MAE6254: Midterm Exam

Due at 9am on Monday March 28, 2016

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Last Name First Name Student ID

Prob. 1	Prob. 2	Prob. 3	Prob. 4	Prob. 5	Prob. 6	Total

Honor Pledge

According to *GWU Code of Academic Integrity*, I pledge that I have neither given nor received unauthorized assistance on this work.

Signature Date

Problem 1 Consider the following system:

$$\dot{x}_1 = -x_1^3 + x_2,$$
$$\dot{x}_2 = x_1 - x_2^3.$$

(a) Find three equilibria of this system.

(b) Find the type of each equilibrium, e.g., center, saddle, stable focus, etc.

Problem 2 Consider the following system:

$$\dot{x}_1 = (1 + x_1^2)^2(-x_1 - x_2), \qquad \dot{x}_2 = x_1(1 + x_2^2)^2.$$

(a) Find the equilibrium of this system.

(b) Using the following Lyapunov function, make the *strongest* statement possible about the stability properties of the equilibrium. (For example, asymptotic stability is stronger than stability, and globally exponential stability is stronger than exponential stability.)

$$V(x_1, x_2) = \frac{x_1^2}{1 + x_1^2} + \frac{x_2^2}{1 + x_2^2}.$$

Problem 3 Consider the following system:

$$\dot{x}_1 = -x_2,$$
$$\dot{x}_2 = x_1 + (x_1^2 - 1)x_2.$$

(a) Show that the following Lyapunov function is positive-definite:

$$V(x_1, x_2) = \frac{3}{2}x_1^2 - x_1x_2 + x_2^2.$$

(b) Show that the equilibrium $x = 0$ is locally asymptotically stable by using the above Lyapunov function. (Hint: let $D = \{x_1, x_2 \in \mathbb{R} \,|\, 1 + x_1x_2 > 0,\ x_1^2 < \frac{1}{2}\}$).

(c) A sublevel set of V is defined as follows:

$$\Omega_c = \{x \in \mathbb{R}^2 \,|\, V(x) \leq c\},$$

and it is described by an ellipse. For a fixed constant c, find the shape of the Ω_c by specifying (i) the direction of the semi-major axis, (ii) the direction of the semi-minor axis, (iii) the length of the semi-major axis, and (iv) the length of the semi-minor axis.

(d) Recall that an estimate of the region of attraction is given by a sublevel set Ω_c, where c is chosen such that $\Omega_c \subset D$. Find an estimate of the region of attraction by using the above Lyapunov function. (You will get more credit as the area of the region of attraction is larger.)

Problem 4 Consider a spring-mass-damper system with a time-varying damping coefficient given by

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -2x_1 - b(t)x_2,\end{aligned}$$

where the damping coefficient $b(t)$ satisfies

$$2 \le b(t) \le 4, \quad \frac{1}{2} \le \dot{b}(t) \le 1.$$

Consider the following Lyapunov function:

$$V(t, x_1, x_2) = \frac{1}{2}(x_1 + x_2)^2 + \frac{1}{2}(1 + b(t))x_1^2.$$

(a) Show that V is positive definite and decrescent.

(b) Show that the origin of this systems is globally exponential stable. Also find the constants k and γ for the following bound:

$$\|x(t)\| \le k\|x(t_0)\| \exp[-\gamma(t - t_0)].$$

Problem 5 Consider a particle m moving under a uniform gravity g, actuated by a thrust $u \in \mathbb{R}^3$. Define an inertial frame $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, where the third axis $\vec{e}_3$ is pointing downward along the gravity. Let the position of the particle with respect to the inertial frame be $p \in \mathbb{R}^3$. According to Newton's second law, the equation of motion is given by

$$m\ddot{p} = mge_3 + u, \tag{1}$$

where $e_3 = [0, 0, 1]^T \in \mathbb{R}^3$.

Suppose that a desired trajectory of the mass $p_d(t) : \mathbb{R} \to \mathbb{R}^3$ is given as a smooth function of time. We wish to design a control input $u(t)$ such that $p(t) \equiv p_d(t)$ becomes an exponentially stable equilibrium of the controlled system. (The subsequent development can be applied to any fully-actuated aerial vehicle.)

(a) Define tracking error variables as

$$e_p = p - p_d, \quad e_v = \dot{p} - \dot{p}_d.$$

Let the state vector be $x = [e_p^T, e_v^T]^T \in \mathbb{R}^6$. Rewrite the equations of motion as

$$\dot{x} = f(t, x, u).$$

(b) For positive constants $k_p, k_v \in \mathbb{R}$, the following control input is proposed:

$$u = -k_p e_p - k_v e_v + m\ddot{p}_d - mge_3.$$

Show that the zero state of the tracking error, namely $x = 0$ is an equilibrium of the controlled system.

(c) Consider the following Lyapunov function.

$$V_0 = \frac{1}{2} m e_v^T e_v + \frac{1}{2} k_p e_p^T e_p.$$

Find the strongest stability property that can be obtained by the above Lyapunov function V_0.

(d) Consider another Lyapunov function.

$$V = \frac{1}{2} m e_v^T e_v + c e_p^T e_v + \frac{1}{2} k_p e_p^T e_p,$$

where c is a positive constant. Find the range of the constant c such that V becomes positive-definite and decrescent. (Note that the constant c does not appear in the controlled systems, and it is required only for stability analysis.)

(e) Show that the equilibrium of the controlled systems is globally exponentially stable by using the above Lyapunov function V. It is also required to show the range of the constant c such that $\dot{V}$ becomes negative definite.

(f) Assume that $m = 0.5\,\mathrm{kg}$, $g = 9.81\,\mathrm{m/s^2}$, and the desired trajectory is given by

$$p_d(t) = [\sin(3\pi t + \pi/2),\ \sin 2\pi t,\ -1]^T.$$

The initial condition is $p(0) = \dot{p}(0) = [0, 0, 0]^T$. Numerically simulate the controlled system for $0 \leq t \leq 20$ seconds with some controller gains k_p, k_v chosen such that the absolute value of each element of $u(t)$ is less than $50\,\mathrm{N}$. Generate the following plots: (i) $p(t)$ and $p_d(t)$ in a 3D space, (ii) e_p, e_v with respect to t, and (iii) u with respect to t.

Problem 6 Here, we consider a control problem of aligning an antenna fixed to a satellite to another antenna in a ground station. The attitude dynamics of a rigid satellite can be written as

$$J\dot{\Omega}(t) + \Omega(t) \times J\Omega(t) = u,$$

where $\Omega \in \mathbb{R}^3$ is the angular velocity of the satellite, $u \in \mathbb{R}^3$ is the control moment, and $J = \mathrm{diag}[J_1, J_2, J_3] \in \mathbb{R}^{3\times 3}$ is the moment of inertia for $J_1, J_2, J_3 > 0$. Let $s \in \mathbb{R}^3$ be the *unit* vector representing the direction of an antenna fixed to the satellite, i.e. $\|s\| = 1$. Note that s is a fixed vector, i.e. $\dot{s} = 0$, since it is observed from the satellite.

Let $g(t) \in \mathbb{R}^3$ be the *unit* vector representing the direction of a ground antenna with respect to the satellite fixed frame, i.e., $\|g(t)\| = 1$. Even though the direction of the ground antenna is fixed, the unit-vector $g(t)$ changes over time when observed from the satellite. In fact, $g(t)$ rotates with angular velocity $-\Omega$, and the equation of motion for $g(t)$ is given by

$$\dot{g}(t) = -\Omega(t) \times g(t).$$

Let the state vector of this satellite be $x(t) = [\Omega(t), g(t)] \in \mathbb{R}^6$.

We wish that the direction of the antenna on the satellite is aligned to the direction of the ground antenna, and they point each other, i.e. $g(t) \to -s$ as $t \to \infty$. Consider the following control input:

$$u = -k\Omega(t) + g(t) \times s$$

for a fixed constant $k > 0$.

(a) Find all of the equilibrium points.
(Hint: $x \times y = 0$ implies that either $x = 0$, or $y = 0$, or $x = cy$ for a constant c.
Note that $g(t)$ cannot be zero since it is a unit vector.
Recall that $x \times x = 0$, $x \cdot (y \times z) = y \cdot (z \times x) = z \cdot (x \times y)$ for any $x, y, z \in \mathbb{R}^3$.)

(b) Using the following Lyapunov function, show that the given control input makes $g(t)$ line up with $-s$. Estimate the region of attraction.

$$V = \frac{1}{2}\Omega^T J\Omega + \frac{1}{2}(g+s)^T(g+s).$$