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Real and Functional Analysis

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Contents

Chapter 1

Set Theory

1.1 Basic notions

Definition 1.1.1. Let X, Y be sets. We say:

- X, Y are **equipotent** if there exists a bijection $f : X \to Y$.
- X has a **cardinality greater or equal** to Y if there exists an surjection $f : X \to Y$.
- X is **finite** if it is equipotent to $\{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$. X is infinite otherwise.

Remark: X is infinite $\iff$ it is equipotent to a proper subset of itself.

E.g.: The set of natural numbers $\mathbb{N}$ is infinite. In fact, the set of even natural numbers $E = \{2, 4, 6, \dots\} \subset \mathbb{N}$ is equipotent to $\mathbb{N}$, as we can define the bijection $f : \mathbb{N} \to E$ as $f(n) = 2n$.

Definition 1.1.2. Let X be an infinite set. We say X is **countable** if it is equipotent to $\mathbb{N}$. X is **uncountable** otherwise, in which case it is **more than countable**.

Definition 1.1.3. X has the **cardinality of the continuum** if it is equipotent to $[0, 1] \subset \mathbb{R}$. Any such set is uncountable.

E.g.: We have that:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable.
- $\mathbb{R}, \mathbb{R}^n, (0, 1), [0, 1]$ are uncountable.
- Countable union of countable sets is countable.

1.2 Families of subsets

Let X be a set. The "Power set" of X is the set of all subsets of X, denoted by $\mathcal{P}(X)$.

$$\mathcal{P}(X) = \{E : \ E \subseteq X\}$$

Note that $\mathcal{P}(X)$ has always a cardinality greater than X. For example, if $X = \mathbb{N}$, then $\mathcal{P}(X)$ has the cardinality of the continuum.

Definition 1.2.1. Let X be a set. A **family of subsets** of X is a set E such that:

$$E \subseteq \mathcal{P}(X)$$

We usually denote $E = \{E_i\}_{i \in I}$, where I is an index set.

Definition 1.2.2. Let $E = \{E_i\}_{i \in I}$ be a family of subsets of X. We define:

- The **union** of E as:

$$\bigcup_{i \in I} E_i = \{x \in X : \ x \in E_i \text{ for some } i \in I\}$$

- The **intersection** of E as:

$$\bigcap_{i \in I} E_i = \{x \in X : \ x \in E_i \text{ for all } i \in I\}$$

Definition 1.2.3. Let $E = \{E_i\}_{i \in I}$ be a family of subsets of X. We say F is **pairwise disjoint** if:

$$E_i \cap E_j = \emptyset \ \ \forall i, j \in I, i \neq j$$

Definition 1.2.4. We say that the family $E = \{E_i\}_{i \in I}$ of subsets of X is a **covering** of X if:

$$X = \bigcup_{i \in I} E_i$$

Any subfamily of E, $E' = \{E_i\}_{i \in I'}$ is a **subcovering** of X if it is a covering of X itself.

E.g.: Let $X = \mathbb{R}$. We define:

$$\mathcal{T} = \{E \subset X : E \text{ is open}\}$$

We say that $\mathcal{T}$ is the standard topology of X. More generally, this can be done in "metric spaces" (X, d).

Properties of $\mathcal{T}$ (open sets):

- $\emptyset, X \in \mathcal{T}$.
- Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$.
- Arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$.

We can also define **sequences of sets**. Let X be a set. A sequence of sets in X is a family of sets $\{E_n\}_{n\in\mathbb{N}}$.

Definition 1.2.5. Let X be a set. A sequence of sets $\{E_n\}_{n\in\mathbb{N}}$ is said to be:

- **Increasing** if:
$$E_n \subseteq E_{n+1} \quad \forall n \in \mathbb{N}$$
It is denoted by $\{E_n\} \uparrow$.
- **Decreasing** if:
$$E_n \supseteq E_{n+1} \quad \forall n \in \mathbb{N}$$
It is denoted by $\{E_n\} \downarrow$.

Let now $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ be a sequence of sets in X:

Definition 1.2.6. We define the following:

- The **limit superior** of $\{E_n\}$ as:
$$\limsup_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$$
- The **limit inferior** of $\{E_n\}$ as:
$$\liminf_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} E_k$$
- If the limit superior and limit inferior are equal, we say that
$$\lim_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n$$

Exercise: Let X be a set and $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ be a sequence of sets in X. Prove that:

$$(i) \quad \{E_n\} \uparrow \Rightarrow \lim_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} E_n \qquad (ii) \quad \{E_n\} \downarrow \Rightarrow \lim_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} E_n$$

1.3 Characteristic functions

Definition 1.3.1. Let X be a set and $E \subseteq X$. The **characteristic function** of E is the function $\mathbb{1}_E : X \to \{0,1\}$ defined as:

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

This function is also called the **indicator function** of E.

Remark: Let $E, F \subseteq X$. We have that:

- $\mathbb{1}_{E\cap F} = \mathbb{1}_E \cdot \mathbb{1}_F$.
- $\mathbb{1}_{E\cup F} = \mathbb{1}_E + \mathbb{1}_F - \mathbb{1}_{E\cap F}$.
- $\mathbb{1}_{E^c} = 1 - \mathbb{1}_E$.
- $\mathbb{1}_{\limsup_{n\to\infty} E_n} = \limsup_{n\to\infty} \mathbb{1}_{E_n}$.
- $\mathbb{1}_{\liminf_{n\to\infty} E_n} = \liminf_{n\to\infty} \mathbb{1}_{E_n}$.

1.4 Equivalence relations and Quotient sets

Definition 1.4.1. A relation R on a set X is a subset of $X \times X$. For any $x, y \in X$, we say that x is related to y if $(x, y) \in R$. We denote this as xRy.

Definition 1.4.2. A relation R on a set X is an **equivalence relation** if it satisfies:

- **Reflexivity:**
$$xRx \ \ \forall x \in X$$
- **Symmetry:**
$$xRy \ \Rightarrow \ yRx \ \ \forall x, y \in X$$
- **Transitivity:**
$$xRy, yRz \ \Rightarrow \ xRz \ \ \forall x, y, z \in X$$

Every equivalence relation on X induces a partition of X. We define the **equivalence class** of $x \in X$ as:

$$[x] = \{y \in X : \ xRy\}$$

The set of all equivalence classes is called the **quotient set** of X by R, denoted by X/R.

$$X/R = \{[x] : \ x \in X\}$$

E.g.: Let $X = \mathbb{Z} \times \mathbb{Z}_0$ such that $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. We define the relation R on X as:

$$(a, b)R(c, d) \iff ad = bc$$

We can prove that R is an equivalence relation. The equivalence classes are:

$$[(a, b)] = \{(c, d) \in X : \ ad = bc\}$$

Notice that:

$$[(a, b)] = \{(a, b), (2a, 2b), (3a, 3b), \ldots\}$$

If we denote a class $[(a, b)]$ as $[a/b]$, then we have that:

$$X/R = \{[a/b] : \ a, b \in \mathbb{Z}_0\} = \mathbb{Q}$$

Chapter 2

Measure Theory

2.1 Measure spaces

Definition 2.1.1. Let X be a non-empty set. A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is a **σ-algebra** if:

(i) $\emptyset \in \mathcal{M}$.

(ii) $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.

(iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M} \implies \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{M}$.

If instead of (iii) we have that $E_1, E_2 \in \mathcal{M} \implies \mathbb{E}_1 \cup E_2 \in \mathcal{M}$, then $\mathcal{M}$ is called an **algebra**.

Remark: If $\mathcal{M}$ is a σ-algebra, then we say that $(X, \mathcal{M})$ is a **measurable space**. Any set $E \in \mathcal{M}$ is called a **measurable set**.

E.g.: Let $X \neq \emptyset$. Then:

- $\mathcal{P}(X)$ is a σ-algebra.
- $\{\emptyset, X\}$ is a σ-algebra.
- $\{\emptyset, E, E^c, X\}$ is a σ-algebra for any $E \subseteq X$.
- $X = \mathbb{R}$, $\mathcal{M} = \{E \subseteq \mathbb{R} : E \text{ is open}\}$ is NOT a σ-algebra.

Properties 2.1.1. Let $(X, \mathcal{M})$ be a measurable space. Then:

(i) $X = \emptyset^c \in \mathcal{M}$

(ii) $\mathcal{M}$ is also an algebra. Indeed, if $\{E_1, E_2\} \subseteq \mathcal{M},\ E_n = \emptyset\ \forall n \geq 3$, then $E_1 \cup E_2 = \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{M}$.

(iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M} \implies \bigcap_n E_n \in \mathcal{M}$.

(iv) $E, F \in \mathcal{M} \implies E \setminus F \in \mathcal{M}$

(v) $\Omega \subseteq X$. Then, the **restriction** of $\mathcal{M}$ to Ω is:

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, E \in \mathcal{M}\}$$

Then, $(\Omega, \mathcal{M}|_\Omega)$ is a measurable space.

2.2 Generation of a σ-algebra

Theorem 2.2.1. *Take any family $\mathcal{A} \subseteq \mathcal{P}(X)$. Then, it is well-defined the σ-algebra generated by $\mathcal{A}$, denoted by $\sigma_0(\mathcal{A})$, as the smallest σ-algebra containing $\mathcal{A}$. It is characterized by:*

(i) $\sigma_0(\mathcal{A})$ is a σ-algebra.

(ii) $\mathcal{A} \subseteq \sigma_0(\mathcal{A})$.

(iii) If $\mathcal{M}$ is a σ-algebra and $\mathcal{A} \subseteq \mathcal{M}$, then $\sigma_0(\mathcal{A}) \subseteq \mathcal{M}$.

Sketch of proof. Define $V = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is } \sigma\text{-algebra}, \mathcal{A} \subseteq \mathcal{M}\}$. Notice that $V \neq \emptyset$ because $\mathcal{P}(X) \in V$. Then, define:

$$\sigma_0(\mathcal{A}) = \bigcap_{\mathcal{M}\in V} \mathcal{M}$$

Then, $\sigma_0(\mathcal{A})$ is a σ-algebra as it satisfies the properties of a σ-algebra, denoted in definition 2.1.1.

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Remark: This is relevant. Often, to check that a σ-algebra has certain properties, it is enough to check the property on a set of generators.

2.3 Borel sets

Take (X, d) as a metric space, so that open sets are defined. Recall that:

$$\mathcal{T} = \{E \subseteq X : E \text{ is open}\}$$

Definition 2.3.1. The σ-algebra generated by $\mathcal{T}$ is called the **Borel σ-algebra** of X, denoted by:

$$\mathcal{B}(X) := \sigma_0(\mathcal{T})$$

Any set $E \in \mathcal{B}(X)$ is a **Borel set**.

Remark: The following are Borel sets:

- Open sets
- Closed sets
- Countable intersections of open sets (G_δ-sets)
- Countable unions of closed sets (F_σ-sets)

We will deal with:

$$X = \mathbb{R} = (-\infty, \infty)$$

but also:

$$X = \overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$$

Let us define the arithmetic operations on $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$:

- $a \pm \infty = \pm\infty$
- $a > 0 : a \cdot \pm\infty = \pm\infty$
- $a < 0 : a \cdot \pm\infty = \mp\infty$
- $a = 0 : 0 \cdot \pm\infty = 0$
- $\infty - \infty$, ∞/∞, $0/0$ are not defined.

Also, the open intervals in $\overline{\mathbb{R}}$ are the following:

- (a, b), with $a, b \in \mathbb{R}$
- $[-\infty, b)$
- $(a, \infty]$

Remark: We have that:

$$\begin{aligned}\mathcal{B}(\mathbb{R}) &:= \sigma_0(\{\text{open sets}\}) \\ &= \sigma_0(\{(a,b) : a < b\}) \\ &= \sigma_0(\{[a,b] : a < b\}) \\ &= \sigma_0(\{(a,\infty) : a \in \mathbb{R}\})\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\overline{\mathbb{R}}) &:= \sigma_0(\{\text{open sets}\}) \\ &= \sigma_0(\{(a,\infty] : a \in \mathbb{R}\})\end{aligned}$$

$$\mathcal{B}(\mathbb{R}^N) = \sigma_0(\{\text{open rectangles}\})$$

2.4 Measures

Let $(X, \mathcal{M})$ be a measurable space.

Definition 2.4.1. A function $\mu : \mathcal{M} \to [0, \infty]$ is a (positive) **measure** on $\mathcal{M}$ if:

(i) $\mu(\emptyset) = 0$

(ii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$, disjoint $\implies \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$

Note: To avoid nonsenses, we always assume that $\exists E \in \mathcal{M}\ s.t.\ \mu(E) < \infty$

Terminology: Let $X, \mathcal{M}, \mu$ defined as above:

- $(X, \mathcal{M}, \mu)$ is a **measure space**.
- If $\mu(X) = 1$, then $(X, \mathcal{M}, \mu)$ is a **probability space** and μ is a **probability measure**.

Definition 2.4.2. A measure μ is:

1. **Finite** if $\mu(X) < \infty$
2. **σ-finite** if $\exists \{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ s.t.

$$\mu(E_n) < \infty \ \ \forall n \in \mathbb{N} \quad \wedge \quad X = \bigcup_{n\in\mathbb{N}} E_n$$

E.g.: Some examples of measures are:

1. (Trivial measure): For any $(X, \mathcal{M})$, define μ as $\mu(E) = 0 \;\; \forall E \in \mathcal{M}$
2. (Counting measure): For any $(X, \mathcal{M})$, typically $\mathcal{M} = \mathcal{P}(X)$, define $\mu_\#$ as:

$$\mu_\#(E) = \begin{cases} \#\{\text{elements of } E\} & E \text{ finite} \\ \infty & \text{otherwise} \end{cases}$$

3. (Dirac measure): For any $(X, \mathcal{M})$, pick $x_0 \in X$. Then, define δ_{x_0} as:

$$\delta_{x_0}(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 \notin E \end{cases}$$

2.4.1 Properties of measures

Theorem 2.4.1 (Basic properties)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:*

(i) *μ is finitely additive:* $E, F \in \mathcal{M}, E \cap F = \emptyset \implies \mu(E \cup F) = \mu(E) + \mu(F)$

(ii) *(Monotonicity):* $E, F \in \mathcal{M}, E \subseteq F \implies \mu(E) \leq \mu(F)$

(iii) *(Excision property):* $E, F \in \mathcal{M}, E \subseteq F, \mu(E) < \infty \implies \mu(F \backslash E) = \mu(F) - \mu(E)$

Proof. The proof is straightforward:

(i) Let $E, F \in \mathcal{M}, E \cap F = \emptyset$. Then:

$$\mu(E \cup F) = \mu(E) + \mu(F)$$

Proof. Obvious, using $E_n = \emptyset$ for $n \geq 3$. ■

(ii) Let $E, F \in \mathcal{M}, E \subseteq F$. Then:

$$\mu(E) \leq \mu(F)$$

Proof. Let $F = E \cup (F \setminus E)$. Then:

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$$

■

(iii) Let $E, F \in \mathcal{M}, E \subseteq F, \mu(E) < \infty$. Then:

$$\mu(F \setminus E) = \mu(F) - \mu(E)$$

Proof. Let $F = E \cup (F \setminus E)$. Then:

$$\mu(F) = \mu(E) + \mu(F \setminus E)$$

■

This concludes the proof.

■

Theorem 2.4.2 (Continuity among monotone sequences)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be a sequence of measurable sets. Then:*

(i) If $\{E_n\} \uparrow$, $E := \lim_n E_n = \bigcup_n E_n$, then:

$$\mu(E) = \lim_n \mu(E_n)$$

(ii) If $\{E_n\} \downarrow$, $E := \lim_n E_n = \bigcap_n E_n$, and $\mu(E_1) < \infty$, then:

$$\mu(E) = \lim_n \mu(E_n)$$

Proof. The proof goes as follows:

(i) If $\mu(E_n) = \infty$ for some n, then the proof is trivial. Otherwise, let $F_1 = E_1$ and $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Then, we can check that:

- $F_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $\{F_n\}$ is a disjoint sequence.
- $E_n = \bigcup_{k=1}^n F_k$
- Because of the disjointness, we have that:

$$\mu(E_n) = \sum_{k=1}^{n} \mu(F_k)$$

Then, we have that:

$$\begin{aligned}\mu(E) = \mu\left(\lim_n E_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \\ = \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) = \\ = \sum_{n=1}^{\infty} (\mu(E_n) - \mu(E_{n-1})) = \lim_n \mu(E_n)\end{aligned}$$

(ii) Define $G_n = E_1 \setminus E_n$. Then, check that:

- $G_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $\{G_n\} \uparrow$

By the previous result, we have that:

$$\mu\left(\bigcup_{n=1}^{\infty} G_n\right) = \lim_n \mu(G_n)$$

Then, on the right-hand side:

$$\begin{aligned}\lim_n \mu(G_n) &= \lim_n \mu(E_1 \setminus E_n) = \\ &= \mu(E_1) - \lim_n \mu(E_n)\end{aligned}$$

On the left-hand side:

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} G_n\right) &= \mu\left(\bigcup_{n=1}^{\infty}(E_1 \setminus E_n)\right) = \\ &= \mu\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right) = \\ &= \mu(E_1) - \mu(E)\end{aligned}$$

Finally, we have that:

$$\mu(E_1) - \mu(E) = \mu(E_1) - \lim_n \mu(E_n)$$

And because $\mu(E_1) < \infty$, we have that:

$$\mu(E) = \lim_n \mu(E_n)$$

■

Remark: In (ii), the condition $\mu(E_1) < \infty$ is essential. Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{\#})$. Then, define the following sequence:

$$E_n = \{n, n+1, n+2, \ldots\}$$

Note that $E_n \subseteq E_{n-1}$. Also, note that for any $n \in \mathbb{N}$, we have that:

$$\mu_{\#}(E_n) = \infty$$

Then, we have that:

$$\mu_{\#}\left(\bigcap_n E_n\right) = \mu_{\#}(\emptyset) = 0$$

But:

$$\lim_n \mu_{\#}(E_n) = \infty$$

This shows that the condition $\mu(E_1) < \infty$ is essential.

Theorem 2.4.3 (σ-subadditivity)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be a sequence of measurable sets. Then:*

$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$$

Proof. Let $F_1 = E_1$ and $F_n = E_n \setminus \left(\bigcup_{k=1}^{n-1} E_k\right)$ for $n \geq 2$. Then, we have that:

- $F_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $F_n \subseteq E_n, \forall n \in \mathbb{N}$
- $\{F_n\}$ is a disjoint sequence.
- $\bigcup_n E_n = \bigcup_n F_n$

Then, we have that:

$$\mu\left(\bigcup_n E_n\right) = \mu\left(\bigcup_n F_n\right) = $$
$$= \sum_n \mu(F_n) \leq \sum_n \mu(E_n)$$

■

2.5 Sets of measure zero, negligible sets, complete measures

Definition 2.5.1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

1. A set $E \in \mathcal{M}$ is a **set of measure zero** if $\mu(E) = 0$.
2. A set $F \in X$ (not necessarily measurable) is a **negligible set** if $\exists E \in \mathcal{M}$ s.t. $F \subseteq E$ and E is a set of measure zero.

Definition 2.5.2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then, we say that μ is a **complete measure** (alternatively, that $(X, \mathcal{M}, \mu)$ is a **complete measure space**) all negligible sets are measurable.

Remark (Completion of a measure space)**:** A measure space $(X, \mathcal{M}, \mu)$ may not be complete. However, we can define the following:

$$\overline{\mathcal{M}} = \{E \subseteq X : \exists F_1, F_2 \in \mathcal{M} : F_1 \subseteq E \subseteq F_2, \mu(F_2 \setminus F_1) = 0\}$$

One can show that $\overline{\mathcal{M}}$ is a σ-algebra, and that $\mathcal{M} \subseteq \overline{\mathcal{M}}$. Moreover, if E, F_1, F_2 are as above, define:

$$\overline{\mu}(E) = \mu(F_1) = \mu(F_2)$$

One can check that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is a complete measure space.

2.6 Towards the Lebesgue measure

We would like to define a measure λ with $X = \mathbb{R}$ (or $X = \mathbb{R}^N$) s.t. $\forall a < b$:

- $\lambda((a, b)) = b - a$ **(length of the interval)**
- $\forall E, \lambda(E + x) = \lambda(E)$ **(translation invariance)**

In principle, we would like to define it in $\mathcal{P}(\mathbb{R})$. Such a measure should satisfy $\lambda(\{a\}) = 0$.

Theorem 2.6.1 (Ulam)**.** *The only measure on $\mathcal{P}(\mathbb{R})$ that satisfies $\lambda(\{a\}) = 0\ \forall a \in \mathbb{R}$ is the trivial measure.*

Therefore, we need to choose an $\mathcal{M} \subsetneq \mathcal{P}(\mathbb{R})$. We can construct one as follows:

- Starting family with a "measure", e.g., $\mathcal{T} = \{(a, b) : a < b\}$ and $f((a, b)) = b - a$.
- Construct an "outer measure" μ^* on $\mathcal{P}(\mathbb{R})$.
- Restrict μ^* to a well-chosen σ-algebra $\mathcal{M} \subsetneq \mathcal{P}(\mathbb{R})$.

Definition 2.6.1. Let X be a set. An **outer measure** μ^* on X is a function

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that:

1. $\mu^*(\emptyset) = 0$
2. (Monotonicity) $E \subseteq F \subseteq X \implies \mu^*(E) \leq \mu^*(F)$
3. (σ-subadditivity) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X) \implies \mu^*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(E_n)$

Remark: Any measure μ is an outer measure. However, the converse is not true.

Proposition 2.6.2. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$, $f : \mathcal{E} \to [0, \infty]$. Assume that $\emptyset, X \in \mathcal{E}, f(\emptyset) = 0$. Then, $\forall E \subseteq X$ define:*

$$\mu^*(E) = \inf \left\{ \sum_{n=1}^{\infty} f(A_n) : A_n \in \mathcal{E}, E \subseteq \bigcup_n A_n \right\}$$

Then, μ^* is an outer measure.

Proof. The proof is omitted. ∎

Remark: In this generality, if $E \in \mathcal{E}$, then $f(E)$ and $\mu^*(E)$ may not be equal. We can only guarantee that $\mu^*(E) \leq f(E)$.

E.g.: There are some important examples:

- $X = \mathbb{R}$, $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$

$$f((a, b)) = \text{length}((a, b)) = b - a$$

- $X = \mathbb{R}^N$, $\mathcal{E} = \{(a_1, b_1) \times \ldots \times (a_N, b_N) : a_i, b_i \in \overline{\mathbb{R}}, a_i \leq b_i\}$

$$f((\underline{a}, \underline{b})) = \text{volume}((\underline{a}, \underline{b})) = \prod_{i=1}^{N} (b_i - a_i)$$

In both cases, the outer measure μ^* is called the **Lebesgue outer measure**. We will denote it by λ^* (or λ^*_N in the second case). Note that in this case, $\lambda^*(E) = f(E)$ for any $E \in \mathcal{E}$.

Remark: Any μ measure on $\mathcal{P}(X)$ is an outer measure. However, the converse is not true. In particular, $\exists A, B \subseteq \mathbb{R}$ s.t. $A \cap B = \emptyset$ and $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$.

2.6.1 Carathéodory's criterion

Definition 2.6.2 (Carathéodory's condition)**.** Let μ^* be an outer measure on $\mathcal{P}(X)$. A ser $E \subseteq X$ is **μ^*-measurable** if $\forall A \subseteq X$:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Lemma 2.6.3 (Equivalence of Carathéodory's condition)**.** *E is μ^*-measurable $\iff$ $\forall A \subseteq X$, $\mu^*(A) < \infty$:*

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Proof. The proof is as follows:

$(\Rightarrow)$: Trivial

$(\Leftarrow)$: Let $A \subseteq X$, such that $\mu^*(A) < \infty$ and:

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Then, notice that $\{A \cap E, A \cap E^c\}$ is a covering of A. By subadditivity:

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Therefore, we have that:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Meaning that E is μ^*-measurable. This concludes the proof.

∎

Theorem 2.6.4 (Carathéodory)**.** *Let μ^* be an outer measure on $\mathcal{P}(X)$. The family:*

$$\mathcal{M} = \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra, and μ^* restricted to $\mathcal{M}$ (denoted $\mu = \mu^*|_{\mathcal{M}}$) is a complete measure.

Remark: $(X, \mathcal{M}, \mu)$ as in the above theorem is sometimes called the "abstract Lebesgue measure space". We will only prove the completeness of μ.

Lemma 2.6.5. *Let $(X, \mathcal{M}, \mu)$ be the measure space as in Carathéodory's theorem. Then, any $N \subseteq X$ s.t. $\mu^*(N) = 0$ is μ-measurable, i.e., $N \in \mathcal{M}$, and $\mu(N) = 0$.*

Proof. We have to show that N satisfies Carathéodory's condition, or equivalently, that it satisfies the lemma 2.6.3. Let $A \subseteq X$ be arbitrary. Then, notice that:

$$\mu^*(A \cap N) \leq \mu^*(N) = 0$$

Also, notice that:

$$\mu^*(A \cap N^c) \leq \mu^*(A)$$

Therefore, we have that:

$$\mu^*(A \cap N) + \mu^*(A \cap N^c) \leq 0 + \mu^*(A) = \mu^*(A)$$

By lemma 2.6.3, we have that N is μ^*-measurable. By Carathéodory's theorem, we have that N is μ-measurable. Finally, we have that $\mu(N) = \mu^*(N) = 0$.

■

Corollary 2.6.5.1. *μ as in Carathéodory's theorem is a complete measure.*

Proof. Let $N \subseteq E$, and $\mu(E) = 0$ ($E \in \mathcal{M}$). Then, we have that:

$$\mu(E) = 0 \implies \mu^*(E) = 0$$

By monotonicity, we have that:

$$\mu^*(N) \leq \mu^*(E) = 0$$

Then, $\mu(N) = \mu^*(N) = 0$, thus $N \in \mathcal{M}$. This concludes the proof.

■

2.7 Lebesgue measure

Definition 2.7.1. Let $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$. Define:

$$\lambda^*((a, b)) = b - a$$

Then, λ^* is the **Lebesgue outer measure** on $\mathbb{R}$.

Theorem 2.7.1. *Let λ^* be the Lebesgue outer measure on $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$. Then, by Carathéodory's theorem, the family:*

$$\mathcal{L}(\mathbb{R}) = \{E \subseteq \mathbb{R} : E \text{ is } \lambda^*\text{-measurable}\}$$

*is a σ-algebra, called the **Lebesgue σ-algebra**, and λ^* restricted to $\mathcal{L}(\mathbb{R})$ (denoted $\lambda = \lambda^*|_{\mathcal{L}(\mathbb{R})}$) is a complete measure, called the **Lebesgue measure**.*

Proof. The proof is omitted.

■

Remark: The measure space $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is called the **Lebesgue measure space**.

Proposition 2.7.2. *Let λ be the Lebesgue measure on $\mathbb{R}$. Then:*

(i) $a \in \mathbb{R} \implies \{a\} \in \mathcal{L}(\mathbb{R})$ *and* $\lambda(\{a\}) = 0$

(ii) $E \subset \mathbb{R}$ *at most countable* $\implies E \in \mathcal{L}(\mathbb{R})$ *and* $\lambda(E) = 0$

Proof. The proof is as follows:

(i) Let $a \in \mathbb{R}$. Then, we have that, for any $\varepsilon > 0$:

$$E_1 = (a - \varepsilon, a + \varepsilon), \quad , E_2 = E_3 = \dots = \emptyset$$

is a covering of $\{a\}$. Then, by definition of λ^*:

$$0 \leq \lambda^*(\{a\}) \leq \sum_{n=1}^{\infty} f(E_n) = 2\varepsilon$$

As ε is arbitrary, we have that $\lambda^*(\{a\}) = 0$. By Lemma 2.6.5, we then have that $\{a\} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\{a\}) = 0$.

(ii) Let $E \subseteq \mathbb{R}$ be at most countable. Then, we have that:

$$E = \bigcup_{a \in E} \{a\}$$

Because $\{a\} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\{a\}) = 0$, we have that $E \in \mathcal{L}(\mathbb{R})$ and:

$$\lambda(E) = \lambda\left(\bigcup_{a \in E} \{a\}\right) = \sum_{a \in E} \lambda(\{a\}) = 0$$

■

Remark: We can point out 2 important consequences of the above proposition:

1. Arguing as in (i), we can show that the Lebesgue measure is translation invariant. That is, $\forall E \in \mathcal{L}(\mathbb{R})$, $\forall x \in \mathbb{R}$:

$$\lambda(E + x) = \lambda(E)$$

2. In particular, since $\mathbb{Q}$ is countable, we have that $\mathbb{Q} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\mathbb{Q}) = 0$. In the measure sense, $\mathbb{Q}$ has very few elements with respect to $\mathbb{R}$. On the other hand, $\mathbb{Q}$ is dense in $\mathbb{R}$. In the topology sense, $\mathbb{Q}$ has a lot of points.

Proposition 2.7.3. *We have that:* $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R})$

Proof. Since $\mathcal{B}(\mathbb{R}) = \sigma_0(\{(a, \infty) : a \in \mathbb{R}\})$, if we show that $(a, \infty) \in \mathcal{L}(\mathbb{R}), \quad \forall a \in \mathbb{R}$, then the prop. follows.

Take $A \subset \mathbb{R}$, s.t. $\lambda^*(A) < \infty$. Then, we must show that:

$$\lambda^*(A) \geq \lambda^*(A \cap (a, \infty)) + \lambda^*(A \cap (-\infty, a])$$

Moreover, by a previous remark, one can assume that $a \notin A$. Then, take any countable covering of A by open intervals:

$$A \subseteq \bigcup_n I_n$$

Then, let us define $A_{left} = A \cap (-\infty, a]$ and $I_{n,left} = I_n \cap (-\infty, a]$. Then, we notice that $\{I_{n,left}\}$ is a covering of A_{left}.

In the same way, we define $A_{right} = A \cap (a, \infty)$ and $I_{n,right} = I_n \cap (a, \infty)$. Then, we notice that $\{I_{n,right}\}$ is a covering of A_{right}.

Then, we have that:

$$\lambda^*(A_{left}) \leq \sum_n \lambda^*(I_{n,left})$$

$$\lambda^*(A_{right}) \leq \sum_n \lambda^*(I_{n,right})$$

Summing both inequalities, we have that:

$$\begin{aligned} \lambda^*(A_{left}) + \lambda^*(A_{right}) &\leq \sum_n \lambda^*(I_{n,left}) + \sum_n \lambda^*(I_{n,right}) \\ &= \sum_n \lambda^*(I_n) \end{aligned}$$

Taking the infimum over all countable coverings of A, we have that:

$$\lambda^*(A_{left}) + \lambda^*(A_{right}) \leq \lambda^*(A)$$

■

Remark: In particular, we have that $\forall (a, b) \subset \mathbb{R}$:

$$(a, b) \in \mathcal{L}(\mathbb{R}), \quad \lambda((a, b)) = b - a$$

We know that:

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$$

Are these inclusions strict? We already know that $\mathcal{L}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$, by Ulam's theorem. In particular, $\exists E \subset \mathbb{R}$ not Lebesgue measurable (**Vitali sets**). More precisely:

$$\forall E \in \mathcal{L}(\mathbb{R}), \text{ s.t } \lambda(E) > 0, \exists V \subset E, \text{ s.t } V \notin \mathcal{L}(\mathbb{R})$$

The relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ is more subtle. It is clarified by the following proposition:

Proposition 2.7.4 (Regularity of the Lebesgue measure). *Let $E \in \mathbb{R}$. Then, the following are equivalent:*

(i) $E \in \mathcal{B}(\mathbb{R})$

(ii) $\forall \varepsilon > 0, \exists A \subset \mathbb{R}$ *open set s.t.*

$$E \subset A \quad \textit{and} \quad \lambda^*(A \setminus E) < \varepsilon$$

(iii) $\forall \varepsilon > 0, \exists G \subset \mathbb{R}$ *of class* G_δ *s.t.*

$$E \subset G \quad \textit{and} \quad \lambda^*(G \setminus E) = 0$$

(iv) $\forall \varepsilon > 0, \exists C \subset \mathbb{R}$ *closed set s.t.*

$$C \subset E \quad \textit{and} \quad \lambda^*(E \setminus C) < \varepsilon$$

(v) $\forall \varepsilon > 0, \exists F \subset \mathbb{R}$ *of class* F_σ *s.t.*

$$F \subset E \quad \textit{and} \quad \lambda^*(E \setminus F) = 0$$

We get as a consequence the following:

Corollary 2.7.4.1. $\forall E \in \mathcal{L}(\mathbb{R})$, $\exists F, G \in \mathcal{B}(\mathbb{R})$ *s.t.* $F \subset E \subset G$ *and*

$$\lambda(G \setminus F) = \lambda(G \setminus E) + \lambda(E \setminus F) = 0$$

In other words:

$$\mathcal{L}(\mathbb{R}) = \overline{\mathcal{B}(\mathbb{R})}$$

(But $\mathcal{L}(\mathbb{R}) \neq \mathcal{B}(\mathbb{R})$*).*

Proof. (Regularity of the Lebesgue measure). The proof goes as follows:

(i) $\Rightarrow$ (ii) :

Let $E \in \mathcal{B}(\mathbb{R})$. Note that, since $A \in \mathcal{L}(\mathbb{R})$ for all A open, we have that:

$$\lambda^*(A \setminus E) = \lambda(A \setminus E)$$

By definition of λ^*, we have that $\forall \varepsilon > 0$, $\exists \{I_n\}_{n \in \mathbb{N}}$ s.t.

$$E \subset \bigcup_n I_n \quad \text{and} \quad \sum_n \lambda(I_n) < \lambda^*(E) + \varepsilon$$

Then, set $A = \bigcup_n I_n$. We have that A is open, $E \subset A$ and:

$$\lambda(A) \leq \sum_n \lambda(I_n) < \lambda(E) + \varepsilon$$
$$\implies \lambda(A \setminus E) = \lambda(A) - \lambda(E) < \varepsilon$$

(ii) $\Rightarrow$ (iii) :

Assume $\forall \varepsilon > 0$, $\exists A_\varepsilon$ open s.t. $E \subset A_\varepsilon$ and $\lambda(A_\varepsilon \setminus E) < \varepsilon$. Then, set $\varepsilon = 1/n$, $n \geq 1$ (for ease of notation, $A_n = A_{1/n}$) and define:

$$G = \bigcap_n A_n$$

Then, G is a G_δ set, $E \subset G$ and:

$$0 \leq \lambda^*(G \setminus E) \leq \lambda^*(A_n \setminus E) < 1/n$$

Taking the limit, we have that $\lambda(G \setminus E) = 0$.

(iii) $\Rightarrow$ (i) :

We know that $E \subset G$, $G \in \mathcal{L}(\mathbb{R})$ with $\lambda(G \setminus E) = 0$. Then, we have that:

$$E = G \setminus (G \setminus E) \in \mathcal{L}(\mathbb{R})$$

since $G \in \mathcal{L}(\mathbb{R})$ and $G \setminus E \in \mathcal{L}(\mathbb{R})$. The last is because it is a negligible set and λ is complete.

■

E.g. (Cantor set)**:** Let $T_0 = [0, 1]$. Then, construct T_{n+1} from T_n (recursively) by removing the inner third part of every interval in T_n:

$$T_0 = [0, 1],$$
$$T_1 = [0, 1/3] \cup [2/3, 1],$$
$$T_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \ \ldots$$

Then, define the **Cantor set** as:

$$C = \bigcap_n T_n$$

It can be proven that:

- C has the cardinality of $\mathbb{R}$
- $\lambda(C) = 0$
- C is compact
- C is nowhere dense (has no interior points), i.e., $\text{int}(C) = \emptyset$
- $\exists E \subset C$ s.t. $E \in \mathcal{L}(\mathbb{R})$ but $E \notin \mathcal{B}(\mathbb{R})$

2.8 Measurable functions

Definition 2.8.1. Given $f : X \to Y$, it is well-defined the **preimage** (or counterimage) of f as:

$$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X), \quad f^{-1}(E) = \{x \in X : f(x) \in E\}$$

Remark: Preimages work well with set operations:

- $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$
- $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$
- $f^{-1}(E \setminus F) = f^{-1}(E) \setminus f^{-1}(F)$

Note this is not true for images.

Definition 2.8.2. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f : X \to Y$ is **measurable** if $\forall E \in \mathcal{N}$, we have that $f^{-1}(E) \in \mathcal{M}$. We also say that f is **($\mathcal{M}$,$\mathcal{N}$)-measurable**.

Proposition 2.8.1. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, and $\rho \subset \mathcal{N}$ s.t. $\mathcal{N} = \sigma_0(\rho)$. Then, $f : X \to Y$ is measurable $\iff$ $\forall E \in \rho$, we have that $f^{-1}(E) \in \mathcal{M}$.*

Proof. The proofs goes as follows:

$(\Rightarrow)$: Trivial

$(\Leftarrow)$: Define $\Sigma := \{E \subset Y : f^{-1}(E) \in \mathcal{M}\}$. We have:

- $\rho \subset \Sigma$ as a consecuence of $(\forall E \in \rho, f^{-1}(E) \in \mathcal{M})$

- Σ is a σ-algebra (check as an exercise)

Then, we have that $\mathcal{N} = \sigma_0(\rho) \subset \Sigma$. Therefore, f is measurable.

$\blacksquare$

Definition 2.8.3. Suppose that $\mathcal{M} \supseteq \mathcal{B}(X)$ and $\mathcal{N} = \mathcal{B}(Y)$. We say that $f : X \to Y$ is:

- **Borel measurable** if f is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable.
- **Lebesgue measurable** if it is $(\mathcal{M}, \mathcal{B}(Y))$-measurable.

Remark: If $f : X \to Y$ is Borel measurable, then it is Lebesgue measurable. The converse is not true.

Also, we never deal with $\mathcal{L}(Y)$.

Corollary 2.8.1.1. *f is Borel measurable $\iff$ $f^{-1}(E) \in \mathcal{B}(X)$, $\forall E \in Y$ open. Also, f is Lebesgue measurable $\iff$ $f^{-1}(E) \in \mathcal{M}$, $\forall E \in Y$ open.*

Proof. It follows from the previous proposition, since $\mathcal{B}(Y) = \sigma_0(\{E \subset Y : E \text{ open}\})$. $\blacksquare$

Definition 2.8.4. We say that f is **continuous** $\iff$ $f^{-1}(E) \subset X$ is open $\forall E \subset Y$ open.

Proposition 2.8.2. *If $f : X \to Y$ is continuous, then f is Borel measurable (and thus Lebesgue measurable).*

Proof. Let $E \subset Y$ be open. By continuity of f, we have that $f^{-1}(E)$ is open. Then $f^{-1}(E) \in \mathcal{B}(X)$, and thus f is Borel measurable.

Note that the proposition is false when $\mathcal{N} \supsetneq \mathcal{B}(Y)$. $\blacksquare$

2.8.1 Operations on measurable functions

Proposition 2.8.3. *Let $f : X \to Y$ be Lebesgue measurable, and $g : Y \to Z$ be continuous. Then:*

$$g \circ f : X \to Z \text{ is Lebesgue measurable}$$

Corollary 2.8.3.1. *Let $f : X \to Y$ be Lebesgue measurable. Then:*

- $f^+(x) = \max\{f(x), 0\}$ *is Lebesgue measurable*
- $f^-(x) = \max\{-f(x), 0\}$ *is Lebesgue measurable*
- $|f(x)| = f^+(x) + f^-(x)$ *is Lebesgue measurable*

Proof. Let f be Lebesgue measurable, and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then, take $E \subset Z$ open. We have that:

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$$

Since g is continuous, $g^{-1}(E)$ is open. Then, $f^{-1}(g^{-1}(E)) \in \mathcal{M}$ ■

Proposition 2.8.4. *Let $f, g : X \to \mathbb{R}$ be Lebesgue measurable, and $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then, $h(x) = \Phi(f(x), g(x))$ is Lebesgue measurable.*

Proof. Let $h(x) = \Phi(f(x), g(x)) = (\Phi \circ \Psi)(x)$, where $\Psi : X \to \mathbb{R}^2$ is defined as

$$\Psi(x) = (f(x), g(x))$$

Then, we should prove that Ψ is Lebesgue measurable for applying the previous proposition. For this, we have to show that $\forall (a, b) \times (c, d) \subset \mathbb{R}^2$, we have that:

$$\Psi^{-1}((a, b) \times (c, d)) = \{x \in X : f(x) \in (a, b), g(x) \in (c, d)\} \in \mathcal{M}$$

This can be done using the fact that f and g are Lebesgue measurable.

■

Corollary 2.8.4.1. *Let $f, g : X \to \mathbb{R}$ be Lebesgue measurable. Then:*

- *$f + g$ is Lebesgue measurable*
- *$f \cdot g$ is Lebesgue measurable*

Proposition 2.8.5. *Let $(X, \mathcal{M})$ be a measurable space (with $\mathcal{M} \supseteq \mathcal{B}(X)$), and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Lebesgue measurable functions $f_n : X \to \mathbb{R}$. Then, the following functions are Lebesgue measurable:*

1. $\sup_n f_n$
2. $\inf_n f_n$
3. $\limsup_n f_n$
4. $\liminf_n f_n$

In particular, if $\lim_n f_n$ exists, then it is Lebesgue measurable.

Proof. The proof goes as follows:

1. Since $\mathcal{B}(\mathbb{R}) = \sigma_0(\{(a, \infty) : a \in \mathbb{R}\})$, it is enough to show that $\forall a \in \mathbb{R}$, we have that:

$$(\sup_n f_n)^{-1}((a, \infty)) = \{x \in X : \sup_n f_n(x) > a\} \in \mathcal{M}$$

This can be done by using the fact that f_n is Lebesgue measurable. Indeed, we have that:

$$\begin{aligned}\{x \in X : \sup_n f_n(x) > a\} &= \bigcup_n \{x \in X : f_n(x) > a\} \\ &= \bigcup_n f_n^{-1}((a, \infty)) \in \mathcal{M}\end{aligned}$$

because $f_n^{-1}((a, \infty)) \in \mathcal{M}$ for all n.

2. The proof is analogous to the previous case, taking that:

$$\inf_n f_n = -\sup_n(-f_n)$$

3. We have that:

$$\limsup_n f_n = \inf_n \sup_{k \geq n} f_k$$

4. We have that:

$$\liminf_n f_n = \sup_n \inf_{k \geq n} f_k$$

■

2.8.2 Properties holding almost everywhere

Definition 2.8.5. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. We say that a property $P(x)$ holds **μ-almost everywhere** (a.e) if:

$$\mu(\{x \in X : P(x) \text{ is false}\}) = 0$$

In other words, $P(x)$ holds μ-almost everywhere if it holds everywhere except for a set of measure zero.

E.g.: Let $f(x) = x^2$. Is it true that $f(x) > 0$ a.e.?

We have that $\{x : x^2 \leq 0\} = \{0\}$

- In $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, the property is true a.e., since $\lambda(\{0\}) = 0$
- In $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_\#)$ (counting measure), the property is false a.e., since $\mu_\#(\{0\}) = 1$

Proposition 2.8.6. *Let $(X, \mathcal{M}, \mu)$ be a measure space:*

1. *$f : X \to \overline{\mathbb{R}}$ s.t. $f = g$ a.e, with g measurable $\implies$ f is measurable*
2. *$\{f_n\}_{n \in \mathbb{N}}$ a sequence of measurable functions s.t. $f_n \to f$ a.e., then f is measurable.*

2.8.3 Simple functions

Definition 2.8.6. Let $(X, \mathcal{M})$ be a measurable space. A function $s : X \to \overline{\mathbb{R}}$ is measurable and **simple** if s is measurable and $s(X)$ is a finite set:

$$s(X) = \{a_1, a_2, ..., a_k\}$$

where $a_i \in \overline{\mathbb{R}}\ \forall i$, with $a_i \neq a_j$ for $i \neq j$. Then, s can be written as:

$$s(x) = \sum_{i=1}^{k} a_i \cdot \chi_{A_i}(x)$$

where $A_i = s^{-1}(\{a_i\})$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{k} A_i = X$ and $A_i \in \mathcal{M}$, $\forall i$.

Particular case:

If $X = \mathbb{R}$ (or $(a, b) \subset \mathbb{R}$) and A_i is an interval $\forall i$, then s is called a **step function**.

On the other hand, $\chi_{\mathbb{Q}}$ is a simple function, but not a step function:

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Remark: One may define simple functions without measurability requirements.

Goal:
Approximate any measurable function $f : X \to \overline{\mathbb{R}}$ with (measurable and) simple functions.

Theorem 2.8.7 (Simple approximation theorem (SAT))**.** *Take $(X, \mathcal{M})$ measurable space and $f : X \to [0, \infty]$, measurable. Then $\exists \{s_n\}_{n \in \mathbb{N}}$ a sequence of measurable, simple functions s.t. $s_1 \leq s_2 \leq ... \leq f$ pointwise (i.e., $\forall x \in X$) and:*

$$\lim_{n \to \infty} s_n(x) = f(x) \quad \forall x \in X$$

Moreover, if f is bounded, the convergence is uniform:

$$\lim_{n \to \infty} \sup_{x \in X} |s_n(x) - f(x)| = 0$$

Proof. In case f is bounded, say $0 \leq f < 1$.

For any $n \geq 1$, divide $[0, 1)$ into 2^n intervals of length 2^{-n}, and define:

$$A_n^{(i)} = \{x \in X : \frac{i}{2^n} \leq f(x) < \frac{i+1}{2^n}\}$$

and:

$$s_n(x) = \sum_{n=0}^{2^n-1} \frac{i}{2^n} \cdot \chi_{A_n^{(i)}}(x)$$

One can show that such a sequence has the required properties ■

2.9 Lebesgue integral

- First, we will define the integral for non-negative simple (measurable) functions.
- Second, we will extend the definition to non-negative measurable functions.
- Finally, we will define the integral for general measurable functions.

2.9.1 Integral of non-negative simple functions

Definition 2.9.1. Let $s : X \to [0, \infty]$ be a measurable and simple function:

$$s = \sum_{i=1}^{k} a_i \cdot \chi_{A_i}$$

where $a_i \geq 0$ and $A_i \in \mathcal{M}$. Let $E \in \mathcal{M}$. Then, we define the **(Lebesgue) integral** of s over E as:

$$\int_E s \, d\mu = \sum_{i=1}^{k} a_i \cdot \mu(A_i \cap E)$$

Remark: There are some remarks:

1. $s : [a, b] \to [0, \infty)$, $\mu, \mu = \lambda$ (Lebesgue measure)
   Then, $\int_{[a,b]} s \, d\mu =$ area under the graph of s in $[a, b]$
2. We are already using $0 \cdot \infty = 0$ in the definition. In particular,

$$a_i \cdot \mu(A_i \cap E) = \begin{cases} 0 & a_i = 0 \\ \infty & a_i > 0 \end{cases}$$

if $\mu(A_i \cap E) = \infty$.

3. $D \in \mathcal{M}$, then χ_D is a simple function, and:

$$\int_E \chi_D \, d\mu = \mu(D \cap E)$$

4. More generally, s simple and measurable, $E \in \mathcal{M}$, then:

$$\int_E s \, d\mu = \int_X s \cdot \chi_E \, d\mu$$

Properties 2.9.1 (Basic properties)**.** Let $N, E, F \in \mathcal{M}$, $s_1, s_2 : X \to [0, \infty)$ simple and measurable functions. Then:

(i) If $\mu(N) = 0$, then:

$$\int_N s_1 \, d\mu = 0$$

(ii) If $0 \leq c \leq \infty$, then:

$$\int_E c \cdot s_1 \, d\mu = c \cdot \int_E s_1 \, d\mu$$

(iii) $\int_E (s_1 + s_2) \, d\mu = \int_E s_1 \, d\mu + \int_E s_2 \, d\mu$

(iv) If $s_1 \leq s_2$, then:

$$\int_E s_1 \, d\mu \leq \int_E s_2 \, d\mu$$

(v) if $E \subset F$, then:

$$\int_E s_1 \, d\mu \leq \int_F s_1 \, d\mu$$

The properties (ii) and (iii) are called **linearity** of the integral. The properties (iv) and (v) are called **monotonicity** of the integral.

Proposition 2.9.1. *Let $s : X \to [0, \infty)$ be a simple measurable function. Then, the function:*

$$\phi(E) := \int_E s\, d\mu : \mathcal{M} \to [0, \infty]$$

is a measure on $(X, \mathcal{M})$.

Proof. Let $s = \sum_{i=1}^k a_i \cdot \chi_{A_i}$, $0 \le a_i \le \infty$. We have to show that:

1. $\phi : \mathcal{M} \to [0, \infty]$?: Yes, since $s \ge 0$, $\phi(E) \ge 0$, $\forall E \in \mathcal{M}$.
2. $\phi(\emptyset) = 0$?: Yes, since $\int_\emptyset s\, d\mu = 0$, as $\mu(\emptyset) = 0$.
3. σ-additivity?: Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{M}$, and $E = \bigcup_n E_n$. Then, we have that:

$$\begin{aligned}\phi(E) = \int_E s\, d\mu = \int_X s \cdot \chi_E\, d\mu &= \sum_{i=1}^k a_i \cdot \mu(A_i \cap E) \\ &= \sum_{i=1}^k a_i \cdot \mu\left(\bigcup_n A_i \cap E_n\right)\end{aligned}$$

Since μ is σ-additive, we have that:

$$\begin{aligned}&= \sum_{i=1}^k a_i \sum_n \cdot \mu(A_i \cap E_n) \\ &= \sum_n \sum_{i=1}^k a_i \cdot \mu(A_i \cap E_n) \\ &= \sum_n \int_{E_n} s\, d\mu = \sum_n \phi(E_n)\end{aligned}$$

■

2.9.2 Integral of non-negative measurable functions

Definition 2.9.2. Let $f : X \to [0, \infty]$ be a measurable function, $E \in \mathcal{M}$. Then, we define the **(Lebesgue) integral** of f over E as:

$$\int_E f\, d\mu = \sup\left\{\int_E s\, d\mu : s \text{ simple, measurable and } 0 \le s \le f\right\}$$

Remark: There are some remarks:

1. If f is simple, then the definition coincides with the previous one.
2. $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{\#})$. Then $f : \mathbb{N} \to [0, \infty]$ is a sequence. Indeed, if we name $f_n = f(n)$, then:

$$\int_{\mathbb{N}} f \, d\mu_{\#} = \sum_n f_n$$

3. All the basic properties of the integral for simple functions above hold for this new definition.

Note: The following propositions assume that $(X, \mathcal{M}, \mu)$ is a complete measure space (needed for a.e. properties).

Proposition 2.9.2 (Chebychev's inequality)**.** *Let $f : X \to [0, \infty]$ be a measurable function, and $0 < c < \infty$. Then:*

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f \geq c\}} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu$$

where $\{f \geq c\} = \{x \in X : f(x) \geq c\}$.

Proof.

$$\int_X f \, d\mu \geq \int_{\{f<c\}} f \, d\mu \geq \int_{\{f<c\}} c \, d\mu = c \cdot \mu(\{f < c\})$$

Now we just divide by c. ■

Note: We have as a consequence the following lemmas:

Lemma 2.9.3 (Vanishing lemma)**.** *Let $f : X \to [0, \infty]$ be a measurable function, $E \in \mathcal{M}$:*

$$\int_E f \, d\mu = 0 \iff f = 0 \text{ a.e. on } E$$

Proof. The proof goes as follows:

$(\Leftarrow)$: Trivial

$(\Rightarrow)$: We have to show that:

$$\mu(\{x \in E : f(x) > 0\}) = 0$$

Let us define $F = \{x : f(x) > 0\} = \bigcup_n F_n$, where $F_n = \{x : f(x) \geq 1/n\}$. Then, we have that:

$$F_n \subset F_{n+1} \quad \forall n$$

so $F_n \uparrow F$. Then, we have that:

$$\mu(F_n) \to \mu(F)$$

and:

$$0 \leq \mu(F_n) = \mu(\{f \geq \frac{1}{n}\}) \leq \frac{1}{1/n} \int_E f \, d\mu = 0$$

Then, $\mu(F) = 0$.

■

Remark: The vanishing lemma applies to **every f** once $\mu(E) = 0$, indeed, every property is true a.e. on negligible sets. "The Lebesgue integral does not see negligible sets".

Lemma 2.9.4. *Let $f : X \to [0, \infty]$ be a measurable function. Then:*

$$\int_X f \, d\mu < \infty \implies \mu(\{f = \infty\}) = 0$$

Proof. Exercise. (Hint: $\{f = \infty\} = \bigcap_n \{f \geq n\}$) ■

Theorem 2.9.5 (Monotone Convergence Theorem (MCT))**.** *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable functions* $f_n : X \to [0,\infty]$*. Assume that:*

(i) $f_n \le f_{n+1} \quad \forall n$

(ii) $\lim_{n\to\infty} f_n(x) = f(x) \quad$ *for a.e.* $x \in X$

Then, we have that:

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

Remark: All assumptions are essential

Proof. The proof goes as follows:

<u>Part 1:</u>

Assume that assumptions (i) and (ii) hold $\forall x \in X$. We have some basic facts:

- $f(x) = \lim_{n\to\infty} f_n(x) \implies f(x) \ge 0$ and measurable.
- $\int_X f_n \, d\mu \le \int_X f_{n+1} \, d\mu$. Then, if we define:

$$\alpha_n = \int_X f_n \, d\mu, \quad \alpha = \lim_{n\to\infty} \alpha_n$$

we have that $\alpha_n \le \alpha_{n+1}$, so $\alpha_n \uparrow \alpha$. Moreover, we have that:

$$f_n(x) \le f(x) \implies \int_X f_n \, d\mu \le \int_X f \, d\mu$$
$$\implies \alpha \le \int_X f \, d\mu$$

So, to complete part 1, we have to show that $\alpha \ge \int_X f \, d\mu$.

We use the definition of $\int_X f \, d\mu$:

Take any $s : X \to [0,\infty)$ simple, measurable and $0 \le s \le f$. Take also $0 \le c < 1$. Then, we have that:

$$0 < c \cdot s \leq f$$

Take $f_n(x) \uparrow f(x)$ $\forall x \in X$. Consider $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\} \in \mathcal{M}$. Then, we have that:

(a) $E_n \subset E_{n+1}$: indeed, $x \in E_n \iff f_n(x) \geq c \cdot s(x) \implies f_{n+1}(x) \geq c \cdot s(x) \iff x \in E_{n+1}$

(b) $\bigcup_n E_n = X$: indeed, either $f(x) = 0 \implies x \in E_n$ $\forall n$ or $f(x) > 0$ and $c \cdot s(x) < f(x)$. Since $f_n(x) \uparrow f(x)$, we have that $\exists N_0$ s.t. $f_{N_0}(x) \geq c \cdot s(x)$. Then $x \in E_{N_0}$.

Then, we have that:

$$\begin{aligned} \alpha \geq \alpha_n = \int_X f_n \, d\mu \geq \int_{E_n} c \cdot s \, d\mu = c \cdot \int_{E_n} s \, d\mu \\ = c \cdot \phi(E_n) \end{aligned}$$

(where $\phi(E) = \int_E s \, d\mu$ is a measure). Then, notice that $E_n \uparrow X$, so $\phi(E_n) \to \phi(X)$.

Then, we have that:

$$\alpha \geq c \cdot \phi(X) = c \cdot \int_X s \, d\mu$$

Then, $\forall c < 1$, $\forall s$:

$$\alpha \geq c \int_X s \, d\mu$$

If we take the limit $c \to 1$, we have that $\alpha \geq \int_X s \, d\mu$. And if we take the supremum over all s, we have that:

$$\alpha \geq \int_X f \, d\mu$$

Part 2:

Now, we have to show that the result holds for $a.e.$ $x \in X$. Define

$$F = \{x \in X : \text{either } (i) \text{ or } (ii) \text{ fails}\}$$

Then we have that $\mu(F) = 0$, and $E = X \setminus F$. For any g (non-negative, measurable), we have that:

$$g - \chi_E \cdot g = 0 \quad \text{a.e. on } X$$

Then, we use the vanishing lemma to show that:

$$\int_X (g - \chi_E \cdot g)\, d\mu = 0$$
$$\iff \int_X g\, d\mu = \int_E g\, d\mu$$

Finally:

$$\int_X f\, d\mu = \int_E f\, d\mu = \lim_{n\to\infty} \int_E f_n\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$$

■

Remark: Note that we now have 2 ways to compute the integral of a non-negative measurable function:

- $\int_X f\, d\mu = \sup \left\{ \int_X s\, d\mu : s \text{ simple, measurable and } 0 \le s \le f \right\}$
- $\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$ where $f_n \uparrow f$ simple and measurable functions.

Corollary 2.9.5.1 (Monotone convergence for series)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : X \to [0, \infty]$. Then, we have that:*

$$\int_X \sum_n f_n\, d\mu = \sum_n \int_X f_n\, d\mu$$

Proposition 2.9.6. *Take $\Phi : X \to [0, \infty]$ measurable, $E \in \mathcal{M}$. Define:*

$$\nu(E) = \int_E \Phi\, d\mu$$

Then, ν is a measure on $(X, \mathcal{M})$. Moreover, for $f : X \to [0, \infty]$ measurable:

$$\int_X f\, d\nu = \int_X f \cdot \Phi\, d\mu$$