HOMEWORK 7 SOLUTIONS

1. a. The system can be decoupled if and only if $\mathbf{CM}^{-1}\mathbf{K} = \mathbf{KM}^{-1}\mathbf{C}$. So we have to show that this equation holds if and only if $\frac{c_1}{c_2} = \frac{k_1}{k_2}$.

$$\mathbf{CM}^{-1}\mathbf{K} = \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}^{-1}\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}\begin{bmatrix} \frac{k_1+k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2}{m_2} \end{bmatrix} = \begin{bmatrix} \frac{(c_1+c_2)(k_1+k_2)}{m_1} + \frac{c_2k_2}{m_2} & -\frac{(c_1+c_2)k_2}{m_1} - \frac{c_2k_2}{m_2} \\ -\frac{c_2(k_1+k_2)}{m_1} - \frac{c_2k_2}{m_2} & \frac{c_2k_2}{m_1} + \frac{c_2k_2}{m_2} \end{bmatrix}$$

$$\mathbf{KM}^{-1}\mathbf{C} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}^{-1}\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}\begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}$$

$$= \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} \frac{c_1+c_2}{m_1} & -\frac{c_2}{m_1} \\ -\frac{c_2}{m_2} & \frac{c_2}{m_2} \end{bmatrix} = \begin{bmatrix} \frac{(c_1+c_2)(k_1+k_2)}{m_1} + \frac{c_2k_2}{m_2} & -\frac{c_2(k_1+k_2)}{m_1} - \frac{c_2k_2}{m_2} \\ -\frac{(c_1+c_2)k_2}{m_1} - \frac{c_2k_2}{m_2} & \frac{c_2k_2}{m_1} + \frac{c_2k_2}{m_2} \end{bmatrix}$$

We see here that the diagonal components are equal already; however, the off-diagonal components are transposed. For them to be equal, we must have

$$\begin{aligned} -\frac{(c_1+c_2)k_2}{m_1} - \frac{c_2k_2}{m_2} &= -\frac{c_2(k_1+k_2)}{m_1} - \frac{c_2k_2}{m_2} \\ -m_2(c_1+c_2)k_2 - m_1c_2k_2 &= -m_2c_2(k_1+k_2) - m_1c_2k_2 \\ -m_2c_1k_2 &= -m_2c_2k_1 \\ \frac{c_1}{c_2} &= \frac{k_1}{k_2} \end{aligned}$$

b. We need the conditions under which the matrix $\tilde{\mathbf{C}}^2 - 4\tilde{\mathbf{K}}$ is negative definite, which are $\left[\tilde{\mathbf{C}}^2 - 4\tilde{\mathbf{K}}\right]_{11} < 0$ and $\det\left(\tilde{\mathbf{C}}^2 - 4\tilde{\mathbf{K}}\right) > 0$.

$$\widetilde{\mathbf{C}} = \mathbf{L}^{-1}\mathbf{C}\mathbf{L}^{-T} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} c_1 + c_2 & -\frac{c_2}{2} \\ -c_2 & \frac{c_2}{2} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 & -\frac{c_2}{2} \\ -\frac{c_2}{2} & \frac{c_2}{4} \end{bmatrix}$$

$$\widetilde{\mathbf{K}} = \mathbf{L}^{-1}\mathbf{K}\mathbf{L}^{-T} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 9 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 9 & -2 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ -2 & 1 \end{bmatrix}$$

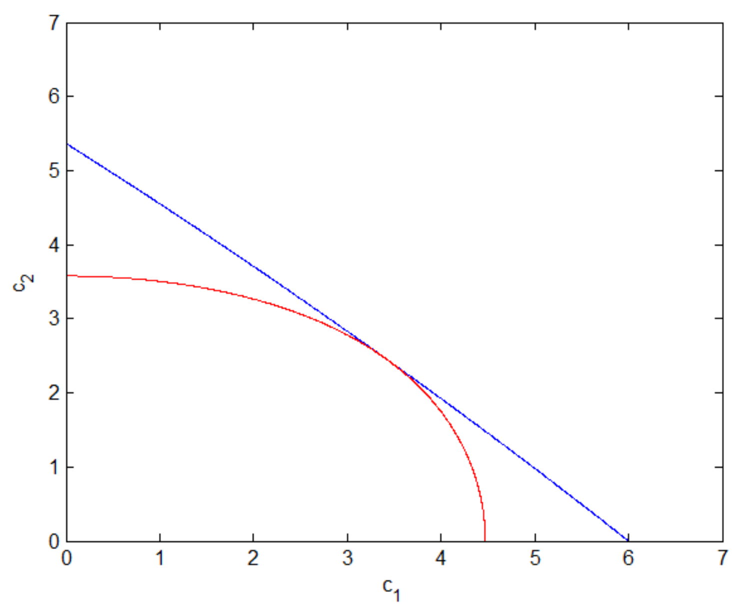
$$\widetilde{\mathbf{C}}^2 - 4\widetilde{\mathbf{K}} = \begin{bmatrix} (c_1 + c_2)^2 + \frac{c_2^2}{4} & -\frac{c_2(c_1 + c_2)}{2} - \frac{c_2^2}{8} \\ -\frac{c_2(c_1 + c_2)}{2} - \frac{c_2^2}{8} & \frac{5c_2^2}{16} \end{bmatrix} - 4\begin{bmatrix} 9 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} (c_1 + c_2)^2 + \frac{c_2^2}{4} - 36 & -\frac{c_2(c_1 + c_2)}{2} - \frac{c_2^2}{8} + 8 \\ -\frac{c_2(c_1 + c_2)}{2} - \frac{c_2^2}{8} + 8 & \frac{5c_2^2}{16} - 4 \end{bmatrix}$$

$$\left[\widetilde{\mathbf{C}}^2 - 4\widetilde{\mathbf{K}}\right]_{11} = (c_1 + c_2)^2 + \frac{c_2^2}{4} - 36 < 0$$

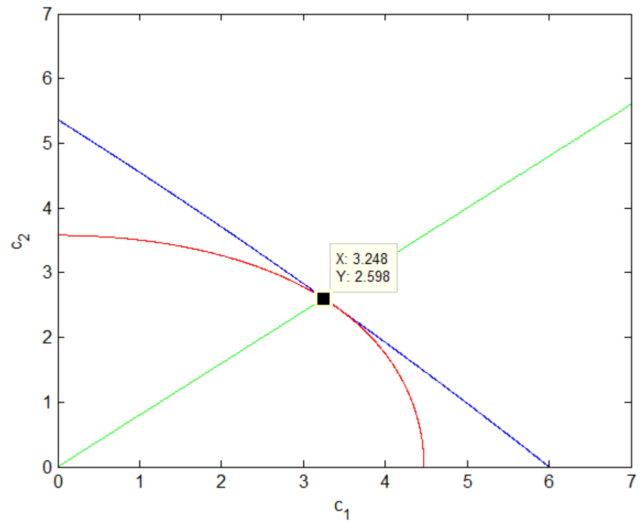
$$\det\left(\widetilde{\mathbf{C}}^2 - 4\widetilde{\mathbf{K}}\right) = -4c_1^2 + \frac{c_1^2 c_2^2}{16} - \frac{25c_2^2}{4} + 80 > 0$$

These two inequalities are plotted below, blue and red respectively. As seen, the second inequality is more restrictive.



c. Inspection of the above two inequalities shows that small values of c_1 and c_2 will satisfy them.

To find the exact condition, we add the condition $\frac{c_1}{c_2} = \frac{5}{4}$ to the plot:



Any values along the green line less than those shown in the plot will suffice, so we choose $c_1 = 1.25$ N/(m/s) and $c_2 = 1$ N/(m/s). Plugging these values into the inequalities yields $-30.69 < 0$ and $67.6 > 0$, respectively.

d. These values do not satisfy the ratio of part (a), so the equations cannot be decoupled; however, we can still compute the eigenvalues of

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}$$

They are $\lambda_{1,2} = -0.0858 \pm 0.723i$ and $\lambda_{3,4} = -1.54 \pm 2.66i$, which indicate that both modes are underdamped (since the roots are complex conjugates).

Complex roots can be written in terms of their natural frequencies and damping ratios:

$$\lambda_i = -\zeta_i \omega_i \pm i\omega_i \sqrt{1-\zeta_i^2}$$

which can be rearranged as

$$\omega_i = \sqrt{[\mathrm{Re}(\lambda_i)]^2 + [\mathrm{Im}(\lambda_i)]^2} \quad \text{and} \quad \zeta_i = \frac{-\mathrm{Re}(\lambda_i)}{\sqrt{[\mathrm{Re}(\lambda_i)]^2 + [\mathrm{Im}(\lambda_i)]^2}}$$

Thus, $\omega_1 = 0.728$ rad/s, $\omega_2 = 3.07$ rad/s, $\zeta_1 = 0.118$, and $\zeta_2 = 0.501$.

Plugging $c_1 = 2\,\mathrm{N/(m/s)}$ and $c_2 = 1\,\mathrm{N/(m/s)}$ into the inequalities found in part (b) yields $-26 < 0$ and $58 > 0$, respectively.

The previous plots are generated with the following Matlab code:

```
c1 = linspace(0,7,1001);
c2 = linspace(0,7,1001);
[C1,C2] = meshgrid(c1,c2);

% contour plot of first condition
Z = (C1+C2).^2 + C2.^2/4 - 36;
contour(c1,c2,Z,[0 0],'b');

% contour plot of second condition
Z = -4*C1.^2 + C1.^2.*C2.^2/16 - 25*C2.^2/4 + 80;
hold on;
contour(c1,c2,Z,[0 0],'r');
ylabel('c_2');
xlabel('c_1');

% plot decoupling condition
plot(c1,4/5*c1,'g');
```

2. a. The EOMs for each of the masses respectively are

$$m_1\ddot{x}_1 = \sum F_x = -k_1x_1 - c_1\dot{x}_1 + k_2(x_2 - x_1) + c_2(\dot{x}_2 - \dot{x}_1) + f_1$$
$$m_1\ddot{x}_1 + (c_1 + c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2 = f_1$$

$$m_2\ddot{x}_2 = \sum F_x = -k_2(x_2 - x_1) - c_2(\dot{x}_2 - \dot{x}_1) + f_2$$
$$m_2\ddot{x}_2 - c_2\dot{x}_1 + c_2\dot{x}_2 - k_2x_1 + k_2x_2 = f_2$$

Arranging these into matrix-vector form yields

$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\mathbf{M}}\underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}}_{\mathbf{C}}\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{\mathbf{q}}} + \underbrace{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}}_{\mathbf{K}}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}}\underbrace{\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}}_{\mathbf{f}}$$

From Matlab (see below), we calculate the natural frequencies $\omega_1 = 0.55\,\mathrm{rad/s}$ and $\omega_2 = 2.6\,\mathrm{rad/s}$.

Also, $\mathbf{P}^T\mathbf{L}^{-1}\mathbf{B} = \begin{bmatrix} -0.36 & -0.43 \\ -0.61 & 0.26 \end{bmatrix}$.

Each modal EOM has the form $\ddot{z}_k + 2\zeta_k\omega_k\dot{z}_k + \omega_k^2 z_k = \alpha_{k,1}f_1 + \alpha_{k,2}f_2$. Thus,

$$\ddot{z}_1 + 0.022\dot{z}_1 + 0.30z_1 = -0.36f_1 - 0.43f_2 \text{ and } \ddot{z}_2 + 0.52\dot{z}_2 + 6.70z_2 = -0.61f_1 + 0.26f_2$$

b. This can be solved using the Laplace transform method with $F_1(s) = \frac{5}{s}$ and $F_2(s) = 0$:

$$Z_1(s) = \frac{-0.36}{s^2 + 0.022s + 0.30}\frac{5}{s},$$

which yields

$$z_1(t) = 6.11e^{-0.01t}\cos(0.55t) + 0.12e^{-0.01t}\sin(0.55t) - 6.11$$

and

$$Z_2(s) = \frac{-0.61}{s^2 + 0.52s + 6.70}\frac{5}{s},$$

which yields

$$z_2(t) = 0.45e^{-0.26t}\cos(2.58t) + 0.045e^{-0.26t}\sin(2.58t) - 0.45$$

c. This can be solved using the Laplace transform method with $F_1(s) = 0$ and $F_2(s) = 2$:

$Z_1(s) = \frac{-0.36}{s^2 + 0.022s + 0.30}2$, which yields $z_1(t) = -1.57e^{-0.01t}\sin(0.55t)$

and

$Z_2(s) = \frac{-0.61}{s^2 + 0.52s + 6.70}2$, which yields $z_2(t) = 0.20e^{-0.26t}\sin(2.58t)$

d. The physical responses are given by $\mathbf{q}(t) = \mathbf{L}^{-T}\mathbf{P}[\mathbf{z}_1(t) + \mathbf{z}_2(t)]$, where $\mathbf{z}_1(t)$ indicates the responses found in part (b), and $\mathbf{z}_2(t)$ indicates the responses found in part (c).

$$\begin{aligned} q_1(t) = &-2.2e^{-0.01t}\cos(0.55t) + 0.91e^{-0.01t}\sin(0.55t) \\ &-0.16e^{-0.26t}\cos(2.58t) - 0.14e^{-0.26t}\sin(2.58t) + 2.4 \end{aligned}$$

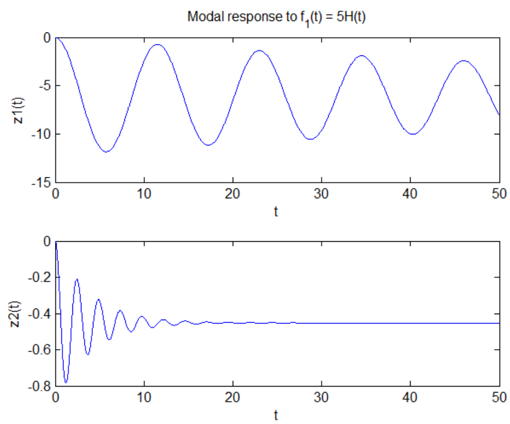
and

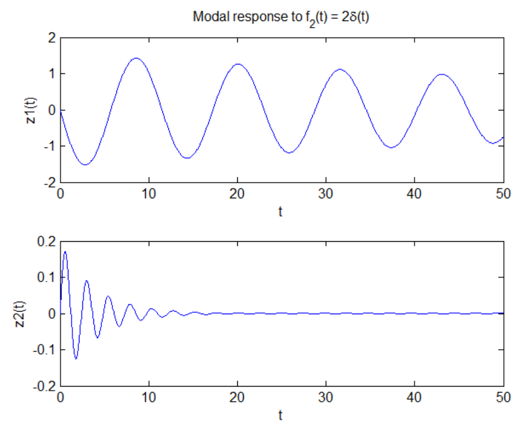
$$\begin{aligned} q_2(t) = &-0.46e^{-0.01t}\cos(0.55t) + 0.032e^{-0.01t}\sin(0.55t) \\ &-0.19e^{-0.26t}\cos(2.58t) - 2.6e^{-0.26t}\sin(2.58t) + 2.8 \end{aligned}$$

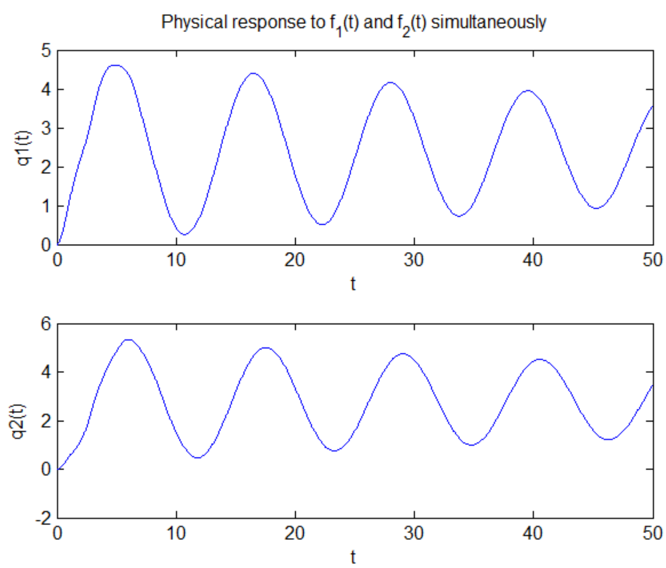
The following Matlab code computes each of these responses symbolically:

```
% system constants
m1 = 2;
m2 = 4;
k1 = 2;
k2 = 8;

% populate matrices
M = diag([m1 m2])
K = [k1+k2 -k2; -k2 k2]
B = [1 0; 0 1];

% compute mass-normalized modes
L = chol(M,'lower')
Ktilde = (L\K)/L'           % same as Ktilde = inv(L)*K*inv(L')
[P,D] = eig(Ktilde)
w = sqrt(diag(D))
N = length(w);

% compute transfer functions
alpha = P'*(L\B)            % same as alpha = P'*inv(L)*B
zeta = [0.02 0.1]';         % modal damping ratios
syms s;
for i = 1:N
    for j = 1:size(B,1)
        G(i,j) = alpha(i,j)/(s^2 + 2*zeta(i)*w(i)*s + w(i)^2);
    end
end
vpa(G,4)

% compute modal responses to f_1 = 5*H(t)
F1 = 5/s;
z11 = vpa(expand(ilaplace(G(1,1)*F1)),4)
z21 = vpa(expand(ilaplace(G(2,1)*F1)),4)
t = linspace(0,50,1001);
figure;
subplot(2,1,1);
plot(t,eval(subs(z11,t)));
title('Modal response to f_1(t) = 5H(t)');
ylabel('z1(t)');
xlabel('t');
subplot(2,1,2);
plot(t,eval(subs(z21,t)));
ylabel('z2(t)');
xlabel('t');

% compute modal responses to f_2 = 2*delta(t)
F2 = 2;
z12 = vpa(expand(ilaplace(G(1,2)*F2)),4)
z22 = vpa(expand(ilaplace(G(2,2)*F2)),4)
figure;
subplot(2,1,1);
plot(t,eval(subs(z12,t)));
title('Modal response to f_2(t) = 2\delta(t)');
ylabel('z1(t)');
xlabel('t');
subplot(2,1,2);
plot(t,eval(subs(z22,t)));
ylabel('z2(t)');
xlabel('t');

% compute physical responses to f_1 and f_2 applied simultaneously
temp = L'\P;                            % same as temp = inv(L')*P
q1 = vpa(temp(1,1)*(z11+z21) + temp(1,2)*(z12+z22),4)
q2 = vpa(temp(2,1)*(z11+z21) + temp(2,2)*(z12+z22),4)
```

```
figure;
subplot(2,1,1);
plot(t,eval(subs(q1,t)));
title('Physical response to f_1(t) and f_2(t) simultaneously');
ylabel('q1(t)');
xlabel('t');
subplot(2,1,2);
plot(t,eval(subs(q2,t)));
ylabel('q2(t)');
xlabel('t');
```







Note that the first mode is lower frequency (by definition) and more weakly damped (compare $\zeta_1 = 0.02$ to $\zeta_2 = 0.1$). You could make the case that the second mode could be ignored in the physical free response, since it decays so quickly relative to the first mode: note that the responses $q_1(t), q_2(t)$ more closely resemble the first modal response $z_1(t)$, other than a small "hiccup" at the beginning.

3. a. In terms of the given variables, the compression of the linear spring is $x + e\sin(\theta) \approx x + e\theta$. Thus, the balance of forces in the vertical direction is

$$m\ddot{x} = \sum F_x = -k(x + e\theta) + f$$
$$m\ddot{x} + kx + ke\theta = f$$

Balance of moments about the C.G. yields

$$J_0\ddot{\theta} = \sum M_0 = -k(x + e\theta)e - k_t\theta + M$$
$$J_0\ddot{\theta} + kex + (ke^2 + k_t)\theta = M$$

Arranging these into matrix-vector form yields

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} k & ke \\ ke & ke^2 + k_t \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} f \\ M \end{bmatrix}}_{\mathbf{f}}$$

or, plugging the values in,

$$\underbrace{\begin{bmatrix} 200 & 0 \\ 0 & 50 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}}_{\ddot{\mathbf{q}}} + \underbrace{\begin{bmatrix} 1000 & 500 \\ 500 & 750 \end{bmatrix}}_{\mathbf{K}} \underbrace{\begin{bmatrix} x \\ \theta \end{bmatrix}}_{\mathbf{q}} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} f \\ M \end{bmatrix}}_{\mathbf{f}}$$

b. From Matlab (see below), we calculate the natural frequencies $\omega_1 = 1.71\ \text{rad/s}$ and $\omega_2 = 4.1\ \text{rad/s}$.

Also, $\mathbf{P}^T\mathbf{L}^{-1}\mathbf{B} = \begin{bmatrix} -0.065 & 0.054 \\ 0.027 & 0.13 \end{bmatrix}$.

Each modal EOM has the form $\ddot{z}_k + 2\zeta_k\omega_k\dot{z}_k + \omega_k^2 z_k = \alpha_{k,1}f + \alpha_{k,2}M$. Thus,

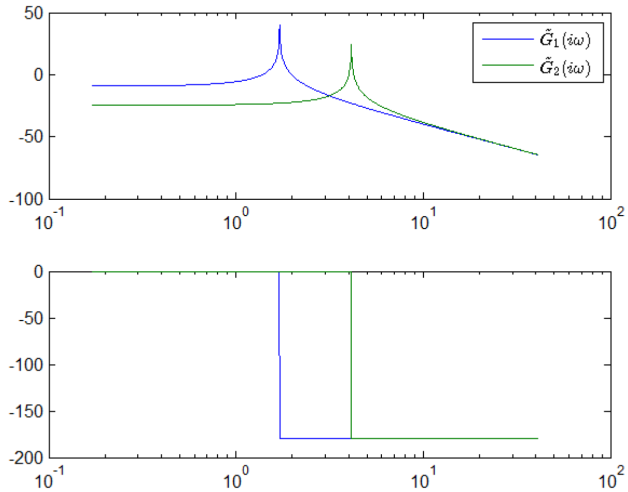
$$\ddot{z}_1 + 2.9z_1 = -0.065f + 0.054M \quad \text{and} \quad \ddot{z}_2 + 17.1z_2 = 0.027f + 0.13M$$

c. The modal dynamic amplification factors are, in general, given by

$$\tilde{G}_k(i\omega) = \frac{1}{-\omega^2 + i2\zeta_k\omega_k\omega + \omega_k^2}$$

So, in this case

$$\tilde{G}_1(i\omega) = \frac{1}{-\omega^2 + 2.9} \quad \text{and} \quad \tilde{G}_2(i\omega) = \frac{1}{-\omega^2 + 17.1}$$



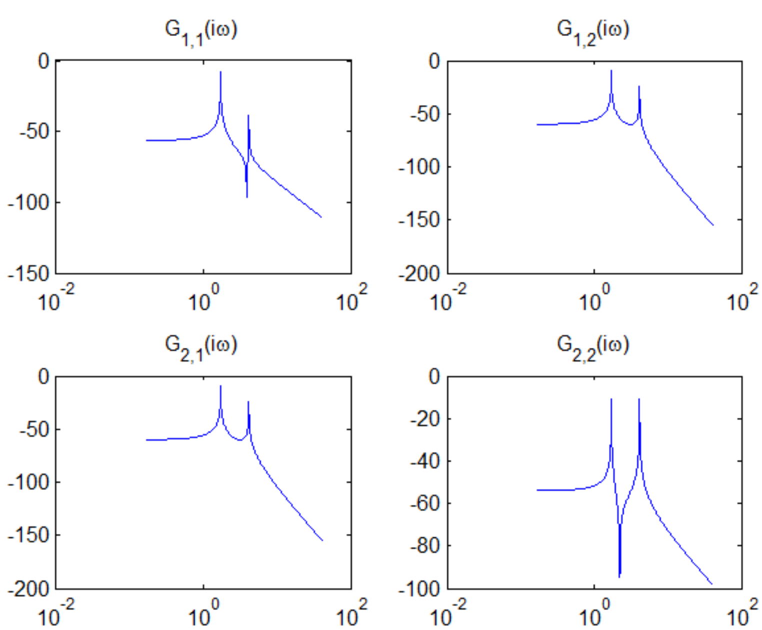
d. In general, the matrix frequency transfer function can be found from

$$\mathbf{G}(i\omega) = \left(-\omega^2\mathbf{M} + i\omega\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{B}$$

However, in this case, since the system can be decoupled, we can write $\mathbf{G}(i\omega)$ in terms of the modal dynamic amplification factors:

$$\mathbf{G}(i\omega) = \mathbf{L}^{-T}\left(\sum_{k=1}^{N}\widetilde{G}_k(i\omega)\mathbf{p}_k\mathbf{p}_k^T\right)\mathbf{L}^{-1}\mathbf{B}$$

Computationally, the latter equation is preferred, but it also shows directly how the modal frequency responses affect the overall system input-output frequency response.



The following Matlab code produces the above plots:

```
m = 200;
J = 50;
e = 0.5;
k = 1000;
kt = 500;
M = diag([m J]);
K = [k k*e; k*e k*e^2+kt];
B = [1 0; 0 1];

% compute mass-normalized modes
L = chol(M,'lower')
Ktilde = (L\K)/L'           % same as Ktilde = inv(L)*K*inv(L')
[P,D] = eig(Ktilde)
wn = sqrt(diag(D))
zeta = 0*wn;

% modal frequency responses
N = length(wn);
w = logspace(log10(0.1*wn(1)),log10(10*wn(end)),1001);
Gtilde = zeros(N,length(w));        % modal dynamic amplification factors
alpha = P'*(L\B)                    % same as alpha = P'*inv(L)*B
for i = 1:N
    Gtilde(i,:) = 1./(-w.^2+(1i*2*zeta(i)*wn(i))*w+wn(i)^2);
end

% physical frequency responses
G = zeros(N,size(B,2),length(w));
for j = 1:length(w)
    temp = zeros(N,N);
    for i = 1:N
        temp = temp + Gtilde(i,j)*P(:,i)*P(:,i)';
    end
    G(:,:,j) = (L'\temp)*(L\B);     % same as G(:,:,j) = inv(L')*temp*inv(L)*B
end

% Bode plots
figure;
subplot(2,1,1);
semilogx(w,20*log10(abs(Gtilde)));
h = legend('$\tilde{G}_1(i\omega)$','$\tilde{G}_2(i\omega)$');
set(h,'Interpreter','latex');
subplot(2,1,2);
semilogx(w,-180/pi*angle(Gtilde));

figure;
for i = 1:N
    for j = 1:size(B,2)
        subplot(N,size(B,2),N*(i-1)+j);
        semilogx(w,20*log10(abs(squeeze(G(i,j,:)))));
        title(sprintf('G_{%d,%d}(i\\omega)',i,j));
    end
end
```