Part 1

Main constraints

- each client is visited (served) once
- all providers start at 14HS (depot) and return to it
- providers must return to 14HS by hour 'l'.
- service times of clients must be respected
  → provider can arrive earlier, or at, but no later than the start time of a client's service.

Data

- l : end of work (all providers must be back to 14HS by hour l)
- $N = \{2, \dots, N+1\}$ clients (N of them)
  - s_i : service start time
  - d_i : service duration
- $M = \{1, \dots, M\}$ providers (M of them)
  - f_i : hiring cost (one-off)
  - w_i : earliest available starting hour.
- $N' = \{1, \underbrace{2, \dots, N, N+1}_{N \text{ clients}}\}$ = all locations

  1 ↑ 14HS
- t_{ij} : travel time between location $i, j \in N'$.

Variables

- $x_{ijk} = 1$ if provider k moves from location i to j, $i, j \in N'$.

Objective (1)

$$\min \sum_{k \in M} \sum_{j \in N} f_k \cdot x_{1jk}$$ ($\sum$ hiring cost × provider leaves 14HS)

Constraints

routing constraints

(2) $$\sum_{i \in N'} \sum_{k \in M} x_{ijk} = 1 \quad , j \in N$$ (clients are served once, by one vehicle)

(3) $$\sum_{i \in N'} x_{ipk} = \sum_{j \in N'} x_{pjk} \quad , k \in M, p \in N$$ (route continuity)

(4) $$\sum_{j \in N} x_{1jk} \leq 1, \quad k \in M$$ (not all providers need to work)

(7) $$\sum_{j \in N} \sum_{k \in M} x_{1jk} \geq 1$$ (at least 1 provider must work)

(9) $$u_{ik} - u_{jk} + |N| x_{ijk} \leq |N| - 1 \quad , i, j \in N, k \in M$$ (subtour elimination constraints.)

time constraints

(10) $x_{1jk} \cdot (w_k + t_{1j}) \leq s_j$, $k \in M, j \in N$ (provider must not arrive later than start time of first client)

(11) $x_{ijk} \cdot (s_i + t_{ij} + d_i) \leq s_j$, $i \in N, j \in N, k \in M$ (service times respected)

(12) $x_{i1k}(s_i + t_{i1} + d_i) \leq l$, $i \in N, k \in M$ (come back to 14HS by hour l)

(13) $x_{ijk} \in \{0,1\}$, $i,j \in N'$, $k \in M$ (integrality constraints)

$1 \leq u_{ik} \leq |N|$, $i \in N'$, $k \in M$

→ Example 1: (5 providers, 6 clients)

Providers	Routes
1	1 → 2 → 1
2	1 → 7 → 1
3	1 → 6 → 1
4	1 → 3 → 4 → 5 → 1

$z^* = 45$. $\left(= \sum_{i=1}^{4} f_i \right)$

- Provider 1 hired for \$15. (0 → 1 → 0)
  - 0h: available, leave to client 1
  - 3h: arrives at client 1, waits 1hr.
  - 4h: starts service, for 2 hours.
  - 6h: leave to 14HS.
  - 8h: arrives at 14HS.
- Provider 2 hired for \$10. (0 → 6 → 0)
  - 3h: available, leave to client 6
  - 5h: arrives at client 6, starts service, for 2hrs.
  - 7h: leave to 14 HS.
  - 8h: arrives at 14HS.
- Provider 3 hired for \$8. (0 → 5 → 0)
  - 4h: avail, leave to c5.
  - 5h: arrive at C5, serve for 1hr
  - 6h: leave to 14HS
  - 7h: return to 14HS

Part 2 - extensions

- make providers to be paid on hourly basis.
  (so f_K := hiring cost of provider K per hour)

→ change objective function to minimise hiring costs based on # hours worked for each vehicle.

$$\cancel{\min \sum_{K \in M} \sum_{j \in N} f_K \cdot x_{1jK}} \longrightarrow \text{change to:}$$

$$\min \sum_{K \in M} f_K \times (\text{\# worked hours for provider } K) \cdot \mathbb{I}(\text{provider } K \text{ worked})$$

$$= \min \sum_{K \in M} \sum_{j \in N} x_{1jK} \cdot f_K \left[\sum_{i \in N'} \sum_{j \in N'} x_{ijK} \left(\underbrace{t_{ij}}_{\text{travel time}} + \underbrace{d_i}_{\text{service time}} + \underbrace{\max(0, s_j - s_i - t_{ij} - d_i)}_{\text{waiting time: 0 or 1,2,3 etc.}} \right) \right]$$

3:

4: $\xrightarrow{1}$ 5:

6: serve.

$s_j - (s_i + d_i + t_{ij})$

$6 - 3 - 1 - 1 = 1$

data 1.

1.

2.

P1: 8 hrs — waiting: 3hrs.

P2: 5hrs

P3: 3hrs

P4: 8hrs

} \$290

P2: 5hrs

P3: 3

P4: 8

P5: 6 hrs.

} \$260