Learning Opportunity 3-3

3-3-1 Use the matrix below to understand Jordan Canonical Form.

$$A = \begin{bmatrix} -44 & 30 & -62 & 10 & 15 & -25 & 35 & -37 & -97 & -53 & -35 \\ -4 & -15 & -5 & -20 & 8 & -20 & -35 & -17 & -36 & -8 & -128 \\ -16 & 14 & -15 & 16 & -9 & 1 & 61 & 23 & 2 & 13 & 47 \\ 22 & -4 & 19 & 5 & -4 & 24 & -38 & -10 & 23 & -14 & 62 \\ 25 & 15 & 26 & 26 & -22 & 46 & 64 & 57 & 109 & 51 & 232 \\ 12 & 20 & 16 & 40 & -29 & 47 & 89 & 62 & 92 & 39 & 248 \\ -41 & 38 & -48 & 34 & -3 & -4 & 92 & 4 & -43 & -25 & 92 \\ 32 & -32 & 35 & -39 & 8 & -5 & -76 & -2 & 33 & 30 & -117 \\ -14 & 3 & -15 & 2 & 8 & -11 & -15 & -26 & -45 & -36 & -38 \\ 30 & -10 & 35 & 3 & -14 & 27 & -6 & 27 & 70 & 34 & 86 \\ 8 & -8 & 10 & -9 & 1 & -2 & -15 & 2 & 11 & 12 & -26 \end{bmatrix}$$

```
[[-44, 30, -62, 10, 15, -25, 35, -37, -97, -53, -35],
[-4, -15, -5, -20, 8, -20, -35, -17, -36, -8, -128],
[-16, 14, -15, 16, -9, 1, 61, 23, 2, 13, 47],
[22, -4, 19, 5, -4, 24, -38, -10, 23, -14, 62],
[25, 15, 26, 26, -22, 46, 64, 57, 109, 51, 232],
[12, 20, 16, 40, -29, 47, 89, 62, 92, 39, 248],
[-41, 38, -48, 34, -3, -4, 92, 4, -43, -25, 92],
[32, -32, 35, -39, 8, -5, -76, -2, 33, 30, -117],
[-14, 3, -15, 2, 8, -11, -15, -26, -45, -36, -38],
[30, -10, 35, 3, -14, 27, -6, 27, 70, 34, 86],
[8, -8, 10, -9, 1, -2, -15, 2, 11, 12, -26]])
```

1. First, compute the eigenvalues and note their algebraic multiplicities.
2. Construct the generalized eigenspace for each eigenvalue.
3. Build a block diagonal matrix representation based on the generalized eigenspace direct sum. (SCLA, Theorem 3.1.10).
4. Improve your matrix representation by constructing basis vectors of each generalized eigenspace which provide the nearly-diagonal Jordan Canonical Form.
5. Build the Jordan Canonical Form again, but this time do it "combinatorially." In other words, from just eigenvalues and dimensions of kernels, construct the Jordan Canonical Form without computing the basis vectors that provide the change-of-basis matrix.
6. Check your work with Sage's `.jordan_form()` matrix method. Note that if you use `transformation=True` you will get basis vectors, which will almost certainly be different from your's (but your's and Sage's can both be correct).

Full marks for a block diagoonal matrix representation via the generalized eigenspaces and the Jordan Canonical Form two ways: via a basis and combinatorially.

3-3-2 Challenge: implement reduced row-echelon form in Python/Sage.

1. Pseudo-code early in FCLA may be helpful, or you may not want to look at it right away.

2. Input to your Python function should be a Sage matrix that could have entries from any field. You may want to find out what this field is early in your routine with `.base_ring()`, especially if you plan to compare entries with zero and one. (Create the *correct* versions of zero and one, according to the matrix the user gives your routine.)

3. Check your work with random matrices over `QQ`. keywords `algorithm='echelonizable'` and `rank=` could be useful here.

4. Check your work with a matrix over the field of integers mod p, `GF(p)` in Sage. You can begin with a random integer matrix and use `.change_ring()` to quickly convert the entries to this field.