Example of Family-Wise Error Rate

Consider an example where three treatments are involved, and three pairwise α-level t-tests are to be conducted. Further, suppose that the treatment means are equal, $\mu_1 = \mu_2 = \mu_3$, and define the following events:

1. Let A be the event that $\mu_1 = \mu_2$ is rejected in the first t-test. Then $P(A) = \alpha$ is the Type I error rate for the first test.
2. Let B be the event that $\mu_1 = \mu_3$ is rejected in the next t-test. Then $P(B) = \alpha$ is the Type I error rate for the second test.
3. Let C be the event that $\mu_2 = \mu_3$ is rejected in the last t-test. Then $P(C) = \alpha$ is the Type I error rate for the third test.

Let D be the event one of the three pairwise t-tests leads to a Type I error for the test. Then, $D = A \cup B \cup C$ is the event at a Type I error occurs in the family of three pairwise t-tests, and $P(D)$ is the **Family-Wise Error Rate**, i.e., the probability that a difference of means is detected when none exists (the probability of a false positive). The goal is to determine the bounds on $P(D)$.

A Lower Bound on $P(D)$

$P(D) = P(A \cup B \cup C) \geq P(A) = \alpha$, so $\boxed{P(D) \geq \alpha}$

A Conservative Upper Bound on $P(D)$

$P(D) = P(A \cup B \cup C) \leq P(A) + P(B) + P(C) = \alpha + \alpha + \alpha = 3\alpha$, so $\boxed{P(D) \leq 3\alpha}$

A Tighter Upper Bound on $P(D)$

$P(D^c) = P(A^c \cap B^c \cap C^c)$, by De Morgan's Law.

$$P(A^c \cap B^c \cap C^c) = P(A^c)P(B^c \mid A^c)P(C^c \mid A^c \cap B^c) \geq P(A^c)P(B^c)P(C^c) = (1-\alpha)(1-\alpha)(1-\alpha),$$

because the coverage probabilities are positively correlated. So, $P(D^c) \geq (1-\alpha)^3$.

Finally, $\boxed{P(D) = 1 - P(D^c) \leq 1-(1-\alpha)^3}$. A little algebra shows that $1-(1-\alpha)^3 \leq 3\alpha$, so this upper bound is not as conservative as the $P(D) \leq 3\alpha$ upper bound (but it is considerably more difficult to prove).

Example: If all pairwise t tests are conducted at $\alpha = 0.05$, then $0.05 \leq P(D) \leq 0.143 \leq 0.15$, and the family-wise error rate for the three 0.05 level tests may be much higher that the nominal 0.05 value.

In general, for k treatments, and $r = \binom{k}{2} = \frac{k!}{2!(k-2)!}$ pairwise α-level t-tests, the family-wise error rate is

$$\boxed{\alpha \leq P(\text{Family of tests Type I Error}) \leq 1-(1-\alpha)^r \leq r\alpha}$$