Learning Opportunity 2-4

AIMS-ZA Advanced Linear Algebra with Sage

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2-4-1 This is a Sage exercise that builds a pair of complementary orthogonal projectors.

1. Create two non-trivial vectors by hand, each with six integer entries. Choose the entries so that (a) the vectors are orthogonal, and (b) the length of each vector is an integer, (c) there are not too many zeros in each vector, and (d) the vectors are not too similar.
2. Let Q be a matrix over the rationals whose columns are your orthogonal vectors, but scaled by their length so that they are now a set of two orthonormal vectors.
3. Create $P = QQ^*$ and $C = I_6 - P$.
4. Use Sage to check that P and C are projector matrices.
5. Compute the rank of P and of C. Is this what you would expect for complementary projectors? Why?
6. For each projector, compute the column space (i.e. its range). These column spaces (ranges) are vector spaces, but you can use Sage's `.basis_matrix()` command to get the basis vectors as rows of a matrix. Perform a matrix computation with the basis matrices from these two projector's ranges that will clearly demonstrate that the ranges are orthogonal vector spaces.
7. Create a random vector, v, with six entries from the rationals. Find vectors v_1 and v_2, from the two orthogonal vector spaces of the previous part, so that

   $$v = v_1 + v_2.$$

   In other words, form a decomposition of v.

Full marks for two orthogonal vectors, the matrices P and Q, and a correct check on the two orthogonal vector spaces and a decomposition of a random vector.

2-4-2 This exercise will demonstrate another curious feature of the singular value decomposition.

1. Begin by constructing A, a random matrix of size 10×14 initially full of single-digit integers, but then converted to `RDF`.
2. Form the two products A^*A and AA^*. Do the following steps twice, once for each product, which we will reference as P below.
3. Compute an SVD: $P = USV^*$.
4. Construct V^*PV and U^*PU.
5. Comment on what you observe.
6. Suppose A was square to begin with. How would you interpret the result of expressions like V^*PV and U^*PU?
7. Challenge: can you give a general proof of your observations? Hint: compute V^*U.

Full marks for four correct products and excellent commentary, along with a good answer to the question about the square case.