Learning Opportunity 2-1

AIMS-ZA Advanced Linear Algebra with Sage

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2-1-1 This exercise works with the **indefinite factorization**, which is similar to the Cholesky decomposition. If you read the source code, you will see that to compute a Cholesky decomposition, first Sage computes an indefinite factorization.

1. Construct a 12×10 random matrix of positive, zero and negative single-digit integers. You will need to delve into Sage's documentation to figure out the right arguments to provide to the `random_matrix()` constructor. Navigating the documentation is part of the exercise. Do not ask me how to do it. Call the matrix A.
2. Construct $B = A^* A$, which is guaranteed to be positive semi-definite. Use Sage to check anyway.
3. Construct the **indefinite factorization** of B in Sage. From this construct an LU factorization of B.
4. Have Sage compute an LU factorization (*not* a PLU factorization) of B. How does this version compare to the previous one? Why? Which method might be superior computationally?

Full marks for creating a random positive semidefinite matrix, two computations of its LU decomposition, and well-written commentary.

2-1-2 This exercise connects positive semidefinite matrices with quadratic forms.
Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 6 & 0 \\ -1 & 0 & 9 \end{bmatrix}$$

1. Check that A is positive definite.
2. Let $x = (x_1, x_2, x_3)^t$ be a column vector of unknowns. The expression $x^t A x$ is called a **quadratic form**. Form this expression. Declare the symbolic variables `x1, x2, x3` using Sage's `var()` function.
3. Form the indefinite factorization of A. Discover how to use this factorization to express the quadratic form as a sum of obviously nonnegative terms. This is the hard part of the problem and you should discover it yourself.
4. From your answer to the previous part, give a well-written argument that A is positive semi-definite.
5. Now build on your argument and prove that A is positive definite.

Full marks for forming the quadratic form, converting it to a sum of nonnegative expressions, and two well-written arguments.