

# CSE 191: Discrete Structures

## Predicate Logic

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# Outline

- Predicates and Quantifiers
  - From Propositions to Predicates
  - Quantifiers

# Predicates and Quantifiers

# From Propositions to Predicates

Consider the statement “X is even.”

$$f(x): x^2 + x + 1$$

- Contains the variable  $X$ , so it is not a proposition.
  - Given a value for  $X$ , we can determine the truth value.
  - Once  $X$  is filled, sentence is TRUE or FALSE, but not both.
- Sentences whose truth value is based on variables are **predicates**.
  - P(X):  $X$  is even.

# From Propositions to Predicates

## Definition

A **predicate** is a **function** that takes some **variable(s) as arguments**; it returns either TRUE or FALSE (but never both) for each combination of the argument values.

- In contrast, a **proposition** is a function of 0 variables.
  - Propositions have no variables.
  - Each proposition is either TRUE or FALSE (but not both).

# From Propositions to Predicates

## Definition

A **predicate** is a **function** that takes some **variable(s) as arguments**; it returns either TRUE or FALSE (but never both) for each combination of the argument values.

- Predicate variables are derived from an associated **domain of discourse**.

- Domain of discourse describes all allowable argument values

$P(x)$ :  $x$  is even

- E.g.,  $x$  is an even number.

- We may want to ask, which set does  $x$  come from?
  - $P(x)$ :  $x$  is an even number, where  $x$  takes a value among the integers.
    - $P$ : “is an even number”
    - $x$ : the variable
    - Integers: the domain
  - $P(x)$  is the value of the **propositional function**  $P$  at  $x$ .

# Outline

- Predicates and Quantifiers
  - From Propositions to Predicates
    - Predicate definition
    - Domain of Discourse definition
    - Predicate examples
  - Quantifiers

# From Propositions to Predicates

## Definition

Given a predicate  $P(x)$ , the **domain of discourse** (often referred to as the **domain**) is a set of all possible values for the variable  $x$ .

- Predicates with multiple variables may have:
  - multiple domains of discourse, one for each variable, or
  - a single domain of discourse for all variables



# From Propositions to Predicates

## Example 1

Consider the predicate  $Q(x, y)$  defined by:

$Q(x, y) : y$  is enrolled in recitation  $x$ .

where

- the domain of discourse for  $x$  is  $\{A1, A2, A3, A4, B1, B2, B3, B4\}$ .
- the domain of discourse for  $y$  is  $\{\text{all students in CSE 191}\}$ .

## Example 2

Consider the predicate  $R(x, y)$  defined by:

$R(x, y) : x$  and  $y$  are friends.

where

the domain for  $R$  is  $\{\text{all students in CSE 191}\}$ .

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# Predicate vs. Proposition

is  $P(1,4)$  a true proposition?

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(1,4): 2 \times 1 = 4$$

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{1, 2, 3\}$
- the domain for  $y$  is  $\{4, 5, 6\}$ .

Statement	Is Proposition?	Truth Value
$P(1, 4): 2(1) = 4$	Yes	FALSE

# Predicate vs. Proposition

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{1, 2, 3\}$
- the domain for  $y$  is  $\{4, 5, 6\}$ .

Statement	Is Proposition?	Truth Value
$P(1, 4): 2(1) = 4$	Yes	FALSE
$P(2, 4): 2(2) = 4$		

# Predicate vs. Proposition

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{1, 2, 3\}$
- the domain for  $y$  is  $\{4, 5, 6\}$ .

Statement	Is Proposition?	Truth Value
$P(1, 4): 2(1) = 4$	Yes	FALSE
$P(2, 4): 2(2) = 4$	Yes	TRUE

# Predicate vs. Proposition

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{1, 2, 3\}$
- the domain for  $y$  is  $\{4, 5, 6\}$ .

Statement	Is Proposition?	Truth Value
$P(1, 4): 2(1) = 4$	Yes	FALSE
$P(2, 4): 2(2) = 4$	Yes	TRUE
$P(x, 4): 2(x) = 4$	No	---

# Predicate vs. Proposition

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{1, 2, 3\}$
- the domain for  $y$  is  $\{4, 5, 6\}$ .

Statement	Is Proposition?	Truth Value
$P(1, 4): 2(1) = 4$	Yes	FALSE
$P(2, 4): 2(2) = 4$	Yes	TRUE
$P(x, 4): 2(x) = 4$	No	---
$P(3, y): 2(3) = y$		

# Predicate vs. Proposition

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{1, 2, 3\}$
- the domain for  $y$  is  $\{4, 5, 6\}$ .

Statement	Is Proposition?	Truth Value
$P(1, 4): 2(1) = 4$	Yes	FALSE
$P(2, 4): 2(2) = 4$	Yes	TRUE
$P(x, 4): 2(x) = 4$	No	---
$P(3, y): 2(3) = y$	No	---



# Predicate vs. Proposition

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{1, 2, 3\}$
- the domain for  $y$  is  $\{4, 5, 6\}$ .

Statement	Is Proposition?	Truth Value
$P(1, 4) : 2(1) = 4$	Yes	FALSE
$P(2, 4) : 2(2) = 4$	Yes	TRUE
$P(x, 4) : 2(x) = 4$	No	---
$P(3, y) : 2(3) = y$	No	---

- What about  $P(2, \underline{3}) : 2(2) = 3$ ?
  - $P(2, 3)$  is meaningless (in this example).
  - 3 is not in the specified domain for  $y$ .

# Predicate vs. Proposition

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{\underline{1}, 2, 3\}$
- the domain for  $y$  is  $\{\underline{4}, 5, 6\}$ .

$\frac{2 \times 1 = 4}{F} \quad \vee \quad \frac{2 \times 3 = 6}{T} \equiv T$

Statement	Is Proposition?	Truth Value
$P(\underline{1}, \underline{4}) \vee P(\underline{3}, \underline{6})$	$\vee$	$T$
$P(1, 4) \vee \neg P(3, 6)$		
$P(2, 4) \rightarrow P(2, 5)$		
$P(2, 4) \wedge P(\underline{x}, \underline{4})$		

# Predicate vs. Proposition

## Example

Let the predicate  $P(x, y)$  be defined by

$$P(x, y) : 2x = y \quad \text{where}$$

- the domain for  $x$  is  $\{1, 2, 3\}$
- the domain for  $y$  is  $\{4, 5, 6\}$ .

Statement	Is Proposition?	Truth Value
$P(1, 4) \vee P(3, 6)$	Yes	TRUE
$P(1, 4) \vee \neg P(3, 6)$	Yes	FALSE
$P(2, 4) \rightarrow P(2, 5)$	Yes	FALSE
$P(2, 4) \wedge P(x, 4)$	No	---

# More Predicate Examples

## Example

Let the predicate  $Q(x, y)$  be defined by:

$$Q(x, y) : x + y > 4$$

- the domain of discourse for  $x$  and  $y$  is **all integers**.
- Which of the following are predicates? Which are propositions?
  - $Q(1, 2)$             proposition
  - $Q(x, 2)$             predicate since  $x$  is left as a variable
  - $Q(1000, y)$         predicate since  $y$  is left as a variable
  - $Q(1000, 2)$         proposition
  - $Q(x, y)$             predicate since  $x$  and  $y$  are left as a variable

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$P(x)$ :  $x$  is even

Domain: all integers

$P(x)$  is true

# Quantifiers

- Quantification expresses the extent to which a predicate is true over a range of elements.
  - English examples: all, some, none, many, few.

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# Universal Quantification

## Definition

Suppose  $P(x)$  is a predicate on some domain.

The **universal quantification** of  $P(x)$  is the proposition:  
“ $P(x)$  is true for all  $x$  in the domain of discourse  $D$ .”

Written as:  $\forall x, P(x)$

Read as: “**For all**  $x, P(x)$ ”, or “**For every**  $x, P(x)$ ”

Here  $\forall$  is called the universal quantifier.

- $\forall x, P(x)$  is TRUE **if**  $P(x)$  is TRUE for every  $x$  in  $D$ .
- $\forall x, P(x)$  is FALSE **if**  $P(x)$  is FALSE for some  $x$  in  $D$ .



# Universal Quantification Example

## Example

$P(x) : x + 2 = 5$ , domain of discourse:  $\{1, 2, 3\}$ .

- $\forall x, P(x)$  means: “for all  $x$  in  $\{1, 2, 3\}$ ,  $x + 2 = 5$ .”
    - $\forall x, P(x) \equiv P(1) \wedge P(2) \wedge P(3)$ , or
    - $\forall x, P(x) \equiv (\underline{1} + 2 = 5) \wedge (\underline{2} + 2 = 5) \wedge (\underline{3} + 2 = 5)$
- $\therefore \forall x, P(x)$  is FALSE (since  $\underline{1} + 2 = 5$  and  $\underline{2} + 2 = 5$  are both FALSE).  
*counterexample = 1*

Note: The symbol  $\therefore$  denotes “therefore”.

$P(x) : x + 2 = 5$   
 $D : \{3\}$   
 $\forall x P(x)$  true?  
 $P(3) : 3 + 2 = 5$   
 $\top$   
 $\rightarrow y$

- An input that causes a universally quantified statement to evaluate to FALSE is called a **counterexample**.
- The meaning of the universal quantification of  $P(x)$  changes if the domain is changed.

# Universal Quantification Example

## Example

$A(x) : x$  is even

$B(x) : x^2 > 0$

$C(x) : x < 2$

where the domain of discourse for  $A$ ,  $B$ , and  $C$  is  $\{0, 1, 2, 3\}$ .

True or False?

$x=0$   $C(0) \rightarrow A(0) \equiv 0 < 2 \rightarrow 0 \text{ is even} \equiv T \rightarrow T \equiv T$   
 $x=1$   $C(1) \rightarrow A(1) \equiv 1 < 2 \rightarrow 1 \text{ is even} \equiv T \rightarrow F \equiv F$

Counterexample 1

- ✓ •  $\forall x, (C(x) \rightarrow A(x))$ . - FALSE. A counterexample is  $x = 1$ .

- $\forall x, (B(x) \vee C(x))$ . - TRUE. Consider the table

$x$	$B(x)$	$C(x)$
0	F	T
1	T	T
2	T	F
3	T	F

T  
T  
T  
T

# Universal Quantification Example

## Example

$S(x)$  :  $x$  is a student in CSE191.

$T(x)$  :  $x$  is a computer science major.

the domain of discourse for  $S$  and  $T$  is {all students enrolled in CSE191}.

True or False?

- $\forall x, S(x)$  TRUE. Everyone enrolled is a student in CSE191.
- $\forall x, T(x)$  FALSE. Some of you are not majoring in CS.
- $\forall x, (S(x) \rightarrow T(x))$  FALSE. There are students in CSE 191 not majoring in computer science.
- $\forall x, (T(x) \rightarrow S(x))$  TRUE. Think carefully about the domain.

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# Existential Quantification

## Definition

Suppose  $P(x)$  is a predicate on some domain of discourse.

The **existential quantification** of  $P(x)$  is the proposition:

**“ $P(x)$  is true for some  $x$  in the domain of discourse  $D$ .”**

Written as:  $\exists x, P(x)$

Read as: **“There exists  $x$ ,  $P(x)$ ”**, or **“For some  $x$ ,  $P(x)$ ”**

Here  $\exists$  is called the existential quantifier.

- $\exists x, P(x)$  is TRUE **if**  $P(x)$  is TRUE for some  $x$  in  $D$ .
- $\exists x, P(x)$  is FALSE **if** for every  $x$  in  $D$ ,  $P(x)$  is FALSE.

# Existential Quantification Example

## Example

$P(x) : x + 2 = 5$ , domain of discourse:  $\{1, 2, 3\}$ .

- $\exists x, P(x)$  means: “for some  $x$  in  $\{1, 2, 3\}$ ,  $x + 2 = 5$ .”

- $\exists x, P(x) \equiv P(1) \vee P(2) \vee P(3)$ , or

- $\exists x, P(x)$   $\equiv$  (1 + 2 = 5)  $\vee$  (2 + 2 = 5)  $\vee$  (3 + 2 = 5)  
 $\text{F} \quad \vee \quad \text{F} \quad \vee \quad \text{T} \quad \equiv \quad \text{T}$

$\therefore \exists x, P(x)$  is TRUE (since 3 + 2 = 5 is TRUE).

*satisfying assignment: 3*

- The meaning of the existential quantification of  $P(x)$  may change if the domain is changed.
- An input that causes a predicate to evaluate to TRUE is called a **satisfying assignment**.

$P(x) : x + 2 = 5$   
 $D : \{4, 5, 6\}$   
 $\exists x P(x)$  true? **NO**  
**False**

# Existential Quantification Example

## Example

$$A(x) : x = 1$$

$$B(x) : x > 5$$

$$C(x) : x < 5$$

where the domain of discourse for  $A$ ,  $B$ , and  $C$  is  $\{1, 2, 3\}$ .

True or False?

$$x=1 \quad C(1) \rightarrow A(1) \equiv 1 < 5 \rightarrow 1 = 1 \equiv T \rightarrow T \equiv T$$

$\top$  • *satisfying assignment: 1*  
 $\exists x, (C(x) \rightarrow A(x)).$

$F$  •  $\exists x, B(x).$

$$x=1 \quad B(1) \equiv 1 > 5 \quad F$$

$$x=2 \quad B(2) \equiv 2 > 5 \quad F$$

$$x=3 \quad B(3) \equiv 3 > 5 \quad F$$

# Existential Quantification Example

## Example

$S(x)$  :  $x$  is a student in CSE191.

$T(x)$  :  $x$  is a computer science major.

Domain of discourse is {all students enrolled in CSE191}.

True or False?

✓ •  $\exists x, S(x)$ .

✓ •  $\exists x, T(x)$ .

Note: The quantifiers  $\forall$  and  $\exists$  have higher precedence than all logical operators.



# Universal Quantification Example

## Example

$$P(x) : x^2 > 9$$

$$Q(x) : x^2 > 0$$

where the domain of discourse is *all Integers*.

True or False?

- $\text{F} \bullet \forall x, P(x)$  counterexample: 1
- $\text{T} \bullet \exists x, P(x)$  satisfying assignment: 4
- $\text{F} \bullet \forall x, Q(x)$  counterexample: 0
- $\text{T} \bullet \exists x, Q(x)$  satisfying assignment: 2

# Binding Variables

- The occurrence of a variable  $x$  is said to be **bound** when a quantifier is used on that variable.
- An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.
- The part of a logical expression to which a quantifier is applied is called the **scope** of this quantifier.
  - A variable is free if it is outside the scope of all quantifiers in the formula that specify this variable.
- Example:
  - $\exists x (x + y = 1)$ 
    - The variable  $x$  is bound by the existential quantifier  $\exists x$ .
    - $y$  is free.

$$\forall x \, P(x) \rightarrow Q(y)$$

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# Quantified Statements and English

Example:  $L(s, t): s \text{ loves } t$

Suppose  $L(x, y): x \text{ loves } y$ , where

- the domain of  $x$  is all CSE 191 students, and
- the domain of  $y$  is the courses offered by UB CSE.

- $\exists x, (L(x, \text{CSE 191}) \wedge L(x, \text{CSE 250}))$ :
  - A CSE 191 student loves both CSE 191 and CSE 250.

- $\exists x \exists y \forall z, ((x \neq y) \wedge (L(x, z) \rightarrow L(y, z)))$ :
  - There are two different students  $x$  and  $y$  in CSE 191 such that if  $x$  loves a CSE course, then  $y$  loves it as well.

- Every CSE course is loved by some student in CSE 191:
  - $\forall y \exists x, L(x, y)$ .

- No student in CSE 191 loves CSE 191 and CSE 250:
  - $\neg \exists x, (L(x, \text{CSE 191}) \wedge L(x, \text{CSE 250}))$ .

$$\exists x (L(x, 191) \wedge L(x, 250))$$

$$\forall y \exists x L(x, y) \quad \exists x \forall y (L(x, y) \rightarrow L(x, 191) \wedge L(x, 250))$$

$$\neg \exists x, (L(x, 191) \wedge L(x, 250))$$

$$\forall x (L(x, 191) \rightarrow \neg L(x, 250))$$

# Quantified Statements and English

## Example:

$A(x)$ :  $x$  lives in Amherst.

$D(x)$ :  $x$  majors in computer science.

$B(x)$ :  $x$  is a CSE 191 student.

$C(x)$ :  $x$  has a good GPA.

Domain of discourse: all UB students.

- ~~$\forall x$~~   $B(x)$   
All UB students are CSE 191 students:
    - $\forall x, B(x)$
  - ~~$\forall x$~~   
All CSE 191 students have a good GPA:  ~~$\forall x (B(x) \rightarrow B(x))$~~   $\forall x (B(x) \rightarrow C(x))$ 
    - $\forall x, (B(x) \rightarrow C(x))$
  - CSE 191 students not living in Amherst major in CS:  $\forall x ((B(x) \wedge \neg A(x)) \rightarrow D(x))$ 
    - $\forall x, ((B(x) \wedge \neg A(x)) \rightarrow D(x))$
  - No CSE 191 student lives in Amherst:  $\forall x (B(x) \rightarrow \neg A(x))$ 
    - $\forall x, (B(x) \rightarrow \neg A(x))$
- $\neg B(x)$

# English to Quantified Statements

Translate the following theorems to quantified statements:

- $\forall x$   $P(x)$   $\rightarrow$   $Q(x+1)$
- If  $x$  is an even number, then  $x + 1$  is odd.

- Identify a domain and predicates:

- Domain: all integers
- $P(x)$  :  $x$  is an even number.
- $Q(x)$  :  $x$  is an odd number.

Quantified statement:

$$\forall x, (P(x) \rightarrow Q(x + 1)).$$

- $\forall y$   $R(y)$   $\rightarrow$   $S(y)$
- Every even number is a multiple of 2.

- Domain and predicates:

- Domain: all integers.
- $R(y)$  :  $y$  is an even number.
- $S(y)$  :  $y$  is a multiple of 2.

Quantified statement:

$$\forall y, (R(y) \rightarrow S(y)).$$

# English to Quantified Statements

Translate the following theorems to quantified statements:

- If  $x$  is an even number, then  $x + 1$  is odd.

- Identify a domain and predicates:

- Domain: all integers
- $P(x)$  :  $x$  is an even number.
- $Q(x)$  :  $x$  is an odd number.

Quantified statement:

$$\forall x, (P(x) \rightarrow Q(x + 1)).$$

- Every even number is a multiple of 2. (Alternative solution).

- Domain and predicates:

- Domain: all even integers.
- $T(z)$  :  $z$  is a multiple of 2.

Quantified statement:

$$\forall z, T(z).$$

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# Quantifier Negation

## Example:

Suppose the domain of discourse is all UB students.

$A(x)$ :  $x$  lives in Amherst.

$D(x)$ :  $x$  majors in computer science.

- Consider the following two propositions:

- Not every UB student majors in computer science:

- $\neg \forall x, D(x)$ .

$$\neg \forall x D(x)$$

- Some UB students do not major in computer science:

- $\exists x, \neg D(x)$ .

$$\exists x \neg D(x)$$

- Similarly,

- There is no UB student living in Amherst:

- $\neg \exists x, A(x)$ .

$$\neg \exists x A(x)$$

- Every UB student lives outside of Amherst:

- $\forall x, \neg A(x)$ .

$$\forall x \neg A(x)$$

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# Quantifier Negation

## Quantifier negation:

In general we have for any predicate  $P(x)$ :

$$\neg \forall x, P(x) \equiv \exists x, \neg P(x) \text{ and } \neg \exists x, P(x) \equiv \forall x, \neg P(x)$$

De Morgan's Law for Quantifiers			
Negation	Equivalent Statement	When Is Negation True?	When False?
$\neg \exists x P(x)$	$\forall x \neg P(x)$	For every x, P(x) is false	There is an x for which P(x) is true
$\neg \forall x P(x)$	$\exists x \neg P(x)$	There is an x for which P(x) is false	P(x) is true for every x

# Quantifier Negation

## Quantifier negation:

In general we have for any predicate  $P(x)$ :

$$\neg \forall x, P(x) \equiv \exists x, \neg P(x) \text{ and } \neg \exists x, P(x) \equiv \forall x, \neg P(x)$$

## Negation Rule:

- Move the negation over a quantifier. Flip the quantifier passed.
  - $\exists$  flips to  $\forall$ .
    - $\neg \exists x, (...) \text{ becomes } \forall x, \neg(...)$
  - $\forall$  flips to  $\exists$ .
    - $\neg \forall x, (...) \text{ becomes } \exists x, \neg(...)$

## Example:

- No CSE 191 student lives in Amherst:

$$\neg \exists x (B(x) \wedge A(x)) \equiv \forall x, \neg (B(x) \wedge A(x)).$$

$A(x)$ :  $x$  lives in Amherst.

$B(x)$ :  $x$  is a CSE 191 student.

$$P \rightarrow Q \\ \equiv \neg P \vee Q$$

$$\neg(S \vee t) \equiv \neg S \wedge \neg t$$

# Quantifier Negation Examples

$$\exists x (P(x) \rightarrow \neg Q(x))$$

$$\neg \exists x (P(x) \rightarrow \neg Q(x))$$

$$\equiv \forall x \neg (P(x) \rightarrow \neg Q(x))$$

$$\equiv \forall x \neg (\neg P(x) \vee \neg Q(x))$$

$$\equiv \forall x (\neg \neg P(x) \wedge \neg \neg Q(x))$$

$$\equiv \forall x (P(x) \wedge Q(x))$$

$$\rightarrow \forall x (P(x) \rightarrow \exists y (P(y) \vee Q(y)))$$

$$\neg \forall x (P(x) \rightarrow \exists y (P(y) \vee Q(y)))$$

$$\equiv \exists x \neg (P(x) \rightarrow \exists y (P(y) \vee Q(y)))$$

$$\equiv \exists x \neg (\neg P(x) \vee \exists y (P(y) \vee Q(y)))$$

$$\equiv \exists x (\neg \neg P(x) \wedge \neg \exists y (P(y) \vee Q(y)))$$

$$\equiv \exists x (P(x) \wedge \forall y \neg (P(y) \vee Q(y)))$$

$$\equiv \exists x (P(x) \wedge \forall y (\neg P(y) \wedge \neg Q(y)))$$

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    - Existential Quantifier
    - Examples: Quantifiers and Natural Language (e.g., English)
    - Quantifier Negation
    - Quantifier Negation Rule
    - **Nested Quantifiers**
    - Nested Quantifier Ordering
    - Nested Quantifier Scoping

# Nested Quantifiers

How do sentences with multiple quantifiers work?

**Definition:**

A logical expression with more than one quantifier that bind different variables in the same predicate is said to have **nested quantifiers**.

Need to consider their **ordering** and **scope**.

# Outline

- Predicates and Quantifiers
  - From Propositions to Predicates
  - Quantifiers
    - Universal Quantifier
    - Existential Quantifier
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# Nested Quantifiers: Ordering

- The order of the quantifiers is important, unless all are universal or all are existential quantifiers.
  - $P(x, y): x + y = y + x$ , where the domain is “all real numbers”.
  - $\forall x \forall y P(x, y)$ : “for all real number  $x$ , for all real numbers  $y$ ,  $x + y = y + x$ ”.

## Example:

$Q(x, y): x + y = 0$ , where the domain is “all real numbers”.

- What is the truth value of  $\exists y \forall x Q(x, y)$  ?
  - $\exists y \forall x Q(x, y)$ : “there is a real number  $y$  such that for every real number  $x$ ,  $Q(x, y)$ ”.
  - True or false? [False]
  - Does switching the ordering of quantifiers maintain the meaning?
  - $\forall x \exists y Q(x, y)$ : “for every real number  $x$  there is a real number  $y$  such that  $Q(x, y)$ ”.
  - True or false? [True]
- In general, we **cannot** switch the ordering and guarantee equivalence.

$$5 \cdot 5 + (-5 \cdot 5) = 0$$

# Nested Quantifiers: Ordering

- Consecutive quantifiers of the same type can be reordered and maintain equivalence.
- Suppose  $Q(x, y, z)$  is an arbitrary predicate:
  - $\forall i \forall j \forall k, Q(i, j, k) \equiv \forall j \forall i \forall k, Q(i, j, k) \equiv \forall k \forall j \forall i, Q(i, j, k) \equiv \dots$
  - $\exists i \exists j \exists k, Q(i, j, k) \equiv \exists j \exists k \exists i, Q(i, j, k) \equiv \dots$
- We usually simplify consecutive variables with the same quantifier:
  - $\forall i \forall j \forall k, Q(i, j, k) \equiv \forall i, j, k, Q(i, j, k).$
  - $\exists i \exists j \exists k, Q(i, j, k) \equiv \exists i, j, k, Q(i, j, k).$
- **Note:** the order variables enter  $Q(\dots)$  does not change

# Nested Quantifiers: Ordering

- Consider the following:
  - $\forall x_1 \exists x_2 \forall x_3 \exists x_4 \dots$
  - $\exists x_2 \forall x_1 \exists x_4 \forall x_3 \dots$
  - $\forall x_1, x_3 \exists x_2, x_4 \dots$
  - None of the above are guaranteed to be equivalent.
- Double-check you don't jump over a different quantifier when simplifying.

# Outline

- Predicates and Quantifiers
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# Nested Quantifiers: Scope

## Definition:

The portion of the formula a quantifier is covering is called the **scope of the quantifier**.

- The scope of the quantifier is the predicate immediately following.
- Precedence is just below parenthesis.
- Any variable that is not covered by a quantifier is called a **free variable**.

Consider the formula:

$$\forall i \exists j, (P(i, j) \rightarrow \forall k, Q(k))$$

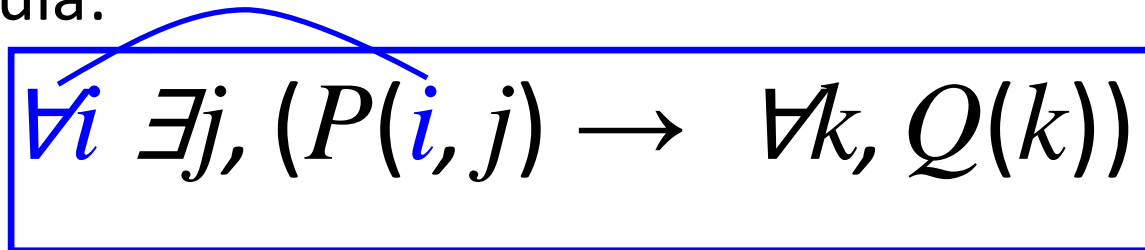
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$$\forall i, (\exists j, (P(i, j) \rightarrow \forall k, Q(k)))$$

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- The scope of  $\exists j$  is the entire formula, minus  $\forall i$ .

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Consider the formula:

$$\forall i \exists j, (P(i, j) \rightarrow \boxed{\forall k, Q(k)})$$

- The scope of  $\forall i$  is the entire formula.
- The scope of  $\exists j$  is the entire formula, minus  $\forall i$ .
- The scope of  $\forall k$  is limited to  $Q(k)$ .



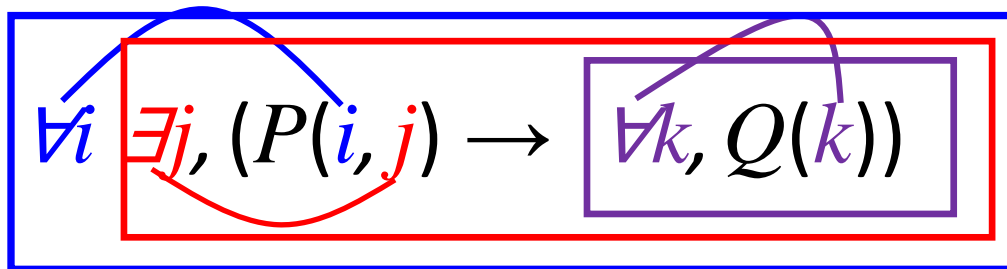
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# Nested Quantifiers: Scope

Quantifiers can move as long as their scope doesn't encompass additional quantifiers of a different type.

Consider

$$\forall i \exists j, (P(i, j) \rightarrow \forall k, Q(k))$$

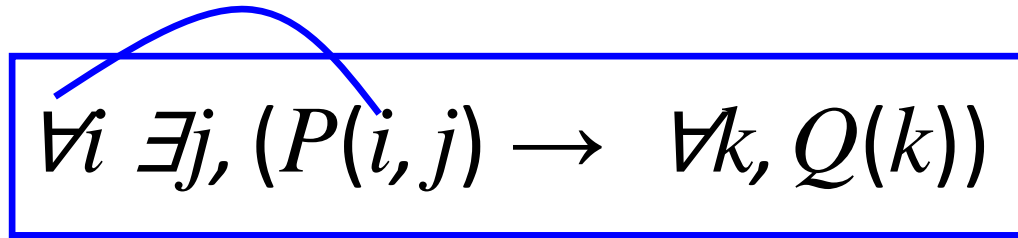
vs

$$\forall i \exists j \forall k, (P(i, j) \rightarrow Q(k))$$

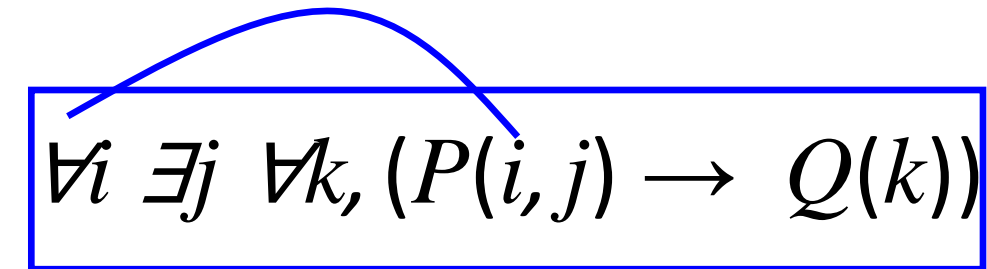
# Nested Quantifiers: Scope

Quantifiers can move as long as their scope doesn't encompass additional quantifiers of a different type.

Consider


$$\forall i \exists j, (P(i, j) \rightarrow \forall k, Q(k))$$

vs


$$\forall i \exists j \forall k, (P(i, j) \rightarrow Q(k))$$

# Nested Quantifiers: Scope

Quantifiers can move as long as their scope doesn't encompass additional quantifiers of a different type.

Consider

$$\forall i \, \boxed{\exists j, (P(i, j) \rightarrow \forall k, Q(k))}$$

vs

$$\forall i \, \boxed{\exists j \, \forall k, (P(i, j) \rightarrow Q(k))}$$

# Nested Quantifiers: Scope

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vs

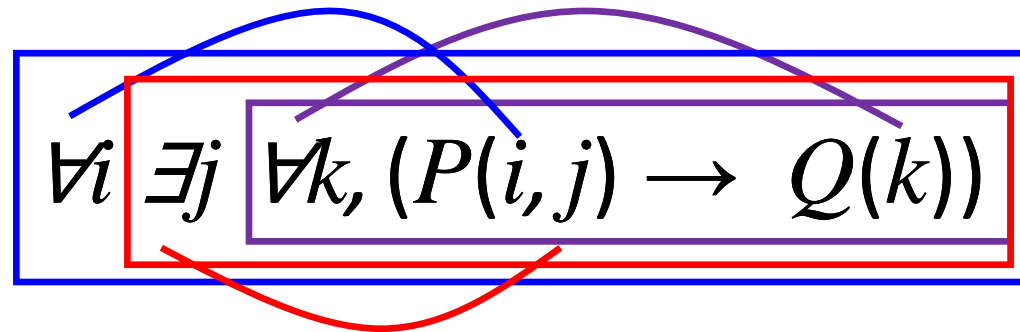
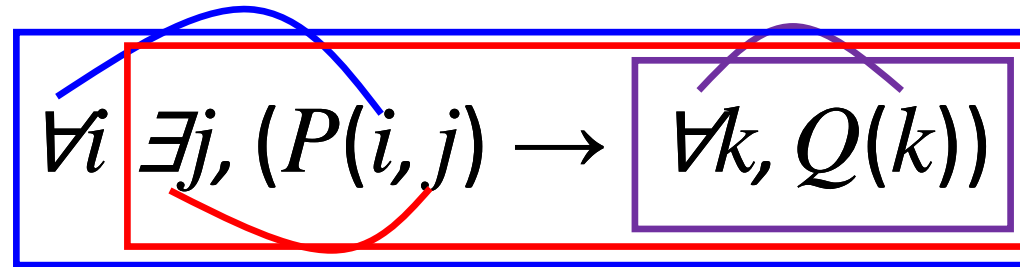
$$\forall i \exists j \boxed{\forall k, (P(i, j) \rightarrow Q(k))}$$

# Nested Quantifiers: Scope

Quantifiers can move as long as their scope doesn't encompass additional quantifiers of a different type.

Consider

vs



# Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables originally covered.

Consider

$$\forall i \exists j, (P(i, j) \rightarrow \forall k, Q(k))$$

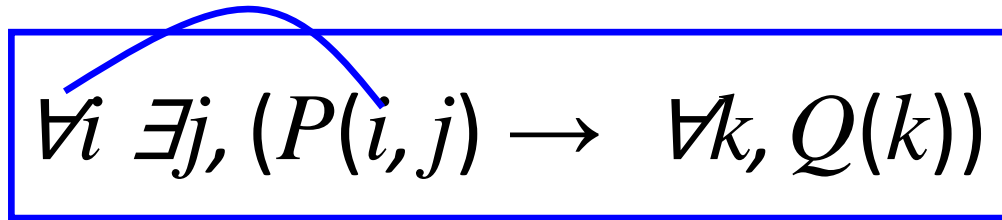
vs

$$\forall i \exists j, ( \forall k, P(i, j) \rightarrow Q(k))$$

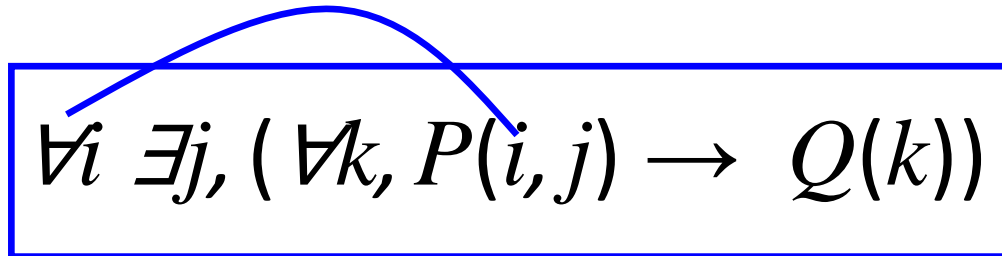
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vs

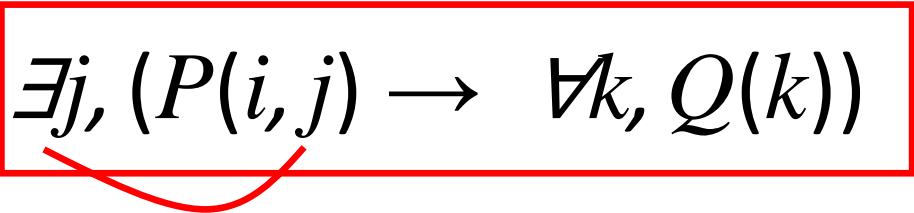

$$\forall i \exists j, (\forall k, P(i, j) \rightarrow Q(k))$$



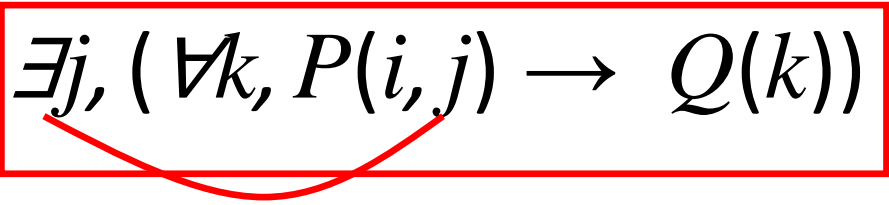
# Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables originally covered.

Consider

$$\forall i \, \exists j, (P(i, j) \rightarrow \forall k, Q(k))$$


vs

$$\forall i \, \exists j, ( \forall k, P(i, j) \rightarrow Q(k) )$$


# Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables originally covered.

Consider

$$\forall i \exists j, (P(i, j) \rightarrow \boxed{\forall k, Q(k)}) \rightarrow \text{proposition}$$

vs

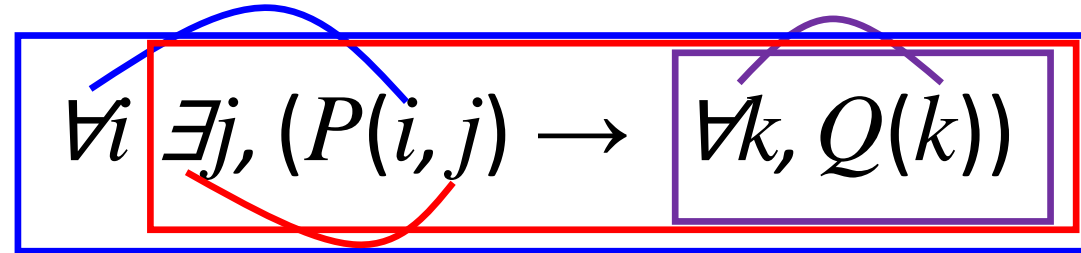
$$\forall i \exists j, (\boxed{\forall k, P(i, j)} \rightarrow Q(k)) \rightarrow \text{Not a proposition}$$

- In the second formula,  $k$  in  $Q(k)$  is no longer **bound** by any quantifier.
  - $k$  is a free variable.
- These two formulas are not equivalent.

# Nested Quantifiers: Scope

Ensure that any reordering doesn't free variables originally covered.

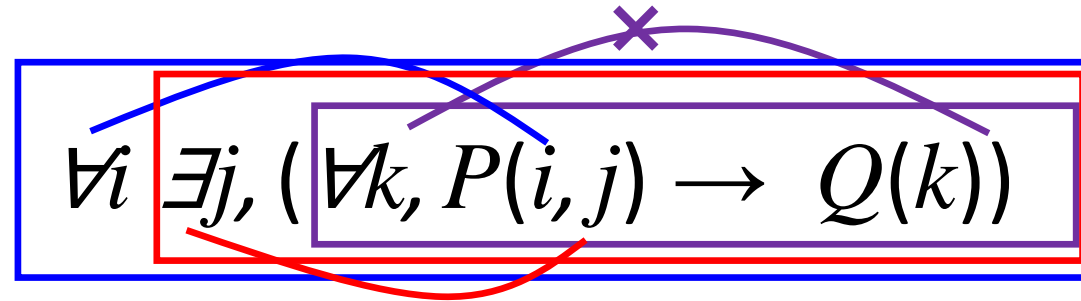
Consider



The diagram shows the formula  $\forall i \exists j, (P(i, j) \rightarrow \forall k, Q(k))$ . It is enclosed in a blue box. Inside, a red box encloses  $\exists j, (P(i, j) \rightarrow \forall k, Q(k))$ , and a purple box encloses  $\forall k, Q(k)$ . A blue arc connects the  $i$  in  $P(i, j)$  to the  $\forall i$  quantifier. A red arc connects the  $j$  in  $P(i, j)$  to the  $\exists j$  quantifier. A purple arc connects the  $k$  in  $Q(k)$  to the  $\forall k$  quantifier.

$$\forall i \exists j, (P(i, j) \rightarrow \forall k, Q(k))$$

vs



The diagram shows the formula  $\forall i \exists j, (\forall k, P(i, j) \rightarrow Q(k))$ . It is enclosed in a blue box. Inside, a red box encloses  $\exists j, (\forall k, P(i, j) \rightarrow Q(k))$ , and a purple box encloses  $\forall k, P(i, j) \rightarrow Q(k)$ . A blue arc connects the  $i$  in  $P(i, j)$  to the  $\forall i$  quantifier. A red arc connects the  $j$  in  $P(i, j)$  to the  $\exists j$  quantifier. A purple arc connects the  $k$  in  $Q(k)$  to the  $\forall k$  quantifier. A purple 'X' is placed over the arc connecting  $k$  to  $Q(k)$ , indicating that  $k$  is no longer bound by the  $\forall k$  quantifier in this formula.

$$\forall i \exists j, (\forall k, P(i, j) \rightarrow Q(k))$$

- In the second formula,  $k$  in  $Q(k)$  is no longer **bound** by any quantifier.
  - $k$  is a free variable.
- These two formulas are not equivalent.
  - $\forall i \exists j, (P(i, j) \rightarrow \forall k, Q(k))$  is a proposition.
  - $\forall i \exists j, (\forall k, P(i, j) \rightarrow Q(k))$  is a predicate (with free variable  $k$ ).