Learning Opportunity 3-1

3-1-1 This exercise will help you become familiar with vector and matrix representations. See FCLA, Chapter R for help.

1. Consider the linear transformation $T\colon \mathbb{C}^{10} \to \mathbb{C}^{8}$ defined by $T(x) = Ax$, with A given below.
2. For the vector v below (given in Sage syntax), compute $T(v)$.
3. Build a vector representation of v relative to the basis B below (given in Sage syntax).
4. Build a matrix representation of T relative to the bases B and C below (given in Sage syntax).
5. Repeat the computation of $T(v)$, but now use your matrix representation. Of course, you should get the same answer as before, though there will be a few more steps. Done correctly, you may be able to express this as one matrix-vector product and two matrix products (with a matrix inverse or two).

$$A = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 0 & -1 & -6 & 2 & 0 \\ 0 & 1 & 2 & 8 & 1 & -2 & -3 & -6 & -4 & -5 \\ -2 & -3 & -3 & -8 & 1 & 5 & 4 & 3 & -2 & -1 \\ -2 & -2 & -3 & -4 & 3 & 6 & 3 & 3 & -3 & -4 \\ 3 & 3 & 5 & 7 & -2 & -6 & -2 & 2 & -1 & 2 \\ -2 & -1 & -1 & 4 & 0 & -1 & -2 & -8 & -2 & -2 \\ 2 & 1 & 1 & -4 & -3 & -4 & -1 & 1 & 6 & 7 \\ 2 & 2 & 2 & 2 & 0 & -2 & 0 & 6 & 0 & 2 \end{bmatrix}$$

```
A: matrix(QQ, [[-1, -1, -1, -1, -1, 0, -1, -6, 2, 0],
[0, 1, 2, 8, 1, -2, -3, -6, -4, -5],
[-2, -3, -3, -8, 1, 5, 4, 3, -2, -1],
[-2, -2, -3, -4, 3, 6, 3, 3, -3, -4],
[3, 3, 5, 7, -2, -6, -2, 2, -1, 2],
[-2, -1, -1, 4, 0, -1, -2, -8, -2, -2],
[2, 1, 1, -4, -3, -4, -1, 1, 6, 7],
[2, 2, 2, 2, 0, -2, 0, 6, 0, 2]])
```

```
v: (-2, 8, -4, 3, -4, -4, -2, 4, -6, 0)
```

```
B: [(1, 0, -2, 1, 1, 0, 0, -1, 1, 0),
(-1, 1, 1, -1, -1, 0, 0, 1, -1, 0),
(0, 1, 0, 1, -1, 0, 0, 0, 0, 0),
(0, -1, 0, 0, 1, 1, -1, -1, -1, 0),
(-1, 1, 2, 2, -1, 1, -1, 0, -1, 0),
(-3, 5, 4, -2, -6, -1, 0, 4, 0, 3),
(1, 7, -1, 8, -7, 1, -2, -4, 3, 7),
(1, 5, 0, 5, -7, 0, -1, -2, 3, 7),
(7, -7, -8, 6, 6, 2, -1, -8, 3, 0),
(7, -8, -8, -1, 4, -2, 4, 0, 5, -5)]
```

```
   C: [(-1, 1, 1, 1, 0, 0, 0, -1),
(-2, 1, 1, 1, 0, -1, 0, -1),
(2, -2, -1, -1, 0, 0, 1, 1),
(-4, 1, 1, 2, 2, -2, 0, -2),
(7, 1, 3, -1, -7, 6, 1, 1),
(4, 1, 0, -2, -4, 4, 0, 4),
(2, 0, 6, 3, -4, 0, 2, -7),
(-6, -2, -3, 2, 8, -2, 3, 3)]
```

Full marks for two evaluations of T and two representations (one vector, one matrix).

3-1-2 This exercise will familiarize you with the Schur decomposition and orthonormal diagonalization.

1. Build a random 15×15 matrix of single digit integers (positive and negative) and call it A.
2. Convert A to a numerical matrix over `CDF`, and compute the Schur decomposition of this matrix. You will likely see some complex numbers on the diagonal of the upper-traingular matrix. If you do not, go back and make a new A and try again.
3. Begin with A and convert to a numerical matrix over `RDF`. Then compute the Schur decomposition of this matrix.
4. Form AA^*, convert this matrix to a numerical matrix over `CDF`, and compute the Schur decomposition of this matrix.
5. Obviously all three matrices have different entries. Other than this, observe and comment on the difference in the *form* of the output. Reading the documention of $.schur()$ might help.

Full marks for all three decompositions, but mostly for observations of differences and explanations of these differences.