EXERCISES (1): analysis of stochastic processes

Exercise 1

Consider the following process:

$$y(t) = e(t) + \frac{1}{5}e(t-1), \qquad e(t) \sim WN(0,2)$$

1) What kind of process is this?
2) Is the process a SSP?
3) Compute the process mean m_y and covariance function $\gamma_y(\tau)$.
4) Compute the spectrum $\Gamma_y(\omega)$ of the process.
5) What happens to the process mean, covariance function and spectrum when $e(t) \sim WN(1,2)$?

1)

The process is a MA(1):

$$y(t) = c_0 e(t) + c_1 e(t-1)$$

where, in this case, $c_0 = 1, c_1 = \frac{1}{5}$.

2)

Since the process is MA(1) we can conclude that it is a SSP too (recall the basic proprieties of MA(n) processes) . However, let's verify the stationarity.

OPERATORIAL REPRESENTATION (in positive powers of z):

$$y(t) = e(t) + \frac{1}{5}z^{-1}e(t)$$

$$y(t) = \left(1 + \frac{1}{5}z^{-1}\right)e(t)$$

$$y(t) = \left(\frac{z+\frac{1}{5}}{z}\right)e(t)$$

W(z)

Notice that the pole of *W(z)* is located in *z=0*. Thus *W(z)* is asymptotically stable. Since *e(t)* is a SSP (it is a WN indeed), then we can conclude that *y(t)* is a SSP too.

We can now compute its mean and covariance function.

3)

MEAN m_y

Since *W(z)* is asymptotically stable and *E[e(t)]=0*, we can conclude that *E[y(t)]=0* . Let's verify it.

$$E[y(t)] = E\left[e(t) + \frac{1}{5}e(t-1)\right] = E[e(t)] + \frac{1}{5}E[e(t-1)] = 0 + \frac{1}{5}\cdot 0 = 0$$

COVARIANCE FUNCTION $\gamma_y(\tau)$

- $\tau = 0$ (VARIANCE OF THE PROCESS)

$$\gamma_y(0) = E\left[\left(y(t) - m_y\right)^2\right] = E[y(t)^2] = E\left[\left(e(t) + \frac{1}{5}e(t-1)\right)^2\right]$$

$$= E\left[e(t)^2 + \frac{2}{5}e(t)e(t-1) + \frac{1}{25}e(t-1)^2\right]$$

$$= E[e(t)^2] + \cancel{\frac{2}{5}E[e(t)e(t-1)]} + \frac{1}{25}E[e(t-1)^2] = 2 + \frac{1}{25}\cdot 2$$

$$= \frac{52}{25}$$

- $\tau = 1$

$$\gamma_y(1) = E\big[\big(y(t) - m_y\big)\big(y(t-1) - m_y\big)\big] = E[y(t)y(t-1)]$$
$$= E\left[\left(e(t) + \frac{1}{5}e(t-1)\right)\left(e(t-1) + \frac{1}{5}e(t-2)\right)\right]$$
$$= E\left[\cancel{e(t)e(t-1)} + \frac{1}{5}\cancel{e(t)e(t-2)} + \frac{1}{5}e(t-1)^2 + \frac{1}{25}\cancel{e(t-1)e(t-2)}\right] = \frac{1}{5}\cdot 2 = \frac{2}{5}$$

- $\tau = 2$

$$\gamma_y(2) = E\big[\big(y(t) - m_y\big)\big(y(t-2) - m_y\big)\big] = E[y(t)y(t-2)]$$
$$= E\left[\left(e(t) + \frac{1}{5}e(t-1)\right)\left(e(t-2) + \frac{1}{5}e(t-3)\right)\right]$$
$$= E\left[\cancel{e(t)e(t-2)} + \frac{1}{5}\cancel{e(t)e(t-3)} + \frac{1}{5}\cancel{e(t-1)e(t-2)} + \frac{1}{25}\cancel{e(t-1)e(t-3)}\right] = 0$$

- $\tau = 3$

$$\gamma_y(3) = 0$$

$\ldots$

Notice that the following theorem holds: since the process is MA(1), the covariance function must be null for τ greater than 1.

Observe that, for a generic MA(1) process:

$$y(t) = c_0 e(t) + c_1 e(t-1), \qquad e(t) \sim WN(0, \lambda^2)$$

the covariance function can be simply computed as:

$$\gamma_y(0) = E\left[\left(y(t) - m_y\right)^2\right] = E[y(t)^2]$$
$$= c_0^2 E[e(t)^2] + 2c_0c_1E[e(t)e(t-1)] + c_1^2E[e(t-1)^2]$$
$$= \lambda^2(c_0^2 + c_1^2)$$

$$\gamma_y(1) = E\left[\left(y(t) - m_y\right)\left(y(t-1) - m_y\right)\right] = E[y(t)y(t-1)]$$
$$= E\left[\left(c_0e(t) + c_1e(t-1)\right)\left(c_0e(t-1) + c_1e(t-2)\right)\right] = c_0c_1\lambda^2$$

$$\gamma_y(\tau) = 0, \quad \forall\tau > 1$$

4)

SPECTRAL DENSITY $\Gamma_y(\omega)$

We exploit two methods:

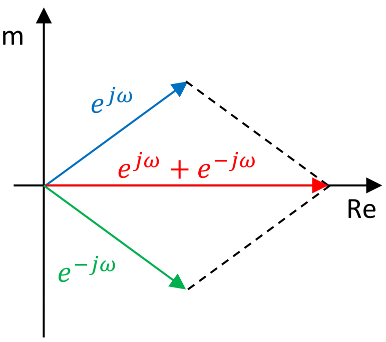
- FROM THE DEFINITION:

$$\Gamma_y(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau)e^{-j\omega\tau}$$
$$= \gamma_y(0)e^{-j\omega\cdot 0} + \gamma_y(1)e^{-j\omega\cdot 1} + \gamma_y(-1)e^{-j\omega\cdot(-1)} + 0$$
$$= \frac{52}{25} + \gamma_y(1)(e^{-j\omega} + e^{j\omega}) = \frac{52}{25} + \frac{2}{5}(e^{-j\omega} + e^{j\omega})$$

Recall that: $Ae^{j\theta} = Acos(\theta) + jAsin(\theta)$ (Euler representation of the complex exponential).

Thus:

$$e^{j\omega} + e^{-j\omega} = cos(\omega) + jsin(\omega) + \cos(\omega) - jsin(\omega) = 2\cos(\omega)$$



So:

$$\Gamma_y(\omega) = \frac{52}{25} + \frac{4}{5}\cos(\omega)$$

- FROM THE TRANSFER FUNCTION:

$$\Gamma_y(\omega) = W(e^{j\omega})W(e^{-j\omega})\lambda^2 = \frac{e^{j\omega} + \frac{1}{5}}{e^{j\omega}} \cdot \frac{e^{-j\omega} + \frac{1}{5}}{e^{-j\omega}} \cdot 2$$

$$= 2 \cdot \left(1 + \frac{1}{25} + \frac{1}{5}(e^{j\omega} + e^{-j\omega})\right) = \frac{52}{25} + \frac{4}{5}\cos(\omega)$$

Notice that the two methods lead to the same result.

The obtained spectrum is:

- a real function;
- a positive function;
- an even function;
- a periodic function (with period 2π, it depends on the cosine).

We draw the spectrum plot based on some simply computable points:

$$\Gamma_y(0) = \frac{52}{25} + \frac{4}{5} = \frac{72}{25}$$

$$\Gamma_y\left(\pm\frac{\pi}{2}\right) = \frac{52}{25}$$

$$\Gamma_y(\pm\pi) = \frac{52}{25} - \frac{4}{5} = \frac{32}{25}$$

Low frequencies are more relevant. We expect a very slow-varying signal.

5)

When $e(t) \sim WN(1,2)$ the stationarity propriety does not change (*e(t)* is again a white noise and the filter W(z) is asymptotically stable): *y(t)* is a SSP. But…

MEAN m_y

$$E[y(t)] = E\left[e(t) + \frac{1}{5}e(t-1)\right] = E[e(t)] + \frac{1}{5}E[e(t-1)] = 1 + \frac{1}{5}\cdot 1 = \frac{6}{5}$$

The process is MA(1) with non null-mean!!

COVARIANCE FUNCTION $\gamma_y(\tau)$

- $\tau = 0$ (VARIANCE OF THE PROCESS)

$$\gamma_y(0) = E\left[\left(y(t) - m_y\right)^2\right] = E\left[\left(y(t) - \frac{6}{5}\right)^2\right]$$

$$= E\left[\left(e(t) + \frac{1}{5}e(t-1) - \frac{6}{5}\right)^2\right]$$

$$= E\left[e(t)^2 + \frac{1}{25}e(t-1)^2 + \frac{36}{25} - \frac{12}{5}e(t) + \frac{2}{5}e(t)e(t-1) - \frac{12}{25}e(t-1)\right]$$

$$= E[e(t)^2] + \frac{1}{25}E[e(t-1)^2] + \frac{36}{25} - \frac{12}{5}E[e(t)] + \frac{2}{5}E[e(t)e(t-1)] - \frac{12}{25}E[e(t-1)]$$

Observe that:

$$E[e(t)^2] = E[e(t-1)^2] = E[(e(t) - m_e)^2] + m_e^2 = \gamma_e(0) + m_e^2 = 2 + 1 = 3$$

$$E[e(t)e(t-1)] = \gamma_e(1) + m_e^2 = 0 + 1 = 1$$

So:

$$\gamma_y(0) = \frac{26}{25}3 + \frac{36}{25} - \frac{12}{5} + \frac{2}{5} - \frac{12}{25} = \frac{52}{25}$$

The variance of the non-null mean MA(1) process is equal to the variance of the same MA(1) process with null mean. But using the classical formula to compute the covariance function of non-null mean processes involves too much computations!!!

So we solve the problem considering the **UNBIASED PROCESSES**.

Recall that:

$$m_y = \frac{6}{5}$$

$$m_e = 1$$

Define the new processes:

$$\tilde{y}(t) = y(t) - m_y = y(t) - \frac{6}{5}$$

$$\tilde{e}(t) = e(t) - m_e = e(t) - 1$$

Observe that:

$$m_{\tilde{y}} = E[\tilde{y}(t)] = E[y(t)] - m_y = 0$$

$$m_{\tilde{e}} = E[\tilde{e}(t)] = E[e(t)] - m_e = 0$$

The processes $\tilde{y}$ and $\tilde{e}$ are the unbiased processes. We have to derive the dynamical relation between these new signals:

$$\begin{cases} y(t) = \tilde{y}(t) + \frac{6}{5} \\ e(t) = \tilde{e}(t) + 1 \end{cases} \xRightarrow{y(t)=e(t)+\frac{1}{5}e(t-1)} \tilde{y}(t) + \frac{6}{5} = \tilde{e}(t) + 1 + \frac{1}{5}(\tilde{e}(t-1) + 1)$$

$$\tilde{y}(t) = \tilde{e}(t) + \frac{1}{5}\tilde{e}(t-1)$$

We can now evaluate the covariance function $\gamma_{\tilde{y}}(\tau)$. Observe that the structure of the process is equivalent to the zero-mean MA(1) process, so that:

$$\gamma_{\tilde{y}}(0) = \frac{52}{25}$$

$$\gamma_{\tilde{y}}(1) = \frac{2}{5}$$

$$\gamma_{\tilde{y}}(\tau) = 0, \qquad \forall \tau > 1$$

The following relation holds:

$$\gamma_{\tilde{y}}(\tau) = \gamma_y(\tau) \qquad \forall \tau$$

The covariance function of the process $y(t)$ is equal to the covariance function of the unbiased process $\tilde{y}(t)$. Thus, also <u>the spectrum does not change</u>!

In conclusion, **when you work with non-null mean processes**:

Process

y(t)=f(e(t))

Compute their means

$m_y = E[y(t)] \quad m_e = E[e(t)]$ (normally, the input mean is given)

CONSIDER THE UNBIASED PROCESSES

$\tilde{y}(t) = y(t) - m_y \quad \tilde{e}(t) = e(t) - m_e$

Derive the dynamical relation between $\tilde{y}(t)$ and $\tilde{e}(t)$

Exploit the propriety:

$$\gamma_{\tilde{y}}(\tau) = \gamma_y(\tau)$$

Exercise 2

Consider the following covariance functions:

1) $\gamma_1(0) = -1 \quad \gamma_1(1) = 0.5 \quad \gamma_1(-1) = 0.5 \quad \gamma_1(\tau) = 0, \ |\tau| \geq 2$

2) $\gamma_2(0) = 1 \quad \gamma_2(1) = 0.5 \quad \gamma_2(-1) = -0.5 \quad \gamma_2(\tau) = 0, \ |\tau| \geq 2$

3) $\gamma_3(0) = 2 \quad \gamma_3(1) = -0.2 \quad \gamma_3(-1) = -0.2 \quad \gamma_3(\tau) = 0, \ |\tau| \geq 2$

Find the underlying zero-mean SSP.

Obviously just the third one satisfies the proprieties for the covariance function of a SSP. In fact, recall that the covariance function should be:

- a positive function (not satisfied by number 1);
- an even function (not satisfied by number 2);
- less or equal than the variance

Since $\gamma_3(\tau) = 0, \quad \tau \geq 2$ we can conclude that the underlying process is a MA(1).

In the previous exercise we have proven that, for a MA(1) process, we have:

$$\gamma_y(0) = \lambda^2(c_0^2 + c_1^2)$$

$$\gamma_y(1) = c_0 c_1 \lambda^2$$

We have to find the coefficients c_0, c_1 of the MA(1) and the white noise variance λ^2. So we solve the following system of equations:

$$\begin{cases} \lambda^2(c_0^2 + c_1^2) = 2 \\ c_0 c_1 \lambda^2 = -0.2 \end{cases}$$

Notice that this system has three unknowns and two equations. There are infinite solutions, i.e. there are infinite MA(1) processes that have the same covariance function!!

So we fix one parameter, e.g. $c_0 = 1$. Thus (assuming $c_1 \neq 0$):

$$\begin{cases} \lambda^2(1 + c_1^2) = 2 \\ c_1\lambda^2 = -0.2 \end{cases} \Rightarrow \begin{cases} \lambda^2 = -\dfrac{0.2}{c_1} \\ c_1^2 + 10c_1 + 1 = 0 \end{cases} \Rightarrow \begin{cases} \lambda^2 = -\dfrac{0.2}{c_1} \\ c_1 = -5 \pm 2\sqrt{6} \end{cases}$$

We now choose for example $c_1 = -5 + 2\sqrt{6}$, so we get:

$$\begin{cases} \lambda^2 \approx 1.980 \\ c_1 = -5 + 2\sqrt{6} \approx -0.101 \end{cases}$$

Exercise 3

Consider the process:

$$y(t) = \theta y(t-1) + e(t), \qquad e(t) \sim WN(0,1), \quad \theta \in \mathbf{R}$$

1) What kind of process is this?
2) What are the values of θ which make the process a SSP?
   For $\theta = -1/4$:
3) Compute the process mean m_y and covariance function $\gamma_y(\tau)$ for ($|\tau| \leq$ 2).
4) Compute the spectrum $\Gamma_y(\omega)$ of the process.
5) What happens if $e(t) \sim WN(2,1)$?

1)

The process is an AR(1):

$$y(t) = a_1 y(t-1) + c_0 e(t)$$

where $a_1 = \theta, \;\; c_0 = 1.$

2)

Immediately, we cannot conclude anything about the stationarity of *y(t)*, since the process has 1 non trivial pole. We must verify the stationarity by writing its operatorial representation.

OPERATORIAL REPRESENTATION

$$y(t) = \theta z^{-1} y(t) + e(t)$$

$$(1 - \theta z^{-1}) y(t) = e(t)$$

$$y(t) = \frac{z}{z-\theta} e(t) \longrightarrow W(z)$$

The system has 1 trivial zero located at the origin. What about the pole?

Obviously (root of the denominator) it is located in: $z - \theta = 0 \Rightarrow z = \theta$.

The digital filter is asymptotically stable when the pole is inside the unit circle, so:

$$|\theta| < 1 \;\; (i.e. -1 < \theta < 1) \qquad \Leftrightarrow \qquad W(z) \text{ asymptotically stable}$$

Since *e(t)* is a SSP, then *y(t)* is a SSP if *W(z)* is asymptotically stable. Hence:

$$|\theta| < 1 \;\; (i.e. -1 < \theta < 1) \qquad \Leftrightarrow \qquad y(t) \text{ SSP}$$

In particular, when $\theta = -1/4$ *y(t)* is a SSP.

3)

Analysis for $\theta = -1/4$

MEAN m_y

Since *e(t)* is a zero-mean WN and *W(z)* is asymptotically stable, then*:*

$$m_y = E[y(t)] = 0$$

COVARIANCE FUNCTION $\gamma_y(\tau)$

$$\begin{aligned}\gamma_y(0) = E[y(t)^2] &= E\left[\left(-\frac{1}{4}y(t-1) + e(t)\right)^2\right] \\ &= \frac{1}{16}E[y(t-1)^2] + E[e(t)^2] - \frac{1}{2}E[y(t-1)e(t)] \\ &= \frac{1}{16}\gamma_y(0) + 1\end{aligned}$$

$$\left(1-\frac{1}{16}\right)\gamma_y(0)=1 \quad \Rightarrow \quad \gamma_y(0)=\frac{16}{15}$$

$$\gamma_y(1)=E[y(t)y(t-1)]=E\left[\left(-\frac{1}{4}y(t-1)+e(t)\right)y(t-1)\right]$$

$$=-\frac{1}{4}E[y(t-1)^2]+E[y(t-1)e(t)]=-\frac{1}{4}\gamma_y(0)=-\frac{4}{15}$$

$$\gamma_y(2)=E[y(t)y(t-2)]=E\left[\left(-\frac{1}{4}y(t-1)+e(t)\right)y(t-2)\right]$$

$$=-\frac{1}{4}E[y(t-1)y(t-2)]+E[y(t-2)e(t)]=-\frac{1}{4}\gamma_y(1)=\frac{1}{15}$$

[…]

Recursive equations…?

YULE-WALKER EQUATIONS (for AR(1) processes):

$$\gamma_y(0)=\frac{\lambda^2}{1-a_1^2}=\frac{1}{1-\frac{1}{16}}=\frac{16}{15}$$

$$\gamma_y(\tau)=a_1\gamma_y(\tau-1), \quad \tau\geq 1$$

$$\Downarrow$$

$$\gamma_y(1)=-\frac{1}{4}\gamma_y(0)=-\frac{4}{15}$$

$$\gamma_y(2)=-\frac{1}{4}\gamma_y(1)=\frac{1}{15}$$

[…]

Yule-Walker equations (for AR(1) processes) provide the same results of the computation using the definition of covariance function.

4)

SPECTRAL DENSITY $\Gamma_y(\omega)$

The better way of evaluating the spectral density of an AR(1) process is via the transfer function:

- FROM THE TRANSFER FUNCTION:

$$\Gamma_y(\omega) = W(e^{j\omega})W(e^{-j\omega})\lambda^2 = \frac{e^{j\omega}}{e^{j\omega}+\frac{1}{4}} \cdot \frac{e^{-j\omega}}{e^{-j\omega}+\frac{1}{4}} \cdot 1$$
$$= \frac{1}{1+\frac{1}{4}(e^{j\omega}+e^{-j\omega})+\frac{1}{16}} = \frac{1}{1+\frac{1}{2}\cos(\omega)+\frac{1}{16}}$$
$$= \frac{16}{16+8\cos(\omega)+1} = \frac{16}{17+8\cos(\omega)}$$

The obtained spectrum is:

- a real function;
- a positive function;
- an even function;
- a periodic function (with period 2π, it depends on the cosine).

We draw the spectrum plot based on some simply computable points:

$$\Gamma_y(0) = \frac{16}{17+8} = \frac{16}{25}$$

$$\Gamma_y\left(\pm\frac{\pi}{2}\right) = \frac{16}{17} = \frac{16}{17}$$

$$\Gamma_y(\pm\pi) = \frac{16}{17-8} = \frac{16}{9}$$

High frequencies are more relevant. We expect a rapidly-varying signal.

5)

Analysis of the process:

$$y(t) = -\frac{1}{4}y(t-1) + e(t), \quad e(t) \sim WN(2,1)$$

MEAN m_y

$$m_y = E[y(t)] = -\frac{1}{4}E[y(t-1)] + E[e(t)] = -\frac{1}{4}m_y + 2$$

$$\left(1 + \frac{1}{4}\right)m_y = 2 \quad \Rightarrow \quad m_y = 2 \cdot \frac{4}{5} = \frac{8}{5}$$

UNBIASED PROCESSES

Define the new processes:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y = y(t) - \frac{8}{5} \\ \tilde{e}(t) = e(t) - m_e = e(t) - 2 \end{cases} \Rightarrow \begin{cases} y(t) = \tilde{y} + \frac{8}{5} \\ e(t) = \tilde{e}(t) + 2 \end{cases}$$

Thus:

$$\tilde{y}(t) + \frac{8}{5} = -\frac{1}{4}\left(\tilde{y}(t-1) + \frac{8}{5}\right) + \tilde{e}(t) + 2, \quad \tilde{e}(t) \sim WN(0,1)$$

$$\tilde{y}(t) = -\frac{1}{4}\tilde{y}(t-1) + \tilde{e}(t) - \frac{8}{5} - \frac{2}{5} + 2, \quad \tilde{e}(t) \sim WN(0,1)$$

$$\tilde{y}(t) = -\frac{1}{4}\tilde{y}(t-1) + \tilde{e}(t), \quad \tilde{e}(t) \sim WN(0,1)$$

which is exactly the previous process.

COVARIANCE FUNCTION $\gamma_{\tilde{y}}(\tau)$ AND SPECTRUM $\Gamma_{\tilde{y}}(\omega)$

Notice that the structure of the process is the same of the initial case. So the covariance function of $\tilde{y}(t)$ is equal to the covariance function of the initial case. As a consequence the spectrum does not change.

Exercise 4

Consider the following process:

$$y(t) = \frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t), \quad e(t) \sim WN(0,1)$$

1) What kind of process is this?
2) Is the process a SSP?
3) Compute the process mean m_y and covariance function $\gamma_y(\tau)$ for $|\tau| \leq 3$.

1)

The process is an AR(2):

$$y(t) = a_1 y(t-1) + a_2 y(t-2) + c_0 e(t)$$

where $a_1 = \frac{1}{2}, \quad a_2 = -\frac{1}{4}, \quad c_0 = 1.$

2)

Immediately, we cannot conclude anything about the stationarity of *y(t)*, since the process has *2* non trivial poles. We must verify the stationarity by writing its operatorial representation.

OPERATORIAL REPRESENTATION

$$y(t) = \frac{1}{2}z^{-1}y(t) - \frac{1}{4}z^{-2}y(t) + e(t)$$

$$\left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)y(t) = e(t)$$

$$y(t) = \underbrace{\frac{z^2}{z^2 - \frac{1}{2}z + \frac{1}{4}}}_{W(z)} e(t)$$

We have 2 trivial zeroes in the origin ($z^2 = 0$). What about the poles?

$$z^2 - \frac{1}{2}z + \frac{1}{4} = 0$$

$$z_{1,2} = \frac{1}{4} \pm \frac{\sqrt{\frac{1}{4} - 1}}{2} = \frac{1}{4} \pm i\frac{\sqrt{3}}{4}$$

Two **complex conjugate poles**! In order to prove the asymptotical stability of *W(z)* we have to verify that:

$$|z_1| = |z_2| < 1$$

Hence:

$$|z_1| = |z_2| = \left|\frac{1}{4} \pm i\frac{\sqrt{3}}{4}\right| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{4}{16}} = \frac{1}{2} < 1$$

So the digital filter *W(z)* is asymptotically stable.

Since *e(t)* is a SSP (it's a WN) and *W(z)* is asymptotically stable, then *y(t)* is a S.S.P.

3)

MEAN m_y

Since *e(t)* is a zero-mean White Noise and *W(z)* is asymptotically stable, we can conclude that:

$$m_y = E[y(t)] = 0$$

COVARIANCE FUNCTION $\gamma_y(\tau)$

- $\tau = 0$

$$\begin{aligned}\gamma_y(0) = E[y(t)^2] = E\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right)^2\right] \\ = \frac{1}{4}E[y(t-1)^2] + \frac{1}{16}E[y(t-2)^2] + E[e(t)^2] \\ - \frac{1}{4}E[y(t-1)y(t-2)] + \cancel{E[y(t-1)e(t)]} - \frac{1}{2}\cancel{E[y(t-2)e(t)]} \\ = \frac{1}{4}\gamma_y(0) + \frac{1}{16}\gamma_y(0) + 1 - \frac{1}{4}\gamma_y(1)\end{aligned}$$

Notice that $\gamma_y(0)$ depends on $\gamma_y(1)$. We cannot solve this equation alone. But if we write the expression for $\gamma_y(1)$:

- $\tau = 1$

$$\begin{aligned}\gamma_y(1) = E[y(t)y(t-1)] = E\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right)y(t-1)\right] \\ = \frac{1}{2}E[y(t-1)^2] - \frac{1}{4}E[y(t-2)y(t-1)] + \cancel{E[e(t)y(t-1)]} \\ = \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)\end{aligned}$$

we observe that $\gamma_y(1)$ depends on $\gamma_y(0)$. So this system of equations:

$$\begin{cases}\gamma_y(0) = \frac{1}{4}\gamma_y(0) + \frac{1}{16}\gamma_y(0) + 1 - \frac{1}{4}\gamma_y(1) \\ \gamma_y(1) = \frac{1}{2}\gamma_y(0) - \frac{1}{4}\gamma_y(1)\end{cases}$$

has just one possible solution, since we have two equations and two unknowns, i.e. $\gamma_y(1)$ and $\gamma_y(0)$. Hence:

$$\begin{cases} \gamma_y(1) = \frac{2}{5}\gamma_y(0) \\ \gamma_y(0) = \frac{1}{4}\gamma_y(0) + \frac{1}{16}\gamma_y(0) + 1 - \frac{1}{4}\cdot\frac{2}{5}\gamma_y(0) \end{cases}$$

$$\begin{cases} \gamma_y(1) = \frac{2}{5}\gamma_y(0) \\ \left(1 - \frac{1}{4} - \frac{1}{16} + \frac{1}{10}\right)\gamma_y(0) = 1 \end{cases}$$

$$\begin{cases} \gamma_y(1) = \frac{2}{5}\cdot\frac{80}{63} = \frac{32}{63} \\ \gamma_y(0) = \frac{80}{63} \end{cases}$$

The computation of the next terms of the covariance function is simpler:

- $\tau = 2$

$$\gamma_y(2) = E[y(t)y(t-2)] = E\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right)y(t-2)\right]$$
$$= \frac{1}{2}E[y(t-1)y(t-2)] - \frac{1}{4}E[y(t-2)^2] + \cancel{E[e(t)y(t-2)]}$$
$$= \frac{1}{2}\gamma_y(1) - \frac{1}{4}\gamma_y(0) = \frac{16}{63} - \frac{20}{63} = -\frac{4}{63}$$

- $\tau = 3$

$$\gamma_y(3) = E[y(t)y(t-3)] = E\left[\left(\frac{1}{2}y(t-1) - \frac{1}{4}y(t-2) + e(t)\right)y(t-3)\right]$$
$$= \frac{1}{2}\gamma_y(2) - \frac{1}{4}\gamma_y(1) + \cancel{E[e(t)y(t-3)]} = -\frac{2}{63} - \frac{8}{63} = -\frac{10}{63}$$

Recursive equations…?

YULE-WALKER EQUATIONS (for AR(2) processes):

Generic AR(2)process:

$$y(t) = a_1 y(t-1) + a_2 y(t-2) + e(t) \quad \text{TIME DOMAIN}$$

$$y(t) = \frac{z^2}{z^2 - a_1 z - a_2} e(t) \quad \text{OPERATORIAL REPRESENTATION}$$

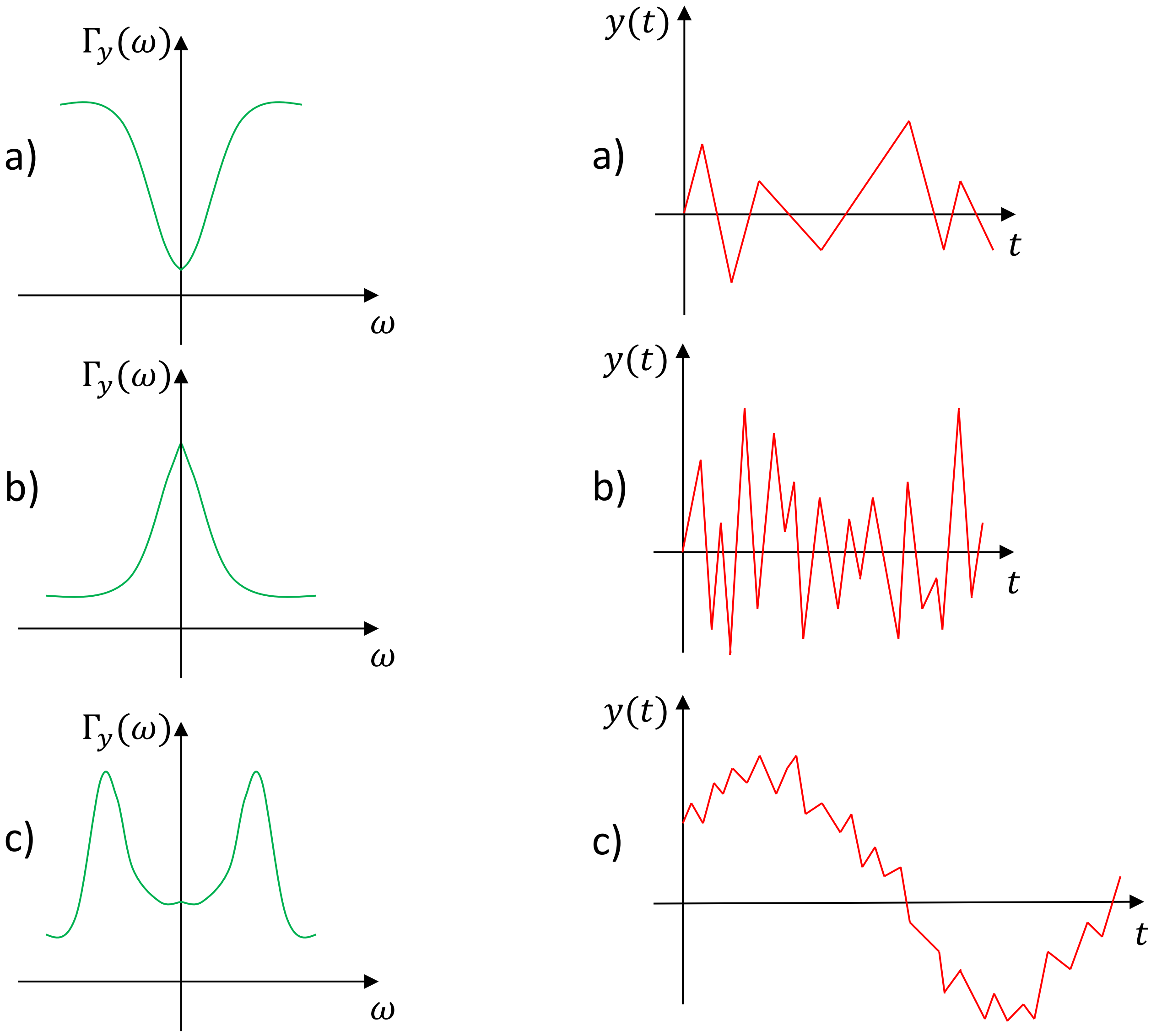
$$\Downarrow$$

$\gamma_y(0)$: write it as a function of $\gamma_y(1)$, starting from the process time domain representation

$$\gamma_y(\tau) = a_1 \gamma_y(\tau - 1) + a_2 \gamma_y(\tau - 2), \quad \tau \geq 1$$

Exercise 5

Relate the spectra of the signals (on the left) to their realization in time (on the right):



The TIME-BEHAVIOUR b) is related to the SPECTRUM a): the high frequencies prevail on the low frequencies.
The TIME-BEHAVIOUR a) is related to the SPECTRUM b): the high frequencies are damped, so that the signal is characterized by low-frequencies components.
The TIME-BEHAVIOUR c) is related to the spectrum c): the resonance causes one frequency to prevail on the others. The time behaviour is then a sort of sinuosidal signal dipped in noise.

Exercise 6

Consider the process:

$$y(t) = \frac{1}{3}y(t-1) + \eta(t) - 2\eta(t-1), \qquad \eta(t) \sim WN(1,9)$$

1) What kind of process is this?
2) Is the process a SSP?
3) Compute the process mean m_y and covariance function $\gamma_y(\tau)$ for $|\tau| \leq 3$.
4) Compute the spectrum $\Gamma_y(\omega)$ of the process.

1)

The process is an ARMA(1,1) with non-null mean:

$$y(t) = a_1 y(t-1) + c_0\eta(t) + c_1\eta(t-1), \quad \eta(t) \sim WN(1,9)$$

where $a_1 = \frac{1}{3}, \ \ c_0 = 1, \ \ c_1 = -2$. Notice that the noise is a white noise with non-null mean.

2)

OPERATORIAL REPRESENTATION

$$y(t) = \frac{1}{3}z^{-1}y(t) + \eta(t) - 2z^{-1}\eta(t)$$

$$\left(1 - \frac{1}{3}z^{-1}\right)y(t) = (1 - 2z^{-1})\eta(t)$$

$$y(t) = \underbrace{\frac{z-2}{z-\frac{1}{3}}}_{W(z)}\eta(t)$$

The zero (z=2) is located outside the unit circle: non-minimum phase system.

The pole ($z = 1/3$) is inside the unit circle: **W(z) is asymptotically stable**.

Since $\eta(t)$ is a SSP and *W(z)* is asymptotically stable, then *y(t)* is a SSP.

3)

MEAN m_y

$$m_y = E[y(t)] = E\left[\frac{1}{3}y(t-1) + \eta(t) - 2\eta(t-1)\right] = \frac{1}{3}m_y + 1 - 2$$

$$\left(1 - \frac{1}{3}\right)m_y = -1 \quad \Rightarrow \quad m_y = -\frac{3}{2}$$

UNBIASED PROCESS

We will work with the unbiased processes $\tilde{y}(t)$ and $\tilde{\eta}(t)$ defined as:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y = y(t) + \dfrac{3}{2} \\ \tilde{\eta}(t) = \eta(t) - m_\eta = \eta(t) - 1 \end{cases}$$

$$\Downarrow$$

$$\begin{cases} y(t) = \tilde{y}(t) - \dfrac{3}{2} \\ \eta(t) = \tilde{\eta}(t) + 1 \end{cases}$$

$$\Downarrow \quad y(t) = \frac{1}{3}y(t-1) + \eta(t) - 2\eta(t-1)$$

$$\tilde{y}(t) - \frac{3}{2} = \frac{1}{3}\left(\tilde{y}(t-1) - \frac{3}{2}\right) + \tilde{\eta}(t) + 1 - 2(\tilde{\eta}(t-1) + 1)$$

$$\tilde{y}(t) = \frac{1}{3}\tilde{y}(t-1) + \tilde{\eta}(t) - 2\tilde{\eta}(t-1) \cancel{+ \frac{3}{2} - \frac{1}{2} + 1 - 2}$$

$$\boxed{\Rightarrow \quad \tilde{y}(t) = \frac{1}{3}\tilde{y}(t-1) + \tilde{\eta}(t) - 2\tilde{\eta}(t-1), \quad \eta(t) \sim WN(0,9)}$$

COVARIANCE FUNCTION $\gamma_{\tilde{y}}(\tau)$

- $\tau = 0$

$$\begin{aligned}\gamma_{\tilde{y}}(0) &= E[\tilde{y}(t)^2] \\ &= \frac{1}{9}E[\tilde{y}(t-1)^2] + E[\tilde{\eta}(t)^2] + 4E[\tilde{\eta}(t-1)^2] \\ &+ \frac{2}{3}\cancel{E[\tilde{y}(t-1)\tilde{\eta}(t)]} - \frac{4}{3}E[\tilde{y}(t-1)\tilde{\eta}(t-1)] - \cancel{4E[\tilde{\eta}(t)\tilde{\eta}(t-1)]} \\ &= \frac{1}{9}\gamma_{\tilde{y}}(0) + 9 + 36 - \frac{4}{3}E[\tilde{y}(t-1)\tilde{\eta}(t-1)]\end{aligned}$$

We compute the term $E[\tilde{y}(t-1)\tilde{\eta}(t-1)]$ separately:

$$\begin{aligned}E[\tilde{y}(t-1)\tilde{\eta}(t-1)] &= E\left[\left(\frac{1}{3}\tilde{y}(t-2) + \tilde{\eta}(t-1) - 2\tilde{\eta}(t-2)\right)\tilde{\eta}(t-1)\right] \\ &= \cancel{\frac{1}{3}E[\tilde{y}(t-2)\tilde{\eta}(t-1)]} + E[\tilde{\eta}(t-1)\tilde{\eta}(t-1)] \\ &- \cancel{2E[\tilde{\eta}(t-2)\tilde{\eta}(t-1)]} = 9\end{aligned}$$

So:

$$\gamma_{\tilde{y}}(0) = \frac{1}{9}\gamma_{\tilde{y}}(0) + 9 + 36 - \frac{4}{3}\cdot 9$$

$$\left(1 - \frac{1}{9}\right)\gamma_{\tilde{y}}(0) = 33$$

$$\gamma_{\tilde{y}}(0) = \frac{297}{8}$$

- $\tau = 1$

$$\begin{aligned}
\gamma_{\tilde{y}}(1) = E[\tilde{y}(t)\tilde{y}(t-1)] &= E\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{\eta}(t) - 2\tilde{\eta}(t-1)\right)\tilde{y}(t-1)\right] \\
&= \frac{1}{3}E[\tilde{y}(t-1)^2] + \cancel{E[\tilde{\eta}(t)\tilde{y}(t-1)]} - 2E[\tilde{\eta}(t-1)\tilde{y}(t-1)] \\
&= \frac{1}{3}\gamma_{\tilde{y}}(0) - 2\cdot 9 = \frac{1}{3}\frac{297}{8} - 18 = -\frac{45}{8}
\end{aligned}$$

- $\tau = 2$

$$\begin{aligned}
\gamma_{\tilde{y}}(2) = E[\tilde{y}(t)\tilde{y}(t-2)] &= E\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{\eta}(t) - 2\tilde{\eta}(t-1)\right)\tilde{y}(t-2)\right] \\
&= \frac{1}{3}E[\tilde{y}(t-1)\tilde{y}(t-2)] + \cancel{E[\tilde{\eta}(t)\tilde{y}(t-2)]} \\
&- 2\cancel{E[\tilde{\eta}(t-1)\tilde{y}(t-2)]} = \frac{1}{3}\gamma_{\tilde{y}}(1) = -\frac{15}{8}
\end{aligned}$$

- $\tau = 3$

$$\begin{aligned}
\gamma_{\tilde{y}}(3) = E[\tilde{y}(t)\tilde{y}(t-3)] &= E\left[\left(\frac{1}{3}\tilde{y}(t-1) + \tilde{\eta}(t) - 2\tilde{\eta}(t-1)\right)\tilde{y}(t-3)\right] \\
&= \frac{1}{3}E[\tilde{y}(t-1)\tilde{y}(t-3)] + \cancel{E[\tilde{\eta}(t)\tilde{y}(t-3)]} \\
&- 2\cancel{E[\tilde{\eta}(t-1)\tilde{y}(t-3)]} = \frac{1}{3}\gamma_{\tilde{y}}(2) = -\frac{5}{8}
\end{aligned}$$

[…]

4)

SPECTRAL DENSITY $\Gamma_y(\omega)$

Since $\gamma_{\tilde{y}}(\tau) = \gamma_y(\tau)$ then:

$$\Gamma_{\tilde{y}}(\omega) = \Gamma_y(\omega)$$

Thus we can work with the undiased processes $\tilde{y}(t), \tilde{\eta}(t)$.

- FROM THE TRANSFER FUNCTION $W(z)$

$$\begin{aligned}\Gamma_y(\omega) = \Gamma_{\tilde{y}}(\omega) = \left|W(e^{j\omega})\right|^2 \cdot \Gamma_{\tilde{\eta}}(\omega) = W(e^{j\omega})W(e^{-j\omega}) \cdot 9 \\ = \frac{e^{j\omega} - 2}{e^{j\omega} - \frac{1}{3}} \cdot \frac{e^{-j\omega} - 2}{e^{-j\omega} - \frac{1}{3}} \cdot 9 = \frac{1 + 4 - 2(e^{j\omega} + e^{-j\omega})}{1 + \frac{1}{9} - \frac{1}{3}(e^{j\omega} + e^{-j\omega})} \cdot 9 \\ = \frac{5 - 4\cos(\omega)}{\frac{10}{9} - \frac{2}{3}\cos(\omega)} \cdot 9 = 81 \cdot \frac{5 - 4\cos(\omega)}{10 - 6\cos(\omega)}\end{aligned}$$

We draw the spectrum plot based on some simply computable points:

$$\Gamma_y(0) = 81 \cdot \frac{5-4}{10-6} = \frac{81}{4} = 20.25$$

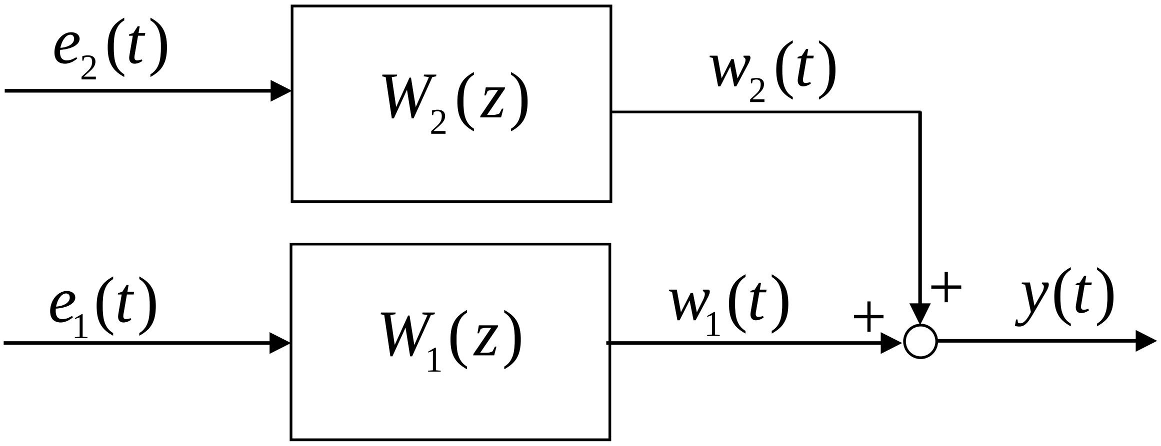
$$\Gamma_y\left(\pm\frac{\pi}{2}\right) = 81 \cdot \frac{5}{10} = \frac{81}{2} = 40.5$$

$$\Gamma_y(\pm\pi) = 81 \cdot \frac{5+4}{10+6} = \frac{729}{16} \approx 45.56$$

High frequencies are more relevant. We expect a very rapidly-varying signal.

Exercise 7

Let us consider the process (generated as the sum of two asymptotically stable processes):



where:

$e_1(t) \sim WN(0, \lambda_1^2)$

$e_2(t) \sim WN(0, \lambda_2^2)$

$e_1(t) \perp e_2(t) \Leftrightarrow \quad E[e_1(t)e_2(t-\tau)] = 0, \quad \forall t, \tau$

Evaluate $\gamma_y(\tau)$ and $\Gamma_y(\omega)$.

The process is characterized by two inputs.

First of all, we evaluate $\gamma_y(\tau)$ by applying straightforwardly the definition:

$$\begin{aligned}\gamma_y(\tau) = E[y(t)y(t-\tau)] &= E\big[\big(w_1(t) + w_2(t)\big)\big(w_1(t-\tau) + w_2(t-\tau)\big)\big] \\ &= E[w_1(t)w_1(t-\tau)] + E[w_1(t)w_2(t-\tau)] + E[w_2(t)w_1(t-\tau)] \\ &\quad + E[w_2(t)w_2(t-\tau)]\end{aligned}$$

Recall that:

$$E[w_1(t)w_1(t-\tau)] = \gamma_{w_1}(\tau)$$

$$E[w_2(t)w_2(t-\tau)] = \gamma_{w_2}(\tau)$$

Hence:

$$\gamma_y(\tau) = \gamma_{w_1}(\tau) + \gamma_{w_2}(\tau) + E[w_1(t)w_2(t-\tau)] + E[w_2(t)w_1(t-\tau)]$$

Notice that: $e_1(t) \perp e_2(t) \quad \Leftrightarrow \quad w_1(t) \perp w_2(t)$

Hypothesis: the processes are MA(∞):

$$\begin{aligned} w_1(t) &= c_{01}e_1(t) + c_{11}e_1(t-1) + c_{21}e_1(t-2) + \cdots \\ w_2(t) &= c_{02}e_2(t) + c_{12}e_2(t-1) + c_{22}e_2(t-2) + \cdots \end{aligned}$$

Then:

$$\begin{aligned} &E[w_1(t)w_2(t-\tau)] \\ &\qquad = E[(c_{01}e_1(t) + c_{11}e_1(t-1) + \cdots)(c_{02}e_2(t-\tau) \\ &\qquad + c_{12}e_2(t-1-\tau) + \cdots)] = 0 \end{aligned}$$

If these terms are null, the covariance function of *y(t)* is:

$$\gamma_y(\tau) = \gamma_{w_1}(\tau) + \gamma_{w_2}(\tau)$$

Turn now to the problem of the spectral density.

By definition we have:

$$\Gamma_y(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau) e^{-j\omega\tau}$$

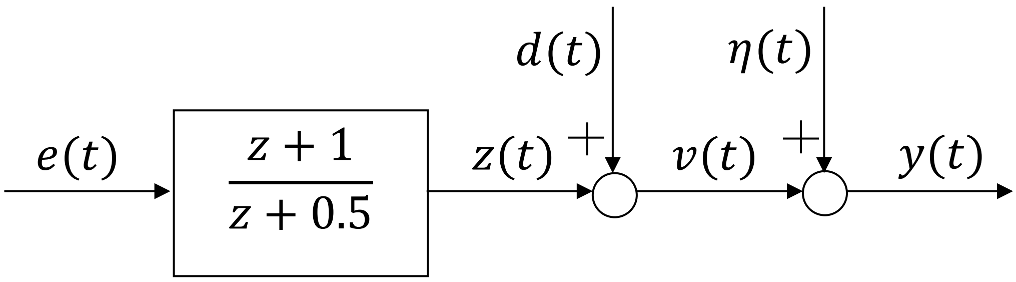
But, as proven before, the covariance function of *y(t)* is just the sum of the covariance functions of the two inputs, so that:

$$\Gamma_y(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau) e^{-j\omega\tau} = \sum_{\tau=-\infty}^{+\infty} \gamma_{w_1}(\tau) e^{-j\omega\tau} + \sum_{\tau=-\infty}^{+\infty} \gamma_{w_2}(\tau) e^{-j\omega\tau}$$

$$= \Gamma_{w_1}(\omega) + \Gamma_{w_2}(\omega) = \left|W_1(e^{j\omega})\right|^2 \lambda_1^2 + \left|W_2(e^{j\omega})\right|^2 \lambda_2^2$$

Exercise 8

Consider the process $y(t)$ generated according to the following scheme:



Evaluate the spectrum when:

a) $d(t) \perp e(t) \perp \eta(t),\;\; d(t) \sim WN(0,1),\; e(t) \sim WN(0,2),\; \eta(t) \sim WN(0,1)$

b) $d(t) = -e(t-1),\;\; d(t) \perp \eta(t),\;\; e(t) \sim WN(0,1),\;\; \eta(t) \sim WN(0,1)$

a)

First of all, we verify that *z(t)* is a SSP. *z(t)* is the steady-state output of the digital filter fed by the WN *e(t)*. So we verify that the digital filter is asymptotically stable:

$$W(z) = \frac{z+1}{z+0.5} \quad \xrightarrow{\textbf{POLES}} \quad z = -0.5 \Rightarrow |z| < 1$$

Thus the digital filter W(z) is asymptotically stable. We can conclude that *z(t)* is a SSP.

Since $d(t) \perp e(t) \perp \eta(t)$, we can conclude that:

$$\Gamma_y(\omega) = \Gamma_z(\omega) + \Gamma_d(\omega) + \Gamma_\eta(\omega)$$

That is:

$$\Gamma_y(\omega) = \Gamma_z(\omega) + 1 + 1 = \Gamma_z(\omega) + 2$$

We have to compute the spectrum of z. This is quite simple:

$$\Gamma_z(\omega) = W(e^{j\omega})W(e^{-j\omega})\Gamma_e(\omega) = \frac{e^{j\omega}+1}{e^{j\omega}+0.5} \cdot \frac{e^{-j\omega}+1}{e^{-j\omega}+0.5} \cdot 2$$
$$= \frac{1+1+(e^{j\omega}+e^{-j\omega})}{1+0.25+0.5(e^{j\omega}+e^{-j\omega})} \cdot 2 = \frac{4+4\cos(\omega)}{1.25+\cos(\omega)}$$

Hence:

$$\Gamma_y(\omega) = \frac{4+4\cos(\omega)}{1.25+\cos(\omega)} + 2 = \frac{6.5+6\cos(\omega)}{1.25+\cos(\omega)}$$

We draw the spectrum plot based on some simply computable points:

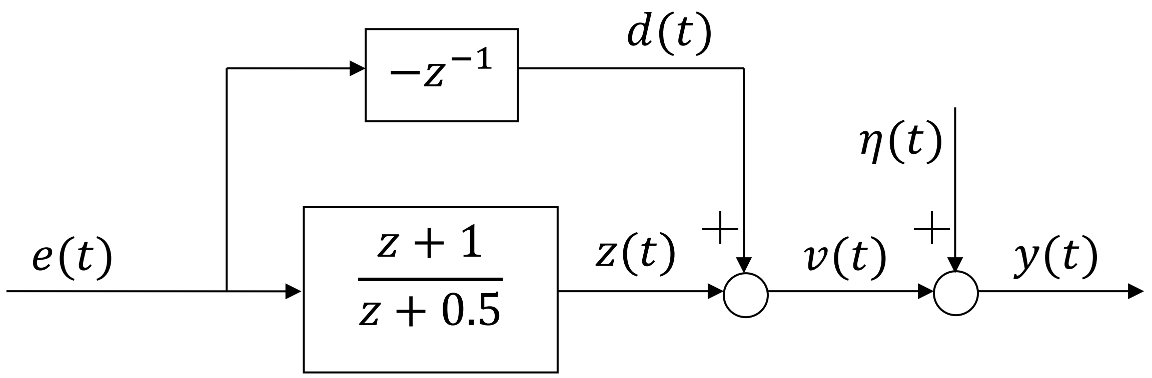
$$\Gamma_y(0) = \frac{6.5+6}{1.25+1} \approx 5.6$$

$$\Gamma_y\left(\pm\frac{\pi}{2}\right) = \frac{6.5}{1.25} = 5.2$$

$$\Gamma_y(\pm\pi) = \frac{6.5-6}{1.25-1} = 2$$

b)

Notice that, if $d(t) = -e(t-1)$, the scheme can be seen as:

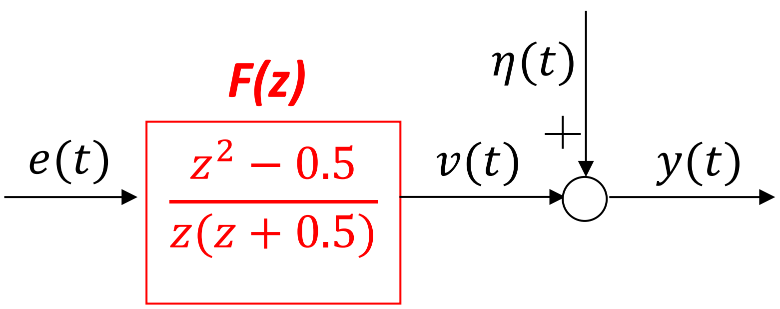


We can compute the transfer function from *e(t)* to *v(t)*:

$$F(z) = \frac{v(t)}{e(t)} = \frac{z+1}{z+0.5} - z^{-1} = \frac{z^2 + z - z - 0.5}{z(z+0.5)} = \frac{z^2 - 0.5}{z(z+0.5)}$$

$$\Rightarrow \quad v(t) = F(z)e(t) = \frac{z^2 - 0.5}{z(z+0.5)}e(t)$$

Thus, the previous scheme becomes:



Notice that *F(z)* is asymptotically stable since its poles z_1 and z_2 are inside the unit circle:

$$z_1 = 0, \quad z_2 = -0.5$$

Thus *v(t)* is a SSP.

Since $e(t) \perp \eta(t)$ and *F(z)* is asymptotically stable, we can conclude that:

$$\Gamma_y(\omega) = \Gamma_v(\omega) + \Gamma_\eta(\omega) = \Gamma_v(\omega) + 1$$

Now we compute $\Gamma_v(\omega)$:

$$\Gamma_v(\omega) = F(e^{j\omega})F(e^{-j\omega})\Gamma_e(\omega) = \frac{e^{j2\omega} - 0.5}{e^{j2\omega} + 0.5e^{j\omega}} \frac{e^{-j2\omega} - 0.5}{e^{-j2\omega} + 0.5e^{-j\omega}} \cdot 1$$

$$= \frac{1 + 0.25 - 0.5(e^{j2\omega} + e^{-j2\omega})}{1 + 0.25 + 0.5(e^{j\omega} + e^{-j\omega})} = \frac{1.25 - \cos(2\omega)}{1.25 + \cos(\omega)}$$

So:

$$\Gamma_y(\omega) = \frac{1.25 - \cos(2\omega)}{1.25 + \cos(\omega)} + 1 = \frac{2.5 + \cos(\omega) - \cos(2\omega)}{1.25 + \cos(\omega)}$$

We draw the spectrum plot based on some simply computable points:

$$\Gamma_y(0) = \frac{2.5 + 1 - 1}{1.25 + 1} \approx 1.1$$

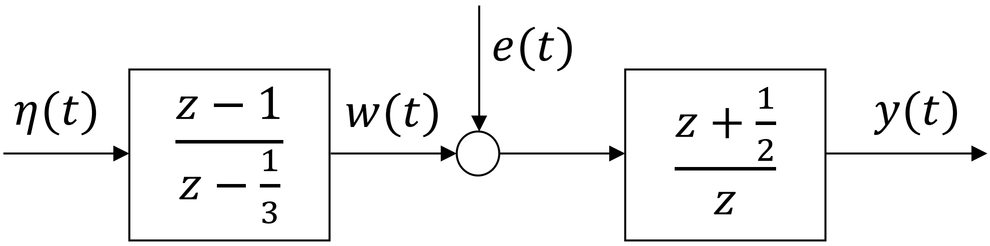
$$\Gamma_y\left(\pm\frac{\pi}{2}\right) = \frac{2.5 + 0 + 1}{1.25} = 2.8$$

$$\Gamma_y(\pm\pi) = \frac{2.5 - 1 - 1}{1.25 - 1} = 2$$

The spectrum has a maximum between $\pm\frac{\pi}{2}$ and $\pm\pi$.

Exercise 9

Consider the process $y(t)$ generated according to the following scheme:



where $\eta(t) = 2 \quad \forall t$ and $e(t) \sim WN(1,1)$.

1) What kind of process is this?
2) Is the process a SSP?
3) Compute the process mean m_y and covariance function $\gamma_y(\tau)$.
4) Compute the spectrum $\Gamma_y(\omega)$ of the process.

1)

For the frequency response theorem, if an asymptotically stable dynamical system is fed by a sinusoid:

$$u(t) = Asin(\Omega t + \varphi)$$

then the steady-state output is a sinusoid, with the same frequency, whose amplitude and phase are "modified" according to the magnitude and phase of the frequency response of the system evaluated at the frequency Ω:

$$y(t) = A\left|F(e^{j\Omega})\right| \sin\left(\Omega t + \varphi + \angle F(e^{j\Omega})\right)$$

In this case, $\boldsymbol{\eta(t)}$ **is a constant exogenous signal filtered by the dynamical system:**

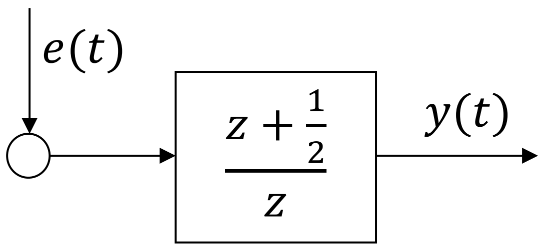
$$F(z) = \frac{z-1}{z-\frac{1}{3}}$$

Notice that $\eta(t) = 2 = 2\sin\left(0 \cdot t + \frac{\pi}{2}\right)$. Then, if we apply the previous theorem, we have that the output *w(t)* can be evaluated as:

$$w(t) = 2\left|\frac{e^{j0}-1}{e^{j0}-\frac{1}{3}}\right| \sin\left(0 \cdot t + \frac{\pi}{2} + \angle\frac{e^{j0}-1}{e^{j0}-\frac{1}{3}}\right) = 0$$

In fact, in this case, the zero of *F(z)* is located on the unitary circle and, in particular, in *z=1* (derivator); this means that the frequency associated to the zero is 0, i.e. the frequency of a constant signal, because $e^{j\omega} = 1 \Longrightarrow \omega = 0$. Since the zeroes are the values of *z* which make the transfer function null, then, in this case, a constant input is totally rejected; in other words, when the input has a frequency component equal to the frequency of a zero of the system, then the output of the system does not have that frequency component.

Since $w(t) = 0$ the system is actually:



We recognize an MA(1) process:

$$zy(t) = \left(z + \frac{1}{2}\right)e(t)$$

$$y(t) = e(t) + \frac{1}{2}e(t-1), \quad e(t) \sim WN(1,1)$$

2)

Since the filter has only 1 trivial pole in the origin, and *e(t)* is a SSP, then *y(t)* is a SSP.

3)

MEAN m_y

$$m_y = E[y(t)] = E[e(t)] + \frac{1}{2}E[e(t-1)] = 1 + \frac{1}{2} = \frac{3}{2}$$

UNBIASED PROCESSES $\tilde{y}(t), \tilde{e}(t)$

$$\begin{cases} \tilde{y}(t) = y(t) - \dfrac{3}{2} \\ \tilde{e}(t) = e(t) - 1 \end{cases}$$

Hence:

$$\tilde{y}(t) = \tilde{e}(t) + \frac{1}{2}\tilde{e}(t-1), \quad \tilde{e}(t) \sim WN(0,1)$$

COVARIANCE FUNCTION $\gamma_y(\tau)$

- $\tau = 0$

$$\gamma_y(0) = \gamma_{\tilde{y}}(0) = E[\tilde{y}(t)^2]$$
$$= E[\tilde{e}(t)^2] + \frac{1}{4}E[\tilde{e}(t-1)^2] + E[\tilde{e}(t)\tilde{e}(t-1)] = 1 + \frac{1}{4}$$
$$= \frac{5}{4}$$

- $\tau = 1$

$$\gamma_y(1) = \gamma_{\tilde{y}}(1) = E[\tilde{y}(t)\tilde{y}(t-1)]$$
$$= E\left[\left(\tilde{e}(t) + \frac{1}{2}\tilde{e}(t-1)\right)\left(\tilde{e}(t-1) + \frac{1}{2}\tilde{e}(t-2)\right)\right]$$
$$= \frac{1}{2}E[\tilde{e}(t-1)^2] = \frac{1}{2}$$

Since the process is a MA(1), then $\gamma_y(\tau) = 0,\ \forall|\tau| > 1$.

4)

SPECTRUM $\Gamma_y(\omega)$

The spectrum can be easily computed from the definition:

$$\Gamma_y(\omega) = \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau)e^{-j\omega\tau} = \gamma_y(0) + \gamma_y(1)e^{-j\omega} + \gamma_y(-1)e^{j\omega}$$
$$= \frac{5}{4} + \frac{1}{2}2\cos(\omega) = \frac{5}{4} + \cos(\omega)$$

Exercise 10

Consider the process *y(t)* generated according to the following scheme:

$$\xrightarrow{e(t)} \boxed{W(z)} \xrightarrow{y(t)}$$

where:

$$W(z) = 1 + az^{-1}, \quad -1 < a < 1$$

$$e(t) \sim WN(0,1)$$

Find the value of *a* which minimizes the variance of *y*.

First of all, we compute the spectrum of the process.

$$\Gamma_y(\omega, a) = (1 + ae^{-j\omega})(1 + ae^{j\omega}) \cdot 1 = 1 + a^2 + 2\text{a}\cos(\omega)$$

By definition, the variance of the process is:

$$\gamma_y(0, a) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \Gamma_y(\omega, a) d\omega = \frac{1}{2\pi} \int_{-\pi}^{+\pi} (1 + a^2 + 2\text{a}\cos(\omega)) d\omega$$
$$= \frac{1}{2\pi} 2\pi(1 + a^2) = 1 + a^2$$

So the value of *a* which minimizes the variance can be computed by solving the equation:

$$\frac{d\gamma_y(0, a)}{da} = 0$$

This leads to:

$$2a = 0 \quad \Rightarrow \quad a = 0 \quad \Rightarrow \quad \gamma_y(0,0) = 1$$

Notice that, for *a=0*, the process becomes:

$$y(t) = e(t)$$

So the minimum value of the process variance can be obtained when the process is equal to the white noise. If the process is generated as a “filtered” white noise, then its variance will be greater than the one of the input for any real value of a.