Learning Opportunity 2-2

2-2-1 The SVD can be used for a variety of different purposes. This exercise is about computing the inverse of a matrix with the SVD.

1. Construct a 6×6 random matrix, A, over the rationals, $\mathbf{Q}$, with rank 6 (i.e. full rank), using a command like `A = random_matrix(QQ, 6, algorithm='echelonizable', rank=6, upper_bound=9)`
2. Compute the SVD of A, though you will need to do this over `QQbar` since Sage still does not have an exact SVD routine.
3. Combine the three parts of the SVD of A to compute the inverse of A using the *simplest* possible collection of Sage and Python commands. "Simplest" here means the least amount of computation, "least expensive." In particular, *do not* use the `.inverse()` command nor any version of a `solve()` command.

Full marks for correct computation, in the simplest possible way.

2-2-2 Pieces of the singular value decomposition can do some wonderful things. This exercise will demonstrate that.

1. Construct a random 4×4 matrix, A, of single-digit integers but convert it to a matrix over `RDF`, as in:
   `B = random_matrix(ZZ, 4, 4, x=-9, y=9).change_ring(RDF)`
2. Build the 8×8 block matrix H using Sage's `block_matrix()` constructor.
   $$H = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$$
   Verify that H is Hermitian.
3. Compute U, S, V from the SVD of A. Build the block matrix T using Sage's `block_matrix()` constructor.
   $$T = \frac{1}{\sqrt{2}} \begin{bmatrix} V & -V \\ U & U \end{bmatrix}$$
   Verify that T is unitary.
4. Compute $T^{-1}HT$, but do not use an inverse, and instead rely on the fact that T is unitary.
5. Report as many interesting observations as you can about the result in the previous part. This is the main purpose of this exercise.

Full marks for correct versions of T and H, and a comprehensive discussion of your observations.