EXERCISES (2): prediction

Exercise 1

Consider the process:

$$y(t) = \frac{1}{4}y(t-1) + e(t) + 2e(t-1), \quad e(t) \sim WN\left(0, \frac{1}{4}\right)$$

1) What kind of process is this? Is *y(t)* a SSP?

2) Compute the process mean m_y and covariance function $\gamma_y(\tau)$ for $\tau = 0, \pm 1$.

3) Derive the expression for the 1-step optimal predictor from the available data.

4) What is the value of the 1-step prediction error variance?

5) Derive the expression for the 2-step optimal predictor from the available data.

6) What is the value of the 2-step prediction error variance?

7) Given the following observations:

$$y(1) = 1, \;\; y(2) = \frac{1}{2}, \quad y(3) = -\frac{1}{2}, \quad y(4) = 0, \quad y(5) = -\frac{1}{2}$$

compute $\hat{y}(6|5)$ and $\hat{y}(7|5)$.

1)

The process is an ARMA(1,1).

OPERATORIAL REPRESENTATION

$$y(t) = W(z)e(t) = \frac{1 + 2z^{-1}}{1 - \frac{1}{4}z^{-1}}e(t) = \frac{z + 2}{z - \frac{1}{4}}e(t)$$

The pole of *W(z)* is $z = \frac{1}{4}$ and it is inside the unit circle, so the digital filter *W(z)* is asymptotically stable. Moreover $e(t)$ is a SSP. Thus *y(t)* is a SSP.

2)

MEAN m_y

$$m_y = E[y(t)] = E\left[\frac{1}{4}y(t-1) + e(t) + 2e(t-1)\right]$$

$$\frac{3}{4}m_y = 0$$

$$\Rightarrow \quad m_y = 0$$

COVARIANCE FUNCTION $\gamma_y(\tau)$

- $\tau = 0$

$$\gamma_y(0) = E[y(t)^2]$$

$$= E\left[\frac{1}{16}y(t-1)^2 + e(t)^2 + 4e(t-1)^2 + \frac{1}{2}y(t-1)e(t) + y(t-1)e(t-1) + 4e(t)e(t-1)\right]$$

$$= \frac{1}{16}\gamma_y(0) + \frac{1}{4} + 4\frac{1}{4} + E[y(t-1)e(t-1)]$$

where:

$$E[y(t-1)e(t-1)] = E\left[\left(\frac{1}{4}y(t-2) + e(t-1) + 2e(t-2)\right)e(t-1)\right] = \frac{1}{4}$$

So:

$$\gamma_y(0) - \frac{1}{16}\gamma_y(0) = \frac{5}{4} + \frac{1}{4} \implies \gamma_y(0) = \frac{6}{4}\frac{16}{15} = \frac{8}{5}$$

- $\tau = \pm 1$

$$\begin{aligned}\gamma_y(-1) = \gamma_y(1) = E[y(t)y(t-1)]\\ &= E\left[\left(\frac{1}{4}y(t-1) + e(t) + 2e(t-1)\right)y(t-1)\right]\\ &= \frac{1}{4}\gamma_y(0) + 2\frac{1}{4} = \frac{2}{5} + \frac{1}{2} = \frac{9}{10}\end{aligned}$$

$\boxed{3)}$

1-STEP OPTIMAL PREDICTOR

First of all, we have to check if *y(t)* is a canonical representation.

CANONICAL REPRESENTATION

y(t) is not expressed in a canonical form; in fact:

$$W(z) = \frac{C(z)}{A(z)} = \frac{z+2}{z-\frac{1}{4}}$$

has the zero outside the unit circle (roots of *C(z)* outside the unit circle).

In order to replace the zero outside the unit circle, we apply the all pass filter:

$$T(z) = 2\frac{z+\frac{1}{2}}{z+2}$$

So:

$$W_1(z) = W(z)T(z) = 2\frac{z+2}{z-\frac{1}{4}}\frac{z+\frac{1}{2}}{z+2} = 2\frac{z+\frac{1}{2}}{z-\frac{1}{4}}$$

Here *C(z)* is still not monic. Thus, we define:

$$\eta(t) = 2e(t) \quad \Rightarrow \quad e(t) = \frac{1}{2}\eta(t)$$

Notice that:

$$m_\eta = E[\eta(t)] = 2E[e(t)] = 0$$

$$\gamma_\eta(0) = E[\eta(t)^2] = E[4e(t)^2] = 4\gamma_e(0) = 1$$

So:

$$\eta(t) \sim WN(0,1)$$

We can conclude that:

$$y(t) = W_1(z)e(t) = 2\frac{z+\frac{1}{2}}{z-\frac{1}{4}}\frac{1}{2}\eta(t) = \frac{z+\frac{1}{2}}{z-\frac{1}{4}}\eta(t) = \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-1}}\eta(t)$$

This is the canonical representation of the process *y(t)*.

Compute 1 step of the polynomial long division:

$$
\begin{array}{cc|cc}
\multicolumn{2}{c}{C(z)} & \multicolumn{2}{c}{A(z)} \\
1 & \frac{1}{2}z^{-1} & 1 & -\frac{1}{4}z^{-1} \\ \cline{3-4}
-1 & \frac{1}{4}z^{-1} & 1 \;\; E(z) & \\ \cline{1-2}
 & \frac{3}{4}z^{-1} & & \\
 & R(z) & &
\end{array}
$$

Thus we have:

$$y(t) = \left(E(z) + \frac{R(z)}{A(z)}\right)\eta(t) = E(z)\eta(t) + \frac{R(z)}{A(z)}\eta(t) = \eta(t) + \frac{\frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}\eta(t)$$

While the first term is unpredictable with the information at time *t-1*, the second term is totally predictable since it depends on $\eta(t-1)$. Thus the 1-step optimal predictor from the noise is:

$$\hat{y}(t|t-1) = \frac{\frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}}\eta(t)$$

Since:

$$y(t) = \frac{C(z)}{A(z)}\eta(t) \quad \Rightarrow \quad \eta(t) = \frac{A(z)}{C(z)}y(t)$$

So:

$$\hat{y}(t|t-1) = \frac{\frac{3}{4}z^{-1}}{1 - \frac{1}{4}z^{-1}} \frac{1 - \frac{1}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}} \; y(t) = \frac{\frac{3}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}}y(t)$$

Time-domain representation:

$$\hat{y}(t|t-1) + \frac{1}{2}\hat{y}(t-1|t-2) = \frac{3}{4}y(t-1)$$

So the 1-step optimal predictor is:

$$\hat{y}(t|t-1) = -\frac{1}{2}\hat{y}(t-1|t-2) + \frac{3}{4}y(t-1)$$

Notice that this is obviously equal to (one-step forward shifting):

$$\hat{y}(t+1|t) = -\frac{1}{2}\hat{y}(t|t-1) + \frac{3}{4}y(t)$$

… a simpler way to find the 1-step predictor of an ARMA process:

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)}\ y(t) = \frac{1 + \frac{1}{2}z^{-1} - 1 + \frac{1}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}} y(t)$$

$$= \frac{\frac{3}{4}z^{-1}}{1 + \frac{1}{2}z^{-1}} y(t)$$

which gives the previously computed predictor.

$\boxed{4)}$

The 1-step prediction error variance is simply given by the unpredictable part:

$$E[\varepsilon(t)^2] = E\left[\left(y(t) - \hat{y}(t|t-1)\right)^2\right] = E\left[\left(E(z)\eta(t)\right)^2\right] = E[\eta(t)^2] = 1$$

$\boxed{5)}$

2-STEP OPTIMAL PREDICTOR

We compute 2 steps of the polynomial long division:

$$\begin{array}{ccc|cc}
1 & \frac{1}{2}z^{-1} & & 1 & -\frac{1}{4}z^{-1} \\ \cline{4-5}
-1 & \frac{1}{4}z^{-1} & & 1 + \frac{3}{4}z^{-1} & \\
 & & & & \textit{E(z)} \\ \cline{1-3}
 & \frac{3}{4}z^{-1} & & & \\
 & -\frac{3}{4}z^{-1} & +\frac{3}{16}z^{-2} & & \\ \cline{1-3}
\textit{R(z)} & +\frac{3}{16}z^{-2} & & &
\end{array}$$

Notice that:

$$R(z) = z^{-k}\tilde{R}(z)$$

Since, in this case, *k=2*, we have:

$$R(z) = z^{-2}\tilde{R}(z) \quad \Rightarrow \quad \tilde{R}(z) = \frac{3}{16}$$

The 2-step optimal predictor is then given by:

$$\hat{y}(t|t-2) = \frac{\tilde{R}(z)}{C(z)}y(t-2) = \frac{\frac{3}{16}}{1+\frac{1}{2}z^{-1}}y(t-2)$$

Time-domain representation:

$$\hat{y}(t|t-2) + \frac{1}{2}\hat{y}(t-1|t-3) = \frac{3}{16}y(t-2)$$

$$\Rightarrow \quad \hat{y}(t|t-2) = -\frac{1}{2}\hat{y}(t-1|t-3) + \frac{3}{16}y(t-2)$$

6)

The 2-step prediction error variance is:

$$E\left[\left(y(t) - \hat{y}(t|t-2)\right)^2\right] = E\left[\left(E(z)\eta(t)\right)^2\right] = E\left[\left(\eta(t) + \frac{3}{4}\eta(t-1)\right)^2\right]$$
$$= 1 + \frac{9}{16} = \frac{25}{16}$$

Observe the variances:

PROCESS VARIANCE: $\gamma_y(0) = \frac{8}{5} = 1.6$

1-STEP PREDICTION ERROR VARIANCE: $E[\varepsilon(t)^2] = 1$

2-STEP PREDICTION ERROR VARIANCE: $E[\varepsilon(t)^2] = \frac{25}{16} \approx 1.56$

The variance of the prediction error tends to the variance of the process, since the k-step predictor tends to the process mean for $k \to \infty$ (the best future prediction when the future is far away is the mean value of the process):

$$\boxed{E\left[\left(y(t)-\hat{y}(t|t-k)\right)^2\right] \xrightarrow[k\to\infty]{} E\left[\left(y(t)-m_y\right)^2\right] = \text{process variance}}$$

7)

COMPUTATION OF $\hat{y}(6|5)$ AND $\hat{y}(7|5)$

Using the previous predictors, we can compute $\hat{y}(6|5)$ and $\hat{y}(7|5)$. Obviously, for the computation of $\hat{y}(6|5)$ we will make use of the 1-step predictor:

$$\hat{y}(t|t-1) = -\frac{1}{2}\hat{y}(t-1|t-2) + \frac{3}{4}y(t-1)$$

while for the computation of $\hat{y}(7|5)$ we will adopt the 2-step predictor:

$$\hat{y}(t|t-2) = -\frac{1}{2}\hat{y}(t-1|t-3) + \frac{3}{16}y(t-2)$$

- $\hat{y}(6|5)$

INITIALIZATION: $\hat{y}(1|0) = E[y(t)] = 0$

$\hat{y}(2\|1) = -\frac{1}{2}\hat{y}(1\|0) + \frac{3}{4}y(1) = \frac{3}{4}$
$\hat{y}(3\|2) = -\frac{1}{2}\hat{y}(2\|1) + \frac{3}{4}y(2) = -\frac{1}{2}\frac{3}{4} + \frac{3}{4}\frac{1}{2} = 0$
$\hat{y}(4\|3) = -\frac{1}{2}\hat{y}(3\|2) + \frac{3}{4}y(3) = -\frac{1}{2}\,0 - \frac{3}{4}\frac{1}{2} = -\frac{3}{8}$
$\hat{y}(5\|4) = -\frac{1}{2}\hat{y}(4\|3) + \frac{3}{4}y(4) = \frac{1}{2}\frac{3}{8} + \frac{3}{4}\,0 = \frac{3}{16}$
$\hat{y}(6\|5) = -\frac{1}{2}\hat{y}(5\|4) + \frac{3}{4}y(5) = -\frac{1}{2}\frac{3}{16} - \frac{3}{4}\frac{1}{2} = -\frac{3}{32} - \frac{3}{8} = -\frac{15}{32}$

- $\hat{y}(7|5)$

INITIALIZATION: $\hat{y}(2|0) = E[y(t)] = 0$

$\hat{y}(3\|1) = -\frac{1}{2}\hat{y}(2\|0) + \frac{3}{16}y(1) = \frac{3}{16}$
$\hat{y}(4\|2) = -\frac{1}{2}\hat{y}(3\|1) + \frac{3}{16}y(2) = -\frac{1}{2}\frac{3}{16} + \frac{3}{16}\frac{1}{2} = 0$
$\hat{y}(5\|3) = -\frac{1}{2}\hat{y}(4\|2) + \frac{3}{16}y(3) = -\frac{1}{2}\ 0 - \frac{3}{16}\frac{1}{2} = -\frac{3}{32}$
$\hat{y}(6\|4) = -\frac{1}{2}\hat{y}(5\|3) + \frac{3}{16}y(4) = \frac{1}{2}\frac{3}{32} + \frac{3}{4}\ 0 = \frac{3}{64}$
$\hat{y}(7\|5) = -\frac{1}{2}\hat{y}(6\|4) + \frac{3}{16}y(5) = -\frac{1}{2}\frac{3}{64} - \frac{3}{16}\frac{1}{2} = -\frac{3}{128} - \frac{3}{32} = -\frac{15}{128}$

The effect of the initialization rapidly vanishes.

Exercise 2

Consider the following ARMA(1,1) process:

$$y(t) = 0.5y(t-1) + e(t) - 2e(t-1), \quad e(t) \sim WN(0,2)$$

1) Is the process a SSP?

2) Compute the predictors $\hat{y}(t|t-k)$ for $k = 1,2$

3) Compute the 1-2 step prediction error variances.

1)

OPERATORIAL REPRESENTATION

$$(1 - 0.5z^{-1})y(t) = (1 - 2z^{-1})e(t)$$

So:

$$\frac{y(t)}{e(t)} = W(z) = \frac{1 - 2z^{-1}}{1 - 0.5z^{-1}} = \frac{z - 2}{z - 0.5}$$

The pole $(z = 0.5)$ is inside the unit circle, so the digital filter is asymptotically stable. Moreover, *e(t)* is a SSP. We can conclude that *y(t)* is a SSP too.

OBSERVATION

Notice that the pole is the reciprocal of the zero: the transfer function *W(z)*, except from the gain, has the structure of an all pass filter:

$$T(z) = \frac{1}{a}\frac{z + a}{z + \frac{1}{a}}$$

It is an all-pass filter with gain equal to -2:

$$y(t) = W(z)e(t) = -2\left[\left(-\frac{1}{2}\right)\frac{z - 2}{z - \frac{1}{2}}\right]e(t) = -2T(z)e(t)$$

where *T(z)* is an all-pass filter with gain 1.

Consider the white noise $\eta(t)$, derived from the original noise *e(t)*:

$$\eta(t) = -2e(t)$$

Its mean and variance are given by:

$$m_\eta = E[\eta(t)] = -2E[e(t)] = 0$$

$$\gamma_\eta(0) = E[\eta(t)^2] = 4E[e(t)^2] = 4 \cdot 2 = 8$$

So:

$$\eta(t) \sim WN(0{,}8)$$

If we replace *e(t)* in the process function with $e(t) = -0.5\,\eta(t)$, we have that:

$$y(t) = 0.5y(t-1) - 0.5\eta(t) + \eta(t-1), \quad \eta(t) \sim WN(0{,}8)$$

The operatorial representation becomes:

$$W_1(z) = -0.5\frac{1-2z^{-1}}{1-0.5z^{-1}} = -0.5\frac{z-2}{z-0.5} = T(z)$$

which is an all pass filter!!!

2)

1-STEP AND 2-STEP PREDICTORS

Since the process *y(t)* is the steady-state output of an all-pass filter fed by the white noise $\eta(t)$, we can conclude that *y(t)* has the same spectrum of the white noise $\eta(t)$. But the white noise is totally unpredictable. So the optimal *k*-step predictor is the trivial predictor, that is the expected value of the process *y(t)*, which is the expected value of the noise $\eta(t)$:

$$\hat{y}(t|t-k) = E[y(t)] = 0 \qquad \forall k$$

3)

Thus, the k-step prediction error variance is simply the variance of the process, i.e. the variance of the white noise $\eta(t)$:

$$E[\varepsilon(t)^2] = E\left[\left(y(t) - \hat{y}(t|t-k)\right)^2\right] = E[y(t)^2] = E[\eta(t)^2] = 8 \quad \forall k$$

Exercise 3

The following discrete-time state-space model is given:

$$\begin{cases} x_1(t+1) = 0.8x_1(t) + u(t) + e(t) \\ x_2(t+1) = x_1(t) + 0.5x_2(t) + 4e(t) \\ y(t) = x_2(t) \end{cases}$$

where $e(t) \sim WN(0,1)$ and $u(t)$ is an exogenous input (e.g. control variable).

1) Find the corresponding ARMAX model:

$$A(z)y(t) = B(z)u(t) + C(z)e(t)$$

2) Find the canonical representation of the process

3) Find the optimal 2-step predictor and the associated prediction error variance.

1)

ARMAX MODEL

The ARMAX model can be simply computed:

$$\begin{cases} (z-0.8)x_1(t) = u(t) + e(t) \\ (z-0.5)x_2(t) = x_1(t) + 4e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} x_1(t) = \dfrac{1}{z-0.8}u(t) + \dfrac{1}{z-0.8}e(t) \\ x_2(t) = \dfrac{1}{z-0.5}x_1(t) + \dfrac{4}{z-0.5}e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} x_1(t) = \dfrac{1}{z-0.8}u(t) + \dfrac{1}{z-0.8}e(t) \\ x_2(t) = \dfrac{1}{(z-0.5)(z-0.8)}u(t) + \dfrac{1}{(z-0.5)(z-0.8)}e(t) + \dfrac{4}{z-0.5}e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} x_1(t) = \dfrac{1}{z-0.8}u(t) + \dfrac{1}{z-0.8}e(t) \\ x_2(t) = \dfrac{1}{(z-0.5)(z-0.8)}u(t) + \dfrac{4z-2.2}{(z-0.5)(z-0.8)}e(t) \\ y(t) = x_2(t) \end{cases}$$

So:

$$y(t) = x_2(t) = \frac{z^{-2}}{1-1.3z^{-1}+0.4z^{-2}}u(t) + \frac{4z^{-1}-2.2z^{-2}}{1-1.3z^{-1}+0.4z^{-2}}e(t)$$

We recognize the three different terms:

$$(1-1.3z^{-1}+0.4z^{-2})y(t) = u(t-2) + (4z^{-1}-2.2z^{-2})e(t)$$

$$A(z) = 1-1.3z^{-1}+0.4z^{-2}$$
$$B(z) = 1$$
$$C(z) = 4z^{-1}-2.2z^{-2}$$

$\boxed{2)}$

CANONICAL REPRESENTATION

Notice that:

$$\frac{y(t)}{e(t)} = W(z) = \frac{C(z)}{A(z)} = \frac{4z^{-1}-2.2z^{-2}}{1-1.3z^{-1}+0.4z^{-2}} = \frac{4z-2.2}{z^2-1.3z+0.4}$$

ZEROES:

$$4z-2.2=0 \quad \Rightarrow \quad z = \frac{2.2}{4} = 0.55$$

Inside the unit circle

POLES:

$$z^2 - 1.3z + 0.4 = 0 \quad \Rightarrow \quad z_{1,2} = \frac{1.3 \pm \sqrt{1.3^2 - 1.6}}{2} = \frac{1.3 \pm 0.3}{2}$$

$$z_1 = 0.5$$

$$z_2 = 0.8$$

Inside the unit circle

But this representation is not canonical since:

- *C(z)* and *A(z)* have not the same degree;
- *C(z)* is not monic.

So we have to multiply/divide by *z* and divide/multiply by 4:

$$y(t) = W(z)e(t) = \frac{4z - 2.2}{z^2 - 1.3z + 0.4}\,\frac{z}{z}\,\frac{4}{4}\,e(t) = \frac{z^2 - 0.55z}{z^2 - 1.3z + 0.4}z^{-1}4e(t)$$
$$= W_1(z)4e(t-1)$$

Where we are considering only the stochastic part of *y(t)* for the moment. Let's introduce this white noise:

$$\eta(t) = 4e(t-1)$$

Notice that:

$$m_\eta = E[\eta(t)] = 4E[e(t-1)] = 0$$

$$\gamma_\eta(0) = E[\eta(t)^2] = 16E[e(t-1)^2] = 16$$

So:

$$\eta(t) \sim WN(0{,}16)$$

So:

$$y(t) = W_1(z)4e(t-1) = \frac{z^2 - 0.55z}{z^2 - 1.3z + 0.4}\eta(t)$$

This is the canonical representation of the process.

3)

2-STEP OPTIMAL PREDICTOR

Write the polynomials $C_1(z)$ and $A_1(z)$ (from $W_1(z)$) in negative powers of z:

$A_1(z) = 1 - 1.3z^{-1} + 0.4z^{-2}$

$C_1(z) = 1 - 0.55z^{-1}$

Compute 2 steps of the polynomial long division:

$$
\begin{array}{rrrr|lll}
1 & -0.55z^{-1} & & & 1 & -1.3z^{-1} & 0.4z^{-2} \\
\cline{5-7}
-1 & 1.3z^{-1} & -0.4z^{-2} & & 1 + 0.75z^{-1} & & \\
\cline{1-4}
 & 0.75z^{-1} & -0.4z^{-2} & & \text{E(z)} & & \\
 & -0.75z^{-1} & +0.975z^{-2} & -0.3z^{-3} & & & \\
\cline{1-4}
 & & 0.575z^{-2} & -0.3z^{-3} & & & \\
 & & & \text{R(z)} & & &
\end{array}
$$

Notice that:

$$R(z) = z^{-k}\tilde{R}(z)$$

Since, in this case, $k=2$, we have that:

$$\tilde{R}(z) = 0.575 - 0.3z^{-1}$$

So:

$$\boldsymbol{\hat{y}(t|t-2) = \frac{B(z)E(z)}{C(z)}u(t-2) + \frac{\tilde{R}(z)}{C(z)}\,y(t-2)}$$

Which yields:

$$\hat{y}(t|t-2) = \frac{1+0.75z^{-1}}{1-0.55z^{-1}}u(t-2) + \frac{0.575-0.3z^{-1}}{1-0.55z^{-1}}y(t-2)$$

That is:

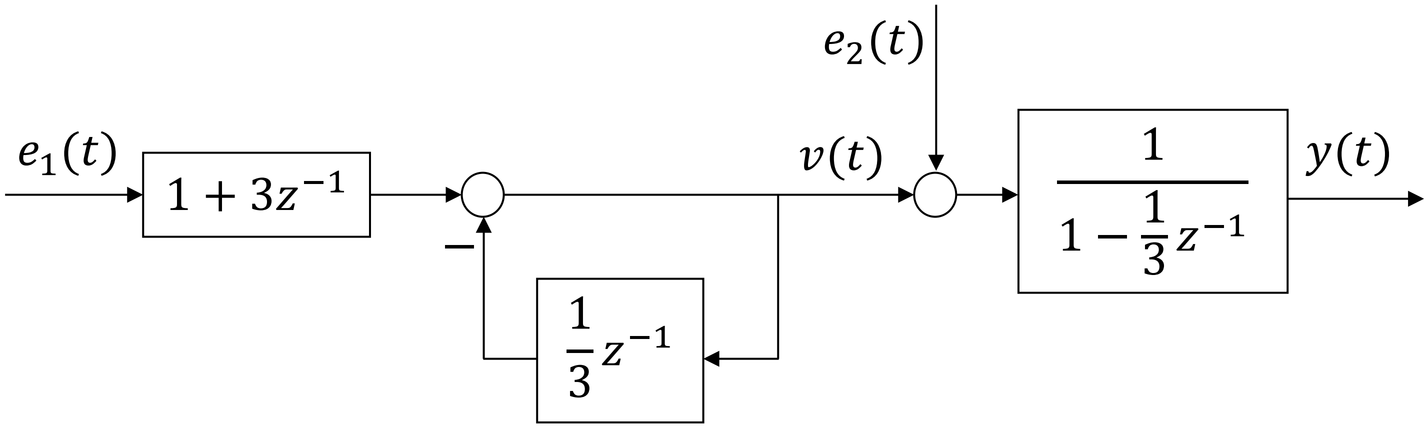
$$\begin{aligned}\hat{y}(t|t-2) = {} & 0.55\hat{y}(t-1|t-3) + u(t-2) + 0.75u(t-3) \\ & + 0.575y(t-2) - 0.3y(t-3)\end{aligned}$$

The 2-step prediction error variance can be simply computed as the variance of the unpredictable part of the process, which is:

$$\begin{aligned}E[\varepsilon(t)^2] &= E\left[\left(y(t) - \hat{y}(t|t-2)\right)^2\right] = E\left[\left(E(z)\eta(t)\right)^2\right] \\ &= E\left[\left(\eta(t) + 0.75\eta(t-1)\right)^2\right] = 16 + 16 \cdot 0.75^2 = 25\end{aligned}$$

Exercise 4

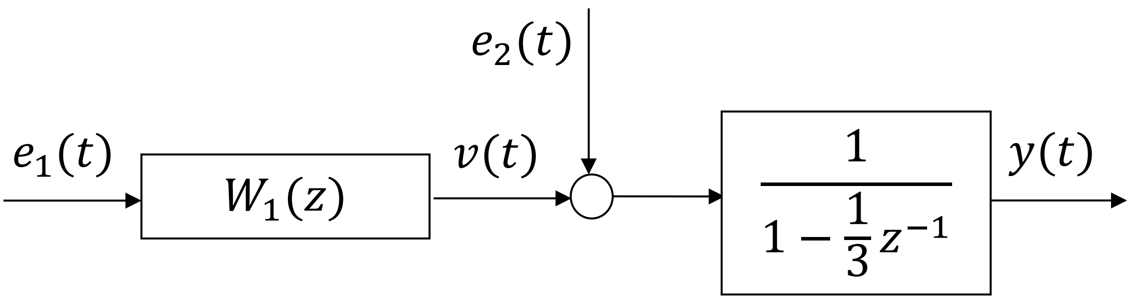
Consider the process *y(t)* generated according to the following scheme:



where $e_1(t) \sim WN(1,1)$ and $e_2(t) \sim WN(1,2)$ are uncorrelated white noises ($e_1(t) \perp e_2(t)$).

Compute the 3-step predictor and the associated prediction error variance and mean.

Notice that the scheme can be seen as:



where the transfer function between $e_1(t)$ and $v(t)$ is:

$$\frac{v(t)}{e_1(t)} = W_1(z) = (1 + 3z^{-1})\frac{1}{1 + \frac{1}{3}z^{-1}} = \frac{z+3}{z+\frac{1}{3}}$$

Notice that the pole is the reciprocal of the zero. The structure is that of an all pass filter of gain 3.

In order to make the gain of the all pass filter 1:

$$v(t) = W_1(z)e_1(t) = \frac{z+3}{z+\frac{1}{3}}e_1(t) = \frac{z+3}{z+\frac{1}{3}}\frac{3}{3}e_1(t) = \frac{1}{3}\frac{z+3}{z+\frac{1}{3}}3e_1(t)$$

We recognize that:

$$v(t) = W_1(z)e_1(t) = T(z)3e_1(t)$$

where:

$$T(z) = \frac{1}{3}\frac{z+3}{z+\frac{1}{3}}$$

is an all-pass filter. So the spectrum of *v(t)* is equal to the spectrum of $3e_1(t)$, which is the same as the spectrum of the noise:

$$\eta(t) = 3e_1(t)$$

whose mean and variance are:

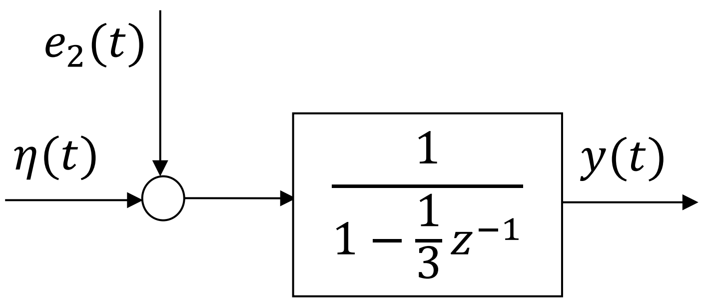
$$m_\eta = E[\eta(t)] = 3E[e_1(t)] = 3$$

$$\gamma_\eta(0) = E[(\eta(t)-3)^2] = E[(3e_1(t)-3)^2] = 9E[(e_1(t)-1)^2] = 9$$

So:

$$\eta(t) \sim WN(3,9)$$

The scheme can be rearranged:



Notice that if we define:

$$e(t) = e_2(t) + \eta(t)$$

This is again a White Noise (sum of uncorrelated white noises); its mean and variance are:

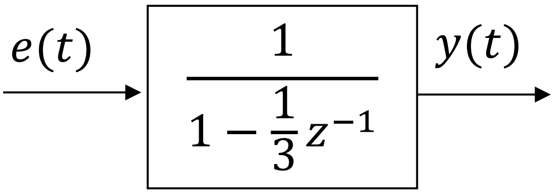
$$m_e = E[e(t)] = E[e_2(t) + \eta(t)] = E[e_2(t)] + E[\eta(t)] = 1 + 3 = 4$$

$$\begin{aligned}\gamma_e(0) = E[(e(t)-4)^2] &= E[(e_2(t)+\eta(t)-4)^2] \\ &= E\left[\left((e_2(t)-1)+(\eta(t)-3)\right)^2\right] \\ &= E[(e_2(t)-1)^2] + 2E[(e_2(t)-1)(\eta(t)-3)] + E[(\eta(t)-3)^2] \\ &= 2+0+9 = 11\end{aligned}$$

So:

$$e(t) \sim WN(4,11)$$

and the scheme can be seen as:



Notice that this process is an AR(1) with non-null mean. In fact, its time-domain representation is:

$$y(t) = \frac{1}{3}y(t-1) + e(t), \quad e(t) \sim WN(4,11)$$

And its mean:

$$m_y = E[y(t)] = \frac{1}{3}m_y + 4$$

$$\Rightarrow \quad m_y = 4\,\frac{3}{2} = 6$$

Variance (unbiased process):

$$\tilde{y}(t) = y(t) - m_y = y(t) - 6$$
$$\tilde{e}(t) = e(t) - m_e = e(t) - 4$$

$$\tilde{y}(t) = \frac{1}{3}\tilde{y}(t-1) + \tilde{e}(t), \quad \tilde{e}(t) \sim WN(0{,}11)$$

$$\gamma_y(0) = \gamma_{\tilde{y}}(0) = E[\tilde{y}(t)^2] = \frac{1}{9}\gamma_y(0) + 11 \quad \Rightarrow \quad \gamma_y(0) = 11\,\frac{9}{8} = \frac{99}{8}$$

The predictor can be found using two different approaches:

FIRST METHOD

1) Consider the unbiased processes:

$$\tilde{y}(t) = y(t) - m_y$$

$$\tilde{e}(t) = e(t) - m_e$$

2) Evaluate the k-step predictor of the unbiased process: $\hat{\tilde{y}}(t|t-k)$

3) Generally, this predictor is different from the predictor of the biased process:

$$\hat{y}(t|t-k) \neq \hat{\tilde{y}}(t|t-k)$$

So turn to the original predictor using the relations:

$$\hat{\tilde{y}}(t|t-k) = \hat{y}(t|t-k) - m_y$$
$$\tilde{e}(t) = e(t) - m_e$$

This method involves a lot of computations.

SECOND METHOD

We translate the biased process into an ARMAX process.

1) Biased process:

$$y(t) = \frac{1}{3}y(t-1) + e(t), \quad e(t) \sim WN(4,11)$$

2) Unbiased error:

$$\tilde{e}(t) = e(t) - m_e \quad \Rightarrow \quad e(t) = \tilde{e}(t) + m_e, \quad \tilde{e}(t) \sim WN(0,11)$$

3) Substitute this unbiased error in the process:

$$y(t) = \frac{1}{3}y(t-1) + \tilde{e}(t) + 4, \quad \tilde{e}(t) \sim WN(0,11)$$

4) Consider the signal:

$$u(t) = u(t-1) = \cdots = 4$$

5) Thus we have translated the biased AR(1) process into the ARMAX(1,0,0,3) process:

$$y(t) = \frac{1}{3}y(t-1) + \tilde{e}(t) + u(t-3), \quad \tilde{e}(t) \sim WN(0,11)$$

i.e.

$$\left(1 - \frac{1}{3}z^{-1}\right)y(t) = \tilde{e}(t) + u(t-3), \quad \tilde{e}(t) \sim WN(0,11)$$

and:

$A(z) = 1 - \frac{1}{3}z^{-1}$ **The denominator of the exogenous and remote terms must be the same!!**

$B(z) = 1$

$C(z) = 1$

6) Compute the 3-step optimal predictor of this ARMAX process:

$$\hat{y}(t|t-3) = \frac{B(z)E(z)}{C(z)}u(t-3) + \frac{\tilde{R}(z)}{C(z)}\,y(t-3)$$

Compute 3 steps of the polynomial long division:

$$
\begin{array}{llll|l}
1 & & & & 1 \quad -\frac{1}{3}z^{-1} \\
\cline{5-5}
-1 & \frac{1}{3}z^{-1} & & & 1+\frac{1}{3}z^{-1}+\frac{1}{9}z^{-2} \\
\cline{1-4}
 & \frac{1}{3}z^{-1} & & & \\
 & -\frac{1}{3}z^{-1} & +\frac{1}{9}z^{-2} & & \\
\cline{1-4}
 & & +\frac{1}{9}z^{-2} & & \\
 & & -\frac{1}{9}z^{-2} & \frac{1}{27}z^{-3} & \\
\cline{1-4}
 & & & \frac{1}{27}z^{-3} &
\end{array}
$$

E(z)

R(z)

The 3-step optimal predictor from the available data is:

$$\hat{y}(t|t-3) = \left(1+\frac{1}{3}z^{-1}+\frac{1}{9}z^{-2}\right)u(t-3) + \frac{1}{27}\,y(t-3)$$

7) Make use of the fact that $u(t) = 4 \quad \forall t$. Thus:

$$\left(1+\frac{1}{3}z^{-1}+\frac{1}{9}z^{-2}\right)u(t-3) = 4+\frac{4}{3}+\frac{4}{9} = \frac{52}{9}$$

So:

$$\hat{y}(t|t-3) = \frac{1}{27}\,y(t-3) + \frac{52}{9}$$

Notice that:

$$E[\hat{y}(t|t-3)] = \frac{1}{27}E[y(t-3)] + \frac{52}{9} = \frac{6}{27} + \frac{52}{9} = 6 = E[y(t)]$$

THE MEAN OF THE PREDICTOR MUST BE EQUAL TO THE MEAN OF THE PROCESS (FOR EVERY PREDICTOR, FOR EVERY PROCESS)!!!

3-step prediction error mean:

$$E[\varepsilon(t)] = E[y(t) - \hat{y}(t|t-3)] = E[E(z)\tilde{e}(t)]$$
$$= E\left[\left(1 + \frac{1}{3}z^{-1} + \frac{1}{9}z^{-2}\right)\tilde{e}(t)\right] = 0$$

THE MEAN OF THE PREDICTION ERROR MUST BE ZERO (FOR EVERY PREDICTOR, FOR EVERY PROCESS)!!!

3-step prediction error variance:

$$E[\varepsilon(t)^2] = E\left[\left(y(t) - \hat{y}(t|t-3)\right)^2\right] = E\left[\left(E(z)\tilde{e}(t)\right)^2\right]$$
$$= E\left[\left(\left(1 + \frac{1}{3}z^{-1} + \frac{1}{9}z^{-2}\right)\tilde{e}(t)\right)^2\right] = \left(1 + \frac{1}{9} + \frac{1}{81}\right)11 = \frac{91}{81}11$$
$$= \frac{1001}{81}$$

THE VARIANCE MUST BE LOWER OR EQUAL TO THE PROCESS VARIANCE (FOR EVERY PREDICTOR, FOR EVERY PROCESS)!!!

$$\gamma_y(0) = \frac{99}{8} \approx 12.38 > 12.36 \approx \frac{1001}{81} = E[\varepsilon(t)^2]$$

Note: the variance of the predictor is EQUAL to the variance of the process when the predictor is the TRIVIAL predictor, i.e. the process mean.

Exercise 5

Consider the following process:

$$y(t) = 0.2y(t-3) + e(t), \quad e(t) \sim WN(0,1)$$

1) Compute the mean and the variance of the process.

2) Compute the 1-step, 2-step, 3-step predictor from the available data.

1)

The process is an AR(3) with a pure delay of three time instants. It is also a SSP (poles inside the unit circle).

MEAN:

$$m_y = 0.2m_y + E[e(t)] = 0.2m_y + 0 \quad \Rightarrow \quad m_y = 0$$

VARIANCE:

$$\gamma_y(0) = E[y(t)^2] = E\left[\left(0.2y(t-3) + e(t)\right)^2\right] = 0.04\gamma_y(0) + E[e(t)^2]$$

$$(1-0.04)\gamma_y(0) = 1 \quad \Rightarrow \quad \gamma_y(0) = \frac{1}{0.96}$$

(Try to compute the covariance function for $|\tau| > 0$, you will see something interesting…)

2)

First of all, we verify if the process is expressed through a canonical representation.

OPERATORIAL REPRESENTATION

$$W(z) = \frac{1}{1-0.2z^{-3}} = \frac{z^3}{z^3 - 0.2}$$

So the three poles are located inside the unit circle: $z_{1,2,3} = \sqrt[3]{0.2} < 1$. Thus the system is asymptotically stable and satisfies the properties of canonical representation (same degree, monic, coprime, poles and zeroes strictly inside the unit circle).

POLYNOMIAL LONG DIVISION

$$
\begin{array}{ll|ll}
1 & & 1 & -0.2z^{-3} \\ \cline{3-4}
-1 & 0.2z^{-3} & 1 & \textit{E(z)} \\ \cline{1-2}
 & 0.2z^{-3} & & \\
 & \textit{R(z)} & &
\end{array}
$$

Notice that:

$$R(z) = z^{-k}\tilde{R}(z)$$

So:

- 1-step predictor → $k=1$ → $\tilde{R}(z) = 0.2z^{-2}$
- 2-step predictor → $k=2$ → $\tilde{R}(z) = 0.2z^{-1}$
- 3-step predictor → $k=3$ → $\tilde{R}(z) = 0.2$

We apply the formula of the predictor:

- 1-step predictor:
$$\hat{y}(t|t-1) = \frac{\tilde{R}(z)}{C(z)}y(t-1) = \frac{0.2z^{-2}}{1}y(t-1) = 0.2y(t-3)$$
- 2-step predictor:
$$\hat{y}(t|t-2) = \frac{\tilde{R}(z)}{C(z)}y(t-2) = \frac{0.2z^{-1}}{1}y(t-2) = 0.2y(t-3)$$
- 3-step predictor:
$$\hat{y}(t|t-3) = \frac{\tilde{R}(z)}{C(z)}y(t-3) = \frac{0.2}{1}y(t-3) = 0.2y(t-3)$$

Obviously, all the predictors must satisfy the constraint of the pure delay. Since the process is characterized by three step delay and it is an AR process, then the 1-step and the 2-step predictors are equal to the 3-step predictor:

$$y(t) = \underbrace{0.2y(t-3)}_{\text{Predictable part}} + \underbrace{e(t)}_{\text{Unpredictable part}}$$

Exercise 6

Consider the process:

$$y(t) = -\frac{1}{9}y(t-2) + e(t) - \frac{1}{4}e(t-2), \quad e(t) \sim WN(0,1)$$

1) What process is *y(t)*? Is *y(t)* a SSP?

2) Compute the predictor $\hat{y}(t|t-k)$ based on the measurements for $k = 1,2$.

3) Compute the mean and the variance of the 1-step/2-step ahead prediction error.

1)

The process is an ARMA(2,2).

OPERATORIAL REPRESENTATION

$$y(t) = W(z)e(t) = \frac{1 - \frac{1}{4}z^{-2}}{1 + \frac{1}{9}z^{-2}}e(t) = \frac{z^2 - \frac{1}{4}}{z^2 + \frac{1}{9}}e(t)$$

Zeroes: $z_{1,2} = \pm\frac{1}{2}$

Poles: $z_{1,2} = \pm j\frac{1}{3}$

The complex conjugate poles lie one the imaginary axis and are inside the unit circle: *W(z)* is asymptotically stable. Since *e(t)* is a SSP, the process *y(t)* is a SSP.

The zeroes are inside the unit circle too. Moreover, notice that:

$$C(z) = 1 - \frac{1}{4}z^{-2}, \ \ A(z) = 1 + \frac{1}{9}z^{-2}$$

have the same degree, are monic and have no common factors. So the process *y(t)* is in the CANONICAL REPRESENTATION.

2.a

1-STEP PREDICTOR

Since the process is an ARMA, the one step predictor can be simply computed:

$$\hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t) = \frac{1 - \frac{1}{4}z^{-2} - 1 - \frac{1}{9}z^{-2}}{1 - \frac{1}{4}z^{-2}} y(t)$$

$$= \frac{-\frac{13}{36}z^{-2}}{1 - \frac{1}{4}z^{-2}} y(t)$$

Thus, the time domain representation of the 1-step ahead predictor is:

$$\hat{y}(t|t-1) = \frac{1}{4}\hat{y}(t-2|t-3) - \frac{13}{36}y(t-2)$$

(Notice that the delay is respected, since the one step predictor does not depend on the measurement at *(t-1)*).

3.a

Prediction error:

$$\varepsilon(t) = y(t) - \hat{y}(t|t-1) = E(z)e(t) = e(t)$$

Prediction error mean and variance:

$$m_\varepsilon = E[e(t)] = 0 = m_y$$

$$\gamma_\varepsilon(0) = E[e(t)^2] = 1$$

2.b

2-STEP PREDICTOR

$$
\begin{array}{cc|cc}
1 & -\frac{1}{4}z^{-2} & 1 & +\frac{1}{9}z^{-2} \\ \cline{3-4}
-1 & -\frac{1}{9}z^{-2} & 1 & \\ \cline{1-2}
 & -\frac{13}{36}z^{-2} & &
\end{array}
$$

E(z)

R(z)

So:

$$\hat{y}(t|t-2) = \frac{\tilde{R}(z)}{C(z)}y(t-2) = \frac{-\frac{13}{36}}{1-\frac{1}{4}z^{-2}}y(t-2)$$

The time domain representation of the 2-step ahead predictor is:

$$\hat{y}(t|t-2) = \frac{1}{4}\hat{y}(t-2|t-4) - \frac{13}{36}y(t-2)$$

(Notice that the term that regards the measurement (*y(t)*) is the same of the 1-step predictor: this is due to the fact that the process has 2 time instants delay, i.e. *y(t)* does not depend on *y(t-1)*!!)

3.b

Prediction error:

$$\varepsilon(t) = y(t) - \hat{y}(t|t-2) = E(z)e(t) = e(t)$$

Prediction error mean and variance:

$$m_\varepsilon = E[e(t)] = 0 = m_y$$

$$\gamma_\varepsilon(0) = E[e(t)^2] = 1$$

Exercise 7

Consider the process:

$$y(t) = \frac{1}{3}y(t-1) + u(t-1) + u(t-2) + 3e(t-1) + e(t-2) + 1 \quad , \quad e(t) \sim WN(1,1)$$

Compute the 1 step ahead predictor and the variance of the prediction error.

First of all, we consider the unbiased error:

$$\tilde{e}(t) = e(t) - m_e = e(t) - 1 \quad \Rightarrow \quad e(t) = \tilde{e}(t) + 1$$

Thus the process becomes:

$$y(t) = \frac{1}{3}y(t-1) + u(t-1) + u(t-2) + 3(\tilde{e}(t-1) + 1) + \tilde{e}(t-2) + 1 + 1, \quad \tilde{e}(t) \sim WN(0,1)$$

$$y(t) = \frac{1}{3}y(t-1) + u(t-1) + u(t-2) + 3\tilde{e}(t-1) + \tilde{e}(t-2) + 5$$

Notice that the terms *u(t-1)+u(t-2)+5* can be gathered in a single exogenous input:

$$\tilde{u}(t-1) = u(t-1) + u(t-2) + 5$$

Thus the process becomes:

$$y(t) = \frac{1}{3}y(t-1) + 3\tilde{e}(t-1) + \tilde{e}(t-2) + \tilde{u}(t-1)$$

That is an ARMAX(1,2,0,1) process.

OPERATORIAL REPRESENTATION

$$y(t)=\frac{3z^{-1}+z^{-2}}{1-\frac{1}{3}z^{-1}}\tilde{e}(t)+\frac{1}{1-\frac{1}{3}z^{-1}}\tilde{u}(t-1)$$
$$=W(z)\tilde{e}(t)+\frac{1}{1-\frac{1}{3}z^{-1}}\tilde{u}(t-1)$$

W(z) is minimum phase and asymptotically stable since its zero is located in *z=-1/3* (inside the unit circle) and its poles are located in *z=0* and *z=+1/3* (inside the unit circle). Nevertheless, *A(z)* and *C(z)* are not monic and of the same degree. So *y(t)* is not a canonical representation.

So:

$$y(t)=\frac{3z+1}{z^2-\frac{1}{3}z}\tilde{e}(t)=\frac{3z+1}{z^2-\frac{1}{3}z}\frac{3}{3}\frac{z^{-1}}{z^{-1}}\tilde{e}(t)=\frac{z+\frac{1}{3}}{z-\frac{1}{3}}3\tilde{e}(t-1)$$

We define the noise:

$$\eta(t)=3\tilde{e}(t-1)$$

$$\eta(t)\sim WN(0,9)$$

And the process is then:

$$y(t)=\frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{3}z^{-1}}\eta(t)+\frac{1}{1-\frac{1}{3}z^{-1}}\tilde{u}(t-1)$$

which is the CANONICAL REPRESENTATION of the process.

Recognize that:

$$A(z)=1-\frac{1}{3}z^{-1}$$

$$B(z)=1$$

$$C(z)=1+\frac{1}{3}z^{-1}$$

1-STEP PREDICTOR

The general formula for ARMAX k-step predictor is:

$$\hat{y}(t|t-k) = \frac{\tilde{R}(z)}{C(z)}y(t-k) + \frac{B(z)E(z)}{C(z)}u(t-k)$$

that can be simplified in the case of 1-step predictor:

$$\hat{y}(t|t-1) = \frac{C(z)-A(z)}{C(z)}y(t) + \frac{B(z)}{C(z)}u(t-1)$$

So:

$$\hat{y}(t|t-1) = \frac{1+\frac{1}{3}z^{-1} - 1 + \frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}y(t) + \frac{1}{1+\frac{1}{3}z^{-1}}\tilde{u}(t-1)$$

Time domain representation of the predictor:

$$\hat{y}(t|t-1) = -\frac{1}{3}\hat{y}(t-1|t-2) + \frac{2}{3}y(t-1) + \tilde{u}(t-1)$$

But recall that:

$$\tilde{u}(t-1) = u(t-1) + u(t-2) + 5$$

Thus:

$$\hat{y}(t|t-1) = -\frac{1}{3}\hat{y}(t-1|t-2) + \frac{2}{3}y(t-1) + u(t-1) + u(t-2) + 5$$

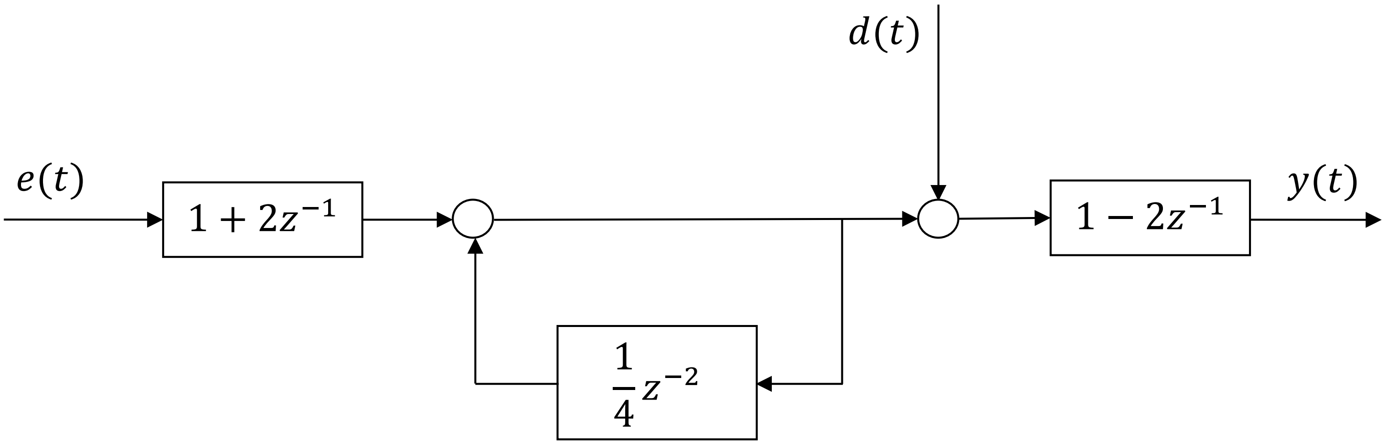
The prediction error variance is:

$$E[\varepsilon(t)^2] = E\left[\left(y(t) - \hat{y}(t|t-1)\right)^2\right] = E[\eta(t)^2] = 9$$

THE 1-STEP PREDICTION ERROR VARIANCE IS ALWAYS EQUAL TO THE VARIANCE OF THE PROCESS NOISE!!!

Exercise 8

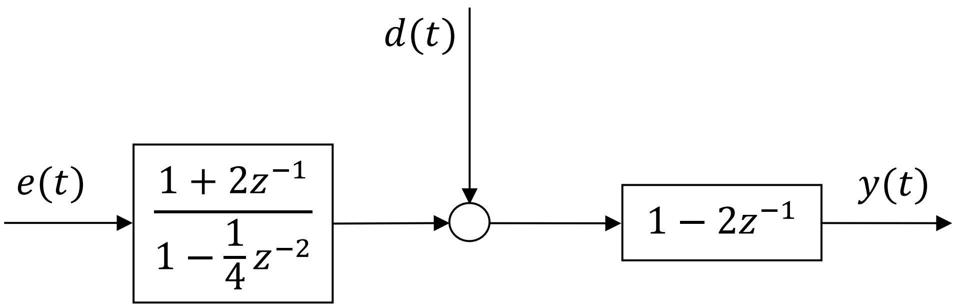
Consider the process *y(t)* generated according to the following scheme:



where $e(t) \sim WN(0,1)$, $\quad d(t) = 2 \quad \forall t$.

Compute the 1-step predictor and the 1-step prediction error mean and variance.

The left side of the block diagram can be simplified as:



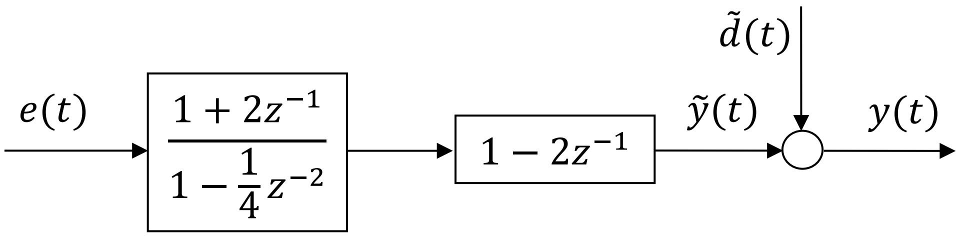
Notice that since *d(t)* is constant then the effect of this signal on *y(t)* can be computed using the frequency response theorem; the steady-state output is:

$$\tilde{d}(t) = (1 - 2e^{-j0})2 = -2$$

In other words:

$$\tilde{d}(t) = (1 - 2z^{-1})d(t) = d(t) - 2d(t-1) = 2 - 4 = -2$$

Which is a constant signal equal to *-2*. So we can move the block *1-2z*$^{-1}$ to the left side of the sum block (be sure to replace $d(t)$ with $\tilde{d}(t) = -2 \quad \forall t$):



So the system can be seen as an ARMAX:

$$y(t) = \tilde{d}(t) + \frac{(1+2z^{-1})(1-2z^{-1})}{1-\frac{1}{4}z^{-2}}e(t), \quad e(t) \sim WN(0,1), \quad \tilde{d}(t) = -2 \quad \forall t$$

or:

$$y(t) = \tilde{d}(t) + W(z)e(t) = \tilde{d}(t) + \tilde{y}(t)$$

where:

$$W(z) = \frac{(1+2z^{-1})(1-2z^{-1})}{1-\frac{1}{4}z^{-2}}$$

Notice that:

$$\begin{aligned}
\tilde{y}(t) = W(z)e(t) &= \frac{1-2z^{-1}}{1-\frac{1}{2}z^{-1}} \cdot \frac{1+2z^{-1}}{1+\frac{1}{2}z^{-1}}e(t) \\
&= \frac{1-2z^{-1}}{1-\frac{1}{2}z^{-1}} \cdot \frac{1+2z^{-1}}{1+\frac{1}{2}z^{-1}}\left(\frac{-4}{-4}\right)e(t) \\
&= \left(-\frac{1}{2}\frac{1-2z^{-1}}{1-\frac{1}{2}z^{-1}}\right) \cdot \left(\frac{1}{2}\frac{1+2z^{-1}}{1+\frac{1}{2}z^{-1}}\right) \cdot (-4)e(t) \\
&= T_1(z)T_2(z)(-4)e(t)
\end{aligned}$$

where $T_1(z)$ and $T_2(z)$ are all pass filters.

Thus, if we define the noise:

$$\eta(t) = -4e(t)$$

$$\eta(t) \sim WN(0,16)$$

The process becomes:

$$y(t) = u(t-1) + \eta(t), \quad \eta(t) \sim WN(0,16)$$

$$A(z) = 1$$

$$B(z) = 1$$

$$C(z) = 1$$

Having introduced the exogenous input $u(t-1) = \tilde{d}(t) = -2 \quad \forall t.$

Since the exogenous input is completely known (constant), the mean value of the process is $m_y = E[u(t-1)] + E[\eta(t)] = -2.$

The optimal 1-step predictor of this process is obviously the trivial predictor:

$$\hat{y}(t|t-1) = u(t-1) = -2$$

Notice that:

$$m_{\hat{y}} = E[\hat{y}(t|t-1)] = -2 = m_y$$

Thus, the prediction error mean is:

$$E[\varepsilon(t)] = E[y(t) - \hat{y}(t|t-1)] = m_y - m_{\hat{y}} = 0$$

And the prediction error variance is:

$$E[\varepsilon(t)^2] = E[\eta(t)^2] = 16$$

Obviously the *k*-step predictor is always the trivial predictor:

$$\hat{y}(t|t-k) = -2$$

And the variance of the *k*-step prediction error is always equal to the variance of the noise (16).

Exercise 9

Consider the process:

$$y(t) = -\frac{7}{12}y(t-1) - \frac{1}{12}y(t-2) + 2d(t) + \frac{1}{2}d(t-1), \quad d(t) \sim WN(1,1)$$

Compute the 1-step predictor and the associated prediction error mean and variance.

First of all, consider the unbiased noise:

$$\tilde{d}(t) = d(t) - 1 \;\Rightarrow\; d(t) = \tilde{d}(t) + 1, \qquad \tilde{d}(t) \sim WN(0,1)$$

The process becomes:

$$\begin{aligned} y(t) &= -\frac{7}{12}y(t-1) - \frac{1}{12}y(t-2) + 2\tilde{d}(t) + 2 + \frac{1}{2}\tilde{d}(t-1) + \frac{1}{2} \\ &= -\frac{7}{12}y(t-1) - \frac{1}{12}y(t-2) + 2\tilde{d}(t) + \frac{1}{2}\tilde{d}(t-1) + \frac{5}{2} \end{aligned}$$

OPERATORIAL REPRESENTATION

$$y(t) = \frac{2 + \frac{1}{2}z^{-1}}{1 + \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}}\tilde{d}(t) + \frac{1}{1 + \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}}\frac{5}{2}$$

Is the process expressed in a CANONICAL FORM?

$$W(z) = \frac{2 + \frac{1}{2}z^{-1}}{1 + \frac{7}{12}z^{-1} + \frac{1}{12}z^{-2}} = \frac{2z^2 + \frac{1}{2}z}{z^2 + \frac{7}{12}z + \frac{1}{12}}$$

Poles:

$$z^2 + \frac{7}{12}z + \frac{1}{12} = 0 \;\Rightarrow\; z_{1,2} = \frac{-\frac{7}{12} \pm \sqrt{\frac{49}{144} - \frac{4}{12}}}{2} = \frac{-\frac{7}{12} \pm \frac{1}{12}}{2} \;\Rightarrow\; \begin{aligned} z_1 &= -\frac{1}{3} \\ z_2 &= -\frac{1}{4} \end{aligned}$$

So:

$$W(z)=\frac{2z\left(z+\frac{1}{4}\right)}{\left(z+\frac{1}{3}\right)\left(z+\frac{1}{4}\right)}$$

Poles inside the unit circle. Zeroes (*z=0*, *z=-1/4*) inside the unit circle. Common factors must be simplified:

$$W(z)=\frac{2z}{z+\frac{1}{3}}$$

Since *A(z)* and *C(z)* had common factors that were simplified, the exogenous part should be written with the same denominator as the stochastic part.

The constant signal 5/2 is filtered with:

$$\frac{1}{1+\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}}$$

For the frequency response theorem, we have that:

$$\frac{1}{1+\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}}\frac{5}{2}=\frac{1}{1+\frac{7}{12}+\frac{1}{12}}\frac{5}{2}=\frac{3}{2}$$

Thus the process is:

$$y(t)=\frac{2z}{z+\frac{1}{3}}\tilde{d}(t)+\frac{3}{2}=W(z)\tilde{d}(t)+\frac{3}{2}=\tilde{y}(t)+\frac{3}{2}$$

We can see the constant signal as an exogenous input:

$$u(t-1)=\frac{3}{2}$$

So that we obtain an ARX(1,0,1) model:

$$y(t)=W(z)\tilde{d}(t)+u(t-1)=\tilde{y}(t)+u(t-1)$$

But the denominator of the exogenous part is not equal to the denominator of the AR term. So we recover it:

$$y(t) = \frac{2}{1+\frac{1}{3}z^{-1}}\tilde{d}(t) + \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}u(t-1)$$

The transfer function *W(z)* is not a canonical representation yet, since the numerator is not monic; so:

$$\tilde{y}(t) = \frac{2}{1+\frac{1}{3}z^{-1}}\frac{2}{2}\tilde{d}(t) = \frac{1}{1+\frac{1}{3}z^{-1}}2\tilde{d}(t)$$

We define the noise:

$$\eta(t) = 2\tilde{d}(t)$$

$$\eta(t) \sim WN(0,4)$$

And the canonical representation of the process is then the ARX(1,1,1) model:

$$y(t) = \frac{1}{1+\frac{1}{3}z^{-1}}\eta(t) + \frac{1+\frac{1}{3}z^{-1}}{1+\frac{1}{3}z^{-1}}u(t-1), \quad \eta(t) \sim WN(0,4), \quad u(t-1) = \frac{3}{2}$$

Notice that:

$$m_y = \frac{3}{2}$$

And we recognize:

$$A(z) = 1 + \frac{1}{3}z^{-1}$$

$$B(z) = 1 + \frac{1}{3}z^{-1}$$

$$C(z) = 1$$

The optimal 1-step predictor of this ARX model is:

$$\hat{y}(t|t-1) = \frac{C(z)-A(z)}{C(z)}y(t) + \frac{B(z)}{C(z)}u(t-1)$$

We substitute the terms and we get:

$$\hat{y}(t|t-1) = \frac{-\frac{1}{3}z^{-1}}{1}y(t) + \frac{1+\frac{1}{3}z^{-1}}{1}u(t-1)$$

The time domain representation of the predictor is:

$$\hat{y}(t|t-1) = -\frac{1}{3}y(t-1) + \frac{3}{2} + \frac{1}{2} = -\frac{1}{3}y(t-1) + 2$$

Mean of the predictor:

$$E[\hat{y}(t|t-1)] = E\left[-\frac{1}{3}y(t-1)+2\right] = -\frac{1}{3}m_y + 2 = -\frac{1}{2} + 2 = \frac{3}{2} = m_y$$

Mean and variance of the prediction error:

$$m_\varepsilon = E[\varepsilon(t)] = E[\eta(t)] = 0$$

$$\gamma_\varepsilon(0) = E[\eta(t)^2] = 4$$

Exercise 10

Consider the following process:

$$y(t) = \frac{1}{2}y(t-2) + e(t-2), \quad e(t) \sim WN(1,1)$$

1) Compute the mean and variance of the process.

2) Draw the plot of the variance of the prediction error as a function of the prediction horizon k (YOU ARE NOT REQUESTED TO COMPUTE THE PREDICTOR FROM THE DATA!)

1)

MEAN m_y

$$m_y = \frac{1}{2}m_y + 1 = 2$$

VARIANCE $\gamma_y(0)$

We consider the unbiased processes:

$$\tilde{y}(t) = y(t) - m_y = y(t) - 2$$

$$\tilde{e}(t) = e(t) - m_e = e(t) - 1$$

So the relation between them is:

$$\tilde{y}(t) = \frac{1}{2}\tilde{y}(t-2) + \tilde{e}(t-2), \quad \tilde{e}(t) \sim WN(0,1)$$

Hence the variance of *y(t)* is:

$$\gamma_y(0) = \gamma_{\tilde{y}}(0) = \frac{1}{4}\gamma_{\tilde{y}}(0) + 1$$

$$\gamma_y(0) = \frac{4}{3}$$

2)

PREDICTION ERROR VARIANCE ANALYSIS

First of all, we translate the ARMA process into an ARMAX process by unbiasing the noise *e(t)*:

$$\tilde{e}(t) = e(t) - 1$$

$$y(t) = \frac{1}{2}y(t-2) + \tilde{e}(t-2) + 1, \quad \tilde{e}(t) \sim WN(0,1)$$

Then we write the operatorial representation of the process

$$\left(1 - \frac{1}{2}z^{-2}\right)y(t) = z^{-2}\tilde{e}(t) + 1$$

$$y(t) = \frac{z^{-2}}{1 - \frac{1}{2}z^{-2}}\tilde{e}(t) + \frac{1}{1 - \frac{1}{2}z^{-2}}u(t-1), \quad u(t) = 1 \quad \forall t$$

We recognize that this process is not written in a canonical form, so we introduce the new noise:

$$\eta(t) = \tilde{e}(t-2), \qquad \eta(t) \sim WN(0,1)$$

So that we restore the canonical form:

$$y(t) = \frac{1}{1 - \frac{1}{2}z^{-2}}\eta(t) + \frac{1}{1 - \frac{1}{2}z^{-2}}u(t-1), \quad u(t) = 1 \quad \forall t$$

$$C(z) = 1, \quad A(z) = 1 - \frac{1}{2}z^{-2}, \quad B(z) = 1$$

QUESTION: Is the canonical representation necessary to compute the predictor from the noise and the variance of the prediction error?
ANSWER: NO! The canonical form is only necessary when we have to compute the predictor from the data and we have to derive the whitening filter.

Notice that the long division leads to:

$$E(z) = 1 + \frac{1}{2}z^{-2} + \frac{1}{4}z^{-4} + \frac{1}{8}z^{-6} + \cdots$$

Thus, for the predictor at *k* steps the following result holds:

$$E_k(z) = \begin{cases} \sum_{i=1}^{j} \left(\frac{1}{2} z^{-2}\right)^{i-1} & k = 2j,\ \ j \in \mathbb{N} \\ E_{k+1}(z) & k = 2j+1,\ \ j \in \mathbb{N}_0 \end{cases}$$

Since the prediction error of the *k*-step predictor is defined as:

$$\varepsilon_k(t) = E_k(z)\eta(t) = \begin{cases} \sum_{i=1}^{j} \left(\frac{1}{2}\right)^{i-1} \eta(t-2i+2) & k = 2j,\ \ j \in \mathbb{N} \\ \varepsilon_{k+1}(t) & k = 2j+1,\ \ j \in \mathbb{N}_0 \end{cases}$$

So we have that the prediction error variance of the *k*-step predictor is simply:

$$\gamma_{\varepsilon_k}(0) = E[\varepsilon_k(t)^2] = \begin{cases} \sum_{i=1}^{j} \left(\frac{1}{2}\right)^{2i-2} E[\eta(t)^2] & k = 2j,\ \ j \in \mathbb{N} \\ \gamma_{\varepsilon_{k+1}}(0) & k = 2j+1,\ \ j \in \mathbb{N}_0 \end{cases}$$

This is a geometric series with common ratio *1/4*. In fact, for *k=2j*, the sum can be written as:

$$\sum_{i=1}^{j} \left(\frac{1}{2}\right)^{2i-2} E[\eta(t)^2] = E[\eta(t)^2] \sum_{i=1}^{j} \left(\frac{1}{4}\right)^{i-1} = E[\eta(t)^2] \sum_{i=0}^{j-1} \left(\frac{1}{4}\right)^{i}$$
$$= E[\eta(t)^2] \sum_{i=0}^{l} \left(\frac{1}{4}\right)^{i}$$

Where *l=j-1=k/2-1*. Asymptotically (i.e. for $k \to \infty$, which implies $l \to \infty$), the variance of the prediction error tends to:

$$\gamma_{\varepsilon_k}(0) = E[\eta(t)^2] \sum_{i=0}^{l} \left(\frac{1}{4}\right)^{i} \xrightarrow[k\to\infty]{} \gamma_{\varepsilon_\infty}(0) = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

which actually IS THE VARIANCE OF THE PROCESS: this means that the predictor tends to the trivial predictor (i.e. the process mean) when the prediction horizon tends to infinity.

Let's compute explicitly the first few values of the prediction error variance:

$$\gamma_{\varepsilon_1}(0) = \gamma_{\varepsilon_2}(0)$$

$$\gamma_{\varepsilon_2}(0) = 1$$

$$\gamma_{\varepsilon_3}(0) = \gamma_{\varepsilon_4}(0)$$

$$\gamma_{\varepsilon_4}(0) = 1 + \frac{1}{4} = \frac{5}{4} = 1.25$$

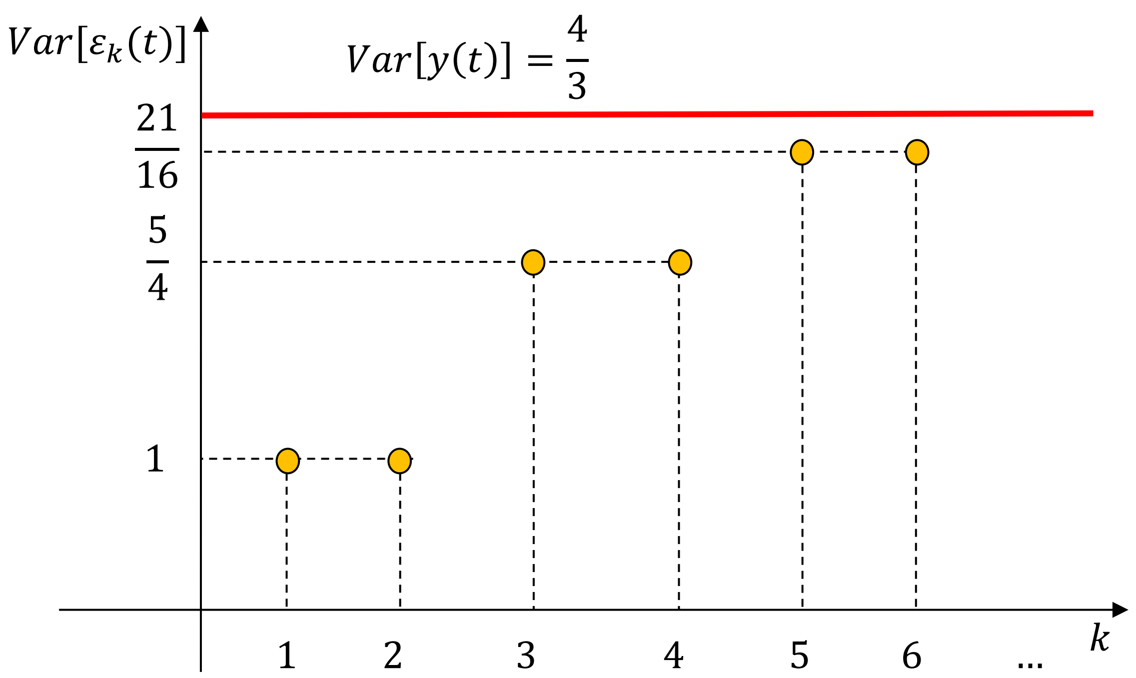
$$\gamma_{\varepsilon_5}(0) = \gamma_{\varepsilon_6}(0)$$

$$\gamma_{\varepsilon_6}(0) = 1 + \frac{1}{4} + \frac{1}{16} = \frac{21}{16} \approx 1.31$$

$$\vdots$$

$$\gamma_{\varepsilon_\infty}(0) = \frac{4}{3} \approx 1.33$$

The prediction error variance has the behaviour depicted in figure (as a function of the prediction horizon *k*).



Exercise 11

Consider the process:

$$y(t) = e(t-1) + 2e(t-2) + e(t-4) + 2e(t-5) + 1, \quad e(t) \sim WN(1,1)$$

1) Compute and draw the plot of the prediction error variance:

$$Var[y(t) - \hat{y}(t|t-k)]$$

as a function of the prediction horizon k.

2) Compute and draw the plot of the mean value of the predictor:

$$m_{\hat{y}} = E[\hat{y}(t|t-k)]$$

as a function of the prediction horizon k.

1)

Notice that the process is a MA(5).

MEAN m_y

$$\begin{aligned} m_y = E[y(t)] &= E[e(t-1) + 2e(t-2) + e(t-4) + 2e(t-5) + 1] \\ &= 1 + 2 + 1 + 2 + 1 = 7 \end{aligned}$$

VARIANCE $\gamma_y(0)$

Consider the unbiased process:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y = y(t) - 7 \\ \tilde{e}(t) = e(t) - m_e = e(t) - 1 \end{cases}$$

$$\Rightarrow \tilde{y}(t) = \tilde{e}(t-1) + 2\tilde{e}(t-2) + \tilde{e}(t-4) + 2\tilde{e}(t-5), \qquad \tilde{e}(t) \sim WN(0,1)$$

So:

$$\gamma_y(0) = \gamma_{\tilde{y}}(0) = E[\tilde{y}(t)^2] = 1 + 4 + 1 + 4 = 10$$

The canonical form of the process is not necessary to express the prediction error variance.

Even if the canonical representation is not necessary, we introduce the noise:

$$\eta(t) = e(t-1)$$

and so:

$$y(t) = \eta(t) + 2\eta(t-1) + \eta(t-3) + 2\eta(t-4) + 1, \quad \eta(t) \sim WN(1,1)$$

Consider the unbiased noise:

$$\tilde{\eta}(t) = \eta(t) - 1$$

And then:

$$y(t) = \tilde{\eta}(t) + 2\tilde{\eta}(t-1) + \tilde{\eta}(t-3) + 2\tilde{\eta}(t-4) + 7, \quad \tilde{\eta}(t) \sim WN(0,1)$$

The prediction error variance of the k-step predictor (we denote the prediction error of the k step predictor with the symbol $\varepsilon_k(t)$), can be computed considering the predictor from the noise.

1-step predictor from the noise:

$$\hat{y}(t|t-1) = 2\tilde{\eta}(t-1) + \tilde{\eta}(t-3) + 2\tilde{\eta}(t-4) + 7$$

$$Var[\varepsilon_1(t)] = E\left[\left(y(t) - \hat{y}(t|t-1)\right)^2\right] = E[\tilde{\eta}(t)^2] = 1$$

2-step predictor from the noise:

$$\hat{y}(t|t-2) = \tilde{\eta}(t-3) + 2\tilde{\eta}(t-4) + 7$$

$$Var[\varepsilon_2(t)] = E\left[\left(y(t) - \hat{y}(t|t-2)\right)^2\right] = E\left[\left(\tilde{\eta}(t) + 2\tilde{\eta}(t-1)\right)^2\right] = 1 + 4 = 5$$

3-step predictor from the noise:

$$\hat{y}(t|t-3) = \tilde{\eta}(t-3) + 2\tilde{\eta}(t-4) + 7$$

$$Var[\varepsilon_3(t)] = E\left[\left(y(t) - \hat{y}(t|t-3)\right)^2\right] = E\left[\left(\tilde{\eta}(t) + 2\tilde{\eta}(t-1)\right)^2\right] = 1 + 4 = 5$$

4-step predictor from the noise:

$$\hat{y}(t|t-4) = 2\tilde{\eta}(t-4) + 7$$

$$Var[\varepsilon_4(t)] = E\left[\left(y(t) - \hat{y}(t|t-4)\right)^2\right] = E\left[\left(\tilde{\eta}(t) + 2\tilde{\eta}(t-1) + \tilde{\eta}(t-3)\right)^2\right] = 1 + 4 + 1 = 6$$

5-step predictor from the noise:

$$\hat{y}(t|t-5) = 7$$

$$\begin{aligned} Var[\varepsilon_5(t)] &= E\left[\left(y(t) - \hat{y}(t|t-5)\right)^2\right] \\ &= E[(\tilde{\eta}(t) + 2\tilde{\eta}(t-1) + \tilde{\eta}(t-3) + 2\eta(t-4))^2] \\ &= 1 + 4 + 1 + 4 = 10 \end{aligned}$$

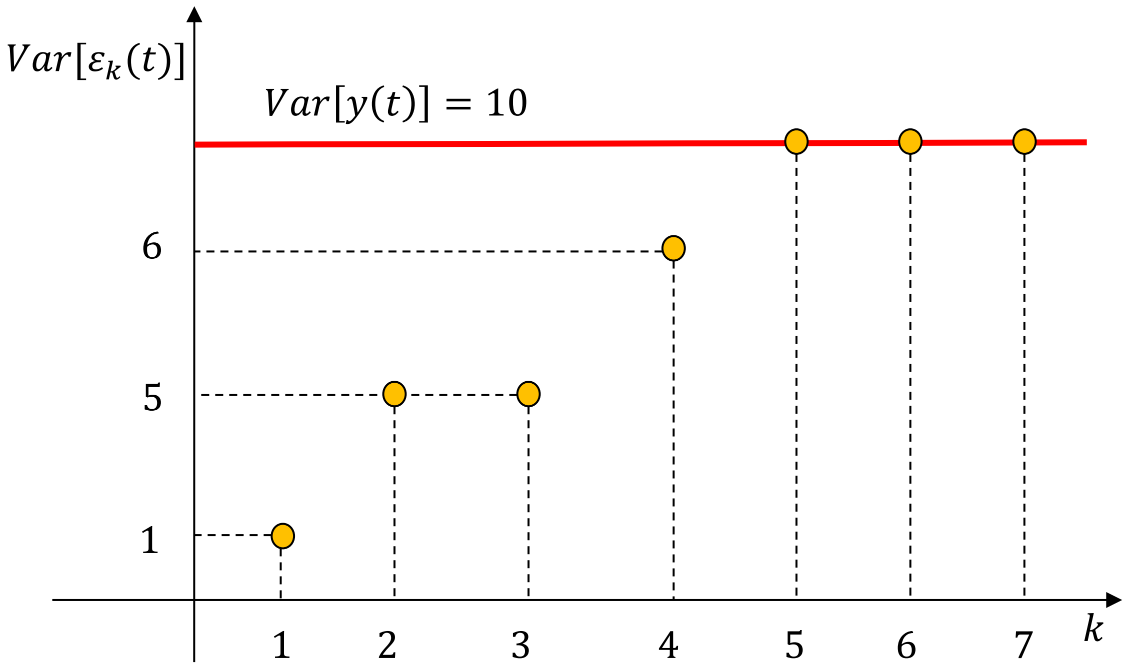
k-step predictor from the noise (for $k \geq 6$)

$$\hat{y}(t|t-k) = 7$$

$$\begin{aligned} Var[\varepsilon_k(t)] &= E\left[\left(y(t) - \hat{y}(t|t-k)\right)^2\right] \\ &= E[(\tilde{\eta}(t) + 2\tilde{\eta}(t-1) + \tilde{\eta}(t-3) + 2\eta(t-4))^2] = 10 \end{aligned}$$

which is the process variance.

So the prediction error variance as a function of the predictor horizon is:



2)

The mean value of the predictor is ALWAYS equal to the mean value of the process, for each prediction horizon k, so:

$$E[\hat{y}(t|t-k)] = m_y = 7, \quad \forall k$$