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Real and Functional Analysis

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Contents

Chapter 1

Set Theory

1.1 Basic notions

Definition 1.1.1. Let X, Y be sets. We say:
- X, Y are **equipotent** if there exists a bijection $f : X \to Y$.
- X has a **cardinality greater or equal** to Y if there exists an surjection $f : X \to Y$.
- X is **finite** if it is equipotent to $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$. X is infinite otherwise.

Remark: X is infinite $\iff$ it is equipotent to a proper subset of itself.

E.g.: The set of natural numbers $\mathbb{N}$ is infinite. In fact, the set of even natural numbers $E = \{2, 4, 6, \ldots\} \subset \mathbb{N}$ is equipotent to $\mathbb{N}$, as we can define the bijection $f : \mathbb{N} \to E$ as $f(n) = 2n$.

Definition 1.1.2. Let X be an infinite set. We say X is **countable** if it is equipotent to $\mathbb{N}$. X is **uncountable** otherwise, in which case it is **more than countable**.

Definition 1.1.3. X has the **cardinality of the continuum** if it is equipotent to $[0, 1] \subset \mathbb{R}$. Any such set is uncountable.

E.g.: We have that:

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are countable.
- $\mathbb{R}, \mathbb{R}^n, (0, 1), [0, 1]$ are uncountable.
- Countable union of countable sets is countable.

1.2 Families of subsets

Let X be a set. The "Power set" of X is the set of all subsets of X, denoted by $\mathcal{P}(X)$.

$$\mathcal{P}(X) = \{E : \ E \subseteq X\}$$

Note that $\mathcal{P}(X)$ has always a cardinality greater than X. For example, if $X = \mathbb{N}$, then $\mathcal{P}(X)$ has the cardinality of the continuum.

Definition 1.2.1. Let X be a set. A **family of subsets** of X is a set E such that:

$$E \subseteq \mathcal{P}(X)$$

We usually denote $E = \{E_i\}_{i \in I}$, where I is an index set.

Definition 1.2.2. Let $E = \{E_i\}_{i \in I}$ be a family of subsets of X. We define:

- The **union** of E as:

$$\bigcup_{i \in I} E_i = \{x \in X : \ x \in E_i \text{ for some } i \in I\}$$

- The **intersection** of E as:

$$\bigcap_{i \in I} E_i = \{x \in X : \ x \in E_i \text{ for all } i \in I\}$$

Definition 1.2.3. Let $E = \{E_i\}_{i \in I}$ be a family of subsets of X. We say F is **pairwise disjoint** if:

$$E_i \cap E_j = \emptyset \ \ \forall i, j \in I, i \neq j$$

Definition 1.2.4. We say that the family $E = \{E_i\}_{i \in I}$ of subsets of X is a **covering** of X if:

$$X = \bigcup_{i \in I} E_i$$

Any subfamily of E, $E' = \{E_i\}_{i \in I'}$ is a **subcovering** of X if it is a covering of X itself.

E.g.: Let $X = \mathbb{R}$. We define:

$$\mathcal{T} = \{E \subset X : E \text{ is open}\}$$

We say that $\mathcal{T}$ is the standard topology of X. More generally, this can be done in "metric spaces" (X, d).

Properties of $\mathcal{T}$ (open sets):

- $\emptyset, X \in \mathcal{T}$.
- Finite intersection of elements in $\mathcal{T}$ is in $\mathcal{T}$.
- Arbitrary union of elements in $\mathcal{T}$ is in $\mathcal{T}$.

We can also define **sequences of sets**. Let X be a set. A sequence of sets in X is a family of sets $\{E_n\}_{n\in\mathbb{N}}$.

Definition 1.2.5. Let X be a set. A sequence of sets $\{E_n\}_{n\in\mathbb{N}}$ is said to be:

- **Increasing** if:
$$E_n \subseteq E_{n+1} \;\; \forall n \in \mathbb{N}$$
It is denoted by $\{E_n\}\uparrow$.
- **Decreasing** if:
$$E_n \supseteq E_{n+1} \;\; \forall n \in \mathbb{N}$$
It is denoted by $\{E_n\}\downarrow$.

Let now $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ be a sequence of sets in X:

Definition 1.2.6. We define the following:

- The **limit superior** of $\{E_n\}$ as:
$$\limsup_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k$$
- The **limit inferior** of $\{E_n\}$ as:
$$\liminf_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} E_k$$
- If the limit superior and limit inferior are equal, we say that
$$\lim_{n\to\infty} E_n = \limsup_{n\to\infty} E_n = \liminf_{n\to\infty} E_n$$

Exercise: Let X be a set and $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X)$ be a sequence of sets in X. Prove that:

$$(i) \quad \{E_n\}\uparrow \;\Rightarrow \lim_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} E_n \qquad (ii) \quad \{E_n\}\downarrow \;\Rightarrow \lim_{n\to\infty} E_n = \bigcap_{n\in\mathbb{N}} E_n$$

1.3 Characteristic functions

Definition 1.3.1. Let X be a set and $E \subseteq X$. The **characteristic function** of E is the function $\mathbb{1}_E : X \to \{0,1\}$ defined as:

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

This function is also called the **indicator function** of E.

Remark: Let $E, F \subseteq X$. We have that:

- $\mathbb{1}_{E \cap F} = \mathbb{1}_E \cdot \mathbb{1}_F$.
- $\mathbb{1}_{E \cup F} = \mathbb{1}_E + \mathbb{1}_F - \mathbb{1}_{E \cap F}$.
- $\mathbb{1}_{E^c} = 1 - \mathbb{1}_E$.
- $\mathbb{1}_{\limsup_{n\to\infty} E_n} = \limsup_{n\to\infty} \mathbb{1}_{E_n}$.
- $\mathbb{1}_{\liminf_{n\to\infty} E_n} = \liminf_{n\to\infty} \mathbb{1}_{E_n}$.

1.4 Equivalence relations and Quotient sets

Definition 1.4.1. A relation R on a set X is a subset of $X \times X$. For any $x, y \in X$, we say that x is related to y if $(x, y) \in R$. We denote this as xRy.

Definition 1.4.2. A relation R on a set X is an **equivalence relation** if it satisfies:

- **Reflexivity:**
$$xRx \ \ \forall x \in X$$
- **Symmetry:**
$$xRy \ \Rightarrow \ yRx \ \ \forall x, y \in X$$
- **Transitivity:**
$$xRy, yRz \ \Rightarrow \ xRz \ \ \forall x, y, z \in X$$

Every equivalence relation on X induces a partition of X. We define the **equivalence class** of $x \in X$ as:

$$[x] = \{y \in X : \ xRy\}$$

The set of all equivalence classes is called the **quotient set** of X by R, denoted by X/R.

$$X/R = \{[x] : \ x \in X\}$$

E.g.: Let $X = \mathbb{Z} \times \mathbb{Z}_0$ such that $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. We define the relation R on X as:

$$(a, b)R(c, d) \iff ad = bc$$

We can prove that R is an equivalence relation. The equivalence classes are:

$$[(a, b)] = \{(c, d) \in X : \ ad = bc\}$$

Notice that:

$$[(a, b)] = \{(a, b), (2a, 2b), (3a, 3b), \dots\}$$

If we denote a class $[(a, b)]$ as $[a/b]$, then we have that:

$$X/R = \{[a/b] : \ a, b \in \mathbb{Z}_0\} = \mathbb{Q}$$

Chapter 2

Measure Spaces

2.1 Measurable spaces

Definition 2.1.1. Let X be a non-empty set. A family $\mathcal{M} \subseteq \mathcal{P}(X)$ is a **σ-algebra** if:

(i) $\emptyset \in \mathcal{M}$.

(ii) $E \in \mathcal{M} \implies E^c \in \mathcal{M}$.

(iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M} \implies \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{M}$.

If instead of (iii) we have that $E_1, E_2 \in \mathcal{M} \implies \mathbb{E}_1 \cup E_2 \in \mathcal{M}$, then $\mathcal{M}$ is called an **algebra**.

Remark: If $\mathcal{M}$ is a σ-algebra, then we say that $(X, \mathcal{M})$ is a **measurable space**. Any set $E \in \mathcal{M}$ is called a **measurable set**.

E.g.: Let $X \neq \emptyset$. Then:

- $\mathcal{P}(X)$ is a σ-algebra.
- $\{\emptyset, X\}$ is a σ-algebra.
- $\{\emptyset, E, E^c, X\}$ is a σ-algebra for any $E \subseteq X$.
- $X = \mathbb{R}$, $\mathcal{M} = \{E \subseteq \mathbb{R} : \ E \text{ is open}\}$ is NOT a σ-algebra.

Properties 2.1.1. Let $(X, \mathcal{M})$ be a measurable space. Then:

(i) $X = \emptyset^c \in \mathcal{M}$

(ii) $\mathcal{M}$ is also an algebra. Indeed, if $\{E_1, E_2\} \subseteq \mathcal{M},\ E_n = \emptyset\ \forall n \geq 3$, then $E_1 \cup E_2 = \bigcup_{n\in\mathbb{N}} E_n \in \mathcal{M}$.

(iii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M} \implies \bigcap_n E_n \in \mathcal{M}$.

(iv) $E, F \in \mathcal{M} \implies E \setminus F \in \mathcal{M}$

(v) $\Omega \subseteq X$. Then, the **restriction** of $\mathcal{M}$ to Ω is:

$$\mathcal{M}|_\Omega := \{F \subseteq \Omega : F = E \cap \Omega, E \in \mathcal{M}\}$$

Then, $(\Omega, \mathcal{M}|_\Omega)$ is a measurable space.

2.2 Generation of a σ-algebra

Theorem 2.2.1. *Take any family $\mathcal{A} \subseteq \mathcal{P}(X)$. Then, it is well-defined the σ-algebra generated by $\mathcal{A}$, denoted by $\sigma_0(\mathcal{A})$, as the smallest σ-algebra containing $\mathcal{A}$. It is characterized by:*

(i) $\sigma_0(\mathcal{A})$ is a σ-algebra.

(ii) $\mathcal{A} \subseteq \sigma_0(\mathcal{A})$.

(iii) If $\mathcal{M}$ is a σ-algebra and $\mathcal{A} \subseteq \mathcal{M}$, then $\sigma_0(\mathcal{A}) \subseteq \mathcal{M}$.

Sketch of proof. Define $V = \{\mathcal{M} \subseteq \mathcal{P}(X) : \mathcal{M} \text{ is } \sigma\text{-algebra}, \mathcal{A} \subseteq \mathcal{M}\}$. Notice that $V \neq \emptyset$ because $\mathcal{P}(X) \in V$. Then, define:

$$\sigma_0(\mathcal{A}) = \bigcap_{\mathcal{M}\in V} \mathcal{M}$$

Then, $\sigma_0(\mathcal{A})$ is a σ-algebra as it satisfies the properties of a σ-algebra, denoted in definition 2.1.1.

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Remark: This is relevant. Often, to check that a σ-algebra has certain properties, it is enough to check the property on a set of generators.

2.3 Borel sets

Take (X, d) as a metric space, so that open sets are defined. Recall that:

$$\mathcal{T} = \{E \subseteq X : E \text{ is open}\}$$

Definition 2.3.1. The σ-algebra generated by $\mathcal{T}$ is called the **Borel σ-algebra** of X, denoted by:

$$\mathcal{B}(X) := \sigma_0(\mathcal{T})$$

Any set $E \in \mathcal{B}(X)$ is a **Borel set**.

Remark: The following are Borel sets:
- Open sets
- Closed sets
- Countable intersections of open sets (G_δ-sets)
- Countable unions of closed sets (F_σ-sets)

We will deal with:

$$X = \mathbb{R} = (-\infty, \infty)$$

but also:

$$X = \overline{\mathbb{R}} = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$$

Let us define the arithmetic operations on $\overline{\mathbb{R}}$. Let $a \in \mathbb{R}$:

- $a \pm \infty = \pm\infty$
- $a > 0 : a \cdot \pm\infty = \pm\infty$
- $a < 0 : a \cdot \pm\infty = \mp\infty$
- $a = 0 : 0 \cdot \pm\infty = 0$
- $\infty - \infty$, ∞/∞, $0/0$ are not defined.

Also, the open intervals in $\overline{\mathbb{R}}$ are the following:

- (a, b), with $a, b \in \mathbb{R}$
- $[-\infty, b)$
- $(a, \infty]$

Remark: We have that:

$$\begin{aligned}\mathcal{B}(\mathbb{R}) &:= \sigma_0(\{\text{open sets}\}) \\ &= \sigma_0(\{(a,b) : a < b\}) \\ &= \sigma_0(\{[a,b] : a < b\}) \\ &= \sigma_0(\{(a,\infty) : a \in \mathbb{R}\})\end{aligned}$$

$$\begin{aligned}\mathcal{B}(\overline{\mathbb{R}}) &:= \sigma_0(\{\text{open sets}\}) \\ &= \sigma_0(\{(a,\infty] : a \in \mathbb{R}\})\end{aligned}$$

$$\mathcal{B}(\mathbb{R}^N) = \sigma_0(\{\text{open rectangles}\})$$

2.4 Measures

Let $(X, \mathcal{M})$ be a measurable space.

Definition 2.4.1. A function $\mu : \mathcal{M} \to [0, \infty]$ is a (positive) **measure** on $\mathcal{M}$ if:

(i) $\mu(\emptyset) = 0$

(ii) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$, disjoint $\implies \mu\left(\bigcup_{n\in\mathbb{N}} E_n\right) = \sum_{n\in\mathbb{N}} \mu(E_n)$

Note: To avoid nonsenses, we always assume that $\exists E \in \mathcal{M}$ *s.t.* $\mu(E) < \infty$

Terminology: Let $X, \mathcal{M}, \mu$ defined as above:

- $(X, \mathcal{M}, \mu)$ is a **measure space**.
- If $\mu(X) = 1$, then $(X, \mathcal{M}, \mu)$ is a **probability space** and μ is a **probability measure**.

Definition 2.4.2. A measure μ is:

1. **Finite** if $\mu(X) < \infty$
2. σ**-finite** if $\exists \{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ s.t.

$$\mu(E_n) < \infty \;\; \forall n \in \mathbb{N} \quad \wedge \quad X = \bigcup_{n\in\mathbb{N}} E_n$$

E.g.: Some examples of measures are:

1. (Trivial measure): For any $(X, \mathcal{M})$, define μ as $\mu(E) = 0 \ \forall E \in \mathcal{M}$
2. (Counting measure): For any $(X, \mathcal{M})$, typically $\mathcal{M} = \mathcal{P}(X)$, define $\mu_\#$ as:

$$\mu_\#(E) = \begin{cases} \#\{\text{elements of } E\} & E \text{ finite} \\ \infty & \text{otherwise} \end{cases}$$

3. (Dirac measure): For any $(X, \mathcal{M})$, pick $x_0 \in X$. Then, define δ_{x_0} as:

$$\delta_{x_0}(E) = \begin{cases} 1 & x_0 \in E \\ 0 & x_0 \notin E \end{cases}$$

2.4.1 Properties of measures

Theorem 2.4.1 (Basic properties)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:*

(i) *μ is finitely additive: $E, F \in \mathcal{M}, E \cap F = \emptyset \implies \mu(E \cup F) = \mu(E) + \mu(F)$*

(ii) *(Monotonicity): $E, F \in \mathcal{M}, E \subseteq F \implies \mu(E) \leq \mu(F)$*

(iii) *(Excision property): $E, F \in \mathcal{M}, E \subseteq F, \mu(E) < \infty \implies \mu(F \backslash E) = \mu(F) - \mu(E)$*

Proof. The proof is straightforward:

(i) Let $E, F \in \mathcal{M}, E \cap F = \emptyset$. Then:

$$\mu(E \cup F) = \mu(E) + \mu(F)$$

Proof. Obvious, using $E_n = \emptyset$ for $n \geq 3$. ∎

(ii) Let $E, F \in \mathcal{M}, E \subseteq F$. Then:

$$\mu(E) \leq \mu(F)$$

Proof. Let $F = E \cup (F \setminus E)$. Then:

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E)$$

∎

(iii) Let $E, F \in \mathcal{M}, E \subseteq F, \mu(E) < \infty$. Then:

$$\mu(F \setminus E) = \mu(F) - \mu(E)$$

Proof. Let $F = E \cup (F \setminus E)$. Then:

$$\mu(F) = \mu(E) + \mu(F \setminus E)$$

■

This concludes the proof.

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Theorem 2.4.2 (Continuity among monotone sequences)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be a sequence of measurable sets. Then:*

(i) If $\{E_n\} \uparrow$, $E := \lim_n E_n = \bigcup_n E_n$, then:

$$\mu(E) = \lim_n \mu(E_n)$$

(ii) If $\{E_n\} \downarrow$, $E := \lim_n E_n = \bigcap_n E_n$, and $\mu(E_1) < \infty$, then:

$$\mu(E) = \lim_n \mu(E_n)$$

Proof. The proof goes as follows:

(i) If $\mu(E_n) = \infty$ for some n, then the proof is trivial. Otherwise, let $F_1 = E_1$ and $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$. Then, we can check that:

- $F_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $\{F_n\}$ is a disjoint sequence.
- $E_n = \bigcup_{k=1}^{n} F_k$
- Because of the disjointness, we have that:

$$\mu(E_n) = \sum_{k=1}^{n} \mu(F_k)$$

Then, we have that:

$$\begin{aligned}\mu(E) = \mu\left(\lim_n E_n\right) &= \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \\ &= \mu\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} \mu(F_n) = \\ &= \sum_{n=1}^{\infty} (\mu(E_n) - \mu(E_{n-1})) = \lim_n \mu(E_n)\end{aligned}$$

(ii) Define $G_n = E_1 \setminus E_n$. Then, check that:

- $G_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $\{G_n\} \uparrow$

By the previous result, we have that:

$$\mu\left(\bigcup_{n=1}^{\infty} G_n\right) = \lim_n \mu(G_n)$$

Then, on the right-hand side:

$$\begin{aligned}\lim_n \mu(G_n) &= \lim_n \mu(E_1 \setminus E_n) = \\ &= \mu(E_1) - \lim_n \mu(E_n)\end{aligned}$$

On the left-hand side:

$$\begin{aligned}\mu\left(\bigcup_{n=1}^{\infty} G_n\right) &= \mu\left(\bigcup_{n=1}^{\infty}(E_1 \setminus E_n)\right) = \\ &= \mu\left(E_1 \setminus \bigcap_{n=1}^{\infty} E_n\right) = \\ &= \mu(E_1) - \mu(E)\end{aligned}$$

Finally, we have that:

$$\mu(E_1) - \mu(E) = \mu(E_1) - \lim_n \mu(E_n)$$

And because $\mu(E_1) < \infty$, we have that:

$$\mu(E) = \lim_n \mu(E_n)$$

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Remark: In (ii), the condition $\mu(E_1) < \infty$ is essential. Consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_{\#})$. Then, define the following sequence:

$$E_n = \{n, n+1, n+2, \ldots\}$$

Note that $E_n \subseteq E_{n-1}$. Also, note that for any $n \in \mathbb{N}$, we have that:

$$\mu_{\#}(E_n) = \infty$$

Then, we have that:

$$\mu_{\#}\left(\bigcap_n E_n\right) = \mu_{\#}(\emptyset) = 0$$

But:

$$\lim_n \mu_{\#}(E_n) = \infty$$

This shows that the condition $\mu(E_1) < \infty$ is essential.

Theorem 2.4.3 (σ-subadditivity)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ be a sequence of measurable sets. Then:*

$$\mu\left(\bigcup_n E_n\right) \leq \sum_n \mu(E_n)$$

Proof. Let $F_1 = E_1$ and $F_n = E_n \setminus \left(\bigcup_{k=1}^{n-1} E_k\right)$ for $n \geq 2$. Then, we have that:

- $F_n \in \mathcal{M}, \forall n \in \mathbb{N}$
- $F_n \subseteq E_n, \forall n \in \mathbb{N}$
- $\{F_n\}$ is a disjoint sequence.
- $\bigcup_n E_n = \bigcup_n F_n$

Then, we have that:

$$\mu\left(\bigcup_n E_n\right) = \mu\left(\bigcup_n F_n\right) =$$
$$= \sum_n \mu(F_n) \leq \sum_n \mu(E_n)$$

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2.5 Sets of measure zero, negligible sets, complete measures

Definition 2.5.1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then:

1. A set $E \in \mathcal{M}$ is a **set of measure zero** if $\mu(E) = 0$.
2. A set $F \in X$ (not necessarily measurable) is a **negligible set** if $\exists E \in \mathcal{M}$ s.t. $F \subseteq E$ and E is a set of measure zero.

Definition 2.5.2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then, we say that μ is a **complete measure** (alternatively, that $(X, \mathcal{M}, \mu)$ is a **complete measure space**) all negligible sets are measurable.

Remark (Completion of a measure space)**:** A measure space $(X, \mathcal{M}, \mu)$ may not be complete. However, we can define the following:

$$\overline{\mathcal{M}} = \{E \subseteq X : \exists F_1, F_2 \in \mathcal{M} : F_1 \subseteq E \subseteq F_2, \mu(F_2 \setminus F_1) = 0\}$$

One can show that $\overline{\mathcal{M}}$ is a σ-algebra, and that $\mathcal{M} \subseteq \overline{\mathcal{M}}$. Moreover, if E, F_1, F_2 are as above, define:

$$\overline{\mu}(E) = \mu(F_1) = \mu(F_2)$$

One can check that $(X, \overline{\mathcal{M}}, \overline{\mu})$ is a complete measure space.

2.6 Towards the Lebesgue measure

We would like to define a measure λ with $X = \mathbb{R}$ (or $X = \mathbb{R}^N$) s.t. $\forall a < b$:

- $\lambda((a, b)) = b - a$ **(length of the interval)**
- $\forall E, \lambda(E + x) = \lambda(E)$ **(translation invariance)**

In principle, we would like to define it in $\mathcal{P}(\mathbb{R})$. Such a measure should satisfy $\lambda(\{a\}) = 0$.

Theorem 2.6.1 (Ulam)**.** *The only measure on $\mathcal{P}(\mathbb{R})$ that satisfies $\lambda(\{a\}) = 0\ \forall a \in \mathbb{R}$ is the trivial measure.*

Therefore, we need to choose an $\mathcal{M} \subsetneq \mathcal{P}(\mathbb{R})$. We can construct one as follows:

- Starting family with a "measure", e.g., $\mathcal{T} = \{(a, b) : a < b\}$ and $f((a, b)) = b - a$.
- Construct an "outer measure" μ^* on $\mathcal{P}(\mathbb{R})$.
- Restrict μ^* to a well-chosen σ-algebra $\mathcal{M} \subsetneq \mathcal{P}(\mathbb{R})$.

Definition 2.6.1. Let X be a set. An **outer measure** μ^* on X is a function

$$\mu^* : \mathcal{P}(X) \to [0, \infty]$$

such that:

1. $\mu^*(\emptyset) = 0$
2. (Monotonicity) $E \subseteq F \subseteq X \implies \mu^*(E) \leq \mu^*(F)$
3. (σ-subadditivity) $\{E_n\}_{n\in\mathbb{N}} \subseteq \mathcal{P}(X) \implies \mu^*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \leq \sum_{n\in\mathbb{N}} \mu^*(E_n)$

Remark: Any measure μ is an outer measure. However, the converse is not true.

Proposition 2.6.2. *Let $\mathcal{E} \subseteq \mathcal{P}(X)$, $f : \mathcal{E} \to [0, \infty]$. Assume that $\emptyset, X \in \mathcal{E}, f(\emptyset) = 0$. Then, $\forall E \subseteq X$ define:*

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} f(A_n) : A_n \in \mathcal{E}, E \subseteq \bigcup_n A_n\right\}$$

Then, μ^* is an outer measure.

Proof. The proof is omitted.

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Remark: In this generality, if $E \in \mathcal{E}$, then $f(E)$ and $\mu^*(E)$ may not be equal. We can only guarantee that $\mu^*(E) \leq f(E)$.

E.g.: There are some important examples:

- $X = \mathbb{R}$, $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$

$$f((a, b)) = \text{length}((a, b)) = b - a$$

- $X = \mathbb{R}^N$, $\mathcal{E} = \{(a_1, b_1) \times \ldots \times (a_N, b_N) : a_i, b_i \in \overline{\mathbb{R}}, a_i \leq b_i\}$

$$f((\underline{a}, \underline{b})) = \text{volume}((\underline{a}, \underline{b})) = \prod_{i=1}^{N} (b_i - a_i)$$

In both cases, the outer measure μ^* is called the **Lebesgue outer measure**. We will denote it by λ^* (or λ^*_N in the second case). Note that in this case, $\lambda^*(E) = f(E)$ for any $E \in \mathcal{E}$.

Remark: Any μ measure on $\mathcal{P}(X)$ is an outer measure. However, the converse is not true. In particular, $\exists A, B \subseteq \mathbb{R}$ s.t. $A \cap B = \emptyset$ and $\lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$.

2.6.1 Carathéodory's criterion

Definition 2.6.2 (Carathéodory's condition)**.** Let μ^* be an outer measure on $\mathcal{P}(X)$. A ser $E \subseteq X$ is **μ^*-measurable** if $\forall A \subseteq X$:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Lemma 2.6.3 (Equivalence of Carathéodory's condition)**.** *E is μ^*-measurable $\iff$ $\forall A \subseteq X$, $\mu^*(A) < \infty$:*

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Proof. The proof is as follows:

$(\Rightarrow)$: Trivial

$(\Leftarrow)$: Let $A \subseteq X$, such that $\mu^*(A) < \infty$ and:

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Then, notice that $\{A \cap E, A \cap E^c\}$ is a covering of A. By subadditivity:

$$\mu^*(A) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Therefore, we have that:

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Meaning that E is μ^*-measurable. This concludes the proof.

∎

Theorem 2.6.4 (Carathéodory)**.** *Let μ^* be an outer measure on $\mathcal{P}(X)$. The family:*

$$\mathcal{M} = \{E \subseteq X : E \text{ is } \mu^*\text{-measurable}\}$$

is a σ-algebra, and μ^* restricted to $\mathcal{M}$ (denoted $\mu = \mu^*|_{\mathcal{M}}$) is a complete measure.

Remark: $(X, \mathcal{M}, \mu)$ as in the above theorem is sometimes called the "abstract Lebesgue measure space". We will only prove the completeness of μ.

Lemma 2.6.5. *Let $(X, \mathcal{M}, \mu)$ be the measure space as in Carathéodory's theorem. Then, any $N \subseteq X$ s.t. $\mu^*(N) = 0$ is μ-measurable, i.e., $N \in \mathcal{M}$, and $\mu(N) = 0$.*

Proof. We have to show that N satisfies Carathéodory's condition, or equivalently, that it satisfies the lemma 2.6.3. Let $A \subseteq X$ be arbitrary. Then, notice that:

$$\mu^*(A \cap N) \leq \mu^*(N) = 0$$

Also, notice that:

$$\mu^*(A \cap N^c) \leq \mu^*(A)$$

Therefore, we have that:

$$\mu^*(A \cap N) + \mu^*(A \cap N^c) \leq 0 + \mu^*(A) = \mu^*(A)$$

By lemma 2.6.3, we have that N is μ^*-measurable. By Carathéodory's theorem, we have that N is μ-measurable. Finally, we have that $\mu(N) = \mu^*(N) = 0$.

■

Corollary 2.6.5.1. *μ as in Carathéodory's theorem is a complete measure.*

Proof. Let $N \subseteq E$, and $\mu(E) = 0$ $(E \in \mathcal{M})$. Then, we have that:

$$\mu(E) = 0 \implies \mu^*(E) = 0$$

By monotonicity, we have that:

$$\mu^*(N) \leq \mu^*(E) = 0$$

Then, $\mu(N) = \mu^*(N) = 0$, thus $N \in \mathcal{M}$. This concludes the proof.

■

2.7 Lebesgue measure

Definition 2.7.1. Let $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$. Define:

$$\lambda^*((a, b)) = b - a$$

Then, λ^* is the **Lebesgue outer measure** on $\mathbb{R}$.

Theorem 2.7.1. *Let λ^* be the Lebesgue outer measure on $\mathcal{E} = \{(a, b) : a, b \in \overline{\mathbb{R}}, a \leq b\}$. Then, by Carathéodory's theorem, the family:*

$$\mathcal{L}(\mathbb{R}) = \{E \subseteq \mathbb{R} : E \text{ is } \lambda^*\text{-measurable}\}$$

*is a σ-algebra, called the **Lebesgue σ-algebra**, and λ^* restricted to $\mathcal{L}(\mathbb{R})$ (denoted $\lambda = \lambda^*|_{\mathcal{L}(\mathbb{R})}$) is a complete measure, called the **Lebesgue measure**.*

Proof. The proof is omitted.

■

Remark: The measure space $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is called the **Lebesgue measure space**.

Proposition 2.7.2. *Let λ be the Lebesgue measure on $\mathbb{R}$. Then:*

(i) $a \in \mathbb{R} \implies \{a\} \in \mathcal{L}(\mathbb{R})$ *and* $\lambda(\{a\}) = 0$

(ii) $E \subset \mathbb{R}$ *at most countable* $\implies E \in \mathcal{L}(\mathbb{R})$ *and* $\lambda(E) = 0$

Proof. The proof is as follows:

(i) Let $a \in \mathbb{R}$. Then, we have that, for any $\varepsilon > 0$:

$$E_1 = (a - \varepsilon, a + \varepsilon), \quad , E_2 = E_3 = ... = \emptyset$$

is a covering of $\{a\}$. Then, by definition of λ^*:

$$0 \leq \lambda^*(\{a\}) \leq \sum_{n=1}^{\infty} f(E_n) = 2\varepsilon$$

As ε is arbitrary, we have that $\lambda^*(\{a\}) = 0$. By Lemma 2.6.5, we then have that $\{a\} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\{a\}) = 0$.

(ii) Let $E \subseteq \mathbb{R}$ be at most countable. Then, we have that:

$$E = \bigcup_{a \in E} \{a\}$$

Because $\{a\} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\{a\}) = 0$, we have that $E \in \mathcal{L}(\mathbb{R})$ and:

$$\lambda(E) = \lambda\left(\bigcup_{a \in E} \{a\}\right) = \sum_{a \in E} \lambda(\{a\}) = 0$$

■

Remark: We can point out 2 important consequences of the above proposition:

1. Arguing as in (i), we can show that the Lebesgue measure is translation invariant. That is, $\forall E \in \mathcal{L}(\mathbb{R})$, $\forall x \in \mathbb{R}$:

$$\lambda(E + x) = \lambda(E)$$

2. In particular, since $\mathbb{Q}$ is countable, we have that $\mathbb{Q} \in \mathcal{L}(\mathbb{R})$ and $\lambda(\mathbb{Q}) = 0$. In the measure sense, $\mathbb{Q}$ has very few elements with respect to $\mathbb{R}$. On the other hand, $\mathbb{Q}$ is dense in $\mathbb{R}$. In the topology sense, $\mathbb{Q}$ has a lot of points.

Proposition 2.7.3. *We have that:* $\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R})$

Proof. Since $\mathcal{B}(\mathbb{R}) = \sigma_0(\{(a, \infty) : a \in \mathbb{R}\})$, if we show that $(a, \infty) \in \mathcal{L}(\mathbb{R}), \quad \forall a \in \mathbb{R}$, then the prop. follows.

Take $A \subset \mathbb{R}$, s.t. $\lambda^*(A) < \infty$. Then, we must show that:

$$\lambda^*(A) \geq \lambda^*(A \cap (a, \infty)) + \lambda^*(A \cap (-\infty, a])$$

Moreover, by a previous remark, one can assume that $a \notin A$. Then, take any countable covering of A by open intervals:

$$A \subseteq \bigcup_n I_n$$

Then, let us define $A_{left} = A \cap (-\infty, a]$ and $I_{n,left} = I_n \cap (-\infty, a]$. Then, we notice that $\{I_{n,left}\}$ is a covering of A_{left}.

In the same way, we define $A_{right} = A \cap (a, \infty)$ and $I_{n,right} = I_n \cap (a, \infty)$. Then, we notice that $\{I_{n,right}\}$ is a covering of A_{right}.

Then, we have that:

$$\lambda^*(A_{left}) \leq \sum_n \lambda^*(I_{n,left})$$

$$\lambda^*(A_{right}) \leq \sum_n \lambda^*(I_{n,right})$$

Summing both inequalities, we have that:

$$\lambda^*(A_{left}) + \lambda^*(A_{right}) \leq \sum_n \lambda^*(I_{n,left}) + \sum_n \lambda^*(I_{n,right})$$

$$= \sum_n \lambda^*(I_n)$$

Taking the infimum over all countable coverings of A, we have that:

$$\lambda^*(A_{left}) + \lambda^*(A_{right}) \leq \lambda^*(A)$$

■

Remark: In particular, we have that $\forall (a, b) \subset \mathbb{R}$:

$$(a, b) \in \mathcal{L}(\mathbb{R}), \quad \lambda((a, b)) = b - a$$

We know that:

$$\mathcal{B}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$$

Are these inclusions strict? We already know that $\mathcal{L}(\mathbb{R}) \neq \mathcal{P}(\mathbb{R})$, by Ulam's theorem. In particular, $\exists E \subset \mathbb{R}$ not Lebesgue measurable (**Vitali sets**). More precisely:

$$\forall E \in \mathcal{L}(\mathbb{R}), \text{ s.t } \lambda(E) > 0, \exists V \subset E, \text{ s.t } V \notin \mathcal{L}(\mathbb{R})$$

The relation between $\mathcal{B}(\mathbb{R})$ and $\mathcal{L}(\mathbb{R})$ is more subtle. It is clarified by the following proposition:

Proposition 2.7.4 (Regularity of the Lebesgue measure)**.** *Let $E \in \mathbb{R}$. Then, the following are equivalent:*

(i) $E \in \mathcal{B}(\mathbb{R})$

(ii) $\forall \varepsilon > 0, \exists A \subset \mathbb{R}$ *open set s.t.*

$$E \subset A \quad \textit{and} \quad \lambda^*(A \setminus E) < \varepsilon$$

(iii) $\forall \varepsilon > 0, \exists G \subset \mathbb{R}$ *of class* G_δ *s.t.*

$$E \subset G \quad \textit{and} \quad \lambda^*(G \setminus E) = 0$$

(iv) $\forall \varepsilon > 0, \exists C \subset \mathbb{R}$ *closed set s.t.*

$$C \subset E \quad \textit{and} \quad \lambda^*(E \setminus C) < \varepsilon$$

(v) $\forall \varepsilon > 0, \exists F \subset \mathbb{R}$ *of class* F_σ *s.t.*

$$F \subset E \quad \textit{and} \quad \lambda^*(E \setminus F) = 0$$

We get as a consequence the following:

Corollary 2.7.4.1. $\forall E \in \mathcal{L}(\mathbb{R})$, $\exists F, G \in \mathcal{B}(\mathbb{R})$ *s.t.* $F \subset E \subset G$ *and*

$$\lambda(G \setminus F) = \lambda(G \setminus E) + \lambda(E \setminus F) = 0$$

In other words:

$$\mathcal{L}(\mathbb{R}) = \overline{\mathcal{B}(\mathbb{R})}$$

(But $\mathcal{L}(\mathbb{R}) \neq \mathcal{B}(\mathbb{R})$*).*

Proof. (Regularity of the Lebesgue measure). The proof goes as follows:

(i) $\Rightarrow$ (ii) :

Let $E \in \mathcal{B}(\mathbb{R})$. Note that, since $A \in \mathcal{L}(\mathbb{R})$ for all A open, we have that:

$$\lambda^*(A \setminus E) = \lambda(A \setminus E)$$

By definition of λ^*, we have that $\forall \varepsilon > 0$, $\exists \{I_n\}_{n \in \mathbb{N}}$ s.t.

$$E \subset \bigcup_n I_n \quad \text{and} \quad \sum_n \lambda(I_n) < \lambda^*(E) + \varepsilon$$

Then, set $A = \bigcup_n I_n$. We have that A is open, $E \subset A$ and:

$$\lambda(A) \leq \sum_n \lambda(I_n) < \lambda(E) + \varepsilon$$
$$\implies \lambda(A \setminus E) = \lambda(A) - \lambda(E) < \varepsilon$$

(ii) $\Rightarrow$ (iii) :

Assume $\forall \varepsilon > 0$, $\exists A_\varepsilon$ open s.t. $E \subset A_\varepsilon$ and $\lambda(A_\varepsilon \setminus E) < \varepsilon$. Then, set $\varepsilon = 1/n$, $n \geq 1$ (for ease of notation, $A_n = A_{1/n}$) and define:

$$G = \bigcap_n A_n$$

Then, G is a G_δ set, $E \subset G$ and:

$$0 \leq \lambda^*(G \setminus E) \leq \lambda^*(A_n \setminus E) < 1/n$$

Taking the limit, we have that $\lambda(G \setminus E) = 0$.

(iii) $\Rightarrow$ (i) :

We know that $E \subset G$, $G \in \mathcal{L}(\mathbb{R})$ with $\lambda(G \setminus E) = 0$. Then, we have that:

$$E = G \setminus (G \setminus E) \in \mathcal{L}(\mathbb{R})$$

since $G \in \mathcal{L}(\mathbb{R})$ and $G \setminus E \in \mathcal{L}(\mathbb{R})$. The last is because it is a negligible set and λ is complete.

■

E.g. (Cantor set)**:** Let $T_0 = [0, 1]$. Then, construct T_{n+1} from T_n (recursively) by removing the inner third part of every interval in T_n:

$$T_0 = [0, 1],$$
$$T_1 = [0, 1/3] \cup [2/3, 1],$$
$$T_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1], \ \ldots$$

Then, define the **Cantor set** as:

$$C = \bigcap_n T_n$$

It can be proven that:

- C has the cardinality of $\mathbb{R}$
- $\lambda(C) = 0$
- C is compact
- C is nowhere dense (has no interior points), i.e., $\text{int}(C) = \emptyset$
- $\exists E \subset C$ s.t. $E \in \mathcal{L}(\mathbb{R})$ but $E \notin \mathcal{B}(\mathbb{R})$

Chapter 3

Measurable functions

Definition 3.0.1. Given $f : X \to Y$, it is well-defined the **preimage** (or counterimage) of f as:

$$f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X), \quad f^{-1}(E) = \{x \in X : f(x) \in E\}$$

Remark: Preimages work well with set operations:
- $f^{-1}(E \cup F) = f^{-1}(E) \cup f^{-1}(F)$
- $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$
- $f^{-1}(E \setminus F) = f^{-1}(E) \setminus f^{-1}(F)$

Note this is not true for images.

Definition 3.0.2. Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f : X \to Y$ is **measurable** if $\forall E \in \mathcal{N}$, we have that $f^{-1}(E) \in \mathcal{M}$. We also say that f is **($\mathcal{M}$,$\mathcal{N}$)-measurable**.

Proposition 3.0.1. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces, and $\rho \subset \mathcal{N}$ s.t. $\mathcal{N} = \sigma_0(\rho)$. Then, $f : X \to Y$ is measurable $\iff$ $\forall E \in \rho$, we have that $f^{-1}(E) \in \mathcal{M}$.*

Proof. The proofs goes as follows:

$(\Rightarrow)$: Trivial

$(\Leftarrow)$: Define $\Sigma := \{E \subset Y : f^{-1}(E) \in \mathcal{M}\}$. We have:

- $\rho \subset \Sigma$ as a consecuence of $(\forall E \in \rho, f^{-1}(E) \in \mathcal{M})$

- Σ is a σ-algebra (check as an exercise)

Then, we have that $\mathcal{N} = \sigma_0(\rho) \subset \Sigma$. Therefore, f is measurable.

■

Definition 3.0.3. Suppose that $\mathcal{M} \supseteq \mathcal{B}(X)$ and $\mathcal{N} = \mathcal{B}(Y)$. We say that $f : X \to Y$ is:

- **Borel measurable** if f is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable.
- **Lebesgue measurable** if it is $(\mathcal{M}, \mathcal{B}(Y))$-measurable.

Remark: If $f : X \to Y$ is Borel measurable, then it is Lebesgue measurable. The converse is not true.

Also, we never deal with $\mathcal{L}(Y)$.

Corollary 3.0.1.1. *f is Borel measurable $\iff$ $f^{-1}(E) \in \mathcal{B}(X)$, $\forall E \in Y$ open. Also, f is Lebesgue measurable $\iff$ $f^{-1}(E) \in \mathcal{M}$, $\forall E \in Y$ open.*

Proof. It follows from the previous proposition, since $\mathcal{B}(Y) = \sigma_0(\{E \subset Y : E \text{ open}\})$. ■

Definition 3.0.4. We say that f is **continuous** $\iff$ $f^{-1}(E) \subset X$ is open $\forall E \subset Y$ open.

Proposition 3.0.2. *If $f : X \to Y$ is continuous, then f is Borel measurable (and thus Lebesgue measurable).*

Proof. Let $E \subset Y$ be open. By continuity of f, we have that $f^{-1}(E)$ is open. Then $f^{-1}(E) \in \mathcal{B}(X)$, and thus f is Borel measurable.

Note that the proposition is false when $\mathcal{N} \supsetneq \mathcal{B}(Y)$. ■

3.1 Operations on measurable functions

Proposition 3.1.1. *Let $f : X \to Y$ be Lebesgue measurable, and $g : Y \to Z$ be continuous. Then:*

$$g \circ f : X \to Z \text{ is Lebesgue measurable}$$

Corollary 3.1.1.1. *Let $f : X \to Y$ be Lebesgue measurable. Then:*

- $f^+(x) = \max\{f(x), 0\}$ *is Lebesgue measurable*
- $f^-(x) = \max\{-f(x), 0\}$ *is Lebesgue measurable*
- $|f(x)| = f^+(x) + f^-(x)$ *is Lebesgue measurable*

Proof. Let f be Lebesgue measurable, and $g : \mathbb{R} \to \mathbb{R}$ be continuous. Then, take $E \subset Z$ open. We have that:

$$(g \circ f)^{-1}(E) = f^{-1}(g^{-1}(E))$$

Since g is continuous, $g^{-1}(E)$ is open. Then, $f^{-1}(g^{-1}(E)) \in \mathcal{M}$ ∎

Proposition 3.1.2. *Let $f, g : X \to \mathbb{R}$ be Lebesgue measurable, and $\Phi : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then, $h(x) = \Phi(f(x), g(x))$ is Lebesgue measurable.*

Proof. Let $h(x) = \Phi(f(x), g(x)) = (\Phi \circ \Psi)(x)$, where $\Psi : X \to \mathbb{R}^2$ is defined as

$$\Psi(x) = (f(x), g(x))$$

Then, we should prove that Ψ is Lebesgue measurable for applying the previous proposition. For this, we have to show that $\forall (a, b) \times (c, d) \subset \mathbb{R}^2$, we have that:

$$\Psi^{-1}((a, b) \times (c, d)) = \{x \in X : f(x) \in (a, b), g(x) \in (c, d)\} \in \mathcal{M}$$

This can be done using the fact that f and g are Lebesgue measurable.

∎

Corollary 3.1.2.1. *Let $f, g : X \to \mathbb{R}$ be Lebesgue measurable. Then:*

- *$f + g$ is Lebesgue measurable*
- *$f \cdot g$ is Lebesgue measurable*

Proposition 3.1.3. *Let $(X, \mathcal{M})$ be a measurable space (with $\mathcal{M} \supseteq \mathcal{B}(X)$), and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Lebesgue measurable functions $f_n : X \to \mathbb{R}$. Then, the following functions are Lebesgue measurable:*

1. $\sup_n f_n$
2. $\inf_n f_n$
3. $\limsup_n f_n$
4. $\liminf_n f_n$

In particular, if $\lim_n f_n$ exists, then it is Lebesgue measurable.

Proof. The proof goes as follows:

1. Since $\mathcal{B}(\mathbb{R}) = \sigma_0(\{(a,\infty) : a \in \mathbb{R}\})$, it is enough to show that $\forall a \in \mathbb{R}$, we have that:

$$(\sup_n f_n)^{-1}((a,\infty)) = \{x \in X : \sup_n f_n(x) > a\} \in \mathcal{M}$$

This can be done by using the fact that f_n is Lebesgue measurable. Indeed, we have that:

$$\begin{aligned}\{x \in X : \sup_n f_n(x) > a\} &= \bigcup_n \{x \in X : f_n(x) > a\} \\ &= \bigcup_n f_n^{-1}((a,\infty)) \in \mathcal{M}\end{aligned}$$

because $f_n^{-1}((a,\infty)) \in \mathcal{M}$ for all n.

2. The proof is analogous to the previous case, taking that:

$$\inf_n f_n = -\sup_n(-f_n)$$

3. We have that:

$$\limsup_n f_n = \inf_n \sup_{k \geq n} f_k$$

4. We have that:

$$\liminf_n f_n = \sup_n \inf_{k \geq n} f_k$$

$\blacksquare$

3.2 Properties holding almost everywhere

Definition 3.2.1. Let $(X, \mathcal{M}, \mu)$ be a complete measure space. We say that a property $P(x)$ holds **μ-almost everywhere** (a.e) if:

$$\mu(\{x \in X : P(x) \text{ is false}\}) = 0$$

In other words, $P(x)$ holds μ-almost everywhere if it holds everywhere except for a set of measure zero.

E.g.: Let $f(x) = x^2$. Is it true that $f(x) > 0$ a.e.?

We have that $\{x : x^2 \leq 0\} = \{0\}$

- In $(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$, the property is true a.e., since $\lambda(\{0\}) = 0$
- In $(\mathbb{R}, \mathcal{P}(\mathbb{R}), \mu_\#)$ (counting measure), the property is false a.e., since $\mu_\#(\{0\}) = 1$

Proposition 3.2.1. *Let $(X, \mathcal{M}, \mu)$ be a measure space:*

1. *$f : X \to \overline{\mathbb{R}}$ s.t. $f = g$ a.e, with g measurable $\implies$ f is measurable*
2. *$\{f_n\}_{n \in \mathbb{N}}$ a sequence of measurable functions s.t. $f_n \to f$ a.e., then f is measurable.*

3.3 Simple functions

Definition 3.3.1. Let $(X, \mathcal{M})$ be a measurable space. A function $s : X \to \overline{\mathbb{R}}$ is measurable and **simple** if s is measurable and $s(X)$ is a finite set:

$$s(X) = \{a_1, a_2, ..., a_k\}$$

where $a_i \in \overline{\mathbb{R}}\ \forall i$, with $a_i \neq a_j$ for $i \neq j$. Then, s can be written as:

$$s(x) = \sum_{i=1}^{k} a_i \cdot \chi_{A_i}(x)$$

where $A_i = s^{-1}(\{a_i\})$, $A_i \cap A_j = \emptyset$ for $i \neq j$, $\bigcup_{i=1}^{k} A_i = X$ and $A_i \in \mathcal{M}$, $\forall i$.

Particular case:

If $X = \mathbb{R}$ (or $(a, b) \subset \mathbb{R}$) and A_i is an interval $\forall i$, then s is called a **step function**.

On the other hand, $\chi_{\mathbb{Q}}$ is a simple function, but not a step function:

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Remark: One may define simple functions without measurability requirements.

Goal:
Approximate any measurable function $f : X \to \overline{\mathbb{R}}$ with (measurable and) simple functions.

Theorem 3.3.1 (Simple approximation theorem (SAT))**.** *Take $(X, \mathcal{M})$ measurable space and $f : X \to [0, \infty]$, measurable. Then $\exists \{s_n\}_{n\in\mathbb{N}}$ a sequence of measurable, simple functions s.t. $s_1 \leq s_2 \leq ... \leq f$ pointwise (i.e., $\forall x \in X$) and:*

$$\lim_{n\to\infty} s_n(x) = f(x) \quad \forall x \in X$$

Moreover, if f is bounded, the convergence is uniform:

$$\lim_{n\to\infty} \sup_{x\in X} |s_n(x) - f(x)| = 0$$

Proof. In case f is bounded, say $0 \leq f < 1$.

For any $n \geq 1$, divide $[0, 1)$ into 2^n intervals of length 2^{-n}, and define:

$$A_n^{(i)} = \{x \in X : \frac{i}{2^n} \leq f(x) < \frac{i+1}{2^n}\}$$

and:

$$s_n(x) = \sum_{n=0}^{2^n-1} \frac{i}{2^n} \cdot \chi_{A_n^{(i)}}(x)$$

One can show that such a sequence has the required properties ■

Chapter 4

Lebesgue integral

- First, we will define the integral for non-negative simple (measurable) functions.
- Second, we will extend the definition to non-negative measurable functions.
- Finally, we will define the integral for general measurable functions.

4.1 Integral of non-negative simple functions

Definition 4.1.1. Let $s : X \to [0, \infty]$ be a measurable and simple function:

$$s = \sum_{i=1}^{k} a_i \cdot \chi_{A_i}$$

where $a_i \geq 0$ and $A_i \in \mathcal{M}$. Let $E \in \mathcal{M}$. Then, we define the **(Lebesgue) integral** of s over E as:

$$\int_E s \, d\mu = \sum_{i=1}^{k} a_i \cdot \mu(A_i \cap E)$$

Remark: There are some remarks:

1. $s : [a, b] \to [0, \infty)$, $\mu, \mu = \lambda$ (Lebesgue measure)
   Then, $\int_{[a,b]} s \, d\mu =$ area under the graph of s in $[a, b]$
2. We are already using $0 \cdot \infty = 0$ in the definition. In particular,

$$a_i \cdot \mu(A_i \cap E) = \begin{cases} 0 & a_i = 0 \\ \infty & a_i > 0 \end{cases}$$

if $\mu(A_i \cap E) = \infty$.

3. $D \in \mathcal{M}$, then χ_D is a simple function, and:

$$\int_E \chi_D \, d\mu = \mu(D \cap E)$$

4. More generally, s simple and measurable, $E \in \mathcal{M}$, then:

$$\int_E s \, d\mu = \int_X s \cdot \chi_E \, d\mu$$

Properties 4.1.1 (Basic properties)**.** Let $N, E, F \in \mathcal{M}$, $s_1, s_2 : X \to [0, \infty)$ simple and measurable functions. Then:

(i) If $\mu(N) = 0$, then:

$$\int_N s_1 \, d\mu = 0$$

(ii) If $0 \leq c \leq \infty$, then:

$$\int_E c \cdot s_1 \, d\mu = c \cdot \int_E s_1 \, d\mu$$

(iii) $\int_E (s_1 + s_2) \, d\mu = \int_E s_1 \, d\mu + \int_E s_2 \, d\mu$

(iv) If $s_1 \leq s_2$, then:

$$\int_E s_1 \, d\mu \leq \int_E s_2 \, d\mu$$

(v) if $E \subset F$, then:

$$\int_E s_1 \, d\mu \leq \int_F s_1 \, d\mu$$

The properties (ii) and (iii) are called **linearity** of the integral. The properties (iv) and (v) are called **monotonicity** of the integral.

Proposition 4.1.1. *Let $s : X \to [0, \infty)$ be a simple measurable function. Then, the function:*

$$\phi(E) := \int_E s \, d\mu : \mathcal{M} \to [0, \infty]$$

is a measure on $(X, \mathcal{M})$.

Proof. Let $s = \sum_{i=1}^k a_i \cdot \chi_{A_i}$, $0 \leq a_i \leq \infty$. We have to show that:

1. $\phi : \mathcal{M} \to [0, \infty]$?: Yes, since $s \geq 0$, $\phi(E) \geq 0$, $\forall E \in \mathcal{M}$.

2. $\phi(\emptyset) = 0$?: Yes, since $\int_\emptyset s \, d\mu = 0$, as $\mu(\emptyset) = 0$.

3. σ-additivity?: Let $\{E_n\}_{n \in \mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{M}$, and $E = \bigcup_n E_n$. Then, we have that:

$$\phi(E) = \int_E s \, d\mu = \int_X s \cdot \chi_E \, d\mu = \sum_{i=1}^k a_i \cdot \mu(A_i \cap E)$$

$$= \sum_{i=1}^k a_i \cdot \mu\left(\bigcup_n A_i \cap E_n\right)$$

Since μ is σ-additive, we have that:

$$= \sum_{i=1}^k a_i \sum_n \cdot \mu(A_i \cap E_n)$$

$$= \sum_n \sum_{i=1}^k a_i \cdot \mu(A_i \cap E_n)$$

$$= \sum_n \int_{E_n} s \, d\mu = \sum_n \phi(E_n)$$

■

4.2 Integral of non-negative measurable functions

Definition 4.2.1. Let $f : X \to [0, \infty]$ be a measurable function, $E \in \mathcal{M}$. Then, we define the **(Lebesgue) integral** of f over E as:

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu : s \text{ simple, measurable and } 0 \leq s \leq f \right\}$$

Remark: There are some remarks:

1. If f is simple, then the definition coincides with the previous one.
2. $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_\#)$. Then $f : \mathbb{N} \to [0, \infty]$ is a sequence. Indeed, if we name $f_n = f(n)$, then:

$$\int_{\mathbb{N}} f \, d\mu_\# = \sum_n f_n$$

3. All the basic properties of the integral for simple functions above hold for this new definition.

Note: The following propositions assume that $(X, \mathcal{M}, \mu)$ is a complete measure space (needed for a.e. properties).

Proposition 4.2.1 (Chebychev's inequality)**.** *Let $f : X \to [0, \infty]$ be a measurable function, and $0 < c < \infty$. Then:*

$$\mu(\{f \geq c\}) \leq \frac{1}{c} \int_{\{f \geq c\}} f \, d\mu \leq \frac{1}{c} \int_X f \, d\mu$$

where $\{f \geq c\} = \{x \in X : f(x) \geq c\}$.

Proof.

$$\int_X f \, d\mu \geq \int_{\{f<c\}} f \, d\mu \geq \int_{\{f<c\}} c \, d\mu = c \cdot \mu(\{f < c\})$$

Now we just divide by c. ∎

Note: We have as a consequence the following lemmas:

Lemma 4.2.2 (Vanishing lemma)**.** *Let $f : X \to [0, \infty]$ be a measurable function, $E \in \mathcal{M}$:*

$$\int_E f \, d\mu = 0 \iff f = 0 \text{ a.e. on } E$$

Proof. The proof goes as follows:

($\Leftarrow$) : Trivial

($\Rightarrow$) : We have to show that:

$$\mu(\{x \in E : f(x) > 0\}) = 0$$

Let us define $F = \{x : f(x) > 0\} = \bigcup_n F_n$, where $F_n = \{x : f(x) \geq 1/n\}$. Then, we have that:

$$F_n \subset F_{n+1} \quad \forall n$$

so $F_n \uparrow F$. Then, we have that:

$$\mu(F_n) \to \mu(F)$$

and:

$$0 \leq \mu(F_n) = \mu(\{f \geq \frac{1}{n}\}) \leq \frac{1}{1/n} \int_E f \, d\mu = 0$$

Then, $\mu(F) = 0$.

∎

Remark: The vanishing lemma applies to **every f** once $\mu(E) = 0$, indeed, every property is true a.e. on negligible sets. "The Lebesgue integral does not see negligible sets".

Lemma 4.2.3. *Let $f : X \to [0, \infty]$ be a measurable function. Then:*

$$\int_X f \, d\mu < \infty \implies \mu(\{f = \infty\}) = 0$$

Proof. Exercise. (Hint: $\{f = \infty\} = \bigcap_n \{f \geq n\}$) ∎

Theorem 4.2.4 (Monotone Convergence Theorem (MCT))**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : X \to [0,\infty]$. Assume that:*

(i) $f_n \leq f_{n+1} \quad \forall n$

(ii) $\lim_{n\to\infty} f_n(x) = f(x) \quad$ *for a.e.*$x \in X$

Then, we have that:

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

Remark: All assumptions are essential

Proof. The proof goes as follows:

Part 1:

Assume that assumptions (i) and (ii) hold $\forall x \in X$. We have some basic facts:

- $f(x) = \lim_{n\to\infty} f_n(x) \implies f(x) \geq 0$ and measurable.
- $\int_X f_n \, d\mu \leq \int_X f_{n+1} \, d\mu$. Then, if we define:

$$\alpha_n = \int_X f_n \, d\mu, \quad \alpha = \lim_{n\to\infty} \alpha_n$$

  we have that $\alpha_n \leq \alpha_{n+1}$, so $\alpha_n \uparrow \alpha$. Moreover, we have that:

$$f_n(x) \leq f(x) \implies \int_X f_n \, d\mu \leq \int_X f \, d\mu$$
$$\implies \alpha \leq \int_X f \, d\mu$$

So, to complete part 1, we have to show that $\alpha \geq \int_X f \, d\mu$.

We use the definition of $\int_X f \, d\mu$:

Take any $s : X \to [0,\infty)$ simple, measurable and $0 \leq s \leq f$. Take also $0 \leq c < 1$. Then, we have that:

$$0 < c \cdot s \leq f$$

Take $f_n(x) \uparrow f(x)\ \forall x \in X$. Consider $E_n = \{x \in X : f_n(x) \geq c \cdot s(x)\} \in \mathcal{M}$. Then, we have that:

(a) $E_n \subset E_{n+1}$: indeed, $x \in E_n \iff f_n(x) \geq c \cdot s(x) \implies f_{n+1}(x) \geq c \cdot s(x) \iff x \in E_{n+1}$

(b) $\bigcup_n E_n = X$: indeed, either $f(x) = 0 \implies x \in E_n\ \forall n$ or $f(x) > 0$ and $c \cdot s(x) < f(x)$. Since $f_n(x) \uparrow f(x)$, we have that $\exists N_0$ s.t. $f_{N_0}(x) \geq c \cdot s(x)$. Then $x \in E_{N_0}$.

Then, we have that:

$$\begin{aligned} \alpha \geq \alpha_n = \int_X f_n \, d\mu \geq \int_{E_n} c \cdot s \, d\mu = c \cdot \int_{E_n} s \, d\mu \\ = c \cdot \phi(E_n) \end{aligned}$$

(where $\phi(E) = \int_E s \, d\mu$ is a measure). Then, notice that $E_n \uparrow X$, so $\phi(E_n) \to \phi(X)$.

Then, we have that:

$$\alpha \geq c \cdot \phi(X) = c \cdot \int_X s \, d\mu$$

Then, $\forall c < 1$, $\forall s$:

$$\alpha \geq c \int_X s \, d\mu$$

If we take the limit $c \to 1$, we have that $\alpha \geq \int_X s \, d\mu$. And if we take the supremum over all s, we have that:

$$\alpha \geq \int_X f \, d\mu$$

Part 2:

Now, we have to show that the result holds for *a.e.* $x \in X$. Define

$$F = \{x \in X : \text{either } (i) \text{ or } (ii) \text{ fails}\}$$

Then we have that $\mu(F) = 0$, and $E = X \setminus F$. For any g (non-negative, measurable), we have that:

$$g - \chi_E \cdot g = 0 \quad \text{a.e. on } X$$

Then, we use the vanishing lemma to show that:

$$\int_X (g - \chi_E \cdot g)\, d\mu = 0$$
$$\iff \int_X g\, d\mu = \int_E g\, d\mu$$

Finally:

$$\int_X f\, d\mu = \int_E f\, d\mu = \lim_{n\to\infty} \int_E f_n\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$$

■

Remark: Note that we now have 2 ways to compute the integral of a non-negative measurable function:

- $\int_X f\, d\mu = \sup \left\{ \int_X s\, d\mu : s \text{ simple, measurable and } 0 \le s \le f \right\}$
- $\int_X f\, d\mu = \lim_{n\to\infty} \int_X f_n\, d\mu$ where $f_n \uparrow f$ simple and measurable functions.

Corollary 4.2.4.1 (Monotone convergence for series)**.** *Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions $f_n : X \to [0, \infty]$. Then, we have that:*

$$\int_X \sum_n f_n\, d\mu = \sum_n \int_X f_n\, d\mu$$

Proposition 4.2.5. *Take $\Phi : X \to [0, \infty]$ measurable, $E \in \mathcal{M}$. Define:*

$$\nu(E) = \int_E \Phi\, d\mu$$

Then, ν is a measure on $(X, \mathcal{M})$. Moreover, for $f : X \to [0, \infty]$ measurable:

$$\int_X f\, d\nu = \int_X f \cdot \Phi\, d\mu$$

Proof. The proof goes as follows:

- $\nu : \mathcal{M} \to [0, \infty]$: Trivial
- $\nu(\emptyset) = 0$: Trivial
- σ-additivity: Let $\{E_n\}_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint sets in $\mathcal{M}$, and $E = \bigcup_n E_n$. Then, we have that:

$$\nu(E) = \int_E \Phi \, d\mu = \int_X \Phi \cdot \chi_E \, d\mu = \sum_n \int_X \Phi \cdot \chi_{E_n} \, d\mu$$

$$= \sum_n \int_{E_n} \Phi \, d\mu = \sum_n \nu(E_n)$$

■

Lemma 4.2.6 (Fatou)**.** *Let $(X, \mathcal{M}, \mu)$ be a complete measure space, and $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of measurable functions. Then:*

$$\int_X \liminf_n f_n \, d\mu \leq \liminf_n \int_X f_n \, d\mu$$

Proof. Recall that:

$$\liminf_n f_n = \lim_{n\to\infty} \left(\inf_{k\geq n} f_k \right)$$

$$= \sup_n \left(\inf_{k\geq n} f_k \right)$$

Then, we define:

$$g_n = \inf_{k\geq n} f_k$$

We have the following properties $\forall n$:

- g_n is measurable.
- $g_n \geq 0$
- $g_n \leq g_{n+1}$
- $g_n \leq f_n$

Then, by the MCT, we have that:

$$\int_X \liminf_n f_n \, d\mu = \int_X \lim_n g_n \, d\mu = \lim_n \int_X g_n \, d\mu$$
$$= \liminf_n \int_X g_n \, d\mu \leq \liminf_n \int_X f_n \, d\mu$$

∎

4.3 Integral of real-valued measurable functions

Let $f : X \to \mathbb{R}$ be a measurable function. Then, we can write $f = f^+ - f^-$, where:

$$f^+(x) = \max\{f(x), 0\} \quad f^-(x) = \max\{-f(x), 0\}$$

Notice that $f^+, f^- \geq 0$ are measurable functions. Then, we define:$<$

$$|f| = f^+ + f^-$$

We also notice that $|f| = f^+ + f^- \geq 0$ is measurable.

Definition 4.3.1. We say $f : X \to \mathbb{R}$ is **integrable** on X if it is measurable and:

$$\int_X |f| \, d\mu < \infty$$

We define the set of **integrable functions** as:

$$\mathcal{L}^1(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{R} : f \text{ is integrable}\}$$

For $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, and $E \in \mathcal{M}$, we define:

$$\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$$

Proposition 4.3.1. *Let $f : X \to \mathbb{R}$ be a measurable function. Then:*

(i) $f \in \mathcal{L}^1 \iff |f| \in \mathcal{L}^1 \iff (f^+ \in \mathcal{L}^1$ *and* $f^- \in \mathcal{L}^1)$

(ii) (Triangular inequality):
$$\left| \int_E f \, d\mu \right| \leq \int_E |f| \, d\mu$$

Proof. The proof goes as follows:

(i) Trivial (but see next remark)

(ii) We have that:

$$\begin{aligned} \left|\int_E f\,d\mu\right| &= \left|\int_E f^+\,d\mu - \int_E f^-\,d\mu\right| \\ &\leq \left|\int_E f^+\,d\mu\right| + \left|\int_E f^-\,d\mu\right| = \int_E f^+\,d\mu + \int_E f^-\,d\mu \\ &= \int_E f^+ + f^-\,d\mu = \int_E |f|\,d\mu \end{aligned}$$

■

Remark: In general, it is not true that $|f|$ measurable $\implies$ f measurable. Take $F \subset X$, $F \notin \mathcal{M}$ and:

$$f(x) = \chi_F(x) - \chi_{X\setminus F}(x)$$

Then, $|f| = 1$ is measurable, but f is not.

Proposition 4.3.2. *We propose two properties:*

(i) $\mathcal{L}^1(X, \mathcal{M}, \mu)$ *is a (real) vector space.*

(ii) The functional

$$I(\cdot) := \int_X \cdot\,d\mu : \mathcal{L}^1(X, \mathcal{M}, \mu) \to \mathbb{R}$$

is a linear functional.

Proof. The proof sketch goes as follows:

Let $u, v \in \mathcal{L}^1(X, \mathcal{M}, \mu)$, $\alpha, \beta \in \mathbb{R}$. We should show that:

$$\alpha u + \beta v \in \mathcal{L}^1(X, \mathcal{M}, \mu)$$

since:

$$|\alpha u + \beta v| \leq |\alpha u| + |\beta v|$$

Then:

$$\int_X (\alpha u + \beta v)\, d\mu \leq \int_X |\alpha u + \beta v|\, d\mu \leq \int_X |\alpha u|\, d\mu + \int_X |\beta v|\, d\mu < \infty$$

since $|\alpha u|, |\beta v| \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Then, we have that $\alpha u + \beta v \in \mathcal{L}^1(X, \mathcal{M}, \mu)$.

For the second property, we have that:

$$I(\alpha u + \beta v) = \int_X (\alpha u + \beta v)\, d\mu = \alpha \int_X u\, d\mu + \beta \int_X v\, d\mu = \alpha I(u) + \beta I(v)$$

■

Remark: All the other basic properties of the integral of non-negative functions can be extended to the integral of real-valued functions.

Theorem 4.3.3 (Vanishing lemma)**.** *Let $(X, \mathcal{M}, \mu)$ be a complete measure space, and $f, g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$. Then:*

$$f = g \text{ a.e.} \iff \int_X |f - g|\, d\mu = 0 \iff \int_E (f - g)\, d\mu = 0\ \forall E \in \mathcal{M}$$

Proof. The "difficult" part of the proof is:

$$\int_E (f - g)\, d\mu, \quad \forall E \in \mathcal{M} \implies f = g \text{ a.e.}$$

The proof goes as follows:

Let $E_1 = \{f \geq g\}$, and $E_2 = X \setminus E_1$. Then, we have that:

$$0 = \int_{E_1} (f - g)\, d\mu = \int_{E_1} (f - g)^+\, d\mu$$
$$0 = \int_{E_2} (f - g)\, d\mu = -\int_{E_2} (f - g)^-\, d\mu$$

Then, we have that:

$$(f - g)^+ = 0 \text{ and } (f - g)^- = 0 \text{ a.e. on } X$$

■

Remark: In particular, for $u \in \mathcal{L}^1$:

$$\int_E u \, d\mu = 0 \; \forall E \in \mathcal{M} \implies u = 0 \text{ a.e.}$$

This is the same as:

$$\int_X u\varphi \, d\mu = 0 \quad \forall \varphi \text{ characteristic function } \implies u = 0 \text{ a.e.}$$

This can be true also replacing φ by "something else". For instance, in the case of $u \in \mathcal{L}^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$:

$$\int_{\mathbb{R}} u\varphi \, d\lambda = 0 \quad \forall \varphi \in V \implies u = 0 \text{ a.e.}$$

where $V = \{C_0^\infty(\mathbb{R})\}$, or $V = \{C_0^0(\mathbb{R})\}$.
This is the "fundamental lemma of calculus of variations".

Theorem 4.3.4 (Dominated convergence theorem (DCT))**.** *Let* $(X, \mathcal{M}.\mu)$ *be a complete measure space and* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable functions* $f_n : X \to \mathbb{R}$*, and* $f : X \to \mathbb{R}$*. Assume that:*

(i) $|f_n| \leq g$ *a.e. on* X*,* $\forall n \in \mathbb{N}$*, where* $g \in \mathcal{L}^1(X, \mathcal{M}, \mu)$

(ii) $\lim_{n\to\infty} f_n(x) = f(x)$ *for a.e.* $x \in X$

Then, $f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$*, and:*

$$\lim_{n\to\infty} \int_E |f_n - f| \, d\mu = 0$$

In particular:

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

Proof. First, we have 2 basic facts:

1. $|f_n| \leq g$ a.e. on X, $\forall n \in \mathbb{N} \implies f_n \in \mathcal{L}^1(X, \mathcal{M}, \mu)$
2. $|f| \leq g$ a.e. on $X \implies f \in \mathcal{L}^1(X, \mathcal{M}, \mu)$

Then, consider the sequence $h_n = 2g - |f_n - f|$. We have that:

- h_n is measurable.
- $h_n \leq 2g$

- $h_n \geq 0$. Indeed:

$$|f_n - f| \leq |f_n| + |f| \leq 2g \implies 2g - |f_n - f| \geq 0$$

We now apply the Fatou's lemma to the sequence h_n:

$$\begin{aligned} \int_X (\liminf_n h_n)\, d\mu &\leq \liminf_n \int_X h_n\, d\mu \\ &= \int_X 2g\, d\mu - \limsup_n \int_X |f_n - f|\, d\mu \end{aligned}$$

Also, notice that:

$$\liminf_n h_n = 2g$$

Then, we have that:

$$\begin{aligned} \int_X 2g\, d\mu \leq \int_X 2g\, d\mu - \limsup_n \int_X |f_n - f|\, d\mu \\ \implies \limsup_n \int_X |f_n - f|\, d\mu \leq 0 \end{aligned}$$

Then, we have that:

$$\limsup_n \int_X |f_n - f|\, d\mu \geq \liminf_n \int_X |f_n - f|\, d\mu \geq 0$$

In the end:

$$\lim_n \int_X |f_n - f|\, d\mu = 0$$

■

Remark: If $\mu(X) < \infty$, then the constants are integrable. Then, if $|f_n(x)| \leq M$ a.e, for some $M \in \mathbb{R}$, then:

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X \lim_{n\to\infty} f_n\, d\mu$$

(We are using the DCT with $g = M$)

Corollary 4.3.4.1 (Dominated Convergence for series)**.** *Let* $\{f_n\}_{n\in\mathbb{N}}$ *be a sequence of measurable functions* $f_n : X \to \mathbb{R}$*, s.t* $f_n \in \mathcal{L}^1(X, \mathcal{M}, \mu)$*. If* $\sum_n \int_X |f_n|\, d\mu < \infty$*, then:*

$$\int_X \sum_n f_n \, d\mu = \sum_n \int_X f_n \, d\mu$$

4.4 Comparison between Riemann and Lebesgue integrals

Theorem 4.4.1. *Let* $I = [a, b] \subset \mathbb{R}$ *be a closed interval, and* $f : I \to \mathbb{R}$*. If* f *is* ***Riemann integrable*** *on* I*, then* f *is* ***Lebesgue integrable*** *on* I*, i.e.,* $f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$*, and the two integrals coincide:*

$$\int_I f \, d\lambda = \int_a^b f(x)\, dx$$

Theorem 4.4.2. *Let* $I = (\alpha, \beta)$*, such that* $-\infty \leq \alpha < \beta \leq \infty$*. If* $|f|$ *is* ***Riemann integrable*** *on* I *(in the generalized sense), then* f *is* ***Lebesgue integrable*** *on* I*:*

$$\int_I f \, d\lambda = \int_\alpha^\beta f(x)\, dx$$

Remark: If the generalized Riemann integral of $|f|$ diverges, then:

$$\int_I |f|\, d\lambda = \infty$$

but $\int_I f \, d\lambda$ is not defined (unless $f = \pm|f|$) and:

$$\int_\alpha^\beta f(x)\, dx \text{ and } \int_I f \, d\lambda$$

are not related.

4.5 Spaces of integrable functions

For a $(X, \mathcal{M}, \mu)$ complete measure space, we already know that $\mathcal{L}^1(X, \mathcal{M}, \mu)$ is a vector space. We can also define a distance in this space:

$$d(f,g) = \int_X |f-g| \, d\mu$$

Immediately, we have that:

- **Symmetry**: $d(f,g) = d(g,f)$
- **Triangle inequality**: $d(f,g) \leq d(f,h) + d(h,g)$
- **Non-negativity**: $d(f,g) \geq 0$

But notice that $d(f,g) = 0$ does not imply $f = g$ (only a.e.). This means that $d(f,g)$ is a **pseudo-distance**.

To solve this, we can define an equivalence relation:

$$f \sim g \iff f = g \text{ a.e.}$$

With this equivalence relation, we can define the following space:

Definition 4.5.1. We define the space $L^1(X, \mathcal{M}, \mu)$ as:

$$L^1(X, \mathcal{M}, \mu) = \{[f] : f \in \mathcal{L}^1(X, \mathcal{M}, \mu)\}$$

where $[f]$ is the equivalence class of f defined as:

$$[f] = \{g : g = f \text{ a.e.}\}$$

Remark: We can define the distance in $L^1(X, \mathcal{M}, \mu)$ as:

$$d([f],[g]) = \int_X |f-g| \, d\mu$$

This distance is well-defined, and it is a true distance. Then, $(L^1(X), d)$ is a metric space.

Note: We understand that elements of L^1 are functions: instead of $[u]$, we work with a representant u, and we can **only** use operations/properties that are **independent of the representant**.

E.g.: $X = (0,1)$, we work on $(X, \mathcal{L}(X), \lambda)$. If we take $u \in L^1(X)$, we have the following:

- $u \geq 0$ in X: **NOT** well-defined
- $u \geq 0$ a.e. on X: **GOOD**
- $u(1/2)$: **NOT** well-defined
- $\int_{[0,1/2]} u \, d\lambda$: **GOOD**

Definition 4.5.2. Let $f : X \to \mathbb{R}$ be a measurable function. We say it is **essentially bounded** if:

$$\exists M \in \mathbb{R} : |f(x)| \leq M \text{ a.e. on } X$$

i.e.:

$$\mu(\{x \in X : |f(x)| > M\}) = 0$$

E.g.: Two examples:

$$f(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \in (0,1] \end{cases} \text{ is essentially bounded}$$

$$g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1/x & \text{if } x \in (0,1] \end{cases} \text{ is not essentially bounded}$$

Definition 4.5.3. If $f : X \to \mathbb{R}$ is essentially bounded, we define the **essential supremum** of f as:

$$\operatorname{ess\,sup} f := \inf\{M \in \mathbb{R} : \mu(\{f > M\}) = 0\}$$

Definition 4.5.4. We define the space $\mathcal{L}^\infty(X, \mathcal{M}, \mu)$ as:

$$\mathcal{L}^\infty(X, \mathcal{M}, \mu) = \{f : X \to \mathbb{R} : f \text{ is essentially bounded}\}$$

We can also define the space $L^\infty(X, \mathcal{M}, \mu)$ as:

$$L^\infty(X, \mathcal{M}, \mu) = \{[f] : f \in \mathcal{L}^\infty(X, \mathcal{M}, \mu)\}$$

where $[f]$ is the equivalence class of f defined as:

$$[f] = \{g : g = f \text{ a.e.}\}$$

Remark: One can prove that $L^\infty(X, \mathcal{M}, \mu)$ is a vector space, with the distance:

$$d([f], [g]) = \operatorname{ess\,sup} |f - g|$$

Chapter 5

Types of convergence

We have various types of convergence for sequences of measurable functions:

Definition 5.0.1. Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions, that converges to a function $f : X \to \overline{\mathbb{R}}$. We say that the convergence is a:

- **Pointwise convergence**:
$$f_n(x) \to f(x) \quad \forall x \in X$$
- **Uniform convergence**:
$$\sup_{x \in X} |f_n(x) - f(x)| \to 0$$
- **Convergence a.e.**:
$$f_n(x) \to f(x) \quad \text{a.e. } x \in X$$
- L^1**-convergence**:
$$\int_X |f_n - f| \, d\mu \to 0$$
- L^∞**-convergence**:
$$\operatorname{ess\,sup}_X |f_n - f| \to 0$$
- **Convergence in measure**:
$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \quad \forall \epsilon > 0$$

Remark: Basic relations:

Uniform convergence $\Rightarrow$ Pointwise convergence $\Rightarrow$ Convergence a.e.

Uniform convergence $\Rightarrow$ L^∞-convergence

Exercise: Let $([0,1], \mathcal{L}([0,1]), \lambda)$ be the Lebesgue measure space. Let:

$$f_n(x) = e^{-nx} \quad 0 \le x \le 1$$

$$g(x) = \begin{cases} 1 & x = 0 \\ 0 & x \in (0,1] \end{cases}$$

$$f(x) = 0 \quad 0 \le x \le 1$$

Show that:

- $f_n \to f$ a.e.
- $f_n \nrightarrow f$ pointwise
- $f_n \to g$ pointwise
- $f_n \nrightarrow g$ uniformly

5.1 a.e. convergence and convergence in measure

Theorem 5.1.1. *Let $\mu(X) < \infty$, f_n, f measurable functions, a.e. finite in X. If $f_n \to f$ a.e., then $f_n \to f$ in measure.*

Remark: if $\mu(X) = \infty$, then the theorem may not hold. For instance, consider $X = \mathbb{R}$, with the Lebesgue measure, and:

$$f_n(x) = \chi_{[n,\infty)}(x) = \begin{cases} 1 & x \ge n \\ 0 & x < n \end{cases}$$

We can show that $f_n(x) \to 0$ a.e., but $\lambda(\{f_n \ge 1/2\}) = \infty \ \forall n$ and thus $f_n \nrightarrow 0$ in measure.

Also notice that convergence in measure **does not imply** convergence a.e., even if $\mu(X) < \infty$. For instance, consider the **"typewriter sequence"**.

Theorem 5.1.2. *Let f_n, f be measurable functions, a.e. finite in X. If $f_n \to f$ in measure, then there exists a subsequence f_{n_k} that converges to f a.e.*

5.2 Convergence in L^1 and convergence in measure

Theorem 5.2.1. *Let f_n, f be measurable functions in $L^1(X, \mathcal{M}, \mu)$. If $f_n \to f$ in L^1, then $f_n \to f$ in measure.*

Proof. Assume by contradiction that $f_n \nrightarrow f$ in measure. Then $\exists \alpha > 0$ s.t.:

$$\mu(\{x \in X : |f_n(x) - f(x)| \geq \alpha\}) \nrightarrow 0$$

I.e., $\exists \epsilon > 0$ and a subsequence f_{n_k} s.t.:

$$\mu(\{x \in X : |f_{n_k}(x) - f(x)| \geq \alpha\}) \geq \epsilon \quad \forall k$$

Let us call $E_k = \{x \in X : |f_{n_k}(x) - f(x)| \geq \alpha\}$. On the other hand, by assumption, $f_{n_k} \to f$ in L^1. But notice that:

$$\int_X |f_{n_k} - f| \, d\mu \geq \int_{E_k} |f_{n_k} - f| \, d\mu \geq \alpha \mu(E_k) \geq \alpha\epsilon > 0$$

Since $f_{n_k} \to f$ in L^1, we have that $\int_X |f_{n_k} - f| \, d\mu \to 0$. But we have just shown that $\int_X |f_{n_k} - f| \, d\mu \geq \alpha\epsilon > 0$. This is a contradiction, and thus $f_n \to f$ in measure. ■

Remark: In general, convergence in measure does not imply convergence in L^1. For instance, consider $X = [0, 1]$, $\mathcal{M} = \mathcal{L}([0, 1])$, μ the Lebesgue measure, and $f_n(x) = n\chi_{[0,1/n]}(x)$. We can show that $f_n \to 0$ in measure, but $\int_X |f_n - 0| \, d\mu = 1 \; \forall n$.

5.3 Convergence in L^1 and a.e. convergence

In general, they are not related. But we have 2 main results: **Dominating convergence theorem** that we already saw, and the **"Reverse Dominating Convergence Theorem"**, that states:

Theorem 5.3.1. *Let $f_n \to f$ in $L^1(X, \mathcal{M}, \mu)$, then there exists a subsequence f_{n_k} that converges to f a.e., and there exists a function $g \in L^1(X, \mathcal{M}, \mu)$ s.t. $|f_{n_k}| \leq g$ a.e. $\forall k$.*

Chapter 6

Absolutely continuous functions and Functions of bounded variations

6.1 Fundamental theorems of calculus

Let $(X, \mathcal{L}(X), \lambda)$ be a complete measure space, such that $X = \mathbb{R}$ or $X = I \subset \mathbb{R}$ an interval. Take $f \in L^1(a, b)$. We can define the **integral function**:

$$F(x) = \int_{[} a, x] f \, d\mu = \int_a^x f(t)\, dt$$

If $f \in C([a, b])$, then:

- $F \in C^1([a, b])$
- $F'(x) = f(x)$
- $F(x) - F(y) = \int_y^x f(t)\, dt$

What if only $f \in L^1(a, b)$?

6.1.1 1st Fundamental Theorem of Calculus

Theorem 6.1.1 (1st Fundamental Theorem of Calculus)**.** *Let $f \in L^1(a,b)$. If we define:*

$$F(x) = \int_a^x f(t)\,dt$$

then:

- *F is differentiable at a.e. $x \in [a,b]$*
- *$F'(x) = f(x)$ a.e. $x \in [a,b]$*

E.g.: Take $[a,b] = [-1,1]$ and:

$$f(x) = \mathcal{H}(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

This is the Heaviside function. Notice that $\mathcal{H} \in L^1(-1,1)$. Now:

$$F(x) = \int_{-1}^x \mathcal{H}(t)\,dt = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}$$

Also, if we define:

$$f(x) = \begin{cases} \mathcal{H}(x) & x \notin \mathbb{Q} \\ \infty & x \in \mathbb{Q} \end{cases}$$

we get the same F.

Note: For the proof, we need a deep result due to Lebesgue. We go back to $\mathcal{L}^1([a,b])$.

Definition 6.1.1. Le t $f \in \mathcal{L}^1([a,b])$. we say $x \in [a,b]$ is a **Lebesgue point** for f if:

$$\lim_{h \to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\,dt = 0$$

Note that if $x = a$ then $h \to 0^+$ and if $x = b$, then $h \to 0^-$.

Remark: If x is a LP, then:

$$
\begin{aligned}
0 &= \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt \\
&\geq \lim_{h\to 0} \left| \frac{1}{h} \int_x^{x+h} (f(t) - f(x))\, dt \right| \\
&= \left| \left(\lim_{h\to 0} \int_x^{x+h} f(t)\, dt \right) - f(x) \right|
\end{aligned}
$$

i.e., LP is related with the validity of a local mean value theorem at x

Remark: We have the following:

- f is continuos $\implies$ x is a LP.
- $f \in C([a,b]) \implies$ every $x \in [a,b]$ is a LP.
- Take $\mathcal{H}(x)$, then $x = 0$ is not a LP.

Theorem 6.1.2 (Lebesgue)**.** *Let $f \in \mathcal{L}^1([a,b])$. Then, a.e. $x \in [a,b]$ is a Lebesgue point.*

Remark: By consequence of the theorem, it makes sense to consider Lebesgue points in L^1. Indeed, changing the representative of the function class in L^1 maintains the same set of Lebesgue points up to a negligible set.

Note: To prove the **1st fund. thm.**, we will show that:

- F is differentiable at x.
- $F'(x) = f(x)$

for all x Lebesgue points for f.

Proof: (1st fund. thm.) Take $x \in [a,b]$ a LP of f. Then:

$$
\begin{aligned}
0 &\leq \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| \\
&= \left| \frac{\int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt}{h} - f(x) \right|
\end{aligned}
$$

$$
= \left| \frac{1}{h} \int_x^{x+h} f(t)\,dt - \frac{1}{h} \int_x^{x+h} f(x)\,dt \right|
$$

$$
\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\,dt \to 0 \quad \text{as } h \to 0
$$

because x is a LP.

■

Remark: Let us try to reverse the point of view: take $g : [a, b] \to \mathbb{R}$, and assume that g is differentiable a.e. in $[a, b]$, and that $g' \in L^1([a, b])$. Is g related with $\int_a^x g'(t)\,dt$?. The answer is **NO!**

E.g.: $\mathcal{H} : [-1, 1] \to \mathbb{R}$ and notice that:

$$
\mathcal{H}'(x) 0 \begin{cases} \nexists & x = 0 \\ 0 & x \neq \end{cases}
$$

We have that $\mathcal{H}' = 0$ a.e. in $[-1, 1]$, and $0 \in L^1([-1, 1])$. But:

$$
\mathcal{H}(1) - \mathcal{H}(0) = 1 - 0 = 1 \neq 0 = \int_{-1}^{1} 0\,dt = \int_{-1}^{1} \mathcal{H}'(t)\,dt
$$

Other example with the Cantor-Vitali function:

$$
g(x) = v(x), \quad \text{s.t. } v(0) = 0, v(1) = 1 \quad \text{and constant outside the Cantor set}
$$

Then, v is differentiable and $v'(x) = 0$ a.e., but we can notice that the same thing as before happens.

Definition 6.1.2. Let I be an interval. We say that $f : I \to \mathbb{R}$ is an **absolutely continuous function**, $f \in AC(I)$, if:

$\forall \varepsilon > 0, \exists \delta$ s.t., $\forall n \in \mathbb{N}$, $\forall$ family of n disjoint subintervals of I, i.e., $(a_i, b_i) \subset I$ s.t. $...b_{i-1} \leq a_i < b_i \leq a_{i+1} < ...$ we have that:

$$
\lambda \left(\bigcup_{i=1}^{n} (a_i, b_i) \right) < \delta \implies \sum_{i=1}^{n} |f(b_i) - f(a_i)| \leq \varepsilon
$$

Remark: Recall that f is uniformly continuous (UC) if:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, y \in I$$
$$|x-y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

(The choice of δ is independent of x, y)

Then:

$$UC(I) \supset AC(I)$$

Recall that f is Lipschitz continuous if $\exists L > 0$ s.t.:

$$\forall x, y \in I, |f(x) - f(y)| \leq L|x-y|$$

Then:

$$Lip(I) \subset AC(I)$$

We will see that:

$$Lip(I) \subsetneq AC(I) \subsetneq UC(I)$$

We will also see that, as $g' \in C \iff g \in C^1$, we have that:

$$g' \in L^1 \iff g \in AC$$

6.1.2 2nd Fundamental Theorem of Calculus

Theorem 6.1.3 (2nd Fundamental Theorem of Calculus)**.** *Let $g : [a,b] \to \mathbb{R}$. The following are equivalent:*

(i) $g \in AC([a,b])$

(ii) *g is differentiable a.e. in $[a,b]$, $g' \in L^1([a,b])$ and:*

$$g(x) - g(y) = \int_y^x g'(t)\, dt \quad \forall x, y \in [a,b]$$

Corollary 6.1.3.1. $f \in L^1([a,b]) \implies F(x) = \int_a^x f(t)\, dt \in AC([a,b])$

Note: To prove one implication of the theorem, we will need some few extra results.

Theorem 6.1.4 (Absolute continuity of the integral function)**.** *Let* $f \in L^1([a,b])$*. Then,* $\forall \varepsilon > 0$*,* $\exists \delta > 0$ *s.t.:*

$$\begin{cases} E \in \mathcal{M} \\ \mu(E) < \delta \end{cases} \implies \int_E |f| \, d\mu < \varepsilon$$

Proof. By contradiction: assume that $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists E \in \mathcal{M}$ s.t. $\mu(E) < \delta$ and $\int_E |f| \, d\mu \geq \varepsilon$.

In particular, $\delta = 1/2^n \to 0$, $E_n = E_{\delta_n}$ and:

$$F_n = \bigcup_{k=n}^{\infty} E_n = E_n \cup F_{n+1}, \quad F = \lim_{n\to\infty} F_n$$

Then:

1. $$(F_{n+1} \subset F_n) \implies \{F_n\} \downarrow F$$

2. $$\forall n, \quad \mu(F_n) \leq \sum_{k=n}^{\infty} \mu(E_k) \leq \sum_{k=n}^{\infty} \delta_k = \sum_{k=n} \frac{1}{2^k} = 2^{-n+1}$$

3. $$\nu(F_n) = \int_{F_n} |f| \, d\mu \geq \int_{E_n} |f| \, d\mu \geq \epsilon \quad \forall n$$

   Moreover:

   $$\nu(F_1) = \int_{F_1} |f| \, d\mu \leq \int_X |f| \, d\mu < \infty$$

Use continuity of measures:

$$(1) + (2) \implies \nu(F) = \lim_{n\to\infty} \nu(F_n) = 0$$

$$(1) + (3) \implies \nu(F) = \lim_{n\to\infty} \nu(F_n) \geq \varepsilon > 0$$

Contradiction, since $\nu(F) = 0$. ■

Remark: As a consequence, we have:

$$f \in L^1([a,b]) \implies F(x) = \int_a^x f(t)\, dt \in AC([a,b])$$

Proof. Take $\varepsilon > 0$, and $\delta = \delta(\varepsilon)$ as in the theorem. I know:

$$\begin{cases} \forall E \in \mathcal{L}([a,b]) \\ \lambda(E) < \delta \end{cases} \implies \int_E |f|\, d\lambda < \varepsilon$$

Take $E = \bigcup_{i=1}^n (a_i, b_i)$, s.t (a_i, b_i) disjoint intervals. Then:

$$\sum_{i=1}^n |F(b_i) - F(a_i)| = \sum_{i=1}^n \left| \int_{a_i}^{b_i} f(t)\, dt \right| \leq \sum_{i=1}^n \int_{a_i}^{b_i} |f|\, dt$$
$$= \int_{\bigcup_{i=1}^n (a_i, b_i)} |f|\, dt < \varepsilon$$

∎

E.g. ((AC $\not\Rightarrow$ Lip))**:** Consider $g(x) = \sqrt{x}$ in $[0,1]$. Then:

$$\sqrt{x} = \int_0^x \frac{1}{2\sqrt{t}}\, dt$$

and $g \in AC([0,1])$. But notice that $g \notin Lip([0,1])$.

$$\left| \frac{\sqrt{x} - \sqrt{0}}{x - 0} \right| \not\leq C$$

for any $C > 0$, as $x \to 0$.

E.g. ((UC $\not\Rightarrow$ AC))**:** Consider:

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

It is continuous in $[0,1] \implies f \in UC([0,1])$. But notice that $f \notin AC([0,1])$. Indeed:

$$f'(x) = \sin(1/x) - \frac{1}{x} \cos(1/x)$$

and $1/x \cos(1/x)$ is not integrable in $[0,1]$, i.e., $f' \notin L^1([0,1])$.

6.2 AC functions and weak derivatives

Proposition 6.2.1 (Integration by parts in AC)**.** *Let* $u : [a, b] \to \mathbb{R}$*. Then,* $u \in AC([a, b])$ *if and only if:*

- $u \in C([a, b])$
- u *is differentiable a.e. in* $[a, b]$
- $u' \in L^1([a, b])$
- $$\int_a^b u'\varphi dx = -\int_a^b u\varphi' dx \quad \forall \varphi \in C_0^\infty([a, b])$$

Definition 6.2.1 (Weak derivative)**.** Let $u \in L^1(a, b)$. We say that $u \in W^{1,1}(a, b) \iff \exists w \in L^1(a, b)$ s.t.:

$$\int_a^b u\varphi' dx = -\int_a^b w\varphi dx \quad \forall \varphi \in C_0^\infty(a, b)$$

Such w is called the **weak derivative** of u.

Remark: Both u and $w = u'$ are equivalence classes of functions, i.e., $u \sim v \iff u = v$ a.e. Properties should be independent of the representative.

Remark: If such a w exists, it is unique (up to a.e. equivalence). Indeed, assume that w_1, w_2 are weak derivatives of u. Then:

$$\int_a^b (w_1 - w_2)\varphi dx = 0 \quad \forall \varphi \in C_0^\infty(a, b)$$

$$\implies w_1 - w_2 = 0 \text{ a.e.}$$

Remark: In principle, the pointwise and weak derivatives are different objects, and the notation u' may be misleading. But we know that if $u \in AC([a, b])$ they coincide.

Remark: In principle, the definition of weak derivatives can be extended (measures, distributions). Take:

$$\mathcal{H}(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Then:

$$\begin{aligned} -\int_{-1}^{1} \mathcal{H}(x)\varphi'(x)\,dx &= -\int_{0}^{1} \varphi'(x)\,dx = \varphi(0) - \varphi(1) \\ &= \varphi(0) = \int_{[-1,1]} \varphi(x)\,d\delta_0 \end{aligned}$$

where δ_0 is the Dirac delta function. This suggest that:

$$\mathcal{H}' = \delta_0 \text{ weakly}$$
$$\mathcal{H}' = 0 \text{ pointwise}$$

Theorem 6.2.2. $u \in AC([a,b]) \iff u \in W^{1,1}(a,b)$

Proof. The proof goes as follows:

($\Rightarrow$) Already proved.

($\Leftarrow$) Assume that u' weak derivative of u, $u' \in L^1(a,b)$. Then:

$$z(x) = \int_a^x u'(t)\,dt, \quad z \in AC$$

We can show that $u = z + c$ for some constant c.

∎

Chapter 7

Derivatives of measures

Let $(X, \mathcal{M}, \mu)$ be a complete measure space. We know that, given $\Phi : X \to [0, \infty]$ measurable, the function:

$$\nu_\Phi(E) := \int_E \Phi d\mu = \int_E d\nu_\Phi$$

is a measure on $(X, \mathcal{M})$. Given μ, ν measures on $(X, \mathcal{M})$, is it true that there exists Φ such that

$$\nu(E) = \int_E \Phi d\mu \quad \forall E \in \mathcal{M}$$

We will study this question in this chapter.

Definition 7.0.1. Let μ, ν measures on $(X, \mathcal{M})$. If $\exists \Phi$ s.t

$$\nu(E) = \int_E \Phi d\mu \quad \forall E \in \mathcal{M}$$

then Φ is the **Radon-Nikodym derivative** of ν with respect to μ and we write:

$$\Phi = \frac{d\nu}{d\mu}$$

Definition 7.0.2. Let μ, ν measures on $(X, \mathcal{M})$. Then ν is **absolutely continuous** with respect to μ ("$\nu << \mu$") if:

$$\forall E \in \mathcal{M}, \quad \mu(E) = 0 \Rightarrow \nu(E) = 0$$

Lemma 7.0.1 (Necessary condition). *Let μ, ν measures on $(X, \mathcal{M})$. If ν has a Radon-Nikodym derivative with respect to μ, then ν is absolutely continuous with respect to μ.*

Proof. Assume ν has a Radon-Nikodym derivative with respect to μ. Then:

$$\nu(E) = \int_E \Phi d\mu = 0$$

∎

Exercise: Take $(X, \mathcal{M}) = (\mathbb{R}, \mathcal{L}(\mathbb{R}))$, $\mu = \lambda$ the Lebesgue measure and $\nu = \delta_0$ the Dirac measure at 0. Show that

$$\nexists \frac{d\nu}{d\mu}$$

7.1 The Radon-Nikodym Theorem

Theorem 7.1.1 (Radon-Nikodym Theorem). *Let $(X, \mathcal{M})$ be a measurable space, μ, ν measures and μ is σ-finite. Then:*

$$\nu << \mu \iff \exists \frac{d\nu}{d\mu}$$

Corollary 7.1.1.1. *Let ν be a measure on $(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N))$ and $\mu << \lambda$. Then:*

$$\exists \Phi := \frac{d\nu}{d\mu} : \quad \nu(E) = \int_E \Phi \, d\lambda \quad \forall E \in \mathcal{L}(\mathbb{R}^N)$$

(Indeed, λ is σ-finite)

Chapter 8

Banach spaces

8.1 Normed and Banach spaces

Definition 8.1.1. Let X be a (real) vector space. A **norm** on X is a function $\|\cdot\| : X \to \mathbb{R}$ such that:

(i) $\|x\| > 0$ for all $x \in X$ and $\|x\| = 0 \iff x = 0$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $x \in X$ and $\alpha \in \mathbb{R}$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a **normed space**.

Proposition 8.1.1. *Let $(X, \|\cdot\|)$ be a normed space. Then:*

$$d(x, y) = \|x - y\|$$

is a metric on X, i.e., (X, d) is a metric space.

Proposition 8.1.2. *Let $\{x_n\}_n$ be a sequence in a normed space $(X, \|\cdot\|)$. Then:*

(i) We say $x_n \to x$ if $\|x_n - x\| \to 0$ as $n \to \infty$.

(ii) For $f : X \to Y$, (X, Y normed spaces), we say f is continuous at $x \in X \iff :$

$$\forall \{x_n\}_n : x_n \to x \in X \implies f(x_n) \to f(x) \in Y$$

Exercise: Show that:

(i) $|\|x\| - \|y\|| \leq \|x - y\|$

(ii) $\|\cdot\| : X \to \mathbb{R}$ is continuous in X.

Definition 8.1.2. We say $\{x_n\}_n$ is a **Cauchy sequence** (or **fundamental sequence**) if $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. I.e., :

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} : n, m \geq N \implies \|x_n - x_m\| < \varepsilon$$

Remark: If $\{x_n\}_n$ converges, then it is a Cauchy sequence. The converse is not true in general.

Definition 8.1.3. A normed vector space $(X, \|\cdot\|)$ is called a **Banach space** if it is complete, i.e., every Cauchy sequence in X converges to a point in X.

E.g.: The following are examples of Banach spaces:

(i) $X = \mathbb{R}^n$ with $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $1 \leq p < \infty$., $\|x\|_\infty = \max_i |x_i|$, are Banach spaces.

(ii) $X = C([a, b])$ with $\|u\| = \max_{x \in [a,b]} |u(x)|$ is a Banach space.

(iii) $X = C^k([a, b])$ with $\|u\| = \sum_{i=0}^k \max_{x \in [a,b]} |u^{(i)}(x)|$ is a Banach space.

Remark: Let $(X, \|\cdot\|)$ normed vector space, $\{x_n\}_n \subset X$. We can deal with series:

$$\sum_{n=1}^{\infty} x_n = y \iff s_k = \sum_{n=1}^{k} x_n, \quad s_k \to y \text{ as } k \to \infty$$

For numerical series, $\{a_n\}_n \subset \mathbb{R}$, we have:

$$\sum_{n=1}^{\infty} |a_n| < \infty \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

This is not true in general for series in normed spaces.

Proposition 8.1.3. $(X, \|\cdot\|)$ *is a Banach space* $\iff$ *every absolutely convergent series in* X *converges. I.e., if:*

$$\forall \{x_n\}_n \subset X : \sum_{n=1}^{\infty} \|x_n\| < \infty \implies \sum_{n=1}^{\infty} x_n \text{ converges}$$

8.2 Equivalent/non equivalent norms

Definition 8.2.1. Let X be a vector space, and $\|\cdot\|_a, \|\cdot\|_b$ be two norms on X. We say $\|\cdot\|_a$ and $\|\cdot\|_b$ are **equivalent** if there exist $0 < c_1 \leq c_2 < \infty$ such that:

$$c_1 \|x\|_a \leq \|x\|_b \leq c_2 \|x\|_a \quad \forall x \in X$$

In particular, we say that they induce the same topology on X.

Theorem 8.2.1. *Let* X *be a vector space, such that* $dim X < \infty$. *Then all norms on* X *are equivalent.*

Proof. Notice that it is enough to prove that any norm $\|\cdot\|$ on X is equivalent to the Euclidean norm $\|\cdot\|_2$.

Moreover, it is enough to prove that $\exists c_1, c_2 > 0$ such that:

$$c_1 \leq \|x\| \leq c_2 \quad \forall x \in X, \|x\|_2 = 1$$

Indeed, if we have this, then:

$$y \in \mathbb{R}^N \setminus \{0\} \implies \left\| \frac{y}{\|y\|_2} \right\|_2 = 1$$

Then, we have:

$$c_1 \leq \left\| \frac{y}{\|y\|_2} \right\| \leq c_2 \implies c_1 \|y\|_2 \leq \|y\| \leq c_2 \|y\|_2$$

Which is what we wanted to prove.

To prove this, let $f(x) = \|x\|$. We will show that f is continuous with respect to the Euclidean norm, i.e.:

$$\|x_n - x\|_2 \to 0 \implies f(x_n - x) \to 0 \iff \|x_n - x\| \to 0$$

Indeed, for $y \in X$, and $\{e_1, ..., e_N\}$ basis of X, we have:

$$\begin{aligned}
\|y\| = \left\| \sum_{i=1}^{N} y_i e_i \right\| \leq \sum_{i=1}^{N} \|y_i e_i\| \\
\leq \sum_{i=1}^{N} |y_i| \, \|e_i\| \leq \left(\max_i |y_i| \right) \sum_{i=1}^{N} \|e_i\| \\
\leq C \, \|y\|_\infty \leq C \, \|y\|_2
\end{aligned}$$

Where $C = \sum_{i=1}^{N} \|e_i\|$. Then, we have:

$$0 < \|x_n - x\| \leq C \, \|x_n - x\|_2 \to 0 \implies \|x_n - x\| \to 0$$

Finally, consider:

$$\min_{\|x\|_2 = 1} f(x) \qquad \max_{\|x\|_2 = 1} f(x)$$

Since f is continuous, and $S = \{x \in X : \|x\|_2 = 1\}$ is compact, we have that f attains its minimum and maximum in S. Let $x_m = \arg\min_{\|x\|_2 = 1} f(x)$, and $x_M = \arg\max_{\|x\|_2 = 1} f(x)$. Then, we have:

$$\begin{aligned}
0 < \|x_m\| \leq f(x) \leq \|x_M\| \quad \forall x \in X, \|x\|_2 = 1 \\
\implies 0 < \|x_m\| \leq \|x\| \leq \|x_M\| \quad \forall x \in X, \|x\|_2 = 1
\end{aligned}$$

■

Note: We postpone more general properties of Banach spaces (in paricular, that in infinite dimension, the theorem above is not true), and we anticipate the Lebesgue spaces.

Chapter 9

Lebesgue spaces $L^p(X)$

9.1 Definition of $L^p(X)$

Definition 9.1.1. Let $(X, \mathcal{M}, \mu)$ be a complete measure space, and $p \in [1, \infty]$. We define the following:

1. $\mathcal{L}^p(X, \mathcal{M}, \mu) := \left\{ f : X \to \mathbb{R} \mid f \text{ is } \mathcal{M}\text{-measurable and } \int_X |f|^p \, d\mu < \infty \right\}$.
2. $u, v \in \mathcal{L}^p(X, \mathcal{M}, \mu)$, $u \sim v \iff u = v$ a.e.
3. $[f]_p := \{g \in \mathcal{L}^p(X, \mathcal{M}, \mu) \mid f \sim g\}$.

Finally, we define the L^p-space as follows:

$$L^p(X, \mathcal{M}, \mu) := \mathcal{L}^p(X, \mathcal{M}, \mu)/\sim = \{[f]_p \mid f \in \mathcal{L}^p(X, \mathcal{M}, \mu)\}$$

where $\sim$ is the equivalence relation defined above. We also define the norm as follows:

$$\|f\|_{L^p} = \|f\|_p = \begin{cases} \left(\int_X |f|^p \, d\mu\right)^{1/p} & \text{if } 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x \in X} |f(x)| & \text{if } p = \infty \end{cases}$$

and $d_p(f, g) = \|f - g\|_p$.

E.g.: Notice that if $(X, \mathcal{M}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_\#)$, then:

$$L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \#) = \ell^p$$

For $1 \leq p < \infty$, we have:

$$\ell^p = \left\{ (a_n)_{n\in\mathbb{N}} \mid \sum_{n\in\mathbb{N}} |a_n|^p < \infty \right\}$$

with norm:

$$\|(a_n)\|_p = \left(\sum_{n\in\mathbb{N}} |a_n|^p \right)^{1/p}$$

For $p = \infty$, we have:

$$\ell^\infty = \left\{ (a_n)_{n\in\mathbb{N}} \mid \sup_{n\in\mathbb{N}} |a_n| < \infty \right\}$$

with norm:

$$\|(a_n)\|_\infty = \sup_{n\in\mathbb{N}} |a_n|$$

Note: Our plan is to show that $L^p(X, \mathcal{M}, \mu)$ is a Banach space, i.e.:

1. $L^p(X, \mathcal{M}, \mu)$ is a vector space.
2. $\|\cdot\|_p$ is a norm.
3. $L^p(X, \mathcal{M}, \mu)$ is complete.

9.2 L^p-spaces are vector spaces

Lemma 9.2.1. *Let $p \in [1, \infty)$, $a, b \in \mathbb{R}, a, b \leq 0$. Then:*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p)$$

Proof (exercise). For $a \neq 0$, $t = b/a$, we have to show that:

$$\frac{(1 + t)^p}{1 + t^p} \leq 2^{p-1} \quad \forall t \leq 0$$

∎

Theorem 9.2.2. *Let $p \in [1, \infty)$, then $L^p(X)$ is a vector space*

Proof. Given $u, v \in L^p(X), \alpha \in \mathbb{R}$, we have to show that:

1. $u+v \in L^p(X)$
2. $\alpha u \in L^p(X)$

1. We have:

$$\int_X |u+v|^p \, d\mu \le \int_X (|u|+|v|)^p \, d\mu \le 2^{p-1} \left(\int_X |u|^p \, d\mu + \int_X |v|^p \, d\mu \right) < \infty$$

2. We have:

$$\int_X |\alpha u|^p \, d\mu = \int_X |\alpha|^p |u|^p \, d\mu = |\alpha|^p \int_X |u|^p \, d\mu < \infty$$

■

9.3 $(L^p(X), \|\cdot\|_p)$ are normed spaces

Definition 9.3.1 (Conjugated exponent)**.** For every $1 \le p \le \infty$, the **conjugated exponent** of p, denoted by $q \in [1, \infty]$, satisfies:

$$\frac{1}{p} + \frac{1}{q} = 1$$

Lemma 9.3.1 (Young's inequality)**.** *Let $p, q \in (1, \infty)$ be conjugated exponents. Then, for every $a, b \ge 0$:*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

Proof. Notice that $\ln(x)$ is a concave function. Then:

$$\ln(\frac{a^p}{p} + \frac{b^q}{q}) \ge \frac{1}{p}\ln(a^p) + \frac{1}{q}\ln(b^q)$$

$$= \ln((a^p)^{1/p}) + \ln((b^q)^{1/q}) = \ln(a) + \ln(b) = \ln(ab)$$

■

Note: As a consequence of Young's inequality, we have the following inequality:

Lemma 9.3.2 (Hölder's inequality)**.** *Let $p, q \in [1, \infty]$ be conjugated exponents, $(X, \mathcal{M}, \mu)$ be a complete measure space, and u, v measurable functions. Then:*

$$\|uv\|_1 \leq \|u\|_p \|v\|_q$$

Proof. We will prove it for $p, q \in (1, \infty)$. For $p = 1, q = \infty$, it is left as an exercise.

We separate in cases:

- If $\|u\|_p = 0$, then $u = 0$ a.e., and $uv = 0$ a.e., meaning that

$$\|uv\|_1 = 0$$

  (The same applies if $\|v\|_q = 0$)

- If $\|u\|_p \cdot \|v\|_q = \infty$, then the inequality is trivial.

- For $0 < \|u\|_p , \|v\|_q < \infty$, we apply the Young inequality for:

$$a = \frac{|u(x)|}{\|u\|_p}, \quad b = \frac{|v(x)|}{\|v\|_q}$$

  We have:

$$\frac{|u(x)| \cdot |v(x)|}{\|u\|_p \|v\|_q} = ab \leq \frac{1}{p} \frac{|u(x)|^p}{\|u\|_p^p} + \frac{1}{q} \frac{|v(x)|^q}{\|v\|_q^q}$$

  We integrate to get:

$$\frac{\|uv\|_1}{\|u\|_p \|v\|_q} \leq \frac{1}{p} \frac{\|u\|_p^p}{\|u\|_p^p} + \frac{1}{q} \frac{\|v\|_q^q}{\|v\|_q^q} = 1$$

$$\implies \|uv\|_1 \leq \|u\|_p \|v\|_q$$

■

9.3.1 Inclusion of L^p spaces

Theorem 9.3.3. *Let $\mu(X) < \infty$, $1 \leq p \leq q \leq \infty$. Then:*

$$L^q(X) \subset L^p(X)$$

More precisely, $\exists C > 0$ s.t.:

$$\|u\|_p \leq C \|u\|_q$$

Theorem 9.3.4 (Interpolation)**.** *Let $1 \leq p < q \leq \infty$. Then:*

$$L^r(X) \subset L^p(X) \cap L^q(X), \quad \forall p \leq r \leq q$$

9.3.2 Minkowski's inequality

Theorem 9.3.5 (Minkowski's inequality)**.** *Let $p \in [1, \infty]$, $(X, \mathcal{M}, \mu)$ be a complete measure space, and $u, v \in L^p(X)$. Then:*

$$\|u + v\|_p \leq \|u\|_p + \|v\|_p$$

Proof. We will prove it for $p \in (1, \infty)$. For $p = 1, p = \infty$, it is left as an exercise.

We have:

$$\begin{aligned} \|u + v\|_p^p = \int_X |u + v|^p \, d\mu &= \int_X |u + v||u + v|^{p-1} \, d\mu \\ &\leq \int_X |u||u + v|^{p-1} \, d\mu + \int_X |v||u + v|^{p-1} \, d\mu \end{aligned}$$

For the first term, we have:

$$\begin{aligned} \int_X |u||u + v|^{p-1} \, d\mu &\leq \|u\|_p \left(\int_X |u + v|^{(p-1)q} \, d\mu \right)^{1/q} \\ &\leq \|u\|_p \|u + v\|_p^{p/q} = \|u\|_p \|u + v\|_p^{p-1} \end{aligned}$$

Analogously, for the second term, we have:

$$\int_X |v||u+v|^{p-1}\,d\mu \leq \|v\|_p \|u+v\|_p^{p-1}$$

and finally, we substitute back to get:

$$\|u+v\|_p^p \leq \|u\|_p \|u+v\|_p^{p-1} + \|v\|_p \|u+v\|_p^{p-1}$$

and we divide by $\|u+v\|_p^{p-1}$ to get:

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p$$

∎

Corollary 9.3.5.1. *$(L^p(X), \|\cdot\|_p)$ is a normed space for $p \in [1, \infty]$*

9.4 Completeness of L^p-spaces

Theorem 9.4.1 (Riesz-Fischer). *Let $p \in [1, \infty]$, $(X, \mathcal{M}, \mu)$ be a complete measure space. Then:*

$$L^p(X) \text{ is a Banach space}$$

Proof. The only thing left to show is that $L^p(X)$ is complete. We will use the characterization of Banach spaces in terms of absolutely convergent series.

Let us suppose that $\{f_n\}_n \subset L^p(X)$ is an absolutely convergent series, i.e.:

$$\sum_{n=1}^{\infty} \|f_n\|_p = M < \infty$$

Introduce $g_k(x) = \sum_{n=1}^{k} |f_n(x)|$. We have that, for every $x \in X$, $\{g_k(x)\}_{k \in \mathbb{N}}$ is a non-decreasing sequence. Then:

$$g(x) = \lim_{k \to \infty} g_k(x) = \sum_{n=1}^{\infty} |f_n(x)|$$

is well-defined for every $x \in X$. We have to show that $g \in L^p(X)$.

Notice that:

$$\|g_k\|_p = \left\| \sum_{n=1}^{k} |f_n| \right\|_p \leq \sum_{n=1}^{k} \|f_n\|_p \leq$$

$$\leq \sum_{n=1}^{\infty} \|f_n\|_p = M$$

where M is a constant (since the series is absolutely convergent). Then, $g_k \in L^p(X)$ for every $k \in \mathbb{N}$.

Then, by the monotone convergence theorem, we have:

$$\int_X g^p \, d\mu = \int_X \left(\lim_{k\to\infty} g_k \right)^p \, d\mu = \lim_{k\to\infty} \int_X g_k^p \, d\mu$$

$$= \lim_{k\to\infty} \|g_k\|_p^p \leq M^p < \infty$$

Then, $g \in L^p(X)$, meaning that $g(x) \leq \infty$ a.e., which implies that:

$$\sum_{n=1}^{\infty} |f_n(x)| < \infty \quad \text{a.e.}$$

Since X is complete, we have that $\sum_{n=1}^{\infty} f_n(x)$ converges a.e. Then:

$$s(x) = \sum_{n=1}^{\infty} f_n(x)$$

is well-defined for every $x \in X$. And we proved that $s_k(x) \to s(x)$ for a.e $x \in X$.

To conclude, we apply the dominated convergence theorem:

- $|s_k(x) - s(x)| \to 0$ a.e.
- $$|s_k - s|^p = \left| \sum_{n=k+1}^{\infty} f_n \right|^p \leq \left(\sum_{n=k+1}^{\infty} |f_n| \right)^p$$

  $$\leq (g)^p \in L^1$$

These conditions imply that:

$$\int_X |s_k - s|^p \, d\mu \to 0$$

that is, convergence in L^p. ■

E.g.: We know that the following are Banach spaces:

1. $(\mathbb{R}^N, \text{any norm})$
2. $(C([a,b]), \|\cdot\|_\infty)$
3. $(L^p(X), \|\cdot\|_p)$
4. $(L^\infty, \|\cdot\|_\infty)$

E.g.: Let $X = C([-1,1])$, $\|u\|_1 = \int_{-1}^{1} |u(x)| \, dx$. Then, let u_n:

$$u_n(x) = \begin{cases} 0 & \text{if } x \in [-1, 0) \\ nx & \text{if } x \in [0, 1/n] \\ 1 & \text{if } x \in [1/n, 1] \end{cases}$$

We have that $\{u_n\}_n \subset X$ is a Cauchy sequence with respect to the norm $\|\cdot\|_1$. On the other hand:

$$\|u_n - u_m\|_\infty = \max_{-1 \le x \le 1} |u_n(x) - u_m(x)| = 1 - \frac{n}{m} \nrightarrow 0$$

Moreover, we have that $\{u_n\}_n \subset L^1([-1,1])$, s.t. $u_n \to \mathcal{H}$, which is not in $C([-1,1])$.

Consequences:

1. $\|\cdot\|_1$ is not equivalent to $\|\cdot\|_\infty$ in $C([-1,1])$.
2. $(C([-1,1]), \|\cdot\|_1)$ is not a Banach space.
3. $C([-1,1])$ is a vector subspace of $L^1([-1,1])$, but it is not closed, since the sequence $\{u_n\}_n \subset C([-1,1])$ converges to a function that is not in $C([-1,1])$.

Chapter 10

Compactness, Density and Separability

10.1 Compactness

We say that (X, d) is a metric space.

Definition 10.1.1. $E \subset X$ is **compact** if from any open covering $\{A_i\}_{i\in I}$ (A_i open $\forall i \in I$, $E \subset \bigcup_{i\in I} A_i$) we can extract a finite subcovering.

Typically, we define it as follows:

Take E, fix $r > 0$ and consider $\{B_r(x)\}_{x\in E}$, the open balls of radius r centered at $x \in E$.
Then, E is compact if there exists $x_1, ..., x_k \in E$ s.t.

$$E \subset \bigcup_{i=1}^{k} B_r(x_i)$$

Definition 10.1.2. E is **sequentially compact** if $\forall\{x_n\}_{n\in\mathbb{N}} \subset E$, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ that converges to some $x \in E$.

Remark: The two definitions are equivalent in metric spaces.

Definition 10.1.3. $E \subset X$ is **relatively compact** if $\overline{E}$ is compact.

Theorem 10.1.1 (Heine-Borel)**.** *Let* $(X, \|\cdot\|)$ *be a normed vector space. If* $dim(X) < \infty$*, then* $E \subset X$ *is compact* $\iff$ E *is closed and bounded.*

Remark: The theorem is not true in infinite-dimensional spaces. In particular, if $E \subset X$ is compact, then E is closed and bounded, but the converse is not true.

Theorem 10.1.2 (Riesz)**.** *Let* $(X, \|\cdot\|)$ *be a normed vector space. Then:*

$$\overline{B_1(0)} \text{ is compact} \iff dim(X) < \infty$$

Proof. $(\Leftarrow)$ Exercise.

$(\Rightarrow)$ Suppose $\overline{B_1(0)} = \{x \in X : \|x\| \leq 1\}$ is compact.

Consider $\{B_{1/2}(x)\}_{x \in \overline{B_1(0)}}$. Then:

$$\overline{B_1(0)} \subset \bigcup_{x \in \overline{B_1(0)}} B_{1/2}(x)$$

By compactness, $\exists x_1, ..., x_k \in \overline{B_1(0)}$ s.t.

$$\overline{B_1(0)} \subset \bigcup_{i=1}^{k} B_{1/2}(x_i)$$

$$\subset \bigcup_{i=1}^{k} \overline{B_{1/2}(x_i)}$$

This means that $\forall x \in \overline{B_1(0)}$, $\exists i \in \{1, ..., k\}$, s.t.

$$x = x_i + z \text{ for some } \|z\| \leq 1/2$$

Define $V = span\{x_1, ..., x_k\}$. Then, $V \subset X$ is a vector subspace and $dim V \leq k < \infty$.

We can then rewrite the previous implication as: $\forall x \in \overline{B_1(0)}$, $\exists v \in V$ s.t.

$$x = v + z \text{ for some } \|z\| \leq 1/2$$

Now, take $y \in X$, s.t. $y \neq 0$. Then, notice that:

$$\frac{y}{\|y\|} \in \overline{B_1(0)}$$

So there exists $v \in V$ and $z : \|z\| \leq 1/2$ s.t.

$$\frac{y}{\|y\|} = v + z$$

Then, $y = \|y\| v + \|y\| z$. We rewrite this as:

$$y = v' + z'$$

where $v' = \|y\| v \in V$ and $\|z'\| \leq \|y\| /2$.

Then, take any $x \in X$ and apply the previous result to $y = x$:

$$x = v_1 + z_1, \quad v_1 \in V, \quad \|z_1\| \leq \|x\| /2$$

Then, apply it again to $y = z_1$:

$$x = v_1 + v' + z_2, \quad v_1, v' \in V, \quad \|z_2\| \leq \|z_1\| /2 \leq \|x\| /4$$

Notice that, because V is a vector space, $v_1 + v' \in V$. Then, we rewrite the previous equation as:

$$x = v_2 + z_2, \quad v_2 \in V, \quad \|z_2\| \leq \|x\| /4$$

By induction:

$$x = v_n + z_n, \quad v_n \in V, \quad \|z_n\| \leq \|x\| /2^n$$

Notice that $z_n \to 0$ as $n \to \infty$. Then:

$$v_n = x - z_n \to x \text{ as } n \to \infty$$

Meaning that the sequence $\{v_n\}_n \subset V$ converges to $x \in X$, and because V is a finite-dimentional vector subspace, it is closed, so $x \in V$.

With this, we have shown that $X = V$, and therefore, $dim X \leq k < \infty$.

$\blacksquare$

10.2 Compactness in $C([a, b])$

Note: We always deal with $(C([a, b]), \|\cdot\|_\infty)$, which is Banach

Definition 10.2.1. Let $\{u_n\}_n \subset C([a, b])$ a sequence of continuous functions. Then, we say that it is **uniformly equicontinuous** if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.,

$$|x - y| < \delta \implies |u_n(x) - u_n(y)| < \varepsilon, \quad \forall x, y \in [a, b], \forall n \in \mathbb{N}$$

(The value of δ only depends on ε)

Theorem 10.2.1 (Ascoli-Arzelà). *Take $\{u_n\}_n \subset C([a, b])$. Assume that:*

(i) $\{u_n\}_{n\in\mathbb{N}}$ is uniformly bounded, i.e.:

$$\exists\, 0 < M < \infty, \quad \|u_n\|_\infty \leq M \quad \forall n \in \mathbb{N}$$

(ii) $\{u_n\}_n$ is uniformly equicontinuous.

Then, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ and $u \in C([a, b])$ s.t. $u_{n_k} \to u$ as $k \to \infty$

E.g.: Let $\{u_n\}_n \subset C^1([a, b]) \subset C([a, b])$. Assume that:

1. $\|u_n\| \leq M.\ \forall n$
2. $\|u_n'\|_n \leq L$, $\forall n$

Then, the theorem applies. Indeed: 1) $\implies$ (i) in Ascoli-Arzelà. To check equicontinuity: $\forall x, y \in [a, b], x \neq y$:

$$|u_n(x) - u_n(y)| = |u_n'(\zeta) \cdot (x - y)| \quad \text{(Mean Value Thm.)}$$

$$\begin{aligned}\implies |u_n(x) - u_n(y)| &\leq |u_n'(\zeta)| \cdot |x - y| \\ &\leq \|u_n'\|_\infty \cdot |x - y| \\ &\leq L|x - y|, \quad \forall n \in \mathbb{N}\end{aligned}$$

$$\implies \text{equicontinuity (take } \delta = \frac{\varepsilon}{L})$$

Roughly, the thm. implies that "boundedness in $C^1 \implies$ compactness in C^0 ".

Remark: The same is true for Lipschitz continuos functions with uniformly bounded Lipschitz constant.

Also, there are similar theorems in L^p with:

$$W^{1,p} = \{L^p \text{ functions having } L^p \text{ weak derivatives}\}$$

and "boundedness in $W^{1,p} \implies$ compactness in L^p ".

10.3 Density, separability

Definition 10.3.1. We say that $D \subset X$ is **dense** if $\overline{D} = X$, i.e.:

$$\forall x \in X, \ \exists \{y_n\}_n \subset D : \ y_n \to x \in X$$

Definition 10.3.2. X is **separable** if $\exists D \subset X$, s.t. D is countable and dense

Remark: Typically, one uses dense subsets because "continuous properties, true on D, are also true on X". When D is separable, you have few elements to check the property.

E.g.: $\mathbb{R}, \mathbb{R}^N, \Omega \subset \mathbb{R}^N$ are all separable: $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}$ is countable.

Theorem 10.3.1. *The following spaces are separable:*

- $(C([a,b]), \|\cdot\|_\infty)$
- $(L^p(\mathbb{R}), \|\cdot\|_p)$ *for* $1 \le p < \infty$

and $(L^\infty(\mathbb{R}), \|\cdot\|_\infty)$ *is* ***NOT*** *separable.*

10.3.1 Dense subspaces

For continuous functions, we have the following result:

Theorem 10.3.2 (Stone-Weierstrass)**.** *Polynomials are dense in* $C([a,b])$*, i.e.:*

$$\forall f \in C([a,b]),\ \forall \varepsilon > 0,\ \exists P(x) \text{ polynomial s.t.}$$
$$\|f - P\|_\infty < \varepsilon$$

Note that polynomials with coefficients in $\mathbb{Q}$ *are countable.*

For L^p spaces, we have the following dense subspaces:

- Simple functions
- Continuous (or more regular) functions

Note (Recall)**:** $s : \mathbb{R} \to \mathbb{R}$ is (measurable and) simple if:

$$s = \sum_{i=1}^{k} \alpha_i \mathbb{1}_{A_i}$$

where $\alpha_i \in \mathbb{R}$ and $A_i \in \mathcal{L}(\mathbb{R})$ are disjoint sets, s.t.:

$$\bigcup_{i=1}^{k} A_i = \mathbb{R}$$

We know that s simple $\implies sinL^\infty(\mathbb{R})$. Does s simple $\implies s \in L^p(\mathbb{R})$? For $p \in [1, \infty)$, we have that:

$$s \in L^p(\mathbb{R}) \iff \lambda(\{x : s(x) \neq 0\}) < \infty$$

Definition 10.3.3. We define $\tilde{\rho}(\mathbb{R})$ as the set of simple functions on $\mathbb{R}$, such that $\lambda(\{x : s(x) \neq 0\}) < \infty$:

$$\tilde{\rho}(\mathbb{R}) = \{s : \mathbb{R} \to \mathbb{R} \text{ simple} \mid \lambda(\{x : s(x) \neq 0\}) < \infty\}$$

Theorem 10.3.3. $\tilde{\rho}(\mathbb{R})$ *is dense in* $L^p(\mathbb{R})$ *for* $1 \leq p < \infty$.

Definition 10.3.4. We define the following concepts:

1. $u : \mathbb{R} \to \mathbb{R}$. The **support** of u is defined as:

$$supp(u) = \overline{\{x : u(x) \neq 0\}}$$

2. $C_c(\mathbb{R}) = \{u \in C(\mathbb{R}) : supp(u) \text{ is compact}\}$

3. $C_c^\infty(\mathbb{R}) = \{u \in C_c(\mathbb{R}) : u \text{ is infinitely differentiable}\} = \mathbb{C}_0^\infty(\mathbb{R}) = \mathcal{D}(\mathbb{R})$

Theorem 10.3.4. *$C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.*

Corollary 10.3.4.1. *$C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$.*

($D \subset X$ dense, $D \subset E \subset X \implies E$ dense in X)

Remark: $C_c^(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$. Indeed, take

$$\mathcal{H}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mathcal{H} \in L^\infty(\mathbb{R})$, but now suppose that we have a function $g \in C_c(\mathbb{R})$ s.t.:

$$\|\mathcal{H} - g\|_\infty \leq 1/3$$

Then:

$$|\mathcal{H}(x) - g(x)| \leq 1/3, \quad \text{a.e. } x \in \mathbb{R}$$
$$\implies \mathcal{H}(x) - 1/3 \leq g(x) \leq \mathcal{H}(x) + 1/3$$

This implies that g cannot be continuous in $x = 0$. Conttradiction.

Note: Let us see that $L^\infty(\mathbb{R})$ is not separable.

Lemma 10.3.5. *Take X Banach. Assume that $\{A_i\}_{i\in I}$ is s.t.:*

(a) $\forall i \in I,\ A_i \subset X$ is open and non-empty

(b) $\forall i \neq j \in I,\ A_i \cap A_j = \emptyset$

(c) I is more than countable.
Then, X is not separable.

Proof. By contradiction. Assume that X is separable. Then, $\exists\{x_n\}_{n\in\mathbb{N}} \subset X$ s.t.:

$$X = \overline{\bigcup_{n\in\mathbb{N}} \{x_n\}}$$

Then, $\forall A_i,\ \exists x_{n_i} \in A_i$. This is because A_i is non-empty, then $\exists z_i \in A_i$, and because $\{x_n\}_n$ dense, $\exists\{x_{n_k}\}_{k\in\mathbb{N}}$ s.t. $x_{n_k} \to z_i$ as $k \to \infty$. Notice that $A_i \subset X$ is open, so the sequence $\{x_{n_k}\}_k$ is eventually in A_i.

Since $A_i \cap A_j = \emptyset$, $x_{n_i} \neq x_{n_j}$, i.e., $n_i \neq n_j$.

Then, we have a map $i \to n_i$ that is injective, so I is at most countable. Contradiction.

■

Theorem 10.3.6. *$L^\infty(\mathbb{R})$ is not separable.*

Proof. We use the previous lemma. $\forall \alpha \in \mathbb{R}^+ = (0,\infty)$, we define:

$$g_\alpha(x) = \chi_{[-\alpha,\alpha]}(x) = \begin{cases} 1 & \text{if } |x| \leq \alpha \\ 0 & \text{otherwise} \end{cases}$$

Notice that, if $\alpha_1 \neq \alpha_2$, then $\|g_{\alpha_1} - g_{\alpha_2}\|_\infty = 1$.

$$\implies B_{1/2}(g_{\alpha_1}) \cap B_{1/2}(g_{\alpha_2}) = \emptyset$$

Indeed, $\forall f \in L^\infty(\mathbb{R})$, we have that:

$$1 = \|g_{\alpha_1} - g_{\alpha_2}\|_\infty \leq \|g_{\alpha_1} - f\|_\infty + \|f - g_{\alpha_2}\|_\infty$$
$$\implies \text{ at least one of the norms is greater than } 1/2$$

Then, we have a family of open sets $\{B_{1/2}(g_\alpha)\}_{\alpha\in\mathbb{R}^+}$ that satisfies the conditions of the lemma.

Then, $L^\infty(\mathbb{R})$ is not separable.

$\blacksquare$

Chapter 11

Linear operators

Note: We will work with $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ normed (Banach) spaces.

Definition 11.0.1. We say that $T : X \to Y$ is a **linear operator** if:

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$\forall x, y \in X$ and $\forall \alpha, \beta \in \mathbb{R}$.

(If $Y = \mathbb{R}$, we say that T is a **linear functional**).

<u>**Notation:**</u> For T linear, $T(u) = Tu$.

E.g.: Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$. Then, $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if:

$$T(x) = Ax$$

where $A \in \mathbb{R}^{m \times n}$.

Remark: T is linear $\implies T(0) = 0$.

Definition 11.0.2. We say that $T : X \to Y$ is **bounded** if $\exists M > 0$ such that:

$$\|Tx\|_Y \leq M \|x\|_X \quad \forall x \in X$$

Note (Recall)**:** We have that:

- T is Lipschitz if $\exists L > 0$ such that $\|Tx - Ty\|_Y \leq L \|x - y\|_X$.
- T is continuous in $x \in X$ if $\forall x_n \to x$ in X, we have that $Tx_n \to Tx$ in Y.

Remark: $T : \mathbb{R}^n \to \mathbb{R}^m$ linear $\implies$ T is continuous and bounded. But notice that if X, Y are infinite-dimensional, then the previous statement is not true.

Theorem 11.0.1. *$T : X \to Y$ linear. Then, the following are equivalent:*

1) T is bounded.

2) T is Lipschitz.

3) T is continuous at any $x_0 \in X$

4) T is continuous at 0.

Proof. The proof goes as follows:

(1 $\implies$ 2) We know that T is bounded, i.e.:

$$\|Tx\|_Y \leq M \|x\|_X, \quad \forall x \in X$$

Take $x = u - v$. Then:

$$\|Tu - Tv\|_Y = \|T(u - v)\|_Y \leq M \|x - y\|_X$$

Then, T is Lipschitz with $L = M$.

(2 $\implies$ 3) Let $L > 0$ be the Lipschitz constant for T. Let $x_n \to x_0$ for some $x_0 \in X$. We have:

$$0 \leq \|Tx_n - Tx_0\|_Y \leq L \|x_n - x_0\|_X \to 0$$

(3 $\implies$ 4) Trivial

(4 $\implies$ 1) By contradiction, assume that T is not bounded:

$$\forall n \in N, \, \exists x_n \in X : \, \|Tx\|_Y \geq n \|x_n\|_X$$

Let $z_n = \frac{1}{n} \frac{x_n}{\|x_n\|_X}$. Then $\|z_n\|_X = 1/n \to 0$ as $n \to \infty$. Since T is continuous at 0, then:

$$Tz_n \to T0 = 0$$

But:

$$\begin{aligned}\|Tz_n\|_Y &= \left\| T\left(\frac{1}{n}\frac{x_n}{\|x_n\|}\right)\right\|_Y \\ &= \frac{1}{n\,\|x_n\|_X}\|Tx_n\|_Y \geq 1 \nrightarrow 0\end{aligned}$$

This is a contradiction.

■

Definition 11.0.3. We define the set $\mathcal{L}(X,Y)$ as:

$$\mathcal{L}(X,Y) := \{T : X \to Y \text{ s.t. } T \text{ linear and bounded}\}$$

If $X = Y$, we write $\mathcal{L}(X)$. If $Y = \mathbb{R}$, then we say that $\mathcal{L}(X,\mathbb{R})$ is the **dual** of X, noted as $X' = X^*$.

Remark: $\mathcal{L}(X,Y)$ is a vector space, i.e., $\forall T, L \in \mathcal{L}(X,Y), \alpha, \beta \in \mathbb{R}$:

$$(\alpha T + \beta L) \in \mathcal{L}(X,Y)$$

$((\alpha T + \beta L)(x) := \alpha Tx + \beta Lx)$

Definition 11.0.4. We define a norm on $\mathcal{L}(X,Y)$, called the **operator norm**, as:

$$\|T\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|\leq 1} \|Tx\|_Y$$

Proposition 11.0.2. *For the operator norm, we have the following equivalences:*

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|=1} \|Tx\|_Y = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf\{M > 0 : \|Tx\|_Y \leq M\,\|x\|_X \;\; \forall x \in X\}$$

Proof. We know that:

$$\sup_{\|x\|\leq 1} \|Tx\|_Y \geq \sup_{\|x\|=1} \|Tx\|_Y$$

The other inequality:

$$\forall x \neq 0,\ \|Tx\|_Y = \|x\|_X \cdot \left\| T\left(\frac{x}{\|x\|_X}\right)\right\|_Y$$

Then, if $z = x/\|x\|_X$:

$$\|Tx\|_Y \leq \|Tz\|_Y\,, \quad \text{with } \|z\|_X = 1$$

obtaining the inequality, so:

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|\leq 1} \|Tx\|_Y = \sup_{\|x\|=1} \|Tx\|_Y$$

For the others, we have:

$$\forall x \neq 0, \quad \|Tx\|_Y \leq M\|x\|_X \iff M \geq \frac{\|Tx\|_Y}{\|x\|_X}$$

$$\iff M \geq \|Tz\|_Y\,, \quad \text{with } \|z\|_X = 1$$

So:

$$\sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \inf\{M > 0 : \|Tx\|_Y \leq M\|x\|_X \ \forall x \in X\}$$

And:

$$\inf(M) \geq \sup_{\|x\|=1} \|Tx\|_Y$$

■

Theorem 11.0.3. *If X is a normed space, and Y is a Banach space, then $\mathcal{L}(X,Y)$ is a Banach space.*

Proof. Omitted. ■

Definition 11.0.5. Let $T : X \to Y$ linear. We define the following:

- **Kernel:** $Ker(T) = \{x \in X : Tx = 0\} \subset X$
- **Range:** $R(T) = \{y \in Y : \exists x \in X, Tx = y\} \subset Y$
- T is **injective** if $Ker(T) = \{0\}$
- T is **surjective** if $R(T) = Y$
- T is **bijective** if T is injective and surjective

Also, if T is bijective, we define the **inverse** of T as $T^{-1} : Y \to X$, s.t. $TT^{-1} = I_Y$ and $T^{-1}T = I_X$. Notice that T^{-1} is linear.

Remark: Let $T : X \to Y$ linear. Then, $Ker(T) \subset X$ and $R(T) \subset Y$ are vector subspaces. Also, if $T \in \mathcal{L}(X, Y)$, then $Ker(T)$ is closed in X. The $R(T)$ may or may not be closed in Y.

Definition 11.0.6 (Isomorphism)**.** We say that X, Y are **isomorphic** if $\exists T \in \mathcal{L}(X, Y)$ bijective and $T^{-1} \in \mathcal{L}(Y, X)$.

In this case, we write $X \cong Y$.

Definition 11.0.7. We say that $T \in \mathcal{L}(X, Y)$ is an **isometry** if:

$$\|Tx\|_Y = \|x\|_X\,, \quad \forall x \in X$$

Definition 11.0.8 (Continuous embedding)**.** Let $X \subset Y$ be a vector subspace. We define the "inclusion" operator $J : X \to Y$ as $Jx = x$. Then, if $J \in \mathcal{L}(X, Y)$, i.e., if:

$$\|x\|_Y \leq M \|x\|_X\,, \quad \forall x \in X$$

Then, we say that X is **continuously embedded** in Y, and we write $X \hookrightarrow Y$.

More generally, if X, Y Banach and $T \in \mathcal{L}(X, Y)$, T injective and $T^{-1} \in \mathcal{L}(R(T), X)$, then we say that X is **continuously embedded** in Y. We call T the **embedding operator**.

E.g.: We have already prove that, for $(X, \mathcal{M}, \mu)$ a measure space, $\mu(X) < \infty$, $1 \leq p < q \leq \infty$, then:

$$L^p(X, \mathcal{M}, \mu) \hookrightarrow L^q(X, \mathcal{M}, \mu)$$

11.1 Uniform boundedness (Banach-Steinhaus theorem)

Theorem 11.1.1 (Uniform boundedness (Banach-Steinhaus theorem))**.** *Let X, Y Banach spaces, and $\mathcal{T} \subset \mathcal{L}(X, Y)$ be a set of linear operators. Suppose that $\mathcal{T}$ is pointwise bounded, i.e., $\forall x \in X$, $\exists M_x > 0$ such that:*

$$\|Tx\|_Y \leq M_x, \quad \forall T \in \mathcal{T}$$

Then, $\mathcal{T}$ is uniformly bounded, i.e., $\exists M > 0$ such that:

$$\|T\|_{\mathcal{L}(X,Y)} \leq M, \quad \forall T \in \mathcal{T}$$

Note: The proof is based on Baire's topological lemma.

Lemma 11.1.2 (Baire's topological lemma)**.** *Let X be a complete metric space, $\{C_n\}_{n\in\mathbb{N}}$ s.t. $C_n \subset X$ is closed and:*

$$X = \bigcup_{n\in\mathbb{N}} C_n$$

Then, $\exists n_0 \in \mathbb{N}$ such that C_{n_0} has non-empty interior.

($\exists r > 0, x_0 \in C_{n_0} :\ \overline{B_r(x_0)} \subset C_{n_0}$)

Uniform boundedness. Define, $\forall n \in \mathbb{N}$,

$$C_n = \{x \in X : \|Tx\|_Y \leq n,\ \forall T \in \mathcal{T}\}$$

We want to apply Baire's lemma to $\{C_n\}_{n\in\mathbb{N}}$. We have:

- (C_n is closed): Indeed, take $\{x_k\}_{k\in\mathbb{N}} \subset C_n$ s.t. $x_k \to \bar{x} \in X$. We have to show that $\bar{x} \in C_n$. We know that $\forall T \in \mathcal{T}$:

$$\|Tx_k\|_Y \leq n, \quad \forall k \in \mathbb{N}$$

Since T is continuous, then $Tx_k \to Tx$ as $k \to \infty$. Then:

$$\|Tx\|_Y \leq n, \quad \forall T \in \mathcal{T}$$

So, $\bar{x} \in C_n$.

- $(X = \bigcup_{n \in \mathbb{N}} C_n)$: Indeed, take any $x \in X$. Since $\mathcal{T}$ is pointwise bounded, then $\exists M_x > 0$ such that:

$$\|Tx\|_Y \leq M_x, \quad \forall T \in \mathcal{T}$$

Then, $x \in C_n$ $\forall n \geq M_x$.

Baire implies that: $\exists n_0 \in \mathbb{N}$, $r > 0$ and $x_0 \in X$ such that:

$$\overline{B_r(x_0)} \subset C_{n_0}$$

Then, we have:

$$\|T(x_0 + rz)\|_Y \leq n_0, \quad \forall T \in \mathcal{T}, \ \forall \|z\|_X \leq 1$$

And notice that:

$$r \|Tz\|_Y - \|Tx_0\|_Y \leq \|T(x_0 + rz)\|_Y \leq n_0$$

Then, we have:

$$\|Tz\|_Y \leq \frac{n_0 + \|Tx_0\|_Y}{r}, \quad \forall T \in \mathcal{T}, \ \forall \|z\|_X \leq 1$$

Taking the supremum over $\|z\|_X \leq 1$, we obtain:

$$\|T\|_{\mathcal{L}(X,Y)} \leq \frac{n_0 + \|Tx_0\|_Y}{r} =: M$$

■

Corollary 11.1.2.1. *Let X, Y Banach spaces, and $\{T_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X,Y)$. Assume that $\forall x \in X$, $\{T_n x\}_{n\in\mathbb{N}} \subset Y$ is a converging sequence. We have:*

$$T(x) := \lim_{n\to\infty} T_n x$$

Then, $T \in \mathcal{L}(X,Y)$.

Proof. The proof goes as follows:

- T **is linear**: $\forall n \in \mathbb{N}$, we have:

$$T_n(\alpha x + \beta y) = \alpha T_n x + \beta T_n y$$

Since T_n is continuous:

$$T(\alpha x + \beta y) = \alpha T x + \beta T y$$

- T **is bounded**: Since $\{T_n x\}_{n\in\mathbb{N}}$ converges, then it is bounded. Then, $\exists M_x > 0$ such that:

$$\|T_n x\|_Y \le M_x, \quad \forall n \in \mathbb{N}$$

Then, $\{T_n\}_{n\in\mathbb{N}}$ is pointwise bounded. By the uniform boundedness theorem, we have that $\{T_n\}_{n\in\mathbb{N}}$ is uniformly bounded, i.e., $\exists M > 0$ such that:

$$\|T_n\|_{\mathcal{L}(X,Y)} \le M, \quad \forall n \in \mathbb{N}$$

I.e.:

$$\|T_n z\| \le M \quad \forall n \in \mathbb{N},\ \forall \|z\|_X \le 1$$

Then, we have:

$$\|Tz\|_Y = \lim_{n\to\infty} \|T_n z\|_Y \le M, \quad \forall \|z\|_X \le 1$$

Then, T is bounded.

■

11.2 Open mapping and closed graph theorems

Definition 11.2.1. We say that $T : X \to Y$ is an **open** if:

$$\forall A \subset X \text{ open, } T(A) \subset Y \text{ is open}$$

Remark: Remember that T is continuous if $T^{-1}(V)$ is open $\forall V \subset Y$ open.

E.g.: Let $f : \mathbb{R} \to \mathbb{R}$, s.t. $f(x) = 0$, $\forall x \in \mathbb{R}$. Then, f is continuous but not open.

Theorem 11.2.1 (Open mapping theorem)**.** *Let X, Y Banach spaces. Then:*

$$T \in \mathcal{L}(X, Y) \textit{ surjective} \implies T \textit{ is open}$$

Proof. Omitted, based on the uniform boundedness theorem and Baire. ∎

Corollary 11.2.1.1. *Let X, Y be Banach spaces, $T \in \mathcal{L}(X, Y)$ bijective. Then*

$$T^{-1} \in \mathcal{L}(Y, X)$$

and $X \cong Y$. Also, if T is injective, then:

$$T \textit{ is embedding, i.e., } X \hookrightarrow Y$$

Corollary 11.2.1.2. *Let $(X, \|\cdot\|_a)$ and $(X, \|\cdot\|_b)$ be Banach spaces, and assume that $\exists c_1 > 0$ s.t. $\|x\|_b \leq c_1 \|x\|_a$. Then,*

$$\exists c_2 > 0 \textit{ s.t. } \|x\|_a \leq c_2 \|x\|_b$$

Proof. Apply previous corollary to $J : (X, \|\cdot\|_a) \to (X, \|\cdot\|_b)$ such that $J(x) = x$. ∎

Definition 11.2.2. We say that $T : X \to Y$ is **closed** if the graph of T is closed in $X \times Y$:

$$\begin{cases} x_n \to x \text{ in } X \\ Tx_n \to y \text{ in } Y \end{cases} \implies y = Tx$$

Theorem 11.2.2 (Closed graph)**.** *Let X, Y be Banach spaces, $T : X \to Y$ linear. Then:*

$$T \text{ is closed} \iff T \in \mathcal{L}(X, Y)$$

Proof. Apply previous corollary to $\|x\|_a = \|x\|_X + \|Tx\|_Y$, $\|x\|_b = \|x\|_X$. ■

Chapter 12

Dual and Reflexive spaces

12.1 Dual spaces

Definition 12.1.1. Let X be a normed space. The **dual space** of X, denoted by X^*, is the set of all bounded linear functionals on X, i.e.:

$$X^* = \mathcal{L}(X, \mathbb{R})$$

It is a Banach space, with norm:

$$\|L\|_{X^*} = \|L\|_* = \sup_{\|x\|_X \leq 1} |Lx|$$

E.g.: Let $X = L^p(\Omega, \mathcal{M}, \mu)$, with $1 \leq p \leq \infty$. Then, take the conjugate exponent q. Take $u \in L^q(\Omega, \mathcal{M}, \mu)$. Define $L_u \in (L^p(\Omega))^*$ as:

$$L_u v = \int_\Omega uv \; d\mu \quad \forall v \in L^p(\Omega)$$

Then, show that:

0) L_u is well-defined:
From Hölder's inequality, we have:

$$|L_u v| = \left| \int_\Omega uv \; d\mu \right| \leq \|u\|_q \|v\|_p$$

So $L_u v \in \mathbb{R}$, $\forall v \in L^p(\Omega)$.

1) L_u is linear:

$$L_u(\alpha_1 v_1 + \alpha_2 v_2) = \int_\Omega u(\alpha_1 v_1 + \alpha_2 v_2)\, d\mu =$$

$$= \alpha_1 \int_\Omega u v_1\, d\mu + \alpha_2 \int_\Omega u v_2\, d\mu = \alpha_1 L_u v_1 + \alpha_2 L_u v_2$$

2) L_u is continuous:
By Hölder's inequality, we also have that $\|L_u\|_* \leq \|u\|_q$. Then, L_u is bounded, so it is continuous.

3) Calculate $\|L_u\|_*$:
Assume that $p \in (1, \infty)$. Then:

$$\|L_u\|_* = \sup_{v \neq 0} \frac{|L_u v|}{\|v\|_p} \geq \frac{|L_u \bar{v}|}{\|\bar{v}\|_p} \quad \text{for any } \bar{v} \neq 0$$

Then we choose a $\bar{v}$ is such a way that $u\bar{v} = |u|^q$.

$$\bar{v} = |u|^{\frac{q}{p}} \cdot sign(u)$$

Notice that $u \in L^q \implies \bar{v} \in L^p$. Then:

$$\|L_u\|_* \geq \frac{|L_u v|}{\|v\|_p} = \frac{\int_\Omega |u|^q\, d\mu}{\left(\int_\Omega |u|^q\, d\mu\right)^{\frac{1}{p}}}$$

$$= \frac{\|u\|_q^q}{\|u\|_q} = \|u\|_q^{\frac{q}{p}} = \|u\|_q$$

So $\|L_u\|_* = \|u\|_q$.

Question: Are all elements of $(L^p)^*$ of the form L_u for some $u \in L^q$?

Answer: Yes, for $p \in (1, \infty)$. This is known as the **Riesz representation theorem**, we will see it later in the course.

Remark: The cases $p = 1$ and $p = \infty$ are more delicate. In any case:

$$p = \infty \implies \|L_u\|_{(L^\infty)^*} = \|u\|_1$$

$$p = 1, X \text{ is } \sigma\text{-finite} \implies \|L_u\|_{(L^1)^*} = \|u\|_\infty$$

12.2 Hahn-Banach theorem and consequences

Theorem 12.2.1 (Hahn-Banach continuous extension theorem)**.** *Let X be a normed space, $Y \subset X$ a subspace, and $L \in Y^*$. Then there exists $\tilde{L} \in X^*$ such that:*

$$\tilde{L}y = Ly \quad \forall y \in Y$$

and $\left\|\tilde{L}\right\|_{X^*} = \|L\|_{Y^*}$.

Proof. Omitted, based on the axiom of choice.

∎

12.2.1 Consequences of H-B

Corollary 12.2.1.1. *Let X be a normed space, $x_0 \in X \setminus \{0\}$. Then $\exists L \in X^*$ s.t.*

$$\|L\|_{X^*} = 1, \quad Lx_0 = \|x_0\|_X$$

Proof. Take $Y = span\{x_0\} = \{tx_0 : t \in \mathbb{R}\}$, and $L_0(tx_0) = t\,\|x_0\|_X$. Notice that:

1. L_0 is well-defined, linear, and continuous.
2. $$\|L_0\|_{Y^*} = \sup_{y \in Y, y \neq 0} \frac{|L_0 y|}{\|y\|} = \sup_{t \neq 0} \frac{|L_0(tx_0)|}{\|tx_0\|} = \sup_{t \neq 0} \frac{|t\,\|x_0\|_X\,|}{|t|\,\|x_0\|_X} = 1$$

Then, by H-B, $\exists L \in X^*$ such that $Lx_0 = \|x_0\|_X$ and $\|L\|_{X^*} = 1$. ∎

Corollary 12.2.1.2 (Bounded linear functions separate points)**.** $\forall x, y \in X$, *normed space, we have:*

$$x \neq y \implies \exists L \in X^* : Lx \neq Ly$$

I.e.:

$$Lx = Ly \quad \forall L \in X^* \implies x = y$$

Proof. Take $x \neq y$ and apply previous corollary to $x_0 = x - y \neq 0$. Then, $\exists L \in X^*$ such that $L(x-y) = \|x-y\|_X \neq 0$, i.e., $Lx \neq Ly$. ∎

Corollary 12.2.1.3 (Bounded linear functionals separate closed subspaces and points)**.** *Let X be a normed space, $Y \subsetneq X$ a closed subspace, $x_0 \in X \setminus Y$. Then, $\exists L \in X^*$ such that:*

$$Ly = 0, \quad \forall y \in Y \quad \textit{and} \quad Lx_0 \neq 0$$

Proof. Take

$$Z = span\{x_0, Y\} = span\{x_0\} \oplus Y = \{z \in X : z = tx_0 + y, t \in \mathbb{R}, y \in Y\}$$

Since $x_0 \notin Y$, for every $z \in Z$, t, y are uniquely defined:

$$t_1 x_0 + y_1 = t_2 x_0 + y_2$$
$$\implies (t_1 - t_2)x_0 = y_1 - y_2$$

and because $y_1 - y_2 \in Y$, but $x_0 \notin Y$, then $t_1 = t_2$ and $y_1 - y_2 = 0 \implies y_1 = y_2$.

Let us define $L_0 : Z \to \mathbb{R}$ as $L_0(tx_0 + y) = t$. We have that $L_0 \in Z^*$, and:

$$L_0 x_0 = L(1 \cdot x_0 + 0) = 1, \quad \text{and} \quad L_0 y = L_0(0 \cdot x_0 + y) = 0$$

And we finally extend it to $L = \tilde{L_0}$ using H-B.

■

12.3 Reflexive spaces

Note: We have X Banach, and $X^* = \mathcal{L}(X, \mathbb{R})$ Banach too. For notation, we will use the following: $L \in X^*, x \in X$:

$$Lx = L(x) = \langle L, x \rangle$$

And notice thar $\langle \cdot, \cdot \rangle$ is a **bilinear form**.

Definition 12.3.1. The bidual of X is the dual of X^*, denoted by:

$$X^{**} = (X^*)^* = \mathcal{L}(X^*, \mathbb{R})$$

Definition 12.3.2. Given $x \in X$ we can construct $\Lambda_x \in X^{**}$ as:

$$\Lambda_x L = Lx \quad \forall L \in X^*$$

Using the notation $\langle \cdot, \cdot \rangle$, we have:

$$\langle \Lambda_x, L \rangle = \langle L, x \rangle$$

The mapping $\tau : X \to X^{**}$ defined by $\tau(x) = \Lambda_x$ is called the **canonical map**.

Proposition 12.3.1. *$\forall x \in X$, $\Lambda_x \in X^{**}$, and the canonical map $\tau : X \to X^{**}$ is an isometry. In other words:*

$$\|\tau(x)\|_{X^{**}} = \|x\|_X \quad \forall x \in X$$

Proof. The proof goes as follows:

- Λ_x is linear: Indeed:

$$\langle \Lambda_x, L \rangle = \langle L, x \rangle$$

  so it follows the linearity of $\langle \cdot, x \rangle$.

- Λ_x is bounded: It is implied by "isometry". Se below

Isometry: We have that:

$$\|\tau(x)\|_{X^{**}} = \sup_{L \neq 0} \frac{|\langle \Lambda_x, L \rangle|}{\|L\|_{X^*}} = \sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}}$$

Upper bounded:

$$\sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}} \leq \sup_{L \neq 0} \frac{\|L\|_* \cdot \|x\|}{\|L\|_*} = \|x\|$$

Lower bounded:

$$\sup_{L \neq 0} \frac{|Lx|}{\|L\|_{X^*}} \geq \frac{|\bar{L}x|}{\|\bar{L}\|_{X^*}} \quad \forall \bar{L} \neq 0$$

By H-B, $\exists \tilde{L}$, s.t. $\tilde{L}x = \|x\|$ and $\left\|\tilde{L}\right\|_{X^*} = 1$. Then, if $\bar{L} = \tilde{L}$, we have:

$$\sup_{L\neq 0} \frac{|Lx|}{\|L\|_{X^*}} \geq \frac{|\tilde{L}x|}{\left\|\tilde{L}\right\|_{X^*}} = \|x\|$$

■

Theorem 12.3.2. *Let $\tau : X \to X^{**}$ be the canonical map. Then:*

- *it is linear and continuous.*
- *it is an isometry.*
- *it is injective.*
- *$R(\tau) \subset X^{**}$ is closed.*
- *it is a continuous embedding*

Remark: This means that we can think that X is a closed subspace of X^{**}, i.e., $X \cong \tau(X)$, and $\tau(X) \subset X^{**}$ is a closed subspace.

Notice that τ may be surjective, in which case $X \cong X^{**}$.

Definition 12.3.3. We say that X is **reflexive** if τ is surjective.

Note: To prove the previous theorem, we will use the following lemma:

Lemma 12.3.3 (Nice properties of linear isometries)**.** *Take X, Y Banach, $T : X \to Y$ linear such that:*

$$\|Tx\|_Y = \|x\|_X \quad \forall x \in X$$

Then:

(i) *T is continuous.*

(ii) *T is injective.*

(iii) *$R(T) \subset Y$ is closed.*

(iv) *$T : X \to R(T)$ is an isomorphism.*

Proof. The proof goes as follows:

(i) T linear $\implies$ T continuous $\iff$ T bounded. Then, notice that:

$$\|Tx\|_Y \leq M \|x\|_X \quad \forall x \in X$$

is true for $M = 1$. Then, T is continuous.

(ii) Let $x, y \in X$ such that $Tx = Ty$. Then:

$$T(x-y) = 0 \implies \|x-y\|_X = 0 \implies x = y$$

(iii) To show that $R(T)$ is closed, take $\{y_n\}_n \subset R(T)$ such that $y_n \to y \in Y$. We want to show that $y \in R(T)$.

Take $\{x_n\}_n$ such that $Tx_n = y_n$. Notice that since $\{y_n\}_n$ is Cauchy, and $\|y_n - y_m\| = \|T(x_n - x_m)\| = \|x_n - x_m\|$, then $\{x_n\}_n$ is Cauchy too. Then, $\exists x \in X$ such that $x_n \to x$ because X is Banach. Now, since T is continuous:

$$Tx = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} y_n = y$$

Then, $y \in R(T)$.

(iv) Since $T \in \mathcal{L}(X, R(T))$ and $R(T)$ is closed (Banach), then T is bijective between X and $R(T)$. by a corollary of the open mappping theorem, T^{-1} is continuous.

■

Proof (theorem for τ). It is enough to check that:

- τ is linear (direct from linearity of $\langle \cdot, \cdot \rangle$)
- τ is an isometry (already proved)

■

12.3.1 Properties of reflexive spaces

Theorem 12.3.4. *Let X be Banach and reflexive. Let $Y \subset X$ closed subspace. Then, Y is reflexive too.*

Theorem 12.3.5. *Let X be Banach. Then:*

$$X \textit{ reflexive} \iff X \textit{ reflexive}$$

Theorem 12.3.6. *Let X be Banach. Then we have:*

$$X^* \text{ separable} \implies X \text{ separable}$$

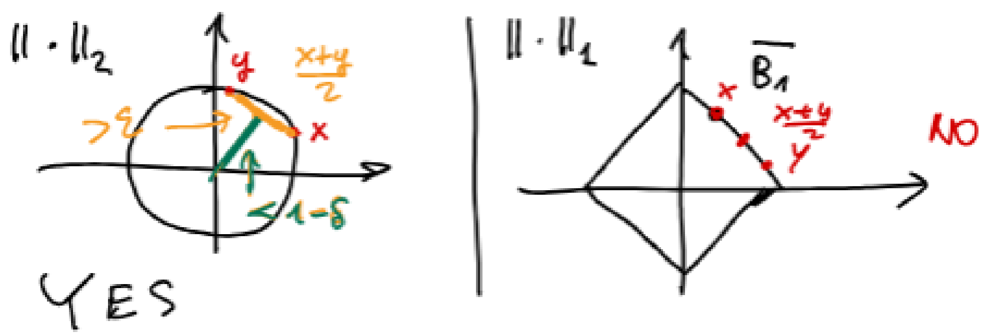
$$X \text{ reflexive and separable} \implies X^* \text{ reflexive and separable}$$

Note: To check reflexivity, it is convenient to introduce the notion of **uniformly convex space**

Definition 12.3.4. We say that X Banach is **uniformly convex** if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.:

$$\begin{cases} x, y \in X \\ \|x\| \leq 1, \|y\| \leq 1 \\ \|x - y\| > \varepsilon \end{cases} \implies \left\| \frac{x+y}{2} < 1 - \delta \right\|$$

Note: This is a "quantitative version" of the strict convexity of $\overline{B_1(0)}$. In $\mathbb{R}^2$, is $\overline{B_1(0)}$ strictly convex?



One can see that, in $(\mathbb{R}^N, \|\cdot\|_p)$, the property is true if and only if $p \in (1, \infty)$, and fails for $p = 1, \infty$.

Theorem 12.3.7 (Millman-Pettis)**.** *X Banach and uniformly convex $\implies$ X reflexive.*

Proof. Omitted. (Very difficult) ∎

Corollary 12.3.7.1. *$L^p(X)$ is reflexive for $p \in (1, \infty)$*

Remark: $L^1(X)$ and $L^\infty(X)$ are **not** reflexive.

12.4 Dual space of L^p

Theorem 12.4.1 (Riesz representation theorem (for $(L^p)^*$))**.** *Let $(X, \mathcal{M}, \mu)$ be a complete measure space, $p \in (1, \infty)$, q the conjugate exponent. Then:*
$\forall L \in (L^p(X))^*, \exists! u \in L^q(X)$ s.t.:

$$Lv = \int_X uv \, d\mu, \quad \forall v \in L^p(X)$$

Moreover, $\|L\|_{(L^p)^*} = \|u\|_{L^q}$.

Remark: We have already seen that $\forall u \in L^q(X)$, L_u defined as:

$$L_u v = \int_X uv \, d\mu$$

is an element of $(L^p)^*$ and $\|L_u\|_{(L^p)^*} = \|u\|_{L^q}$.

Moreover, for $T : L^q \to (L^p)^*$ s.t. $T(u) = L_u$, we obtain that T is an isometric isomorphism.

Proof. By the "properties of isometries" and the example of last time, the only thing left to prove is that T is surjective. This follows by H-B.

■

Theorem 12.4.2. *For $p = 1$, X σ-finite, we have that the $T : L^\infty \to (L^1)^*$ defined as:*

$$T(u)v = \int_X uv \, d\mu \forall v \in L^1$$

is an isometric isomorphism, i.e., $(L^1)^* \cong L^\infty$.

Theorem 12.4.3. *For $p = \infty$, we have that $L^1 \hookrightarrow (L^\infty)^*$, but the embedding is not surjective.*

E.g.: We have that $\forall u \in L^1$, $L_u \in (L^\infty)^*$, but $(L^\infty)^*$ contains elements that are not of the form L_u.

Take $L^\infty([-1,1])$ and $C([-1,1]) \subset L^\infty([-1,1])$ subspace. Then, take $L_0 : C([-1,1]) \to \mathbb{R}$ defined as:

$$L_0 f = f(0)$$

Then, L_0 is linear and bounded, so $L_0 \in (C([-1,1]))^*$.

By H-B, $\exists \tilde{L_0} \in (L^\infty([-1,1]))^*$ such that

$$\tilde{L_0} f = L_0 f \quad \forall f \in C([-1,1]), \qquad \left\| \tilde{L_0} \right\|_{(L^\infty)^*} = \|L_0\|_{(C)^*}$$

Claim: $\nexists u \in L^1([-1,1])$ s.t. $\tilde{L_0} = L_u$.

To show it, by contradiction, assume that $\exists u \in L^1$ s.t:

$$\int_{-1}^{1} uw \, d\mu = \tilde{L_0} w \quad w \in L^\infty$$

Take w_n s.t.:

$$w_n = \begin{cases} 1 - n|x| & |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

We can show that:

- $w_n(x) \to 0$ a.e.
- $|w_n(x)| \leq 1$ a.e.
- $(w_n u)(x) \to 0$ a.e.
- $|w_n(x) u(x)| \leq |u(x)| \in L^1$

Then, by DCT:

$$\int_{-1}^{1} u w_n \, d\lambda \to 0 \qquad \text{as } n \to \infty$$

But:

$$\tilde{L_0} w_n = L_0 w_n = w_n(0) = 1$$

which is a contradiction.

Note (Resuming)**:** For $L^p(\Omega, \mathcal{L}(\Omega), \lambda)$, $\Omega \in \mathcal{L}(\mathbb{R}^N)$ (but also ℓ^p):

Space	**Completeness**	**Separability**	**Reflexivity**	**Dual**
$L^p, p \in (1, \infty)$	Yes	Yes	Yes	$L^q, \frac{1}{p} + \frac{1}{q} = 1$
L^1	Yes	Yes	No	L^∞
L^∞	Yes	No	No	$\supsetneq L^1$

Chapter 13

Weak convergence

Definition 13.0.1. Let X be Banach, $\{x_n\}_{n\in\mathbb{N}} \subset X$, $x \in X$. We say that x_n **weakly converges** to x, denoted:

$$x_n \rightharpoonup x$$

if $\forall L \in X^* : Lx_n \to Lx$

Remark: Suppose that $x_n \to x$ strongly, and $f : X \to Y$ continuous. Then $f(x_n) \to f(x)$. Since $L \in X^*$, then L is continuous, meaning that $Lx_n \to Lx$. In other words:

$$\{x_n\}_n \text{ converges strongly} \implies \{x_n\}_n \text{ converges weakly}$$

Actually, in $\mathbb{R}^N$ strong convergence iff weak convergence.

Remark: For $p \in [1, \infty)$, by using the Riesz rep. thm., we have that:

$$u_n \rightharpoonup u \iff \int_X wu_n \, d\mu \to \int_X wu \, d\mu \quad w \in L^q(X)$$

Proposition 13.0.1. *$u_n \rightharpoonup u$ and $u_n \to v$ a.e., then $u = v$ a.e.*

13.1 Basic properties

Proposition 13.1.1. *If it exists, the weak limit is unique.*

Proof. Let $\{x_n\}_n \subset X$, and suppose that $x_n \rightharpoonup y$ and $x_n \rightharpoonup z$. Then, $\forall L \in X^*$, $Lx_n \to Ly$

and $Lx_n \to Lz$. Then:

$$Ly = Lz \quad \forall L \in X^* \implies y = z$$

by a corollary of H-B.

■

Proposition 13.1.2. *If $x_n \rightharpoonup x$ in X, then $\{x_n\}_n$ is bounded.*

Proof. Use Banach-Steinhaus in X^*. Let us propose the sequence of operators given by $\{\tau(x_n)\}_{n\in\mathbb{N}} \subset X^{**}$.

Notice that $x_n \rightharpoonup x \implies Lx_n \to Lx \quad \forall L \in X^*$.

Then, $\langle \tau(x_n), L\rangle = Lx_n \to Lx = \langle \tau(x), L\rangle \quad \forall L \in X^*$.

This means that $\{\tau(x_n)\}_n$ converges pointwise to $\tau(x)$, i.e.:

$$\langle \tau(x_n), L\rangle \to \langle \tau(x), L\rangle \quad \forall L \in X^*$$

By Banach-Steinhaus, $\{\tau(x_n)\}_n$ is bounded in X^{**}, meaning that:

$$\exists M > 0 : \text{ s.t. } \|\tau(x_n)\|_{X^{**}} \leq M \quad \forall n \in \mathbb{N}$$

Since $\|\tau(x_n)\|_{X^{**}} = \|x_n\|_X$, we have that:

$$\|x_n\|_X \leq M \quad \forall n \in \mathbb{N}$$

which means that $\{x_n\}_n$ is bounded in X.

■

Proposition 13.1.3. *If $x_n \rightharpoonup x$ in X, then:*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|$$

Proof. By a corollary of H-B, if $x \neq 0$, then $\exists L \in X^*$ s.t.:

$$Lx = \|x\|, \quad \|L\| = 1$$

Then:

$$\begin{aligned}\|x\| = Lx = \lim_{n\to\infty} Lx_n = \liminf_{n\to\infty} Lx_n \le \liminf_{n\to\infty} \|L\|_* \|x_n\| \\ = \liminf_{n\to\infty} \|x_n\|\end{aligned}$$

■

Remark: Notice that $\|\cdot\| : X \to \mathbb{R}$ is strongly continuous, but not weakly continuous. It is "weakly lower semicontinuous".

Proposition 13.1.4. *Let $x_n \rightharpoonup x$ in X, and $L_n \to L$ strongly in X^*. Then:*

$$L_n x_n \to Lx$$

Tha same if $L_n \rightharpoonup L$ in X^* and $x_n \to x$ strongly.

If both sequences converge weakly, nothing can be inferred.

Proof. Let $x_n \rightharpoonup x$ in X, and $L_n \to L$ in X^*. Then:

$$0 \le |L_n x_n - Lx| = |L_n x_n - Lx_n + Lx_n - Lx| \le |L_n x_n - Lx_n| + |Lx_n - Lx|$$

Notice that $|Lx_n - Lx| \to 0$ by the strong convergence of x_n. Also, we have that:

$$|L_n x_n - Lx_n| \le \|L_n - L\|_* \|x_n\| \to 0$$

This means that $L_n x_n \to Lx$. ■

Proposition 13.1.5. *Let X be Banach, $V \subset X*$ dense, $\{x_n\}_n \subset X$ bounded. Then:*

$$Lx_n \to Lx \quad \forall L \in V \implies Lx_n \to Lx \quad \forall L \in X^*$$

i.e., $x_n \rightharpoonup x$.

Proof. Omitted (as the one in prop. 4, and use the density of V) ■

E.g.: Recall that $1 < p < \infty$, then $u_n \rightharpoonup u$ in $L^p(\Omega)$ if:

$$\int_\Omega wu_n \, d\mu \to \int_\Omega wu \, d\mu \quad \forall w \in L^q(\Omega)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

By property 5, it is enough to ask: $\{u_n\}_n \subset L^p(\Omega)$ bounded and:

$$\int_\Omega wu_n \, d\mu \to \int_\Omega wu \, d\mu \quad \forall w \in C_c(\Omega)$$

or $\forall w$ simple functions.

Proposition 13.1.6. *Let X, Y Banach, $T \in \mathcal{L}(X,Y)$ and $\{x_n\}_n \subset X$. Then:*

$$x_n \rightharpoonup x \implies Tx_n \rightharpoonup Tx$$

We say that T is "weakly-weakly continuous".

Definition 13.1.1. Let X be Banach, X^* (Banach) dual of X, $\{L_n\}_n \subset X^*$ and $L \in X^*$. We say that L_n **weakly-* converges** to L, denoted:

$$L_n \overset{*}{\rightharpoonup} L$$

if $\forall x \in X$, $L_n x \to Lx$ as $n \to \infty$.

Remark: Note that:

- $L_n \rightharpoonup L$ if $\phi L_n \to \phi L \quad \forall \phi \in X^{**}$.
- $L_n \overset{*}{\rightharpoonup} L$ if $\tau(x) L_n \to \tau(x) L \quad \forall x \in X$.

Proposition 13.1.7. *If X is reflexive, then:*

$$L_n \overset{*}{\rightharpoonup} L \text{ in } X^* \iff L_n \rightharpoonup L \text{ in } X^*$$

E.g.: Weak-* convergence in $L^\infty(\Omega)$ $(\Omega \in \mathcal{L}(\mathbb{R}^N))$:

We know that $L^1(\Omega)$ is Banach and $L^\infty(\Omega) \cong (L^1(\Omega))^*$.
Then, for $\{u_n\}_n \subset L^\infty(\Omega)$, $u \in L^\infty(\Omega)$, we have that $u_n \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$ if:

$$\int_\Omega u_n v \, d\mu \to \int_\Omega uv \, d\mu \quad \forall v \in L^1(\Omega)$$

Remark: In general, weak convergence implies weak-* convergence, but the converse is not true.

Properties 13.1.1 (Weak-* convergence)**.** For weak-* convergence, we have that:

1. If $L_n \overset{*}{\rightharpoonup} L$, then the limit is unique.
2. If $L_n \overset{*}{\rightharpoonup} L$, then $\{L_n\}_n$ is bounded in X^*.
3. If $_n \overset{*}{\rightharpoonup} L$, then:
$$\|L\|_* \leq \liminf_{n\to\infty} \|L_n\|_*$$
4. If $L_n \overset{*}{\rightharpoonup} L$ and $x_n \to x$ strongly, then:
$$L_n x_n \to Lx$$

Remark: The notions of (topological) dual, weak convergence, weak-* convergence, do not need norms, just a topology. E.g., "test functions" $\mathcal{D}(\mathbb{R}^N) = C_c^\infty(\mathbb{R}^N)$, have a topological dual $\mathcal{D}'(\mathbb{R}^N)$, and convergence in $\mathcal{D}'$ is the weak-* convergence.

Remark: We defined weak (weak-*) convergence, not the weak (weak-*) topology. This topology in general is not metrizable and weakly compact sets are not weakly sequentially compact.

13.2 Banach-Alaoglu theorem

Theorem 13.2.1 (Banach-Alaoglu (variant 1))**.** *Let X be Banach and reflexive. Then, every bounded sequence $\{x_n\}_n \subset X$ admits a subsequence $\{x_{n_k}\}_k$ which weakly converges in X.*

Theorem 13.2.2 (Banach-Alaoglu (variant 2)). *Let X be Banach and separable. Then, every bounded sequence $\{L_n\}_n \subset X^*$ admits a subsequence $\{L_{n_k}\}_k$ which weakly-* converges in X^*.*

E.g.: Let $1 < p < \infty$, then we know that $L^p(\Omega)$ is reflexive. Moreover, we know that $f_n \rightharpoonup f$ in $L^p \iff$:

$$\int_\Omega f_n g \, d\mu \to \int_\Omega fg \, d\mu \quad \forall g \in L^q(\Omega)$$

where $\frac{1}{p} + \frac{1}{q} = 1$. If we apply variant 1 of Banach-Alaoglu, we have that:
$\forall \{u_n\}_n \subset L^p(\Omega)$, s.t., $\|u_n\|_p \leq M \ \forall n \in \mathbb{N}$, $\exists \{u_{n_k}\}_k, u \in L^p(\Omega)$ s.t. $u_{n_k} \rightharpoonup u$ in $L^p(\Omega)$, i.e.:

$$\int_\Omega u_{n_k} g \, d\mu \to \int_\Omega ug \, d\mu \quad \forall g \in L^q(\Omega)$$

Also, we know that $L^1(\Omega)$ is separable, and $(L^1(\Omega))^* \cong L^\infty(\Omega)$. Then, if we apply variant 2 of Banach-Alaoglu, we have that:
$\forall \{u_n\}_n \subset L^\infty(\Omega)$, s.t., $\|u_n\|_\infty \leq M \ \forall n \in \mathbb{N}$, $\exists \{u_{n_k}\}_k, u \in L^\infty(\Omega)$ s.t. $u_{n_k} \overset{*}{\rightharpoonup} u$ in $L^\infty(\Omega)$, i.e.:

$$\int_\Omega u_{n_k} g \, d\mu \to \int_\Omega ug \, d\mu \quad \forall g \in L^1(\Omega)$$

Finally, bounded sequences on L^1 have no reason to converge.

Chapter 14

Compact operators

Note: We will work with X, Y Banach spaces.

Definition 14.0.1. Let $K : X \to Y$ be a linear operator. We say that K is **compact** if:

$$\forall E \subset X \text{ bounded }, K(E) \text{ is relatively compact, i.e., } \overline{K(E)} \text{ is compact}$$

Or, equivalently:

$$\forall \{x_n\}_n \subset X \text{ bounded, } \exists \{K(x_{n_k})\}_k \subset Y \text{ (strongly) convergent subsequence}$$

Proposition 14.0.1. *Let $K : X \to Y$ be a linear compact operator. Then K is bounded, i.e., $K \in \mathcal{L}(X, Y)$.*

Proof. We know that $B_1(0) \subset X$ is bounded. Then, $\overline{K(B_1(0))}$ is compact in Y. Therefore, $\overline{K(B_1(0))}$ is bounded in Y.

Then, $\exists M > 0$ such that $\|K(x)\| \leq M \quad \forall x \in B_1(0)$. ■

Remark: The above property is not true for non-linear compact operators.

Exercise (Compactness of the integral map)**:** Let $K : C([0,1]) \to C([0,1])$ be the integral map:

$$K(f)(x) = \int_0^x f(t)\, dt \quad \forall x \in [0,1]$$

and note that it is linear. Prove that K is compact.

(Hint: take $\{u_n\}_n \subset C([0,1])$ bounded, and prove that $\{K(u_n)\}_n$ has a convergent subsequence using the Arzelà-Ascoli theorem).

Definition 14.0.2. We say that $T \in \mathcal{L}(X,Y)$ is a **finite rank operator** if:

$$dim\ R(T) < \infty$$

(Note: $R(T) = T(X)$).

E.g.: As many as you want:

$T \in X^*$: $C^k([a,b]) \to \mathbb{P}^k$ polynomials of degree k:

- Taylor expansion
- Lagrange interpolation
- etc.

Proposition 14.0.2. *Let $T \in \mathcal{L}(X,Y)$ be a finite rank operator. Then T is compact.*

Proof. Let $A \subset X$ be bounded. Then, $T(A)$ is bounded in Y, and $\overline{T(A)}$ is bounded and closed in Y.

Since $dim\ R(T) < \infty$, $\overline{T(A)}$ is compact in Y. ■

Definition 14.0.3. We denote $\mathcal{K}(X,Y)$ as the set of compact operators from X to Y, i.e.:

$$\mathcal{K}(X,Y) = \{K \in \mathcal{L}(X,Y) \mid K \text{ is compact}\}$$

Theorem 14.0.3. *Let X, Y be Banach spaces. Then $\mathcal{K}(X,Y)$ is a closed vector subspace of $\mathcal{L}(X,Y)$.*

Remark: Now, to check that T is compact, it is enough to find a sequence $\{T_n\}_n \subset \mathcal{K}(X,Y)$ such that $T_n \to T$ in $\mathcal{L}(X,Y)$, i.e., $\|T_n - T\|_{\mathcal{L}(X,Y)} \to 0$.

Theorem 14.0.4 (Compact operators vs weak convergence)**.** *Let X, Y be Banach, then:*

(i) If $T \in \mathcal{K}(X,Y)$, then:

$$\{x_n\}_n \subset X \text{ s.t. } x_n \rightharpoonup x \text{ in } X \implies T(x_n) \to T(x) \text{ in } Y$$

(ii) If X is reflexive, then, the converse is also true, i.e., $T \in \mathcal{K}(X,Y)$ if $\forall \{x_n\}_n \subset X$:

$$x_n \rightharpoonup x \text{ in } X \implies T(x_n) \to T(x) \text{ in } Y$$

Proposition 14.0.5. *Let $T \in \mathcal{K}(X,Y)$, and $dim Y = \infty$. Then, T cannot be surjective.*

Proposition 14.0.6. *Take either $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{K}(X,Y)$ or $T \in K(X,Y)$ and $S \in \mathcal{L}(X,Y)$. Then:*

$$S \circ T \in \mathcal{K}(X,Y)$$

Proof. Trivial, because bounded operators map bounded sets to bounded sets, and precompact sets to precompact sets.

∎

Chapter 15

Hilbert spaces

Definition 15.0.1. Let H be a (real) vector space. A function $p : H \times H \to \mathbb{R}$ is called a **scalar product** (or **inner product**) if:

(i) (Positivity): $p(x, x) \geq 0 \; \forall x \in H$, and $p(x, x) = 0 \iff x = 0$.

(ii) (Symmetry): $p(x, y) = p(y, x) \; \forall x, y \in H$.

(iii) (Bilinearity): $p(\alpha x + \beta y, z) = \alpha p(x, z) + \beta p(y, z)$
$\forall x, y, z \in H$ and $\alpha, \beta \in \mathbb{R}$.

Note: For notation, we use the following:

$$p(x, y) = \langle x, y \rangle = (x, y) = x \cdot y$$

Definition 15.0.2. The space $(H, \langle \cdot, \cdot \rangle)$ is called a **pre-Hilbertian** space (inner product space).

Proposition 15.0.1. *Let $(H, \langle \cdot, \cdot \rangle)$ be a pre-Hilbertian space. Then:*

1) $|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$ *(Cauchy-Schwarz inequality).*

2) $\|x\| = \sqrt{\langle x, x \rangle}$ *is a norm on H.*

3) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$ *(parallelogram law).*

Proof. The proof is as follows:

1) Same as $\mathbb{R}^N$.

2) Exercise (Cauchy-Schwarz ineq. $\implies$ triangle ineq.)

3) $\langle x \pm y, x \pm y \rangle = \|x\|^2 \pm 2 \langle x, y \rangle + \|y\|^2$.

■

Remark: Notice that, because an inner product induces a norm, the space (H, d) with $d(x, y) = \|x - y\|$ is a metric space. Then, we can talk about convergence.

Definition 15.0.3. A pre-Hilbertian space $(H, \langle \cdot, \cdot \rangle)$ is called a **Hilbert space** if it is complete with respect to the induced norm $\|x\| = \sqrt{\langle x, x \rangle}$. (I.e., if $(H, \|\cdot\|)$ is a Banach space).

E.g.: We have the following examples of Hilbert spaces:

1) $\mathbb{R}^N$ with the Euclidean scalar product (usual dot product).

2) $L^2(X, \mathcal{M}, \mu)$ with the scalar product:

$$\langle f, g \rangle = \int_X f \cdot g \, d\mu$$

That induces the norm:

$$\|f\| = \left(\int_X f^2 \, d\mu \right)^{1/2}$$

Notice that $(C([a, b]), \langle \cdot, \cdot \rangle_{L^2})$ is an inner product space, but not a Hilbert space.

Proposition 15.0.2. *Let $(X, \|\cdot\|)$ be a Banach space. Then, it is also a Hilbert space if and only if the norm satisfies the parallelogram law. The inner product is then given by:*

$$\langle x, y \rangle = \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right)$$

Remark: With the proposition above, we can check that the space $(C([0, 1]), \|\cdot\|_\infty)$ is not a Hilbert space. Also, $(L^p(X, \mathcal{M}, \mu), \|\cdot\|_p)$ is not a Hilbert space for $p \neq 2$.

Definition 15.0.4. Let H be a Hilbert space. We say that:

(i) $x, y \in H$ are **orthogonal** $\iff \langle x, y\rangle = 0$. We write $x \perp y$.

(ii) Given $V \subset H$, the **orthogonal complement** of V is:

$$V^{\perp} = \{x \in H \mid \langle x, y\rangle = 0 \ \forall y \in V\}$$

15.1 Orthogonal projections

Note: Recall that:

- $S \subset H$ is convex is $\forall x, y \in S$, $\alpha x + (1-\alpha)y \in S$ for all $\alpha \in [0,1]$.
- $S \subset H$, $x \in H$, then the distance from x to S is:

$$d(x, S) = \inf_{y \in S} \|x - y\|$$

Theorem 15.1.1 (Projection theorem on closed convex sets). *Let H be Hilbert, $x \in H$ and $S \subset H$ closed, convex and non-empty. Then:*

$$\exists! h \in S \ s.t. \ d(x, S) = \|x - h\|$$

Moreover, h is characterized by the "variational inequality":

$$\langle x - h, y - h\rangle \leq 0 \quad \forall y \in S$$

(This inequality is equivalent to the first statement)

*We call h the **orthogonal projection** of x onto S.*

Proof. The proof is as follows:

1) Existence: Let $d = d(x, S)$. Then, take a "minimizing sequence", $\{y_n\}_n \subset S$ such that $\|x - y_n\| \to d$.

   Then, we are going to show that $\{y_n\}_n$ is a Cauchy sequence by applying the parallelogram law to $x - y_n$ and $x - y_m$:

$$\|x - v_n + x - v_m\|^2 + \|x - v_n - x + v_m\|^2 = 2\|x - v_n\|^2 + 2\|x - v_m\|$$
$$\implies \|v_m - v_n\|^2 = 2\|x - v_n\|^2 + 2\|x - v_m\|^2 - \|2x - v_n - v_m\|^2$$

Notice that:
$$\|2x - v_n - v_m\|^2 = 4 \left\| x - \frac{v_n + v_m}{2} \right\|^2 \geq 4d^2$$

since $\frac{v_n+v_m}{2} \in S$ (convexity). Then, we have:

$$\|v_m - v_n\|^2 \leq 2\|x - v_n\|^2 + 2\|x - v_m\|^2 - 4d^2 \to 0$$

Then, $\{y_n\}_n$ is Cauchy, and since H is complete, $\exists h \in H$ such that $\|x - h\| = d$. Also, $h \in S$ because S is closed.

2) Uniqueness: Let $h_1, h_2 \in S$ be two orthogonal projections of x onto S. Then, using the parallelogram law, we have:

$$\begin{aligned}\|h_1 - h_2\|^2 &= 2\|x - h_1\|^2 + 2\|x - h_2\|^2 - \|2x - h_1 - h_2\|^2 \\ &\leq 2d^2 + 2d^2 - 4d^2 = 0\end{aligned}$$

Then, $h_1 = h_2$.

$\blacksquare$

Theorem 15.1.2 (Projection theorem on closed subspaces)**.** *Let H be Hilbert, $x \in H$, $V \subset H$ a closed subspace. Then:*

$$\exists! h \in V : \quad \|x - h\| = d(x, V)$$

Moreover, h satisfies the previous implication if and only if:

$$\langle x - h, y \rangle = 0 \quad \forall y \in V$$

Remark: Notice that $x - h \perp y \ \forall y \in V$, meaning that $x - h \in V^\perp$.

We use the following notation:

$$h = P_V x = proj_V x$$

Remark: Let H be a Hilbert space, $V \subset H$ a subspace. Then, it is always closed on the following cases:

- if $dimV < \infty$
- $V = KerL$ for some $L \in \mathcal{L}(H, Y)$

- $V^\perp$ is closed for any V. This implies that:

$$(V^\perp)^\perp = \overline{V}$$

Theorem 15.1.3. *Let H be Hilbert. $V \subset H$ a closed subspace. Then:*

(i) $\forall x \in H, \quad x = P_V x + P_{V^\perp} x$

(ii) $x \in V \iff x = P_V x$

(iii) $\|x\|^2 = \|P_V x\|^2 + \|P_{V^\perp} x\|^2$

(iv) $P_V, P_{V^\perp} \in \mathcal{L}(H)$ *and their norm is 1.*

15.2 Dual of a Hilbert space

Definition 15.2.1. Let H be Hilbert. We define the mapping $i : H \to H^*$ (**Riesz map isometry**) as:

$$i(u) = L_u$$

where L_u is defined as:

$$L_u v := \langle u, v \rangle, \quad \forall v \in H$$

Notice that L_u is linear, and moreover, $\|L_u\|_* = \|u\|$.

Theorem 15.2.1 (Riesz representation theorem)**.** *Let H be Hilbert. Then, $\forall L \in H*$, $\exists! u \in H$ s.t.:*

$$L_v = \langle u, v \rangle, \quad \forall v \in H$$

Moreover, $\|u\| = \|L\|_*$. This means that the mapping i is an isometric isomorphism.

Corollary 15.2.1.1. *Let H be Hilbert, then H is reflexive. Also:*

$$H \cong H^* \implies H^* \cong H^{**}$$

(or: parallelogram $\implies$ H unif. convex)

Remark: We can identify H and H^*, but depending on $\langle \cdot, \cdot \rangle$. So, for a $V \subset H$ subspace dense, we have that:

$$V \subset H \cong H^* \subset V^*$$

Remark: The Riesz rep. thm. is a "well-posedness" theorem.

Proof (Riesz): The proof goes as follows:

- Existence:
  Case 1: $KerL = H$. Then, take $u = 0$. Notice that:

  $$\langle 0, v \rangle = 0 = Lv \quad \forall v \in H$$

  Case 2: $\exists z_0 \in H \setminus KerL$. Since $KerL$ is a closed subspace of H, let:

  $$z := \frac{P_{(KerL)^\perp} z_0}{\left\| P_{(KerL)^\perp} z_0 \right\|}$$

  and notice that $\|z\| = 1$ and $z \in (KerL)^\perp$. Take $v \in H$ and

  $$w = v - \frac{Lv}{Lz} z$$

  so that $Lw = 0$, $w \in KerL$. Now, we have:

  $$0 = \langle w, z \rangle = \left\langle v - \frac{Lv}{Lz} z, z \right\rangle = \langle v, z \rangle - \frac{Lv}{Lz} \langle z, z \rangle$$

  $$= \langle z, v \rangle - \frac{Lv}{Lz}$$

  I.e.:

  $$Lv = Lz \langle z, v \rangle = \langle (Lz) z, v \rangle \quad \forall v \in H$$

  Now, let $u = (Lz)z$

- Uniqueness: Let $u_1, u_2 \in H$ s.t.

  $$Lv = \langle u_1, v \rangle \quad \forall v \in H$$

  $$Lv = \langle u_2, v \rangle \quad \forall v \in H$$

  Then:

  $$\langle u_1 - u_2, v \rangle = 0 \quad \forall v \in H$$

Take $v = u_1 - u_2$, then:

$$\|u_1 - u_2\|^2 = 0$$

Therefore, $u_1 = u_2$.

Finally:

$$\|L\|_* = \sup_{x \neq 0} \frac{|Lx|}{\|x\|} = \sup_{x \neq 0} \frac{|\langle u, x \rangle|}{\|x\|} \leq \sup_{x \neq 0} \frac{\|u\| \|x\|}{\|x\|} = \|u\|$$

$$\|L\|_* = \sup_{x \neq 0} \frac{|Lx|}{\|x\|} = \sup_{x \neq 0} \frac{|\langle u, x \rangle|}{\|x\|} \geq \frac{|\langle u, u \rangle|}{\|u\|} = \|u\|$$

Then, $\|L\|_* = \|u\|$.

■

15.3 Consequences of the Riesz theorem

Theorem 15.3.1. *Let H be Hilbert, $\{x_n\}_n \subset H$. Then, $x_n \rightharpoonup x$ in H if and only if:*

$$\langle u, x_n \rangle \to \langle u, x \rangle \quad \forall u \in H$$

Moreover, by reflexivity, if $\{x_n\}_n \subset H$ is bounded, then $\exists \{x_{n_k}\}_k$ subsequence such that:

$$x_{n_k} \rightharpoonup x$$

Proposition 15.3.2. *Let $\{x_n\}_n \subset H$ and assume that:*

(i) $x_n \rightharpoonup x$ *weakly in* H

(ii) $\|x_n\| \to \|x\|$ *strongly in* H

Then, $x_n \to x$ strongly in H.

Proof. We have that:

$$\|x_n - x\| = \|x_n\|^2 - 2\langle x_n, x \rangle + \|x\|^2$$

Notice that $\|x_n\| \to \|x\|$ and $\langle x_n, x \rangle \to \langle x, x \rangle = \|x\|^2$. Then:

$$\|x_n - x\|^2 \to 0 \implies x_n \to x$$

■

15.4 Orthonormal basis

Note: We will consider H as a Hilbert space.

Definition 15.4.1. A sequence $\{e_n\}_{n\in\mathbb{N}} \subset H$ is an **orthonormal basis** if:

(i) $\|e_n\| = 1$, $\langle e_i, e_j\rangle = 0\ \forall i \neq j$.

(ii) $span(\{e_n\}_{n\in\mathbb{N}})$ is dense in H, i.e. $H = \overline{span(\{e_n\}_{n\in\mathbb{N}})}$.

(Note: the span of an infinite sequence of vectors consists of all the finite linear combinations of them).

E.g.: We have some examples:

- $H = \ell^2$, we have $e_1 = (1,0,0,...)$, $\ e_2 = (0,1,0,...), ...$
- $H = L^2[-\pi,\pi]$ we have:

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos nx}{\sqrt{\pi}}, n = 1,2,3,...\right\}$$

Theorem 15.4.1. *Every separable Hilbert space H has an orthonormal basis.*

Theorem 15.4.2. *Let $\{e_n\}_{n\in\mathbb{N}} \subset H$ be an orthonormal basis. Then:*

(i) $\forall u \in H$,

$$u = \sum_{n\in\mathbb{N}} \langle u, e_n\rangle\, e_n$$

and

$$\|u\|^2 = \sum_{n\in\mathbb{N}} |\langle u, e_n\rangle|^2$$

*(**Parseval-Bessel identity**).*

(ii) *Conversely: $\{\alpha_n\}_{n\in\mathbb{N}} \in \ell^2$, then:*

$$\sum_{n\in\mathbb{N}} \alpha_n e_n = x \in H$$

with $\langle x, e_n\rangle = \alpha_n$.

Proposition 15.4.3. $\{e_n\}_{n\in\mathbb{N}}$ *orthonormal basis. Then:*

$$e_n \rightharpoonup 0 \text{ weakly in } H$$

but $e_n \nrightarrow 0$ *strongly.*

Proof. By the theorem, $\forall u \in H$, the series:

$$\sum_n |\langle u, e_n\rangle|^2 < \infty$$

This implies that $\langle u, e_n\rangle \to 0, \ \ \forall u \in H$. So, $e_n \rightharpoonup 0$. But:

$$\|e_n\| = 1 \nrightarrow 0$$

■

Chapter 16

Spectral theory

Note: We will consider E Banach, $T \in \mathcal{L}(E, E) = \mathcal{L}(E)$, and the problem:

$$Tx = \lambda x \iff (T - \lambda I)x = 0$$

Definition 16.0.1. We define the following concepts:

- The **resolvent set** of T is:

$$\rho(T) = \{\lambda \in \mathbb{R} : T - \lambda I : E \to E \text{ is bijective}\}$$

- The **spectrum** of T is:

$$\sigma(T) = \mathbb{R} \setminus \rho(T)$$

- λ is an **eigenvalue** of T if:

$$Ker(T - \lambda I) \neq \{0\}$$

  where $Ker(T - \lambda I)$ is called the **eigenspace** corresponding to λ. Also:

$$EV(T) = \{\text{eigenvalues of } T\} \subset \mathbb{R}$$

Remark: Note that:

$$EV(T) \subset \sigma(T)$$

as $\lambda \in EV(T) \iff T - \lambda I$ is not injective. Also, note that if $dimE < \infty$, then $EV(T) = \sigma(T)$. If E has infinite dimension, then the inclusion may be strict.

Theorem 16.0.1. *Let E be Banach, $T \in \mathcal{L}(E)$. Then:*

(i) $\sigma(T) \subset [-\|T\|, \|T\|]$

(ii) $\sigma(T)$ *is closed.*

Remark: (i) means that the "spectral radius" is always $\leq$ the operatorial norm of T.

Remark: $|\lambda| > \|T\| \implies T - \lambda I$ is invertible. Moreover, $\rho(T)$ is open.

E.g.: Let $E = \ell^2$. We define the "left shift operator" $T_\ell : \ell^2 \to \ell^2$ as follows:

$$T_\ell x = (x^{(1)}, x^{(2)}, ...)$$

for $x = (x^{(0)}, x^{(1)}, x^{(2)}, ...)$. Then, one can prove that $T_\ell \in \mathcal{L}(\ell^2)$ and $\|T_\ell\| = 1$. By the theorem, $\sigma(T) \subset [-1, 1]$, closed.

Also, notice that (for $\lambda = 0$) T_ℓ is surjective, but not injective. I.e.,

$$R(T_\ell) = \ell^2, \quad Ker\ T_\ell = \{x \in \ell^2 : x^{(k)} = 0\ \forall k \geq 1,\ x^{(0)} \in \mathbb{R}\}$$

Then, $\lambda = 0$ is an eigenvalue of multiplicity 1.

Let us look for more eigenvalues: we know that $\lambda \in EV(T) \iff \exists x \neq 0$ s.t.:

$$T_\ell x = \lambda x$$

$$\iff (T_\ell x)^{(k)} = \lambda x^{(k)} \quad \forall k \geq 0$$
$$\iff x^{(k+1)} = \lambda x^{(k)} \quad \forall k \geq 0$$

Take any $x^{(0)} = x_0 \neq 0$. Then, notice that:

$$\begin{aligned}
x^{(1)} &= \lambda x_0 \\
x^{(2)} &= \lambda x^{(1)} = \lambda^2 x_0 \\
&\vdots \\
x^{(k)} &= \lambda x^{(k-1)} = ... = \lambda^k x_0
\end{aligned}$$

Then, λ is an eigenvalue $\iff x = (x_0, \lambda x_0, \lambda^2 x_0, ...) = x_0(1, \lambda, \lambda^2, ...) \in \ell^2$

$$\iff \sum_{k=0}^{\infty} (\lambda^k)^2 < \infty \iff |\lambda| < 1$$

Then, $EV(T) = (-1, 1)$ and

$$(-1, 1) \subset \sigma(T) \subset [-1, 1]$$

and because $\sigma(T)$ is closed, we conclude that $\sigma(T) = [-1, 1]$.

Exercise: Discuss $T_r : \ell^2 \to \ell^2$ the "right shift operator" such that:

$$T_r x = (0, x^{(0)}, x^{(1)}, x^{(2)}, \dots)$$

Show that $T_r \in \mathcal{L}(\ell^2)$ and $\|T_r\| = 1$. Then, show that $\sigma(T_r) = [-1, 1]$ and $EV(T_r) = \emptyset$.

16.1 Symmetric operators

Note: In what follows, consider:

- $(H, \langle \cdot, \cdot \rangle)$ a Hilbert space.
- $T \in \mathcal{K}(X) = \mathcal{K}(X, X)$ a compact operator.
- T is symmetric (self-adjoint)

Definition 16.1.1. We say that T is **symmetric** $\iff$

$$\langle Tx, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in H$$

Remark: If T is symmetric, then:

$$\|T\|_{\mathcal{L}(H)} = \sup_{x \neq 0} \frac{\langle Tx, x \rangle}{\|x\|^2}$$

This is called the **Rayleigh quotient**.

16.1.1 Fredholm's alternative theorem

Theorem 16.1.1 (Fredholm's alternative theorem)**.** *Let H be Hilbert, $T \in \mathcal{K}(H)$ symmetric. Then:*

(i) $\dim Ker(I-T) < \infty$

(ii) $R(I-T)$ *is closed.*

(iii) $Ker(I-T) = R(I-T)^{\perp}$ *and* $R(I-T) = Ker(I-T)^{\perp}$
(in particular, $I-T$ *is surjective* $\iff$ *is injective)*

(iv) Consider the following problem:

$$(\star) = \begin{cases} \text{Given } f \in H, \text{ find } x \in H, \text{ s.t.:} \\ (I-T)x = f \end{cases}$$

Then, exactly one of the following is true:

- $\forall f,\ \exists! x \in H$ *solving* $(\star)$
- $(\star)$ *is solvable* $\iff$ $f \in Ker(I-T)^{\perp}$*, and because* $\dim Ker(I-T) = N < \infty$*, this means that:*

$$\langle f, u_i \rangle = 0 \quad \forall i = 1, ..., N$$

s.t. $span(\{u_i\}_{i=1}^{N}) = Ker(I-T)$.

Remark: Consider $\lambda \neq 0$, $T - \lambda I$. Then, the FAT applies:

$$T - \lambda I = -\lambda(I - \frac{1}{\lambda}T)$$

where $\frac{1}{\lambda}T \in \mathcal{K}(H)$. As a consequence, we have that, for $T \in \mathcal{K}(H)$ symmetric:

$$\sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$$

Remark: For the theorem, there are some conditions that are not strictly necessary:

- T symmetric is not necessary, as (i), (ii) are true, and (iii), (iv) can be formulated in terms of the adjoint operator T^*: $\langle Tx, y \rangle = \langle x, T^* y \rangle$.
- H Hilbert is not necessary, for E Banach, use duality pairing instead of the scalar

product.

Notice that, without the compactness assumption, the theorem breaks.

16.2 Spectral theorem

Theorem 16.2.1 (Spectral theorem)**.** *Let H be Hilbert and separable, with $dim H = \infty$. Let $T \in \mathcal{K}(H)$ symmetric. Then:*

$$0 \in \sigma(T), \quad \sigma(T) \setminus \{0\} = EV(T) \setminus \{0\}$$

and the following alternative holds:

(i) Either T has finitely many eigenvalues different from 0, and then $0 \in EV(T)$, with dim $Ker(T) = \infty$

(ii) Or $EV(T) \setminus \{0\}$ is a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subset [-\|T\|, \|T\|]$ and $\lambda_n \to 0$ as $n \to \infty$, i.e., $0 \in \sigma(T)$.

Moreover, in both cases, there exists an orthonormal basis of H made by the eigenvectors of T.

Proof. The proof is based on the FAT, and it is omitted.

■