# Learning Opportunity 2-5

**2-5-1** This exercise will demonstrate another curious feature of the singular value decomposition.

1. Begin by constructing $A$, a random matrix of size $10 \times 14$ initially full of single-digit integers, but then converted to `RDF`.
2. Form the two products $A^*A$ and $AA^*$. Do the following steps twice, once for each product, which we will reference as $P$ below.
3. Compute an SVD: $P = USV^*$.
4. Construct $V^*PV$ and $U^*PU$.
5. Comment on what you observe.
6. Suppose $A$ was square to begin with. How would you interpret the result of expressions like $V^*PV$ and $U^*PU$?
7. Challenge: can you give a general proof of your observations? Hint: compute $V^*U$.

Full marks for four correct products and excellent commentary, along with a good answer to the question about the square case.

**2-5-2** This exercise will analyze the $10 \times 10$ matrix $A$ below:

$$A = \begin{bmatrix} -11 & -22 & 24 & -50 & -14 & -24 & -24 & -11 & 27 & 28 \\ 6 & 17 & -12 & 29 & 5 & 16 & 16 & 7 & -14 & -16 \\ -18 & 2 & 26 & -44 & -37 & -14 & -14 & -20 & 31 & 35 \\ -21 & 36 & 15 & -6 & -45 & 11 & 11 & -20 & 15 & 25 \\ 1 & 3 & 3 & -3 & -4 & 0 & 0 & 1 & 3 & 2 \\ 6 & -14 & -18 & 21 & 22 & 0 & 2 & 6 & -18 & -15 \\ 3 & -51 & 30 & -75 & 7 & -42 & -44 & 0 & 34 & 25 \\ 8 & 16 & -26 & 50 & 17 & 22 & 22 & 8 & -29 & -26 \\ -7 & 24 & 4 & 8 & -18 & 11 & 11 & -5 & 1 & 6 \\ -3 & -3 & 1 & -3 & -2 & -2 & -2 & -3 & 1 & 5 \end{bmatrix}$$

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[[-11, -22, 24, -50, -14, -24, -24, -11, 27, 28],
[6, 17, -12, 29, 5, 16, 16, 7, -14, -16],
[-18, 2, 26, -44, -37, -14, -14, -20, 31, 35]
, [-21, 36, 15, -6, -45, 11, 11, -20, 15, 25],
[1, 3, 3, -3, -4, 0, 0, 1, 3, 2],
[6, -14, -18, 21, 22, 0, 2, 6, -18, -15],
[3, -51, 30, -75, 7, -42, -44, 0, 34, 25],
[8, 16, -26, 50, 17, 22, 22, 8, -29, -26],
[-7, 24, 4, 8, -18, 11, 11, -5, 1, 6],
[-3, -3, 1, -3, -2, -2, -2, -3, 1, 5]]
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1. Use Sage and theorems from FCLA to verify that $A$ is deficient.
2. Compute the generalized eigenspace of $A$ for each eigenvalue using the definition and Sage's $.right_kernel()$ matrix method. (Generalized eigenspaces do not appear to be implemented.)
3. Describe a decomposition of $\mathbb{C}^{10}$ into invariant subspaces relative to the linear transformation $T$ given by $T(x) = Ax$.

4. Construct a random vector, $v$, from $\mathbb{C}^{10}$. Express $v$ as a sum of generalized eigenvectors of $A$. Provide computational proof that each of your generalized eigenvectors really is a generalized eigenvector and also really provide a decomposition of $v$.

Full marks for a convincing deficiency check, efficient computation of the generalized eigenspaces, and a decomposition of a random vector with checks that the decomposition is correct.