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Real and Functional Analysis

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Contents

Chapter 1

Set Theory

1.1 Basic notions

Definition 1.1.1. Let X, Y be sets. We say:
- X, Y are **equipotent** if there exists a bijection $f : X \to Y$.
- X has a **cardinality greater or equal** to Y if there exists an surjection $f : X \to Y$.
- X is **finite** if it is equipotent to $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. X is infinite otherwise.

Note: X is infinite $\iff$ it is equipotent to a proper subset of itself.

E.g.: The set of natural numbers $\mathbb{N}$ is infinite. In fact, the set of even natural numbers $E = \{2, 4, 6, \dots\} \subset \mathbb{N}$ is equipotent to $\mathbb{N}$, as we can define the bijection $f : \mathbb{N} \to E$ as $f(n) = 2n$.

Definition 1.1.2. Let X be an infinite set. We say X is **countable** if it is equipotent to $\mathbb{N}$. X is **uncountable** otherwise, in which case it is **more than countable**.

Definition 1.1.3. X has the **cardinality of the continuum** if it is equipotent to $[0, 1] \subset \mathbb{R}$. Any such set is uncountable.

E.g.: We have that:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable.
- $\mathbb{R}, \mathbb{R}^n, (0, 1), [0, 1]$ are uncountable.
- Countable union of countable sets is countable.

1.2 Families of subsets

Let X be a set. The "Power set" of X is the set of all subsets of X, denoted by $\mathcal{P}(X)$.

$$\mathcal{P}(X) = \{E : \ E \subseteq X\}$$

Note that $\mathcal{P}(X)$ has always a cardinality greater than X. For example, if $X = \mathbb{N}$, then $\mathcal{P}(X)$ has the cardinality of the continuum.

Definition 1.2.1. Let X be a set. A **family of subsets** of X is a set E such that:

$$E \subseteq \mathcal{P}(X)$$

We usually denote $E = \{E_i\}_{i \in I}$, where I is an index set.

Definition 1.2.2. Let $E = \{E_i\}_{i \in I}$ be a family of subsets of X. We define:

- The **union** of E as:

$$\bigcup_{i \in I} E_i = \{x \in X : \ x \in E_i \text{ for some } i \in I\}$$

- The **intersection** of E as:

$$\bigcap_{i \in I} E_i = \{x \in X : \ x \in E_i \text{ for all } i \in I\}$$

Definition 1.2.3. Let $E = \{E_i\}_{i \in I}$ be a family of subsets of X. We say F is **pairwise disjoint** if:

$$E_i \cap E_j = \emptyset \ \ \forall i, j \in I, i \neq j$$

Definition 1.2.4. We say that the family $E = \{E_i\}_{i \in I}$ of subsets of X is a **covering** of X if:

$$X = \bigcup_{i \in I} E_i$$

Any subfamily of E, $E' = \{E_i\}_{i \in I'}$ is a **subcovering** of X if it is a covering of X itself.

E.g.: Let $X = \mathbb{R}$. We define:

$$\mathcal{T} = \{E \subset X : E \text{ is open}\}$$

We say that $\mathcal{T}$ is the standard topology of X. More generally, this can be done in "metric spaces" (X, d).

Properties of $\mathcal{T}$ (open sets):

- $\emptyset, X \in \mathcal{T}$.
- Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$.
- Arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$.

We can also define **sequences of sets**. Let X be a set. A sequence of sets in X is a family of sets $\{E_n\}_{n\in\mathbb{N}}$.

Definition 1.2.5. Let X be a set. A sequence of sets $\{E_n\}_{n\in\mathbb{N}}$ is said to be:

- **Increasing** if:
$$E_n \subseteq E_{n+1} \ \ \forall n \in \mathbb{N}$$
It is denoted by $\{E_n\} \uparrow$.
- **Decreasing** if:
$$E_n \supseteq E_{n+1} \ \ \forall n \in \mathbb{N}$$
It is denoted by $\{E_n\} \downarrow$.

Let now $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ be a sequence of sets in X:

Definition 1.2.6. We define the following:

- The **limit superior** of $\{E_n\}$ as:
$$\limsup_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$$
- The **limit inferior** of $\{E_n\}$ as:
$$\liminf_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} E_k$$
- If the limit superior and limit inferior are equal, we say that
$$\lim_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n$$

Exercise: Let X be a set and $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ be a sequence of sets in X. Prove that:

$$(i) \quad \{E_n\} \uparrow \ \Rightarrow \lim_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} E_n \qquad (ii) \quad \{E_n\} \downarrow \ \Rightarrow \lim_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} E_n$$

1.3 Characteristic functions

Definition 1.3.1. Let X be a set and $E \subseteq X$. The **characteristic function** of E is the function $\mathbb{1}_E : X \to \{0, 1\}$ defined as:

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

This function is also called the **indicator function** of E.

Note: Let $E, F \subseteq X$. We have that:

- $\mathbb{1}_{E \cap F} = \mathbb{1}_E \cdot \mathbb{1}_F$.
- $\mathbb{1}_{E \cup F} = \mathbb{1}_E + \mathbb{1}_F - \mathbb{1}_{E \cap F}$.
- $\mathbb{1}_{E^c} = 1 - \mathbb{1}_E$.
- $\mathbb{1}_{\limsup_{n\to\infty} E_n} = \limsup_{n\to\infty} \mathbb{1}_{E_n}$.
- $\mathbb{1}_{\liminf_{n\to\infty} E_n} = \liminf_{n\to\infty} \mathbb{1}_{E_n}$.

1.4 Equivalence relations and Quotient sets

Definition 1.4.1. A relation R on a set X is a subset of $X \times X$. For any $x, y \in X$, we say that x is related to y if $(x, y) \in R$. We denote this as xRy.

Definition 1.4.2. A relation R on a set X is an **equivalence relation** if it satisfies:

- **Reflexivity:**
$$xRx \;\; \forall x \in X$$
- **Symmetry:**
$$xRy \;\; \Rightarrow \;\; yRx \;\; \forall x, y \in X$$
- **Transitivity:**
$$xRy, yRz \;\Rightarrow\; xRz \;\; \forall x, y, z \in X$$

Every equivalence relation on X induces a partition of X. We define the **equivalence class** of $x \in X$ as:

$$[x] = \{y \in X : \; xRy\}$$

The set of all equivalence classes is called the **quotient set** of X by R, denoted by X/R.

$$X/R = \{[x] : \; x \in X\}$$

E.g.: Let $X = \mathbb{Z} \times \mathbb{Z}_0$ such that $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. We define the relation R on X as:

$$(a, b)R(c, d) \iff ad = bc$$

We can prove that R is an equivalence relation. The equivalence classes are:

$$[(a, b)] = \{(c, d) \in X : \ ad = bc\}$$

Notice that:

$$[(a, b)] = \{(a, b), (2a, 2b), (3a, 3b), \ldots\}$$

If we denote a class $[(a, b)]$ as $[a/b]$, then we have that:

$$X/R = \{[a/b] : \ a, b \in \mathbb{Z}_0\} = \mathbb{Q}$$

Chapter 2

Measure Theory

2.1 Measure spaces

Definition 2.1.1. Let X be a non-empty set. A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is a **σ-algebra** if:

(i) $\emptyset \in \mathcal{M}$.
(ii) $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.
(iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M} \implies \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{M}$.

If instead of (iii) we have that $E_1, E_2 \in \mathcal{M} \implies \mathbb{E}_1 \cup E_2 \in \mathcal{M}$, then $\mathcal{M}$ is called an **algebra**.

Note: If $\mathcal{M}$ is a σ-algebra, then we say that $(X, \mathcal{M})$ is a **measurable space**. Any set $E \in \mathcal{M}$ is called a **measurable set**.

E.g.: Let $X \neq \emptyset$. Then:

- $\mathcal{P}(X)$ is a σ-algebra.
- $\{\emptyset, X\}$ is a σ-algebra.
- $\{\emptyset, E, E^c, X\}$ is a σ-algebra for any $E \subseteq X$.
- $X = \mathbb{R}$, $\mathcal{M} = \{E \subseteq \mathbb{R} : \ E \text{ is open}\}$ is NOT a σ-algebra.

Properties 2.1.1. Let $(X, \mathcal{M})$ be a measurable space. Then:

(i) $X = \emptyset^c \in \mathcal{M}$

(ii) $\mathcal{M}$ is also an algebra. Indeed, if $\{E_1, E_2\} \subseteq \mathcal{M},\ E_n = \emptyset\ \forall n \geq 3$, then $E_1 \cup E_2 = \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{M}$.

(iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M} \implies \bigcap_n E_n \in \mathcal{M}$.

(iv) $E, F \in \mathcal{M} \implies E \setminus F \in \mathcal{M}$

(v) $\Omega \subseteq X$. Then, the **restriction** of $\mathcal{M}$ to Ω is:

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, E \in \mathcal{M}\}$$

Then, $(\Omega, \mathcal{M}|_\Omega)$ is a measurable space.