Sets approximating regions of instability

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Overview

In this talk we will sketch a result to approximate *non*-normality regions of one-parameter families of holomorphic functions by solving dynamically defined equations. In particular, this result gives an algorithm to draw bifurcation loci of one parameter families, like the (boundary of) the Mandelbrot set

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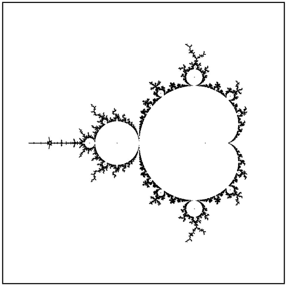

C. Henriksen's algorithm

It is a well known result that

$$\partial\mathcal{M} = \{\text{centres of hyperbolic components}\}',$$

i.e. centres of hyperbolic components are dense in the boundary of the Mandelbrot set. Remember that if $q_c(z) = z^2 + c$, then $c \in \mathbb{C}$ is a centre of a hyperbolic component of the Mandelbrot set of period n if $q_c^n(0) = 0$.

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Henriksen's idea

Numerically find points c such that $q_c^n(0) = 0$ (which are centres of hyperbolic components!) for $n = 1, \ldots, Nmax$.

Finding zeroes is usually done by a Newton method

Newton's method

As you already know, Newton's method is the iteration of Newton's step. Let $n \in \mathbb{N}$ be the target period, then the Newton iteration to find centres of hyperbolic components of period n is

$$N(c) = c - \frac{q_c^n(0)}{\partial_c(q_c^n(0))}$$

Stop the iteration if $\frac{q_c^n(0)}{\partial_c(q_c^n(0))}$ is small enough (or if $q_c^n(N(c))$ is small enough).

Henriksen's idea

Do just one step of Newton's method:

- For each point c (pixel in an image), compute

$$\widetilde{N}(c) = \frac{q_c^n(0)}{\partial_c(q_c^n(0))}$$

- If $\widetilde{N}(c)$ is suitably small (more precisely: if the numerator is very small or the denominator is very large) set this pixel black, white otherwise.
- Points where $|\widetilde{N}|$ is sufficiently small are close to a centre of hyperbolic component of period n (Newton-Kantorovich's theorem).

What can we do with two singular values?

Zakeri's cubic polynomial family

In [Zak99] S. Zakeri studied

$$P_c(z) = \lambda z \left(1 - \frac{1}{2}(1 + \frac{1}{c})z + \frac{1}{3c}z^2\right),$$

which has a persistent Siegel disk at 0 and 2 critical points, 1 and c.

How can we approximate the bifurcation locus of this family?

In [BF10], we studied

$$f_a(z) = \lambda a(e^{z/a}(z+1-a)+1-a), \quad \lambda = exp(\pi i(1+\sqrt{5})).$$

This family has a persistent Siegel disk around $f_a(0) = 0$ and two singularities, $c = -1$ a critical point and $v_a = \lambda a(a-1)$ an asymptotic value.

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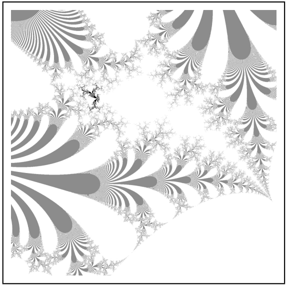
Its parameter plane looks like this:



For some parameters, the orbit of v_a lands inside the persistent Siegel disk, while the orbit of -1 accumulates in the boundary of the persistent Siegel disk

Experiment

Find a such that $f_a^n(v_a) = 0$. I.e. find points such that the asymptotic value lies in the center of a pre-image of the persistent Siegel disk.

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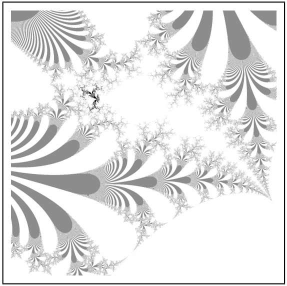
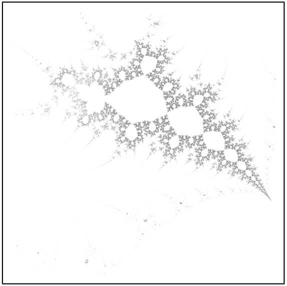
This is what we get.





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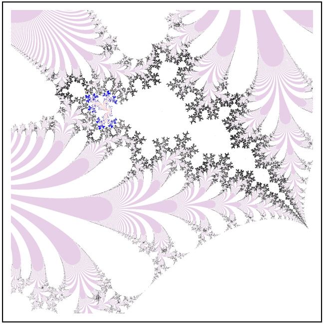

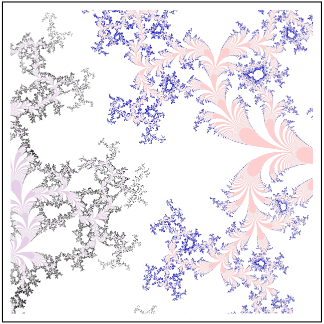

Left: In violet a such that the orbit of v_a escapes.
Right: In red a such that the orbit of -1 escapes.

Set-up: Definitions

Consider a one-parameter family of entire functions of degree at least 2 (except for a discrete set of points) depending analytically on the parameter c, $f_c : \mathbb{C} \to \mathbb{C}$ with a discrete set of singular values.

Given $f_c(z)$ as above and $w(c)$ an analytic function we define the sequence of functions g_n as

$$\{g_n(c)\}_{n\in\mathbb{N}} = \{g_n^{w(c)}(c)\}_{n\in\mathbb{N}} = \{f_c^n(w(c))\}_{n\in\mathbb{N}}$$

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Define the following set

$$\mathcal{B} = \mathcal{B}_{w(c)} = \{c \in \mathbb{C} \mid \{g_n(c)\}_{n\in\mathbb{N}} \text{ not normal in any neighborhood of } c\}$$

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This plays the role of the boundary of the Mandelbrot set when $f_c = q_c$ and the region of instability for a singular value for f_a.

We denote by $\mathrm{sing}_j(f)(c)$, $j = 1, 2, \ldots$ the set of functions with respect to $c \in \mathbb{C}$ which correspond to the singular values of f. Recall that this set is discrete by assumption.

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Given $\beta(c)$ an analytic function and $g_n(c) = f_c^n(w(c))$ as before, define the *set of n-centres* as

$$C_n = C_n^{\beta(c)} = \{c \in \mathbb{C} \,|\, g_n(c) = \beta(c), \quad n \in \mathbb{N}\}$$

and the *set of centres* as

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In the previous examples we choose $\beta(c) = 0$ for the centres of hyperbolic components in the Mandelbrot set and $\beta(c) = 0$ for the centres of Siegel discs and $\beta(c) = -1$ for the hyperbolic components in the BF example.

Define the *j-critical set* as

$$K_j = K_j^{\beta(c)} = \{c \in \mathbb{C} \mid \operatorname{sing}_j(f)(c) = \beta(c)\}$$

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This set comes from a technical requirement in the proof. For the Mandelbrot set, $K = \{0\}$. For the BF example, this set is harder to determine.

Remember:

- $\mathcal{B}$ is the set of not normality,
- C is the "set of centres",
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Theorem

$$\mathcal{B} \backslash (\mathcal{B} \cap K_\varepsilon) \subseteq C'$$

In other words, the set of *not* normality is contained in the limit set of the zeroes of $\{f^n(w(c)) - \beta(c)\}_{n \in \mathbb{N}}$, except at a neighborhood of K.

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In many cases $\mathcal{B} \cap K$ is empty.

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- Since we have two branches $\varphi^{\pm}(c)$ of the inverse function well-defined (as functions of c), we can construct an auxiliar family using these:

$$G_n(c) = \frac{f_c^n(w(c)) - \varphi^+_{\beta(c)}(c)}{f_c^n(w(c)) - \varphi^-_{\beta(c)}(c)}, \quad c \in U,\ n \in \mathbb{N}.$$

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- and this family avoids three points. $\square$

The algorithm is useful

There are very few algorithms that can draw accurate boundaries of bifurcation loci. One of the best known is the DEM/M algorithm (Distance Estimator Method, see [BDM$^+$88]) which uses the Green potential of the *Mandelbrot set* to bound the distance of points in $\mathbb{C}$ to $\mathcal{M}$. This algorithm needs the Green function... Which makes it unusable for entire transcendental families, since usually there is no way to define such a function.

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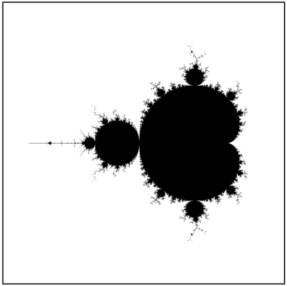

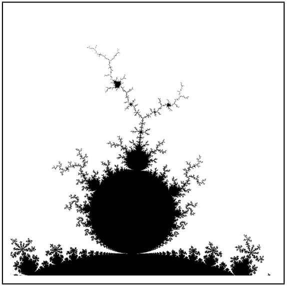

Our algorithm is just the application of a Newton method to the iterated family. We can just do one Newton step for each point or run a complete Newton method starting with each point to find all the zeroes up to a certain depth.

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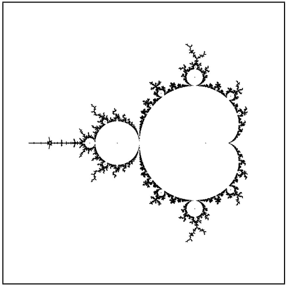

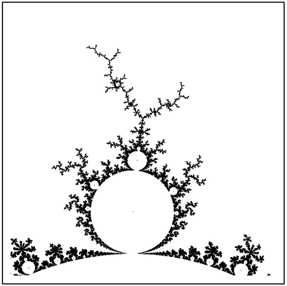

Our algorithm vs the DEM/M algorithm

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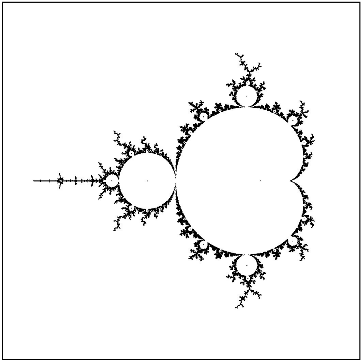

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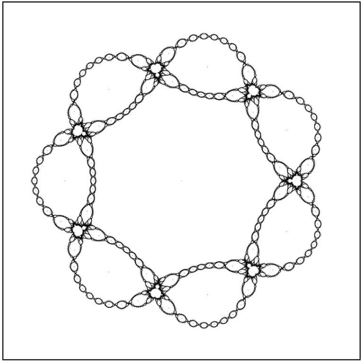

$f_c(z) = z^n + \frac{c}{z^d}$ with $d = n = 7$
(*Checkerboard Julia Sets for Rational Maps*, to appear Devaney, Blanchard, Cilingir, Cuzzocreo, Look and Russell)

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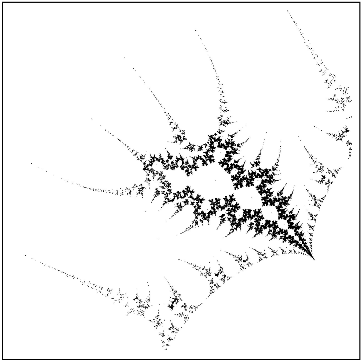

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Application

The theorem can be translated directly into an useful algorithm to approximate bifurcation loci.

Work in progress

- Accumulation properties (speed at which points accumulate to $\mathcal{B}$): can we find a theoretically "best" choice for the function $\beta(c)$ to get a best accumulation?

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Any suggestions will be greatly appreciated!

Further developments

- The multi-parameter case: can we extend this result to families f_A, with $A \in \mathbb{C}^n$? Higher dimensional Montel-like results look harder to apply.

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- Write a generic package: simplify the use of the implementation of the algorithm to make it useful out of the box.

The third one is "easy". But if you know something about higher dimension normality or weak convergence of measures, let's talk.

Thanks for your attention

You can download this presentation from
http://www.maia.ub.es/~ruben/research.html

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