The parametrisation method for invariant manifolds of tori in skew-product systems with spatial decay in lattices

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Physical motivation

Physics

The beginning of the study of lattice systems is found in the first models of chains of particles under the action of a potential, with an interaction to nearest neighbors. These models were first considered by Ludwig Prandtl for deformations in solids in [Pra28] and Ulrich Dehlinger for crystal lattices in [Deh29].

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Later similar models were considered by Yacob Frenkel and Tatiana Kontorova in their study of solid dislocation and deformation ([FK38a], [FK38b], [FK38c], [FK39].)

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Neuroscience

Similar models (see for instance [Hop86] or [Izh07]) can be used for arrays of cells (nodes) in neuroscience, where each neuron is connected to (possibly) all others, all sharing the same dynamics.

One-dimensional models of chains of particles with nearest-neighbour interactions can be described by a formal Hamiltonian,

$$H(p,q) = \sum_{i\in\mathbb{Z}} \left(\frac{1}{2}\|p_i\|^2 + V(q_i) \right) + \sum_{i\in\mathbb{Z}} W(q_{i+1} - q_i), \quad (q_i, p_i) \in \mathbb{R}^{2n}, i \in \mathbb{Z}.$$

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The equations of motion are then

$$\begin{aligned}\dot{q}_i &= p_i\\ \dot{p}_i &= -\nabla V(q_i) + \nabla W(q_{i+1}-q_i) - \nabla W(q_i - q_{i-1}), \quad i\in\mathbb{Z}^d,\end{aligned}$$

or equivalently

$$\ddot{q}_i + \nabla V(q_i) = \nabla W(q_{i+1}-q_i) - \nabla W(q_i - q_{i-1}), \quad i\in\mathbb{Z}.$$

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$$\ell^\infty(\mathbb{R}^n) = \left\{ (x_i)_{i\in\mathbb{Z}^m} \mid x_i \in \mathbb{R}^n, \sup_{i\in\mathbb{Z}^m} \|x_i\| < \infty \right\},$$

where $\|\cdot\|$ is a given norm in $\mathbb{R}^n$. We endow $\ell^\infty(\mathbb{R}^n)$ with the norm $\|x\|_\infty = \sup_{i\in\mathbb{Z}^m} \|x_i\|$ as usual.

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Observe that $\ell^\infty(\mathbb{R}^n)$ is a Banach space but not a Hilbert space.

Decay functions

Definition

We say that a function $\Gamma : \mathbb{Z}^m \to \mathbb{R}^+$ is a decay function when it satisfies:

1. $\sum_{k\in\mathbb{Z}^m} \Gamma(k) \leq 1$,
2. $\sum_{k\in\mathbb{Z}^m} \Gamma(i-k)\Gamma(k-j) \leq \Gamma(i-j), \quad i, j \in \mathbb{Z}^m$.

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This is a family of decay functions in $\mathbb{Z}^m$ (as proved in [JdlL00]) satisfying the previous definition for some $a(\alpha,\theta,m)$:

$$\Gamma(j) = \begin{cases} a, & j = 0, \\ a|j|^{-\alpha}e^{-\theta|j|}, & j \neq 0 \end{cases}.$$

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In [JdlL00] there is also a proof that the standard exponential function $\Gamma(j) = Ce^{-\theta|j|}$ is not a decay function for any $C, \theta > 0$.

Linear maps with decay

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$$L_\Gamma = L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) = \left\{ A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)) \mid \|A\|_\Gamma < \infty \right\},$$

where

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\},$$

with $\|A\|$ the operator norm of A and

$$\gamma(A) = \sup_{i,k \in \mathbb{Z}^m} \sup_{\substack{\|u\| \leq 1, \\ \mathrm{proj}_j u = 0, j \neq k}} \|(Au)_i\| \Gamma(i-k)^{-1}.$$

Useful properties

If we define $A_{ij} = \mathrm{proj}_i A\, \mathrm{emb}_j$ then

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Lemma

Let $A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, and $v = (v_i)_{i\in\mathbb{Z}^m} \in \ell^\infty(\mathbb{R}^n)$ such that

$$\lim_{|j|\to\infty} \|v_j\| = 0.$$

Then

$$(Av)_i = \sum_{k\in\mathbb{Z}^m} A_{ik} v_k.$$

Useful properties

Observe that given an uncoupled linear map (where the map is equal in all nodes,) $A = \mathfrak{a}\delta_{ij}$, with $\mathfrak{a} \in L(\mathbb{R}^n, \mathbb{R}^n)$ we have $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $\gamma(A) = \Gamma(0)^{-1}\|\mathfrak{a}\|$ and

$$\|A\|_\Gamma = \Gamma(0)^{-1}\|\mathfrak{a}\|.$$

Here be dragons

In general, a linear map $A \in L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ is *not* determined by its matrix components A_{ij}.

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An example would be the linear map lim, $\lim : E_0 \to \mathbb{R}$ defined as $\lim(v) = \lim_{|j|\to\infty} v_j$ in the following space:

$$E_0 = \{v \in \ell^\infty(\mathbb{R}) \mid \lim_{|j|\to\infty} v_j \text{ exists}\}.$$

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The matrix elements of $\mathcal{L}$ are all 0 but $\mathcal{L}$ is not the map 0.

Multilinear maps with decay

Definition

We define L_Γ^k as

$$L_\Gamma^k(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n)) = \Big\{A \in L^k(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n)) \mid \iota_p(A) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), L^{k-1}(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n))),\ 1 \le p \le k\Big\},$$

with the norm

$$\|A\|_\Gamma = \max\{\|A\|, \gamma(A)\},$$

where

$$\gamma(A) = \max_{1\le p\le k}\{\gamma(\iota_p(A))\}$$

and

$$\iota_j(A)(w)(v_1,\ldots,v_{k-1}) = A(v_1,\ldots,\overbrace{w}^{j},\ldots,v_{k-1}).$$

Multilinear maps are well-behaved

Proposition (Γ norms of contractions)

Let $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $k \geq 2$, and $u \in \ell^\infty(\mathbb{R}^n)$. Then, for any permutation of k elements $\tau \in S_k$ the map $B_{\tau,u} : \ell^\infty(\mathbb{R}^n) \times \overset{(k-1)}{\cdots} \times \ell^\infty(\mathbb{R}^n) \to \ell^\infty(\mathbb{R}^n)$ defined by

$$B_{\tau,u}(v_1, \ldots, v_{k-1}) = A(\tau(v_1, \ldots, v_{k-1}, u))$$

belongs to $L^{k-1}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$. Moreover

$$\|B_{\tau,u}\|_\Gamma \leq \|A\|_\Gamma \|u\|. \tag{1}$$

Maps in L^k_Γ satisfy...

Proposition

If $A \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and* $B_j \in L^{l_j}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$*, for* $j = 1, \dots, k$*, then the composition* $AB_1 \cdots B_k \in L^{l_1+\cdots+l_k}_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and*

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If $A \in L_\Gamma^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and* $B_j \in L_\Gamma^{l_j}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$*, for* $j = 1, \ldots, k$*, then the composition* $AB_1 \cdots B_k \in L_\Gamma^{l_1+\cdots+l_k}(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and*

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Corollary

If $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and* $B \in L_\Gamma^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *then* $A \cdot B \in L_\Gamma^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *and*

$$\|A\, B\|_\Gamma \leq \|A\|_\Gamma \|B\|_\Gamma.$$

Of course!

Theorem

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We have that $L_\Gamma(E, E) \subset L(E, E)$ as sets, but $L_\Gamma(E, E)$ is not a closed subalgebra of $L(E, E)$, so that it is not a Banach subalgebra of $L(E, E)$.

Γ-spectrum of linear maps on lattices

Consider the lattice $\ell^\infty(\mathbb{R}^n)$, a decay function Γ and the complexified space

$$\ell^\infty(\mathbb{R}^n) \otimes_{\mathbb{R}} \mathbb{C} \sim \ell^\infty(\mathbb{R}^n) \oplus i\ell^\infty(\mathbb{R}^n) \sim \ell^\infty(\mathbb{C}^n).$$

Let $\mathcal{E}$ be a linear subspace of $\ell^\infty(\mathbb{C}^n)$. Given $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$ we define:

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- Γ-resolvent of A as

$$\rho_\Gamma(A) = \{\lambda \in \mathbb{C} \mid A - \lambda \mathrm{Id} \text{ is invertible and } (A - \lambda \mathrm{Id})^{-1} \in L_\Gamma(\mathcal{E}, \mathcal{E})\},$$

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From the definition it is immediate that

$$\rho_\Gamma(A) \subset \rho(A)$$

and therefore

$$\mathrm{Spec}(A) \subset \mathrm{Spec}_\Gamma(A), \quad r(A) \leq r_\Gamma(A).$$

The fact that $L_\Gamma(\mathcal{E},\mathcal{E})$ is a Banach algebra implies that $\mathrm{Spec}_\Gamma(A)$ is a compact subset of $\mathbb{C}$.

The theory of Γ-spectrum is similar to the theory of the spectrum of linear maps between Banach spaces but the proofs have to be adapted to this setting, because the algebra $L(\mathcal{E}, \mathcal{E})$ is different from $L_\Gamma(\mathcal{E}, \mathcal{E})$

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One of the fundamental properties is the continuity (and decay properties) of projections associated to a gap in the Γ-spectrum

Spectral projections associated to a gap in the Γ-spectrum

Assume $\mathrm{Spec}_\Gamma(A) = \sigma_1 \cup \sigma_2$ with σ_1, σ_2 totally disjoint sets, i.e.

$$\sigma_i \subset \omega_i \subset \overline{\omega_i} \subset \Omega_i, \qquad i = 1, 2,$$

with Ω_i disjoint open sets and $\partial\omega_i$ finite union of simple closed curves.

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Lemma

We have

(i) $P \in L_\Gamma(\mathcal{E}, \mathcal{E})$,

(ii) $P^2 = P$,

(iii) $P(\mathcal{E})$ *and* $\mathrm{Ker}(P)$ *are closed and invariant.*

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Theorem

We have that

$$\mathrm{Spec}_\Gamma(A_i) = \sigma_i, \quad i = 1, 2.$$

We need differentiable functions!

Definition

Let U be an open set of $\ell^\infty(\mathbb{R}^n)$. We define

$$\begin{aligned} C^1_\Gamma(U, \ell^\infty(\mathbb{R}^n)) = \{F \in C^1(U, \ell^\infty(\mathbb{R}^n)) \mid & \sup_{x\in U} \|F(x)\|_\infty < \infty, \\ & DF(x) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)), \forall x \in U, \\ & \sup_{x\in U} \|DF(x)\|_\Gamma < \infty\} \end{aligned}$$

with norm

$$\|F\|_{C^1_\Gamma} = \max\left(\|F\|_{C^0}, \sup_{x\in U} \|DF(x)\|_\Gamma\right),$$

where $\|F\|_{C^0} = \sup_{x\in U} \|F(x)\|_\infty$ as usual. We can thus define

$$\begin{aligned} C^1_\Gamma(U, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) = \Big\{F \in C^1(U, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) \mid & \\ & F(x) \in L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n)), \ \forall x \in U, \\ & \sup_{x\in U} \|F(x)\|_\Gamma < \infty\Big\}. \end{aligned}$$

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We can define $C^r_\Gamma(U,\ell^\infty(\mathbb{R}^n))$ in a similar way.

We need a concept of "decay" functions in $\ell^\infty(\mathbb{R}^n)$

Definition

Given $j \in \mathbb{Z}^m$, we define

$$S^0_{j,\Gamma} = S^0_{j,\Gamma}(\mathbb{T}^d) = \left\{\sigma \in C^0(\mathbb{T}^d, U) \mid \sup_{i\in\mathbb{Z}^m} \sup_{\theta\in\mathbb{T}^d} \|\sigma_i(\theta)\|\Gamma(i-j)^{-1} < \infty\right\},$$

with norm

$$\|\sigma\|_{S^0_{j,\Gamma}} = \sup_{i\in\mathbb{Z}^m} \|\sigma_i\|_{C^0}\Gamma(i-j)^{-1} = \sup_{\theta\in\mathbb{T}^d} \|\sigma(\theta)\|_{S_{j,\Gamma}} = \sup_{i\in\mathbb{Z}^m} \sup_{\theta\in\mathbb{T}^d} \|\sigma_i(\theta)\|\Gamma(i-j)^{-1}$$

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Likewise, we can define a differentiable analogue

We need a concept of "decay" functions in $\ell^\infty(\mathbb{R}^n)$

Definition

Given $j \in \mathbb{Z}^m$, we define

$$S^0_{j,\Gamma} = S^0_{j,\Gamma}(\mathbb{T}^d) = \left\{\sigma \in C^0(\mathbb{T}^d, U) \mid \sup_{i\in\mathbb{Z}^m} \sup_{\theta\in\mathbb{T}^d} \|\sigma_i(\theta)\|\Gamma(i-j)^{-1} < \infty\right\},$$

with norm

$$\|\sigma\|_{S^0_{j,\Gamma}} = \sup_{i\in\mathbb{Z}^m} \|\sigma_i\|_{C^0}\Gamma(i-j)^{-1} = \sup_{\theta\in\mathbb{T}^d} \|\sigma(\theta)\|_{S_{j,\Gamma}} = \sup_{i\in\mathbb{Z}^m} \sup_{\theta\in\mathbb{T}^d} \|\sigma_i(\theta)\|\Gamma(i-j)^{-1}$$

Likewise, we can define a differentiable analogue

Definition

$$S^r_{j,\Gamma} = S^r_{j,\Gamma}(\mathbb{T}^d) = \Big\{\sigma \in C^r(\mathbb{T}^d, U) \mid \frac{\partial^k}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}}\sigma \in S^0_{j,\Gamma}(\mathbb{T}^d), \quad \ell_1+\ldots+\ell_d = k,\ 0\le k\le r\Big\}$$

with norm

$$\|\sigma\|_{S^r_{j,\Gamma}} = \max_{0\le k\le r} \max_{\substack{l_1,\ldots,l_d\ge 0\\ l_1+\ldots+l_d=k}} \left\|\frac{\partial^k}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}}\sigma\right\|_{S^0_{j,\Gamma}}$$

We also need functions from the torus

Definition

We define

$$C^0_{L^k_\Gamma} = C^0\left(\mathbb{T}^d, L^k_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))\right)$$
$$= \{F \in C^0(\mathbb{T}^d, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) \mid \sup_{\theta\in\mathbb{T}^d} \|F(\theta)\|_\Gamma < \infty\}.$$

with norm

$$\|F\|_{C^0_{L_\Gamma}} = \sup_{\theta\in\mathbb{T}^d} \|F(\theta)\|_\Gamma$$

and

$$C^r_{L^k_\Gamma} = C^r_\Gamma = \{F \in C^r(\mathbb{T}^d, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))) \mid \|F\|_{C^r_{L^k_\Gamma}} < \infty\}$$

with norm

$$\|F\|_{C^r_{L^k_\Gamma}} = \max_{\substack{0\le j\le r\\ \ell_1+\ldots+\ell_d=j}} \left\| \frac{\partial^j F(\theta)}{\partial\theta_1^{\ell_1}\cdots\partial\theta_d^{\ell_d}} \right\|_{C^0_{L_\Gamma}}$$

And functions differentiable with respect to two variables, with decay...

Definition

Let U be an open set of $\ell^\infty(\mathbb{R}^n)$. We define

$$\begin{aligned} C^{t,r}_\Gamma(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n)) = \{F\in C^{t,r}(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))\mid \\ D^i_x D^j_\theta F\in C^0(U\times\mathbb{T}^d, L^i_\Gamma(\ell^\infty(\mathbb{R}^n),\ell^\infty(\mathbb{R}^n))), \\ 1\le i\le t,\ 0\le j\le r,\ \|F\|_{C^{t,r}_\Gamma}<\infty\}, \end{aligned}$$

with norm

$$\|F\|_{C^{t,r}_\Gamma} = \max\left(\|F\|_{C^0}, \max_{\substack{1\le i\le t\\ 0\le j\le r}} \sup_{\substack{x\in U\\ \theta\in\mathbb{T}^d}} \|D^i_x D^j_\theta F(x,\theta)\|_\Gamma\right),$$

where $\|F\|_{C^0} = \sup_{\substack{x\in U\\ \theta\in\mathbb{T}^d}} \|F(x,\theta)\|$ as usual.

and with centred decay

Definition

Let U be an open set of $\ell^\infty(\mathbb{R}^n)$. We define

$$C^{t,r}_{j,\Gamma}(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n)) = \{F\in C^{t,r}_{\Gamma}(U\times\mathbb{T}^d,\ell^\infty(\mathbb{R}^n))\,|\, D^k_\theta F(x,\cdot)\in S^0_{j,\Gamma},$$
$$0\le k\le r,\, x\in U, \|F\|_{C^{t,r}_{j,\Gamma}}<\infty\},$$

with norm

$$\|F\|_{C^{t,r}_{j,\Gamma}} = \max\left(\|F\|_{C^{t,r}_{\Gamma}}, \max_{0\le k\le r}\sup_{x\in U}\|D^k_\theta F(x,\cdot)\|_{S^0_{j,\Gamma}}\right)$$

Why so many spaces?

The definition of $C^{\Sigma_{t,r}}_{j,\Gamma}$ is needed to study j-localised properties of invariant tori, forced by the chain rule.

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D^2_\theta F(W(\theta),\theta) &= D^2_x F(W(\theta),\theta)(D_\theta W(\theta))^2 + 2 D_\theta D_x F(W(\theta),\theta) D_\theta W(\theta) \\
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Since we need $F(W(\theta),\theta)$ to be in $S^2_{j,\Gamma}$, a necessary condition comes now from the results on compositions between linear maps and centred vectors. An easy lemma proves products/contractions of L^k_Γ and $S_{j,\Gamma}$ are in $S_{j,\Gamma}$.

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Since we need $F(W(\theta),\theta)$ to be in $S^2_{j,\Gamma}$, a necessary condition comes now from the results on compositions between linear maps and centred vectors. An easy lemma proves products/contractions of L^k_Γ and $S_{j,\Gamma}$ are in $S_{j,\Gamma}$. Hence when $D^2_x F$ and $D_x F$ are in $C^0_{L^2_\Gamma}$ and $C^0_{L_\Gamma}$ respectively and $W \in S^1_{j,\Gamma}$, then $D^2_x F(D_\theta W, D_\theta W)$ and $D_\theta D_x F D_\theta W$ are in $S^0_{j,\Gamma}$.Then we also need $D^2_\theta F(W(\theta),\theta)$ to be in $S^0_{j,\Gamma}$, motivating the definition of $C^{t,r}_{j,\Gamma}$ and similar spaces.

Finally, we will need functions with anisotropic differentiability

Consider the following subset of indices

$$\Sigma_{t,r} = \{(k,i) \in \left(\mathbb{Z}^+\right)^2 \mid i \leq r,\, i+k \leq r+t\},$$

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Definition

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$$\begin{aligned} C^{\Sigma_{t,r}}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) = \Big\{ & F : U \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n) \mid \\ & D_\theta^i D_x^k F \in C^0(U \times \mathbb{T}^d, L^k(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))),\, (k,i) \in \Sigma_{t,r}, \\ & \|F\|_{C^{\Sigma_{t,r}}} < \infty \Big\}, \end{aligned}$$

where

$$\|F\|_{C^{\Sigma_{t,r}}} = \max\left(\|F\|_{C^0}, \max_{(k,i)\in\Sigma_{t,r}} \|D_\theta^i D_x^k F\|_{C^0} \right), \tag{2}$$

and

$$\|D_x^k D_\theta^i F\|_{C^0} = \max_{i_1+\ldots+i_d \leq i} \sup_{(x,\theta)\in U\times\mathbb{T}^d} \left\| D_x^k \frac{\partial^i}{\partial\theta_1^{i_1}\ldots\partial\theta_d^{i_d}} F(x,\theta) \right\|_{L^k}$$

Functions with anisotropic differentiability and decay/centred decay

Analogously we can define the sets of functions

Definition

$C_\Gamma^{\Sigma_{t,r}} = \{\ldots\}$

and

Definition

$C_{j,\Gamma}^{\Sigma_{t,r}} = \{\ldots\}$

in a similar way to the definitions of $C_\Gamma^{t,r}$ and $C_{j,\Gamma}^{t,r}$, respectively.

l-flat maps

Definition

Given U an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$, we define for $l \leq t$

$$
\begin{aligned}
C^{\Sigma_{t,r,l}}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) = \{F \in C^{\Sigma_{t,r}}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)) \mid \\
D^j_x F(0,\theta) = 0, \forall \theta \in \mathbb{T}^d, 0 \leq j \leq l, \\
\|F\|_{C^{\Sigma_{t,r,l}}_{j,\Gamma}} < \infty\},
\end{aligned} \tag{3}
$$

where

$$
\|F\|_{C^{\Sigma_{t,r,l}}_{j,\Gamma}} = \max\left(\|F\|_{C^{\Sigma_{t,r}}_{j,\Gamma}}, \max_{i \leq r} \sup_{x \in U \setminus \{0\}} \frac{\|D^i_\theta D^l_x F(x,\cdot)\|_{C^0_{L^l_\Gamma}}}{\|x\|}\right). \tag{4}
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We could also similarly define $C^{t,r,l}$, $C^{t,r,l}_\Gamma$, $C^{t,r,l}_{j,\Gamma}$ and $C^{\Sigma_{t,r,l}}_\Gamma$.

We will consider dynamical systems in $\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d$ which have the form

$$F(x,\theta) = F_0(x) + F_1(x,\theta), \qquad x \in \ell^\infty(\mathbb{R}^n),\, \theta \in \mathbb{T}^d.$$

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The dynamical system considered will then be

$$(x,\theta) \mapsto (F(x,\theta), \theta + \omega).$$

We will write

$$F(x,\theta) = F_0(x) + F_1(x,\theta),$$

with

$$F_0(x) = M_0 x + N_0(x),$$
$$F_1(x,\theta) = M_1(\theta)x + N_1(x,\theta).$$

with $DN_0(0) = D_x N_1(0,\theta) = 0$. We also write

$$M(\theta) = M_0 + M_1(\theta),$$
$$N(x,\theta) = N_0(x) + N_1(x,\theta),$$

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Given a map $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ with $F(0) = 0$, where U open, $0 \in U$ and a linear subspace of $\mathbb{R}^n$, we look for a manifold invariant under the action of F tangent to a subspace $E \subset \mathbb{R}^n$ in 0 as an embedding $K : U_1 \subset E \to \mathbb{R}^n$ and a map $R : U_1 \subset E \to U_1$ such that

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- For invariant tori...
- In Banach spaces...
- Combinations of the above.

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- For flows...
- For invariant tori...
- In Banach spaces...
- Combinations of the above.

Many people have contributed to the development of this method, among them (in alphabetic order) Cabré, Canadell, de la Llave, Figueras, Fontich, Haro, Luque, Martín, Mondelo, Simó, Sire...

We will denote the invariant torus by its parametrisation, $W_0 : \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$. Finding this object is the first step in the parametrisation method as detailed in [CFdlL03], [HdlL06] or [FdlLS09]. We will do this (under suitable smallness assumptions on the perturbing function F_1) by solving the functional equation given by the invariance of the torus with respect to the dynamical system,

$$F(W_0(\theta), \theta) = W_0(\theta + \omega).$$

Theorem

Let F be as above. Assume that $Df(0)$ is hyperbolic and consider the functional equation

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Then...

(i)

Assume $M_1 \in C^0\left(\mathbb{T}^d, L(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))\right)$ and $N(x,\theta) \in C^{0,0}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with N Lipschitz with respect to x for all $\theta \in \mathbb{T}^d$ and assume $\|F_1\|_{C^0}$ and $\mathrm{Lip}_x(N)$ are small enough. Then the functional equation (5) has a unique solution $W_0(\theta) \in C^0(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.

$$F(W_0(\theta), \theta) = W_0(\theta + \omega). \tag{5}$$

(ii)

Assume $F_0 \in C^t(U, \ell^\infty(\mathbb{R}^n))$, $F_1 \in C^{t,r}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $t \geq r + 1$, $r \geq 0$ and $\|F_1\|_{C^{t,r}}$ small enough. Then (5) has a solution $W_0 \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$. Since $C_\Gamma^{t,r} \subset C^{t,r}$, for $F \in C_\Gamma^{t,r}$ we also obtain a solution $W_0 \in C^r(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.

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(iii)

Assume $F_0 \in C^t(U, \ell^\infty(\mathbb{R}^n))$, $F_1 \in C^{t,r}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $t \geq r + 2$, $r \geq 0$ and $\|F_1\|_{C^{t,r}_{j,\Gamma}}$ small enough. Then (5) has a solution $W_0 \in S^r_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.

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(iii)

Assume $F_0 \in C^t(U, \ell^\infty(\mathbb{R}^n))$, $F_1 \in C^{t,r}_{j,\Gamma}(U \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ with $t \geq r+2$, $r \geq 0$ and $\|F_1\|_{C^{t,r}_{j,\Gamma}}$ small enough. Then (5) has a solution $W_0 \in S^r_{j,\Gamma}(\mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ close to 0.

The differentiability $t \geq r+2$ in this last result can be changed for $t \geq r+1$ after some improvements in the techniques involved in the proof.

Sketch of the proof

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Is standard using hyperbolicity and a fixed point theorem.

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(ii) & (iii)

Recall that for an uncoupled map, $\|A\|_\Gamma = \Gamma(0)^{-1}\|A\|$, hence direct hyperbolicity arguments fail, since a normal iteration argument would yield $\Gamma(0)^{-N}\|A\|^N$, which for a contraction A in general does not converge to 0.

Sketch of the proof

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(ii) & (iii)

Recall that for an uncoupled map, $\|A\|_\Gamma = \Gamma(0)^{-1}\|A\|$, hence direct hyperbolicity arguments fail, since a normal iteration argument would yield $\Gamma(0)^{-N}\|A\|^N$, which for a contraction A in general does not converge to 0. We work instead with

$$F^{[n]}(x,\theta) = F(F^{[n-1]}(x,\theta-\omega),\theta-\omega), \qquad F^{[1]}(x,\theta) = F(x,\theta-\omega).$$

Sketch of the proof

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In this case we would have instead

$$\|A^n\|_\Gamma = \Gamma(0)^{-1}\|A\|^n,$$

and hyperbolicity arguments work if we choose a specific $N \in \mathbb{N}$ large enough.

Actually to prove case (iii) we first need to prove (ii), for $F \in C^{t,r}$. Once we know there exists a C^r solution for $F(W(\theta-\omega),\theta) = W(\theta)$ (denote it for now by ${}^1W$), we know this also is a C^r solution for $F^{[n]}(W(\theta - n\omega),\theta) = W(\theta)$, for all $n \geq 1$.

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With the "iteration trick" we find a solution ${}^NW \in S^r_{j,\Gamma}$ of $F^{[N]}(W(\theta-N\omega))=W(\theta)$, which is also a C^r solution of this equation. Hence ${}^NW={}^1W$ and we have found a $S^r_{j,\Gamma}$ solution of $F(W(\theta-\omega),\theta)=W(\theta)$: an invariant torus (with centred decay).

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From now on assume we have translated this torus to 0. This translated system has the same decay and regularity properties as the original system. We will keep denoting the system by $F(x,\theta)$

First steps first

Assume we have a splitting of the spectrum of M_0 such that we can write $\ell^\infty(\mathbb{R}^n) = \mathcal{E}^1 \oplus \mathcal{E}^2$. Write

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We will begin the study of the invariant manifolds of the torus by determining its linear part $W_1(\theta)$. To do so, we will determine it in the form $(\mathrm{Id}, v(\theta))$, where Id denotes the identity in $\mathcal{E}^1$ and $v(\theta) \in \mathcal{E}^2$, $\forall \theta \in \mathbb{T}^d$. In other words, we will try to convert $M(\theta)$ into a block upper triangular matrix.

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Proposition (strong stable case, particular case of the next result)

Let F and $D_x F_0(0)$ as before. Assume

(H1) $\|A_{1,1}\| < 1$ in some norm,

(H2) $A_{2,2}$ is invertible, and $\|A_{1,1}\|\|A_{2,2}^{-1}\| < 1$ in the same norm as the previous hypothesis,

(H3) $\|F_1\|_{C_\Gamma^{\Sigma_{t,r}}}$ is small enough, $t \geq r+1$, $r \geq 0$.

Then we can find $R_1 \in C_\Gamma^r(\mathbb{T}^d, L(\mathcal{E}^1, \mathcal{E}^1))$ and $W_1 \in C_\Gamma^r(\mathbb{T}^d, L(\mathcal{E}^1, \ell^\infty(\mathbb{R}^n)))$ such that

$$F(W_1(\theta)s, \theta) = W_1(\theta+\omega)R_1(\theta)s + o(\|s\|). \tag{6}$$

Moreover

$$R_1 = A_{1,1} + \mathcal{O}(\|F_1\|).$$

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In Hypothesis (H3) we ask for $t \geq r+1$ because this condition is needed to determine the invariant torus of class C^r. However, if we start with $F(x,\theta) = F_0(x) + F_1(x,\theta)$ such that

$$F_0(0) = 0, \quad F_1(0,\theta) = 0, \qquad \forall \theta \in \mathbb{T}^d,$$

to prove these results it would be enough to require $F_1 \in C_\Gamma^{\Sigma_{1,r}}$.

Proposition (Non-resonant case for the linear part)

Let F and $D_x F_0(0)$ as before. Assume

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From the proofs of these results, we can actually transform the linear part of $M(\theta)$ into a block diagonal form by a C_Γ^r transform.

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Note

From the proofs of these results, we can actually transform the linear part of $M(\theta)$ into a block diagonal form by a C_Γ^r transform.

Apply a scaling procedure now to be able to work in the unit ball, moving this smallness requirement to an external parameter, δ.

Theorem 1/2

Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$ and consider a dynamical system $F : U \times \mathbb{T}^d \subseteq \ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$, $F(x,\theta) = M(\theta)x + N_1(x,\theta)$ with $M(\theta) = M_0 + \widetilde{M}(\theta)$ and

$$M_0 = \begin{pmatrix} A_{1,1} & 0 \\ 0 & A_{2,2} \end{pmatrix}, \ \widetilde{M}(\theta) = \begin{pmatrix} B_{1,1}(\theta) & B_{1,2}(\theta) \\ 0 & B_{2,2}(\theta) \end{pmatrix}, \ M(\theta) = \begin{pmatrix} M_{1,1}(\theta) & M_{1,2}(\theta) \\ 0 & M_{2,2}(\theta) \end{pmatrix}.$$

Assume the following hypotheses,

(H1) $F \in C_\Gamma^{\Sigma_{t,r}}\left(\ell^\infty(\mathbb{R}^n) \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n)\right)$, with $t \geq r+1$, M_0, $\widetilde{M}(\theta) \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$, $\sup_{\theta \in \mathbb{T}^d} \|\widetilde{M}(\theta)\|_\Gamma = \mathcal{O}(\varepsilon)$ and the scaling parameter δ are sufficiently small

(H2) $\mathcal{A}\,\mathrm{Spec}(A_{1,1}) \subset \mathbb{D}\backslash\{0\}$,

(H3) $0 \notin \mathrm{Spec}(A_{2,2})$,

(H4) $\mathcal{A}\,\mathrm{Spec}(A_{1,1})^{L+1} \cdot \mathcal{A}\,\mathrm{Spec}(M_0^{-1}) \subset \mathbb{D}$,

(H5) $\mathcal{A}\,\mathrm{Spec}(A_{1,1})^i \cap \mathcal{A}\,\mathrm{Spec}(A_{2,2}) = \emptyset$ for $2 \leq i \leq L$,

(H6) $L+1 \leq t$.

Theorem 2/2

Then

(a) We can determine a polynomial bundle map $R : \mathcal{E}^1 \times \mathbb{T}^d \to \mathcal{E}^1$ of degree not larger than L in $C_\Gamma^{\infty,r}(\mathcal{E}^1 \times \mathbb{T}^d, \mathcal{E}^1)$ such that $R(0,\theta) = 0$, $D_s R(0,\theta) = M_{1,1}(\theta)$ and a bundle map $W : B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d \to \ell^\infty(\mathbb{R}^n)$ in $C_\Gamma^{\Sigma_{t,r}}(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \ell^\infty(\mathbb{R}^n))$ such that

$$F(W(s,\theta),\theta) = W(R(s,\theta),\theta + \omega),$$

where $W(0,\theta) = 0$, $\Pi_{\mathcal{E}^1} D_s W(0,\theta) = \mathrm{Id}_{\mathcal{E}^1}$ and $\Pi_{\mathcal{E}^2} D_s W(0,\theta) = 0$.

(b) Furthermore, if there is $l \geq 2$ such that

$$(\mathcal{A}\,\mathrm{Spec}(A_{1,1}))^i \cap \mathcal{A}\,\mathrm{Spec}(A_{1,1}) = \emptyset, \quad l \leq i \leq L,$$

then we can choose R to be a polynomial bundle map of degree not larger than $l - 1$.

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$$M(\theta)W_k(\theta) = W_1(\theta+\omega)R_k(\theta) + W_k(\theta+\omega)R_1^{\otimes k}(\theta) + \hat{Q}_k(\theta), \quad k \geq 2,$$

where $\hat{Q}_k(\theta)$ comes inductively and depends on $D_x^j F(0,\theta)$, $j \leq k$, and $W_j(\theta)$ and $R_j(\theta)$ for $j < k$.

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After some changes of variables, projecting on $\mathcal{E}^i$ and using the fact that $M_{1,1}(\theta) = A_{1,1} + B_{1,1}(\theta)$, $M_{2,2}(\theta) = A_{2,2} + B_{2,2}(\theta)$ and $M_{1,2}(\theta) = B_{1,2}(\theta)$ we get

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$$\begin{aligned} A_{1,1}W_k^1(\theta)\left(A_{1,1}^{-1}\right)^{\otimes k} &+ \widetilde{\mathcal{T}}_1(W_k^1)(\theta) - W_k^1(\theta+\omega) \\ &= R_k(\theta)\left(M_{1,1}^{-1}(\theta)\right)^{\otimes k} - B_{1,2}(\theta)W_k^2(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} + Q_k^1(\theta), \end{aligned} \tag{8}$$

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We can solve this type of equations using Sylvester operators.

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Definition

Given a linear map $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$, we define the operator $\mathcal{R}_{j,A} : C^r_{L^k_\Gamma} \to C^r_{L^k_\Gamma}$, $1 \leq j \leq k$, by

$$\mathcal{R}_{j,A}(W)(\theta)(z_1, \dots, z_k) = W(\theta)(z_1, \dots, Az_j, \dots, z_k).$$

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Given a linear map $B \in L_\Gamma(\mathcal{F}, \mathcal{F})$, we define the operator $\mathcal{L}_B : C^r_{L^k_\Gamma} \to C^r_{L^k_\Gamma}$ by

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Given $A \in L_\Gamma(\mathcal{E}, \mathcal{E})$, $B \in L_\Gamma(\mathcal{F}, \mathcal{F})$ we define the Sylvester operator $\mathcal{S}_{B,A} : C^r_{L^k_\Gamma} \to C^r_{L^k_\Gamma}$ by

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Observe that $\|\widetilde{\mathcal{T}}_2(W_k^2)\|_{C_\Gamma^{k,r}}$ is small and all operators involved in the l.h.s. of this equation are linear.

Thus we can solve this equation if 1 is not in the spectrum of the Sylvester operator $\mathcal{S}_{A_{2,2},A_{1,1}^{-1}}$.

And solving it is easy

Proposition

We have the following inclusions of spectra

$$\begin{aligned}\operatorname{Spec}\left(\mathcal{S}_{B,A}, C^r_{L^k_\Gamma}\right) &\subseteq \operatorname{Spec}\left(\mathcal{L}_B, C^r_{L^k_\Gamma}\right)\cdot \operatorname{Spec}\left(\mathcal{R}_{1,A}, C^r_{L^k_\Gamma}\right)\cdot\ldots\cdot \operatorname{Spec}\left(\mathcal{R}_{k,A}, C^r_{L^k_\Gamma}\right)\\ &\subseteq \mathcal{A}\operatorname{Spec}_\Gamma(B)\cdot\mathcal{A}\left(\operatorname{Spec}_\Gamma(A)\right)^k.\end{aligned}$$

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$$\subseteq \mathcal{A}\,\mathrm{Spec}_\Gamma(B) \cdot \mathcal{A}\left(\mathrm{Spec}_\Gamma(A)\right)^k.$$

One of the key tools in proving this result is

Theorem 11.23 in [Rud91]

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two commuting elements in a Banach algebra. Then

$$\mathrm{Spec}(\mathfrak{a}\mathfrak{b}) \subseteq \mathrm{Spec}(\mathfrak{a}) \cdot \mathrm{Spec}(\mathfrak{b}).$$

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Hence

$$\operatorname{Spec}_\Gamma \mathcal{S}_{A_{2,2},A_{1,1}^{-1}} \subseteq \mathcal{A}\operatorname{Spec}_\Gamma(A_{2,2}) \cdot \left(\mathcal{A}\operatorname{Spec}_\Gamma(A_{1,1}^{-1})\right)^k,$$

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We have the following inclusions of spectra

$$\operatorname{Spec}\left(\mathcal{S}_{B,A}, C^r_{L^k_\Gamma}\right) \subseteq \operatorname{Spec}\left(\mathcal{L}_B, C^r_{L^k_\Gamma}\right) \cdot \operatorname{Spec}\left(\mathcal{R}_{1,A}, C^r_{L^k_\Gamma}\right) \cdot \ldots \cdot \operatorname{Spec}\left(\mathcal{R}_{k,A}, C^r_{L^k_\Gamma}\right)$$
$$\subseteq \mathcal{A}\operatorname{Spec}_\Gamma(B) \cdot \mathcal{A}\left(\operatorname{Spec}_\Gamma(A)\right)^k .$$

Hence

$$\operatorname{Spec}_\Gamma \mathcal{S}_{A_{2,2}, A_{1,1}^{-1}} \subseteq \mathcal{A}\operatorname{Spec}_\Gamma(A_{2,2}) \cdot \left(\mathcal{A}\operatorname{Spec}_\Gamma(A_{1,1}^{-1})\right)^k,$$

and by by Hypothesis (H5), the r.h.s of this expression does not contain values in the unit circle for $2 \leq k \leq L$.

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\begin{aligned}
(\mathcal{S}_{A_{1,1},A_{1,1}^{-1}} - \mathrm{Id})(W_k^1)(\theta+\omega) + \widetilde{\mathcal{T}}_1(W_k^1)(\theta+\omega)& \\
= -B_{1,2}(\theta)W_k^2(\theta)M_{1,1}^{-1}(\theta)^{\otimes k} + Q_k^1(\theta),& \qquad (10)
\end{aligned}
$$

i.e. if 1 is not in the spectrum of $(\mathcal{S}_{A_{1,1},A_{1,1}^{-1}}, C^r_{L^k_\Gamma})$, we set $W_k^1(\theta)$ equal to the solution of (10), and $R_k \equiv 0$.

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If this is not the case and we can not solve equation (10), we can solve equation (8) by setting $W_k^1 \equiv 0$ and R_k as

$$R_k(\theta) = B_{1,2}(\theta)W_k^2(\theta) - Q_k^1(\theta)(M_{1,1}(\theta))^{\otimes k}.$$

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Obviously there are many other possibilities to solve equations (8) and (9).

If we write $W^{\leq}(s,\theta) = \sum_{j=1}^{L} W_j(\theta) s^{\otimes j}$, we can write the parametrisation as

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So far we have determined $W^{\leq}$, and now it's the turn of finding

$$W^{>} \in C_{\Gamma}^{\Sigma_{t,r},L}\Big(B(0,1) \times \mathbb{T}^d \subset \mathcal{E}^1 \times \mathbb{T}^d, \ell^{\infty}(\mathbb{R}^n)\Big)$$

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We can determine $W^{>} \in C_{\Gamma}^{\Sigma_{t-1,r},L}$ using a fixed point argument, where the key fact is proving this series (or expressions conceptually similar) is convergent for $\eta \in C^{\Sigma_{t,r},L}$:

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$$\mathcal{G}^{-1}(\eta)(s,\theta) = \sum_{k=0}^{\infty}(M^{-1})^{[k]}(\theta)\eta(R^{[k]}(s,\theta),\theta+k\omega)$$

where superscript $[k]$ denotes a kind of iteration with angular shifts.

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And this holds (mostly) because η is L-flat, hence composing and/or differentiating the general term of this series always has enough factors (namely, L) to be contracting by the conditions on the spectrum of M_0.

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And this holds (mostly) because η is L-flat, hence composing and/or differentiating the general term of this series always has enough factors (namely, L) to be contracting by the conditions on the spectrum of M_0.

We lose one derivative when proving the fixed point argument, finding $W^{>} \in C^{\Sigma_{t-1,r,L}}$.

But we recover this derivative using a very similar fixed point argument for $D_x W^>$, proving $D_x W^> \in C^{\Sigma_{t-1,r},L-1}$

But we recover this derivative using a very similar fixed point argument for $D_x W^>$, proving $D_x W^> \in C^{\Sigma_{t-1,r},L-1}$ and thus $W^> \in C_\Gamma^{\Sigma_{t,r},L}$.

$\square$

With very similar tools to the ones introduced previously we can study normal forms for fixed points.

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Consider an open set U of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$ and

$$F : U \to \ell^\infty(\mathbb{R}^n)$$

a map such that $F(0) = 0$ and $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$. Let $A = DF(0)$, with $A \in L_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and consider its Γ-spectrum $\mathrm{Spec}_\Gamma(A)$.

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Theorem

In the previously described setting there exist polynomials $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ and $R \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ of degree at most r such that $H(0) = 0$, $DH(0) = \mathrm{Id}$ and

$$F \circ H(x) - H \circ R(x) = o(\|x\|^r)$$

and $R(x) = Ax + \sum_{j \in J} R_j x^{\otimes j}$ where

$$J = \{2 \le j \le r \mid \mathrm{Spec}_\Gamma(A)^j \cap \mathrm{Spec}_\Gamma(A) \neq \emptyset\}$$

and $R_j \in L^j_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$.

Corollary

Under the conditions of the previous theorem, if

$$\mathrm{Spec}_\Gamma(A)^j \cap \mathrm{Spec}_\Gamma(A) = \emptyset, \quad 2 \leq j \leq r,$$

then there exists a polynomial $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *such that*

$$F \circ H(x) - H \circ Ax = o(\|x\|^r).$$

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Using this normal form we can prove some Sternberg theorems, under several assumptions.

Theorem

Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$. Let $F : U \to \ell^\infty(\mathbb{R}^n)$ be a C^r_Γ map of the form $F = F_0 + F_1$ where F_0 is an uncoupled map and $F_0(0) = F_1(0) = 0$. Let $A = DF_0(0)$, $B = DF_1(0)$ and $M = A + B$. Assume that $A_{ij} = \mathfrak{a}\delta_{ij}$ with $\mathfrak{a} \in L(\mathbb{R}^n, \mathbb{R}^n)$. Let $\operatorname{Spec}(\mathfrak{a}) = \{\lambda_1, \dots, \lambda_n\}$. Assume furthermore

(H1) $0 < |\lambda_i| < 1, \quad 1 \leq i \leq n,$

(H2) $\lambda_i \neq \lambda^k, \quad k \in \mathbb{Z}^n, \quad |k| \geq 2, 1 \leq i \leq n.$

Let $\alpha = \min_i |\lambda_i|$, $\beta = \max_i |\lambda_i|$, $\nu = \frac{\log \alpha}{\log \beta}$ and $r_0 = [\nu] + 1$. Then if $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ with $r \geq r_0$ and $\|B\|_\Gamma$ is small enough, there exists $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ such that $R(0) = 0$, $DR(0) = \mathrm{Id}$ and

$$R \circ F = MR$$

in some neighborhood $U_1 \subseteq U$ of 0 in $\ell^\infty(\mathbb{R}^n)$.

Theorem

Under the conditions and notation of the previous theorem except hypothesis (H2), if $F \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *with* $r \geq r_0$ *and* $\|B\|_\Gamma$ *is small enough there exists a polynomial* $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *of degree not larger than* r_0 *and* $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *such that*

$$R(0) = 0, \quad DR(0) = \mathrm{Id}$$

and

$$R \circ F = H \circ R$$

in some neighborhood $U_1 \subset U$ *of 0.*

Theorem

Let U be an open set of $\ell^\infty(\mathbb{R}^n)$ such that $0 \in U$. Let $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ with $F(0) = 0$. Let $A = DF(0)$. Assume

(H1) $0 \notin \mathrm{Spec}_\Gamma(A)$ *and* $\mathrm{Spec}_\Gamma(A) \subset \mathbb{D}(0,1)$,

(H2) $\mathrm{Spec}_\Gamma(A) \cap (\mathrm{Spec}_\Gamma(A))^j = \emptyset$, $j \geq 2$.

Let $\alpha = \inf\{|\lambda| \mid \lambda \in \mathrm{Spec}_\Gamma(A)\}$, $\beta = \sup\{|\lambda| \mid \lambda \in \mathrm{Spec}_\Gamma(A)\}$, $\nu = \frac{\log \alpha}{\log \beta}$ *and* $r_0 = [\nu] + 1$. *Then if* $F \in C^r_\Gamma(U, \ell^\infty(\mathbb{R}^n))$ *with* $r \geq r_0$ *there exists* $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *such that* $R(0) = 0$, $DR(0) = \mathrm{Id}$ *and*

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Theorem

Under the conditions of the previous theorem, except condition (H2), if $F \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *with* $r \geq r_0$ *there exists a polynomial* $H \in C^\infty_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *of degree not larger than* r_0 *and* $R \in C^r_\Gamma(\ell^\infty(\mathbb{R}^n), \ell^\infty(\mathbb{R}^n))$ *such that*

$$R(0) = 0, \quad DR(0) = \mathrm{Id}$$

and

$$R \circ F = H \circ R$$

in some neighborhood $U_1 \subset U$ *of 0.*

Thanks for your attention

You can download this presentation from
http://www.maia.ub.es/~ruben/research.html

References I

Xavier Cabré, Ernest Fontich, and Rafael de la Llave.
The parameterization method for invariant manifolds. I. Manifolds associated to non-resonant subspaces.
Indiana Univ. Math. J., 52(2):283–328, 2003.

Ulrich Dehlinger.
Zur theorie der rekristallisation reiner metalle.
Annalen der Physik, 394(7):749–793, 1929.

Ernest Fontich, Rafael de la Llave, and Yannick Sire.
Construction of invariant whiskered tori by a parameterization method. I: Maps and flows in finite dimensions.
J. Differential Equations, 246(8):3136–3213, 2009.

Yacov I. Frenkel and Tatiana A. Kontorova.
The model of dislocation in solid body.
Zh. Eksp. Teor. Fiz, 8(1340), 1938.

Yacov I. Frenkel and Tatiana A. Kontorova.
On the theory of plastic deformation and twinning. ii.
Zh. Eksp. Teor. Fiz., 8:1340–1348, 1938.

References II

Yacov I. Frenkel and Tatiana A. Kontorova.
On the theory of plastic deformation and twinning. iii.
Zh. Eksp. Teor. Fiz., 8:1349–1358, 1938.

Yacov I. Frenkel and Tatiana A. Kontorova.
On the theory of plastic deformation and twinning.
Izv. Akad. Nauk, Ser. Fiz., 1:137–149, 1939.

Álex Haro and Rafael de la Llave.
A parameterization method for the computation of invariant tori and their whiskers in quasi-periodic maps: rigorous results.
J. Differential Equations, 228(2):530–579, 2006.

Frank C. Hoppensteadt.
An introduction to the mathematics of neurons, volume 6 of *Cambridge Studies in Mathematical Biology*.
Cambridge University Press, Cambridge, 1986.

Eugene M. Izhikevich.
Dynamical systems in neuroscience: the geometry of excitability and bursting.
Computational Neuroscience Series. MIT Press, Cambridge, MA, 2007.

References III

Miaohua Jiang and Rafael de la Llave.
Smooth dependence of thermodynamic limits of SRB-measures.
Comm. Math. Phys., 211(2):303–333, 2000.

Ludwig Prandtl.
A conceptual model to the kinetic theory of solid bodies.
Z. Angew. Math. Mech, 8:85–106, 1928.

Walter Rudin.
Functional analysis.
International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.