CSE 191: Discrete Structures
Logical Reasoning and Proof Methods

Nasrin Akhter

Slides are either directly used or adapted from Prof. Andrew Hughes's CSE 191 presentations, and are also based on the textbook "Discrete Mathematics and it's applications" (7th edition) by Kenneth H. Rosen.

Outline

- Logical Reasoning
  - Logical Reasoning Definition
  - Invalid Argument
  - Logical Reasoning Rules
  - Logical Reasoning Example
- Introduction to Mathematical Proofs

Logical Reasoning: What is it?

- Suppose the following are true statements:

  You will buy your friend lunch if they drive you to work.

  They drove you to work.

  - What can you conclude?
    - You will buy your friend lunch.
- Differs from logical equivalence
  - Statements derived are not always equivalent.
  - Can be new knowledge
  - Multiple facts can be used to derive a new statement.
- Also known as deductive reasoning.

Logical Reasoning: Arguments

- Arguments are:
  - a list of propositions, called hypotheses (also called premises), and
  - A final proposition, called the conclusion.

Form of the argument

Hypotheses (i.e., premises) → p_1, p_2, …, p_n

Conclusion → $\therefore c$

$$p_1 \wedge p_2 \wedge \ldots \wedge p_n$$

$$\downarrow$$

$$c$$

$$\begin{array}{l} p_1 \\ p_2 \\ \ldots \\ p_n \\ \hline \therefore c \end{array}$$

F→F
F→T
T→T
T

T→F ×
$\begin{array}{c} T \\ \hline \therefore F \end{array}$

Definition

- An argument is valid if:

$$(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \rightarrow c \text{ is a tautology}$$

- An argument is invalid if it is not valid.
- Fallacies are incorrect reasonings which lead to invalid arguments.

Logical Reasoning: What is it?

$P \to Q$

You will buy your friend lunch if they drive you to work.

q p

They drove you to work.

P

- You will buy your friend lunch.

$$\left.\begin{array}{l} P \to Q \quad T \\ P \quad T \\ \hline \therefore Q \end{array}\right.$$

$(T \wedge T) \to T$

$\underline{((P \to Q) \wedge P) \to Q}$

needs to be a Tautology for to be valid

Logical Reasoning: Simplest Example

Example:

$p \to p$

- Prove that the following is a valid argument: $\dfrac{p}{\therefore p}$

Proving this argument is valid is same as proving that $p \to p$ is a tautology.

p	$p \to p$
F	T
T	T

Logical reasoning proof:

1. p Hypothesis

Since we have arrived at our conclusion, our proof is complete.

- Therefore, we have shown that $\dfrac{p}{\therefore p}$ is a valid argument.

Recall Logical Equivalence Rules

Equivalence	Name
$p \land \mathrm{T} \equiv p, \quad p \lor \mathrm{F} \equiv p$	Identity laws
$p \lor \mathrm{T} \equiv \mathrm{T}, \quad p \land \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \lor p \equiv p, \quad p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p, \quad p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r), \quad (p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r), \quad p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \lor q) \equiv \neg p \land \neg q, \quad \neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p, \quad p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathrm{T}, \quad p \land \neg p \equiv \mathrm{F}$	Negation laws

Recall Logical Equivalence Rules

Logical Equivalences Involving Conditional Statements

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditional Statements

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv q \leftrightarrow p$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Logical Reasoning: Another Simple Example

<u>Example</u>: Consider the contrapositive as a logical argument.

$$\begin{array}{l} p \rightarrow q \\ \hline \therefore \neg q \rightarrow \neg p \end{array} \qquad\qquad \begin{array}{l} \neg q \rightarrow \neg p \\ \hline \therefore p \rightarrow q \end{array}$$

<u>Proof of validity:</u>

1. $p \rightarrow q$ Hypothesis
2. $\neg p \vee q$ Conditional law
3. $q \vee \neg p$ Commutative law
4. $\neg\neg q \vee \neg p$ Double negation
5. $\neg q \rightarrow \neg p$ Conditional identity

$\neg L \vee R$
$L: \neg q$
$R: \neg p$

<u>Proof of validity:</u>

1. $\neg q \rightarrow \neg p$ Hypothesis
2. $\neg\neg q \vee \neg p$ Conditional identity
3. $q \vee \neg p$ Double negation
4. $\neg p \vee q$ Commutative law
5. $p \rightarrow q$ Conditional identity

- Note: this is the logical equivalence proof we performed.

Logical Reasoning: Another Simple Example

Example:

$$\frac{p \rightarrow q}{\therefore \neg q \rightarrow \neg p}$$

F

T

$$\frac{\neg q \rightarrow \neg p}{\therefore p \rightarrow q}$$

Proof of validity:

1. $p \rightarrow q$ Hypothesis
2. $\neg p \vee q$ Conditional identity, 1
3. $q \vee \neg p$ Cumulative law, 2
4. $\neg\neg q \vee \neg p$ Double negation, 3
5. $\neg q \rightarrow \neg p$ Conditional identity, 4

Proof of validity:

1. $\neg q \rightarrow \neg p$ Hypothesis
2. $\neg\neg q \vee \neg p$ Conditional identity, 1
3. $q \vee \neg p$ Double negation, 2
4. $\neg p \vee q$ Commutative law, 3
5. $p \rightarrow q$ Conditional identity, 4

- Note: this is the logical equivalence proof we performed.
  - Add line numbers for logical argument proofs.

Outline

- Logical Reasoning
  - Logical Reasoning Definition
  - Invalid Argument
  - Logical Reasoning Rules
  - Logical Reasoning Example
- Introduction to Mathematical Proofs

Logical Reasoning: Proof Definition

Definition

A **logical proof** of an argument is a sequence of steps, each of which consists of a proposition and a justification.

- Each line should contain:
  - a hypothesis (assumption).
  - a proposition that is equivalent to a previous statement
  - a proposition that is derived by applying an argument to previous statements.
- Justifications should state
  - hypothesis.
  - the equivalence law used (and the line it was applied to).
  - the argument used (and the line(s) it was applied to).
- The last line should be the conclusion.

Outline

- Logical Reasoning
  - Logical Reasoning Definition
  - Invalid Argument
  - Logical Reasoning Rules
  - Logical Reasoning Example
- Introduction to Mathematical Proofs

Logical Reasoning: Invalid Argument

- To prove an argument is invalid, we need a counterexample.

<u>Example:</u>

Consider the converse as an argument:

$$\left.\begin{array}{l} \overset{F \to T}{p \rightarrow q} \\ \hline \therefore \underset{T \to F}{q \rightarrow p} \end{array}\right.$$

F→T T F T→F

<u>Proof of validity:</u>

F→T
T

Suppose p : FALSE and q : TRUE.
Then $p \longrightarrow q$ is TRUE, but $q \longrightarrow p$ is FALSE.
Thus, the argument is invalid.

- *Counterexample*: a situation where all hypotheses are TRUE and the conclusion is FALSE.

Outline

- Logical Reasoning
  - Logical Reasoning Definition
  - Invalid Argument
  - Logical Reasoning Rules
  - Logical Reasoning Example
- Introduction to Mathematical Proofs

Logical Reasoning: Rules of Inference

Rule of Inference	Name
p T $p \to q$ T T $\therefore q$ T	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$ $\neg q$ $\neg q \to \neg p$ $\therefore \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$ $p \to q \to r$	Hypothetical syllogism

Logical Reasoning: Rules of Inference

Rule of Inference	Name
$p \lor q$ T $\neg p$ T $\therefore q$	Disjunctive syllogism
p $\therefore p \lor q$	Addition
$p \land q$ $\therefore p$ $\therefore q$	Simplification

Logical Reasoning: Rules of Inference

Rule of Inference	Name
p T q T $\therefore p \wedge q$	Conjunction
$p \vee q$ T $\neg p \vee r$ T $\therefore q \vee r$ T	Resolution

Outline

- Logical Reasoning
  - Logical Reasoning Definition
  - Invalid Argument
  - Logical Reasoning Rules
  - Logical Reasoning Example
- Introduction to Mathematical Proofs

Logical Reasoning Proofs

$T \to T \; \& \; T$

Example:

Using Modus ponens: $T\left\{\begin{array}{l} p \\ p \to q \end{array}\right\} \& \;\; \dfrac{}{\therefore q} \;\; \& \; T$, prove Modus tollens: $\dfrac{\begin{array}{c} \neg q \\ p \to q \end{array}}{\therefore \neg p} \;\; \&$

$$\left\{\begin{array}{l} \neg q \\ \neg q \to \neg p \end{array}\right. \\ \overline{\therefore \neg p}$$

Proof:

1. $\neg q$	Hypothesis
2. $p \to q$	Hypothesis
3. $\neg q \to \neg p$	Contr~~a~~positive, 2
4 ~~5~~. $\neg p$	Modus ponens, 3,1

Logical Reasoning: Proofs

Example: Prove the validity of the following argument.

If you send me an e-mail message, then I will finish writing the program.
If you do not send me an e-mail message, then I will go to sleep early.
If I go to sleep early, then I will wake up feeling refreshed.

∴ If I do not finish writing the program, then I will wake up feeling refreshed.

$p \rightarrow q$
$\neg p \rightarrow r$
$r \rightarrow s$
$\therefore \neg q \rightarrow s$

p: You send me an e-mail message.
q: I will finish writing the program.
r: I will go to sleep early.
s: I will wake up feeling refreshed.

Form of the argument:

$$
\begin{array}{l}
p \rightarrow q \\
\neg p \rightarrow r \\
r \rightarrow s \\
\hline
\therefore \neg q \rightarrow s
\end{array}
$$

Proof:

1. $p \rightarrow q$	Hypothesis
2. $\neg q \rightarrow \neg p$	Contrapositive, 1
3. $\neg p \rightarrow r$	Hypothesis
4. $\neg q \rightarrow r$	Hypothetical syllogism, 2, 3
5. $r \rightarrow s$	Hypothesis
6. $\neg q \rightarrow s$	Hypothetical syllogism, 4, 5

$$
\begin{array}{l}
p \rightarrow \cancel{q} \\
\cancel{q} \rightarrow r \\
\hline
\therefore p \rightarrow r
\end{array}
$$

Logical Reasoning: Another Example

$p \to q \equiv \neg p \vee q$ $\quad \neg(p \vee q) \equiv \neg p \wedge \neg q$

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

Example:

Prove that the following argument is valid:

$$\begin{array}{ll} (\neg f \vee \neg r) \to (h \wedge t) & T \\ \neg t & T \\ \hline \therefore r & \end{array}$$

1. $(\neg f \vee \neg r) \to (h \wedge t)$ H
2. $\neg t$ H
3. $\neg(\neg f \vee \neg r) \vee (h \wedge t)$ Cond., 1
4. $(\neg\neg f \wedge \neg\neg r) \vee (h \wedge t)$ DM, 3
5. $(f \wedge r) \vee (h \wedge t)$ DN, 4
6. $\neg t \vee \neg h$ Addition, 2
7. $\neg h \vee \neg t$ Commutative, 6
8. $\neg(h \wedge t)$ DM, 7
9. $(h \wedge t) \vee (f \wedge r)$ Commutative, 5
10. $f \wedge r$ D.S., 8, 9
11. r Simplification, 10

$$\begin{array}{l} \neg q \\ q \vee p \\ \hline \therefore p \end{array}$$

Outline

- Logical Reasoning
- Introduction to Mathematical Proofs
  - Terminology
  - Proof by Exhaustion
  - Disproof by Counterexample
  - Direct proofs
  - Proof by Contraposition
  - Proof by Exhaustion

Mathematical Proofs

- A mathematical proof is usually “informal”.
- More formal than everyday language, less formal than logical proofs.
  - More than one rule may be used in a step.
  - (**Some**) steps may be skipped.
  - Axioms may be assumed.
  - Rules for inference need not be explicitly stated.
- Proofs must be a self-contained line of reasoning.
  - Statements used must be
    - facts (axioms),
    - theorems, lemmas, corollaries (previously proved statements), or
    - statements that can be derived from the above.
  - You cannot use something as fact within a proof if you are not certain it is.

Outline

- Logical Reasoning
- Introduction to Mathematical Proofs
  - Terminology
  - Proof by Exhaustion
  - Disproof by Counterexample
  - Direct proofs
  - Proof by Contraposition
  - Proof by Exhaustion

Some Terminology

- Theorem: statement that can be shown true.
  - Proposition: less important theorem.
  - Lemma: less important theorem used to prove other theorems.
  - Corollary: theorem that trivially follows another theorem.
- Conjecture: statement that is proposed to be true, but has not been proved.
- Axiom: statement assumed to be true (i.e., true statement that does not need a proof).
- Most axioms, theorems, etc., are properties concerning all elements over some domain.
  - E.g., All perfect squares are non-negative.
- The domain should be clear from context or explicitly stated.

Hidden Universal Quantifier

<u>Example:</u>

$8 > 5, \quad 8 - 5 > 0$ $\qquad$ $8.5 > 5.5, \quad 8.5 - 5.5 > 0$

Theorem: If $a > b$, then $a - b > 0$.

- [For all real numbers a and b], if $a > b$, then $a - b > 0$.
- Defined by the predicates:
  - $P(a, b) : a > b$
  - $Q(a, b) : a - b > 0$
  - Theorem: $\forall a, b, (P(a, b) \rightarrow Q(a, b))$.
- We can assume a general domain: $\mathbb{R}$ (the real numbers).
  - The context doesn't state otherwise.

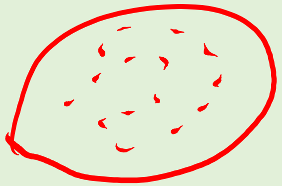
Outline

- Logical Reasoning
- Introduction to Mathematical Proofs
  - Terminology
  - Proof by Exhaustion
  - Disproof by Counterexample
  - Direct proofs
  - Proof by Contraposition
  - Proof by Exhaustion

Proof Method: Proof by Exhaustion

Definition

A proof by exhaustion for $p \rightarrow q$ starts by considering each element of the domain of discourse and showing that the predicate is true.



- A special type of proof by case.

- Only a useful method when dealing with a small domain.
  - Small is relative, but must be finite.
  - Example: $\{2, 4, 6\}$ is a small domain.

Proof by Exhaustion Example

Example:
Prove if n is in the domain $\{2, 4, 6\}$, then $3n \leq 18$.

Proof idea: We need to show this is true for $n = 2$, $n = 4$, and $n = 6$.

Proof:
- Take $n = 2$: $3n = 3(2) = 6$. We know $6 \leq 18$.
- Take $n = 4$: $3n = 3(4) = 12$. We know $12 \leq 18$.
- Take $n = 6$: $3n = 3(6) = 18$. We know $18 \leq 18$.

So for all possible values of n, $3n \leq 18$ is true.

Proof by Exhaustion (non-)Example

__Example:__

Prove that if n has the form x^2 for some integer x, then $n > 0$.

- "n has the form x^2" is the same as "$n = x^2$."

Proof:

- Take $n = 4$: Let $x = 2$, so $x^2 = (2)^2 = 4$.
  - Then $n = x^2$ and we know $4 > 0$.
- Take $n = 625$: Let $x = 25$, so $x^2 = (25)^2 = 625$.
  - Then $n = x^2$ and we know $625 > 0$.
- Take $n = 900$: Let $x = -30$, so $x^2 = (-30)^2 = 900$.
  - Then $n = x^2$ and we know $900 > 0$.

$\therefore$ If $n = x^2$ for some integer x, then $n > 0$.

- Is this true for every n?
  - A proof should handle every possible scenario.

Proof by Exhaustion (non-)Example

<u>Example:</u>

Prove that if n has the form x^2 for some integer x, then $n > 0$.

- "n has the form x^2" is the same as "$n = x^2$."

Proof:

- Take $n = 4$: Let $x = 2$, so $x^2 = (2)^2 = 4$.
  - Then $n = x^2$ and we know $4 > 0$.
- Take $n = 625$: Let $x = 25$, so $x^2 = (25)^2 = 625$.
  - Then $n = x^2$ and we know $625 > 0$.
- Take $n = 900$: Let $x = -30$, so $x^2 = (-30)^2 = 900$.
  - Then $n = x^2$ and we know $900 > 0$.

∴ If $n = x^2$ for some integer x, then $n > 0$.

- Is this true for every n?
  - A proof should handle every possible scenario.

Outline

- Logical Reasoning
- Introduction to Mathematical Proofs
  - Terminology
  - Proof by Exhaustion
  - Disproof by Counterexample
  - Direct proofs
  - Proof by Contraposition
  - Proof by Exhaustion

Disproof by Counterexample

- How can we prove a statement is false?

- The theorems we try to prove are generally universally quantified implications.
  - Prove $\forall x(P(x) \rightarrow Q(x))$ is false by finding a counterexample.
  - One such x where $P(x)$ is TRUE and $Q(x)$ is FALSE ($T \rightarrow F \equiv F$).
  - Recall negation of quantifiers: $\neg \forall x, (\ldots) \equiv \exists x, \neg(\ldots)$.

Disproof by Counterexample

<u>Example:</u>

Find a counter-example for each of the statements below:

- Every month of the year has 30 or 31 days. Feb
- If n is an integer and n^2 is divisible by 4, then n is divisible by 4. 6
- For every positive integer x, $x^3 < 2x$
- Every positive integer can be expressed as the sum of the squares of two integers.
- The multiplicative inverse of a real number x is a real number y such that $xy = 1$. Every real number has a multiplicative inverse.

Outline

- Logical Reasoning
- Introduction to Mathematical Proofs
    - Terminology
    - Proof by Exhaustion
    - Disproof by Counterexample
    - Direct proofs
    - Proof by Contraposition
    - Proof by Exhaustion

Direct Proofs

Definition

A direct proof for $P(x) \rightarrow Q(x)$ starts by assuming $P(x)$ (for x) as fact and finishes by establishing $Q(x)$.

- Make use of axioms, previously proven theorems, inference rules, etc.
- Same approach was used to prove that a logical argument is true.
  - $P(x)$ is the hypothesis.
  - $Q(x)$ is the conclusion.

Direct Proofs

Definition

A direct proof for $P(x) \to Q(x)$ starts by assuming $P(x)$ (for x) as fact and finishes by establishing $Q(x)$.

- Proof Layout

Proof:

Assume $P(x)$, for some x.

: Perform your derivations (using theorems, axioms, etc.)

$\therefore Q(x)$.

Direct Proof Example

Prove that if n is an odd integer, then n^2 is also odd.

- Decomposition of statement to $P(x) \rightarrow Q(x)$:
  - The domain of x is all integers.
  - $P(x) : x$ is an odd integer.
  - $Q(x) : x^2$ is an odd integer.

Proof:

Assume n is an odd integer.

Then there exists an integer k such that $n = 2k + 1$. Hence:

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2 \cdot (2k^2 + 2k) + 1.$$

Since k is an integer, $2k^2 + 2k$ is also an integer (call it j).

So n^2 has the form $2j + 1$.

Therefore, n^2 is an odd integer.

Direct Proof Example

If n is an odd integer then $\frac{n+3}{2}$ is an integer.

- Decomposition of statement to $P(x) \to Q(x)$:
  - The domain of x is all integers.
  - $P(x)$: x is an odd integer.
  - $Q(x)$: $\frac{x+3}{2}$ is an integer.

<u>Proof:</u>

Assume n is an odd integer.

Then there exists an integer k such that $n = 2k + 1$. Hence:

$$n + 3 = (2k + 1) + 3 = 2k + 4 = 2 \cdot (k + 2).$$

Then $\frac{n+3}{2} = \frac{2.(k+2)}{2} = k + 2$, which is an integer.

Therefore, $\frac{n+3}{2}$ is an integer.

Outline

- Logical Reasoning
- Introduction to Mathematical Proofs
  - Terminology
  - Proof by Exhaustion
  - Disproof by Counterexample
  - Direct proofs
  - Proof by Contraposition
  - Proof by Exhaustion

Proof by Contraposition

Recall that $p \rightarrow q$ is logically equivalent to its contrapositive, $\neg q \rightarrow \neg p$.

Definition

A proof by contraposition for $P(x) \rightarrow Q(x)$ is proof $P(x) \rightarrow Q(x)$ where:

- write a direct proof for $\neg Q(x) \rightarrow \neg P(x)$, and
- conclude that the contrapositive of $\neg Q(x) \rightarrow \neg P(x)$ is also true.

- Proof Layout

Proof:

Assume $\neg Q(x)$, for some x.

: Perform your derivations (using theorems, axioms, etc.)

$\therefore \neg P(x)$.

Since $\neg Q(x) \rightarrow \neg P(x)$ is true, we may conclude that our original statement $P(x) \rightarrow Q(x)$ is also true.

Proof by Contraposition Example

Prove that if n is an integer and $3n + 2$ is odd, then n is an odd.

- Decomposition of statement to $P(x) \rightarrow Q(x)$:
  - The domain of x is all integers.
  - $P(x)$: $3x+2$ is an odd integer.
  - $Q(x)$: x is an odd integer.

Proof by contraposition:

Assume n is not an odd integer. So n is even.

Then there exists an integer k such that $n = 2k$. Hence:

$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2 \cdot (3k + 1)$$

Since k is an integer, $3k + 1$ is also an integer (call it j). So $3n + 2$ has the form $2j$.

Therefore, $3n + 2$ is an even integer, so $3n + 2$ is not an odd integer.

Thus, $\neg Q(n) \rightarrow \neg P(n)$.

Then, $P(n) \rightarrow Q(n)$ is also true.

Outline

- Logical Reasoning
- Introduction to Mathematical Proofs
  - Terminology
  - Proof by Exhaustion
  - Disproof by Counterexample
  - Direct proofs
  - Proof by Contraposition
  - Proof by Contradiction
  - Proof Examples

Proof by Contradiction

Note that p is logically equivalent to $\neg p \rightarrow (r \wedge \neg r)$ (for any proposition r).

<u>Definition</u>

A proof by contradiction for P is actually a proof for $\neg P \rightarrow (r \wedge \neg r)$ where:

- write a proof starting with the assumption $\neg P$ and
- find some proposition r where you can derive both r and $\neg r$ to both be TRUE (a contradiction).

<u>Proof Layout</u>

<u>Proof by Contradiction:</u>

Assume $\neg P$.

:Find something that breaks.

$\therefore$ Contradiction.
Therefore, P is true.

Proof by Contradiction Example

Prove that $\sqrt{2}$ is not a rational number.

- Decomposition of statement to *P*:
  - The domain of *x* is all rational numbers.
  - $P: \forall x, x \neq \sqrt{2}$.

Proof by contradiction ($\sqrt{2}$ is not a rational number.):

Assume that $r = \sqrt{2}$ is a rational number.

- Then there exist integers a and b such that $r = a/b$.
  - WLOG, we assume that a and b have no common divisors.
    - (if they had common divisors, reduce the fraction and use that.)
- Then, $2 = r^2 = (a/b)^2 = a^2/b^2$.
- Transforming $2 = a^2/b^2$ gives: $a^2 = 2 \cdot b^2$,
  - so a^2 is an even number, giving that a is even.
  - Thus, there exists integer i such that $a = 2i$.
- Plug into $a^2 = 2b^2$: $(2i)^2 = 2b^2$, so $4i^2 = 2b^2$ or $b^2 = 2i^2$,
  - so b^2 is an even number, giving that b is also even.
  - Thus, there exists integer j such that $b = 2j$.
- Since a and b are both even, they share a common divisor 2.
- A contradiction: a and b share the divisor 2 but share no common divisors.

Therefore, our original assumption is false, so no rational number equals $\sqrt{2}$.

WLOG: Without Loss of Generality

- In this example, when we say **WLOG** – without loss of generality:
  - We are saying we can consider a reduced terms "without loss of generality."
- We can say this when considering a case would be redundant:
  - Suppose we considered an arbitrary fraction a/b.
  - Then we could have our first step reduce a/b to lowest terms and proceed.

Proof by Contradiction Example

Definition

A prime number is a positive integer larger than 1 whose only divisors are 1 and itself.

- The first few prime numbers: 2, 3, 5, 7, 11, 13, 17 ...
- At the beginning, the prime numbers are **dense.**
  - (i.e., there are many of them).
  - E.g., there are 168 prime numbers between 1 and 1000. (~17%)
- When the number gets bigger, the prime numbers are **sparse.**
  - (i.e. there are few of them).
  - E.g., there are 78,498 primes between 1 and 1,000,000. (~8%)
- How many prime numbers are there? Finite? or infinite?

Theorem

There are infinitely many prime numbers.

Proof by contradiction (There are infinitely many prime numbers) (Euclid 325-265 BC):

Assume that there are only finitely many primes: $p_1, p_2, \ldots, p_n$.

- Consider the number $Q = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$.
- We ask: Is Q a prime number?
  - For each i, $1 \leq i \leq n$, $Q > p_i$,
    - $p_1, \ldots, p_n$ are ALL prime numbers (by assumption).
  - So Q IS NOT prime.
- If Q is not a prime, it must have a prime factor.
  - So one of p_i $(1 \leq i \leq n)$ must be a factor of Q.
  - But Q divided by each p_i has remainder 1.
  - So none of p_i $(1 \leq i \leq n)$ is a divisor of Q.
  - Q IS a prime.
- A contradiction: Q is and is not prime.

Therefore, our original assumption is false, so there must exist infinitely many primes.

Outline

- Logical Reasoning
- Introduction to Mathematical Proofs
  - Terminology
  - Proof by Exhaustion
  - Disproof by Counterexample
  - Direct proofs
  - Proof by Contraposition
  - Proof by Contradiction
  - Proof Examples

Proof by Contraposition Example

Prove that for any integers x and y, if both $x + y$ and xy are even, then both x and y are even.

- The contrapositive:

If x and y are not both even, then $x + y$ and xy are not both even.

<u>Proof by contraposition:</u>
Assume that x and y are not both even: Either x is odd, or y is odd (or both).
WLOG, we assume x is odd.

- Case 1: y is even.

$x + y$ = odd + even = odd.
Thus, $x + y$ is odd.
So $x + y$ and xy are not both even.

- Case 2: y is odd.

xy = odd . odd = odd.
Thus, xy is odd.
So $x + y$ and xy are not both even.

In both cases, we get that $x + y$ and xy are not both even.

Proof by Exhaustion Example

Prove that if n is an integer, then $n^2 \geq n$.

Proof:

Let n be an integer.

- Case 1: Assume $n = 0$. Then, $n^2 = 0 = n$.
  - So $n^2 \geq n$.
- Case 2: Assume $n \geq 1$. Then, $n > 0$.
  - Multiply both sides of inequality by n: $n^2 \geq n$.
- Case 3: Assume $n \leq -1$. Hence, $n^2 \geq 0 > -1 \geq n$.
  - So $n^2 \geq n$.
- In all cases, we get that $n^2 \geq n$.

Proof by Cases

Proof by Exhaustion

- For a proof by exhaustion to work, cases must **exhaust**, or consider, the entire domain.
- Overlap is OK, but may introduce redundant work.
  - For the domain of integers,
    - $n \geq 0$, $n = 0$, and $n \leq 0$ are exhaustive cases, but have overlap.
    - Better: $n \geq 0$ and $n < 0$ or $n > 0$ and $n \leq 0$
- Non-exhaustive cases leave the possibility for error:
  - For the domain of integers,
    - n is positive and n is negative are non-exhaustive cases.
    - Missing $n = 0$.