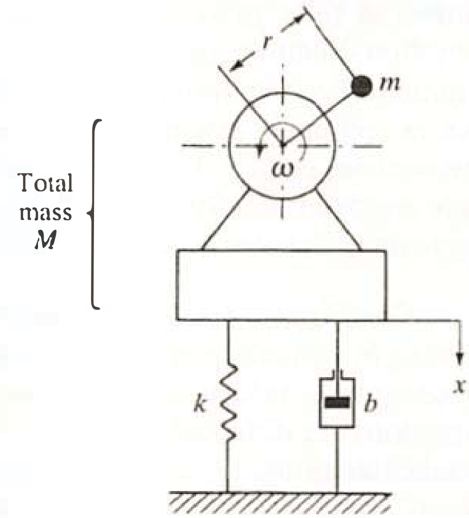

Figure 9–6 Unbalanced machine resting on shock mounts.

This acceleration acts toward the center of rotation, and the centripetal force is $ma = m\omega^2 r$. The *centrifugal force* is the opposing inertia force that acts outward. Its magnitude is also $m\omega^2 r$.

Vibration due to rotating unbalance. Force inputs that excite vibratory motion often arise from rotating unbalance, a condition that arises when the mass center of a rotating rigid body and the center of rotation do not coincide. Figure 9–6 shows an unbalanced machine resting on shock mounts. Assume that the rotor is rotating at a constant speed ω rad/s and that the unbalanced mass m is located a distance r from the center of rotation. Then the unbalanced mass will produce a centrifugal force of magnitude $m\omega^2 r$.

In the present analysis, we limit the motion to the vertical direction only, even though the rotating unbalance produces a horizontal component of force. The vertical component of this force, $m\omega^2 r \sin \omega t$, acts on the bearings and is thus transmitted to the foundation, thereby possibly causing the machine to vibrate excessively. [Note that, for convenience, we arbitrarily choose the time origin $t = 0$, so that the unbalance force applied to the system is $m\omega^2 r \sin \omega t$.]

Let us assume that the total mass of the system is M, which includes the unbalanced mass m. Here, we consider only vertical motion and measure the vertical displacement x from the equilibrium position in the absence of the forcing function. Then the equation of motion for the system becomes

$$M\ddot{x} + b\dot{x} + kx = p(t) \tag{9–11}$$

where

$$p(t) = m\omega^2 r \sin \omega t$$

is the force applied to the system. Taking the Laplace transform of both sides of Equation (9–11), assuming zero initial conditions, we have

$$(Ms^2 + bs + k)X(s) = P(s)$$

or

$$\frac{X(s)}{P(s)} = \frac{1}{Ms^2 + bs + k}$$

The sinusoidal transfer function is

$$\frac{X(j\omega)}{P(j\omega)} = G(j\omega) = \frac{1}{-M\omega^2 + bj\omega + k}$$

For the sinusoidal forcing function $p(t)$, the steady-state output is obtained from Equation (9-6) as

$$\begin{aligned} x(t) &= X\sin(\omega t + \phi) \\ &= |G(j\omega)|m\omega^2 r \sin\left(\omega t - \tan^{-1}\frac{b\omega}{k - M\omega^2}\right) \\ &= \frac{m\omega^2 r}{\sqrt{(k - M\omega^2)^2 + b^2\omega^2}} \sin\left(\omega t - \tan^{-1}\frac{b\omega}{k - M\omega^2}\right) \end{aligned}$$

In this last equation, if we divide the numerator and denominator of the amplitude and those of the phase angle by k and substitute $k/M = \omega_n^2$ and $b/M = 2\zeta\omega_n$ into the result, the steady-state output becomes

$$x(t) = \frac{m\omega^2 r/k}{\sqrt{[1 - (\omega^2/\omega_n^2)]^2 + (2\zeta\omega/\omega_n)^2}} \sin\left[\omega t - \tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - (\omega^2/\omega_n^2)}\right]$$

Thus, the steady-state output is a sinusoidal motion whose amplitude becomes large when the damping ratio ζ is small and the forcing frequency ω is close to the natural frequency ω_n.

9-4 VIBRATION ISOLATION

Vibration isolation is a process by which vibratory effects are minimized or eliminated. The function of a vibration isolator is to reduce the magnitude of force transmitted from a machine to its foundation or to reduce the magnitude of motion transmitted from a vibratory foundation to a machine.

The concept is illustrated in Figures 9-7(a) and (b). The system consists of a rigid body representing a machine connected to a foundation by an isolator that consists of a spring and a damper. Figure 9-7(a) illustrates the case in which the source of vibration is a vibrating force originating within the machine (force excitation). The isolator reduces the force transmitted to the foundation. In Figure 9-7(b), the source of vibration is a vibrating motion of the foundation (motion excitation). The isolator reduces the vibration amplitude of the machine.

The isolator essentially consists of a resilient load-supporting means (such as a spring) and an energy-dissipating means (such as a damper). A typical vibration isolator appears in Figure 9-8. (In a simple vibration isolator, a single element like synthetic rubber can perform the functions of both the load-supporting means and the energy-dissipating means.) In the analysis given here, the machine and the foundation are assumed rigid and the isolator is assumed massless.

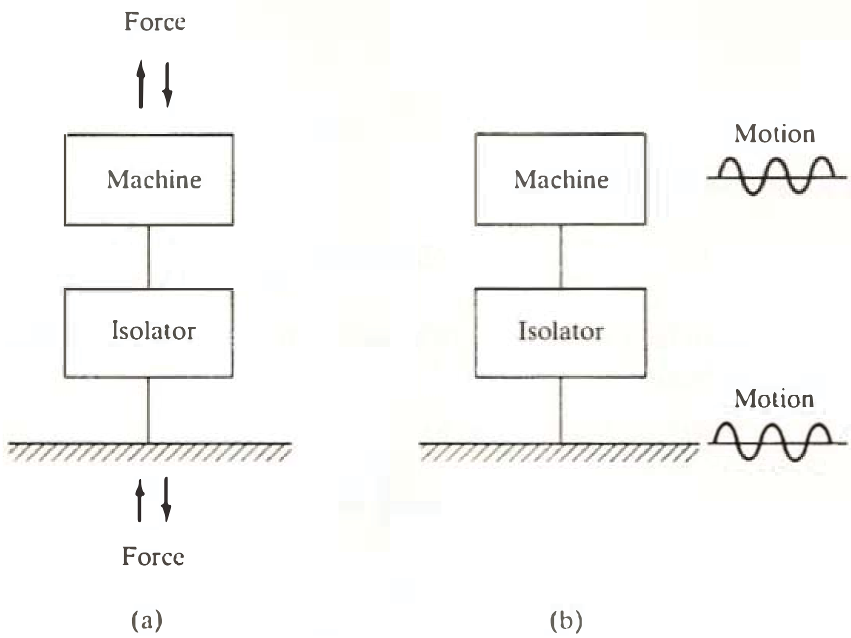

Figure 9–7 Vibration isolation. (a) Force excitation; (b) motion excitation.

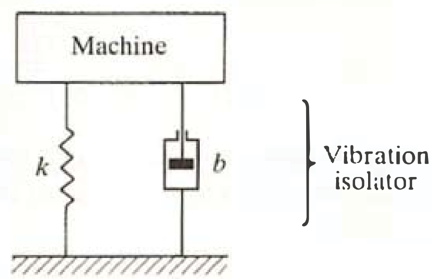

Figure 9–8 Vibration isolator.

Transmissibility. *Transmissibility* is a measure of the reduction of a transmitted force or of motion afforded by an isolator. If the source of vibration is a vibrating force due to the unbalance of the machine (force excitation), transmissibility is the ratio of the amplitude of the force transmitted to the foundation to the amplitude of the excitatory force. If the source of vibration is a vibratory motion of the foundation (motion excitation), transmissibility is the ratio of the vibration amplitude of the machine to the vibration amplitude of the foundation.

Transmissibility for force excitation. For the system shown in Figure 9–6, the source of vibration is a vibrating force resulting from the unbalance of the machine. The transmissibility in this case is the force amplitude ratio and is given by

$$\text{transmissibility} = \text{TR} = \frac{F_t}{F_0} = \frac{\text{amplitude of the transmitted force}}{\text{amplitude of the excitatory force}}$$

Let us find the transmissibility of this system in terms of the damping ratio ζ and the frequency ratio $\beta = \omega/\omega_n$.

The excitatory force (in the vertical direction) is caused by the unbalanced mass of the machine and is

$$p(t) = m\omega^2 r \sin \omega t = F_0 \sin \omega t$$

The equation of motion for the system is Equation (9–11), rewritten here for convenience:

$$M\ddot{x} + b\dot{x} + kx = p(t) \tag{9–12}$$

where M is the total mass of the machine including the unbalance mass m. The force $f(t)$ transmitted to the foundation is the sum of the damper and spring forces, or

$$f(t) = b\dot{x} + kx = F_t \sin(\omega t + \phi) \tag{9–13}$$

Taking the Laplace transforms of Equations (9–12) and (9–13), assuming zero initial conditions, gives

$$(Ms^2 + bs + k)X(s) = P(s)$$
$$(bs + k)X(s) = F(s)$$

where $X(s) = \mathscr{L}[x(t)]$, $P(s) = \mathscr{L}[p(t)]$, and $F(s) = \mathscr{L}[f(t)]$. Hence,

$$\frac{X(s)}{P(s)} = \frac{1}{Ms^2 + bs + k}$$
$$\frac{F(s)}{X(s)} = bs + k$$

Eliminating $X(s)$ from the last two equations yields

$$\frac{F(s)}{P(s)} = \frac{F(s)}{X(s)}\frac{X(s)}{P(s)} = \frac{bs + k}{Ms^2 + bs + k}$$

The sinusoidal transfer function is thus

$$\frac{F(j\omega)}{P(j\omega)} = \frac{bj\omega + k}{-M\omega^2 + bj\omega + k} = \frac{(b/M)j\omega + (k/M)}{-\omega^2 + (b/M)j\omega + (k/M)}$$

Substituting $k/M = \omega_n^2$ and $b/M = 2\zeta\omega_n$ into this last equation and simplifying, we have

$$\frac{F(j\omega)}{P(j\omega)} = \frac{1 + j(2\zeta\omega/\omega_n)}{1 - (\omega^2/\omega_n^2) + j(2\zeta\omega/\omega_n)}$$

from which it follows that

$$\left|\frac{F(j\omega)}{P(j\omega)}\right| = \frac{\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{[1 - (\omega^2/\omega_n^2)]^2 + (2\zeta\omega/\omega_n)^2}} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}}$$

where $\beta = \omega/\omega_n$.

Noting that the amplitude of the excitatory force is $F_0 = |P(j\omega)|$ and that the amplitude of the transmitted force is $F_t = |F(j\omega)|$, we obtain the transmissibility:

$$\mathrm{TR} = \frac{F_t}{F_0} = \frac{|F(j\omega)|}{|P(j\omega)|} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} \tag{9–14}$$

From Equation (9–14), we see that the transmissibility depends on both β and ζ. When $\beta = \sqrt{2}$, however, the transmissibility is equal to unity, regardless of the value of the damping ratio ζ.

Figure 9–9 shows some curves of transmissibility versus $\beta(= \omega/\omega_n)$. We see that all of the curves pass through a critical point where TR = 1 and $\beta = \sqrt{2}$. For $\beta < \sqrt{2}$, as the damping ratio ζ increases, the transmissibility at resonance decreases. For $\beta > \sqrt{2}$, as ζ increases, the transmissibility increases. Therefore, for $\beta < \sqrt{2}$, or $\omega < \sqrt{2}\,\omega_n$ (the forcing frequency ω is smaller than $\sqrt{2}$ times the undamped natural frequency ω_n), increasing damping improves the vibration isolation. For $\beta > \sqrt{2}$, or $\omega > \sqrt{2}\,\omega_n$, increasing damping adversely affects the vibration isolation.

Note that, since $|P(j\omega)| = F_0 = m\omega^2 r$, the amplitude of the force transmitted to the foundation is

$$F_t = |F(j\omega)| = \frac{m\omega^2 r\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} \tag{9–15}$$

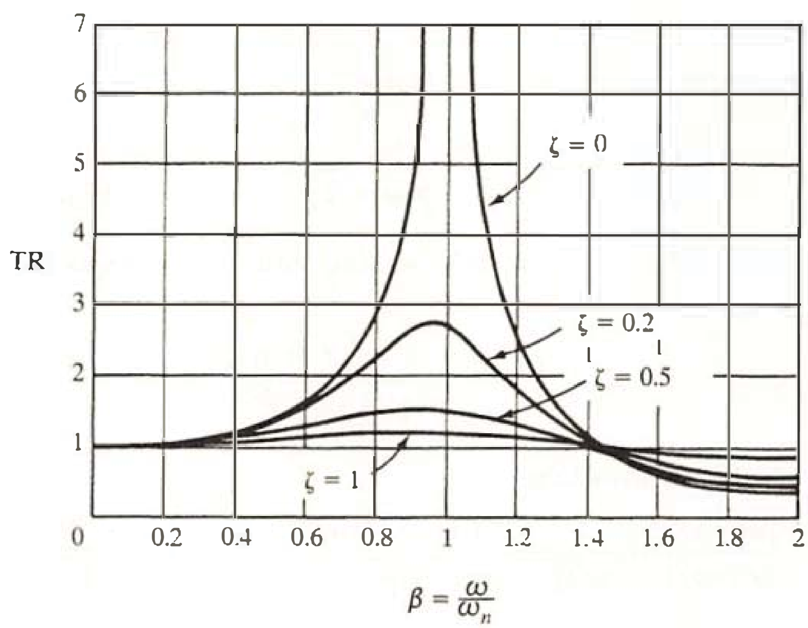

Figure 9–9 Curves of transmissibility TR versus $\beta(= \omega/\omega_n)$.

Example 9–3

In the system shown in Figure 9–6, if $M = 15$ kg, $b = 450$ N-s/m, $k = 6000$ N/m, $m = 0.005$ kg, $r = 0.2$ m, and $\omega = 16$ rad/s, what is the force transmitted to the foundation?

The equation of motion for the system is

$$15\ddot{x} + 450\dot{x} + 6000x = (0.005)(16)^2(0.2)\sin 16t$$

Consequently,

$$\omega_n = 20 \text{ rad/s}, \qquad \zeta = 0.75$$

and we find that $\beta = \omega/\omega_n = 16/20 = 0.8$. From Equation (9–15), we have

$$F_t = \frac{m\omega^2 r\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}}$$
$$= \frac{(0.005)(16)^2(0.2)\sqrt{1 + (2 \times 0.75 \times 0.8)^2}}{\sqrt{(1 - 0.8^2)^2 + (2 \times 0.75 \times 0.8)^2}} = 0.319 \text{ N}$$

The force transmitted to the foundation is sinusoidal with an amplitude of 0.319 N.

Automobile suspension system. Figure 9–10(a) shows an automobile system. Figure 9–10(b) is a schematic diagram of an automobile suspension system. As the car moves along the road, the vertical displacements at the tires act as motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass. A complete analysis of the suspension system would be very involved. A highly simplified version appears in Figure 9–11. Let us analyze this simple model when the motion input is sinusoidal. We shall derive the transmissibility for the motion excitation system. (As a related problem, see **Problem B–9–13.**)

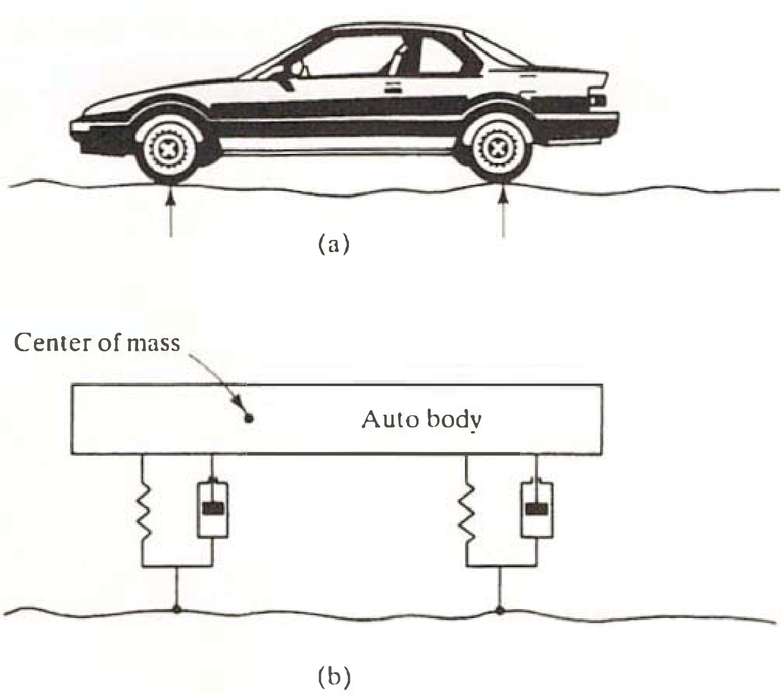

Figure 9–10 (a) Automobile system; (b) schematic diagram of an automobile suspension system.

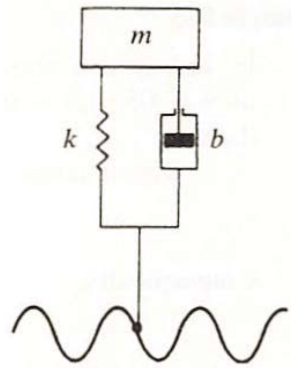

Figure 9–11 Simplified version of the automobile suspension system of Figure 9–10.

Transmissibility for motion excitation. In the mechanical system shown in Figure 9–12, the motion of the body is in the vertical direction only. The motion $p(t)$ at point A is the input to the system; the vertical motion $x(t)$ of the body is the output. The displacement $x(t)$ is measured from the equilibrium position in the absence of input $p(t)$. We assume that $p(t)$ is sinusoidal, or $p(t) = P \sin \omega t$.

The equation of motion for the system is

$$m\ddot{x} + b(\dot{x} - \dot{p}) + k(x - p) = 0$$

or

$$m\ddot{x} + b\dot{x} + kx = b\dot{p} + kp$$

The Laplace transform of this last equation, assuming zero initial conditions, gives

$$(ms^2 + bs + k)X(s) = (bs + k)P(s)$$

Hence,

$$\frac{X(s)}{P(s)} = \frac{bs + k}{ms^2 + bs + k}$$

The sinusoidal transfer function is

$$\frac{X(j\omega)}{P(j\omega)} = \frac{bj\omega + k}{-m\omega^2 + bj\omega + k}$$

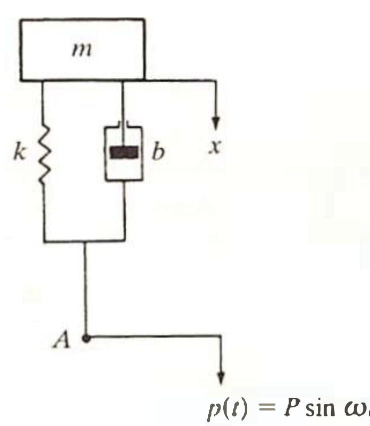

Figure 9–12 Mechanical system.

The steady-state output $x(t)$ has the amplitude $|X(j\omega)|$. The input amplitude is $|P(j\omega)|$. The transmissibility TR in this case is the displacement amplitude ratio and is given by

$$\text{TR} = \frac{\text{amplitude of the output displacement}}{\text{amplitude of the input displacement}}$$

Thus,

$$\text{TR} = \frac{|X(j\omega)|}{|P(j\omega)|} = \frac{\sqrt{b^2\omega^2 + k^2}}{\sqrt{(k - m\omega^2)^2 + b^2\omega^2}}$$

Noting that $k/m = \omega_n^2$ and $b/m = 2\zeta\omega_n$, we see that the transmissibility is given, in terms of the damping ratio ζ and the undamped natural frequency ω_n, by

$$\text{TR} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} \tag{9–16}$$

where $\beta = \omega/\omega_n$. This equation is identical to Equation (9–14).

Example 9–4

A rigid body is mounted on an isolator to reduce vibratory effects. Assume that the mass of the rigid body is 500 kg, the damping coefficient of the isolator is very small ($\zeta = 0.01$), and the effective spring constant of the isolator is 12,500 N/m. Find the percentage of motion transmitted to the body if the frequency of the motion excitation of the base of the isolator is 20 rad/s.

The undamped natural frequency ω_n of the system is

$$\omega_n = \sqrt{\frac{12{,}500}{500}} = 5 \text{ rad/s}$$

so

$$\beta = \frac{\omega}{\omega_n} = \frac{20}{5} = 4$$

Substituting $\zeta = 0.01$ and $\beta = 4$ into Equation (9–16), we have

$$\text{TR} = \frac{\sqrt{1 + (2\zeta\beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2\zeta\beta)^2}} = \frac{\sqrt{1 + (2 \times 0.01 \times 4)^2}}{\sqrt{(1 - 4^2)^2 + (2 \times 0.01 \times 4)^2}} = 0.0669$$

The isolator thus reduces the vibratory motion of the rigid body to 6.69% of the vibratory motion of the base of the isolator.

9–5 DYNAMIC VIBRATION ABSORBERS

If a mechanical system operates near a critical frequency, the amplitude of vibration increases to a degree that cannot be tolerated, because the machine might break down or might transmit too much vibration to the surrounding machines. This section discusses a way to reduce vibrations near a specified operating frequency that is