CSE 191: Discrete Structures
Logical Equivalence

Some of the slides are either directly used or adapted from Prof. Andrew Hughes's CSE 191 presentations and are also based on the textbook "Discrete Mathematics and it's applications" (7th edition) by Kenneth H. Rosen.

Outline

- Logic Rules and Equivalence
  - Tautologies and Contradictions
  - Equivalence Laws
  - Proving Logical Equivalence

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  - Logical Equivalence
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Tautologies and Contradictions

Definition

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a ***tautology***.

T

A compound proposition that is always false is called a ***contradiction***.

F

A compound proposition that is neither a tautology nor a contradiction is called a ***contingency***.

Tautologies and Contradictions Examples

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
F	T	T	F
T	F	T	F

Tautologies and Contradictions Examples

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
F	T	T	F
T	F	T	F

- $p \vee \neg p$ is a tautology
- $p \wedge \neg p$ *is a contradiction*

Tautology Example

Construct the truth table for the proposition ***f_1*** defined by:

$$f_1 : p \vee \neg(q \wedge p)$$

p	q	$p \vee \neg(q \wedge p)$
F	F	
F	T	
T	F	
T	T	

Tautology Example

Construct the truth table for the proposition f_1 defined by:

$$f_1 : p \vee \neg(q \wedge p)$$

p	q	$(q \wedge p)$	$\neg(q \wedge p)$	$p \vee \neg(q \wedge p)$
F	F	**F**		
F	T	**F**		
T	F	**F**		
T	T	**T**		

Tautology Example

Construct the truth table for the proposition $\boldsymbol{f_1}$ defined by:

$$f_1 : p \vee \neg(q \wedge p)$$

p	q	$(q \wedge p)$	$\boldsymbol{\neg(q \wedge p)}$	$p \vee \neg(q \wedge p)$
F	F	F	**T**	
F	T	F	**T**	
T	F	F	**T**	
T	T	T	**F**	

Tautology Example

Construct the truth table for the proposition $\boldsymbol{f_1}$ defined by:

$$f_1 : p \vee \neg(q \wedge p)$$

p	q	$(q \wedge p)$	$\neg(q \wedge p)$	$\boldsymbol{p \vee \neg(q \wedge p)}$
F	F	F	T	**T**
F	T	F	T	**T**
T	F	F	T	**T**
T	T	T	F	**T**

- Since the compound proposition f_1 is true in every case, we have proved that f_1 is a tautology.
  - We can say that $f_1 \equiv$ T

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Logical Equivalence

$p: a \to b$

$q: \neg a \vee b$

$(a \to b) \leftrightarrow (\neg a \vee b)$

Definition

$T \leftrightarrow T$

$F \leftrightarrow F$ } T

Two propositions p and q are called ***logically equivalent*** if the proposition $p \Leftrightarrow q$ is a tautology.

- In other words, p and q are logically equivalent if their truth values in their truth tables are the same.

- Two compound propositions are logically equivalent if their truth values agree for all possible combinations of the truth values of their atomics.

- If p and q are logically equivalent, we write $p \equiv q$
  - ≡ is not a logical connective.
  - p ≡ q is not a compound proposition.

Logical Equivalence Example

Consider these two compound propositions: $p \rightarrow q$ and $q \vee \neg p$.

Are $(p \rightarrow q)$ and $(q \vee \neg p)$ logically equivalent?

p	q	$\boldsymbol{p \rightarrow q}$	$\neg p$	$q \vee \neg p$	$(p \rightarrow q) \Leftrightarrow (q \vee \neg p)$
F	F	**T**		T	
F	T	**T**		T	
T	F	**F**		F	
T	T	**T**		T	

Logical Equivalence Example

Consider these two compound propositions: $p \rightarrow q$ and $q \vee \neg p$.
Are $(p \rightarrow q)$ and $(q \vee \neg p)$ logically equivalent?

p	q	$p \rightarrow q$	$\neg p$	$q \vee \neg p$	$(p \rightarrow q) \Leftrightarrow (q \vee \neg p)$
F	F	T	**T**		
F	T	T	**T**		
T	F	F	**F**		
T	T	T	**F**		

Logical Equivalence Example

Consider these two compound propositions: $p \rightarrow q$ and $q \vee \neg p$.

Are $(p \rightarrow q)$ and $(q \vee \neg p)$ logically equivalent?

p	q	$p \rightarrow q$	$\neg p$	$q \vee \neg p$	$(p \rightarrow q) \Leftrightarrow (q \vee \neg p)$
F	F	T	T	T	
F	T	T	T	T	
T	F	F	F	F	
T	T	T	F	T	

Logical Equivalence Example

Consider these two compound propositions: $p \rightarrow q$ and $q \vee \neg p$.

Are $(p \rightarrow q)$ and $(q \vee \neg p)$ logically equivalent?

p	q	3 $p \rightarrow q$	$\neg p$	5 $q \vee \neg p$	1 $(p \rightarrow q) \Leftrightarrow (q \vee \neg p)$
F	F	T	T	T	T
F	T	T	T	T	T
T	F	F	F	F	T
T	T	T	F	T	T

- The columns (3 & 5) for our propositions in question are identical.
  - $(p \rightarrow q) \leftrightarrow (q \vee \neg p)$ is a tautology.
- Therefore, $(p \rightarrow q)$ and $(q \vee \neg p)$ are logically equivalent.

Logical Equivalence Example

Recall the binary operator $p \oplus q$. Let f_2 be the defined by the following proposition:

$$f_2 : (p \vee q) \wedge (p \rightarrow q)$$

Are $(p \oplus q)$ and f_2 logically equivalent?

p	q	$p \oplus q$	$p \vee q$	$p \rightarrow q$	f_2	$(p \oplus q) \Leftrightarrow f_2$
F	F	**F**				
F	T	**T**				
T	F	**T**				
T	T	**F**				

Logical Equivalence Example

Recall the binary operator $p \oplus q$. Let f_2 be the defined by the following proposition:

$$f_2 : (p \vee q) \wedge (p \rightarrow q)$$

Are $(p \oplus q)$ and f_2 logically equivalent?

p	q	$p \oplus q$	$\boldsymbol{p \vee q}$	$p \rightarrow q$	$\boldsymbol{f_2}$	$(p \oplus q) \Leftrightarrow f_2$
F	F	F	**F**			
F	T	T	**T**			
T	F	T	**T**			
T	T	F	**T**			

Logical Equivalence Example

Recall the binary operator $p \oplus q$. Let f_2 be the defined by the following proposition:

$$f_2 : (p \vee q) \wedge (p \rightarrow q)$$

Are $(p \oplus q)$ and f_2 logically equivalent?

$\wedge$

p	q	$p \oplus q$	$p \vee q$	$\boldsymbol{p \rightarrow q}$	$\boldsymbol{f_2}$	$(p \oplus q) \Leftrightarrow f_2$
F	F	F	F	**T**		
F	T	T	T	**T**		
T	F	T	T	**F**		
T	T	F	T	**T**		

Logical Equivalence Example

Recall the binary operator $p \oplus q$. Let f_2 be the defined by the following proposition:

$$f_2 : (p \vee q) \wedge (p \to q)$$

Are $(p \oplus q)$ and f_2 logically equivalent?

p	q	$p \oplus q$	$p \vee q$	$p \to q$	f_2	$(p \oplus q) \Leftrightarrow f_2$
F	F	F	F	T	**F**	
F	T	T	T	T	**T**	
T	F	T	T	F	**F**	
T	T	F	T	T	**T**	

Logical Equivalence Example

Recall the binary operator $p \oplus q$. Let f_2 be the defined by the following proposition:

$$f_2 : (p \vee q) \wedge (p \rightarrow q)$$

Are $(p \oplus q)$ and f_2 logically equivalent?

p	q	3 $p \oplus q$	$p \vee q$	$p \rightarrow q$	6 f_2	$(p \oplus q) \Leftrightarrow f_2$
F	F	**F**	F	T	**F**	**T**
F	T	**T**	T	T	**T**	**T**
T	F	**T**	T	F	**F**	**F**
T	T	**F**	T	T	**T**	**F**

- The columns (3 & ~~7~~ 6) for our propositions in question are **not** identical.
  - $(p \oplus q) \leftrightarrow f_2$ is not a tautology.
- Therefore, $(p \oplus q)$ and f_2 are **not** logically equivalent.

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De Morgan’s Law

De Morgan’s Law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad (1)$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q		$\neg(p \wedge q)$			$\neg p \vee \neg q$
F	F					
F	T					
T	F					
T	T					

De Morgan's Law

De Morgan's Law:

$$\neg(p \land q) \equiv \neg p \lor \neg q \qquad (1)$$

$$\neg(p \lor q) \equiv \neg p \land \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
F	F					
F	T					
T	F					
T	T					

De Morgan's Law

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$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad (1)$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$\boldsymbol{p \wedge q}$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
F	F	**F**				
F	T	**F**				
T	F	**F**				
T	T	**T**				

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$$\neg(p \land q) \equiv \neg p \lor \neg q \qquad (1)$$

$$\neg(p \lor q) \equiv \neg p \land \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
F	F	F	T			
F	T	F	T			
T	F	F	T			
T	T	T	F			

De Morgan's Law

De Morgan's Law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad (1)$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
F	F	F	T	**T**		
F	T	F	T	**T**		
T	F	F	T	**F**		
T	T	T	F	**F**		

De Morgan's Law

De Morgan's Law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad (1)$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
F	F	F	T	T	**T**	
F	T	F	T	T	**F**	
T	F	F	T	F	**T**	
T	T	T	F	F	**F**	

De Morgan's Law

De Morgan's Law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad (1)$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
F	F	F	T	T	T	**T**
F	T	F	T	T	F	**T**
T	F	F	T	F	T	**T**
T	T	T	F	F	F	**F**

De Morgan’s Law

De Morgan’s Law:

$$\neg(p \land q) \equiv \neg p \lor \neg q \qquad (1)$$

$$\neg(p \lor q) \equiv \neg p \land \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	$p \land q$	$\neg(p \land q)$	$\neg p$	$\neg q$	$\neg p \lor \neg q$
F	F	F	T	T	T	T
F	T	F	T	T	F	T
T	F	F	T	F	T	T
T	T	T	F	F	F	F

De Morgan's Law

De Morgan's Law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q \qquad (1)$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

P	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$
F	F	F	T	T	T	T	T
F	T	F	T	T	F	T	T
T	F	F	T	F	T	T	T
T	T	T	F	F	F	F	T

- The columns (4 & 7) for De Morgan's Law (1) are identical.
  - $\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ a tautology.
- Therefore, $\neg(p \wedge q)$ and $(\neg p \vee \neg q)$ are logically equivalent.
  - $\neg(p \wedge q) \equiv (\neg p \vee \neg q)$.

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Law of Distributivity

$p \vee (q \wedge r)$ $\quad (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r)$

$(p \wedge q) \vee (p \wedge r)$

Distributivity:

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \qquad (1)$$

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \qquad (2)$$

The following is the truth table proof for (1). The proof for (2) is similar.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
F	F	F	F	F	F	F	F
F	F	T	F	F	F	T	F
F	T	F	F	F	T	F	F
F	T	T	T	T	T	T	T
T	F	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	T	F	F	T	T	T	T
T	T	T	T	T	T	T	T

Examples of Logical Equivalence

Exercise:

Prove case (2) for De Morgan’s Law and case (2) of Distributivity.

Note that these are similar to the proof of case (1) for each.

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Law of Contraposition

Contrapositive:

The proposition $\neg q \rightarrow \neg p$ is called the **contrapositive** of the proposition $p \rightarrow q$.

- An implication is always **logically equivalent** to its contrapositive. That is,

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

p	q	$p \rightarrow q$	$\neg p$	$\neg q$	$\neg q \rightarrow \neg p$
F	F	T	T	T	T
F	T	T	T	F	T
T	F	F	F	T	F
T	T	T	F	F	T

- A common proof technique is called proof by contraposition.
  - Prove the contrapositive of a given implication is true
  - Conclude that the given implication is therefore true, as they are equivalent.

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Converse and Inverse

Converse

- The proposition $q \rightarrow p$ is called the **converse** of the proposition $p \rightarrow q$.

Inverse

- The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of the proposition $p \rightarrow q$.

Exercises:

- Prove that the converse $p \rightarrow q$ is not logically equivalent to $p \rightarrow q$.
- Prove that the inverse $p \rightarrow q$ is not logically equivalent to $p \rightarrow q$.

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Logical Equivalence Rules

Equivalence	Name
$p \wedge \mathrm{T} \equiv p, \quad p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}, \quad p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r), \quad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q, \quad \neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan’s laws
$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathrm{T}, \quad p \wedge \neg p \equiv \mathrm{F}$	Negation laws

Logical Equivalence Rules

Logical Equivalences Involving Conditional Statements

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Equivalences Involving Biconditional Statements

$p \Leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \Leftrightarrow q \equiv q \Leftrightarrow p$
$p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$
$p \Leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \Leftrightarrow q) \equiv p \Leftrightarrow \neg q$

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Proving Logical Equivalence

- By using Equivalence laws, we can prove two propositions are logically equivalent without having to construct large truth tables.
- The logical equivalence shown in the tables can be used to construct additional logical equivalences

Proving Logical Equivalence

<u>Example 1:</u> Prove $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$.

Equivalence	Name
$p \wedge \mathrm{T} \equiv p, \quad p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}, \quad p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r), \quad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r), \; p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q, \quad \neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathrm{T}, \quad p \wedge \neg p \equiv \mathrm{F}$	Negation laws

Proving Logical Equivalence

Example 1: Prove $\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg q$

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$	by De Morgan's law
$\equiv \neg p \wedge (\neg\neg p \vee \neg q)$	by De Morgan's law
$\equiv \neg p \wedge (p \vee \neg q)$	by Double Negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by Distributive law
$\equiv (p \wedge \neg p) \vee (\neg p \wedge \neg q)$	by Commutative law
$\equiv \mathrm{F} \vee (\neg p \wedge \neg q)$	by Negation law
$\equiv (\neg p \wedge \neg q) \vee \mathrm{F}$	by Commutative law
$\equiv \neg p \wedge \neg q$	by Identity law

Equivalence	Name
$p \wedge \mathrm{T} \equiv p, \quad p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}, \quad p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r), \quad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q, \quad \neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathrm{T}, \quad p \wedge \neg p \equiv \mathrm{F}$	Negation laws

Proving Logical Equivalence

<u>In general:</u>

- Each line should be equivalent to the previous.
- Each line should list the equivalence rule that led to it.
  - Exactly one rule applied per line.
- Start with LHS and go until you reach the RHS.

<u>**Note:**</u> logical equivalence proofs are very exact. Later proofs will become less restrictive.

Proving Logical Equivalence

<u>Example 1:</u> Prove $(p \wedge q) \rightarrow (p \vee q) \equiv$ T.

Equivalence	Name
$p \wedge \mathrm{T} \equiv p, \quad p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}, \quad p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r), \quad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q, \quad \neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathrm{T}, \quad p \wedge \neg p \equiv \mathrm{F}$	Negation laws

Proving Logical Equivalence

Example 2: Prove $(p \wedge q) \rightarrow (p \vee q) \equiv T$

$(p \wedge q) \rightarrow (p \vee q)$

$\equiv \neg(p \wedge q) \vee (p \vee q)$	by Conditional law
$\equiv (\neg p \vee \neg q) \vee (p \vee q)$	by De Morgan's law
$\equiv \neg p \vee (\neg q \vee (p \vee q))$	by Associative law
$\equiv \neg p \vee ((\neg q \vee p) \vee q)$	by Associative law
$\equiv \neg p \vee ((p \vee \neg q) \vee q)$	by Commutative law
$\equiv \neg p \vee (p \vee (\neg q \vee q))$	by Associative law
$\equiv (\neg p \vee p) \vee (\neg q \vee q)$	by Associative law
$\equiv (p \vee \neg p) \vee (\neg q \vee q)$	by Commutative law
$\equiv (p \vee \neg p) \vee (q \vee \neg q)$	by Commutative law
$\equiv (p \vee \neg p) \vee T$	by Complement law
$\equiv T \vee T$	by Complement law
$\equiv T$	by Domination law

$p \rightarrow q \equiv \neg p \vee q$
$p \rightarrow q \equiv \neg q \rightarrow \neg p$
$p \vee q \equiv \neg p \rightarrow q$
$p \wedge q \equiv \neg(p \rightarrow \neg q)$
$\neg(p \rightarrow q) \equiv p \wedge \neg q$
$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$
$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$
$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$
$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Equivalence	Name
$p \wedge \mathrm{T} \equiv p, \quad p \vee \mathrm{F} \equiv p$	Identity laws
$p \vee \mathrm{T} \equiv \mathrm{T}, \quad p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination laws
$p \vee p \equiv p, \quad p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r), \quad (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r), \quad p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \vee q) \equiv \neg p \wedge \neg q, \quad \neg(p \wedge q) \equiv \neg p \vee \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p, \quad p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathrm{T}, \quad p \wedge \neg p \equiv \mathrm{F}$	Negation laws

Proving Logical Equivalence

$p \to q \equiv \neg p \lor q$
$p \to q \equiv \neg q \to \neg p$
$p \lor q \equiv \neg p \to q$
$p \land q \equiv \neg(p \to \neg q)$
$\neg(p \to q) \equiv p \land \neg q$
$(p \to q) \land (p \to r) \equiv p \to (q \land r)$
$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$
$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$
$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

<u>Example 3:</u> Prove $(p \to q) \land (p \to r) \equiv p \to (q \land r)$.

Note that, by the "conditional law", we have: RHS $\equiv \neg p \lor (q \land r)$.

So, we start from the LHS and try to get RHS.

$(p \to q) \land (p \to r)$

$\equiv (\neg p \lor q) \land (\neg p \lor r)$ by conditional law

$\equiv \neg p \lor (q \land r)$ by distributive law

$\equiv p \to (q \land r)$

Equivalence	Name
$p \land T \equiv p, \quad p \lor F \equiv p$	Identity laws
$p \lor T \equiv T, \quad p \land F \equiv F$	Domination laws
$p \lor p \equiv p, \quad p \land p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \lor q \equiv q \lor p, \quad p \land q \equiv q \land p$	Commutative laws
$(p \lor q) \lor r \equiv p \lor (q \lor r), \quad (p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r), \quad p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \lor q) \equiv \neg p \land \neg q, \quad \neg(p \land q) \equiv \neg p \lor \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p, \quad p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv T, \quad p \land \neg p \equiv F$	Negation laws

Propositional Satisfiability

A compound proposition is **satisfiable** if there is an assignment of truth values to its variables that makes it true.

A compound proposition is **unsatisfiable** when no such assignment exists.

- A compound proposition is unsatisfiable if and only if its negation is a tautology.
- An assignment of truth values that make a compound proposition true, i.e., satisfiable), is called a solution of that satisfiability problem.

- $r = (p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$ is true when p, q, and r have the same truth value.
  - r is satisfiable.