a(t) \end{bmatrix} + \begin{bmatrix} k + k_a & -k_a \\ -k_a & k_a \end{bmatrix}\begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} \sin \omega t \tag{5.27}$$

Note, as was mentioned in Section 4.5, that these equations cannot necessarily be solved by using the modal analysis technique of Chapter 4 because the equations do not decouple $(KM^{-1}C \neq CM^{-1}K)$. The steady-state solution can be calculated, however, by using a combination of the exponential approach discussed in Section 2.3 and the matrix inverse used in previous sections for the undamped case.

To this end, let $F_0 \sin \omega t$ be represented in exponential form by $F_0 e^{j\omega t}$ in equation (5.27) and assume that the steady-state solution is of the form

$$\mathbf{x}(t) = \mathbf{X}e^{j\omega t} = \begin{bmatrix} X \\ X_a \end{bmatrix} e^{j\omega t} \tag{5.28}$$

where X is the amplitude of vibration of the primary mass and X_a is the amplitude of vibration of the absorber mass. Substitution into equation (5.27) yields

$$\begin{bmatrix} (k + k_a - m\omega^2) + (c + c_a)\omega j & -k_a - c_a\omega j \\ -k_a - c_a\omega j & (k_a - m_a\omega^2) + c_a\omega j \end{bmatrix}\begin{bmatrix} X \\ X_a \end{bmatrix} e^{j\omega t} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} e^{j\omega t} \tag{5.29}$$

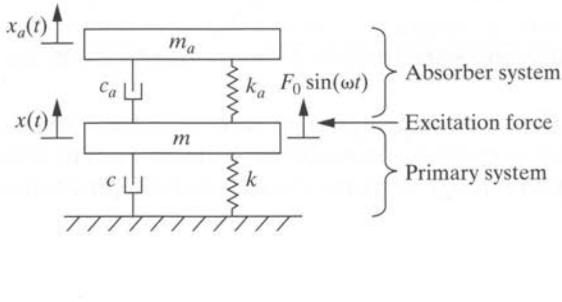

Figure 5.18 Schematic of a vibration absorber with damping in both the primary and absorber system.

Note that the coefficient matrix of the vector **X** has complex elements. Dividing equation (5.29) by the nonzero scalar $e^{j\omega t}$ yields a complex matrix equation in the amplitudes X and X_a. Calculating the matrix inverse using the formula of Example 4.1.4, reviewed in Window 5.3, and multiplying equation (5.29) by the inverse from the right yields

$$\begin{bmatrix} X \\ X_a \end{bmatrix} = \frac{\begin{bmatrix} (k_a - m_a\omega^2) + c_a\omega j & k_a + c_a\omega j \\ k_a + c_a\omega j & k + k_a - m\omega^2 + (c + c_a)\omega j \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}}{\det(K - \omega^2 M + \omega j C)} \tag{5.30}$$

Here the determinant in the denominator is given by (recall Example 4.1.4)

$$\det(K - \omega^2 M + \omega j C) = m m_a \omega^4 - (c_a c + m_a(k + k_a) + k_a m)\omega^2 + k_a k \\ + [(k c_a + c k_a)\omega - (c_a(m + m_a) + c m_a)\omega^3] j \tag{5.31}$$

and the system coefficient matrices M, C, and K are given by

$$M = \begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix} \quad C = \begin{bmatrix} c + c_a & -c_a \\ -c_a & c_a \end{bmatrix} \quad K = \begin{bmatrix} k + k_a & -k_a \\ -k_a & k_a \end{bmatrix}$$

Simplifying the matrix vector product yields

$$X = \frac{[(k_a - m_a\omega^2) + c_a\omega j]F_0}{\det(K - \omega^2 M + \omega j C)} \tag{5.32}$$

$$X_a = \frac{(k_a + c_a\omega j)F_0}{\det(K - \omega^2 M + \omega j C)} \tag{5.33}$$

which expresses the magnitude of the response of the primary mass and absorber mass, respectively. Note that these values are now complex numbers and are multiplied by the complex value $e^{j\omega t}$ to get the time responses.

Equations (5.32) and (5.33) are the two-degree-of-freedom version of the frequency response function given for a single-degree-of-freedom system in equation (2.52). The complex nature of these values reflects a magnitude and phase. The magnitude is calculated following the rules of complex numbers and is best done with a symbolic computer code, or after substitution of numerical values for the various physical constants. It is important to note from equation (5.32) that unlike the tuned undamped absorber, the response of the primary system cannot be exactly zero even if the tuning condition is satisfied. Hence the presence of damping ruins the ability of the absorber system to exactly cancel the motion of the primary system.

Equations (5.32) and (5.33) can be analyzed for several specific cases. First, consider the case for which the internal damping of the primary system is neglected ($c = 0$). If the primary system is made of metal, the internal damping is likely to be very low and it is reasonable to neglect it in many circumstances. In this case the determinant of equation (5.31) reduces to the complex number

$$\det(K - \omega^2 M + \omega C j)$$
$$= \left[(-m\omega^2 + k)(-m_a\omega^2 + k_a) - m_a k_a \omega^2\right] + \left[(k - (m + m_a)\omega^2)c_a\omega\right]j \quad (5.34)$$

The maximum deflection of the primary mass is given by equation (5.32) with the determinant in the denominator evaluated as given in equation (5.34). This is the ratio of two complex numbers and hence is a complex number representing the phase and the amplitude of the response of the primary mass. Using complex arithmetic (see Window 5.4) the amplitude of the motion of the primary mass can be written as the real number

$$\frac{X^2}{F_0^2} = \frac{(k_a - m_a\omega^2)^2 + \omega^2 c_a^2}{\left[(k - m\omega^2)(k_a - m_a\omega^2) - m_a k_a \omega^2\right]^2 + \left[k - (m + m_a)\omega^2\right]^2 c_a^2 \omega^2} \quad (5.35)$$

Window 5.4
Reminder of Complex Arithmetic

The response magnitude given by equation (5.32) can be written as the ratio of two complex numbers:

$$\frac{X}{F_0} = \frac{A_1 + B_1 j}{A_2 + B_2 j}$$

where A_1, A_2, B_1, and B_2 are real numbers and $j = \sqrt{-1}$. Multiplying this by the conjugate of the denominator divided by itself yields

$$\frac{X}{F_0} = \frac{(A_1 + B_1 j)(A_2 - B_2 j)}{(A_2 + B_2 j)(A_2 - B_2 j)} = \frac{(A_1 A_2 + B_1 B_2)}{A_2^2 + B_2^2} + \frac{B_1 A_2 - A_1 B_2}{A_2^2 + B_2^2} j$$

which indicates how X/F_0 is written as a single complex number of the form $X/F_0 = a + bj$. This is interpreted, as indicated, that the response magnitude has two components: one in phase with the applied force and one out of phase. The magnitude of X/F_0 is the length of the preceding complex number (i.e., $|X/F_0| = \sqrt{a^2 + b^2}$. This yields

$$\left|\frac{X}{F_0}\right| = \sqrt{\frac{A_1^2 + B_1^2}{A_2^2 + B_2^2}}$$

which corresponds to the expression given in equation (5.35). (Also see Appendix A.)

It is instructive to examine this amplitude in terms of the dimensionless ratios introduced in Section 5.3 for the undamped vibration absorber. The amplitude X is written in terms of the static deflection $\Delta = F_0/k$ of the primary system. In addition, consider the mixed "damping ratio" defined by

$$\zeta = \frac{c_a}{2m_a\omega_p} \tag{5.36}$$

where $\omega_p = \sqrt{k/m}$ is the original natural frequency of the primary system without the absorber attached. Using the standard frequency ratio $r = \omega/\omega_p$, the ratio of natural frequencies $\beta = \omega_a/\omega_p$ (where $\omega_a = \sqrt{k_a/m_a}$), and the mass ratio $\mu = m_a/m$, equation (5.35) can be rewritten as

$$\frac{X}{\Delta} = \frac{Xk}{F_0} = \sqrt{\frac{(2\zeta r)^2 + (r^2 - \beta^2)^2}{(2\zeta r)^2(r^2 - 1 + \mu r^2)^2 + [\mu r^2\beta^2 - (r^2 - 1)(r^2 - \beta^2)]^2}} \tag{5.37}$$

which expresses the dimensionless amplitude of the primary system. Note from examining equation (5.37) that the amplitude of the primary system response is determined by four physical parameter values:

- μ the ratio of the absorber mass to the primary mass
- β the ratio of the decoupled natural frequencies
- r the ratio of the driving frequency to the primary natural frequency
- ζ the ratio of the absorber damping and $2m_a\omega_p$

These four numbers can be considered as design variables and are chosen to give the smallest possible value of the primary mass's response, X, for a given application. Figure 5.19 illustrates how the damping value, as reflected in ζ, affects the response for a fixed value of $\mu = 0.25$ and $\beta = 1$, as r varies.

As mentioned at the beginning of this section, damping is often added to the absorber to improve the bandwidth of operation. This effect is illustrated in Figure 5.19.

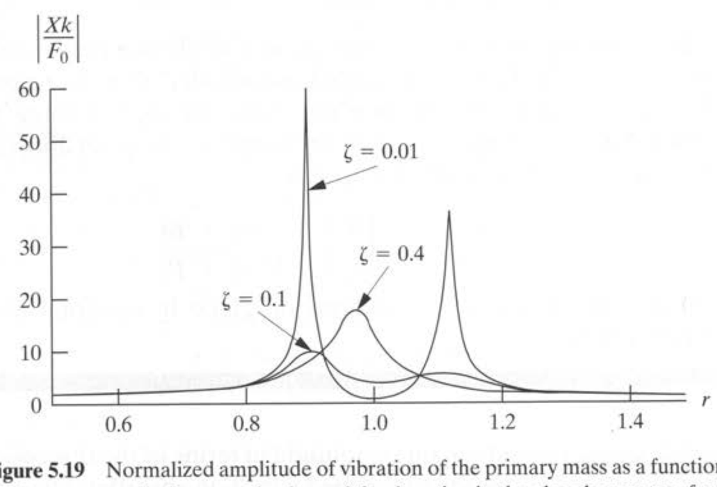

Figure 5.19 Normalized amplitude of vibration of the primary mass as a function of the frequency ratio for several values of the damping in the absorber system for the case of negligible damping in the primary system [i.e., a plot of equation (5.37)].

Recall that if there is no damping in the absorber ($\zeta = 0$), the magnitude of the response of the primary mass as a function of the frequency ratio r is as illustrated in Figure 5.15 (i.e., zero at $r = 1$ but infinite at $r = 0.781$ and $r = 1.281$). Thus the completely undamped absorber has poor bandwidth (i.e., if r changes by a small amount, the amplitude grows). In fact, as noted in Section 5.3, the bandwidth, or useful range of operation of that undamped absorber, is $0.908 \leq r \leq 1.118$. For these values of r, $|Xk/F_0| \leq 1$. However, if damping is added to the absorber ($\zeta \neq 0$), Figure 5.19 results, and the bandwidth, or useful range of operation, is extended. The price for this increased operating region is that $|Xk/F|$ is never zero in the damped case (see Figure 5.19).

Examination of Figure 5.19 shows that as ζ is varied, the amplification of $|Xk/F_0|$ over the range of r can be reduced. The design question now becomes: For what values of the mass ratio μ, the absorber damping ratio ζ, and the frequency ratio β is the magnitude $|Xk/F_0|$ smallest over the region $0 \leq r \leq 2$? Just increasing the damping with μ and β fixed does not necessarily yield the lowest amplitude. Note from Figure 5.19 that $\zeta = 0.1$ produces a smaller amplification over a larger region of r than does the higher ratio, $\zeta = 0.4$. Figures 5.20 and 5.21 yield some hint of how the various parameters affect the magnitude by providing plots of $|Xk/F_0|$ for various combinations of ζ, μ, and β.

A solution of the best choice of μ and ζ is discussed again in Section 5.5. Note from Figure 5.21 that $\mu = 0.25$, $\beta = 0.8$, and $\zeta = 0.27$ yield a minimum value of $|Xk/F_0|$ over a large range of values of r. However, amplification of the response X still occurs (i.e., $|Xk/F_0| > 1$ for values of $r < \sqrt{2}$), but no order-of-magnitude increase in $|X|$ occurs as in the case of the undamped absorber.

Next consider the case of an appended absorber mass connected to an undamped primary mass only by a dashpot, an arrangement illustrated in Figure 5.22. Systems of this form arise in the design of vibration reduction devices for rotating systems such as engines, where the operating speed (and hence the driving frequency)

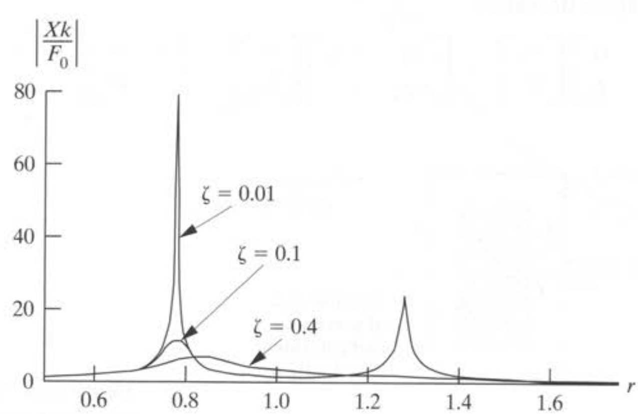

Figure 5.20 Repeat of the plot of Figure 5.19 with $\mu = 0.25$ and $\beta = 1$ for several values of ζ. Note that in this case, $\zeta = 0.4$ yields a lower magnitude than does $\zeta = 0.1$.

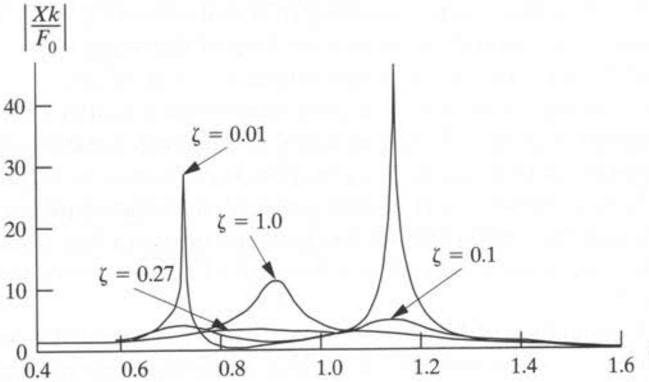

Figure 5.21 Repeat of the plots of Figure 5.19 with $\mu = 0.25$, $\beta = 0.8$ for several values of ζ. In this case $\zeta = 0.27$ yields the lowest amplification over the largest bandwidth.

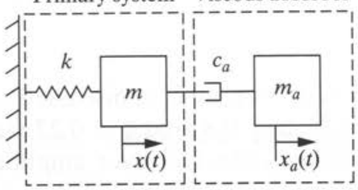

Figure 5.22 Damper–mass system added to a primary mass (with no damping) to form a viscous vibration absorber.

varies over a wide range. In such cases a viscous damper is added to the end of the crankshaft (or other rotating device) as indicated in Figure 5.23. The shaft spins through an angle θ_1 with torsional stiffness k and inertia J_1. The damping inertia J_2 spins through an angle θ_2 in a viscous film providing a damping force $c_a(\dot{\theta}_1 - \dot{\theta}_2)$. If an external harmonic torque is applied of the form $M_0 e^{\omega t j}$, the equation of motion of this system becomes

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_a & -c_a \\ -c_a & c_a \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_0 \\ 0 \end{bmatrix} e^{\omega t j} \quad (5.38)$$

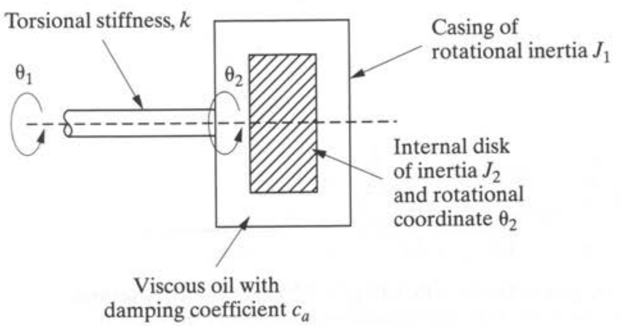

Figure 5.23 Viscous damper and mass added to a rotating shaft for broadband vibration absorption. Often called a *Houdaille damper*.

This is a rotational equivalent to the translational model given in Figure 5.22. It is easy to calculate the undamped natural frequencies of this two-degree-of-freedom system. They are

$$\omega_p = \sqrt{\frac{k}{J_1}} \text{ and } \omega_a = 0$$

The solution of this set of equations is given by equations (5.32) and (5.33) with m and m_a replaced by J_1 and J_2, respectively, $c = 0, k_a = 0$, and F_0 replaced by M_0. Equation (5.32) is given in nondimensional form as equation (5.37). Hence letting $\beta = \omega_a/\omega_p = 0$ in equation (5.37) yields that amplitude of vibration of the primary inertia J_1 [i.e., the amplitude of $\theta_1(t)$] is described by

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}} \tag{5.39}$$

where $\zeta = c/(2J_2\omega_p)$, $r = \omega/\omega_p$, and $\mu = J_2/J_1$. Figure 5.24 illustrates several plots of Xk/M_0 for various values of ζ for a fixed μ as a function of r. Note again that the highest damping does not correspond to the largest-amplitude reduction.

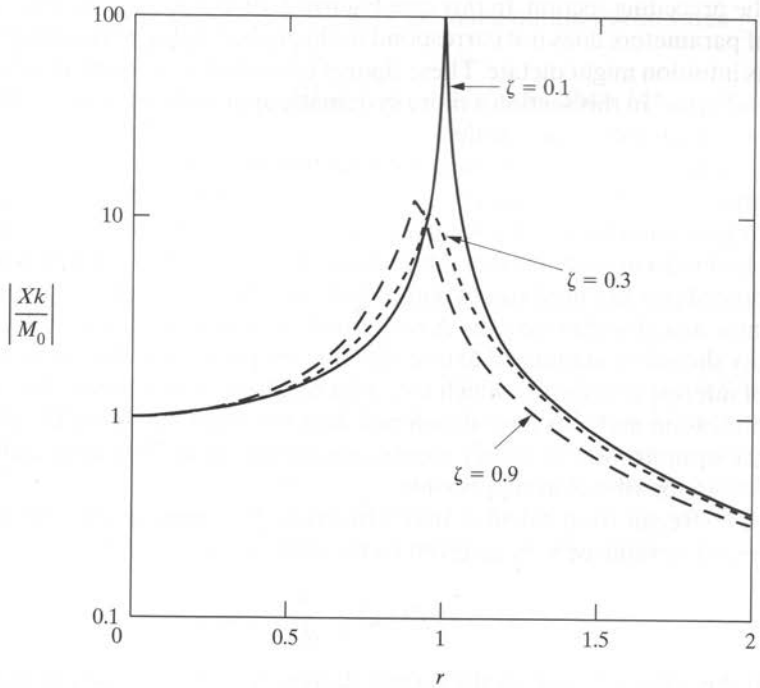

Figure 5.24 Amplitude curves for a system with a viscous absorber, a plot of equation (5.39), for the case $\mu = 0.25$ and for three different values of ζ.

The various absorber designs discussed previously, excluding the undamped case, result in a number of possible "good" choices for the various design parameters. When faced with a number of good choices, it is natural to ask which is the best choice. Looking for the best possible choice among a number of acceptable or good choices can be made systematic by using methods of optimization introduced in the next section.

5.5 OPTIMIZATION

In the design of vibration systems the best selection of system parameters is often sought. In the case of the undamped vibration absorber of Section 5.3 the best selection for values of mass and stiffness of the absorber system is obvious from examining the expression for the amplitude of vibration of the primary system. In this case the amplitude could be driven to zero by tuning the absorber mass and stiffness to the driving frequency. In the other cases, especially when damping is included, the choice of parameters to produce the best response is not obvious. In such cases optimization methods can often be used to help select the best performance. Optimization techniques often produce results that are not obvious. An example is in the case of the undamped primary system or the damped absorber system discussed in the preceding section. In this case Figures 5.19 to 5.21 indicate that the best selection of parameters does not correspond to the highest value of the damping in the system as intuition might dictate. These figures essentially represent an optimization by trial and error. In this section a more systematic approach to optimization is suggested by taking advantage of calculus.

Recall from elementary calculus that minimums and maximums of particular functions can be obtained by examining certain derivatives. Namely, if the first derivative vanishes and the second derivative of the function is positive, the function has obtained a minimum value. This section presents a few examples where optimization procedures are used to obtain the best possible vibration reduction for various isolator and absorber systems. A major task of optimization is first deciding what quantity should be minimized to best describe the problem under study. The next question of interest is to decide which variables to allow to vary during the optimization. Optimization methods have developed over the years that allow the parameters during the optimization to satisfy constraints, for example. This approach is often used in design for vibration suppression.

Recall from calculus that a function $f(x)$ experiences a maximum (or minimum) at value of $x = x_m$ given by the solution of

$$f'(x_m) = \frac{d}{dx}[f(x_m)] = 0 \tag{5.40}$$

If this value of x causes the second derivative, $f''(x_m)$, to be less than zero, the value of $f(x)$ at $x = x_m$ is the maximum value that $f(x)$ takes on in the region near $x = x_m$. Similarly, if $f''(x_m)$ is greater than zero, the value of $f(x_m)$ is the smallest or minimum value that $f(x)$ obtains in the interval near x_m. Note that if $f''(x) = 0$, at $x = x_m$, the value $f(x_m)$ is neither a minimum or maximum for $f(x)$. The points where $f'(x)$ vanish are called *critical points*.

These simple rules were used in Section 2.2, Example 2.2.3, for computing the value (r_{peak}) where the maximum value of normalized magnitude of the steady-state response of a harmonically driven single-degree-of-freedom system occurs. The second derivative test was not checked because several plots of the function clearly indicated that the curve contains a global maximum value rather than a minimum. In both absorber and isolator design, plots of the magnitude of the response can be used to avoid having to calculate the second derivative (second derivatives are often unpleasant to calculate).

If the function f to be minimized (or maximized) is a function of two variables [i.e., $f = f(x, y)$], the preceding derivative tests become slightly more complicated and involve examining the various partial derivatives of the function $f(x, y)$. In this case the critical points are determined from the equations

$$f_x(x, y) = \frac{\partial f(x, y)}{\partial x} = 0$$
$$f_y(x, y) = \frac{\partial f(x, y)}{\partial y} = 0 \tag{5.41}$$

Whether or not these critical points (x, y) are a maximum of the value $f(x, y)$ or a minimum depends on the following:

1. If $f_{xx}(x, y) > 0$ and $f_{xx}(x, y)f_{yy}(x, y) > f_{xy}^2(x, y)$, then $f(x, y)$ has a relative minimum value at x, y.
2. If $f_{xx}(x, y) < 0$ and $f_{xx}(x, y)f_{yy}(x, y) > f_{xy}^2(x, y)$, then $f(x, y)$ has a relative maximum value at x, y.
3. If $f_{xy}^2(x, y) > f_{xx}(x, y)f_{yy}(x, y)$, then $f(x, y)$ is neither a maximum nor a minimum value; the point x, y is a *saddle* point.
4. If $f_{xy}^2(x, y) = f_{xx}(x, y)f_{yy}(x, y)$, the test fails and the point x, y could be any or none of the above.

Plots of $f(x, y)$ can also be used to determine whether or not a given critical point is a maximum, minimum, saddle point, or none of these. These rules can be used to help solve vibration design problems in some circumstances. As an example of using these optimization formulations for designing a vibration suppression system, recall the damped absorber system of Section 5.4. In this case the magnitude of the primary mass normalized with respect to the input force (moment) magnitude is given in equation (5.39) to be

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}} = f(r, \zeta) \tag{5.42}$$

which is considered to be a function of the mixed damping ratio ζ and the frequency ratio r for a fixed mass ratio μ.

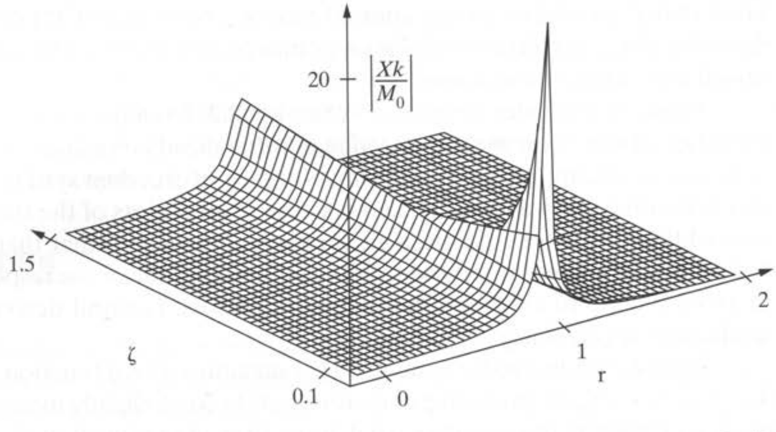

Figure 5.25 Plot of the normalized magnitude of the primary system versus both ζ and r [i.e., a two-dimensional plot of equation (5.42) for $\mu = 0.25$]. This illustrates that the most desirable response is obtained at the *saddle point*.

In Section 5.4, values of $f(r)$ are plotted versus r for several values of ζ in an attempt to find the value of ζ that yields the smallest maximum value of $f(r, \zeta)$. This is illustrated in Figure 5.24. Figure 5.25 illustrates the magnitude as a function of both ζ and r. From the figure it can be concluded that the derivative $\partial f/\partial r = 0$ yields the maximum value of the magnitude for each fixed ζ.

Looking along the ζ axis, the partial derivative $\partial f/\partial \zeta = 0$ yields the minimum value of $f(r, \zeta)$ for each fixed value of r. The best design, corresponding to the smallest of the largest amplitudes, is thus illustrated in Figure 5.25. This point corresponds to a saddle point and can be calculated by evaluating the appropriate first partial derivatives.

First consider $\partial(Xk/M_0)/\partial\zeta$. From equation (5.42), the function to be differentiated is of the form

$$f = \frac{A^{1/2}}{B^{1/2}} \tag{5.43}$$

where $A = 4\zeta^2 + r^2$ and $B = 4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2$. Differentiating and equating the resulting derivatives to zero yields

$$\frac{\partial f}{\partial \zeta} = \frac{1}{2}\frac{A^{-1/2}dA}{B^{1/2}} - \frac{1}{2}A^{1/2}\frac{dB}{B^{3/2}} = 0 \tag{5.44}$$

Solving this yields the form $[B\,dA - A\,dB]/2B^{3/2} = 0$ or

$$B\,dA = A\,dB \tag{5.45}$$

where A and B are as defined previously and

$$dA = 8\zeta \quad \text{and} \quad dB = 8\zeta(r^2 + \mu r^2 - 1)^2 \tag{5.46}$$

Substitution of these values of A, dA, B, and dB into equation (5.45) yields

$$(1 - r^2)^2 = (1 - r^2 - \mu r^2)^2 \tag{5.47}$$

For $\mu \neq 0, r > 0$, this has the solution

$$r = \sqrt{\frac{2}{2 + \mu}} \tag{5.48}$$

Similarly, differentiating equation (5.42) with respect to r and substituting the value for r obtained previously yields

$$\zeta_{\text{op}} = \frac{1}{\sqrt{2(\mu + 1)(\mu + 2)}} \tag{5.49}$$

Equation (5.49) reveals the value of ζ that yields the smallest amplitude at the point of largest amplitude (resonance) for the response of the primary mass. The maximum value of the displacement for the optimal damping is given by

$$\left(\frac{Xk}{M_0}\right)_{\max} = 1 + \frac{2}{\mu} \tag{5.50}$$

which is obtained by substitution of equations (5.48) and (5.49) into equation (5.42). This last expression suggests that μ should be as large as possible. However, the practical consideration that the absorber mass should be smaller than the primary mass requires $\mu \leq 1$. The value $\mu = 0.25$ is fairly common.

The second derivative conditions for the function f to have a saddle point (condition 3 in the preceding list) are too cumbersome to calculate. However, the plot of Figure 5.25 clearly illustrates that these conditions are satisfied. Furthermore, the plot indicates that f as a function of ζ is convex and f as a function of r is concave so that the saddle point condition is also the solution of minimizing the maximum value $f(r, \zeta)$, called the *min-max problem* in applied mathematics and optimization.

Example 5.5.1

A viscous damper–mass absorber is added to the shaft of an engine. The mass moment of inertia of the shaft system is 1.5 kg · m²/rad and has a torsional stiffness of 6×10^3 N · m/rad. The nominal running speed of the engine is 2000 rpm. Calculate the values of the added damper and mass moment of inertia such that the primary system has a magnification (Xk/M_0) of less than 5 for all speeds and is as small as possible at the running speed.

Solution Since $\omega_p = \sqrt{k/J}$, the natural frequency of the engine system is

$$\omega_p = \sqrt{\frac{6.0 \times 10^3 \text{ N} \cdot \text{m/rad}}{1.5 \text{ kg} \cdot \text{m}^2/\text{rad}}} = 63.24 \text{ rad/s}$$

The running speed of the engine is 2000 rpm or 209.4 rad/s, which is assumed to be the driving frequency (actually, it is a function of the number of cylinders). Hence the frequency ratio is

$$r = \frac{\omega}{\omega_p} = \frac{209.4}{63.24} = 3.31$$

so that the running speed is well away from the maximum amplification as illustrated in Figures 5.24 and 5.25 and the absorber is not needed to protect the shaft at its running speed. However, the engine spends some time getting to the running speed and often runs at lower speeds. The peak response occurs at

$$r_{\text{peak}} = \frac{\omega}{\omega_p} = \sqrt{\frac{2}{2 + \mu}}$$

as given by equation (5.48) and has a value of

$$\left(\frac{Xk}{M_0}\right)_{\max} = 1 + \frac{2}{\mu}$$

as given by equation (5.50). The magnification is restricted to be 5, so that

$$1 + \frac{2}{\mu} \le 5, \quad \text{or} \quad \mu \ge 0.5$$

Thus $\mu = 0.5$ is chosen for the design. Since the mass of the primary system is $J_1 = 1.5\ \text{kg}\cdot\text{m}^2/\text{rad}$ and $\mu = J_2/J_1$, the mass of the absorber is

$$J_2 = \mu J_1 = \frac{1}{2}(1.5)\ \text{kg}\cdot\text{m}^2/\text{rad} = 0.75\ \text{kg}\cdot\text{m}^2\cdot\text{rad}$$

The damping value required for equation (5.50) to hold is given by equation (5.49) or

$$\zeta_{\text{op}} = \frac{1}{\sqrt{2(\mu + 1)(\mu + 2)}} = \frac{1}{\sqrt{2(1.5)(2.5)}} = 0.3651$$

Recall from Section 5.4 [just following equation (5.39)] that $\zeta = c/(2J_2\omega_p)$, so that the optimal damping constant becomes

$$c_{\text{op}} = 2\zeta_{\text{op}}J_2\omega_{\text{p}} = 2(0.3651)(0.75)(63.24) = 34.638\ \text{N}\cdot\text{m}\cdot\text{s/rad}$$

The two values of J_2 and c given here form an optimal solution to the problem of designing a viscous damper–mass absorber system so that the maximum deflection of the primary shaft is satisfied $|Xk/M_0| < 5$. This solution is optimal in terms of a choice of ζ, which corresponds to the saddle point of Figure 5.25 and yields a minimum value of all maximum amplifications.

□

Optimization methods can also be useful in the design of certain types of vibration isolation systems. For example, consider the model of a machine mounted