Learning Opportunity 2-1

2-1-1 This exercise works with the **indefinite factorization**, which is similar to the Cholesky decomposition. If you read the source code, you will see that to compute a Cholesky decomposition, first Sage computes an indefinite factorization.

1. Construct a 12×10 random matrix of positive, zero and negative single-digit integers. You will need to delve into Sage's documentation to figure out how to do this—this is part of the exercise. Do not ask me how to do it. Call the matrix A.
2. Construct $B = A^*A$, which is guaranteed to be positive semi-definite. Use Sage to check anyway.
3. Construct the **indefinite factorization** of Bin Sage. From this construct an LU factorization of B.
4. Have Sage compute an LU factorization (*not* a PLU factorization) of B. How does this version compare to the previous one? Why? Which method might be superior computationally?

Full marks for creating a random positive semidefinite matrix, two computations of its LU decomposition, and well-written commentary.

2-1-2 This exercise connects positive semidefinite matrices with quadratic forms.

Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 9 \end{bmatrix}$$

1. Check that A is positive definite.
2. Let $x = (x_1, x_2, x_3)^t$ be a column vector of unknowns. The expression x^tAx is called a **quadratic form**. Form this expression. Declare the symbolic variables `x1, x2, x3` using Sage's `var()` function.
3. Form the indefinite factorization of A. Discover how to use this factorization to express the quadratic form as a sum of obviously nonnegative terms. This is the hard part of the problem and you should discover it yourself.
4. From your answer to the previous part, give a well-written argument that A is positive semi-definite.
5. Now build on your argument and prove that A is positive definite.

Full marks for forming the quadratic form, converting it to a sum of nonnegative expressions, and two well-written arguments.