Fast Matrix Multiplication

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Standard Matrix Multiplication

Two matrices A and B of size $n \times n$.
We want $A \times B$. Example when $n = 2$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2} \quad and \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_{2\times 2}$$

$$AB = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}_{2\times 2} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{21}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}_{2\times 2}$$

$$(AB)_{ij} = C_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}$$

8 matrix multiplications and **4** matrix additions

Example:

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 23 \\ 16 & 32 \end{pmatrix}$$

Standard Matrix Multiplication

Algorithm:

```
for i:= 1 to n do
    for j:=1 to n do
        C[i,j] := 0;
        for k := 1 to n do
            C[i,j] := C[i,j] + A[i,k] * B[k,j]
```

Time analysis:

$$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} c = cn^3 = \mathcal{O}(n^3)$$

Strassen's Algorithm for Matrix Multiplication **(MM)**

- 1968 **Volker Strassen** developed **MM**
- **MM** asymptotically faster than standard matrix multiplication $\mathcal{O}(n^3)$
- Each level of **MM** algorithm uses: 7 matrix multiplications (instead of 8), and 18 matrix additions (instead of 4)

Strassen's Algorithm for Matrix Multiplication**(MM)**

2×2 cases :

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$
$$= \begin{pmatrix} m_1 + m_4 - m_5 + m_7 & m_3 + m_5 \\ m_2 + m_4 & m_1 + m_3 - m_2 + m_6 \end{pmatrix}$$

where,

$$\begin{aligned} m_1 &= (a_{11} + a_{22})(b_{11} + b_{22}) \\ m_2 &= (a_{21} + a_{22})b_{11} \\ m_3 &= a_{11}(b_{12} - b_{22}) \\ m_4 &= a_{22}(b_{21} - b_{11}) \\ m_5 &= (a_{11} + a_{12})b_{22} \\ m_6 &= (a_{21} - a_{11})(b_{11} + b_{12}) \\ m_7 &= (a_{12} - a_{22})(b_{21} + b_{22}) \end{aligned}$$

Example:

$$\begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 2 & 5 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 14 & 23 \\ 16 & 32 \end{pmatrix}$$

$$\begin{aligned} m_1 &= (1+2)(2+6) = 24 & m_2 &= (4+2) * 2 = 12 \\ m_3 &= 1 * (5-6) = -1 & m_4 &= 2 * (4-2) = 4 \\ m_5 &= (1+3) * 6 = 24 & m_6 &= (4-1)(2+5) = 21 \\ m_7 &= (3-2)(4+6) = 10 & & \\ c_{11} &= m_1 + m_4 - m_5 + m_7 = 14 & c_{12} &= m_3 + m_5 = 23 \\ c_{21} &= m_2 + m_4 = 16 & c_{22} &= m_1 + m_3 - m_2 + m_6 = 32 \end{aligned}$$

Strassen's Algorithm for Matrix Multiplication(**MM**)

Block matrix :

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

$$= \begin{pmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 + M_3 - M_2 + M_6 \end{pmatrix}$$

where,

$$\begin{aligned}
M_1 =& (A_{11} + A_{22})(B_{11} + B_{22}) \\
M_2 =& (A_{21} + A_{22})B_{11} \\
M_3 =& A_{11}(B_{12} - B_{22}) \\
M_4 =& A_{22}(B_{21} - B_{11}) \\
M_5 =& (A_{11} + B_{12})B_{22} \\
M_6 =& (A_{21} - B_{11})(B_{11} + B_{12}) \\
M_7 =& (A_{12} - B_{22})(B_{21} + B_{22})
\end{aligned}$$

Strassen's Algorithm for Matrix Multiplication(MM)

```
function C = strass(A,B,n,n_min)
    if n <= n_min
        C=A*B
    else
        m=n/2; u=1:m; v=m+1:n
        m1=strass(A(u,u)+A(v,v),B(u,u)+B(v,v),m,n_min)
        m2=strass(A(u,u)+A(v,v),B(u,u),m,n_min)
        m3=strass(A(u,u),A(u,v)-B(v,v),m,n_min)
        m4=strass(A(v,v),B(v,u)-B(u,u),m,n_min)
        m5=strass(A(u,u)+A(u,v),B(v,v),m,n_min)
        m6=strass(A(v,u)-A(u,u),B(u,u)+B(u,v),m,n_min)
        m7=strass(A(u,v)-A(v,v),B(v,u)+B(v,v),m,n_min)
        C(u,u)=m1+m4-m5+m7
        C(u,v)=m3+m5
        C(v,u)=m2+m4
        C(v,v)=m1+m3-m2+m6
```

Time Analysis

$$T(n) = \begin{cases} \mathcal{O}(1) & \text{if n=1} \\ 7T(\frac{n}{2}) + \mathcal{O}(n^2) & \text{otherwise} \end{cases}$$

then

$$T(n) = \mathcal{O}(n^{log_2 7}) = \mathcal{O}(n^{2.81})$$

Note: $n^{2.81} < n^3$ hence **MM** faster method.

Thank You!