STATIONARY MARKOV CHAINS

Filister Chinake

AIMS

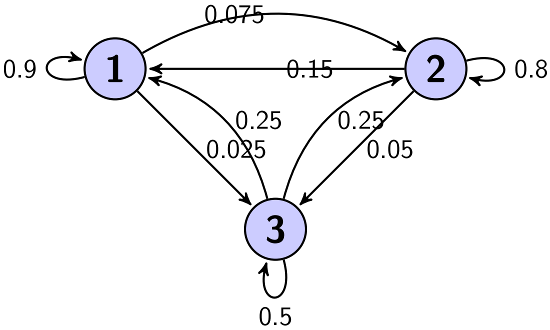
January 18, 2015

DEFINITION

A Markov Process is a random process within a system that has a finite number of states .The system also undergoes changes from state to state with a probability for each state transition that depends solely upon the current state.

- Transition matrix
  This matrix denoted P where P $= (p_{ij} : i, j \in I$ with $pij \geq 0 \forall i, j)$ describes the probabilistic motion of our objects of study from one state to the other.
- State Vector
  This is a column vector $\vec{x}$ whose k^{th} component ,x_k is the probability that the system is in state "k" at that time.
  Distribution over states is written with the following relation:
  $\vec{x}^{n+1} = [P]\vec{x}^{n}$

STATE TRANSITION DIAGRAM



The label for the state space is $\{1 = bull, 2 = bear, 3 = stagnant\}$

Extract the Transition Matrix from the transition diagram above.

$$P = \begin{pmatrix} 0.9 & 0.075 & 0.025 \\ 0.15 & 0.8 & 0.05 \\ 0.25 & 0.25 & 0.5 \end{pmatrix}$$

MATRIX COMPUTATIONS

- The naive computation of matrix powers is multiplying n times the matrix [P] to itself, starting with the identity matrix just like the scalar case.
- A better method is to use the eigenvalue decomposition of [P]

EIGENVALUE AND EIGENVECTOR CALCULATIONS

The eigenvalues corresponding to the transition matrix P are $[1.0000000000000002, 0.7414213562373095, 0.4585786437626903]$

Where:

$\lambda_1 = 1$

$\lambda_2 = 0.7414213562373095$

$\lambda_3 = 0.4585786437626903$

The right eigenvectors corresponding to the eigenvalues are:

$$\lambda_1 \text{ is } \begin{pmatrix} -0.577350269189625 \\ -0.577350269189627 \\ -0.577350269189626 \end{pmatrix} = \vec{v_1}$$

$$\lambda_2 \text{ is } \begin{pmatrix} -0.443718565113635 \\ 0.811307060207225 \\ 0.380650350100204 \end{pmatrix} = \vec{v_2}$$

$$\lambda_3 \text{ is } \begin{pmatrix} -0.0340025743143045 \\ -0.130176377816082 \\ 0.990907632223451 \end{pmatrix} = \vec{v_3}$$

We will use the above eigenvectors to construct a matrix we will denote E and the columns of this matrix are vectors$= \vec{v_2}, = \vec{v_2}, = \vec{v_3}$

$$E = \begin{pmatrix} -0.577350269189625 & -0.443718565113635 & -0.0340025743143045 \\ -0.577350269189627 & 0.811307060207225 & -0.130176377816082 \\ -0.577350269189626 & 0.380650350100204 & 0.990907632223451 \end{pmatrix}$$

Multiplying $E^{-1}PE = D$ a diagonal matrix whose elements are the eigenvalues of P.

Thus $D = \begin{pmatrix} 1.0 & 0.0 & 0.0 \\ 0.0 & 0.741421356237 & 0.0 \\ 0.0 & 0.0 & 0.458578643763 \end{pmatrix}$

$P = EDE^{-1}$

$P^2 = ED^2E^{-1}$

$P^n = ED^nE^{-1}$

Below is a table of state vectors for this system for different time periods.

i	$\vec{x}^{n+i}$
0	(0,1,0)
1	(0.15,0.8,0.05)
2	(0.2675)
3	(0.3575,0.56825,0.07425)
4	(0.42555,0.499975,0.74475)
5	(0.47761,0.450515,0.072875)
10	(0.5915850692,0.343030961394,0.06531874914)
20	(0.623321933853,0.314034130471,0.0626439356755)
21	(0.623755843958,0.313637433335,0.0626067227076)
30	(0.624915770975,0.312577003325,0.0625072256999)
50	(0.624999787791,0.312500194004,0.0625000182046)

STEADY STATE VECTORS

- The transition matrix eventually reaches its limit after a number of iterations.
- Our transition matrix converges to $\begin{pmatrix} 0.625 & 0.3125 & 0.0625 \\ 0.625 & 0.3125 & 0.0625 \\ 0.625 & 0.3125 & 0.0625 \end{pmatrix}$ as i$\rightarrow \infty$
- Steady state vectors are of the form $\vec{x} = [P]\vec{x}$
- The above equation is clearly an eigenvalue equation of the form $[P - \lambda\mathbb{I}]\vec{x} = \vec{0}$
$\Longrightarrow [P]\vec{x} = \lambda\vec{x}$
- Thus $\lambda = 1$ is an eigenvalue of an arbitrary stochastic matrix [P] with right eigenvector $\vec{x}$.