Soluzione tema esame imad

giovedì 7 gennaio 2021 14:29

Esercizio $\boxed{1}$

$$y(t) = \frac{3}{2} y(t-1) - \frac{3}{4} y(t-2) + e(t) \qquad e(t) \sim WN(0,1)$$

$$y(t) = \frac{1}{1 - \frac{3}{2} z^{-1} + \frac{3}{4} z^{-2}} e(t)$$

$\boxed{1.a}$ Dire perché $y(t)$ è stazionario

Denominatore: $1 - \frac{3}{2} z^{-1} + \frac{3}{4} z^{-2} = 0$

$$z^2 - \frac{3}{2} z + \frac{3}{4} = 0 \Rightarrow \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4} - \cancel{4} \cdot \frac{3}{\cancel{4}}}}{2}$$

$$= \frac{\frac{3}{2} \pm \sqrt{\frac{9-12}{4}}}{2} = \frac{\frac{3}{2} \pm j\sqrt{\frac{3}{4}}}{2}$$

$$|z_1| = |z_2| = \left| \frac{3}{4} + \frac{1}{2} j \sqrt{\frac{3}{4}} \right| = \sqrt{\frac{9}{16} + \frac{3}{16}}$$

$$= \sqrt{\frac{12}{16}} = \frac{2}{4}\sqrt{3} = \boxed{\frac{\sqrt{3}}{2}}$$

Dato che $e(t)$ è PSS e $|z_1| = |z_2| < 1$, allora $y(t)$ è PSS

$\boxed{1.b}$ Calcolare media m_y e varianza $\gamma_y(0)$ di $y(t)$

- $m_y = 0$ poiché $\mathbb{E}[e(t)] = 0$ e $y(t)$ è PSS

- $\gamma_y(0) = \mathbb{E}\left[(y(t) - m_y)^2\right] = \mathbb{E}\left[y(t)^2\right] =$

$$= \mathbb{E}\left[\left(\frac{3}{2} y(t-1) - \frac{3}{4} y(t-2) + e(t)\right)^2\right] =$$

$$= \mathbb{E}\left[\left(\frac{3}{2}y(t-1) - \frac{3}{4}y(t-2) + e(t)\right)^2\right] =$$

$$= \mathbb{E}\left[\frac{9}{4}y(t-1)^2 + \frac{9}{16}y(t-2)^2 + e(t)^2 - \frac{18}{8}y(t-1)y(t-2) + \cancel{\frac{6}{2}y(t-1)e(t)} - \cancel{\frac{6}{4}y(t-2)e(t)}\right] =$$

$$= \frac{9}{4}\gamma_y(0) + \frac{9}{16}\gamma_y(0) + 1 - \frac{9}{4}\gamma_y(1) =$$

$$\Rightarrow \gamma_y(0)\left[1 - \frac{9}{4} - \frac{9}{16}\right] = 1 - \frac{9}{4}\gamma_y(1)$$

$$\gamma_y(0)\left[\frac{16-36-9}{16}\right] = 1 - \frac{9}{4}\gamma_y(1)$$

$$\gamma_y(0)\cdot\left(-\frac{29}{16}\right) = 1 - \frac{9}{4}\gamma_y(1) \Rightarrow$$

$$\boxed{\gamma_y(0) = -\frac{16}{29} + \frac{36}{29}\gamma_y(1)}$$

- $\gamma_y(1) = \mathbb{E}\left[\left(\frac{3}{2}y(t-1) - \frac{3}{4}y(t-2) + e(t)\right)\cdot y(t-1)\right] =$

$$= \mathbb{E}\left[\frac{3}{2}y(t-1)^2 - \frac{3}{4}y(t-1)y(t-2) + \cancel{y(t-1)e(t)}\right]$$

$$= \frac{3}{2}\gamma_y(0) - \frac{3}{4}\gamma_y(1)$$

$$\Rightarrow \gamma_y(1)\left[1 + \frac{3}{4}\right] = \frac{3}{2}\gamma_y(0) \Rightarrow \gamma_y(1)\cdot\frac{7}{4} = \frac{3}{2}\gamma_y(0)$$

$$\Rightarrow \boxed{\gamma_y(1) = \frac{6}{7}\gamma_y(0)}$$

$$\begin{cases} \gamma_y(0) = -\frac{16}{25} + \frac{36}{25}\gamma_y(1) \\ \gamma_y(1) = \frac{6}{7}\gamma_y(0) \end{cases} \qquad \begin{cases} \gamma_y(0) = -\frac{16}{25} + \frac{36}{25}\cdot\frac{6}{7}\gamma_y(0) \\ " \end{cases}$$

$$\begin{cases} \gamma_y(0)\left[1 - \frac{216}{203}\right] = \frac{-16}{25} \Rightarrow \gamma_y(0) = \frac{-16}{25}\cdot\left(-\frac{203}{13}\right) \\ " \end{cases}$$

$$\boxed{\approx 8,6194}$$

$\boxed{1.c}$ Calcolare lo spettro in $\omega = 0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \pi$

$$y(t) = \frac{3}{2}y(t-1) - \frac{3}{4}y(t-2) + e(t) \qquad e(t) \sim WN(0,1)$$

$$y(t) = \frac{1}{1 - \frac{3}{2}z^{-1} + \frac{3}{4}z^{-2}}\, e(t)$$

$$\Gamma_y(\omega) = \left|\frac{1}{1 - \frac{3}{2}z^{-1} + \frac{3}{4}z^{-2}}\right|^2_{z=e^{j\omega}} \cdot \Gamma_e(\omega) =$$

$$= \left(\frac{1}{1 - \frac{3}{2}e^{-j\omega} + \frac{3}{4}e^{-2j\omega}}\right)\cdot\left(\frac{1}{1 - \frac{3}{2}e^{j\omega} + \frac{3}{4}e^{2j\omega}}\right)\cdot 1$$

$$= \frac{1}{1 - \frac{3}{2}e^{+j\omega} + \frac{3}{4}e^{2j\omega} - \frac{3}{2}e^{-j\omega} + \frac{9}{4} - \frac{9}{8}e^{j\omega} + \frac{3}{4}e^{-2j\omega} - \frac{9}{8}e^{-j\omega} + \frac{9}{16}}$$

$$= \frac{1}{1 + \frac{9}{4} + \frac{9}{16} - \frac{3}{2}\left(e^{j\omega} + e^{-j\omega}\right) - \frac{9}{8}\left(e^{j\omega} + e^{-j\omega}\right) + }$$

$$1+\frac{9}{4}+\frac{9}{16}-\frac{3}{2}\left(e^{j\omega}+e^{-j\omega}\right)-\frac{9}{8}\left(e^{j\omega}+e^{-j\omega}\right)+$$
$$+\frac{3}{4}\left(e^{+2j\omega}+e^{-2j\omega}\right)$$

$$=\frac{1}{\frac{16+36+9}{16}-3\cos\omega-\frac{9}{4}\cos\omega+\frac{3}{2}\cos 2\omega}$$

$$=\boxed{\frac{1}{\frac{61}{16}-\frac{21}{4}\cos\omega+\frac{3}{2}\cos 2\omega}}$$

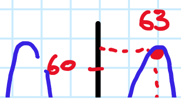
- $$\Gamma_y(0)=\frac{1}{\frac{61}{16}-\frac{21}{4}+\frac{3}{2}}=\boxed{16}$$

- $$\Gamma_y\left(\frac{\pi}{6}\right)=\frac{1}{\frac{61}{16}-\frac{21}{4}\cdot\frac{\sqrt{3}}{2}+\frac{3}{2}\cdot\frac{1}{2}}=\boxed{63}$$

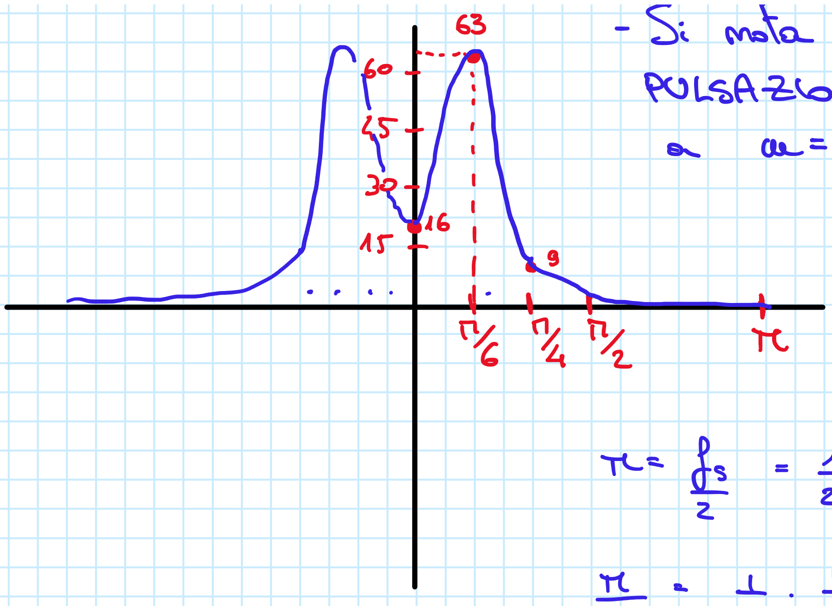
- $$\Gamma_y\left(\frac{\pi}{4}\right)=\frac{1}{\frac{61}{16}-\frac{21}{4}\cdot\frac{\sqrt{2}}{2}+\frac{3}{2}\cdot 0}=\boxed{9,88}$$

- $$\Gamma_y(\pi)=\frac{1}{\frac{61}{16}-\frac{21}{4}(-1)+\frac{3}{2}\cdot 1}=\boxed{0,0947}$$

- $$\Gamma_y(\pi/2)=\frac{1}{\frac{61}{16}-\frac{21}{4}\cdot 0+\frac{3}{2}\cdot(-1)}=\boxed{0,4324}$$



- Si nota una
PULSAZIONE DOMINANTE



- Si nota una PULSAZIONE DOMINANTE a $\omega = \frac{\pi}{6}$

$$\pi = \frac{f_s}{2} = \frac{1}{2}$$

$$\frac{\pi}{6} = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}$$

$\Downarrow$

Dovremmo osservare una sinusoide dominante con periodo di 12 campioni

$\Downarrow$

La risposta è il grafico di figura $\boxed{A}$

Esercizio $\boxed{2}$

$$y(t) = -0{,}3\, y(t-2) + e(t) - 2\, e(t-2) \qquad e(t) \sim WN(0,1)$$

$\boxed{2.a}$ Predittore ottimo a 1 passo dai dati

$$y(t) = \frac{1 - 2z^{-2}}{1 + 0{,}3 z^{-2}}\, e(t)$$ $\Rightarrow$ NON È IN FORMA CANONICA (zero $| \; | > 1$)

$$\Rightarrow \quad y(t) = \frac{1 - \cancel{2z^{-2}}}{1 + 0{,}3 z^{-2}} \cdot \frac{1 - \frac{1}{2} z^{-2}}{1 - \cancel{2z^{-2}}} \cdot (-2) \cdot e(t)$$

$$= \frac{1 - \frac{1}{2} z^{-2}}{1 + 0{,}3 z^{-2}}\, \eta(t), \qquad \eta(t) \sim WN(0, 4)$$

$$\bullet\ \hat{y}(t|t-1) = \frac{C(z) - A(z)}{C(z)} y(t) = \frac{1 - \frac{1}{2} z^{-2} - 1 - 0{,}3 z^{-2}}{1 - \frac{1}{2} z^{-2}} y(t)$$

$$= \frac{-0{,}8 z^{-2}}{1 - 0{,}5 z^{-2}} y(t)$$

$$\hat{y}(t|t-1) = -0{,}8\, y(t-2) + 0{,}5\, \hat{y}(t-2|t-3)$$

$\boxed{2.b}$ Calcolare $\hat{y}(8|7)$

$$\hat{y}(8|7) = -0{,}8\, y(6) + 0{,}5\, \hat{y}(6|5) = \boxed{-1{,}2}$$

$$\hat{y}(6|5) = -0{,}8\, y(4) + 0.5\, \hat{y}(4|3) = \boxed{2{,}2680}$$

$$\hat{y}(4|3) = -0{,}8\, y(2) + 0.5\, \hat{y}(2|1) = -0.8 \cdot 0{,}73 = \boxed{-0{,}5840}$$

$$\hat{y}(2|1) = -0.8\, \underbrace{y(0)}_{0} + 0.5\, \underbrace{\hat{y}(0|-1)}_{0} = \boxed{0}$$

$\boxed{2.c}$ Varianza errore predizione ad un passo

$$Var\left[y(t) - \hat{y}(t|t-1)\right] = Var\left[\eta(t)\right] = 4$$

Esercizio $\boxed{3}$

$$M(\theta):\ y(t) = a\, y(t-1) + e(t)$$

Troviamo il predittore $\hat{y}(t|t-1)$ del modello $M(\theta)$:

$$\hat{y}(t|t-1) = a\, y(t-1)$$

$$\hat{a} = \arg\min J(a)$$

$$J(a) = \mathbb{E}\left[\left(y(t) - \hat{y}(t|t-1)\right)^2\right] = \mathbb{E}\left[\left(y(t) - a\, y(t-1)\right)^2\right]$$

$$= \mathbb{E}\left[y(t)^2 + a^2 y(t-1)^2 - 2a\, y(t)\, y(t-1)\right] =$$

$$= \gamma_y(0) + a^2 \gamma_y(0) - 2a\, \gamma_y(1)$$

$$\frac{dJ(a)}{da} = 0 \Rightarrow 2a\gamma_y(0) - 2\gamma_y(1) = 0$$

$$\Rightarrow \boxed{\hat{a} = \frac{\gamma_y(1)}{\gamma_y(0)}}$$

Questa è la stima che il nostro modello $M(\theta)$ ci fornisce

3.a

$$S: \quad y(t) = 0.3\, y(t-1) + \eta(t), \quad \eta(t) \sim WN(0, 1)$$

- $\gamma_y(0) = \mathbb{E}\left[(y(t) - m_y)^2\right] = \mathbb{E}\left[y(t)^2\right] =$

$$= \mathbb{E}\left[\left(0{,}3\, y(t-1) + \eta(t)\right)^2\right] =$$

$$= \mathbb{E}\left[0{,}09\, y(t-1)^2 + \eta^2(t) + 0{,}6\, y(t-1)\,\eta(t)\right] =$$

$$= 0{,}09\, \gamma_y(0) + 1 \Rightarrow \gamma_y(0)\left[1 - 0{,}09\right] = 1$$

$$\Rightarrow \gamma_y(0) = \frac{1}{0{,}91} = 1{,}098$$

- $$\gamma_y(1) = \mathbb{E}\left[y(t)\, y(t-1)\right] = \mathbb{E}\left[\left(0{,}3\, y(t-1) + \eta(t)\right) y(t-1)\right] =$$

$$= \mathbb{E}\left[0{,}3\, y(t-1)^2 + y(t-1)\,\eta(t)\right] = 0{,}3\, \gamma_y(0)$$

$$\Rightarrow \hat{a} = \frac{\gamma_y(1)}{\gamma_y(0)} = \frac{0{,}3\, \gamma_y(0)}{\gamma_y(0)} = \boxed{0{,}3}$$

È IL VALORE VERO DEL PARAMETRO!

Questo lo abbiamo poiché $S \in \mathcal{M}(\theta)$

3.b

$$S:\ y(t) = 0{,}3\, y(t-1) + \eta(t) + 0{,}5\, \eta(t-1), \quad \eta(t) \sim WN(0,1)$$

- $$\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] = \mathbb{E}\left[\left(0{,}3\, y(t-1) + \eta(t) + 0{,}5\, \eta(t-1)\right)^2\right]$$

$$= \mathbb{E}\Big[0{,}09\, y(t-1)^2 + \eta(t)^2 + 0{,}25\, \eta(t-1)^2 + 0{,}6\, \cancel{y(t-1)\,\eta(t)} + \underbrace{0{,}3 \cdot 2 \cdot 0{,}5}_{0{,}3} \cdot y(t-1)\,\eta(t-1) + 0{,}5 \cdot 2\, \cancel{\eta(t)\,\eta(t-1)}\Big] =$$

$$= 0{,}09\, \gamma_y(0) + 1 + 0{,}25 + \mathbb{E}\left[0{,}3\, y(t-1)\,\eta(t-1)\right] =$$

$$\mathbb{E}\left[0{,}3\, y(t-1)\,\eta(t-1)\right] =$$

$$\mathbb{E}\left[\left(0{,}09\, y(t-2) + 0{,}3\, \eta(t-1) + 0{,}15\, \eta(t-2)\right) \eta(t-1)\right]$$

$$= \mathbb{E}\left[\left(0{,}09\, y(t-2) + 0{,}3\, \eta(t-1) + 0{,}15\, \eta(t-2)\right)\cdot \eta(t-1)\right]$$

$$= 0{,}3$$

$$= 0{,}09\, \gamma_y(0) + 1{,}55 \Rightarrow \gamma_y(0)\left[1 - 0{,}09\right] = 1{,}55$$

$$\Rightarrow \gamma_y(0) \simeq \boxed{1{,}7}$$

- $$\gamma_y(1) = \mathbb{E}\left[y(t)\, y(t-1)\right] = \mathbb{E}\left[\left(0{,}3\, y(t-1) + \eta(t) + 0{,}5\, \eta(t-1)\right) y(t-1)\right]$$

$$= \mathbb{E}\left[0{,}3\, y(t-1)^2 + \cancel{y(t-1)\, \eta(t)} + 0{,}5\, y(t-1)\, \eta(t-1)\right]$$

$$= 0{,}3\, \gamma_y(0) + 0{,}5 \Rightarrow \gamma_y(1) \simeq 1{,}7 \cdot 0{,}3 + 0{,}5$$

$$= \boxed{1{,}01}$$

$$\Rightarrow \hat{a} = \frac{\gamma_y(1)}{\gamma_y(0)} \simeq \frac{1{,}01}{1{,}7} = \boxed{0{,}59}$$

$\hat{a} \neq 0{,}3$ in quanto $S \notin \mathcal{M}(\theta)$

Esercizio $\boxed{5}$

Dataset a disposizione

W ← | E   C | | P   S | | AP   AS |

ANNO 2020

ANNO 2019

|

ANNO 2000

Quantità da predire: W. Se W > 95, la squadra va ai playoff, altrimenti non ci va

Da cosa dipende W? Da E e da C

↓

Un modo per stimare W è fare una regressione lineare usando E e C, per esempio definendo la quantità D = E - C:

$$\hat{W} = \theta_0^w + \theta_1^w \cdot D$$

Come faccio ad ottenere una stima di D? Devo conoscere E e C.

- E dipende da P e S
- C dipende da AP e AS

Posso stimare E e C con altre due regressioni lineari:

$$\hat{E} = \theta_0^E + \theta_1^E \cdot P + \theta_2^E \cdot S$$

$$\hat{C} = \theta_0^C + \theta_1^C \cdot AP + \theta_2^C \cdot AS$$

Così che:

$$\hat{W} = \theta_0^w + \theta_1^w \cdot \hat{D}$$

$$= \theta_0^w + \theta_1^w \cdot (\hat{E} - \hat{C})$$

e vedere se $\hat{W} > 95$.

e vedere se $\hat{W} > 55$.

Un'alternativa è quella di creare una nuova variabile:

$$W_D = \begin{cases} 1 & \text{se } W > 55 \\ 0 & \text{altrimenti} \end{cases}$$

e usare una regressione logistica per stimare la probabilità che $W_D = 1$