CSE 191: Discrete Structures

Predicate Logic

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Some of the slides are either directly used or adapted from Prof. Andrew Hughes's CSE 191 presentations and are also based on the textbook "Discrete Mathematics and it's applications" (7th edition) by Kenneth H. Rosen.

Outline

- Predicates and Quantifiers
  - From Propositions to Predicates
  - Quantifiers

Predicates and Quantifiers

From Propositions to Predicates

Consider the statement "*X* is even."

$f(x): x^2+x+1$

- Contains the variable *X*, so it is not a proposition.
  - Given a value for *X*, we can determine the truth value.
  - Once *X* is filled, sentence is TRUE or FALSE, but not both.
- Sentences whose truth value is based on variables are predicates.
  - P(X): X is even.

From Propositions to Predicates

Definition

A **predicate** is a **function** that takes some **variable(s) as arguments**; it returns either TRUE or FALSE (but never both) for each combination of the argument values.

- In contrast, a proposition is a function of 0 variables.
  - Propositions have no variables.
  - Each proposition is either TRUE or FALSE (but not both).

From Propositions to Predicates

Definition

A **predicate** is a **function** that takes some **variable(s) as arguments**; it returns either TRUE or FALSE (but never both) for each combination of the argument values.

- Predicate variables are derived from an associated domain of discourse.
  - Domain of discourse describes all allowable argument values

P(x): x is even

- E.g., *x* is an even number.
  - We may want to ask, which set does x come from?
  - P(*x*): *x* is an even number, where x takes a value among the integers.
    - P: "is an even number"
    - x: the variable
    - Integers: the domain
  - P(*x*) is the value of the **propositional function** P at *x*.

Outline

- Predicates and Quantifiers
  - From Propositions to Predicates
    - Predicate definition
    - Domain of Discourse definition
    - Predicate examples
  - Quantifiers

From Propositions to Predicates

<u>Definition</u>

Given a predicate $P(x)$, the **domain of discourse** (often referred to as the **domain**) is a set of all possible values for the variable x.

- Predicates with multiple variables may have:
  - multiple domains of discourse, one for each variable, or
  - a single domain of discourse for all variables

From Propositions to Predicates

Example 1

Consider the predicate $Q(x, y)$ defined by:

$Q(x, y)$: y is enrolled in recitation x.

where

- the domain of discourse for x is {A1, A2, A3, *A4*, *B1*, *B2*, *B3*, *B4*}.
- the domain of discourse for y is {all students in CSE 191}.

Example 2

Consider the predicate $R(x, y)$ defined by:

$R(x, y)$: x and y are friends.

where

the domain for R is {all students in CSE 191}.

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Predicate vs. Proposition

is $P(1,4)$ a true proposition?

Example

Let the predicate $P(x, y)$ be defined by

$P(1,4): 2\times 1 = 4$

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

Statement	Is Proposition?	Truth Value
$P(1, 4)$: $2(1) = 4$	Yes	FALSE

Predicate vs. Proposition

Example

Let the predicate $P(x, y)$ be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

Statement	Is Proposition?	Truth Value
P(1, 4): 2(1) = 4	Yes	FALSE
P(2, 4): 2(2) = 4		

Predicate vs. Proposition

<u>Example</u>

Let the predicate $P(x, y)$ be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

Statement	Is Proposition?	Truth Value
P(1, 4): 2(1) = 4	Yes	FALSE
P(2, 4): 2(2) = 4	Yes	TRUE

Predicate vs. Proposition

<u>Example</u>

Let the predicate $P(x, y)$ be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

Statement	Is Proposition?	Truth Value
P(1, 4): 2(1) = 4	Yes	FALSE
P(2, 4): 2(2) = 4	Yes	TRUE
P(x, 4): 2(x) = 4	No	---

Predicate vs. Proposition

<u>Example</u>

Let the predicate $P(x, y)$ be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

Statement	Is Proposition?	Truth Value
P(1, 4): 2(1) = 4	Yes	FALSE
P(2, 4): 2(2) = 4	Yes	TRUE
P(x, 4): 2(x) = 4	No	---
P(3, y): 2(3) = y		

Predicate vs. Proposition

<u>Example</u>

Let the predicate $P(x, y)$ be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

Statement	Is Proposition?	Truth Value
P(1, 4): 2(1) = 4	Yes	FALSE
P(2, 4): 2(2) = 4	Yes	TRUE
P(x, 4): 2(x) = 4	No	---
P(3, y): 2(3) = y	No	---

Predicate vs. Proposition

<u>Example</u>

Let the predicate $P(x, y)$ be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

Statement	Is Proposition?	Truth Value
P(1, 4): 2(1) = 4	Yes	FALSE
P(2, 4): 2(2) = 4	Yes	TRUE
P(x, 4): 2(x) = 4	No	---
P(3, y): 2(3) = y	No	---

- What about $P(2, 3) : 2(2) = 3$?
  - $P(2, 3)$ is meaningless (in this example).
  - 3 is not in the specified domain for y.

Predicate vs. Proposition

<u>Example</u>

Let the predicate $P(x, y)$ be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

$\underline{2\times1=4} \lor 2\times3=6$

$F \lor T \equiv T$

Statement	Is Proposition?	Truth Value
P(1, 4) ∨ P(3, 6)	Y	T
P(1, 4) ∨ ¬P(3, 6)		
P(2, 4) → P(2, 5)		
P(2, 4) ∧ P(x, 4)		

Predicate vs. Proposition

<u>Example</u>

Let the predicate $P(x, y)$ be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for x is $\{1, 2, 3\}$
- the domain for y is $\{4, 5, 6\}$.

Statement	Is Proposition?	Truth Value
$P(1, 4) \vee P(3, 6)$	Yes	TRUE
$P(1, 4) \vee \neg P(3, 6)$	Yes	FALSE
$P(2, 4) \rightarrow P(2, 5)$	Yes	FALSE
$P(2, 4) \wedge P(x, 4)$	No	---

More Predicate Examples

<u>Example</u>

Let the predicate $Q(x, y)$ be defined by:

$$Q(x, y) : x + y > 4$$

- the domain of discourse for x and y is **all integers.**

- Which of the following are predicates? Which are propositions?
  - $Q(1, 2)$ proposition
  - $Q(x, 2)$ predicate since x is left as a variable
  - $Q(1000, y)$ predicate since y is left as a variable
  - $Q(1000, 2)$ proposition
  - $Q(x, y)$ predicate since x and y are left as a variable

Outline

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$P(x)$: x is even

Domain: all integers

$P(x)$ is true

Quantifiers

- Quantification expresses the extent to which a predicate is true over a range of elements.
  - English examples: all, some, none, many, few.

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        - Existential Quantifier
        - Examples: Quantifiers and English
        - Quantifier Negation
        - Quantifier Negation Rule
        - Nested Quantifiers
        - Nested Quantifier Ordering
        - Nested Quantifier Scoping

Universal Quantification

Definition

Suppose $P(x)$ is a predicate on some domain.

The **universal quantification of** $P(x)$ is the proposition:
"$P(x)$ is true for all x in the domain of discourse D."

Written as: $\forall x, P(x)$

Read as: **"For all** $x,\ P(x)$**"**, or **"For every** $x,\ P(x)$**"**

Here $\forall$ is called the universal quantifier.

- $\forall x, P(x)$ is TRUE **if** $P(x)$ is TRUE for every x in D.
- $\forall x, P(x)$ is FALSE **if** $P(x)$ is FALSE for some x in D.

Universal Quantification Example

Example

$P(x) : x + 2 = 5$, domain of discourse: $\{1, 2, 3\}$.

- $\forall x, P(x)$ means: "for all x in $\{1, 2, 3\}$, $x + 2 = 5$."
  - $\forall x, P(x) \equiv P(1) \wedge P(2) \wedge P(3)$, or
  - $\forall x, P(x) \equiv (1 + 2 = 5) \wedge (2 + 2 = 5) \wedge (3 + 2 = 5)$

F ∧ F ∧ T = F

$\therefore$ $\forall x, P(x)$ is FALSE (since $1 + 2 = 5$ and $2 + 2 = 5$ are both FALSE).

counterexample = 1

Note: The symbol $\therefore$ denotes "therefore".

$P(x) : x+2=5$

$D : \{3\}$

$\forall x\ P(x)$ true?

$P(3) : 3+2=5$

T

Y

- An input that causes a universally quantified statement to evaluate to FALSE is called a **counterexample**.
- The meaning of the universal quantification of P(x) changes if the domain is changed.

Universal Quantification Example

Example

$$A(x) : x \text{ is even}$$
$$B(x) : x^2 > 0$$
$$C(x) : x < 2$$

where the domain of discourse for A, B, and C is $\{0, 1, 2, 3\}$.

True or False?

$x=0 \quad C(0) \to A(0) \equiv 0<2 \to 0$ is even $\equiv T \to T \equiv T$

$x=1 \quad C(1) \to A(1) \equiv 1<2 \to 1$ is even $\equiv T \to F \equiv F$

counterexample: 1

F
- $\forall x, (C(x) \to A(x))$. - FALSE. A counterexample is $x = 1$.

∨

- $\forall x, (B(x) \vee C(x))$. – TRUE. Consider the table

x	$B(x)$	$C(x)$	
0	F	T	T
1	T	T	T
2	T	F	T
3	T	F	T

Universal Quantification Example

Example

$S(x)$: x is a student in CSE191.
$T(x)$: x is a computer science major.
the domain of discourse for S and T is {all students enrolled in CSE191}.

True or False?

- $\forall x, S(x)$ TRUE. Everyone enrolled is a student in CSE191.
- $\forall x, T(x)$ FALSE. Some of you are not majoring in CS.
- $\forall x, (S(x) \rightarrow T(x))$ FALSE. There are students in CSE 191 not majoring in computer science.
- $\forall x, (T(x) \rightarrow S(x))$ TRUE. Think carefully about the domain.

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Existential Quantification

Definition

Suppose $P(x)$ is a predicate on some domain of discourse.

The **existential quantification of** $P(x)$ is the proposition:
"$P(x)$ is true for some x in the domain of discourse D."

Written as: $\exists x, P(x)$

Read as: **"There exists x, $P(x)$", or "For some x, $P(x)$"**

Here $\exists$ is called the existential quantifier.

- $\exists x, P(x)$ is TRUE **if** $P(x)$ is TRUE for some x in D.
- $\exists x, P(x)$ is FALSE **if** for every x in D, $P(x)$ is FALSE.

Existential Quantification Example

Example

$P(x)$: $x + 2 = 5$, domain of discourse: $\{1, 2, 3\}$.

- $\exists x, P(x)$ means: "for some x in $\{1, 2, 3\}$, $x + 2 = 5$."
    - $\exists x, P(x) \equiv P(1) \vee P(2) \vee P(3)$, or
    - $\exists x, P(x) \equiv (\underset{F}{1 + 2 = 5}) \vee (\underset{F}{2 + 2 = 5}) \vee (\underset{T}{3 + 2 = 5}) \equiv T$

$\therefore$ $\exists x, P(x)$ is TRUE (since $3 + 2 = 5$ is TRUE).

satisfying assignment : 3

P(x): x+2=5

D: {4,5,6}

$\exists x\, P(x)$ true? NO

False

- The meaning of the existential quantification of P(x) may change if the domain is changed.
- An input that causes a predicate to evaluate to TRUE is called a **satisfying assignment**.

Existential Quantification Example

<u>Example</u>

$$A(x) : x = 1$$
$$B(x) : x > 5$$
$$C(x) : x < 5$$

where the domain of discourse for A, B, and C is $\{1, 2, 3\}$.

$x=1 \quad C(1) \to A(1) \equiv 1<5 \to 1=1 \equiv T \to T \equiv T$

True or False?

satisfying assignment: 1

T • $\exists x, (C(x) \to A(x))$.

F • $\exists x, B(x)$.

$x=1 \quad B(1) \equiv 1>5 \quad F$

$x=2 \quad B(2) \equiv 2>5 \quad F$

$x=3 \quad B(3) \equiv 3>5 \quad F$

Existential Quantification Example

<u>Example</u>

$S(x)$: x is a student in CSE191.
$T(x)$: x is a computer science major.

Domain of discourse is {all students enrolled in CSE191}.

True or False?

- T $\exists x, S(x)$.
- T $\exists x, T(x)$.

Note: The quantifiers ∀ and ∃ have higher precedence than all logical operators.

Universal Quantification Example

<u>Example</u>

$$P(x) : x^2 > 9$$
$$Q(x) : x^2 > 0$$

where the domain of discourse is *all Integers.*

True or False?

- F $\forall x, P(x)$ counterexample: 1
- T $\exists x, P(x)$ satisfying assignment: 4
- F $\forall x, Q(x)$ counterexample: 0
- T $\exists x, Q(x)$ satisfying assignment: 2

Binding Variables

- The occurrence of a variable x is said to be **bound** when a quantifier is used on that variable.

$\forall x\, P(x) \to Q(y)$

- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.
- The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.
  - A variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.
- <u>Example:</u>
  - $\exists x\,(x + y = 1)$
    - The variable x is bound by the existential quantifier $\exists x$.
    - y is free.

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Quantified Statements and English

<u>Example:</u> $L(s,t)$: s Loves t

Suppose $L(x, y)$: x loves y, where

- the domain of ~~x~~ s is all CSE 191 students, and
- the domain of ~~y~~ t is the courses offered by UB CSE.

~~$\exists x\ L(x,191) \wedge L(x,250)$~~

- $\exists x, (L(x, \text{CSE } 191) \wedge L(x, \text{CSE } 250))$:
  - A CSE 191 student loves both CSE 191 and CSE 250.
- $\exists x\ \exists y\ \forall z, ((x \neq y) \wedge (\overset{L(a,b)}{L(x, z)} \rightarrow L(y, z)))$:
  - There are two different students x and y in CSE 191 such that if x loves a CSE course, then y loves it as well.
- Every CSE course is loved by some student in CSE 191: ($\forall y$, $\exists x$)

  $\forall y\ \exists x\ L(x, y)$
  - $\forall y\ \exists x, L(x, y)$.
- No student in CSE 191 loves CSE 191 and CSE 250:

  $\equiv \forall x \neg(L(x,191) \wedge L(x,250))$

  $\neg \exists x, (L(x,191) \wedge L(x,250))$

  $\forall x\ (L(x,191) \rightarrow \neg L(x,250))$
  - $\neg\exists x, (L(x, \text{CSE } 191) \wedge L(x, \text{CSE } 250))$.

Quantified Statements and English

Example:

$A(x)$: x lives in Amherst.

$D(x)$: x majors in computer science.

$B(x)$: x is a CSE 191 student.

$C(x)$: x has a good GPA.

Domain of discourse: all UB students.

$\forall x \quad B(x)$

- All UB students are CSE 191 students:
  - $\forall x,\ B(x)$

$\forall x$

- All CSE 191 students have a good GPA: $\cancel{\forall x(B(x) \to B(x))}$ $\forall x(B(x) \to C(x))$
  - $\forall x,\ (B(x) \to C(x))$
- CSE 191 students not living in Amherst major in CS: $\forall x((B(x) \land \neg A(x)) \to D(x))$
  - $\forall x,\ ((B(x) \land \neg A(x)) \to D(x))$
- No CSE 191 student lives in Amherst: $\cancel{\exists}\ \forall x\ (B(x) \to \neg A(x))$
  - $\forall x,\ (B(x) \to \neg A(x))$

$\neg B(x)$

English to Quantified Statements

Translate the following theorems to quantified statements:

$\forall x$ $P(x)$ $\rightarrow$ $Q(x+1)$

- If x is an even number, then $x + 1$ is odd.
  - Identify a domain and predicates:
    - Domain: all integers
    - $P(x)$: x is an even number.
    - $Q(x)$: x is an odd number.

Quantified statement:
$\forall x, (P(x) \rightarrow Q(x+1))$.

$\forall y$ $R(y)$ $\rightarrow$ $S(y)$

- Every even number is a multiple of 2.
  - Domain and predicates:
    - Domain: all integers.
    - $R(y)$: y is an even number.
    - $S(y)$: y is a multiple of 2.

Quantified statement:
$\forall y, (R(y) \rightarrow S(y))$.

English to Quantified Statements

Translate the following theorems to quantified statements:

- If x is an even number, then $x + 1$ is odd.
  - Identify a domain and predicates:
    - Domain: all integers
    - $P(x)$: x is an even number.
    - $Q(x)$: x is an odd number.

Quantified statement:
$\forall x, (P(x) \rightarrow Q(x+1))$.

- Every even number is a multiple of 2. (Alternative solution).
  - Domain and predicates:
    - Domain: all even integers.
    - $T(z)$: z is a multiple of 2.

Quantified statement:
$\forall z, T(z)$.

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Quantifier Negation

Example:
Suppose the domain of discourse is all UB students.
$A(x)$: x lives in Amherst. $D(x)$: x majors in computer science.

- Consider the following two propositions:
  - Not every UB student majors in computer science: $\neg \forall x\ D(x)$
    - $\neg \forall x, D(x)$.

    $\equiv$
  - Some UB students do not major in computer science: $\exists x\ \neg D(x)$
    - $\exists x, \neg D(x)$.
- Similarly,
  - There is no UB student living in Amherst: $\neg \exists x\ A(x)$
    - $\neg \exists x, A(x)$.
  - Every UB student lives outside of Amherst: $\forall x\ \neg A(x)$
    - $\forall x, \neg A(x)$.

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Quantifier Negation

Quantifier negation:

In general we have for any predicate $P(x)$:

$$\neg \forall x, P(x) \equiv \exists x, \neg P(x) \text{ and } \neg \exists x, P(x) \equiv \forall x, \neg P(x)$$

De Morgan's Law for Quantifiers			
Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, P(x) is false	There is an x for which P(x) is true
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which P(x) is false	P(x) is true for every x

Quantifier Negation

Quantifier negation:

In general we have for any predicate $P(x)$:

$$\neg \forall x, P(x) \equiv \exists x, \neg P(x) \text{ and } \neg \exists, P(x) \equiv \forall x, \neg P(x)$$

Negation Rule:

- Move the negation over a quantifier. Flip the quantifier passed.
  - $\exists$ flips to $\forall$.
    - $\neg \exists x, (\ldots)$ becomes $\forall x, \neg(\ldots)$
  - $\forall$ flips to $\exists$.
    - $\neg \forall x, (\ldots)$ becomes $\exists x, \neg(\ldots)$

Example:

- No CSE 191 student lives in Amherst:

$A(x)$: x lives in Amherst.

$B(x)$: x is a CSE 191 student.

$$\neg \exists x\, (B(x) \wedge A(x)) \equiv \forall x, \neg(B(x) \wedge A(x)).$$

$p \to q$
$\equiv \neg p \vee q$

$\neg(s \vee t) \equiv \neg s \wedge \neg t$

Quantifier Negation Examples

$\exists x\,(P(x) \to \neg Q(x))$

$\neg \exists x\,(P(x) \to \neg Q(x))$

$\equiv \forall x\, \neg(P(x) \to \underline{\neg Q(x)})$

$\equiv \forall x\, \neg(\neg P(x) \underline{\vee} \neg Q(x))$

$\equiv \forall x\,(\neg\neg P(x) \wedge \neg\neg Q(x))$

$\equiv \forall x\,(P(x) \wedge Q(x))$

$\forall x\,(P(x) \to \exists y\,(P(y) \vee Q(y)))$

$\neg \forall x\,(P(x) \to \exists y\,(P(y) \vee Q(y)))$

$\equiv \exists x\, \neg(P(x) \to \exists y\,(P(y) \vee Q(y)))$

$\equiv \exists x\, \neg(\neg P(x) \vee \exists y\,(P(y) \vee Q(y)))$

$\equiv \exists x\,(\neg\neg P(x) \wedge \neg \exists y\,(P(y) \vee Q(y)))$

$\equiv \exists x\,(P(x) \wedge \forall y\, \neg(P(y) \vee Q(y)))$

$\equiv \exists x\,(P(x) \wedge \forall y\,(\neg P(y) \wedge \neg Q(y)))$

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Nested Quantifiers

How do sentences with multiple quantifiers work?

Definition:

A logical expression with more than one quantifier that bind different variables in the same predicate is said to have **nested quantifiers**.

Need to consider their **ordering** and **scope**.

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Nested Quantifiers: Ordering

- The order of the quantifiers is important, unless all are universal or all are existential quantifiers.
  - P(x, y): x + y = y + x, where the domain is "all real numbers".
  - $\forall x\ \forall y\ P(x, y)$: "for all real number x, for all real numbers y, x + y = y + x".

<u>Example:</u>

Q(x, y): x + y = 0, where the domain is "all real numbers".

- What is the truth value of $\exists y\ \forall x\ Q(x, y)$?
  - $\exists y\ \forall x\ Q(x, y)$: "there is a real number y such that for every real number x, Q(x, y)".
  - True or false? [False]
  - Does switching the ordering of quantifiers maintain the meaning?
  - $\forall x\ \exists y\ Q(x, y)$: "for every real number x there is a real number y such that Q(x, y)".
  - True or false? [True]
- In general, we **cannot** switch the ordering and guarantee equivalence.

$5.5 + (-5.5) = 0$

Nested Quantifiers: Ordering

- Consecutive quantifiers of the same type can be reordered and maintain equivalence.
- Suppose $Q(x, y, z)$ is an arbitrary predicate:
  - $\forall i \ \forall j \ \forall k, Q(i, j, k) \equiv \ \forall j \ \forall i \ \forall k, Q(i, j, k) \equiv \ \forall k \ \forall j \ \forall i, Q(i, j, k) \equiv \ \ldots$
  - $\exists i \ \exists j \ \exists k, Q(i, j, k) \equiv \ \exists j \ \exists k \ \exists i, Q(i, j, k) \equiv \ \ldots$
- We usually simplify consecutive variables with the same quantifier:
  - $\forall i \ \forall j \ \forall k, Q(i, j, k) \equiv \ \forall i, j, k, Q(i, j, k).$
  - $\exists i \ \exists j \ \exists k, Q(i, j, k) \equiv \ \exists i, j, k, Q(i, j, k).$
- **Note:** the order variables enter $Q(\ldots)$ does not change

Nested Quantifiers: Ordering

- Consider the following:
  - $\forall x_1 \; \exists x_2 \; \forall x_3 \; \exists x_4 \ldots$
  - $\exists x_2 \; \forall x_1 \; \exists x_4 \; \forall x_3 \ldots$
  - $\forall x_1, x_3 \; \exists x_2, x_4 \ldots$
  - None of the above are guaranteed to be equivalent.

- Double-check you don’t jump over a different quantifier when simplifying.

Outline

- Predicates and Quantifiers
  - From Propositions to Predicates
  - Quantifiers
    - Universal Quantifier
    - Existential Quantifier
    - Examples: Quantifiers and Natural Language (e.g., English)
    - Quantifier Negation
    - Quantifier Negation Rule
    - Nested Quantifiers
    - Nested Quantifier Ordering
    - Nested Quantifier Scoping

Nested Quantifiers: Scope

Definition:

The portion of the formula a quantifier is covering is called the **scope of the quantifier.**

- The scope of the quantifier is the predicate immediately following.
- Precedence is just below parenthesis.

- Any variable that is not covered by a quantifier is called a **free variable**.

Consider the formula:

$$\forall i\ \exists j, (P(i,j) \rightarrow \ \forall k, Q(k))$$

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$\exists j\, P(i,j)$ $(\longrightarrow)$

Nested Quantifiers: Scope

Quantifiers can move as long as their scope doesn't encompass additional quantifiers of a different type.

Consider

$$\forall i\ \exists j, (P(i,j) \to \forall k, Q(k))$$

vs

$$\forall i\ \exists j\ \forall k, (P(i,j) \to Q(k))$$

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$$\boxed{\forall i\ \exists j\ \forall k, (P(i,j) \rightarrow\ Q(k))}$$

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Consider

$$\forall i \; \boxed{\exists j, (P(i,j) \longrightarrow \; \forall k, Q(k))}$$

vs

$$\forall i \; \boxed{\exists j \;\; \forall k, (P(i,j) \longrightarrow \; Q(k))}$$

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Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables originally covered.

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$$\forall i\ \exists j, (P(i,j) \rightarrow\ \forall k, Q(k))$$

vs

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vs

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Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables originally covered.

Consider

$$\forall i\ \exists j, (P(i,j) \longrightarrow \boxed{\forall k, Q(k))}$$

→ proposition

vs

$$\forall i\ \exists j, (\boxed{\boxed{\forall k, P(i,j)} \longrightarrow\ Q(k))}$$

→ Not a proposition

- In the second formula, *k* in *Q*(*k*) is no longer **bound** by any quantifier.
  - *k* is a free variable.
- These two formulas are not equivalent.

Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables originally covered.

Consider

$$\forall i\ \exists j, (P(i,j) \rightarrow \forall k, Q(k))$$

vs

$$\forall i\ \exists j, (\forall k, P(i,j) \rightarrow Q(k))$$

- In the second formula, *k* in *Q*(*k*) is no longer **bound** by any quantifier.
  - *k* is a free variable.
- These two formulas are not equivalent.
  - $\forall i\ \exists j, (P(i,j) \rightarrow \forall k, Q(k))$ is a proposition.
  - $\forall i\ \exists j, (\forall k, P(i,j) \rightarrow Q(k))$ is a predicate (with free variable *k*).