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Real and Functional Analysis

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These are unreviewed notes and may contain errors.**

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Chapter 1

Set Theory

1.1 Basic notions

Definition 1.1.1. Let X, Y be sets. We say:

- X, Y are **equipotent** if there exists a bijection $f : X \rightarrow Y$.
- X has a **cardinality greater or equal** to Y if there exists an surjection $f : X \rightarrow Y$.
- X is **finite** if it is equipotent to $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. X is infinite otherwise.

Remark: X is infinite \iff it is equipotent to a proper subset of itself.

E.g.: The set of natural numbers \mathbb{N} is infinite. In fact, the set of even natural numbers $E = \{2, 4, 6, \dots\} \subset \mathbb{N}$ is equipotent to \mathbb{N} , as we can define the bijection $f : \mathbb{N} \rightarrow E$ as $f(n) = 2n$.

Definition 1.1.2. Let X be an infinite set. We say X is **countable** if it is equipotent to \mathbb{N} . X is **uncountable** otherwise, in which case it is **more than countable**.

Definition 1.1.3. X has the **cardinality of the continuum** if it is equipotent to $[0, 1] \subset \mathbb{R}$. Any such set is uncountable.

E.g.: We have that:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable.
- $\mathbb{R}, \mathbb{R}^n, (0, 1), [0, 1]$ are uncountable.
- Countable union of countable sets is countable.

1.2 Families of subsets

Let X be a set. The “Power set” of X is the set of all subsets of X , denoted by $\mathcal{P}(X)$.

$$\mathcal{P}(X) = \{E : E \subseteq X\}$$

Note that $\mathcal{P}(X)$ has always a cardinality greater than X . For example, if $X = \mathbb{N}$, then $\mathcal{P}(X)$ has the cardinality of the continuum.

Definition 1.2.1. Let X be a set. A **family of subsets** of X is a set E such that:

$$E \subseteq \mathcal{P}(X)$$

We usually denote $E = \{E_i\}_{i \in I}$, where I is an index set.

Definition 1.2.2. Let $E = \{E_i\}_{i \in I}$ be a family of subsets of X . We define:

- The **union** of E as:

$$\bigcup_{i \in I} E_i = \{x \in X : x \in E_i \text{ for some } i \in I\}$$

- The **intersection** of E as:

$$\bigcap_{i \in I} E_i = \{x \in X : x \in E_i \text{ for all } i \in I\}$$

Definition 1.2.3. Let $E = \{E_i\}_{i \in I}$ be a family of subsets of X . We say F is **pairwise disjoint** if:

$$E_i \cap E_j = \emptyset \quad \forall i, j \in I, i \neq j$$

Definition 1.2.4. We say that the family $E = \{E_i\}_{i \in I}$ of subsets of X is a **covering** of X if:

$$X = \bigcup_{i \in I} E_i$$

Any subfamily of E , $E' = \{E_i\}_{i \in I'}$ is a **subcovering** of X if it is a covering of X itself.

E.g.: Let $X = \mathbb{R}$. We define:

$$\mathcal{T} = \{E \subset X : E \text{ is open}\}$$

We say that \mathcal{T} is the standard topology of X . More generally, this can be done in

“metric spaces” (X, d) .

Properties of \mathcal{T} (open sets):

- $\emptyset, X \in \mathcal{T}$.
- Finite intersection of elements in \mathcal{T} is in \mathcal{T} .
- Arbitrary union of elements in \mathcal{T} is in \mathcal{T} .

We can also define **sequences of sets**. Let X be a set. A sequence of sets in X is a family of sets $\{E_n\}_{n \in \mathbb{N}}$.

Definition 1.2.5. Let X be a set. A sequence of sets $\{E_n\}_{n \in \mathbb{N}}$ is said to be:

- **Increasing** if:

$$E_n \subseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

It is denoted by $\{E_n\} \uparrow$.

- **Decreasing** if:

$$E_n \supseteq E_{n+1} \quad \forall n \in \mathbb{N}$$

It is denoted by $\{E_n\} \downarrow$.

Let now $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)$ be a sequence of sets in X :

Definition 1.2.6. We define the following:

- The **limit superior** of $\{E_n\}$ as:

$$\limsup_{n \rightarrow \infty} E_n = \bigcap_{n \in \mathbb{N}} \bigcup_{k \geq n} E_k$$

- The **limit inferior** of $\{E_n\}$ as:

$$\liminf_{n \rightarrow \infty} E_n = \bigcup_{n \in \mathbb{N}} \bigcap_{k \geq n} E_k$$

- If the limit superior and limit inferior are equal, we say that

$$\lim_{n \rightarrow \infty} E_n = \limsup_{n \rightarrow \infty} E_n = \liminf_{n \rightarrow \infty} E_n$$

Exercise: Let X be a set and $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X)$ be a sequence of sets in X . Prove that:

$$(i) \quad \{E_n\} \uparrow \Rightarrow \lim_{n \rightarrow \infty} E_n = \bigcup_{n \in \mathbb{N}} E_n \quad (ii) \quad \{E_n\} \downarrow \Rightarrow \lim_{n \rightarrow \infty} E_n = \bigcap_{n \in \mathbb{N}} E_n$$

1.3 Characteristic functions

Definition 1.3.1. Let X be a set and $E \subseteq X$. The **characteristic function** of E is the function $\mathbb{1}_E : X \rightarrow \{0, 1\}$ defined as:

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

This function is also called the **indicator function** of E .

Remark: Let $E, F \subseteq X$. We have that:

- $\mathbb{1}_{E \cap F} = \mathbb{1}_E \cdot \mathbb{1}_F$.
- $\mathbb{1}_{E \cup F} = \mathbb{1}_E + \mathbb{1}_F - \mathbb{1}_{E \cap F}$.
- $\mathbb{1}_{E^c} = 1 - \mathbb{1}_E$.
- $\mathbb{1}_{\limsup_{n \rightarrow \infty} E_n} = \limsup_{n \rightarrow \infty} \mathbb{1}_{E_n}$.
- $\mathbb{1}_{\liminf_{n \rightarrow \infty} E_n} = \liminf_{n \rightarrow \infty} \mathbb{1}_{E_n}$.

1.4 Equivalence relations and Quotient sets

Definition 1.4.1. A relation R on a set X is a subset of $X \times X$. For any $x, y \in X$, we say that x is related to y if $(x, y) \in R$. We denote this as xRy .

Definition 1.4.2. A relation R on a set X is an **equivalence relation** if it satisfies:

- **Reflexivity:**

$$xRx \quad \forall x \in X$$

- **Symmetry:**

$$xRy \Rightarrow yRx \quad \forall x, y \in X$$

- **Transitivity:**

$$xRy, yRz \Rightarrow xRz \quad \forall x, y, z \in X$$

Every equivalence relation on X induces a partition of X . We define the **equivalence class** of $x \in X$ as:

$$[x] = \{y \in X : xRy\}$$

The set of all equivalence classes is called the **quotient set** of X by R , denoted by X/R .

$$X/R = \{[x] : x \in X\}$$

E.g.: Let $X = \mathbb{Z} \times \mathbb{Z}_0$ such that $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. We define the relation R on X as:

$$(a, b)R(c, d) \iff ad = bc$$

We can prove that R is an equivalence relation. The equivalence classes are:

$$[(a, b)] = \{(c, d) \in X : ad = bc\}$$

Notice that:

$$[(a, b)] = \{(a, b), (2a, 2b), (3a, 3b), \dots\}$$

If we denote a class $[(a, b)]$ as $[a/b]$, then we have that:

$$X/R = \{[a/b] : a, b \in \mathbb{Z}_0\} = \mathbb{Q}$$

Chapter 2

Measure Spaces

2.1 Measurable spaces

Definition 2.1.1. Let X be a non-empty set. A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is a **σ -algebra** if:

- (i) $\emptyset \in \mathcal{M}$.
- (ii) $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.
- (iii) $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M} \implies \bigcup_{n \in \mathbb{N}} E_n \in \mathcal{M}$.

If instead of (iii) we have that $E_1, E_2 \in \mathcal{M} \implies E_1 \cup E_2 \in \mathcal{M}$, then \mathcal{M} is called an **algebra**.

Remark: If \mathcal{M} is a σ -algebra, then we say that (X, \mathcal{M}) is a **measurable space**. Any set $E \in \mathcal{M}$ is called a **measurable set**.

E.g.: Let $X \neq \emptyset$. Then:

- $\mathcal{P}(X)$ is a σ -algebra.
- $\{\emptyset, X\}$ is a σ -algebra.
- $\{\emptyset, E, E^c, X\}$ is a σ -algebra for any $E \subseteq X$.
- $X = \mathbb{R}$, $\mathcal{M} = \{E \subseteq \mathbb{R} : E \text{ is open}\}$ is NOT a σ -algebra.

Properties 2.1.1. Let (X, \mathcal{M}) be a measurable space. Then:

- (i) $X = \emptyset^c \in \mathcal{M}$
- (ii) \mathcal{M} is also an algebra. Indeed, if $\{E_1, E_2\} \subseteq \mathcal{M}$, $E_n = \emptyset \forall n \geq 3$, then $E_1 \cup E_2 = \bigcup_{n \in \mathbb{N}} E_n \in \mathcal{M}$.
- (iii) $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M} \implies \bigcap_n E_n \in \mathcal{M}$.
- (iv) $E, F \in \mathcal{M} \implies E \setminus F \in \mathcal{M}$
- (v) $\Omega \subseteq X$. Then, the **restriction** of \mathcal{M} to Ω is:

$$\mathcal{M}|_{\Omega} := \{F \subseteq \Omega : F = E \cap \Omega, E \in \mathcal{M}\}$$

Then, $(\Omega, \mathcal{M}|_{\Omega})$ is a measurable space.

2.2 Generation of a σ -algebra

Theorem 2.2.1. Take any family $\mathcal{A} \subseteq \mathcal{P}(X)$. Then, it is well-defined the σ -algebra generated by \mathcal{A} , denoted by $\sigma_0(\mathcal{A})$, as the smallest σ -algebra containing \mathcal{A} . It is characterized by:

- (i) $\sigma_0(\mathcal{A})$ is a σ -algebra.
- (ii) $\mathcal{A} \subseteq \sigma_0(\mathcal{A})$.
- (iii) If \mathcal{M} is a σ -algebra and $\mathcal{A} \subseteq \mathcal{M}$, then $\sigma_0(\mathcal{A}) \subseteq \mathcal{M}$.

Sketch of proof. Define $V = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is } \sigma\text{-algebra, } \mathcal{A} \subseteq \mathcal{M}\}$. Notice that $V \neq \emptyset$ because $\mathcal{P}(X) \in V$. Then, define:

$$\sigma_0(\mathcal{A}) = \bigcap_{\mathcal{M} \in V} \mathcal{M}$$

Then, $\sigma_0(\mathcal{A})$ is a σ -algebra as it satisfies the properties of a σ -algebra, denoted in definition 2.1.1. ■

Remark: This is relevant. Often, to check that a σ -algebra has certain properties, it is enough to check the property on a set of generators.

2.3 Borel sets

Take (X, d) as a metric space, so that open sets are defined. Recall that:

$$\mathcal{T} = \{E \subseteq X : E \text{ is open}\}$$

Definition 2.3.1. The σ -algebra generated by \mathcal{T} is called the **Borel σ -algebra** of X , denoted by:

$$\mathcal{B}(X) := \sigma_0(\mathcal{T})$$

Any set $E \in \mathcal{B}(X)$ is a **Borel set**.

Remark: The following are Borel sets:

- Open sets
- Closed sets
- Countable intersections of open sets (G_δ -sets)
- Countable unions of closed sets (F_σ -sets)

We will deal with:

$$X = \mathbb{R} = (-\infty, \infty)$$

but also:

$$X = \overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$$

Let us define the arithmetic operations on $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$:

- $a \pm \infty = \pm\infty$
- $a > 0 : a \cdot \pm\infty = \pm\infty$
- $a < 0 : a \cdot \pm\infty = \mp\infty$
- $a = 0 : 0 \cdot \pm\infty = 0$
- $\infty - \infty, \infty/\infty, 0/0$ are not defined.

Also, the open intervals in $\overline{\mathbb{R}}$ are the following:

- (a, b) , with $a, b \in \mathbb{R}$
- $[-\infty, b)$
- $(a, \infty]$

Remark: We have that:

$$\begin{aligned}\mathcal{B}(\mathbb{R}) &:= \sigma_0(\{\text{open sets}\}) \\ &= \sigma_0(\{(a, b) : a < b\}) \\ &= \sigma_0(\{[a, b] : a < b\}) \\ &= \sigma_0(\{(a, \infty) : a \in \mathbb{R}\})\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\overline{\mathbb{R}}) &:= \sigma_0(\{\text{open sets}\}) \\ &= \sigma_0(\{(a, \infty] : a \in \mathbb{R}\})\end{aligned}$$

$$\mathcal{B}(\mathbb{R}^N) = \sigma_0(\{\text{open rectangles}\})$$

2.4 Measures

Let (X, \mathcal{M}) be a measurable space.

Definition 2.4.1. A function $\mu : \mathcal{M} \rightarrow [0, \infty]$ is a (positive) **measure** on \mathcal{M} if:

- (i) $\mu(\emptyset) = 0$
- (ii) $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}, \text{ disjoint} \implies \mu(\bigcup_{n \in \mathbb{N}} E_n) = \sum_{n \in \mathbb{N}} \mu(E_n)$

Note: To avoid nonsenses, we always assume that $\exists E \in \mathcal{M} \text{ s.t. } \mu(E) < \infty$

Terminology: Let X, \mathcal{M}, μ defined as above:

- (X, \mathcal{M}, μ) is a **measure space**.
- If $\mu(X) = 1$, then (X, \mathcal{M}, μ) is a **probability space** and μ is a **probability measure**.

Definition 2.4.2. A measure μ is:

1. **Finite** if $\mu(X) < \infty$
2. **σ -finite** if $\exists \{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M} \text{ s.t.}$

$$\mu(E_n) < \infty \quad \forall n \in \mathbb{N} \quad \wedge \quad X = \bigcup_{n \in \mathbb{N}} E_n$$

E.g.: Some examples of measures are:

1. (Trivial measure): For any (X, \mathcal{M}) , define μ as $\mu(E) = 0 \quad \forall E \in \mathcal{M}$
2. (Counting measure): For any (X, \mathcal{M}) , typically $\mathcal{M} = \mathcal{P}(X)$, define $\mu_{\#}$ as:

$$\mu_{\#}(E) = \begin{cases} \#\{\text{elements of } E\} & E \text{ finite} \\ \infty & \text{otherwise} \end{cases}$$

3. (Dirac measure): For any (X, \mathcal{M}) , pick $x_0 \in X$. Then, define δ_{x_0} as:

$$\delta_{x_0}(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 \notin E \end{cases}$$

2.4.1 Properties of measures

Theorem 2.4.1 (Basic properties). *Let (X, \mathcal{M}, μ) be a measure space. Then:*

- (i) μ is finitely additive: $E, F \in \mathcal{M}, E \cap F = \emptyset \implies \mu(E \cup F) = \mu(E) + \mu(F)$
- (ii) (Monotonicity): $E, F \in \mathcal{M}, E \subseteq F \implies \mu(E) \leq \mu(F)$
- (iii) (Excision property): $E, F \in \mathcal{M}, E \subseteq F, \mu(E) < \infty \implies \mu(F \setminus E) = \mu(F) - \mu(E)$

Proof. The proof is straightforward:

- (i) Let $E, F \in \mathcal{M}, E \cap F = \emptyset$. Then:

$$\mu(E \cup F) = \mu(E) + \mu(F)$$

Proof. Obvious, using $E_n = \emptyset$ for $n \geq 3$. ■

- (ii) Let $E, F \in \mathcal{M}, E \subseteq F$. Then:

$$\mu(E) \leq \mu(F)$$

Proof. Let $F = E \cup (F \setminus E)$. Then:

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$$
■

- (iii) Let $E, F \in \mathcal{M}, E \subseteq F, \mu(E) < \infty$. Then:

$$\mu(F \setminus E) = \mu(F) - \mu(E)$$

Proof. Let $F = E \cup (F \setminus E)$. Then:

$$\mu(F) = \mu(E) + \mu(F \setminus E)$$

This concludes the proof. ■

Theorem 2.4.2 (Continuity among monotone sequences). *Let (X, \mathcal{M}, μ) be a measure space. Let $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ be a sequence of measurable sets. Then:*

(i) *If $\{E_n\} \uparrow$, $E := \lim_n E_n = \bigcup_n E_n$, then:*

$$\mu(E) = \lim_n \mu(E_n)$$

(ii) *If $\{E_n\} \downarrow$, $E := \lim_n E_n = \bigcap_n E_n$, and $\mu(E_1) < \infty$, then:*

$$\mu(E) = \lim_n \mu(E_n)$$

Proof. The proof goes as follows:

(i) If $\mu(E_n) = \infty$ for some n , then the proof is trivial. Otherwise, let $F_1 = E_1$ and $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Then, we can check that:

- $F_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $\{F_n\}$ is a disjoint sequence.
- $E_n = \bigcup_{k=1}^n F_k$
- Because of the disjointness, we have that:

$$\mu(E_n) = \sum_{k=1}^n \mu(F_k)$$

Then, we have that:

$$\begin{aligned} \mu(E) &= \mu\left(\lim_n E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \\ &= \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) = \\ &= \sum_{n=1}^{\infty} (\mu(E_n) - \mu(E_{n-1})) = \lim_n \mu(E_n) \end{aligned}$$

(ii) Define $G_n = E_1 \setminus E_n$. Then, check that:

- $G_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $\{G_n\} \uparrow$

By the previous result, we have that:

$$\mu \left(\bigcup_{n=1}^{\infty} G_n \right) = \lim_n \mu(G_n)$$

Then, on the right-hand side:

$$\begin{aligned} \lim_n \mu(G_n) &= \lim_n \mu(E_1 \setminus E_n) = \\ &= \mu(E_1) - \lim_n \mu(E_n) \end{aligned}$$

On the left-hand side:

$$\begin{aligned} \mu \left(\bigcup_{n=1}^{\infty} G_n \right) &= \mu \left(\bigcup_{n=1}^{\infty} (E_1 \setminus E_n) \right) = \\ &= \mu \left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n \right) = \\ &= \mu(E_1) - \mu(E) \end{aligned}$$

Finally, we have that:

$$\mu(E_1) - \mu(E) = \mu(E_1) - \lim_n \mu(E_n)$$

And because $\mu(E_1) < \infty$, we have that:

$$\mu(E) = \lim_n \mu(E_n)$$

■

Remark: In (ii), the condition $\mu(E_1) < \infty$ is essential. Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{\#})$. Then, define the following sequence:

$$E_n = \{n, n+1, n+2, \dots\}$$

Note that $E_n \subseteq E_{n-1}$. Also, note that for any $n \in \mathbb{N}$, we have that:

$$\mu_{\#}(E_n) = \infty$$

Then, we have that:

$$\mu_{\#} \left(\bigcap_n E_n \right) = \mu_{\#}(\emptyset) = 0$$

But:

$$\lim_n \mu_{\#}(E_n) = \infty$$

This shows that the condition $\mu(E_1) < \infty$ is essential.

Theorem 2.4.3 (σ -subadditivity). *Let (X, \mathcal{M}, μ) be a measure space. Let $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{M}$ be a sequence of measurable sets. Then:*

$$\mu \left(\bigcup_n E_n \right) \leq \sum_n \mu(E_n)$$

Proof. Let $F_1 = E_1$ and $F_n = E_n \setminus (\bigcup_{k=1}^{n-1} E_k)$ for $n \geq 2$. Then, we have that:

- $F_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $F_n \subseteq E_n, \forall n \in \mathbb{N}$
- $\{F_n\}$ is a disjoint sequence.
- $\bigcup_n E_n = \bigcup_n F_n$

Then, we have that:

$$\begin{aligned} \mu \left(\bigcup_n E_n \right) &= \mu \left(\bigcup_n F_n \right) = \\ &= \sum_n \mu(F_n) \leq \sum_n \mu(E_n) \end{aligned}$$

■

2.5 Sets of measure zero, negligible sets, complete measures

Definition 2.5.1. Let (X, \mathcal{M}, μ) be a measure space. Then:

1. A set $E \in \mathcal{M}$ is a **set of measure zero** if $\mu(E) = 0$.
2. A set $F \in X$ (not necessarily measurable) is a **negligible set** if $\exists E \in \mathcal{M}$ s.t. $F \subseteq E$ and E is a set of measure zero.

Definition 2.5.2. Let (X, \mathcal{M}, μ) be a measure space. Then, we say that μ is a **complete measure** (alternatively, that (X, \mathcal{M}, μ) is a **complete measure space**) all negligible sets are measurable.

Remark (Completion of a measure space): A measure space (X, \mathcal{M}, μ) may not be complete. However, we can define the following:

$$\overline{\mathcal{M}} = \{E \subseteq X : \exists F_1, F_2 \in \mathcal{M} : F_1 \subseteq E \subseteq F_2, \mu(F_2 \setminus F_1) = 0\}$$

One can show that $\overline{\mathcal{M}}$ is a σ -algebra, and that $\mathcal{M} \subseteq \overline{\mathcal{M}}$. Moreover, if E, F_1, F_2 are as above, define:

$$\overline{\mu}(E) = \mu(F_1) = \mu(F_2)$$

One can check that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is a complete measure space.

2.6 Towards the Lebesgue measure

We would like to define a measure λ with $X = \mathbb{R}$ (or $X = \mathbb{R}^N$) s.t. $\forall a < b$:

- $\lambda((a, b)) = b - a$ (**length of the interval**)
- $\forall E, \lambda(E + x) = \lambda(E)$ (**translation invariance**)

In principle, we would like to define it in $\mathcal{P}(\mathbb{R})$. Such a measure should satisfy $\lambda(\{a\}) = 0$.

Theorem 2.6.1 (Ulam). *The only measure on $\mathcal{P}(\mathbb{R})$ that satisfies $\lambda(\{a\}) = 0 \forall a \in \mathbb{R}$ is the trivial measure.*

Therefore, we need to choose an $\mathcal{M} \subsetneq \mathcal{P}(\mathbb{R})$. We can construct one as follows:

- Starting family with a “measure”, e.g., $\mathcal{T} = \{(a, b) : a < b\}$ and $f((a, b)) = b - a$.
- Construct an “outer measure” μ^* on $\mathcal{P}(\mathbb{R})$.
- Restrict μ^* to a well-chosen σ -algebra $\mathcal{M} \subsetneq \mathcal{P}(\mathbb{R})$.

Definition 2.6.1. Let X be a set. An **outer measure** μ^* on X is a function

$$\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$$

such that:

1. $\mu^*(\emptyset) = 0$
2. (Monotonicity) $E \subseteq F \subseteq X \implies \mu^*(E) \leq \mu^*(F)$
3. (σ -subadditivity) $\{E_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(X) \implies \mu^*\left(\bigcup_{n \in \mathbb{N}} E_n\right) \leq \sum_{n \in \mathbb{N}} \mu^*(E_n)$

Remark: Any measure μ is an outer measure. However, the converse is not true.

Proposition 2.6.2. Let $\mathcal{E} \subseteq \mathcal{P}(X)$, $f : \mathcal{E} \rightarrow [0, \infty]$. Assume that $\emptyset, X \in \mathcal{E}$, $f(\emptyset) = 0$. Then, $\forall E \subseteq X$ define:

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} f(A_n) : A_n \in \mathcal{E}, E \subseteq \bigcup_n A_n \right\}$$

Then, μ^* is an outer measure.

Proof. The proof is omitted. ■

Remark: In this generality, if $E \in \mathcal{E}$, then $f(E)$ and $\mu^*(E)$ may not be equal. We can only guarantee that $\mu^*(E) \leq f(E)$.

E.g.: There are some important examples:

- $X = \mathbb{R}$, $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$

$$f((a, b)) = \text{length}((a, b)) = b - a$$

- $X = \mathbb{R}^N$, $\mathcal{E} = \{(a_1, b_1) \times \dots \times (a_N, b_N) : a_i, b_i \in \overline{\mathbb{R}}, a_i \leq b_i\}$

$$f(\underline{a}, \underline{b}) = \text{volume}(\underline{a}, \underline{b}) = \prod_{i=1}^N (b_i - a_i)$$

In both cases, the outer measure μ^* is called the **Lebesgue outer measure**. We will denote it by λ^* (or λ_N^* in the second case). Note that in this case, $\lambda^*(E) = f(E)$ for any $E \in \mathcal{E}$.

Remark: Any μ measure on $\mathcal{P}(X)$ is an outer measure. However, the converse is not true. In particular, $\exists A, B \subseteq \mathbb{R}$ s.t. $A \cap B = \emptyset$ and $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$.

2.6.1 Carathéodory's criterion

Definition 2.6.2 (Carathéodory's condition). Let μ^* be an outer measure on $\mathcal{P}(X)$. A set $E \subseteq X$ is μ^* -**measurable** if $\forall A \subseteq X$:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Lemma 2.6.3 (Equivalence of Carathéodory's condition). E is μ^* -measurable $\iff \forall A \subseteq X, \mu^*(A) < \infty$:

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Proof. The proof is as follows:

(\Rightarrow) : Trivial

(\Leftarrow) : Let $A \subseteq X$, such that $\mu^*(A) < \infty$ and:

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Then, notice that $\{A \cap E, A \cap E^c\}$ is a covering of A . By subadditivity:

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Therefore, we have that:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Meaning that E is μ^* -measurable. This concludes the proof. ■

Theorem 2.6.4 (Carathéodory). *Let μ^* be an outer measure on $\mathcal{P}(X)$. The family:*

$$\mathcal{M} = \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$$

is a σ -algebra, and μ^ restricted to \mathcal{M} (denoted $\mu = \mu^*|_{\mathcal{M}}$) is a complete measure.*

Remark: (X, \mathcal{M}, μ) as in the above theorem is sometimes called the “abstract Lebesgue measure space”. We will only prove the completeness of μ .

Lemma 2.6.5. *Let (X, \mathcal{M}, μ) be the measure space as in Carathéodory’s theorem. Then, any $N \subseteq X$ s.t. $\mu^*(N) = 0$ is μ -measurable, i.e., $N \in \mathcal{M}$, and $\mu(N) = 0$.*

Proof. We have to show that N satisfies Carathéodory’s condition, or equivalently, that it satisfies the lemma 2.6.3. Let $A \subseteq X$ be arbitrary. Then, notice that:

$$\mu^*(A \cap N) \leq \mu^*(N) = 0$$

Also, notice that:

$$\mu^*(A \cap N^c) \leq \mu^*(A)$$

Therefore, we have that:

$$\mu^*(A \cap N) + \mu^*(A \cap N^c) \leq 0 + \mu^*(A) = \mu^*(A)$$

By lemma 2.6.3, we have that N is μ^* -measurable. By Carathéodory’s theorem, we have that N is μ -measurable. Finally, we have that $\mu(N) = \mu^*(N) = 0$. ■

Corollary 2.6.5.1. *μ as in Carathéodory’s theorem is a complete measure.*

Proof. Let $N \subseteq E$, and $\mu(E) = 0$ ($E \in \mathcal{M}$). Then, we have that:

$$\mu(E) = 0 \implies \mu^*(E) = 0$$

By monotonicity, we have that:

$$\mu^*(N) \leq \mu^*(E) = 0$$

Then, $\mu(N) = \mu^*(N) = 0$, thus $N \in \mathcal{M}$. This concludes the proof. ■

2.7 Lebesgue measure

Definition 2.7.1. Let $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$. Define:

$$\lambda^*((a, b)) = b - a$$

Then, λ^* is the **Lebesgue outer measure** on \mathbb{R} .

Theorem 2.7.1. Let λ^* be the Lebesgue outer measure on $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$. Then, by Carathéodory's theorem, the family:

$$\mathcal{L}(\mathbb{R}) = \{E \subseteq \mathbb{R} : E \text{ is } \lambda^*\text{-measurable}\}$$

is a σ -algebra, called the **Lebesgue σ -algebra**, and λ^* restricted to $\mathcal{L}(\mathbb{R})$ (denoted $\lambda = \lambda^*|_{\mathcal{L}(\mathbb{R})}$) is a complete measure, called the **Lebesgue measure**.

Proof. The proof is omitted. ■

Remark: The measure space $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is called the **Lebesgue measure space**.

Proposition 2.7.2. Let λ be the Lebesgue measure on \mathbb{R} . Then:

- (i) $a \in \mathbb{R} \implies \{a\} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\{a\}) = 0$
- (ii) $E \subset \mathbb{R}$ at most countable $\implies E \in \mathcal{L}(\mathbb{R})$ and $\lambda(E) = 0$

Proof. The proof is as follows:

- (i) Let $a \in \mathbb{R}$. Then, we have that, for any $\varepsilon > 0$:

$$E_1 = (a - \varepsilon, a + \varepsilon), \quad E_2 = E_3 = \dots = \emptyset$$

is a covering of $\{a\}$. Then, by definition of λ^* :

$$0 \leq \lambda^*(\{a\}) \leq \sum_{n=1}^{\infty} \lambda(E_n) = 2\varepsilon$$

As ε is arbitrary, we have that $\lambda^*(\{a\}) = 0$. By Lemma 2.6.5, we then have that $\{a\} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\{a\}) = 0$.

(ii) Let $E \subseteq \mathbb{R}$ be at most countable. Then, we have that:

$$E = \bigcup_{a \in E} \{a\}$$

Because $\{a\} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\{a\}) = 0$, we have that $E \in \mathcal{L}(\mathbb{R})$ and:

$$\lambda(E) = \lambda\left(\bigcup_{a \in E} \{a\}\right) = \sum_{a \in E} \lambda(\{a\}) = 0$$

■

Remark: We can point out 2 important consequences of the above proposition:

1. Arguing as in (i), we can show that the Lebesgue measure is translation invariant. That is, $\forall E \in \mathcal{L}(\mathbb{R}), \forall x \in \mathbb{R}$:

$$\lambda(E + x) = \lambda(E)$$

2. In particular, since \mathbb{Q} is countable, we have that $\mathbb{Q} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\mathbb{Q}) = 0$. In the measure sense, \mathbb{Q} has very few elements with respect to \mathbb{R} . On the other hand, \mathbb{Q} is dense in \mathbb{R} . In the topology sense, \mathbb{Q} has a lot of points.

Proposition 2.7.3. *We have that: $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R})$*

Proof. Since $\mathcal{B}(\mathbb{R}) = \sigma_0(\{(a, \infty) : a \in \mathbb{R}\})$, if we show that $(a, \infty) \in \mathcal{L}(\mathbb{R}), \forall a \in \mathbb{R}$, then the prop. follows.

Take $A \subset \mathbb{R}$, s.t. $\lambda^*(A) < \infty$. Then, we must show that:

$$\lambda^*(A) \geq \lambda^*(A \cap (a, \infty)) + \lambda^*(A \cap (-\infty, a])$$

Moreover, by a previous remark, one can assume that $a \notin A$. Then, take any countable covering of A by open intervals:

$$A \subseteq \bigcup_n I_n$$

Then, let us define $A_{left} = A \cap (-\infty, a]$ and $I_{n,left} = I_n \cap (-\infty, a]$. Then, we notice that $\{I_{n,left}\}$ is a covering of A_{left} .

In the same way, we define $A_{right} = A \cap (a, \infty)$ and $I_{n,right} = I_n \cap (a, \infty)$. Then, we notice that $\{I_{n,right}\}$ is a covering of A_{right} .

Then, we have that:

$$\begin{aligned}\lambda^*(A_{left}) &\leq \sum_n \lambda^*(I_{n,left}) \\ \lambda^*(A_{right}) &\leq \sum_n \lambda^*(I_{n,right})\end{aligned}$$

Summing both inequalities, we have that:

$$\begin{aligned}\lambda^*(A_{left}) + \lambda^*(A_{right}) &\leq \sum_n \lambda^*(I_{n,left}) + \sum_n \lambda^*(I_{n,right}) \\ &= \sum_n \lambda^*(I_n)\end{aligned}$$

Taking the infimum over all countable coverings of A , we have that:

$$\lambda^*(A_{left}) + \lambda^*(A_{right}) \leq \lambda^*(A)$$

■

Remark: In particular, we have that $\forall (a, b) \subset \mathbb{R}$:

$$(a, b) \in \mathcal{L}(\mathbb{R}), \quad \lambda((a, b)) = b - a$$

We know that:

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$$

Are these inclusions strict? We already know that $\mathcal{L}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$, by Ulam's theorem. In particular, $\exists E \subset \mathbb{R}$ not Lebesgue measurable (**Vitali sets**). More precisely:

$$\forall E \in \mathcal{L}(\mathbb{R}), \text{ s.t } \lambda(E) > 0, \exists V \subset E, \text{ s.t } V \notin \mathcal{L}(\mathbb{R})$$

The relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ is more subtle. It is clarified by the following proposition:

Proposition 2.7.4 (Regularity of the Lebesgue measure). *Let $E \in \mathbb{R}$. Then, the following are equivalent:*

(i) $E \in \mathcal{B}(\mathbb{R})$

(ii) $\forall \varepsilon > 0, \exists A \subset \mathbb{R}$ open set s.t.

$$E \subset A \quad \text{and} \quad \lambda^*(A \setminus E) < \varepsilon$$

(iii) $\forall \varepsilon > 0, \exists G \subset \mathbb{R}$ of class G_δ s.t.

$$E \subset G \quad \text{and} \quad \lambda^*(G \setminus E) = 0$$

(iv) $\forall \varepsilon > 0, \exists C \subset \mathbb{R}$ closed set s.t.

$$C \subset E \quad \text{and} \quad \lambda^*(E \setminus C) < \varepsilon$$

(v) $\forall \varepsilon > 0, \exists F \subset \mathbb{R}$ of class F_σ s.t.

$$F \subset E \quad \text{and} \quad \lambda^*(E \setminus F) = 0$$

We get as a consequence the following:

Corollary 2.7.4.1. $\forall E \in \mathcal{L}(\mathbb{R}), \exists F, G \in \mathcal{B}(\mathbb{R})$ s.t. $F \subset E \subset G$ and

$$\lambda(G \setminus F) = \lambda(G \setminus E) + \lambda(E \setminus F) = 0$$

In other words:

$$\mathcal{L}(\mathbb{R}) = \overline{\mathcal{B}(\mathbb{R})}$$

(But $\mathcal{L}(\mathbb{R}) \neq \mathcal{B}(\mathbb{R})$).

Proof. (Regularity of the Lebesgue measure). The proof goes as follows:

(i) \Rightarrow (ii) :

Let $E \in \mathcal{B}(\mathbb{R})$. Note that, since $A \in \mathcal{L}(\mathbb{R})$ for all A open, we have that:

$$\lambda^*(A \setminus E) = \lambda(A \setminus E)$$

By definition of λ^* , we have that $\forall \varepsilon > 0, \exists \{I_n\}_{n \in \mathbb{N}}$ s.t.

$$E \subset \bigcup_n I_n \quad \text{and} \quad \sum_n \lambda(I_n) < \lambda^*(E) + \varepsilon$$

Then, set $A = \bigcup_n I_n$. We have that A is open, $E \subset A$ and:

$$\begin{aligned} \lambda(A) &\leq \sum_n \lambda(I_n) < \lambda(E) + \varepsilon \\ \implies \lambda(A \setminus E) &= \lambda(A) - \lambda(E) < \varepsilon \end{aligned}$$

(ii) \Rightarrow (iii) :

Assume $\forall \varepsilon > 0$, $\exists A_\varepsilon$ open s.t. $E \subset A_\varepsilon$ and $\lambda(A_\varepsilon \setminus E) < \varepsilon$. Then, set $\varepsilon = 1/n$, $n \geq 1$ (for ease of notation, $A_n = A_{1/n}$) and define:

$$G = \bigcap_n A_n$$

Then, G is a G_δ set, $E \subset G$ and:

$$0 \leq \lambda^*(G \setminus E) \leq \lambda^*(A_n \setminus E) < 1/n$$

Taking the limit, we have that $\lambda(G \setminus E) = 0$.

(iii) \Rightarrow (i) :

We know that $E \subset G$, $G \in \mathcal{L}(\mathbb{R})$ with $\lambda(G \setminus E) = 0$. Then, we have that:

$$E = G \setminus (G \setminus E) \in \mathcal{L}(\mathbb{R})$$

since $G \in \mathcal{L}(\mathbb{R})$ and $G \setminus E \in \mathcal{L}(\mathbb{R})$. The last is because it is a negligible set and λ is complete. ■

E.g. (Cantor set): Let $T_0 = [0, 1]$. Then, construct T_{n+1} from T_n (recursively) by removing the inner third part of every interval in T_n :

$$\begin{aligned} T_0 &= [0, 1], \\ T_1 &= [0, 1/3] \cup [2/3, 1], \\ T_2 &= [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \dots \end{aligned}$$

Then, define the **Cantor set** as:

$$C = \bigcap_n T_n$$

It can be proven that:

- C has the cardinality of \mathbb{R}
- $\lambda(C) = 0$
- C is compact
- C is nowhere dense (has no interior points), i.e., $\text{int}(C) = \emptyset$
- $\exists E \subset C$ s.t. $E \in \mathcal{L}(\mathbb{R})$ but $E \notin \mathcal{B}(\mathbb{R})$

Chapter 3

Measurable functions

Definition 3.0.1. Given $f : X \rightarrow Y$, it is well-defined the **preimage** (or counterimage) of f as:

$$f^{-1} : \mathcal{P}(Y) \rightarrow \mathcal{P}(X), \quad f^{-1}(E) = \{x \in X : f(x) \in E\}$$

Remark: Preimages work well with set operations:

- $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$
- $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$
- $f^{-1}(E \setminus F) = f^{-1}(E) \setminus f^{-1}(F)$

Note this is not true for images.

Definition 3.0.2. Let (X, \mathcal{M}) and (Y, \mathcal{N}) be measurable spaces. A function $f : X \rightarrow Y$ is **measurable** if $\forall E \in \mathcal{N}$, we have that $f^{-1}(E) \in \mathcal{M}$. We also say that f is **$(\mathcal{M}, \mathcal{N})$ -measurable**.

Proposition 3.0.1. Let (X, \mathcal{M}) and (Y, \mathcal{N}) be measurable spaces, and $\rho \subset \mathcal{N}$ s.t. $\mathcal{N} = \sigma_0(\rho)$. Then, $f : X \rightarrow Y$ is measurable $\iff \forall E \in \rho$, we have that $f^{-1}(E) \in \mathcal{M}$.

Proof. The proofs goes as follows:

(\Rightarrow) : Trivial

(\Leftarrow) : Define $\Sigma := \{E \subset Y : f^{-1}(E) \in \mathcal{M}\}$. We have:

- $\rho \subset \Sigma$ as a consequence of $(\forall E \in \rho, f^{-1}(E) \in \mathcal{M})$

- Σ is a σ -algebra (check as an exercise)

Then, we have that $\mathcal{N} = \sigma_0(\rho) \subset \Sigma$. Therefore, f is measurable. ■

Definition 3.0.3. Suppose that $\mathcal{M} \supseteq \mathcal{B}(X)$ and $\mathcal{N} = \mathcal{B}(Y)$. We say that $f : X \rightarrow Y$ is:

- **Borel measurable** if f is $(\mathcal{B}(X), \mathcal{B}(Y))$ -measurable.
- **Lebesgue measurable** if it is $(\mathcal{M}, \mathcal{B}(Y))$ -measurable.

Remark: If $f : X \rightarrow Y$ is Borel measurable, then it is Lebesgue measurable. The converse is not true.

Also, we never deal with $\mathcal{L}(Y)$.

Corollary 3.0.1.1. f is Borel measurable $\iff f^{-1}(E) \in \mathcal{B}(X), \forall E \in Y$ open.
Also, f is Lebesgue measurable $\iff f^{-1}(E) \in \mathcal{M}, \forall E \in Y$ open.

Proof. It follows from the previous proposition, since $\mathcal{B}(Y) = \sigma_0(\{E \subset Y : E \text{ open}\})$. ■

Definition 3.0.4. We say that f is **continuous** $\iff f^{-1}(E) \subset X$ is open $\forall E \subset Y$ open.

Proposition 3.0.2. If $f : X \rightarrow Y$ is continuous, then f is Borel measurable (and thus Lebesgue measurable).

Proof. Let $E \subset Y$ be open. By continuity of f , we have that $f^{-1}(E)$ is open. Then $f^{-1}(E) \in \mathcal{B}(X)$, and thus f is Borel measurable.

Note that the proposition is false when $\mathcal{N} \supsetneq \mathcal{B}(Y)$. ■

3.1 Operations on measurable functions

Proposition 3.1.1. *Let $f : X \rightarrow Y$ be Lebesgue measurable, and $g : Y \rightarrow Z$ be continuous. Then:*

$$g \circ f : X \rightarrow Z \text{ is Lebesgue measurable}$$

Corollary 3.1.1.1. *Let $f : X \rightarrow Y$ be Lebesgue measurable. Then:*

- $f^+(x) = \max\{f(x), 0\}$ is Lebesgue measurable
- $f^-(x) = \max\{-f(x), 0\}$ is Lebesgue measurable
- $|f(x)| = f^+(x) + f^-(x)$ is Lebesgue measurable

Proof. Let f be Lebesgue measurable, and $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. Then, take $E \subset Z$ open. We have that:

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$$

Since g is continuous, $g^{-1}(E)$ is open. Then, $f^{-1}(g^{-1}(E)) \in \mathcal{M}$ ■

Proposition 3.1.2. *Let $f, g : X \rightarrow \mathbb{R}$ be Lebesgue measurable, and $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuous. Then, $h(x) = \Phi(f(x), g(x))$ is Lebesgue measurable.*

Proof. Let $h(x) = \Phi(f(x), g(x)) = (\Phi \circ \Psi)(x)$, where $\Psi : X \rightarrow \mathbb{R}^2$ is defined as

$$\Psi(x) = (f(x), g(x))$$

Then, we should prove that Ψ is Lebesgue measurable for applying the previous proposition. For this, we have to show that $\forall (a, b) \times (c, d) \subset \mathbb{R}^2$, we have that:

$$\Psi^{-1}((a, b) \times (c, d)) = \{x \in X : f(x) \in (a, b), g(x) \in (c, d)\} \in \mathcal{M}$$

This can be done using the fact that f and g are Lebesgue measurable. ■

Corollary 3.1.2.1. *Let $f, g : X \rightarrow \mathbb{R}$ be Lebesgue measurable. Then:*

- $f + g$ is Lebesgue measurable
- $f \cdot g$ is Lebesgue measurable

Proposition 3.1.3. *Let (X, \mathcal{M}) be a measurable space (with $\mathcal{M} \supseteq \mathcal{B}(X)$), and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of Lebesgue measurable functions $f_n : X \rightarrow \mathbb{R}$. Then, the following functions are Lebesgue measurable:*

1. $\sup_n f_n$
2. $\inf_n f_n$
3. $\limsup_n f_n$
4. $\liminf_n f_n$

In particular, if $\lim_n f_n$ exists, then it is Lebesgue measurable.

Proof. The proof goes as follows:

1. Since $\mathcal{B}(\mathbb{R}) = \sigma_0(\{(a, \infty) : a \in \mathbb{R}\})$, it is enough to show that $\forall a \in \mathbb{R}$, we have that:

$$(\sup_n f_n)^{-1}((a, \infty)) = \{x \in X : \sup_n f_n(x) > a\} \in \mathcal{M}$$

This can be done by using the fact that f_n is Lebesgue measurable. Indeed, we have that:

$$\begin{aligned} \{x \in X : \sup_n f_n(x) > a\} &= \bigcup_n \{x \in X : f_n(x) > a\} \\ &= \bigcup_n f_n^{-1}((a, \infty)) \in \mathcal{M} \end{aligned}$$

because $f_n^{-1}((a, \infty)) \in \mathcal{M}$ for all n .

2. The proof is analogous to the previous case, taking that:

$$\inf_n f_n = -\sup_n (-f_n)$$

3. We have that:

$$\limsup_n f_n = \inf_n \sup_{k \geq n} f_k$$

4. We have that:

$$\liminf_n f_n = \sup_n \inf_{k \geq n} f_k$$

■

3.2 Properties holding almost everywhere

Definition 3.2.1. Let (X, \mathcal{M}, μ) be a complete measure space. We say that a property $P(x)$ holds μ -almost everywhere (a.e) if:

$$\mu(\{x \in X : P(x) \text{ is false}\}) = 0$$

In other words, $P(x)$ holds μ -almost everywhere if it holds everywhere except for a set of measure zero.

E.g.: Let $f(x) = x^2$. Is it true that $f(x) > 0$ a.e.?

We have that $\{x : x^2 \leq 0\} = \{0\}$

- In $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, the property is true a.e., since $\lambda(\{0\}) = 0$
- In $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_{\#})$ (counting measure), the property is false a.e., since $\mu_{\#}(\{0\}) = 1$

Proposition 3.2.1. Let (X, \mathcal{M}, μ) be a measure space:

1. $f : X \rightarrow \overline{\mathbb{R}}$ s.t. $f = g$ a.e, with g measurable $\implies f$ is measurable
2. $\{f_n\}_{n \in \mathbb{N}}$ a sequence of measurable functions s.t. $f_n \rightarrow f$ a.e., then f is measurable.

3.3 Simple functions

Definition 3.3.1. Let (X, \mathcal{M}) be a measurable space. A function $s : X \rightarrow \overline{\mathbb{R}}$ is measurable and **simple** if s is measurable and $s(X)$ is a finite set:

$$s(X) = \{a_1, a_2, \dots, a_k\}$$

where $a_i \in \overline{\mathbb{R}} \forall i$, with $a_i \neq a_j$ for $i \neq j$. Then, s can be written as:

$$s(x) = \sum_{i=1}^k a_i \cdot \chi_{A_i}(x)$$

where $A_i = s^{-1}(\{a_i\})$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^k A_i = X$ and $A_i \in \mathcal{M}$, $\forall i$.

Particular case:

If $X = \mathbb{R}$ (or $(a, b) \subset \mathbb{R}$) and A_i is an interval $\forall i$, then s is called a **step function**.

On the other hand, $\chi_{\mathbb{Q}}$ is a simple function, but not a step function:

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Remark: One may define simple functions without measurability requirements.

Goal:

Approximate any measurable function $f : X \rightarrow \overline{\mathbb{R}}$ with (measurable and) simple functions.

Theorem 3.3.1 (Simple approximation theorem (SAT)). *Take (X, \mathcal{M}) measurable space and $f : X \rightarrow [0, \infty]$, measurable. Then $\exists \{s_n\}_{n \in \mathbb{N}}$ a sequence of measurable, simple functions s.t. $s_1 \leq s_2 \leq \dots \leq f$ pointwise (i.e., $\forall x \in X$) and:*

$$\lim_{n \rightarrow \infty} s_n(x) = f(x) \quad \forall x \in X$$

Moreover, if f is bounded, the convergence is uniform:

$$\lim_{n \rightarrow \infty} \sup_{x \in X} |s_n(x) - f(x)| = 0$$

Proof. In case f is bounded, say $0 \leq f < 1$.

For any $n \geq 1$, divide $[0, 1)$ into 2^n intervals of length 2^{-n} , and define:

$$A_n^{(i)} = \{x \in X : \frac{i}{2^n} \leq f(x) < \frac{i+1}{2^n}\}$$

and:

$$s_n(x) = \sum_{i=0}^{2^n-1} \frac{i}{2^n} \cdot \chi_{A_n^{(i)}}(x)$$

One can show that such a sequence has the required properties

■

Chapter 4

Lebesgue integral

- First, we will define the integral for non-negative simple (measurable) functions.
- Second, we will extend the definition to non-negative measurable functions.
- Finally, we will define the integral for general measurable functions.

4.1 Integral of non-negative simple functions

Definition 4.1.1. Let $s : X \rightarrow [0, \infty]$ be a measurable and simple function:

$$s = \sum_{i=1}^k a_i \cdot \chi_{A_i}$$

where $a_i \geq 0$ and $A_i \in \mathcal{M}$. Let $E \in \mathcal{M}$. Then, we define the **(Lebesgue) integral** of s over E as:

$$\int_E s \, d\mu = \sum_{i=1}^k a_i \cdot \mu(A_i \cap E)$$

Remark: There are some remarks:

1. $s : [a, b] \rightarrow [0, \infty)$, $\mu, \mu = \lambda$ (Lebesgue measure)
Then, $\int_{[a,b]} s \, d\mu = \text{area under the graph of } s \text{ in } [a, b]$
2. We are already using $0 \cdot \infty = 0$ in the definition. In particular,

$$a_i \cdot \mu(A_i \cap E) = \begin{cases} 0 & a_i = 0 \\ \infty & a_i > 0 \end{cases}$$

if $\mu(A_i \cap E) = \infty$.

3. $D \in \mathcal{M}$, then χ_D is a simple function, and:

$$\int_E \chi_D d\mu = \mu(D \cap E)$$

4. More generally, s simple and measurable, $E \in \mathcal{M}$, then:

$$\int_E s d\mu = \int_X s \cdot \chi_E d\mu$$

Properties 4.1.1 (Basic properties). Let $N, E, F \in \mathcal{M}$, $s_1, s_2 : X \rightarrow [0, \infty)$ simple and measurable functions. Then:

(i) If $\mu(N) = 0$, then:

$$\int_N s_1 d\mu = 0$$

(ii) If $0 \leq c \leq \infty$, then:

$$\int_E c \cdot s_1 d\mu = c \cdot \int_E s_1 d\mu$$

(iii) $\int_E (s_1 + s_2) d\mu = \int_E s_1 d\mu + \int_E s_2 d\mu$

(iv) If $s_1 \leq s_2$, then:

$$\int_E s_1 d\mu \leq \int_E s_2 d\mu$$

(v) if $E \subset F$, then:

$$\int_E s_1 d\mu \leq \int_F s_1 d\mu$$

The properties (ii) and (iii) are called **linearity** of the integral. The properties (iv) and (v) are called **monotonicity** of the integral.

Proposition 4.1.1. *Let $s : X \rightarrow [0, \infty)$ be a simple measurable function. Then, the function:*

$$\phi(E) := \int_E s \, d\mu : \mathcal{M} \rightarrow [0, \infty]$$

is a measure on (X, \mathcal{M}) .

Proof. Let $s = \sum_{i=1}^k a_i \cdot \chi_{A_i}$, $0 \leq a_i \leq \infty$. We have to show that:

1. $\phi : \mathcal{M} \rightarrow [0, \infty]$?: Yes, since $s \geq 0$, $\phi(E) \geq 0$, $\forall E \in \mathcal{M}$.
2. $\phi(\emptyset) = 0$?: Yes, since $\int_{\emptyset} s \, d\mu = 0$, as $\mu(\emptyset) = 0$.
3. σ -additivity?: Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint sets in \mathcal{M} , and $E = \bigcup_n E_n$. Then, we have that:

$$\begin{aligned} \phi(E) &= \int_E s \, d\mu = \int_X s \cdot \chi_E \, d\mu = \sum_{i=1}^k a_i \cdot \mu(A_i \cap E) \\ &= \sum_{i=1}^k a_i \cdot \mu\left(\bigcup_n A_i \cap E_n\right) \end{aligned}$$

Since μ is σ -additive, we have that:

$$\begin{aligned} &= \sum_{i=1}^k a_i \sum_n \mu(A_i \cap E_n) \\ &= \sum_n \sum_{i=1}^k a_i \cdot \mu(A_i \cap E_n) \\ &= \sum_n \int_{E_n} s \, d\mu = \sum_n \phi(E_n) \end{aligned}$$

■

4.2 Integral of non-negative measurable functions

Definition 4.2.1. Let $f : X \rightarrow [0, \infty]$ be a measurable function, $E \in \mathcal{M}$. Then, we define the **(Lebesgue) integral** of f over E as:

$$\int_E f d\mu = \sup \left\{ \int_E s d\mu : s \text{ simple, measurable and } 0 \leq s \leq f \right\}$$

Remark: There are some remarks:

1. If f is simple, then the definition coincides with the previous one.
2. $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{\#})$. Then $f : \mathbb{N} \rightarrow [0, \infty]$ is a sequence. Indeed, if we name $f_n = f(n)$, then:

$$\int_{\mathbb{N}} f d\mu_{\#} = \sum_n f_n$$

3. All the basic properties of the integral for simple functions above hold for this new definition.

Note: The following propositions assume that (X, \mathcal{M}, μ) is a complete measure space (needed for a.e. properties).

Proposition 4.2.1 (Chebychev's inequality). *Let $f : X \rightarrow [0, \infty]$ be a measurable function, and $0 < c < \infty$. Then:*

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f \geq c\}} f d\mu \leq \frac{1}{c} \int_X f d\mu$$

where $\{f \geq c\} = \{x \in X : f(x) \geq c\}$.

Proof.

$$\int_X f d\mu \geq \int_{\{f < c\}} f d\mu \geq \int_{\{f < c\}} c d\mu = c \cdot \mu(\{f < c\})$$

Now we just divide by c . ■

Note: We have as a consequence the following lemmas:

Lemma 4.2.2 (Vanishing lemma). *Let $f : X \rightarrow [0, \infty]$ be a measurable function, $E \in \mathcal{M}$:*

$$\int_E f d\mu = 0 \iff f = 0 \text{ a.e. on } E$$

Proof. The proof goes as follows:

(\Leftarrow) : Trivial

(\Rightarrow) : We have to show that:

$$\mu(\{x \in E : f(x) > 0\}) = 0$$

Let us define $F = \{x : f(x) > 0\} = \bigcup_n F_n$, where $F_n = \{x : f(x) \geq 1/n\}$. Then, we have that:

$$F_n \subset F_{n+1} \quad \forall n$$

so $F_n \uparrow F$. Then, we have that:

$$\mu(F_n) \rightarrow \mu(F)$$

and:

$$0 \leq \mu(F_n) = \mu(\{f \geq \frac{1}{n}\}) \leq \frac{1}{1/n} \int_E f d\mu = 0$$

Then, $\mu(F) = 0$. ■

Remark: The vanishing lemma applies to **every** f once $\mu(E) = 0$, indeed, every property is true a.e. on negligible sets. “The Lebesgue integral does not see negligible sets”.

Lemma 4.2.3. *Let $f : X \rightarrow [0, \infty]$ be a measurable function. Then:*

$$\int_X f d\mu < \infty \implies \mu(\{f = \infty\}) = 0$$

Proof. Exercise. (Hint: $\{f = \infty\} = \bigcap_n \{f \geq n\}$) ■

Theorem 4.2.4 (Monotone Convergence Theorem (MCT)). *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions $f_n : X \rightarrow [0, \infty]$. Assume that:*

$$(i) \quad f_n \leq f_{n+1} \quad \forall n$$

$$(ii) \quad \lim_{n \rightarrow \infty} f_n(x) = f(x) \quad \text{for a.e. } x \in X$$

Then, we have that:

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

Remark: All assumptions are essential

Proof. The proof goes as follows:

Part 1:

Assume that assumptions (i) and (ii) hold $\forall x \in X$. We have some basic facts:

- $f(x) = \lim_{n \rightarrow \infty} f_n(x) \implies f(x) \geq 0$ and measurable.
- $\int_X f_n d\mu \leq \int_X f_{n+1} d\mu$. Then, if we define:

$$\alpha_n = \int_X f_n d\mu, \quad \alpha = \lim_{n \rightarrow \infty} \alpha_n$$

we have that $\alpha_n \leq \alpha_{n+1}$, so $\alpha_n \uparrow \alpha$. Moreover, we have that:

$$\begin{aligned} f_n(x) \leq f(x) &\implies \int_X f_n d\mu \leq \int_X f d\mu \\ &\implies \alpha \leq \int_X f d\mu \end{aligned}$$

So, to complete part 1, we have to show that $\alpha \geq \int_X f d\mu$.

We use the definition of $\int_X f d\mu$:

Take any $s : X \rightarrow [0, \infty)$ simple, measurable and $0 \leq s \leq f$. Take also $0 \leq c < 1$. Then, we have that:

$$0 < c \cdot s \leq f$$

Take $f_n(x) \uparrow f(x) \forall x \in X$. Consider $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\} \in \mathcal{M}$. Then, we have that:

- (a) $E_n \subset E_{n+1}$: indeed, $x \in E_n \iff f_n(x) \geq c \cdot s(x) \implies f_{n+1}(x) \geq c \cdot s(x) \iff x \in E_{n+1}$
- (b) $\bigcup_n E_n = X$: indeed, either $f(x) = 0 \implies x \in E_n \forall n$ or $f(x) > 0$ and $c \cdot s(x) < f(x)$. Since $f_n(x) \uparrow f(x)$, we have that $\exists N_0$ s.t. $f_{N_0}(x) \geq c \cdot s(x)$. Then $x \in E_{N_0}$.

Then, we have that:

$$\begin{aligned} \alpha \geq \alpha_n &= \int_X f_n d\mu \geq \int_{E_n} c \cdot s d\mu = c \cdot \int_{E_n} s d\mu \\ &= c \cdot \phi(E_n) \end{aligned}$$

(where $\phi(E) = \int_E s d\mu$ is a measure). Then, notice that $E_n \uparrow X$, so $\phi(E_n) \rightarrow \phi(X)$.

Then, we have that:

$$\alpha \geq c \cdot \phi(X) = c \cdot \int_X s d\mu$$

Then, $\forall c < 1, \forall s$:

$$\alpha \geq c \int_X s d\mu$$

If we take the limit $c \rightarrow 1$, we have that $\alpha \geq \int_X s d\mu$. And if we take the supremum over all s , we have that:

$$\alpha \geq \int_X f d\mu$$

Part 2:

Now, we have to show that the result holds for *a.e.* $x \in X$. Define

$$F = \{x \in X : \text{either (i) or (ii) fails}\}$$

Then we have that $\mu(F) = 0$, and $E = X \setminus F$. For any g (non-negative, measurable), we have that:

$$g - \chi_E \cdot g = 0 \quad \text{a.e. on } X$$

Then, we use the vanishing lemma to show that:

$$\begin{aligned} \int_X (g - \chi_E \cdot g) d\mu &= 0 \\ \iff \int_X g d\mu &= \int_E g d\mu \end{aligned}$$

Finally:

$$\int_X f d\mu = \int_E f d\mu = \lim_{n \rightarrow \infty} \int_E f_n d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$$

■

Remark: Note that we now have 2 ways to compute the integral of a non-negative measurable function:

- $\int_X f d\mu = \sup \left\{ \int_X s d\mu : s \text{ simple, measurable and } 0 \leq s \leq f \right\}$
- $\int_X f d\mu = \lim_{n \rightarrow \infty} \int_X f_n d\mu$ where $f_n \uparrow f$ simple and measurable functions.

Corollary 4.2.4.1 (Monotone convergence for series). *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions $f_n : X \rightarrow [0, \infty]$. Then, we have that:*

$$\int_X \sum_n f_n d\mu = \sum_n \int_X f_n d\mu$$

Proposition 4.2.5. *Take $\Phi : X \rightarrow [0, \infty]$ measurable, $E \in \mathcal{M}$. Define:*

$$\nu(E) = \int_E \Phi d\mu$$

Then, ν is a measure on (X, \mathcal{M}) . Moreover, for $f : X \rightarrow [0, \infty]$ measurable:

$$\int_X f d\nu = \int_X f \cdot \Phi d\mu$$

Proof. The proof goes as follows:

- $\nu : \mathcal{M} \rightarrow [0, \infty]$: Trivial
- $\nu(\emptyset) = 0$: Trivial
- σ -additivity: Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint sets in \mathcal{M} , and $E = \bigcup_n E_n$. Then, we have that:

$$\begin{aligned} \nu(E) &= \int_E \Phi \, d\mu = \int_X \Phi \cdot \chi_E \, d\mu = \sum_n \int_X \Phi \cdot \chi_{E_n} \, d\mu \\ &= \sum_n \int_{E_n} \Phi \, d\mu = \sum_n \nu(E_n) \end{aligned}$$

■

Lemma 4.2.6 (Fatou). *Let (X, \mathcal{M}, μ) be a complete measure space, and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions. Then:*

$$\int_X \liminf_n f_n \, d\mu \leq \liminf_n \int_X f_n \, d\mu$$

Proof. Recall that:

$$\begin{aligned} \liminf_n f_n &= \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} f_k \right) \\ &= \sup_n \left(\inf_{k \geq n} f_k \right) \end{aligned}$$

Then, we define:

$$g_n = \inf_{k \geq n} f_k$$

We have the following properties $\forall n$:

- g_n is measurable.
- $g_n \geq 0$
- $g_n \leq g_{n+1}$
- $g_n \leq f_n$

Then, by the MCT, we have that:

$$\begin{aligned}\int_X \liminf_n f_n d\mu &= \int_X \lim_n g_n d\mu = \lim_n \int_X g_n d\mu \\ &= \liminf_n \int_X g_n d\mu \leq \liminf_n \int_X f_n d\mu\end{aligned}$$

■

4.3 Integral of real-valued measurable functions

Let $f : X \rightarrow \mathbb{R}$ be a measurable function. Then, we can write $f = f^+ - f^-$, where:

$$f^+(x) = \max\{f(x), 0\} \quad f^-(x) = \max\{-f(x), 0\}$$

Notice that $f^+, f^- \geq 0$ are measurable functions. Then, we define:

$$|f| = f^+ + f^-$$

We also notice that $|f| = f^+ + f^- \geq 0$ is measurable.

Definition 4.3.1. We say $f : X \rightarrow \mathbb{R}$ is **integrable** on X if it is measurable and:

$$\int_X |f| d\mu < \infty$$

We define the set of **integrable functions** as:

$$\mathcal{L}^1(X, \mathcal{M}, \mu) = \{f : X \rightarrow \mathbb{R} : f \text{ is integrable}\}$$

For $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, and $E \in \mathcal{M}$, we define:

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu$$

Proposition 4.3.1. Let $f : X \rightarrow \mathbb{R}$ be a measurable function. Then:

$$(i) \quad f \in \mathcal{L}^1 \iff |f| \in \mathcal{L}^1 \iff (f^+ \in \mathcal{L}^1 \text{ and } f^- \in \mathcal{L}^1)$$

(ii) (Triangular inequality):

$$\left| \int_E f d\mu \right| \leq \int_E |f| d\mu$$

Proof. The proof goes as follows:

- (i) Trivial (but see next remark)
- (ii) We have that:

$$\begin{aligned}
 \left| \int_E f \, d\mu \right| &= \left| \int_E f^+ \, d\mu - \int_E f^- \, d\mu \right| \\
 &\leq \left| \int_E f^+ \, d\mu \right| + \left| \int_E f^- \, d\mu \right| = \int_E f^+ \, d\mu + \int_E f^- \, d\mu \\
 &= \int_E f^+ + f^- \, d\mu = \int_E |f| \, d\mu
 \end{aligned}$$

■

Remark: In general, it is not true that $|f|$ measurable $\implies f$ measurable. Take $F \subset X$, $F \notin \mathcal{M}$ and:

$$f(x) = \chi_F(x) - \chi_{X \setminus F}(x)$$

Then, $|f| = 1$ is measurable, but f is not.

Proposition 4.3.2. *We propose two properties:*

- (i) $\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a (real) vector space.
- (ii) The functional

$$I(\cdot) := \int_X \cdot \, d\mu : \mathcal{L}^1(X, \mathcal{M}, \mu) \rightarrow \mathbb{R}$$

is a linear functional.

Proof. The proof sketch goes as follows:

Let $u, v \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, $\alpha, \beta \in \mathbb{R}$. We should show that:

$$\alpha u + \beta v \in \mathcal{L}^1(X, \mathcal{M}, \mu)$$

since:

$$|\alpha u + \beta v| \leq |\alpha u| + |\beta v|$$

Then:

$$\int_X (\alpha u + \beta v) d\mu \leq \int_X |\alpha u + \beta v| d\mu \leq \int_X |\alpha u| d\mu + \int_X |\beta v| d\mu < \infty$$

since $|\alpha u|, |\beta v| \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Then, we have that $\alpha u + \beta v \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.

For the second property, we have that:

$$I(\alpha u + \beta v) = \int_X (\alpha u + \beta v) d\mu = \alpha \int_X u d\mu + \beta \int_X v d\mu = \alpha I(u) + \beta I(v)$$

■

Remark: All the other basic properties of the integral of non-negative functions can be extended to the integral of real-valued functions.

Theorem 4.3.3 (Vanishing lemma). *Let (X, \mathcal{M}, μ) be a complete measure space, and $f, g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Then:*

$$f = g \text{ a.e.} \iff \int_X |f - g| d\mu = 0 \iff \int_E (f - g) d\mu = 0 \forall E \in \mathcal{M}$$

Proof. The “difficult” part of the proof is:

$$\int_E (f - g) d\mu = 0, \quad \forall E \in \mathcal{M} \implies f = g \text{ a.e.}$$

The proof goes as follows:

Let $E_1 = \{f \geq g\}$, and $E_2 = X \setminus E_1$. Then, we have that:

$$\begin{aligned} 0 &= \int_{E_1} (f - g) d\mu = \int_{E_1} (f - g)^+ d\mu \\ 0 &= \int_{E_2} (f - g) d\mu = - \int_{E_2} (f - g)^- d\mu \end{aligned}$$

Then, we have that:

$$(f - g)^+ = 0 \text{ and } (f - g)^- = 0 \text{ a.e. on } X$$

■

Remark: In particular, for $u \in \mathcal{L}^1$:

$$\int_E u d\mu = 0 \quad \forall E \in \mathcal{M} \implies u = 0 \text{ a.e.}$$

This is the same as:

$$\int_X u \varphi d\mu = 0 \quad \forall \varphi \text{ characteristic function} \implies u = 0 \text{ a.e.}$$

This can be true also replacing φ by “something else”. For instance, in the case of $u \in \mathcal{L}^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$:

$$\int_{\mathbb{R}} u \varphi d\lambda = 0 \quad \forall \varphi \in V \implies u = 0 \text{ a.e.}$$

where $V = \{C_0^\infty(\mathbb{R})\}$, or $V = \{C_0^0(\mathbb{R})\}$.

This is the “fundamental lemma of calculus of variations”.

Theorem 4.3.4 (Dominated convergence theorem (DCT)). *Let (X, \mathcal{M}, μ) be a complete measure space and $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions $f_n : X \rightarrow \mathbb{R}$, and $f : X \rightarrow \mathbb{R}$. Assume that:*

$$(i) \quad |f_n| \leq g \text{ a.e. on } X, \quad \forall n \in \mathbb{N}, \text{ where } g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$$

$$(ii) \quad \lim_{n \rightarrow \infty} f_n(x) = f(x) \text{ for a.e. } x \in X$$

Then, $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, and:

$$\lim_{n \rightarrow \infty} \int_X |f_n - f| d\mu = 0$$

In particular:

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X f d\mu$$

Proof. First, we have 2 basic facts:

1. $|f_n| \leq g$ a.e. on X , $\forall n \in \mathbb{N} \implies f_n \in \mathcal{L}^1(X, \mathcal{M}, \mu)$
2. $|f| \leq g$ a.e. on $X \implies f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$

Then, consider the sequence $h_n = 2g - |f_n - f|$. We have that:

- h_n is measurable.
- $h_n \leq 2g$

- $h_n \geq 0$. Indeed:

$$|f_n - f| \leq |f_n| + |f| \leq 2g \implies 2g - |f_n - f| \geq 0$$

We now apply the Fatou's lemma to the sequence h_n :

$$\begin{aligned} \int_X (\liminf_n h_n) d\mu &\leq \liminf_n \int_X h_n d\mu \\ &= \int_X 2g d\mu - \limsup_n \int_X |f_n - f| d\mu \end{aligned}$$

Also, notice that:

$$\liminf_n h_n = 2g$$

Then, we have that:

$$\begin{aligned} \int_X 2g d\mu &\leq \int_X 2g d\mu - \limsup_n \int_X |f_n - f| d\mu \\ \implies \limsup_n \int_X |f_n - f| d\mu &\leq 0 \end{aligned}$$

Then, we have that:

$$\limsup_n \int_X |f_n - f| d\mu \geq \liminf_n \int_X |f_n - f| d\mu \geq 0$$

In the end:

$$\lim_n \int_X |f_n - f| d\mu = 0$$

■

Remark: If $\mu(X) < \infty$, then the constants are integrable. Then, if $|f_n(x)| \leq M$ a.e, for some $M \in \mathbb{R}$, then:

$$\lim_{n \rightarrow \infty} \int_X f_n d\mu = \int_X \lim_{n \rightarrow \infty} f_n d\mu$$

(We are using the DCT with $g = M$)

Corollary 4.3.4.1 (Dominated Convergence for series). *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of measurable functions $f_n : X \rightarrow \mathbb{R}$, s.t. $f_n \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. If $\sum_n \int_X |f_n| d\mu < \infty$, then:*

$$\int_X \sum_n f_n d\mu = \sum_n \int_X f_n d\mu$$

4.4 Comparison between Riemann and Lebesgue integrals

Theorem 4.4.1. *Let $I = [a, b] \subset \mathbb{R}$ be a closed interval, and $f : I \rightarrow \mathbb{R}$. If f is **Riemann integrable** on I , then f is **Lebesgue integrable** on I , i.e., $f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$, and the two integrals coincide:*

$$\int_I f d\lambda = \int_a^b f(x) dx$$

Theorem 4.4.2. *Let $I = (\alpha, \beta)$, such that $-\infty \leq \alpha < \beta \leq \infty$. If $|f|$ is **Riemann integrable** on I (in the generalized sense), then f is **Lebesgue integrable** on I :*

$$\int_I f d\lambda = \int_\alpha^\beta f(x) dx$$

Remark: If the generalized Riemann integral of $|f|$ diverges, then:

$$\int_I |f| d\lambda = \infty$$

but $\int_I f d\lambda$ is not defined (unless $f = \pm|f|$) and:

$$\int_\alpha^\beta f(x) dx \text{ and } \int_I f d\lambda$$

are not related.

4.5 Spaces of integrable functions

For a (X, \mathcal{M}, μ) complete measure space, we already know that $\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a vector space. We can also define a distance in this space:

$$d(f, g) = \int_X |f - g| d\mu$$

Immediately, we have that:

- **Symmetry:** $d(f, g) = d(g, f)$
- **Triangle inequality:** $d(f, g) \leq d(f, h) + d(h, g)$
- **Non-negativity:** $d(f, g) \geq 0$

But notice that $d(f, g) = 0$ does not imply $f = g$ (only a.e.). This means that $d(f, g)$ is a **pseudo-distance**.

To solve this, we can define an equivalence relation:

$$f \sim g \iff f = g \text{ a.e.}$$

With this equivalence relation, we can define the following space:

Definition 4.5.1. We define the space $L^1(X, \mathcal{M}, \mu)$ as:

$$L^1(X, \mathcal{M}, \mu) = \{[f] : f \in \mathcal{L}^1(X, \mathcal{M}, \mu)\}$$

where $[f]$ is the equivalence class of f defined as:

$$[f] = \{g : g = f \text{ a.e.}\}$$

Remark: We can define the distance in $L^1(X, \mathcal{M}, \mu)$ as:

$$d([f], [g]) = \int_X |f - g| d\mu$$

This distance is well-defined, and it is a true distance. Then, $(L^1(X), d)$ is a metric space.

Note: We understand that elements of L^1 are functions: instead of $[u]$, we work with a representant u , and we can **only** use operations/properties that are **independent of the representant**.

E.g.: $X = (0, 1)$, we work on $(X, \mathcal{L}(X), \lambda)$. If we take $u \in L^1(X)$, we have the following:

- $u \geq 0$ in X : **NOT** well-defined
- $u \geq 0$ a.e. on X : **GOOD**
- $u(1/2)$: **NOT** well-defined
- $\int_{[0,1/2]} u \, d\lambda$: **GOOD**

Definition 4.5.2. Let $f : X \rightarrow \mathbb{R}$ be a measurable function. We say it is **essentially bounded** if:

$$\exists M \in \mathbb{R} : |f(x)| \leq M \text{ a.e. on } X$$

i.e.:

$$\mu(\{x \in X : |f(x)| > M\}) = 0$$

E.g.: Two examples:

$$f(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \in (0, 1] \end{cases} \text{ is essentially bounded}$$

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x \in (0, 1] \end{cases} \text{ is not essentially bounded}$$

Definition 4.5.3. If $f : X \rightarrow \mathbb{R}$ is essentially bounded, we define the **essential supremum** of f as:

$$\text{ess sup } f := \inf\{M \in \mathbb{R} : \mu(\{f > M\}) = 0\}$$

Definition 4.5.4. We define the space $\mathcal{L}^\infty(X, \mathcal{M}, \mu)$ as:

$$\mathcal{L}^\infty(X, \mathcal{M}, \mu) = \{f : X \rightarrow \mathbb{R} : f \text{ is essentially bounded}\}$$

We can also define the space $L^\infty(X, \mathcal{M}, \mu)$ as:

$$L^\infty(X, \mathcal{M}, \mu) = \{[f] : f \in \mathcal{L}^\infty(X, \mathcal{M}, \mu)\}$$

where $[f]$ is the equivalence class of f defined as:

$$[f] = \{g : g = f \text{ a.e.}\}$$

Remark: One can prove that $L^\infty(X, \mathcal{M}, \mu)$ is a vector space, with the distance:

$$d([f], [g]) = \text{ess sup } |f - g|$$

Chapter 5

Types of convergence

We have various types of convergence for sequences of measurable functions:

Definition 5.0.1. Let $f_n : X \rightarrow \overline{\mathbb{R}}$ be a sequence of measurable functions, that converges to a function $f : X \rightarrow \overline{\mathbb{R}}$. We say that the convergence is a:

- **Pointwise convergence:**

$$f_n(x) \rightarrow f(x) \quad \forall x \in X$$

- **Uniform convergence:**

$$\sup_{x \in X} |f_n(x) - f(x)| \rightarrow 0$$

- **Convergence a.e.:**

$$f_n(x) \rightarrow f(x) \quad \text{a.e. } x \in X$$

- **L^1 -convergence:**

$$\int_X |f_n - f| d\mu \rightarrow 0$$

- **L^∞ -convergence:**

$$\text{ess sup}_X |f_n - f| \rightarrow 0$$

- **Convergence in measure:**

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \rightarrow 0 \quad \forall \epsilon > 0$$

Remark: Basic relations:

Uniform convergence \Rightarrow Pointwise convergence \Rightarrow Convergence a.e.

Exercise: Let $([0, 1], \mathcal{L}([0, 1]), \lambda)$ be the Lebesgue measure space. Let:

$$f_n(x) = e^{-nx} \quad 0 \leq x \leq 1$$

$$g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0, 1] \end{cases}$$

$$f(x) = 0 \quad 0 \leq x \leq 1$$

Show that:

- $f_n \rightarrow f$ a.e.
- $f_n \not\rightarrow f$ pointwise
- $f_n \rightarrow g$ pointwise
- $f_n \not\rightarrow g$ uniformly

5.1 a.e. convergence and convergence in measure

Theorem 5.1.1. Let $\mu(X) < \infty$, f_n, f measurable functions, a.e. finite in X . If $f_n \rightarrow f$ a.e., then $f_n \rightarrow f$ in measure.

Remark: if $\mu(X) = \infty$, then the theorem may not hold. For instance, consider $X = \mathbb{R}$, with the Lebesgue measure, and:

$$f_n(x) = \chi_{[n, \infty)}(x) = \begin{cases} 1 & x \geq n \\ 0 & x < n \end{cases}$$

We can show that $f_n(x) \rightarrow 0$ a.e., but $\lambda(\{f_n \geq 1/2\}) = \infty \forall n$ and thus $f_n \not\rightarrow 0$ in measure.

Also notice that convergence in measure **does not imply** convergence a.e., even if $\mu(X) < \infty$. For instance, consider the “**typewriter sequence**”.

Theorem 5.1.2. Let f_n, f be measurable functions, a.e. finite in X . If $f_n \rightarrow f$ in measure, then there exists a subsequence f_{n_k} that converges to f a.e.

5.2 Convergence in L^1 and convergence in measure

Theorem 5.2.1. *Let f_n, f be measurable functions in $L^1(X, \mathcal{M}, \mu)$. If $f_n \rightarrow f$ in L^1 , then $f_n \rightarrow f$ in measure.*

Proof. Assume by contradiction that $f_n \not\rightarrow f$ in measure. Then $\exists \alpha > 0$ s.t.:

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \alpha\}) \not\rightarrow 0$$

I.e., $\exists \epsilon > 0$ and a subsequence f_{n_k} s.t.:

$$\mu(\{x \in X : |f_{n_k}(x) - f(x)| \geq \alpha\}) \geq \epsilon \quad \forall k$$

Let us call $E_k = \{x \in X : |f_{n_k}(x) - f(x)| \geq \alpha\}$. On the other hand, by assumption, $f_{n_k} \rightarrow f$ in L^1 . But notice that:

$$\int_X |f_{n_k} - f| d\mu \geq \int_{E_k} |f_{n_k} - f| d\mu \geq \alpha \mu(E_k) \geq \alpha \epsilon > 0$$

Since $f_{n_k} \rightarrow f$ in L^1 , we have that $\int_X |f_{n_k} - f| d\mu \rightarrow 0$. But we have just shown that $\int_X |f_{n_k} - f| d\mu \geq \alpha \epsilon > 0$. This is a contradiction, and thus $f_n \rightarrow f$ in measure. ■

Remark: In general, convergence in measure does not imply convergence in L^1 . For instance, consider $X = [0, 1]$, $\mathcal{M} = \mathcal{L}([0, 1])$, μ the Lebesgue measure, and $f_n(x) = n\chi_{[0, 1/n]}(x)$. We can show that $f_n \rightarrow 0$ in measure, but $\int_X |f_n - 0| d\mu = 1 \quad \forall n$.

5.3 Convergence in L^1 and a.e. convergence

In general, they are not related. But we have 2 main results: **Dominating convergence theorem** that we already saw, and the “**Reverse Dominating Convergence Theorem**”, that states:

Theorem 5.3.1. *Let $f_n \rightarrow f$ in $L^1(X, \mathcal{M}, \mu)$, then there exists a subsequence f_{n_k} that converges to f a.e., and there exists a function $g \in L^1(X, \mathcal{M}, \mu)$ s.t. $|f_{n_k}| \leq g$ a.e. $\forall k$.*

Chapter 6

Absolutely continuous functions and Functions of bounded variations

6.1 Fundamental theorems of calculus

Let $(X, \mathcal{L}(X), \lambda)$ be a complete measure space, such that $X = \mathbb{R}$ or $X = I \subset \mathbb{R}$ an interval. Take $f \in L^1(a, b)$. We can define the **integral function**:

$$F(x) = \int_{[a, x]} f \, d\mu = \int_a^x f(t) \, dt$$

If $f \in C([a, b])$, then:

- $F \in C^1([a, b])$
- $F'(x) = f(x)$
- $F(x) - F(y) = \int_y^x f(t) \, dt$

What if only $f \in L^1(a, b)$?

6.1.1 1st Fundamental Theorem of Calculus

Theorem 6.1.1 (1st Fundamental Theorem of Calculus). *Let $f \in L^1(a, b)$. If we define:*

$$F(x) = \int_a^x f(t) dt$$

then:

- F is differentiable at a.e. $x \in [a, b]$
- $F'(x) = f(x)$ a.e. $x \in [a, b]$

E.g.: Take $[a, b] = [-1, 1]$ and:

$$f(x) = \mathcal{H}(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

This is the Heaviside function. Notice that $\mathcal{H} \in L^1(-1, 1)$. Now:

$$F(x) = \int_{-1}^x \mathcal{H}(t) dt = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}$$

Also, if we define:

$$f(x) = \begin{cases} \mathcal{H}(x) & x \notin \mathbb{Q} \\ \infty & x \in \mathbb{Q} \end{cases}$$

we get the same F .

Note: For the proof, we need a deep result due to Lebesgue. We go back to $\mathcal{L}^1([a, b])$.

Definition 6.1.1. Let $f \in \mathcal{L}^1([a, b])$. we say $x \in [a, b]$ is a **Lebesgue point** for f if:

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt = 0$$

Note that if $x = a$ then $h \rightarrow 0^+$ and if $x = b$, then $h \rightarrow 0^-$.

Remark: If x is a LP, then:

$$\begin{aligned}
0 &= \lim_{h \rightarrow 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt \\
&\geq \lim_{h \rightarrow 0} \left| \frac{1}{h} \int_x^{x+h} (f(t) - f(x)) dt \right| \\
&= \left| \left(\lim_{h \rightarrow 0} \int_x^{x+h} f(t) dt \right) - f(x) \right|
\end{aligned}$$

i.e., LP is related with the validity of a local mean value theorem at x

Remark: We have the following:

- f is continuous $\implies x$ is a LP.
- $f \in C([a, b]) \implies$ every $x \in [a, b]$ is a LP.
- Take $\mathcal{H}(x)$, then $x = 0$ is not a LP.

Theorem 6.1.2 (Lebesgue). *Let $f \in \mathcal{L}^1([a, b])$. Then, a.e. $x \in [a, b]$ is a Lebesgue point.*

Remark: By consequence of the theorem, it makes sense to consider Lebesgue points in L^1 . Indeed, changing the representative of the function class in L^1 maintains the same set of Lebesgue points up to a negligible set.

Note: To prove the **1st fund. thm.**, we will show that:

- F is differentiable at x .
- $F'(x) = f(x)$

for all x Lebesgue points for f .

Proof: (1st fund. thm.) Take $x \in [a, b]$ a LP of f . Then:

$$\begin{aligned}
0 &\leq \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \\
&= \left| \frac{\int_a^{x+h} f(t) dt - \int_a^x f(t) dt}{h} - f(x) \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| \frac{1}{h} \int_x^{x+h} f(t) dt - \frac{1}{h} \int_x^{x+h} f(x) dt \right| \\
&\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt \rightarrow 0 \quad \text{as } h \rightarrow 0
\end{aligned}$$

because x is a LP. ■

Remark: Let us try to reverse the point of view: take $g : [a, b] \rightarrow \mathbb{R}$, and assume that g is differentiable a.e. in $[a, b]$, and that $g' \in L^1([a, b])$. Is g related with $\int_a^x g'(t) dt$? The answer is **NO!**

E.g.: $\mathcal{H} : [-1, 1] \rightarrow \mathbb{R}$ and notice that:

$$\mathcal{H}'(x) = \begin{cases} \nexists & x = 0 \\ 0 & x \neq 0 \end{cases}$$

We have that $\mathcal{H}' = 0$ a.e. in $[-1, 1]$, and $0 \in L^1([-1, 1])$. But:

$$\mathcal{H}(1) - \mathcal{H}(0) = 1 - 0 = 1 \neq 0 = \int_{-1}^1 0 dt = \int_{-1}^1 \mathcal{H}'(t) dt$$

Other example with the Cantor-Vitali function:

$$g(x) = v(x), \quad \text{s.t. } v(0) = 0, v(1) = 1 \quad \text{and constant outside the Cantor set}$$

Then, v is differentiable and $v'(x) = 0$ a.e., but we can notice that the same thing as before happens.

Definition 6.1.2. Let I be an interval. We say that $f : I \rightarrow \mathbb{R}$ is an **absolutely continuous function**, $f \in AC(I)$, if:

$\forall \varepsilon > 0, \exists \delta$ s.t., $\forall n \in \mathbb{N}, \forall$ family of n disjoint subintervals of I , i.e., $(a_i, b_i) \subset I$ s.t. $\dots b_{i-1} \leq a_i < b_i \leq a_{i+1} < \dots$ we have that:

$$\lambda \left(\bigcup_{i=1}^n (a_i, b_i) \right) < \delta \implies \sum_{i=1}^n |f(b_i) - f(a_i)| \leq \varepsilon$$

Remark: Recall that f is uniformly continuous (UC) if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, y \in I$$

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

(The choice of δ is independent of x, y)

Then:

$$UC(I) \supset AC(I)$$

Recall that f is Lipschitz continuous if $\exists L > 0$ s.t.:

$$\forall x, y \in I, |f(x) - f(y)| \leq L|x - y|$$

Then:

$$Lip(I) \subset AC(I)$$

We will see that:

$$Lip(I) \subsetneq AC(I) \subsetneq UC(I)$$

We will also see that, as $g' \in C \iff g \in C^1$, we have that:

$$g' \in L^1 \iff g \in AC$$

6.1.2 2nd Fundamental Theorem of Calculus

Theorem 6.1.3 (2nd Fundamental Theorem of Calculus). *Let $g : [a, b] \rightarrow \mathbb{R}$. The following are equivalent:*

$$(i) \ g \in AC([a, b])$$

(ii) g is differentiable a.e. in $[a, b]$, $g' \in L^1([a, b])$ and:

$$g(x) - g(y) = \int_y^x g'(t) dt \quad \forall x, y \in [a, b]$$

Corollary 6.1.3.1. $f \in L^1([a, b]) \implies F(x) = \int_a^x f(t) dt \in AC([a, b])$

Note: To prove one implication of the theorem, we will need some few extra results.

Theorem 6.1.4 (Absolute continuity of the integral function). *Let $f \in L^1([a, b])$. Then, $\forall \varepsilon > 0, \exists \delta > 0$ s.t.:*

$$\begin{cases} E \in \mathcal{M} \\ \mu(E) < \delta \end{cases} \implies \int_E |f| d\mu < \varepsilon$$

Proof. By contradiction: assume that $\exists \varepsilon > 0$ s.t. $\forall \delta > 0, \exists E \in \mathcal{M}$ s.t. $\mu(E) < \delta$ and $\int_E |f| d\mu \geq \varepsilon$.

In particular, $\delta = 1/2^n \rightarrow 0, E_n = E_{\delta_n}$ and:

$$F_n = \bigcup_{k=n}^{\infty} E_k = E_n \cup F_{n+1}, \quad F = \lim_{n \rightarrow \infty} F_n$$

Then:

1.

$$(F_{n+1} \subset F_n) \implies \{F_n\} \downarrow F$$

2.

$$\forall n, \quad \mu(F_n) \leq \sum_{k=n}^{\infty} \mu(E_k) \leq \sum_{k=n}^{\infty} \delta_k = \sum_{k=n}^{\infty} \frac{1}{2^k} = 2^{-n+1}$$

3.

$$\nu(F_n) = \int_{F_n} |f| d\mu \geq \int_{E_n} |f| d\mu \geq \varepsilon \quad \forall n$$

Moreover:

$$\nu(F_1) = \int_{F_1} |f| d\mu \leq \int_X |f| d\mu < \infty$$

Use continuity of measures:

$$(1) + (2) \implies \nu(F) = \lim_{n \rightarrow \infty} \nu(F_n) = 0$$

$$(1) + (3) \implies \nu(F) = \lim_{n \rightarrow \infty} \nu(F_n) \geq \varepsilon > 0$$

Contradiction, since $\nu(F) = 0$. ■

Remark: As a consequence, we have:

$$f \in L^1([a, b]) \implies F(x) = \int_a^x f(t) dt \in AC([a, b])$$

Proof. Take $\varepsilon > 0$, and $\delta = \delta(\varepsilon)$ as in the theorem. I know:

$$\begin{cases} \forall E \in \mathcal{L}([a, b]) \\ \lambda(E) < \delta \end{cases} \implies \int_E |f| d\lambda < \varepsilon$$

Take $E = \bigcup_{i=1}^n (a_i, b_i)$, s.t. (a_i, b_i) disjoint intervals. Then:

$$\begin{aligned} \sum_{i=1}^n |F(b_i) - F(a_i)| &= \sum_{i=1}^n \left| \int_{a_i}^{b_i} f(t) dt \right| \leq \sum_{i=1}^n \int_{a_i}^{b_i} |f| dt \\ &= \int_{\bigcup_{i=1}^n (a_i, b_i)} |f| dt < \varepsilon \end{aligned}$$

■

E.g. ((AC \nRightarrow Lip)): Consider $g(x) = \sqrt{x}$ in $[0, 1]$. Then:

$$\sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}} dt$$

and $g \in AC([0, 1])$. But notice that $g \notin Lip([0, 1])$.

$$\left| \frac{\sqrt{x} - \sqrt{0}}{x - 0} \right| \not\leq C$$

for any $C > 0$, as $x \rightarrow 0$.

E.g. ((UC \nRightarrow AC)): Consider:

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is continuous in $[0, 1] \implies f \in UC([0, 1])$. But notice that $f \notin AC([0, 1])$. Indeed:

$$f'(x) = \sin(1/x) - \frac{1}{x} \cos(1/x)$$

and $1/x \cos(1/x)$ is not integrable in $[0, 1]$, i.e., $f' \notin L^1([0, 1])$.

6.2 AC functions and weak derivatives

Proposition 6.2.1 (Integration by parts in AC). *Let $u : [a, b] \rightarrow \mathbb{R}$. Then, $u \in AC([a, b])$ if and only if:*

- $u \in C([a, b])$
- u is differentiable a.e. in $[a, b]$
- $u' \in L^1([a, b])$
-

$$\int_a^b u' \varphi dx = - \int_a^b u \varphi' dx \quad \forall \varphi \in C_0^\infty([a, b])$$

Definition 6.2.1 (Weak derivative). Let $u \in L^1(a, b)$. We say that $u \in W^{1,1}(a, b) \iff \exists w \in L^1(a, b)$ s.t.:

$$\int_a^b u \varphi' dx = - \int_a^b w \varphi dx \quad \forall \varphi \in C_0^\infty(a, b)$$

Such w is called the **weak derivative** of u .

Remark: Both u and $w = u'$ are equivalence classes of functions, i.e., $u \sim v \iff u = v$ a.e. Properties should be independent of the representative.

Remark: If such a w exists, it is unique (up to a.e. equivalence). Indeed, assume that w_1, w_2 are weak derivatives of u . Then:

$$\begin{aligned} \int_a^b (w_1 - w_2) \varphi dx &= 0 \quad \forall \varphi \in C_0^\infty(a, b) \\ \implies w_1 - w_2 &= 0 \text{ a.e.} \end{aligned}$$

Remark: In principle, the pointwise and weak derivatives are different objects, and the notation u' may be misleading. But we know that if $u \in AC([a, b])$ they coincide.

Remark: In principle, the definition of weak derivatives can be extended (measures, distributions). Take:

$$\mathcal{H}(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Then:

$$\begin{aligned} - \int_{-1}^1 \mathcal{H}(x) \varphi'(x) dx &= - \int_0^1 \varphi'(x) dx = \varphi(0) - \varphi(1) \\ &= \varphi(0) = \int_{[-1,1]} \varphi(x) d\delta_0 \end{aligned}$$

where δ_0 is the Dirac delta function. This suggest that:

$$\begin{aligned} \mathcal{H}' &= \delta_0 \text{ weakly} \\ \mathcal{H}' &= 0 \text{ pointwise} \end{aligned}$$

Theorem 6.2.2. $u \in AC([a, b]) \iff u \in W^{1,1}(a, b)$

Proof. The proof goes as follows:

(\Rightarrow) Already proved.

(\Leftarrow) Assume that u' weak derivative of u , $u' \in L^1(a, b)$. Then:

$$z(x) = \int_a^x u'(t) dt, \quad z \in AC$$

We can show that $u = z + c$ for some constant c .

■

Chapter 7

Derivatives of measures

Let (X, \mathcal{M}, μ) be a complete measure space. We know that, given $\Phi : X \rightarrow [0, \infty]$ measurable, the function:

$$\nu_\Phi(E) := \int_E \Phi d\mu = \int_E d\nu_\Phi$$

is a measure on (X, \mathcal{M}) . Given μ, ν measures on (X, \mathcal{M}) , is it true that there exists Φ such that

$$\nu(E) = \int_E \Phi d\mu \quad \forall E \in \mathcal{M}$$

We will study this question in this chapter.

Definition 7.0.1. Let μ, ν measures on (X, \mathcal{M}) . If $\exists \Phi$ s.t

$$\nu(E) = \int_E \Phi d\mu \quad \forall E \in \mathcal{M}$$

then Φ is the **Radon-Nikodym derivative** of ν with respect to μ and we write:

$$\Phi = \frac{d\nu}{d\mu}$$

Definition 7.0.2. Let μ, ν measures on (X, \mathcal{M}) . Then ν is **absolutely continuous** with respect to μ (" $\nu << \mu$ ") if:

$$\forall E \in \mathcal{M}, \quad \mu(E) = 0 \Rightarrow \nu(E) = 0$$

Lemma 7.0.1 (Necessary condition). *Let μ, ν measures on (X, \mathcal{M}) . If ν has a Radon-Nikodym derivative with respect to μ , then ν is absolutely continuous with respect to μ .*

Proof. Assume ν has a Radon-Nikodym derivative with respect to μ . Then:

$$\nu(E) = \int_E \Phi d\mu = 0$$

■

Exercise: Take $(X, \mathcal{M}) = (\mathbb{R}, \mathcal{L}(\mathbb{R}))$, $\mu = \lambda$ the Lebesgue measure and $\nu = \delta_0$ the Dirac measure at 0. Show that

$$\nexists \frac{d\nu}{d\mu}$$

7.1 The Radon-Nikodym Theorem

Theorem 7.1.1 (Radon-Nikodym Theorem). *Let (X, \mathcal{M}) be a measurable space, μ, ν measures and μ is σ -finite. Then:*

$$\nu \ll \mu \iff \exists \frac{d\nu}{d\mu}$$

Corollary 7.1.1.1. *Let ν be a measure on $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N))$ and $\mu \ll \lambda$. Then:*

$$\exists \Phi := \frac{d\nu}{d\mu} : \quad \nu(E) = \int_E \Phi d\lambda \quad \forall E \in \mathcal{L}(\mathbb{R}^N)$$

(Indeed, λ is σ -finite)

Chapter 8

Banach spaces

8.1 Normed and Banach spaces

Definition 8.1.1. Let X be a (real) vector space. A **norm** on X is a function $\|\cdot\| : X \rightarrow \mathbb{R}$ such that:

- (i) $\|x\| > 0$ for all $x \in X$ and $\|x\| = 0 \iff x = 0$.
- (ii) $\|\alpha x\| = |\alpha| \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$.
- (iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a **normed space**.

Proposition 8.1.1. Let $(X, \|\cdot\|)$ be a normed space. Then:

$$d(x, y) = \|x - y\|$$

is a metric on X , i.e., (X, d) is a metric space.

Proposition 8.1.2. Let $\{x_n\}_n$ be a sequence in a normed space $(X, \|\cdot\|)$. Then:

- (i) We say $x_n \rightarrow x$ if $\|x_n - x\| \rightarrow 0$ as $n \rightarrow \infty$.
- (ii) For $f : X \rightarrow Y$, $(X, Y$ normed spaces), we say f is continuous at $x \in X \iff :$

$$\forall \{x_n\}_n : x_n \rightarrow x \in X \implies f(x_n) \rightarrow f(x) \in Y$$

Exercise: Show that:

- (i) $|||x|| - ||y||| \leq \|x - y\|$
- (ii) $\|\cdot\| : X \rightarrow \mathbb{R}$ is continuous in X .

Definition 8.1.2. We say $\{x_n\}_n$ is a **Cauchy sequence** (or **fundamental sequence**) if $\|x_n - x_m\| \rightarrow 0$ as $n, m \rightarrow \infty$. I.e., :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : n, m \geq N \implies \|x_n - x_m\| < \varepsilon$$

Remark: If $\{x_n\}_n$ converges, then it is a Cauchy sequence. The converse is not true in general.

Definition 8.1.3. A normed vector space $(X, \|\cdot\|)$ is called a **Banach space** if it is complete, i.e., every Cauchy sequence in X converges to a point in X .

E.g.: The following are examples of Banach spaces:

- (i) $X = \mathbb{R}^n$ with $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $1 \leq p < \infty$, $\|x\|_\infty = \max_i |x_i|$, are Banach spaces.
- (ii) $X = C([a, b])$ with $\|u\| = \max_{x \in [a, b]} |u(x)|$ is a Banach space.
- (iii) $X = C^k([a, b])$ with $\|u\| = \sum_{i=0}^k \max_{x \in [a, b]} |u^{(i)}(x)|$ is a Banach space.

Remark: Let $(X, \|\cdot\|)$ normed vector space, $\{x_n\}_n \subset X$. We can deal with series:

$$\sum_{n=1}^{\infty} x_n = y \iff s_k = \sum_{n=1}^k x_n, \quad s_k \rightarrow y \text{ as } k \rightarrow \infty$$

For numerical series, $\{a_n\}_n \subset \mathbb{R}$, we have:

$$\sum_{n=1}^{\infty} |a_n| < \infty \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

This is not true in general for series in normed spaces.

Proposition 8.1.3. $(X, \|\cdot\|)$ is a Banach space \iff every absolutely convergent series in X converges. I.e., if:

$$\forall \{x_n\}_n \subset X : \sum_{n=1}^{\infty} \|x_n\| < \infty \implies \sum_{n=1}^{\infty} x_n \text{ converges}$$

8.2 Equivalent/non equivalent norms

Definition 8.2.1. Let X be a vector space, and $\|\cdot\|_a, \|\cdot\|_b$ be two norms on X . We say $\|\cdot\|_a$ and $\|\cdot\|_b$ are **equivalent** if there exist $0 < c_1 \leq c_2 < \infty$ such that:

$$c_1 \|x\|_a \leq \|x\|_b \leq c_2 \|x\|_a \quad \forall x \in X$$

In particular, we say that they induce the same topology on X .

Theorem 8.2.1. Let X be a vector space, such that $\dim X < \infty$. Then all norms on X are equivalent.

Proof. Notice that it is enough to prove that any norm $\|\cdot\|$ on X is equivalent to the Euclidean norm $\|\cdot\|_2$.

Moreover, it is enough to prove that $\exists c_1, c_2 > 0$ such that:

$$c_1 \leq \|x\| \leq c_2 \quad \forall x \in X, \|x\|_2 = 1$$

Indeed, if we have this, then:

$$y \in \mathbb{R}^N \setminus \{0\} \implies \left\| \frac{y}{\|y\|_2} \right\|_2 = 1$$

Then, we have:

$$c_1 \leq \left\| \frac{y}{\|y\|_2} \right\| \leq c_2 \implies c_1 \|y\|_2 \leq \|y\| \leq c_2 \|y\|_2$$

Which is what we wanted to prove.

To prove this, let $f(x) = \|x\|$. We will show that f is continuous with respect to the Euclidean norm, i.e.:

$$\|x_n - x\|_2 \rightarrow 0 \implies f(x_n - x) \rightarrow 0 \iff \|x_n - x\| \rightarrow 0$$

Indeed, for $y \in X$, and $\{e_1, \dots, e_N\}$ basis of X , we have:

$$\begin{aligned} \|y\| &= \left\| \sum_{i=1}^N y_i e_i \right\| \leq \sum_{i=1}^N \|y_i e_i\| \\ &\leq \sum_{i=1}^N |y_i| \|e_i\| \leq \left(\max_i |y_i| \right) \sum_{i=1}^N \|e_i\| \\ &\leq C \|y\|_\infty \leq C \|y\|_2 \end{aligned}$$

Where $C = \sum_{i=1}^N \|e_i\|$. Then, we have:

$$0 < \|x_n - x\| \leq C \|x_n - x\|_2 \rightarrow 0 \implies \|x_n - x\| \rightarrow 0$$

Finally, consider:

$$\min_{\|x\|_2=1} f(x) \quad \max_{\|x\|_2=1} f(x)$$

Since f is continuous, and $S = \{x \in X : \|x\|_2 = 1\}$ is compact, we have that f attains its minimum and maximum in S . Let $x_m = \arg \min_{\|x\|_2=1} f(x)$, and $x_M = \arg \max_{\|x\|_2=1} f(x)$. Then, we have:

$$\begin{aligned} 0 < \|x_m\| \leq f(x) \leq \|x_M\| \quad \forall x \in X, \|x\|_2 = 1 \\ \implies 0 < \|x_m\| \leq \|x\| \leq \|x_M\| \quad \forall x \in X, \|x\|_2 = 1 \end{aligned}$$

■

Note: We postpone more general properties of Banach spaces (in particular, that in infinite dimension, the theorem above is not true), and we anticipate the Lebesgue spaces.

Chapter 9

Lebesgue spaces $L^p(X)$

9.1 Definition of $L^p(X)$

Definition 9.1.1. Let (X, \mathcal{M}, μ) be a complete measure space, and $p \in [1, \infty]$. We define the following:

1. $\mathcal{L}^p(X, \mathcal{M}, \mu) := \{f : X \rightarrow \mathbb{R} \mid f \text{ is } \mathcal{M}\text{-measurable and } \int_X |f|^p d\mu < \infty\}$.
2. $u, v \in \mathcal{L}^p(X, \mathcal{M}, \mu), u \sim v \iff u = v \text{ a.e.}$
3. $[f]_p := \{g \in \mathcal{L}^p(X, \mathcal{M}, \mu) \mid f \sim g\}$.

Finally, we define the L^p -space as follows:

$$L^p(X, \mathcal{M}, \mu) := \mathcal{L}^p(X, \mathcal{M}, \mu) / \sim = \{[f]_p \mid f \in \mathcal{L}^p(X, \mathcal{M}, \mu)\}$$

where \sim is the equivalence relation defined above. We also define the norm as follows:

$$\|f\|_{L^p} = \|f\|_p = \begin{cases} \left(\int_X |f|^p d\mu\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \text{ess sup}_{x \in X} |f(x)| & \text{if } p = \infty \end{cases}$$

and $d_p(f, g) = \|f - g\|_p$.

E.g.: Notice that if $(X, \mathcal{M}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{\#})$, then:

$$L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{\#}) = \ell^p$$

For $1 \leq p < \infty$, we have:

$$\ell^p = \left\{ (a_n)_{n \in \mathbb{N}} \mid \sum_{n \in \mathbb{N}} |a_n|^p < \infty \right\}$$

with norm:

$$\|(a_n)\|_p = \left(\sum_{n \in \mathbb{N}} |a_n|^p \right)^{1/p}$$

For $p = \infty$, we have:

$$\ell^\infty = \left\{ (a_n)_{n \in \mathbb{N}} \mid \sup_{n \in \mathbb{N}} |a_n| < \infty \right\}$$

with norm:

$$\|(a_n)\|_\infty = \sup_{n \in \mathbb{N}} |a_n|$$

Note: Our plan is to show that $L^p(X, \mathcal{M}, \mu)$ is a Banach space, i.e.:

1. $L^p(X, \mathcal{M}, \mu)$ is a vector space.
2. $\|\cdot\|_p$ is a norm.
3. $L^p(X, \mathcal{M}, \mu)$ is complete.

9.2 L^p -spaces are vector spaces

Lemma 9.2.1. *Let $p \in [1, \infty)$, $a, b \in \mathbb{R}, a, b \leq 0$. Then:*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p)$$

Proof (exercise). For $a \neq 0$, $t = b/a$, we have to show that:

$$\frac{(1+t)^p}{1+t^p} \leq 2^{p-1} \quad \forall t \leq 0$$

■

Theorem 9.2.2. *Let $p \in [1, \infty)$, then $L^p(X)$ is a vector space*

Proof. Given $u, v \in L^p(X), \alpha \in \mathbb{R}$, we have to show that:

1. $u + v \in L^p(X)$

2. $\alpha u \in L^p(X)$

1. We have:

$$\int_X |u + v|^p d\mu \leq \int_X (|u| + |v|)^p d\mu \leq 2^{p-1} \left(\int_X |u|^p d\mu + \int_X |v|^p d\mu \right) < \infty$$

2. We have:

$$\int_X |\alpha u|^p d\mu = \int_X |\alpha|^p |u|^p d\mu = |\alpha|^p \int_X |u|^p d\mu < \infty$$

■

9.3 $(L^p(X), \|\cdot\|_p)$ are normed spaces

Definition 9.3.1 (Conjugated exponent). For every $1 \leq p \leq \infty$, the **conjugated exponent** of p , denoted by $q \in [1, \infty]$, satisfies:

$$\frac{1}{p} + \frac{1}{q} = 1$$

Lemma 9.3.1 (Young's inequality). Let $p, q \in (1, \infty)$ be conjugated exponents. Then, for every $a, b \geq 0$:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Notice that $\ln(x)$ is a concave function. Then:

$$\begin{aligned} \ln\left(\frac{a^p}{p} + \frac{b^q}{q}\right) &\geq \frac{1}{p} \ln(a^p) + \frac{1}{q} \ln(b^q) \\ &= \ln((a^p)^{1/p}) + \ln((b^q)^{1/q}) = \ln(a) + \ln(b) = \ln(ab) \end{aligned}$$

■

Note: As a consequence of Young's inequality, we have the following inequality:

Lemma 9.3.2 (Hölder's inequality). *Let $p, q \in [1, \infty]$ be conjugated exponents, (X, \mathcal{M}, μ) be a complete measure space, and u, v measurable functions. Then:*

$$\|uv\|_1 \leq \|u\|_p \|v\|_q$$

Proof. We will prove it for $p, q \in (1, \infty)$. For $p = 1, q = \infty$, it is left as an exercise.

We separate in cases:

- If $\|u\|_p = 0$, then $u = 0$ a.e., and $uv = 0$ a.e., meaning that

$$\|uv\|_1 = 0$$

(The same applies if $\|v\|_q = 0$)

- If $\|u\|_p \cdot \|v\|_q = \infty$, then the inequality is trivial.
- For $0 < \|u\|_p, \|v\|_q < \infty$, we apply the Young inequality for:

$$a = \frac{|u(x)|}{\|u\|_p}, \quad b = \frac{|v(x)|}{\|v\|_q}$$

We have:

$$\frac{|u(x)| \cdot |v(x)|}{\|u\|_p \|v\|_q} = ab \leq \frac{1}{p} \frac{|u(x)|^p}{\|u\|_p^p} + \frac{1}{q} \frac{|v(x)|^q}{\|v\|_q^q}$$

We integrate to get:

$$\frac{\|uv\|_1}{\|u\|_p \|v\|_q} \leq \frac{1}{p} \frac{\|u\|_p^p}{\|u\|_p^p} + \frac{1}{q} \frac{\|v\|_q^q}{\|v\|_q^q} = 1$$

$$\implies \|uv\|_1 \leq \|u\|_p \|v\|_q$$

■

9.3.1 Inclusion of L^p spaces

Theorem 9.3.3. *Let $\mu(X) < \infty$, $1 \leq p \leq q \leq \infty$. Then:*

$$L^q(X) \subset L^p(X)$$

More precisely, $\exists C > 0$ s.t.:

$$\|u\|_p \leq C \|u\|_q$$

Theorem 9.3.4 (Interpolation). *Let $1 \leq p < q \leq \infty$. Then:*

$$L^r(X) \subset L^p(X) \cap L^q(X), \quad \forall p \leq r \leq q$$

9.3.2 Minkowski's inequality

Theorem 9.3.5 (Minkowski's inequality). *Let $p \in [1, \infty]$, (X, \mathcal{M}, μ) be a complete measure space, and $u, v \in L^p(X)$. Then:*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p$$

Proof. We will prove it for $p \in (1, \infty)$. For $p = 1, p = \infty$, it is left as an exercise.

We have:

$$\begin{aligned} \|u + v\|_p^p &= \int_X |u + v|^p d\mu = \int_X |u + v| |u + v|^{p-1} d\mu \\ &\leq \int_X |u| |u + v|^{p-1} d\mu + \int_X |v| |u + v|^{p-1} d\mu \end{aligned}$$

For the first term, we have:

$$\begin{aligned} \int_X |u| |u + v|^{p-1} d\mu &\leq \|u\|_p \left(\int_X |u + v|^{(p-1)q} d\mu \right)^{1/q} \\ &\leq \|u\|_p \|u + v\|_p^{p/q} = \|u\|_p \|u + v\|_p^{p-1} \end{aligned}$$

Analogously, for the second term, we have:

$$\int_X |v| |u + v|^{p-1} d\mu \leq \|v\|_p \|u + v\|_p^{p-1}$$

and finally, we substitute back to get:

$$\|u + v\|_p^p \leq \|u\|_p \|u + v\|_p^{p-1} + \|v\|_p \|u + v\|_p^{p-1}$$

and we divide by $\|u + v\|_p^{p-1}$ to get:

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p$$

■

Corollary 9.3.5.1. *$(L^p(X), \|\cdot\|_p)$ is a normed space for $p \in [1, \infty]$*

9.4 Completeness of L^p -spaces

Theorem 9.4.1 (Riesz-Fischer). *Let $p \in [1, \infty]$, (X, \mathcal{M}, μ) be a complete measure space. Then:*

$L^p(X)$ is a Banach space

Proof. The only thing left to show is that $L^p(X)$ is complete. We will use the characterization of Banach spaces in terms of absolutely convergent series.

Let us suppose that $\{f_n\}_n \subset L^p(X)$ is an absolutely convergent series, i.e.:

$$\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty$$

Introduce $g_k(x) = \sum_{n=1}^k |f_n(x)|$. We have that, for every $x \in X$, $\{g_k(x)\}_{k \in \mathbb{N}}$ is a non-decreasing sequence. Then:

$$g(x) = \lim_{k \rightarrow \infty} g_k(x) = \sum_{n=1}^{\infty} |f_n(x)|$$

is well-defined for every $x \in X$. We have to show that $g \in L^p(X)$.

Notice that:

$$\begin{aligned}\|g_k\|_p &= \left\| \sum_{n=1}^k |f_n| \right\|_p \leq \sum_{n=1}^k \|f_n\|_p \leq \\ &\leq \sum_{n=1}^{\infty} \|f_n\|_p = M\end{aligned}$$

where M is a constant (since the series is absolutely convergent). Then, $g_k \in L^p(X)$ for every $k \in \mathbb{N}$.

Then, by the monotone convergence theorem, we have:

$$\begin{aligned}\int_X g^p d\mu &= \int_X \left(\lim_{k \rightarrow \infty} g_k \right)^p d\mu = \lim_{k \rightarrow \infty} \int_X g_k^p d\mu \\ &= \lim_{k \rightarrow \infty} \|g_k\|_p^p \leq M^p < \infty\end{aligned}$$

Then, $g \in L^p(X)$, meaning that $g(x) \leq \infty$ a.e., which implies that:

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty \quad \text{a.e.}$$

Since X is complete, we have that $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. Then:

$$s(x) = \sum_{n=1}^{\infty} f_n(x)$$

is well-defined for every $x \in X$. And we proved that $s_k(x) \rightarrow s(x)$ for a.e $x \in X$.

To conclude, we apply the dominated convergence theorem:

- $|s_k(x) - s(x)| \rightarrow 0$ a.e.

-

$$\begin{aligned}|s_k - s|^p &= \left| \sum_{n=k+1}^{\infty} f_n \right|^p \leq \left(\sum_{n=k+1}^{\infty} |f_n| \right)^p \\ &\leq (g)^p \in L^1\end{aligned}$$

These conditions imply that:

$$\int_X |s_k - s|^p d\mu \rightarrow 0$$

that is, convergence in L^p . ■

E.g.: We know that the following are Banach spaces:

1. $(\mathbb{R}^N, \text{any norm})$
2. $(C([a, b]), \|\cdot\|_\infty)$
3. $(L^p(X), \|\cdot\|_p)$
4. $(L^\infty, \|\cdot\|_\infty)$

E.g.: Let $X = C([-1, 1])$, $\|u\|_1 = \int_{-1}^1 |u(x)| dx$. Then, let u_n :

$$u_n(x) = \begin{cases} 0 & \text{if } x \in [-1, 0) \\ nx & \text{if } x \in [0, 1/n] \\ 1 & \text{if } x \in [1/n, 1] \end{cases}$$

We have that $\{u_n\}_n \subset X$ is a Cauchy sequence with respect to the norm $\|\cdot\|_1$. On the other hand:

$$\|u_n - u_m\|_\infty = \max_{-1 \leq x \leq 1} |u_n(x) - u_m(x)| = 1 - \frac{n}{m} \not\rightarrow 0$$

Moreover, we have that $\{u_n\}_n \subset L^1([-1, 1])$, s.t. $u_n \rightarrow \mathcal{H}$, which is not in $C([-1, 1])$.

Consequences:

1. $\|\cdot\|_1$ is not equivalent to $\|\cdot\|_\infty$ in $C([-1, 1])$.
2. $(C([-1, 1]), \|\cdot\|_1)$ is not a Banach space.
3. $C([-1, 1])$ is a vector subspace of $L^1([-1, 1])$, but it is not closed, since the sequence $\{u_n\}_n \subset C([-1, 1])$ converges to a function that is not in $C([-1, 1])$.

Chapter 10

Compactness, Density and Separability

10.1 Compactness

We say that (X, d) is a metric space.

Definition 10.1.1. $E \subset X$ is **compact** if from any open covering $\{A_i\}_{i \in I}$ (A_i open $\forall i \in I$, $E \subset \bigcup_{i \in I} A_i$) we can extract a finite subcovering.

Typically, we define it as follows:

Take E , fix $r > 0$ and consider $\{B_r(x)\}_{x \in E}$, the open balls of radius r centered at $x \in E$.

Then, E is compact if there exists $x_1, \dots, x_k \in E$ s.t.

$$E \subset \bigcup_{i=1}^k B_r(x_i)$$

Definition 10.1.2. E is **sequentially compact** if $\forall \{x_n\}_{n \in \mathbb{N}} \subset E$, there exists a subsequence $\{x_{n_k}\}_{k \in \mathbb{N}}$ that converges to some $x \in E$.

Remark: The two definitions are equivalent in metric spaces.

Definition 10.1.3. $E \subset X$ is **relatively compact** if \overline{E} is compact.

Theorem 10.1.1 (Heine-Borel). *Let $(X, \|\cdot\|)$ be a normed vector space. If $\dim(X) < \infty$, then $E \subset X$ is compact $\iff E$ is closed and bounded.*

Remark: The theorem is not true in infinite-dimensional spaces. In particular, if $E \subset X$ is compact, then E is closed and bounded, but the converse is not true.

Theorem 10.1.2 (Riesz). *Let $(X, \|\cdot\|)$ be a normed vector space. Then:*

$$\overline{B_1(0)} \text{ is compact } \iff \dim(X) < \infty$$

Proof. (\Leftarrow) Exercise.

(\Rightarrow) Suppose $\overline{B_1(0)} = \{x \in X : \|x\| \leq 1\}$ is compact.

Consider $\{B_{1/2}(x)\}_{x \in \overline{B_1(0)}}$. Then:

$$\overline{B_1(0)} \subset \bigcup_{x \in \overline{B_1(0)}} B_{1/2}(x)$$

By compactness, $\exists x_1, \dots, x_k \in \overline{B_1(0)}$ s.t.

$$\begin{aligned} \overline{B_1(0)} &\subset \bigcup_{i=1}^k B_{1/2}(x_i) \\ &\subset \bigcup_{i=1}^k \overline{B_{1/2}(x_i)} \end{aligned}$$

This means that $\forall x \in \overline{B_1(0)}$, $\exists i \in \{1, \dots, k\}$, s.t.

$$x = x_i + z \text{ for some } \|z\| \leq 1/2$$

Define $V = \text{span}\{x_1, \dots, x_k\}$. Then, $V \subset X$ is a vector subspace and $\dim V \leq k < \infty$.

We can then rewrite the previous implication as: $\forall x \in \overline{B_1(0)}$, $\exists v \in V$ s.t.

$$x = v + z \text{ for some } \|z\| \leq 1/2$$

Now, take $y \in X$, s.t. $y \neq 0$. Then, notice that:

$$\frac{y}{\|y\|} \in \overline{B_1(0)}$$

So there exists $v \in V$ and $z : \|z\| \leq 1/2$ s.t.

$$\frac{y}{\|y\|} = v + z$$

Then, $y = \|y\| v + \|y\| z$. We rewrite this as:

$$y = v' + z'$$

where $v' = \|y\| v \in V$ and $\|z'\| \leq \|y\| / 2$.

Then, take any $x \in X$ and apply the previous result to $y = x$:

$$x = v_1 + z_1, \quad v_1 \in V, \quad \|z_1\| \leq \|x\| / 2$$

Then, apply it again to $y = z_1$:

$$x = v_1 + v' + z_2, \quad v_1, v' \in V, \quad \|z_2\| \leq \|z_1\| / 2 \leq \|x\| / 4$$

Notice that, because V is a vector space, $v_1 + v' \in V$. Then, we rewrite the previous equation as:

$$x = v_2 + z_2, \quad v_2 \in V, \quad \|z_2\| \leq \|x\| / 4$$

By induction:

$$x = v_n + z_n, \quad v_n \in V, \quad \|z_n\| \leq \|x\| / 2^n$$

Notice that $z_n \rightarrow 0$ as $n \rightarrow \infty$. Then:

$$v_n = x - z_n \rightarrow x \text{ as } n \rightarrow \infty$$

Meaning that the sequence $\{v_n\}_n \subset V$ converges to $x \in X$, and because V is a finite-dimensional vector subspace, it is closed, so $x \in V$.

With this, we have shown that $X = V$, and therefore, $\dim X \leq k < \infty$.

■

10.2 Compactness in $C([a, b])$

Note: We always deal with $(C([a, b]), \|\cdot\|_\infty)$, which is Banach

Definition 10.2.1. Let $\{u_n\}_n \subset C([a, b])$ a sequence of continuous functions. Then, we say that it is **uniformly equicontinuous** if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.,

$$|x - y| < \delta \implies |u_n(x) - u_n(y)| < \varepsilon, \quad \forall x, y \in [a, b], \forall n \in \mathbb{N}$$

(The value of δ only depends on ε)

Theorem 10.2.1 (Ascoli-Arzelà). Take $\{u_n\}_n \subset C([a, b])$. Assume that:

(i) $\{u_n\}_{n \in \mathbb{N}}$ is uniformly bounded, i.e.:

$$\exists 0 < M < \infty, \quad \|u_n\|_\infty \leq M \quad \forall n \in \mathbb{N}$$

(ii) $\{u_n\}_n$ is uniformly equicontinuous.

Then, there exists a subsequence $\{u_{n_k}\}_{k \in \mathbb{N}}$ and $u \in C([a, b])$ s.t. $u_{n_k} \rightarrow u$ as $k \rightarrow \infty$

E.g.: Let $\{u_n\}_n \subset C^1([a, b]) \subset C([a, b])$. Assume that:

$$1. \quad \|u_n\| \leq M. \quad \forall n$$

$$2. \quad \|u'_n\|_n \leq L, \quad \forall n$$

Then, the theorem applies. Indeed: 1) \implies (i) in Ascoli-Arzelà. To check equicontinuity: $\forall x, y \in [a, b], x \neq y$:

$$|u_n(x) - u_n(y)| = |u'_n(\zeta) \cdot (x - y)| \quad (\text{Mean Value Thm.})$$

$$\begin{aligned} \implies |u_n(x) - u_n(y)| &\leq |u'_n(\zeta)| \cdot |x - y| \\ &\leq \|u'_n\|_\infty \cdot |x - y| \\ &\leq L|x - y|, \quad \forall n \in \mathbb{N} \end{aligned}$$

$$\implies \text{equicontinuity (take } \delta = \frac{\varepsilon}{L} \text{)}$$

Roughly, the thm. implies that “boundedness in $C^1 \implies$ compactness in C^0 ”.

Remark: The same is true for Lipschitz continuous functions with uniformly bounded Lipschitz constant.

Also, there are similar theorems in L^p with:

$$W^{1,p} = \{L^p \text{ functions having } L^p \text{ weak derivatives}\}$$

and “boundedness in $W^{1,p} \implies$ compactness in L^p ”.

10.3 Density, separability

Definition 10.3.1. We say that $D \subset X$ is **dense** if $\overline{D} = X$, i.e.:

$$\forall x \in X, \exists \{y_n\}_n \subset D : y_n \rightarrow x \in X$$

Definition 10.3.2. X is **separable** if $\exists D \subset X$, s.t. D is countable and dense

Remark: Typically, one uses dense subsets because “continuous properties, true on D , are also true on X ”. When D is separable, you have few elements to check the property.

E.g.: $\mathbb{R}, \mathbb{R}^N, \Omega \subset \mathbb{R}^N$ are all separable: $\overline{\mathbb{Q}} = \mathbb{R}$ and \mathbb{Q} is countable.

Theorem 10.3.1. *The following spaces are separable:*

- $(C([a, b]), \|\cdot\|_\infty)$
- $(L^p(\mathbb{R}), \|\cdot\|_p)$ for $1 \leq p < \infty$

and $(L^\infty(\mathbb{R}), \|\cdot\|_\infty)$ is **NOT** separable.

10.3.1 Dense subspaces

For continuous functions, we have the following result:

Theorem 10.3.2 (Stone-Weierstrass). *Polynomials are dense in $C([a, b])$, i.e.:*

$$\forall f \in C([a, b]), \forall \varepsilon > 0, \exists P(x) \text{ polynomial s.t.}$$

$$\|f - P\|_\infty < \varepsilon$$

Note that polynomials with coefficients in \mathbb{Q} are countable.

For L^p spaces, we have the following dense subspaces:

- Simple functions
- Continuous (or more regular) functions

Note (Recall): $s : \mathbb{R} \rightarrow \mathbb{R}$ is (measurable and) simple if:

$$s = \sum_{i=1}^k \alpha_i \mathbb{1}_{A_i}$$

where $\alpha_i \in \mathbb{R}$ and $A_i \in \mathcal{L}(\mathbb{R})$ are disjoint sets, s.t.:

$$\bigcup_{i=1}^k A_i = \mathbb{R}$$

We know that $s \text{ simple} \implies s \in L^\infty(\mathbb{R})$. Does $s \text{ simple} \implies s \in L^p(\mathbb{R})$? For $p \in [1, \infty)$, we have that:

$$s \in L^p(\mathbb{R}) \iff \lambda(\{x : s(x) \neq 0\}) < \infty$$

Definition 10.3.3. We define $\tilde{\rho}(\mathbb{R})$ as the set of simple functions on \mathbb{R} , such that $\lambda(\{x : s(x) \neq 0\}) < \infty$:

$$\tilde{\rho}(\mathbb{R}) = \{s : \mathbb{R} \rightarrow \mathbb{R} \text{ simple} \mid \lambda(\{x : s(x) \neq 0\}) < \infty\}$$

Theorem 10.3.3. $\tilde{\rho}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Definition 10.3.4. We define the following concepts:

1. $u : \mathbb{R} \rightarrow \mathbb{R}$. The **support** of u is defined as:

$$\text{supp}(u) = \overline{\{x : u(x) \neq 0\}}$$

2. $C_c(\mathbb{R}) = \{u \in C(\mathbb{R}) : \text{supp}(u) \text{ is compact}\}$
3. $C_c^\infty(\mathbb{R}) = \{u \in C_c(\mathbb{R}) : u \text{ is infinitely differentiable}\} = \mathbb{C}_0^\infty(\mathbb{R}) = \mathcal{D}(\mathbb{R})$

Theorem 10.3.4. $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

Corollary 10.3.4.1. $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.

$(D \subset X \text{ dense}, D \subset E \subset X \implies E \text{ dense in } X)$

Remark: $C_c^\infty(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$. Indeed, take

$$\mathcal{H}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mathcal{H} \in L^\infty(\mathbb{R})$, but now suppose that we have a function $g \in C_c(\mathbb{R})$ s.t.:

$$\|\mathcal{H} - g\|_\infty \leq 1/3$$

Then:

$$\begin{aligned} |\mathcal{H}(x) - g(x)| &\leq 1/3, \quad \text{a.e. } x \in \mathbb{R} \\ \implies \mathcal{H}(x) - 1/3 &\leq g(x) \leq \mathcal{H}(x) + 1/3 \end{aligned}$$

This implies that g cannot be continuous in $x = 0$. Contradiction.

Note: Let us see that $L^\infty(\mathbb{R})$ is not separable.

Lemma 10.3.5. Take X Banach. Assume that $\{A_i\}_{i \in I}$ is s.t.:

(a) $\forall i \in I, A_i \subset X$ is open and non-empty

(b) $\forall i \neq j \in I, A_i \cap A_j = \emptyset$

(c) I is more than countable.

Then, X is not separable.

Proof. By contradiction. Assume that X is separable. Then, $\exists \{x_n\}_{n \in \mathbb{N}} \subset X$ s.t.:

$$X = \overline{\bigcup_{n \in \mathbb{N}} \{x_n\}}$$

Then, $\forall A_i, \exists x_{n_i} \in A_i$. This is because A_i is non-empty, then $\exists z_i \in A_i$, and because $\{x_n\}_n$ dense, $\exists \{x_{n_k}\}_{k \in \mathbb{N}}$ s.t. $x_{n_k} \rightarrow z_i$ as $k \rightarrow \infty$. Notice that $A_i \subset X$ is open, so the sequence $\{x_{n_k}\}_k$ is eventually in A_i .

Since $A_i \cap A_j = \emptyset, x_{n_i} \neq x_{n_j}$, i.e., $n_i \neq n_j$.

Then, we have a map $i \rightarrow n_i$ that is injective, so I is at most countable. Contradiction. ■

Theorem 10.3.6. $L^\infty(\mathbb{R})$ is not separable.

Proof. We use the previous lemma. $\forall \alpha \in \mathbb{R}^+ = (0, \infty)$, we define:

$$g_\alpha(x) = \chi_{[-\alpha, \alpha]}(x) = \begin{cases} 1 & \text{if } |x| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

Notice that, if $\alpha_1 \neq \alpha_2$, then $\|g_{\alpha_1} - g_{\alpha_2}\|_\infty = 1$.

$$\implies B_{1/2}(g_{\alpha_1}) \cap B_{1/2}(g_{\alpha_2}) = \emptyset$$

Indeed, $\forall f \in L^\infty(\mathbb{R})$, we have that:

$$\begin{aligned} 1 &= \|g_{\alpha_1} - g_{\alpha_2}\|_\infty \leq \|g_{\alpha_1} - f\|_\infty + \|f - g_{\alpha_2}\|_\infty \\ \implies &\text{ at least one of the norms is greater than } 1/2 \end{aligned}$$

Then, we have a family of open sets $\{B_{1/2}(g_\alpha)\}_{\alpha \in \mathbb{R}^+}$ that satisfies the conditions of the lemma.

Then, $L^\infty(\mathbb{R})$ is not separable.

■

Chapter 11

Linear operators

Note: We will work with $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ normed (Banach) spaces.

Definition 11.0.1. We say that $T : X \rightarrow Y$ is a **linear operator** if:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$\forall x, y \in X$ and $\forall \alpha, \beta \in \mathbb{R}$.

(If $Y = \mathbb{R}$, we say that T is a **linear functional**).

Notation: For T linear, $T(u) = Tu$.

E.g.: Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. Then, $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear if:

$$T(x) = Ax$$

where $A \in \mathbb{R}^{m \times n}$.

Remark: T is linear $\implies T(0) = 0$.

Definition 11.0.2. We say that $T : X \rightarrow Y$ is **bounded** if $\exists M > 0$ such that:

$$\|Tx\|_Y \leq M \|x\|_X \quad \forall x \in X$$

Note (Recall): We have that:

- T is Lipschitz if $\exists L > 0$ such that $\|Tx - Ty\|_Y \leq L \|x - y\|_X$.
- T is continuous in $x \in X$ if $\forall x_n \rightarrow x$ in X , we have that $Tx_n \rightarrow Tx$ in Y .

Remark: $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ linear $\implies T$ is continuous and bounded. But notice that if X, Y are infinite-dimensional, then the previous statement is not true.

Theorem 11.0.1. $T : X \rightarrow Y$ linear. Then, the following are equivalent:

- 1) T is bounded.
- 2) T is Lipschitz.
- 3) T is continuous at any $x_0 \in X$
- 4) T is continuous at 0.

Proof. The proof goes as follows:

(1 \implies 2) We know that T is bounded, i.e.:

$$\|Tx\|_Y \leq M \|x\|_X, \quad \forall x \in X$$

Take $x = u - v$. Then:

$$\|Tu - Tv\|_Y = \|T(u - v)\|_Y \leq M \|u - v\|_X$$

Then, T is Lipschitz with $L = M$.

(2 \implies 3) Let $L > 0$ be the Lipschitz constant for T . Let $x_n \rightarrow x_0$ for some $x_0 \in X$. We have:

$$0 \leq \|Tx_n - Tx_0\|_Y \leq L \|x_n - x_0\|_X \rightarrow 0$$

(3 \implies 4) Trivial

(4 \implies 1) By contradiction, assume that T is not bounded:

$$\forall n \in \mathbb{N}, \exists x_n \in X : \|Tx_n\|_Y \geq n \|x_n\|_X$$

Let $z_n = \frac{1}{n} \frac{x_n}{\|x_n\|_X}$. Then $\|z_n\|_X = 1/n \rightarrow 0$ as $n \rightarrow \infty$. Since T is continuous at 0, then:

$$Tz_n \rightarrow T0 = 0$$

But:

$$\begin{aligned} \|Tz_n\|_Y &= \left\| T \left(\frac{1}{n} \frac{x_n}{\|x_n\|} \right) \right\|_Y \\ &= \frac{1}{n \|x_n\|_X} \|Tx_n\|_Y \geq 1 \not\rightarrow 0 \end{aligned}$$

This is a contradiction. ■

Definition 11.0.3. We define the set $\mathcal{L}(X, Y)$ as:

$$\mathcal{L}(X, Y) := \{T : X \rightarrow Y \text{ s.t. } T \text{ linear and bounded}\}$$

If $X = Y$, we write $\mathcal{L}(X)$. If $Y = \mathbb{R}$, then we say that $\mathcal{L}(X, \mathbb{R})$ is the **dual** of X , noted as $X' = X^*$.

Remark: $\mathcal{L}(X, Y)$ is a vector space, i.e., $\forall T, L \in \mathcal{L}(X, Y), \alpha, \beta \in \mathbb{R}$:

$$(\alpha T + \beta L) \in \mathcal{L}(X, Y)$$

$$((\alpha T + \beta L)(x) := \alpha Tx + \beta Lx)$$

Definition 11.0.4. We define a norm on $\mathcal{L}(X, Y)$, called the **operator norm**, as:

$$\|T\|_{\mathcal{L}(X, Y)} := \sup_{\|x\| \leq 1} \|Tx\|_Y$$

Proposition 11.0.2. *For the operator norm, we have the following equivalences:*

$$\|T\|_{\mathcal{L}(X, Y)} = \sup_{\|x\|=1} \|Tx\|_Y = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf\{M > 0 : \|Tx\|_Y \leq M \|x\|_X \ \forall x \in X\}$$

Proof. We know that:

$$\sup_{\|x\| \leq 1} \|Tx\|_Y \geq \sup_{\|x\|=1} \|Tx\|_Y$$

The other inequality:

$$\forall x \neq 0, \|Tx\|_Y = \|x\|_X \cdot \left\| T \left(\frac{x}{\|x\|_X} \right) \right\|_Y$$

Then, if $z = x/\|x\|_X$:

$$\|Tx\|_Y \leq \|Tz\|_Y, \quad \text{with } \|z\|_X = 1$$

obtaining the inequality, so:

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\| \leq 1} \|Tx\|_Y = \sup_{\|x\|=1} \|Tx\|_Y$$

For the others, we have:

$$\begin{aligned} \forall x \neq 0, \quad \|Tx\|_Y \leq M \|x\|_X &\iff M \geq \frac{\|Tx\|_Y}{\|x\|_X} \\ &\iff M \geq \|Tz\|_Y, \quad \text{with } \|z\|_X = 1 \end{aligned}$$

So:

$$\sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf \{ M > 0 : \|Tx\|_Y \leq M \|x\|_X \quad \forall x \in X \}$$

And:

$$\inf(M) \geq \sup_{\|x\|=1} \|Tx\|_Y$$

■

Theorem 11.0.3. *If X is a normed space, and Y is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space.*

Proof. Omitted. ■

Definition 11.0.5. Let $T : X \rightarrow Y$ linear. We define the following:

- **Kernel:** $\text{Ker}(T) = \{x \in X : Tx = 0\} \subset X$
- **Range:** $R(T) = \{y \in Y : \exists x \in X, Tx = y\} \subset Y$
- T is **injective** if $\text{Ker}(T) = \{0\}$
- T is **surjective** if $R(T) = Y$
- T is **bijective** if T is injective and surjective

Also, if T is bijective, we define the **inverse** of T as $T^{-1} : Y \rightarrow X$, s.t. $TT^{-1} = I_Y$ and $T^{-1}T = I_X$. Notice that T^{-1} is linear.

Remark: Let $T : X \rightarrow Y$ linear. Then, $\text{Ker}(T) \subset X$ and $R(T) \subset Y$ are vector subspaces. Also, if $T \in \mathcal{L}(X, Y)$, then $\text{Ker}(T)$ is closed in X . The $R(T)$ may or may not be closed in Y .

Definition 11.0.6 (Isomorphism). We say that X, Y are **isomorphic** if $\exists T \in \mathcal{L}(X, Y)$ bijective and $T^{-1} \in \mathcal{L}(Y, X)$.

In this case, we write $X \cong Y$.

Definition 11.0.7. We say that $T \in \mathcal{L}(X, Y)$ is an **isometry** if:

$$\|Tx\|_Y = \|x\|_X, \quad \forall x \in X$$

Definition 11.0.8 (Continuous embedding). Let $X \subset Y$ be a vector subspace. We define the “inclusion” operator $J : X \rightarrow Y$ as $Jx = x$. Then, if $J \in \mathcal{L}(X, Y)$, i.e., if:

$$\|x\|_Y \leq M \|x\|_X, \quad \forall x \in X$$

Then, we say that X is **continuously embedded** in Y , and we write $X \hookrightarrow Y$.

More generally, if X, Y Banach and $T \in \mathcal{L}(X, Y)$, T injective and $T^{-1} \in \mathcal{L}(R(T), X)$, then we say that X is **continuously embedded** in Y . We call T the **embedding operator**.

E.g.: We have already prove that, for (X, \mathcal{M}, μ) a measure space, $\mu(X) < \infty$, $1 \leq p < q \leq \infty$, then:

$$L^p(X, \mathcal{M}, \mu) \hookrightarrow L^q(X, \mathcal{M}, \mu)$$

11.1 Uniform boundedness (Banach-Steinhaus theorem)

Theorem 11.1.1 (Uniform boundedness (Banach-Steinhaus theorem)). *Let X, Y Banach spaces, and $\mathcal{T} \subset \mathcal{L}(X, Y)$ be a set of linear operators. Suppose that \mathcal{T} is pointwise bounded, i.e., $\forall x \in X, \exists M_x > 0$ such that:*

$$\|Tx\|_Y \leq M_x, \quad \forall T \in \mathcal{T}$$

Then, \mathcal{T} is uniformly bounded, i.e., $\exists M > 0$ such that:

$$\|T\|_{\mathcal{L}(X, Y)} \leq M, \quad \forall T \in \mathcal{T}$$

Note: The proof is based on Baire's topological lemma.

Lemma 11.1.2 (Baire's topological lemma). *Let X be a complete metric space, $\{C_n\}_{n \in \mathbb{N}}$ s.t. $C_n \subset X$ is closed and:*

$$X = \bigcup_{n \in \mathbb{N}} C_n$$

Then, $\exists n_0 \in \mathbb{N}$ such that C_{n_0} has non-empty interior.

$$(\exists r > 0, x_0 \in C_{n_0} : \overline{B_r(x_0)} \subset C_{n_0})$$

Uniform boundedness. Define, $\forall n \in \mathbb{N}$,

$$C_n = \{x \in X : \|Tx\|_Y \leq n, \forall T \in \mathcal{T}\}$$

We want to apply Baire's lemma to $\{C_n\}_{n \in \mathbb{N}}$. We have:

- (C_n is closed): Indeed, take $\{x_k\}_{k \in \mathbb{N}} \subset C_n$ s.t. $x_k \rightarrow \bar{x} \in X$. We have to show that $\bar{x} \in C_n$. We know that $\forall T \in \mathcal{T}$:

$$\|Tx_k\|_Y \leq n, \quad \forall k \in \mathbb{N}$$

Since T is continuous, then $Tx_k \rightarrow Tx$ as $k \rightarrow \infty$. Then:

$$\|Tx\|_Y \leq n, \quad \forall T \in \mathcal{T}$$

So, $\bar{x} \in C_n$.

- $(X = \bigcup_{n \in \mathbb{N}} C_n)$: Indeed, take any $x \in X$. Since \mathcal{T} is pointwise bounded, then $\exists M_x > 0$ such that:

$$\|Tx\|_Y \leq M_x, \quad \forall T \in \mathcal{T}$$

Then, $x \in C_n \forall n \geq M_x$.

Baire implies that: $\exists n_0 \in \mathbb{N}$, $r > 0$ and $x_0 \in X$ such that:

$$\overline{B_r(x_0)} \subset C_{n_0}$$

Then, we have:

$$\|T(x_0 + rz)\|_Y \leq n_0, \quad \forall T \in \mathcal{T}, \forall \|z\|_X \leq 1$$

And notice that:

$$r \|Tz\|_Y - \|Tx_0\|_Y \leq \|T(x_0 + rz)\|_Y \leq n_0$$

Then, we have:

$$\|Tz\|_Y \leq \frac{n_0 + \|Tx_0\|_Y}{r}, \quad \forall T \in \mathcal{T}, \forall \|z\|_X \leq 1$$

Taking the supremum over $\|z\|_X \leq 1$, we obtain:

$$\|T\|_{\mathcal{L}(X,Y)} \leq \frac{n_0 + \|Tx_0\|_Y}{r} =: M$$

■

Corollary 11.1.2.1. *Let X, Y Banach spaces, and $\{T_n\}_{n \in \mathbb{N}} \subset \mathcal{L}(X, Y)$. Assume that $\forall x \in X, \{T_n x\}_{n \in \mathbb{N}} \subset Y$ is a converging sequence. We have:*

$$T(x) := \lim_{n \rightarrow \infty} T_n x$$

Then, $T \in \mathcal{L}(X, Y)$.

Proof. The proof goes as follows:

- **T is linear:** $\forall n \in \mathbb{N}$, we have:

$$T_n(\alpha x + \beta y) = \alpha T_n x + \beta T_n y$$

Since T_n is continuous:

$$T(\alpha x + \beta y) = \alpha T x + \beta T y$$

- **T is bounded:** Since $\{T_n x\}_{n \in \mathbb{N}}$ converges, then it is bounded. Then, $\exists M_x > 0$ such that:

$$\|T_n x\|_Y \leq M_x, \quad \forall n \in \mathbb{N}$$

Then, $\{T_n\}_{n \in \mathbb{N}}$ is pointwise bounded. By the uniform boundedness theorem, we have that $\{T_n\}_{n \in \mathbb{N}}$ is uniformly bounded, i.e., $\exists M > 0$ such that:

$$\|T_n\|_{\mathcal{L}(X, Y)} \leq M, \quad \forall n \in \mathbb{N}$$

I.e.:

$$\|T_n z\| \leq M \quad \forall n \in \mathbb{N}, \quad \forall \|z\|_X \leq 1$$

Then, we have:

$$\|T z\|_Y = \lim_{n \rightarrow \infty} \|T_n z\|_Y \leq M, \quad \forall \|z\|_X \leq 1$$

Then, T is bounded. ■

11.2 Open mapping and closed graph theorems

Definition 11.2.1. We say that $T : X \rightarrow Y$ is an **open** if:

$$\forall A \subset X \text{ open, } T(A) \subset Y \text{ is open}$$

Remark: Remember that T is continuous if $T^{-1}(V)$ is open $\forall V \subset Y$ open.

E.g.: Let $f : \mathbb{R} \rightarrow \mathbb{R}$, s.t. $f(x) = 0, \forall x \in \mathbb{R}$. Then, f is continuous but not open.

Theorem 11.2.1 (Open mapping theorem). *Let X, Y Banach spaces. Then:*

$$T \in \mathcal{L}(X, Y) \text{ surjective} \implies T \text{ is open}$$

Proof. Omitted, based on the uniform boundedness theorem and Baire. ■

Corollary 11.2.1.1. *Let X, Y be Banach spaces, $T \in \mathcal{L}(X, Y)$ bijective. Then*

$$T^{-1} \in \mathcal{L}(Y, X)$$

and $X \cong Y$. Also, if T is injective, then:

$$T \text{ is embedding, i.e., } X \hookrightarrow Y$$

Corollary 11.2.1.2. *Let $(X, \|\cdot\|_a)$ and $(X, \|\cdot\|_b)$ be Banach spaces, and assume that $\exists c_1 > 0$ s.t. $\|x\|_b \leq c_1 \|x\|_a$. Then,*

$$\exists c_2 > 0 \text{ s.t. } \|x\|_a \leq c_2 \|x\|_b$$

Proof. Apply previous corollary to $J : (X, \|\cdot\|_a) \rightarrow (X, \|\cdot\|_b)$ such that $J(x) = x$. ■

Definition 11.2.2. We say that $T : X \rightarrow Y$ is **closed** if the graph of T is closed in $X \times Y$:

$$\begin{cases} x_n \rightarrow x \text{ in } X \\ Tx_n \rightarrow y \text{ in } Y \end{cases} \implies y = Tx$$

Theorem 11.2.2 (Closed graph). *Let X, Y be Banach spaces, $T : X \rightarrow Y$ linear. Then:*

$$T \text{ is closed} \iff T \in \mathcal{L}(X, Y)$$

Proof. Apply previous corollary to $\|x\|_a = \|x\|_X + \|Tx\|_Y$, $\|x\|_b = \|x\|_X$. ■

Chapter 12

Dual and Reflexive spaces

12.1 Dual spaces

Definition 12.1.1. Let X be a normed space. The **dual space** of X , denoted by X^* , is the set of all bounded linear functionals on X , i.e.:

$$X^* = \mathcal{L}(X, \mathbb{R})$$

It is a Banach space, with norm:

$$\|L\|_{X^*} = \|L\|_* = \sup_{\|x\|_X \leq 1} |Lx|$$

E.g.: Let $X = L^p(\Omega, \mathcal{M}, \mu)$, with $1 \leq p \leq \infty$. Then, take the conjugate exponent q . Take $u \in L^q(\Omega, \mathcal{M}, \mu)$. Define $L_u \in (L^p(\Omega))^*$ as:

$$L_u v = \int_{\Omega} uv \, d\mu \quad \forall v \in L^p(\Omega)$$

Then, show that:

0) L_u is well-defined:

From Hölder's inequality, we have:

$$|L_u v| = \left| \int_{\Omega} uv \, d\mu \right| \leq \|u\|_q \|v\|_p$$

So $L_u v \in \mathbb{R}$, $\forall v \in L^p(\Omega)$.

1) L_u is linear:

$$\begin{aligned} L_u(\alpha_1 v_1 + \alpha_2 v_2) &= \int_{\Omega} u(\alpha_1 v_1 + \alpha_2 v_2) d\mu = \\ &= \alpha_1 \int_{\Omega} u v_1 d\mu + \alpha_2 \int_{\Omega} u v_2 d\mu = \alpha_1 L_u v_1 + \alpha_2 L_u v_2 \end{aligned}$$

2) L_u is continuous:

By Hölder's inequality, we also have that $\|L_u\|_* \leq \|u\|_q$. Then, L_u is bounded, so it is continuous.

3) Calculate $\|L_u\|_*$:

Assume that $p \in (1, \infty)$. Then:

$$\|L_u\|_* = \sup_{v \neq 0} \frac{|L_u v|}{\|v\|_p} \geq \frac{|L_u \bar{v}|}{\|\bar{v}\|_p} \quad \text{for any } \bar{v} \neq 0$$

Then we choose a \bar{v} is such a way that $u\bar{v} = |u|^q$.

$$\bar{v} = |u|^{\frac{q}{p}} \cdot \text{sign}(u)$$

Notice that $u \in L^q \implies \bar{v} \in L^p$. Then:

$$\begin{aligned} \|L_u\|_* &\geq \frac{|L_u \bar{v}|}{\|\bar{v}\|_p} = \frac{\int_{\Omega} |u|^q d\mu}{\left(\int_{\Omega} |u|^q d\mu\right)^{\frac{1}{p}}} \\ &= \frac{\|u\|_q^q}{\|u\|_q} = \|u\|_q^{\frac{q}{p}} = \|u\|_q \end{aligned}$$

So $\|L_u\|_* = \|u\|_q$.

Question: Are all elements of $(L^p)^*$ of the form L_u for some $u \in L^q$?

Answer: Yes, for $p \in (1, \infty)$. This is known as the **Riesz representation theorem**, we will see it later in the course.

Remark: The cases $p = 1$ and $p = \infty$ are more delicate. In any case:

$$p = \infty \implies \|L_u\|_{(L^\infty)^*} = \|u\|_1$$

$$p = 1, X \text{ is } \sigma\text{-finite} \implies \|L_u\|_{(L^1)^*} = \|u\|_\infty$$

12.2 Hahn-Banach theorem and consequences

Theorem 12.2.1 (Hahn-Banach continuous extension theorem). *Let X be a normed space, $Y \subset X$ a subspace, and $L \in Y^*$. Then there exists $\tilde{L} \in X^*$ such that:*

$$\tilde{L}y = Ly \quad \forall y \in Y$$

$$\text{and } \|\tilde{L}\|_{X^*} = \|L\|_{Y^*}.$$

Proof. Omitted, based on the axiom of choice. ■

12.2.1 Consequences of H-B

Corollary 12.2.1.1. *Let X be a normed space, $x_0 \in X \setminus \{0\}$. Then $\exists L \in X^*$ s.t.*

$$\|L\|_{X^*} = 1, \quad Lx_0 = \|x_0\|_X$$

Proof. Take $Y = \text{span}\{x_0\} = \{tx_0 : t \in \mathbb{R}\}$, and $L_0(tx_0) = t\|x_0\|_X$. Notice that:

1. L_0 is well-defined, linear, and continuous.
- 2.

$$\|L_0\|_{Y^*} = \sup_{y \in Y, y \neq 0} \frac{|L_0 y|}{\|y\|} = \sup_{t \neq 0} \frac{|L_0(tx_0)|}{\|tx_0\|} = \sup_{t \neq 0} \frac{|t\|x_0\|_X|}{|t|\|x_0\|_X} = 1$$

Then, by H-B, $\exists L \in X^*$ such that $Lx_0 = \|x_0\|_X$ and $\|L\|_{X^*} = 1$. ■

Corollary 12.2.1.2 (Bounded linear functions separate points). *$\forall x, y \in X$, normed space, we have:*

$$x \neq y \implies \exists L \in X^* : Lx \neq Ly$$

I.e.:

$$Lx = Ly \quad \forall L \in X^* \implies x = y$$

Proof. Take $x \neq y$ and apply previous corollary to $x_0 = x - y \neq 0$. Then, $\exists L \in X^*$ such that $L(x - y) = \|x - y\|_X \neq 0$, i.e., $Lx \neq Ly$. ■

Corollary 12.2.1.3 (Bounded linear functionals separate closed subspaces and points). *Let X be a normed space, $Y \subsetneq X$ a closed subspace, $x_0 \in X \setminus Y$. Then, $\exists L \in X^*$ such that:*

$$Ly = 0, \quad \forall y \in Y \quad \text{and} \quad Lx_0 \neq 0$$

Proof. Take

$$Z = \text{span}\{x_0, Y\} = \text{span}\{x_0\} \oplus Y = \{z \in X : z = tx_0 + y, t \in \mathbb{R}, y \in Y\}$$

Since $x_0 \notin Y$, for every $z \in Z$, t, y are uniquely defined:

$$\begin{aligned} t_1x_0 + y_1 &= t_2x_0 + y_2 \\ \implies (t_1 - t_2)x_0 &= y_1 - y_2 \end{aligned}$$

and because $y_1 - y_2 \in Y$, but $x_0 \notin Y$, then $t_1 = t_2$ and $y_1 - y_2 = 0 \implies y_1 = y_2$.

Let us define $L_0 : Z \rightarrow \mathbb{R}$ as $L_0(tx_0 + y) = t$. We have that $L_0 \in Z^*$, and:

$$L_0x_0 = L_0(1 \cdot x_0 + 0) = 1, \quad \text{and} \quad L_0y = L_0(0 \cdot x_0 + y) = 0$$

And we finally extend it to $L = \tilde{L}_0$ using H-B. ■

12.3 Reflexive spaces

Note: We have X Banach, and $X^* = \mathcal{L}(X, \mathbb{R})$ Banach too. For notation, we will use the following: $L \in X^*, x \in X$:

$$Lx = L(x) = \langle L, x \rangle$$

And notice that $\langle \cdot, \cdot \rangle$ is a **bilinear form**.

Definition 12.3.1. The bidual of X is the dual of X^* , denoted by:

$$X^{**} = (X^*)^* = \mathcal{L}(X^*, \mathbb{R})$$

Definition 12.3.2. Given $x \in X$ we can construct $\Lambda_x \in X^{**}$ as:

$$\Lambda_x L = Lx \quad \forall L \in X^*$$

Using the notation $\langle \cdot, \cdot \rangle$, we have:

$$\langle \Lambda_x, L \rangle = \langle L, x \rangle$$

The mapping $\tau : X \rightarrow X^{**}$ defined by $\tau(x) = \Lambda_x$ is called the **canonical map**.

Proposition 12.3.1. $\forall x \in X$, $\Lambda_x \in X^{**}$, and the canonical map $\tau : X \rightarrow X^{**}$ is an isometry. In other words:

$$\|\tau(x)\|_{X^{**}} = \|x\|_X \quad \forall x \in X$$

Proof. The proof goes as follows:

- Λ_x is linear: Indeed:

$$\langle \Lambda_x, L \rangle = \langle L, x \rangle$$

so it follows the linearity of $\langle \cdot, x \rangle$.

- Λ_x is bounded: It is implied by “isometry”. See below

Isometry: We have that:

$$\|\tau(x)\|_{X^{**}} = \sup_{L \neq 0} \frac{|\langle \Lambda_x, L \rangle|}{\|L\|_{X^*}} = \sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}}$$

Upper bounded:

$$\sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}} \leq \sup_{L \neq 0} \frac{\|L\|_* \cdot \|x\|}{\|L\|_*} = \|x\|$$

Lower bounded:

$$\sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}} \geq \frac{|\bar{L}x|}{\|\bar{L}\|_{X^*}} \quad \forall \bar{L} \neq 0$$

By H-B, $\exists \tilde{L}$, s.t. $\tilde{L}x = \|x\|$ and $\|\tilde{L}\|_{X^*} = 1$. Then, if $\bar{L} = \tilde{L}$, we have:

$$\sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}} \geq \frac{|\tilde{L}x|}{\|\tilde{L}\|_{X^*}} = \|x\|$$

■

Theorem 12.3.2. *Let $\tau : X \rightarrow X^{**}$ be the canonical map. Then:*

- *it is linear and continuous.*
- *it is an isometry.*
- *it is injective.*
- *$R(\tau) \subset X^{**}$ is closed.*
- *it is a continuous embedding*

Remark: This means that we can think that X is a closed subspace of X^{**} , i.e., $X \cong \tau(X)$, and $\tau(X) \subset X^{**}$ is a closed subspace.

Notice that τ may be surjective, in which case $X \cong X^{**}$.

Definition 12.3.3. We say that X is **reflexive** if τ is surjective.

Note: To prove the previous theorem, we will use the following lemma:

Lemma 12.3.3 (Nice properties of linear isometries). *Take X, Y Banach, $T : X \rightarrow Y$ linear such that:*

$$\|Tx\|_Y = \|x\|_X \quad \forall x \in X$$

Then:

- (i) *T is continuous.*
- (ii) *T is injective.*
- (iii) *$R(T) \subset Y$ is closed.*
- (iv) *$T : X \rightarrow R(T)$ is an isomorphism.*

Proof. The proof goes as follows:

(i) T linear $\implies T$ continuous $\iff T$ bounded. Then, notice that:

$$\|Tx\|_Y \leq M \|x\|_X \quad \forall x \in X$$

is true for $M = 1$. Then, T is continuous.

(ii) Let $x, y \in X$ such that $Tx = Ty$. Then:

$$T(x - y) = 0 \implies \|x - y\|_X = 0 \implies x = y$$

(iii) To show that $R(T)$ is closed, take $\{y_n\}_n \subset R(T)$ such that $y_n \rightarrow y \in Y$. We want to show that $y \in R(T)$.

Take $\{x_n\}_n$ such that $Tx_n = y_n$. Notice that since $\{y_n\}_n$ is Cauchy, and $\|y_n - y_m\| = \|T(x_n - x_m)\| = \|x_n - x_m\|$, then $\{x_n\}_n$ is Cauchy too. Then, $\exists x \in X$ such that $x_n \rightarrow x$ because X is Banach. Now, since T is continuous:

$$Tx = T(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} y_n = y$$

Then, $y \in R(T)$.

(iv) Since $T \in \mathcal{L}(X, R(T))$ and $R(T)$ is closed (Banach), then T is bijective between X and $R(T)$. by a corollary of the open mapping theorem, T^{-1} is continuous. ■

Proof (theorem for τ). It is enough to check that:

- τ is linear (direct from linearity of $\langle \cdot, \cdot \rangle$)
- τ is an isometry (already proved) ■

12.3.1 Properties of reflexive spaces

Theorem 12.3.4. *Let X be Banach and reflexive. Let $Y \subset X$ closed subspace. Then, Y is reflexive too.*

Theorem 12.3.5. *Let X be Banach. Then:*

$$X \text{ reflexive} \iff X^* \text{ reflexive}$$

Theorem 12.3.6. *Let X be Banach. Then we have:*

$$X^* \text{ separable} \implies X \text{ separable}$$

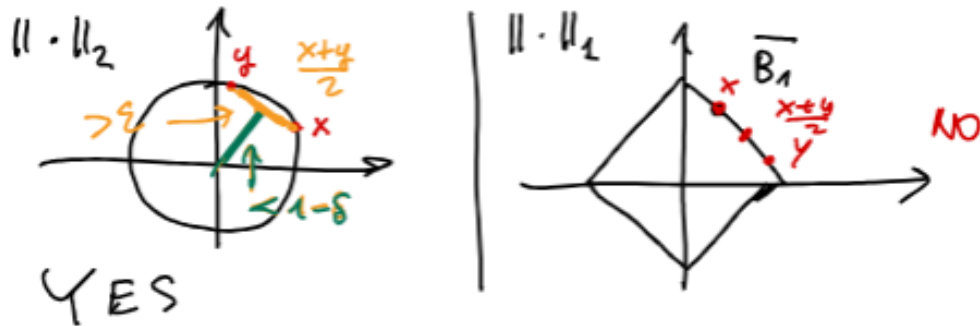
$$X \text{ reflexive and separable} \implies X^* \text{ reflexive and separable}$$

Note: To check reflexivity, it is convenient to introduce the notion of **uniformly convex space**

Definition 12.3.4. We say that X Banach is **uniformly convex** if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.:

$$\begin{cases} x, y \in X \\ \|x\| \leq 1, \|y\| \leq 1 \\ \|x - y\| > \varepsilon \end{cases} \implies \left\| \frac{x + y}{2} \right\| < 1 - \delta$$

Note: This is a “quantitative version” of the strict convexity of $\overline{B_1(0)}$. In \mathbb{R}^2 , is $\overline{B_1(0)}$ strictly convex?



One can see that, in $(\mathbb{R}^N, \|\cdot\|_p)$, the property is true if and only if $p \in (1, \infty)$, and fails for $p = 1, \infty$.

Theorem 12.3.7 (Millman-Pettis). X Banach and uniformly convex $\implies X$ reflexive.

Proof. Omitted. (Very difficult) ■

Corollary 12.3.7.1. $L^p(X)$ is reflexive for $p \in (1, \infty)$

Remark: $L^1(X)$ and $L^\infty(X)$ are **not** reflexive.

12.4 Dual space of L^p

Theorem 12.4.1 (Riesz representation theorem (for $(L^p)^*$)). *Let (X, \mathcal{M}, μ) be a complete measure space, $p \in (1, \infty)$, q the conjugate exponent. Then:*
 $\forall L \in (L^p(X))^*, \exists! u \in L^q(X)$ s.t.:

$$Lv = \int_X uv \, d\mu, \quad \forall v \in L^p(X)$$

Moreover, $\|L\|_{(L^p)^*} = \|u\|_{L^q}$.

Remark: We have already seen that $\forall u \in L^q(X)$, L_u defined as:

$$L_u v = \int_X uv \, d\mu$$

is an element of $(L^p)^*$ and $\|L_u\|_{(L^p)^*} = \|u\|_{L^q}$.

Moreover, for $T : L^q \rightarrow (L^p)^*$ s.t. $T(u) = L_u$, we obtain that T is an isometric isomorphism.

Proof. By the “properties of isometries” and the example of last time, the only thing left to prove is that T is surjective. This follows by H-B. ■

Theorem 12.4.2. *For $p = 1$, X σ -finite, we have that the $T : L^\infty \rightarrow (L^1)^*$ defined as:*

$$T(u)v = \int_X uv \, d\mu \quad \forall v \in L^1$$

is an isometric isomorphism, i.e., $(L^1)^ \cong L^\infty$.*

Theorem 12.4.3. *For $p = \infty$, we have that $L^1 \hookrightarrow (L^\infty)^*$, but the embedding is not surjective.*

E.g.: We have that $\forall u \in L^1$, $L_u \in (L^\infty)^*$, but $(L^\infty)^*$ contains elements that are not of the form L_u .

Take $L^\infty([-1, 1])$ and $C([-1, 1]) \subset L^\infty([-1, 1])$ subspace. Then, take $L_0 : C([-1, 1]) \rightarrow \mathbb{R}$ defined as:

$$L_0 f = f(0)$$

Then, L_0 is linear and bounded, so $L_0 \in (C([-1, 1]))^*$.

By H-B, $\exists \tilde{L}_0 \in (L^\infty([-1, 1]))^*$ such that

$$\tilde{L}_0 f = L_0 f \quad \forall f \in C([-1, 1]), \quad \left\| \tilde{L}_0 \right\|_{(L^\infty)^*} = \|L_0\|_{(C)^*}$$

Claim: $\nexists u \in L^1([-1, 1])$ s.t. $\tilde{L}_0 = L_u$.

To show it, by contradiction, assume that $\exists u \in L^1$ s.t:

$$\int_{-1}^1 uw \, d\mu = \tilde{L}_0 w \quad w \in L^\infty$$

Take w_n s.t.:

$$w_n = \begin{cases} 1 - n|x| & |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

We can show that:

- $w_n(x) \rightarrow 0$ a.e.
- $|w_n(x)| \leq 1$ a.e.
- $(w_n u)(x) \rightarrow 0$ a.e.
- $|w_n(x)u(x)| \leq |u(x)| \in L^1$

Then, by DCT:

$$\int_{-1}^1 uw_n \, d\lambda \rightarrow 0 \quad \text{as } n \rightarrow \infty$$

But:

$$\tilde{L}_0 w_n = L_0 w_n = w_n(0) = 1$$

which is a contradiction.

Note (Resuming): For $L^p(\Omega, \mathcal{L}(\Omega), \lambda)$, $\Omega \in \mathcal{L}(\mathbb{R}^N)$ (but also ℓ^p):

Space	Completeness	Separability	Reflexivity	Dual
$L^p, p \in (1, \infty)$	Yes	Yes	Yes	$L^q, \frac{1}{p} + \frac{1}{q} = 1$
L^1	Yes	Yes	No	L^∞
L^∞	Yes	No	No	$\supsetneq L^1$

Chapter 13

Weak convergence

Definition 13.0.1. Let X be Banach, $\{x_n\}_{n \in \mathbb{N}} \subset X$, $x \in X$. We say that x_n **weakly converges** to x , denoted:

$$x_n \rightharpoonup x$$

if $\forall L \in X^* : Lx_n \rightarrow Lx$

Remark: Suppose that $x_n \rightarrow x$ strongly, and $f : X \rightarrow Y$ continuous. Then $f(x_n) \rightarrow f(x)$. Since $L \in X^*$, then L is continuous, meaning that $Lx_n \rightarrow Lx$. In other words:

$$\{x_n\}_n \text{ converges strongly} \implies \{x_n\}_n \text{ converges weakly}$$

Actually, in \mathbb{R}^N strong convergence *iff* weak convergence.

Remark: For $p \in [1, \infty)$, by using the Riesz rep. thm., we have that:

$$u_n \rightharpoonup u \iff \int_X w u_n d\mu \rightarrow \int_X w u d\mu \quad w \in L^q(X)$$

Proposition 13.0.1. $u_n \rightharpoonup u$ and $u_n \rightarrow v$ a.e., then $u = v$ a.e.

13.1 Basic properties

Proposition 13.1.1. *If it exists, the weak limit is unique.*

Proof. Let $\{x_n\}_n \subset X$, and suppose that $x_n \rightharpoonup y$ and $x_n \rightharpoonup z$. Then, $\forall L \in X^*$, $Lx_n \rightarrow Ly$

and $Lx_n \rightarrow Lz$. Then:

$$Ly = Lz \quad \forall L \in X^* \implies y = z$$

by a corollary of H-B. ■

Proposition 13.1.2. *If $x_n \rightharpoonup x$ in X , then $\{x_n\}_n$ is bounded.*

Proof. Use Banach-Steinhaus in X^* . Let us propose the sequence of operators given by $\{\tau(x_n)\}_{n \in \mathbb{N}} \subset X^{**}$.

Notice that $x_n \rightharpoonup x \implies Lx_n \rightarrow Lx \quad \forall L \in X^*$.

Then, $\langle \tau(x_n), L \rangle = Lx_n \rightarrow Lx = \langle \tau(x), L \rangle \quad \forall L \in X^*$.

This means that $\{\tau(x_n)\}_n$ converges pointwise to $\tau(x)$, i.e.:

$$\langle \tau(x_n), L \rangle \rightarrow \langle \tau(x), L \rangle \quad \forall L \in X^*$$

By Banach-Steinhaus, $\{\tau(x_n)\}_n$ is bounded in X^{**} , meaning that:

$$\exists M > 0 : \text{ s.t. } \|\tau(x_n)\|_{X^{**}} \leq M \quad \forall n \in \mathbb{N}$$

Since $\|\tau(x_n)\|_{X^{**}} = \|x_n\|_X$, we have that:

$$\|x_n\|_X \leq M \quad \forall n \in \mathbb{N}$$

which means that $\{x_n\}_n$ is bounded in X . ■

Proposition 13.1.3. *If $x_n \rightharpoonup x$ in X , then:*

$$\|x\| \leq \liminf_{n \rightarrow \infty} \|x_n\|$$

Proof. By a corollary of H-B, if $x \neq 0$, then $\exists L \in X^*$ s.t.:

$$Lx = \|x\|, \quad \|L\| = 1$$

Then:

$$\begin{aligned} \|x\| = Lx &= \lim_{n \rightarrow \infty} Lx_n = \liminf_{n \rightarrow \infty} Lx_n \leq \liminf_{n \rightarrow \infty} \|L\|_* \|x_n\| \\ &= \liminf_{n \rightarrow \infty} \|x_n\| \end{aligned}$$

■

Remark: Notice that $\|\cdot\| : X \rightarrow \mathbb{R}$ is strongly continuous, but not weakly continuous. It is “weakly lower semicontinuous”.

Proposition 13.1.4. *Let $x_n \rightharpoonup x$ in X , and $L_n \rightarrow L$ strongly in X^* . Then:*

$$L_n x_n \rightarrow Lx$$

The same if $L_n \rightharpoonup L$ in X^ and $x_n \rightarrow x$ strongly.*

If both sequences converge weakly, nothing can be inferred.

Proof. Let $x_n \rightharpoonup x$ in X , and $L_n \rightarrow L$ in X^* . Then:

$$0 \leq |L_n x_n - Lx| = |L_n x_n - Lx_n + Lx_n - Lx| \leq |L_n x_n - Lx_n| + |Lx_n - Lx|$$

Notice that $|Lx_n - Lx| \rightarrow 0$ by the strong convergence of x_n . Also, we have that:

$$|L_n x_n - Lx_n| \leq \|L_n - L\|_* \|x_n\| \rightarrow 0$$

This means that $L_n x_n \rightarrow Lx$.

■

Proposition 13.1.5. *Let X be Banach, $V \subset X^*$ dense, $\{x_n\}_n \subset X$ bounded. Then:*

$$Lx_n \rightarrow Lx \quad \forall L \in V \implies Lx_n \rightarrow Lx \quad \forall L \in X^*$$

i.e., $x_n \rightharpoonup x$.

Proof. Omitted (as the one in prop. 4, and use the density of V)

■

E.g.: Recall that $1 < p < \infty$, then $u_n \rightharpoonup u$ in $L^p(\Omega)$ if:

$$\int_{\Omega} w u_n d\mu \rightarrow \int_{\Omega} w u d\mu \quad \forall w \in L^q(\Omega)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

By property 5, it is enough to ask: $\{u_n\}_n \subset L^p(\Omega)$ bounded and:

$$\int_{\Omega} w u_n d\mu \rightarrow \int_{\Omega} w u d\mu \quad \forall w \in C_c(\Omega)$$

or $\forall w$ simple functions.

Proposition 13.1.6. Let X, Y Banach, $T \in \mathcal{L}(X, Y)$ and $\{x_n\}_n \subset X$. Then:

$$x_n \rightharpoonup x \implies T x_n \rightharpoonup T x$$

We say that T is “weakly-weakly continuous”.

Definition 13.1.1. Let X be Banach, X^* (Banach) dual of X , $\{L_n\}_n \subset X^*$ and $L \in X^*$. We say that L_n **weakly-* converges** to L , denoted:

$$L_n \xrightarrow{*} L$$

if $\forall x \in X$, $L_n x \rightarrow L x$ as $n \rightarrow \infty$.

Remark: Note that:

- $L_n \rightharpoonup L$ if $\phi L_n \rightarrow \phi L \quad \forall \phi \in X^{**}$.
- $L_n \xrightarrow{*} L$ if $\tau(x) L_n \rightarrow \tau(x) L \quad \forall x \in X$.

Proposition 13.1.7. If X is reflexive, then:

$$L_n \xrightarrow{*} L \text{ in } X^* \iff L_n \rightharpoonup L \text{ in } X^*$$

E.g.: Weak-* convergence in $L^\infty(\Omega)$ ($\Omega \in \mathcal{L}(\mathbb{R}^N)$):

We know that $L^1(\Omega)$ is Banach and $L^\infty(\Omega) \cong (L^1(\Omega))^*$.

Then, for $\{u_n\}_n \subset L^\infty(\Omega)$, $u \in L^\infty(\Omega)$, we have that $u_n \xrightarrow{*} u$ in $L^\infty(\Omega)$ if:

$$\int_{\Omega} u_n v \, d\mu \rightarrow \int_{\Omega} u v \, d\mu \quad \forall v \in L^1(\Omega)$$

Remark: In general, weak convergence implies weak-* convergence, but the converse is not true.

Properties 13.1.1 (Weak-* convergence). For weak-* convergence, we have that:

1. If $L_n \xrightarrow{*} L$, then the limit is unique.
2. If $L_n \xrightarrow{*} L$, then $\{L_n\}_n$ is bounded in X^* .
3. If $L_n \xrightarrow{*} L$, then:

$$\|L\|_* \leq \liminf_{n \rightarrow \infty} \|L_n\|_*$$

4. If $L_n \xrightarrow{*} L$ and $x_n \rightarrow x$ strongly, then:

$$L_n x_n \rightarrow Lx$$

Remark: The notions of (topological) dual, weak convergence, weak-* convergence, do not need norms, just a topology. E.g., “test functions” $\mathcal{D}(\mathbb{R}^N) = C_c^\infty(\mathbb{R}^N)$, have a topological dual $\mathcal{D}'(\mathbb{R}^N)$, and convergence in \mathcal{D}' is the weak-* convergence.

Remark: We defined weak (weak-*) convergence, not the weak (weak-*) topology. This topology in general is not metrizable and weakly compact sets are not weakly sequentially compact.

13.2 Banach-Alaoglu theorem

Theorem 13.2.1 (Banach-Alaoglu (variant 1)). *Let X be Banach and reflexive. Then, every bounded sequence $\{x_n\}_n \subset X$ admits a subsequence $\{x_{n_k}\}_k$ which weakly converges in X .*

Theorem 13.2.2 (Banach-Alaoglu (variant 2)). *Let X be Banach and separable. Then, every bounded sequence $\{L_n\}_n \subset X^*$ admits a subsequence $\{L_{n_k}\}_k$ which weakly- $*$ converges in X^* .*

E.g.: Let $1 < p < \infty$, then we know that $L^p(\Omega)$ is reflexive. Moreover, we know that $f_n \rightharpoonup f$ in $L^p \iff$:

$$\int_{\Omega} f_n g \, d\mu \rightarrow \int_{\Omega} f g \, d\mu \quad \forall g \in L^q(\Omega)$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If we apply variant 1 of Banach-Alaoglu, we have that:

$\forall \{u_n\}_n \subset L^p(\Omega)$, s.t., $\|u_n\|_p \leq M \, \forall n \in \mathbb{N}$, $\exists \{u_{n_k}\}_k, u \in L^p(\Omega)$ s.t. $u_{n_k} \rightharpoonup u$ in $L^p(\Omega)$, i.e.:

$$\int_{\Omega} u_{n_k} g \, d\mu \rightarrow \int_{\Omega} u g \, d\mu \quad \forall g \in L^q(\Omega)$$

Also, we know that $L^1(\Omega)$ is separable, and $(L^1(\Omega))^* \cong L^\infty(\Omega)$. Then, if we apply variant 2 of Banach-Alaoglu, we have that:

$\forall \{u_n\}_n \subset L^\infty(\Omega)$, s.t., $\|u_n\|_\infty \leq M \, \forall n \in \mathbb{N}$, $\exists \{u_{n_k}\}_k, u \in L^\infty(\Omega)$ s.t. $u_{n_k} \xrightarrow{*} u$ in $L^\infty(\Omega)$, i.e.:

$$\int_{\Omega} u_{n_k} g \, d\mu \rightarrow \int_{\Omega} u g \, d\mu \quad \forall g \in L^1(\Omega)$$

Finally, bounded sequences on L^1 have no reason to converge.

Chapter 14

Compact operators

Note: We will work with X, Y Banach spaces.

Definition 14.0.1. Let $K : X \rightarrow Y$ be a linear operator. We say that K is **compact** if:

$\forall E \subset X$ bounded, $K(E)$ is relatively compact, i.e., $\overline{K(E)}$ is compact

Or, equivalently:

$\forall \{x_n\}_n \subset X$ bounded, $\exists \{K(x_{n_k})\}_k \subset Y$ (strongly) convergent subsequence

Proposition 14.0.1. Let $K : X \rightarrow Y$ be a linear compact operator. Then K is bounded, i.e., $K \in \mathcal{L}(X, Y)$.

Proof. We know that $B_1(0) \subset X$ is bounded. Then, $\overline{K(B_1(0))}$ is compact in Y . Therefore, $\overline{K(B_1(0))}$ is bounded in Y .

Then, $\exists M > 0$ such that $\|K(x)\| \leq M \quad \forall x \in B_1(0)$. ■

Remark: The above property is not true for non-linear compact operators.

Exercise (Compactness of the integral map): Let $K : C([0, 1]) \rightarrow C([0, 1])$ be the integral map:

$$K(f)(x) = \int_0^x f(t) dt \quad \forall x \in [0, 1]$$

and note that it is linear. Prove that K is compact.

(Hint: take $\{u_n\}_n \subset C([0, 1])$ bounded, and prove that $\{K(u_n)\}_n$ has a convergent subsequence using the Arzelà-Ascoli theorem).

Definition 14.0.2. We say that $T \in \mathcal{L}(X, Y)$ is a **finite rank operator** if:

$$\dim R(T) < \infty$$

(Note: $R(T) = T(X)$).

E.g.: As many as you want:

$T \in X^*: C^k([a, b]) \rightarrow \mathbb{P}^k$ polynomials of degree k :

- Taylor expansion
- Lagrange interpolation
- etc.

Proposition 14.0.2. Let $T \in \mathcal{L}(X, Y)$ be a finite rank operator. Then T is compact.

Proof. Let $A \subset X$ be bounded. Then, $T(A)$ is bounded in Y , and $\overline{T(A)}$ is bounded and closed in Y .

Since $\dim R(T) < \infty$, $\overline{T(A)}$ is compact in Y . ■

Definition 14.0.3. We denote $\mathcal{K}(X, Y)$ as the set of compact operators from X to Y , i.e.:

$$\mathcal{K}(X, Y) = \{K \in \mathcal{L}(X, Y) \mid K \text{ is compact}\}$$

Theorem 14.0.3. Let X, Y be Banach spaces. Then $\mathcal{K}(X, Y)$ is a closed vector subspace of $\mathcal{L}(X, Y)$.

Remark: Now, to check that T is compact, it is enough to find a sequence $\{T_n\}_n \subset \mathcal{K}(X, Y)$ such that $T_n \rightarrow T$ in $\mathcal{L}(X, Y)$, i.e., $\|T_n - T\|_{\mathcal{L}(X, Y)} \rightarrow 0$.

Theorem 14.0.4 (Compact operators vs weak convergence). *Let X, Y be Banach, then:*

(i) *If $T \in \mathcal{K}(X, Y)$, then:*

$$\{x_n\}_n \subset X \text{ s.t. } x_n \rightharpoonup x \text{ in } X \implies T(x_n) \rightarrow T(x) \text{ in } Y$$

(ii) *If X is reflexive, then, the converse is also true, i.e., $T \in \mathcal{K}(X, Y)$ if $\forall \{x_n\}_n \subset X$:*

$$x_n \rightharpoonup x \text{ in } X \implies T(x_n) \rightarrow T(x) \text{ in } Y$$

Proposition 14.0.5. *Let $T \in \mathcal{K}(X, Y)$, and $\dim Y = \infty$. Then, T cannot be surjective.*

Proposition 14.0.6. *Take either $T \in \mathcal{L}(X, Y)$, $S \in \mathcal{K}(X, Y)$ or $T \in \mathcal{K}(X, Y)$ and $S \in \mathcal{L}(X, Y)$. Then:*

$$S \circ T \in \mathcal{K}(X, Y)$$

Proof. Trivial, because bounded operators map bounded sets to bounded sets, and precompact sets to precompact sets. ■

Chapter 15

Hilbert spaces

Definition 15.0.1. Let H be a (real) vector space. A function $p : H \times H \rightarrow \mathbb{R}$ is called a **scalar product** (or **inner product**) if:

- (i) (Positivity): $p(x, x) \geq 0 \forall x \in H$, and $p(x, x) = 0 \iff x = 0$.
- (ii) (Symmetry): $p(x, y) = p(y, x) \forall x, y \in H$.
- (iii) (Bilinearity): $p(\alpha x + \beta y, z) = \alpha p(x, z) + \beta p(y, z)$
 $\forall x, y, z \in H$ and $\alpha, \beta \in \mathbb{R}$.

Note: For notation, we use the following:

$$p(x, y) = \langle x, y \rangle = (x, y) = x \cdot y$$

Definition 15.0.2. The space $(H, \langle \cdot, \cdot \rangle)$ is called a **pre-Hilbertian** space (inner product space).

Proposition 15.0.1. Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbertian space. Then:

- 1) $|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$ (Cauchy-Schwarz inequality).
- 2) $\|x\| = \sqrt{\langle x, x \rangle}$ is a norm on H .
- 3) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (parallelogram law).

Proof. The proof is as follows:

- 1) Same as \mathbb{R}^N .

2) Exercise (Cauchy-Schwarz ineq. \implies triangle ineq.)

3) $\langle x \pm y, x \pm y \rangle = \|x\|^2 \pm 2 \langle x, y \rangle + \|y\|^2$.

■

Remark: Notice that, because an inner product induces a norm, the space (H, d) with $d(x, y) = \|x - y\|$ is a metric space. Then, we can talk about convergence.

Definition 15.0.3. A pre-Hilbertian space $(H, \langle \cdot, \cdot \rangle)$ is called a **Hilbert space** if it is complete with respect to the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$. (I.e., if $(H, \|\cdot\|)$ is a Banach space).

E.g.: We have the following examples of Hilbert spaces:

1) \mathbb{R}^N with the Euclidean scalar product (usual dot product).

2) $L^2(X, \mathcal{M}, \mu)$ with the scalar product:

$$\langle f, g \rangle = \int_X f \cdot g \, d\mu$$

That induces the norm:

$$\|f\| = \left(\int_X f^2 \, d\mu \right)^{1/2}$$

Notice that $(C([a, b]), \langle \cdot, \cdot \rangle_{L^2})$ is an inner product space, but not a Hilbert space.

Proposition 15.0.2. Let $(X, \|\cdot\|)$ be a Banach space. Then, it is also a Hilbert space if and only if the norm satisfies the parallelogram law. The inner product is then given by:

$$\langle x, y \rangle = \frac{1}{4} (\|x + y\|^2 - \|x - y\|^2)$$

Remark: With the proposition above, we can check that the space $(C([0, 1]), \|\cdot\|_\infty)$ is not a Hilbert space. Also, $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_p)$ is not a Hilbert space for $p \neq 2$.

Definition 15.0.4. Let H be a Hilbert space. We say that:

- (i) $x, y \in H$ are **orthogonal** $\iff \langle x, y \rangle = 0$. We write $x \perp y$.
- (ii) Given $V \subset H$, the **orthogonal complement** of V is:

$$V^\perp = \{x \in H \mid \langle x, y \rangle = 0 \ \forall y \in V\}$$

15.1 Orthogonal projections

Note: Recall that:

- $S \subset H$ is convex is $\forall x, y \in S, \alpha x + (1 - \alpha)y \in S$ for all $\alpha \in [0, 1]$.
- $S \subset H, x \in H$, then the distance from x to S is:

$$d(x, S) = \inf_{y \in S} \|x - y\|$$

Theorem 15.1.1 (Projection theorem on closed convex sets). *Let H be Hilbert, $x \in H$ and $S \subset H$ closed, convex and non-empty. Then:*

$$\exists! h \in S \text{ s.t. } d(x, S) = \|x - h\|$$

Moreover, h is characterized by the “variational inequality”:

$$\langle x - h, y - h \rangle \leq 0 \quad \forall y \in S$$

(This inequality is equivalent to the first statement)

We call h the **orthogonal projection** of x onto S .

Proof. The proof is as follows:

- 1) Existence: Let $d = d(x, S)$. Then, take a “minimizing sequence”, $\{y_n\}_n \subset S$ such that $\|x - y_n\| \rightarrow d$.

Then, we are going to show that $\{y_n\}_n$ is a Cauchy sequence by applying the parallelogram law to $x - y_n$ and $x - y_m$:

$$\begin{aligned} \|x - y_n + x - y_m\|^2 + \|x - y_n - x + y_m\|^2 &= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 \\ \implies \|y_m - y_n\|^2 &= 2\|x - y_n\|^2 + 2\|x - y_m\|^2 - \|2x - y_n - y_m\|^2 \end{aligned}$$

Notice that:

$$\|2x - v_n - v_m\|^2 = 4 \left\| x - \frac{v_n + v_m}{2} \right\|^2 \geq 4d^2$$

since $\frac{v_n + v_m}{2} \in S$ (convexity). Then, we have:

$$\|v_m - v_n\|^2 \leq 2\|x - v_n\|^2 + 2\|x - v_m\|^2 - 4d^2 \rightarrow 0$$

Then, $\{y_n\}_n$ is Cauchy, and since H is complete, $\exists h \in H$ such that $\|x - h\| = d$. Also, $h \in S$ because S is closed.

- 2) Uniqueness: Let $h_1, h_2 \in S$ be two orthogonal projections of x onto S . Then, using the parallelogram law, we have:

$$\begin{aligned} \|h_1 - h_2\|^2 &= 2\|x - h_1\|^2 + 2\|x - h_2\|^2 - \|2x - h_1 - h_2\|^2 \\ &\leq 2d^2 + 2d^2 - 4d^2 = 0 \end{aligned}$$

Then, $h_1 = h_2$.

■

Theorem 15.1.2 (Projection theorem on closed subspaces). *Let H be Hilbert, $x \in H$, $V \subset H$ a closed subspace. Then:*

$$\exists! h \in V : \|x - h\| = d(x, V)$$

Moreover, h satisfies the previous implication if and only if:

$$\langle x - h, y \rangle = 0 \quad \forall y \in V$$

Remark: Notice that $x - h \perp y \quad \forall y \in V$, meaning that $x - h \in V^\perp$.

We use the following notation:

$$h = P_V x = \text{proj}_V x$$

Remark: Let H be a Hilbert space, $V \subset H$ a subspace. Then, it is always closed on the following cases:

- if $\dim V < \infty$
- $V = \text{Ker} L$ for some $L \in \mathcal{L}(H, Y)$

- V^\perp is closed for any V . This implies that:

$$(V^\perp)^\perp = \overline{V}$$

Theorem 15.1.3. *Let H be Hilbert. $V \subset H$ a closed subspace. Then:*

$$(i) \quad \forall x \in H, \quad x = P_V x + P_{V^\perp} x$$

$$(ii) \quad x \in V \iff x = P_V x$$

$$(iii) \quad \|x\|^2 = \|P_V x\|^2 + \|P_{V^\perp} x\|^2$$

$$(iv) \quad P_V, P_{V^\perp} \in \mathcal{L}(H) \text{ and their norm is } 1.$$

15.2 Dual of a Hilbert space

Definition 15.2.1. Let H be Hilbert. We define the mapping $i : H \rightarrow H^*$ (**Riesz map isometry**) as:

$$i(u) = L_u$$

where L_u is defined as:

$$L_u v := \langle u, v \rangle, \quad \forall v \in H$$

Notice that L_u is linear, and moreover, $\|L_u\|_* = \|u\|$.

Theorem 15.2.1 (Riesz representation theorem). *Let H be Hilbert. Then, $\forall L \in H^*$, $\exists! u \in H$ s.t.:*

$$L_v = \langle u, v \rangle, \quad \forall v \in H$$

Moreover, $\|u\| = \|L\|_$. This means that the mapping i is an isometric isomorphism.*

Corollary 15.2.1.1. *Let H be Hilbert, then H is reflexive. Also:*

$$H \cong H^* \implies H^* \cong H^{**}$$

(or: parallelogram $\implies H$ unif. convex)

Remark: We can identify H and H^* , but depending on $\langle \cdot, \cdot \rangle$. So, for a $V \subset H$ subspace dense, we have that:

$$V \subset H \cong H^* \subset V^*$$

Remark: The Riesz rep. thm. is a “well-posedness” theorem.

Proof (Riesz): The proof goes as follows:

- Existence:

Case 1: $\text{Ker} L = H$. Then, take $u = 0$. Notice that:

$$\langle 0, v \rangle = 0 = Lv \quad \forall v \in H$$

Case 2: $\exists z_0 \in H \setminus \text{Ker} L$. Since $\text{Ker} L$ is a closed subspace of H , let:

$$z := \frac{P_{(\text{Ker} L)^\perp} z_0}{\|P_{(\text{Ker} L)^\perp} z_0\|}$$

and notice that $\|z\| = 1$ and $z \in (\text{Ker} L)^\perp$. Take $v \in H$ and

$$w = v - \frac{Lv}{Lz} z$$

so that $Lw = 0$, $w \in \text{Ker} L$. Now, we have:

$$\begin{aligned} 0 = \langle w, z \rangle &= \left\langle v - \frac{Lv}{Lz} z, z \right\rangle = \langle v, z \rangle - \frac{Lv}{Lz} \langle z, z \rangle \\ &= \langle z, v \rangle - \frac{Lv}{Lz} \end{aligned}$$

I.e.:

$$Lv = Lz \langle z, v \rangle = \langle (Lz)z, v \rangle \quad \forall v \in H$$

Now, let $u = (Lz)z$

- Uniqueness: Let $u_1, u_2 \in H$ s.t.

$$Lv = \langle u_1, v \rangle \quad \forall v \in H$$

$$Lv = \langle u_2, v \rangle \quad \forall v \in H$$

Then:

$$\langle u_1 - u_2, v \rangle = 0 \quad \forall v \in H$$

Take $v = u_1 - u_2$, then:

$$\|u_1 - u_2\|^2 = 0$$

Therefore, $u_1 = u_2$.

Finally:

$$\|L\|_* = \sup_{x \neq 0} \frac{|Lx|}{\|x\|} = \sup_{x \neq 0} \frac{|\langle u, x \rangle|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|u\| \|x\|}{\|x\|} = \|u\|$$

$$\|L\|_* = \sup_{x \neq 0} \frac{|Lx|}{\|x\|} = \sup_{x \neq 0} \frac{|\langle u, x \rangle|}{\|x\|} \geq \frac{|\langle u, u \rangle|}{\|u\|} = \|u\|$$

Then, $\|L\|_* = \|u\|$. ■

15.3 Consequences of the Riesz theorem

Theorem 15.3.1. *Let H be Hilbert, $\{x_n\}_n \subset H$. Then, $x_n \rightharpoonup x$ in H if and only if:*

$$\langle u, x_n \rangle \rightarrow \langle u, x \rangle \quad \forall u \in H$$

Moreover, by reflexivity, if $\{x_n\}_n \subset H$ is bounded, then $\exists \{x_{n_k}\}_k$ subsequence such that:

$$x_{n_k} \rightharpoonup x$$

Proposition 15.3.2. *Let $\{x_n\}_n \subset H$ and assume that:*

(i) $x_n \rightharpoonup x$ weakly in H

(ii) $\|x_n\| \rightarrow \|x\|$ strongly in H

Then, $x_n \rightarrow x$ strongly in H .

Proof. We have that:

$$\|x_n - x\|^2 = \|x_n\|^2 - 2\langle x_n, x \rangle + \|x\|^2$$

Notice that $\|x_n\| \rightarrow \|x\|$ and $\langle x_n, x \rangle \rightarrow \langle x, x \rangle = \|x\|^2$. Then:

$$\|x_n - x\|^2 \rightarrow 0 \implies x_n \rightarrow x$$
■

15.4 Orthonormal basis

Note: We will consider H as a Hilbert space.

Definition 15.4.1. A sequence $\{e_n\}_{n \in \mathbb{N}} \subset H$ is an **orthonormal basis** if:

- (i) $\|e_n\| = 1, \langle e_i, e_j \rangle = 0 \ \forall i \neq j$.
- (ii) $\text{span}(\{e_n\}_{n \in \mathbb{N}})$ is dense in H , i.e. $H = \overline{\text{span}(\{e_n\}_{n \in \mathbb{N}})}$.

(Note: the span of an infinite sequence of vectors consists of all the finite linear combinations of them).

E.g.: We have some examples:

- $H = \ell^2$, we have $e_1 = (1, 0, 0, \dots), \ e_2 = (0, 1, 0, \dots), \dots$
- $H = L^2[-\pi, \pi]$ we have:

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}}, n = 1, 2, 3, \dots \right\}$$

Theorem 15.4.1. Every separable Hilbert space H has an orthonormal basis.

Theorem 15.4.2. Let $\{e_n\}_{n \in \mathbb{N}} \subset H$ be an orthonormal basis. Then:

(i) $\forall u \in H$,

$$u = \sum_{n \in \mathbb{N}} \langle u, e_n \rangle e_n$$

and

$$\|u\|^2 = \sum_{n \in \mathbb{N}} |\langle u, e_n \rangle|^2$$

(Parseval-Bessel identity).

(ii) Conversely: $\{\alpha_n\}_{n \in \mathbb{N}} \in \ell^2$, then:

$$\sum_{n \in \mathbb{N}} \alpha_n e_n = x \in H$$

with $\langle x, e_n \rangle = \alpha_n$.

Proposition 15.4.3. $\{e_n\}_{n \in \mathbb{N}}$ orthonormal basis. Then:

$$e_n \rightharpoonup 0 \text{ weakly in } H$$

but $e_n \not\rightarrow 0$ strongly.

Proof. By the theorem, $\forall u \in H$, the series:

$$\sum_n |\langle u, e_n \rangle|^2 < \infty$$

This implies that $\langle u, e_n \rangle \rightarrow 0$, $\forall u \in H$. So, $e_n \rightharpoonup 0$. But:

$$\|e_n\| = 1 \not\rightarrow 0$$

■

Chapter 16

Spectral theory

Note: We will consider E Banach, $T \in \mathcal{L}(E, E) = \mathcal{L}(E)$, and the problem:

$$Tx = \lambda x \iff (T - \lambda I)x = 0$$

Definition 16.0.1. We define the following concepts:

- The **resolvent set** of T is:

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I : E \rightarrow E \text{ is bijective}\}$$

- The **spectrum** of T is:

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

- λ is an **eigenvalue** of T if:

$$\text{Ker}(T - \lambda I) \neq \{0\}$$

where $\text{Ker}(T - \lambda I)$ is called the **eigenspace** corresponding to λ . Also:

$$EV(T) = \{\text{eigenvalues of } T\} \subset \mathbb{R}$$

Remark: Note that:

$$EV(T) \subset \sigma(T)$$

as $\lambda \in EV(T) \iff T - \lambda I$ is not injective. Also, note that if $\dim E < \infty$, then $EV(T) = \sigma(T)$. If E has infinite dimension, then the inclusion may be strict.

Theorem 16.0.1. Let E be Banach, $T \in \mathcal{L}(E)$. Then:

$$(i) \sigma(T) \subset [-\|T\|, \|T\|]$$

(ii) $\sigma(T)$ is closed.

Remark: (i) means that the “spectral radius” is always \leq the operatorial norm of T .

Remark: $|\lambda| > \|T\| \implies T - \lambda I$ is invertible. Moreover, $\rho(T)$ is open.

E.g.: Let $E = \ell^2$. We define the “left shift operator” $T_\ell : \ell^2 \rightarrow \ell^2$ as follows:

$$T_\ell x = (x^{(1)}, x^{(2)}, \dots)$$

for $x = (x^{(0)}, x^{(1)}, x^{(2)}, \dots)$. Then, one can prove that $T_\ell \in \mathcal{L}(\ell^2)$ and $\|T_\ell\| = 1$. By the theorem, $\sigma(T) \subset [-1, 1]$, closed.

Also, notice that (for $\lambda = 0$) T_ℓ is surjective, but not injective. I.e.,

$$R(T_\ell) = \ell^2, \quad \text{Ker } T_\ell = \{x \in \ell^2 : x^{(k)} = 0 \ \forall k \geq 1, \ x^{(0)} \in \mathbb{R}\}$$

Then, $\lambda = 0$ is an eigenvalue of multiplicity 1.

Let us look for more eigenvalues: we know that $\lambda \in EV(T) \iff \exists x \neq 0$ s.t.:

$$\begin{aligned} T_\ell x &= \lambda x \\ \iff (T_\ell x)^{(k)} &= \lambda x^{(k)} \quad \forall k \geq 0 \\ \iff x^{(k+1)} &= \lambda x^{(k)} \quad \forall k \geq 0 \end{aligned}$$

Take any $x^{(0)} = x_0 \neq 0$. Then, notice that:

$$\begin{aligned} x^{(1)} &= \lambda x_0 \\ x^{(2)} &= \lambda x^{(1)} = \lambda^2 x_0 \\ &\vdots \\ x^{(k)} &= \lambda x^{(k-1)} = \dots = \lambda^k x_0 \end{aligned}$$

Then, λ is an eigenvalue $\iff x = (x_0, \lambda x_0, \lambda^2 x_0, \dots) = x_0(1, \lambda, \lambda^2, \dots) \in \ell^2$

$$\iff \sum_{k=0}^{\infty} (\lambda^k)^2 < \infty \iff |\lambda| < 1$$

Then, $EV(T) = (-1, 1)$ and

$$(-1, 1) \subset \sigma(T) \subset [-1, 1]$$

and because $\sigma(T)$ is closed, we conclude that $\sigma(T) = [-1, 1]$.

Exercise: Discuss $T_r : \ell^2 \rightarrow \ell^2$ the “right shift operator” such that:

$$T_r x = (0, x^{(0)}, x^{(1)}, x^{(2)}, \dots)$$

Show that $T_r \in \mathcal{L}(\ell^2)$ and $\|T_r\| = 1$. Then, show that $\sigma(T_r) = [-1, 1]$ and $EV(T_r) = \emptyset$.

16.1 Symmetric operators

Note: In what follows, consider:

- $(H, \langle \cdot, \cdot \rangle)$ a Hilbert space.
- $T \in \mathcal{K}(X) = \mathcal{K}(X, X)$ a compact operator.
- T is symmetric (self-adjoint)

Definition 16.1.1. We say that T is **symmetric** \iff

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in H$$

Remark: If T is symmetric, then:

$$\|T\|_{\mathcal{L}(H)} = \sup_{x \neq 0} \frac{\langle Tx, x \rangle}{\|x\|^2}$$

This is called the **Rayleigh quotient**.

16.1.1 Fredholm's alternative theorem

Theorem 16.1.1 (Fredholm's alternative theorem). *Let H be Hilbert, $T \in \mathcal{K}(H)$ symmetric. Then:*

- (i) $\dim \text{Ker}(I - T) < \infty$
- (ii) $R(I - T)$ is closed.
- (iii) $\text{Ker}(I - T) = R(I - T)^\perp$ and $R(I - T) = \text{Ker}(I - T)^\perp$
(in particular, $I - T$ is surjective \iff is injective)
- (iv) Consider the following problem:

$$(\star) = \begin{cases} \text{Given } f \in H, \text{ find } x \in H, \text{ s.t.:} \\ (I - T)x = f \end{cases}$$

Then, exactly one of the following is true:

- $\forall f, \exists! x \in H$ solving (\star)
- (\star) is solvable $\iff f \in \text{Ker}(I - T)^\perp$, and because $\dim \text{Ker}(I - T) = N < \infty$, this means that:

$$\langle f, u_i \rangle = 0 \quad \forall i = 1, \dots, N$$

$$\text{s.t. } \text{span}(\{u_i\}_{i=1}^N) = \text{Ker}(I - T).$$

Remark: Consider $\lambda \neq 0$, $T - \lambda I$. Then, the FAT applies:

$$T - \lambda I = -\lambda(I - \frac{1}{\lambda}T)$$

where $\frac{1}{\lambda}T \in \mathcal{K}(H)$. As a consequence, we have that, for $T \in \mathcal{K}(H)$ symmetric:

$$\sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$$

Remark: For the theorem, there are some conditions that are not strictly necessary:

- T symmetric is not necessary, as (i), (ii) are true, and (iii), (iv) can be formulated in terms of the adjoint operator T^* : $\langle Tx, y \rangle = \langle x, T^*y \rangle$.
- H Hilbert is not necessary, for E Banach, use duality pairing instead of the scalar

product.

Notice that, without the compactness assumption, the theorem breaks.

16.2 Spectral theorem

Theorem 16.2.1 (Spectral theorem). *Let H be Hilbert and separable, with $\dim H = \infty$. Let $T \in \mathcal{K}(H)$ symmetric. Then:*

$$0 \in \sigma(T), \quad \sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$$

and the following alternative holds:

- (i) *Either T has finitely many eigenvalues different from 0, and then $0 \in EV(T)$, with $\dim \text{Ker}(T) = \infty$*
- (ii) *Or $EV(T) \setminus \{0\}$ is a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset [-\|T\|, \|T\|]$ and $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, i.e., $0 \in \sigma(T)$.*

Moreover, in both cases, there exists an orthonormal basis of H made by the eigenvectors of T .

Proof. The proof is based on the FAT, and it is omitted.

■