CSE 191: Discrete Structures
Introduction to Set Theory

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Some of the slides are either directly used or adapted from Prof. Andrew Hughes's CSE 191 presentations and are also based on the textbook "Discrete Mathematics and it's applications" (7^{th} edition) by Kenneth H. Rosen.

Outline

- Set Basics
  - Definition
  - Universal Set
  - Cardinality
- Set Equality and Subsets
- Set Operations

Sets

Definition

A **set** is a **collection of objects** that do NOT have an order.

- Each object is called an **element or a member** of the set.
- We write
  - $e \in S$ if e **is an element of** S; and
  - $e \notin S$ if e **is not an element of** S.

Sets

<u>How to describe a set:</u>

- List all elements.
  - E.g., $\{1, 2, 3\}$.
    - This is called **roster notation** – list all contents.
- Provide a description of what the elements look like.
  - E.g., $\{a \mid a > 2, a \in \mathbb{Z}\}$.
    - This is called **set builder notation** – describe contained elements.

Common Sets

- $\mathbb{N} = \{1, 2, \ldots\}$: the set of **natural numbers**.
  - Sometimes 0 is considered a member, which some people do not agree with.
- $\mathbb{Z} = \{0, -1, 1, -2, 2, \ldots\}$: the set of **integers**.
- $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$: the set of **positive integers**.
- $\mathbb{Q} = \{p/q \mid p \in Z, q \in Z, q \neq 0\}$: the set of **rational numbers**.
  - numbers that can be written as a fraction of integers.
- $\mathbb{Q}^+ = \{x \mid x \in Q, x > 0\}$: the set of **positive rational numbers**.
- $\mathbb{R}$: the set of **real numbers**.
- $\mathbb{R}^+ = \{x \mid x \in R, x > 0\}$: the set of **positive real numbers**.
- $\mathbb{C}$: the set of **complex numbers**.

More Examples

- $A = \{$Orange, Apple, Banana$\}$ is a set containing the names of three fruits.
- $B = \{$Red, Blue, Black, White, Grey$\}$ is a set containing five colors.
- $\{x \mid x$ takes CSE191 at UB in Fall 2021$\}$ is a set of 202 students.
- $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$ is **a set containing four sets**.
- $\{x \mid x \in \{1, 2, 3\}$ and $x > 1\}$ is a set of two numbers.

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Universal Set

- When discussing sets, there is always a **universal set** *U* involved, which contains all objects under consideration.
  - E.g., for A=*{*Orange, Apple, Banana*}*, the universal set might be the set of names of all fruits.
  - E.g., for B=*{*Red, Blue, Black, White, Grey*}*, the universal set might be the set of all colors.
- In many cases, the universal set is **implicit and omitted from discussion**.

Universal *Universal Set*

Is there a universal set covering all universes? (Russell's Paradox)

- Consider a book ***Book Titles*** containing a list of titles of all books not containing their own title.
- Then does ***Book Titles*** contain a line for ***Book Titles***?
  - If ***Book Titles*** lists the title "***Book Titles***" in its pages, then ***Book Titles*** is a book containing its own title and therefore it should not be listed.
  - If *Book Titles* doesn't list the title "*Book Titles*" in its pages, then ***Book Titles*** is a book not containing it's own title and therefore it should be listed.

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Cardinality (for Finite Sets)

Definition

If a set A contains exactly n elements, where n is a non-negative integer, then A **is a finite set**.

- n is called **the cardinality of** A.
- Denoted by $|A| = n$.

Definition

The **empty set or null set** is the set that **contains no elements**.

- Denoted by $\emptyset$ or **{}**.
- Has size 0.

Example:

The set of all positive integers that are greater than their squares.

Cardinality (for Finite Sets)

Definition

If a set *A* contains exactly *n* elements where *n* is a non-negative integer, then *A* **is a finite set**, and *n* is called **the cardinality of** *A*. We write |A| = *n*.

- Do we count duplicate items?
  - **NO**. We only count unique items for cardinality.

- The following sets are the same:
  - C = *{*Apple, Banana, Apple, Orange, Orange, Apple*},* and
  - A = *{*Orange, Apple, Banana*}*
- Removing duplicates from C gives:
  - C = *{*Apple, Banana, Orange*}*.

Cardinality (for Finite Sets)

- $|\{x \mid -2 < x < 5, x \in \mathbb{Z}\}|$ = 6, the elements −1, 0, 1, 2, 3, 4
- $|\varnothing|$ = 0, no elements in the empty set
- $|\{x \mid x \in \varnothing \text{ and } x < 3\}|$ = 0, because no x satisfies $x \in \varnothing$
- $|\{x \mid x \in \{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\}\}|$ = 4, the 4 sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$
- $|\{0, 0, 0, 1, 1, 1, 2, 2, 2, 3, 3, 4\}|$ = 5 (only count unique elements)

Cardinality: $\emptyset$ vs $\{\emptyset\}$

- Consider this shopping cart's contents:

$|ShoppingCart| = 0$ (represents $\emptyset$ or $\{\}$)

Cardinality: $\emptyset$ vs $\{\emptyset\}$

- Consider this shopping basket's contents:

$|ShoppingBasket| = 0$ (also represents $\emptyset$ or $\{\}$)

Cardinality: $\emptyset$ vs $\{\emptyset\}$

- Now, consider this shopping cart's contents:

This is representative of: $\{\emptyset\}$

$|\text{ShoppingCart}| = |\{\text{ShoppingBasket}\}| = 1$

$\{\emptyset\}$ does not indicate an empty set; it contains an empty set as a member and thus has a cardinality of 1.

Cardinality (for Infinite Sets)

Definition

If A is not finite, then it is an **infinite set**.

- What is the cardinality (i.e., the size) of an infinite set?
- Do all infinite sets have the same size (i.e., ∞)?
  - Appears to not be the case.
    - Are there more rational numbers than integers?
    - Are there are more real numbers than rational numbers?
    - Only one of these is true.

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Equal Sets

Definition

Two sets are **equal** if and only if **they have the same elements**.

Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.

- Denoted by $A = B$.
- Order of elements is irrelevant.

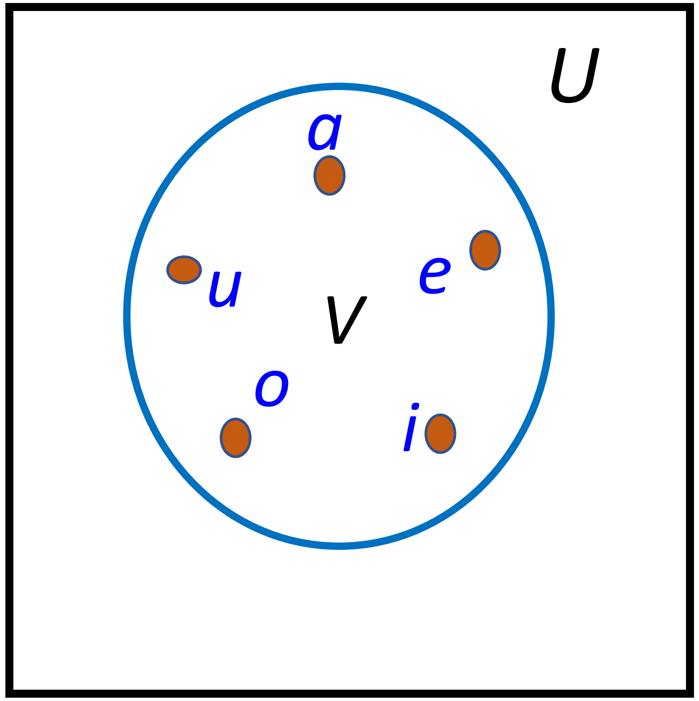
Examples:

- $\{1,2,3\} = \{2,1,3\}$
- $\{1, 2, 3, 4\} = \{x \mid x \in \mathbb{Z} \text{ and } 1 \leq x < 5\}$
- $\{\{\}\} = \{\emptyset, \{\}\}$.

Venn Diagrams

Venn diagrams are a graphical way to represent sets.

- The universal set U is represented by a rectangle.
- Inside the rectangle, circles and other geometrical figures are used to represent sets.
- Sometimes points are used to represent the particular elements of the set.
- Venn diagrams are often used to indicate relationships between sets.



Venn diagram for the set of vowels

Outline

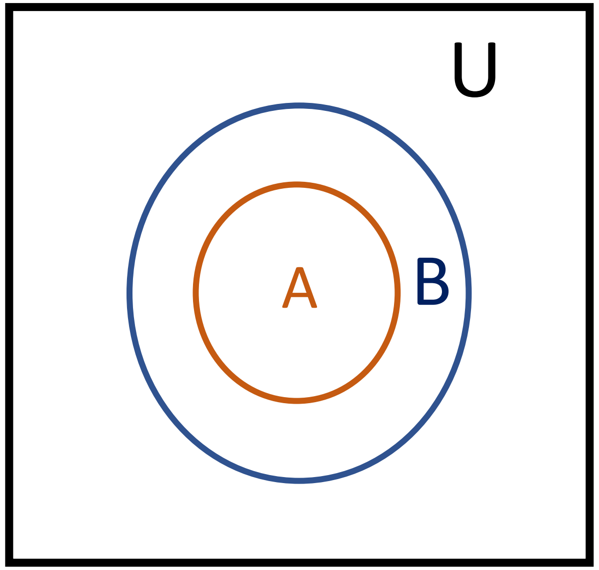
- Set Basics
- Set Equality and Subsets
    - Subsets
    - Set Equality
- Set Operations

Subsets

Definition

A set A is **a subset of** B if and only if every element of A is also in B.

- Denoted by $A \subseteq B$.
- If $A \subseteq B$, then $\forall x \in A,\ x \in B$.



Venn Diagram showing that A is a subset of B

- For any set A,
  - $\emptyset \subseteq A$ and
  - $A \subseteq A$.

Definition

If $A \subseteq B$ but $A \neq B$, then A is **a proper subset** of B.

- Denoted by $A \subset B$ or $A \subsetneq B$.

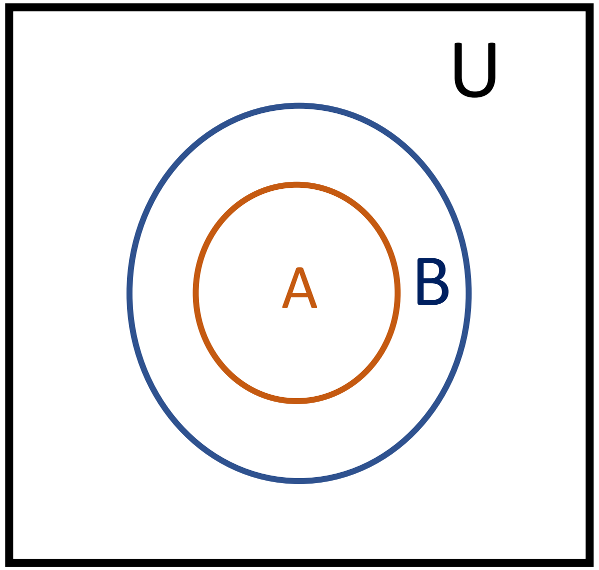
Subsets

Showing that A is a subset of B:

- To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.

Showing that A is not a subset of B:

- To show that $A \nsubseteq B$, find a single $x \in A$, such that $x \notin B$.



Venn Diagram showing that A is a subset of B

Subset examples

- $\{1, 2\} \subseteq \{2, 1, 3\}$.
  - Also, $\{1, 2\} \subset \{2, 1, 3\}$.
- $\{x \in \mathbb{Z} \mid x \text{ is even}\} \subseteq \{x \mid x \in \mathbb{Z}\}$.
  - Every even integer is an integer.
- $\{x \in \mathbb{Z} \mid x \text{ is even}\} \nsubseteq \{x \mid x \in \mathbb{Z} \text{ and } 1 \leq x < 5\}$.
  - We use $A \nsubseteq B$ to denote **not a subset of**.
  - Can still have overlap.
  - Both sets share 2 and 4.

“Continue pattern”

- $\{2, 4, 6, 8, \boxed{...}\} \subseteq \{n \in \mathbb{N} \mid n \text{ is even}\}$.
  - We can use $A \subseteq B$ even in the case of equality.

Subset examples

A = {a, b, c}
B = {a, b, e}
C = {a, e}

True of False?

1. $A \subseteq B$
2. $A \subset B$
3. $C \subset B$
4. $C \subseteq C$
5. $\{b\} \subseteq \{a, \{b\}, c\}$
6. $\emptyset \subseteq \{a, \{b\}, c\}$

Outline

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  - Subsets
  - Set Equality
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Equal Sets

Fact

Suppose A and B are sets. Then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Proof of set equality:

Prove $A \subseteq B$:	Next prove $B \subseteq A$:
Assume x in A.	Assume y in B.
……..	……..
……..	……..
$\therefore$ x belongs to B.	$\therefore$ y belongs to A.
Conclude that $A \subseteq B$.	Conclude that $B \subseteq A$.

Conclude that since $A \subseteq B$ and $B \subseteq A$, $A = B$.

Equality via Subsets

Let $A = \{1, 2, 3, 4\}$ and $B = \{x \mid x \in \mathbb{Z}$ and $1 \leq x < 5\}$. Prove that $A = B$.

<u>Proof of $A \subseteq B$:</u>

Assume $x \in A$.

- Case $x = 1$: $1 \in \mathbb{Z}$ and $1 \leq 1 < 5$.
  - $\therefore x \in B$.
- Case $x = 2$: $2 \in \mathbb{Z}$ and $1 \leq 2 < 5$.
  - $\therefore x \in B$.
- Case $x = 3$: $3 \in \mathbb{Z}$ and $1 \leq 3 < 5$.
  - $\therefore x \in B$.
- Case $x = 4$: $4 \in \mathbb{Z}$ and $1 \leq 4 < 5$.
  - $\therefore x \in B$.

$\therefore x \in B$. Thus, $A \subseteq B$.

<u>Proof of ($B \subseteq A$):</u>

Assume $x \in B$.

- Then $x \in \mathbb{Z}$ and $1 \leq x < 5$
  - So, x must be 1, 2, 3, or 4.
- If x is 1, 2, 3, or 4, then $x \in A$.

$\therefore x \in A$. Thus, $B \subseteq A$.

Since $A \subseteq B$ and $B \subseteq A$, we get that $A = B$.

Equality vis Subsets

Let $E_1 = \{\{\}\}$ and $E_2 = \{\emptyset, \{\}\}$. Prove that $E_1 = E_2$.

<u>Proof of ($E_1 \subseteq E_2$):</u>

Assume $x \in E_1$.

- $x = \{\}$: $\{\} \in E_2$, so $x \in E_2$.

This is the only element in E_1.

Thus, $E_1 \subseteq E_2$.

<u>Proof of ($E_2 \subseteq E_1$):</u>

Assume $x \in E_2$.

- $x = \emptyset$: $\emptyset = \{\} \in E_1$, so $x \in E_1$.
- $x = \{\}$: $\{\} \in E_1$, so $x \in E_1$.

Thus, $E_2 \subseteq E_1$.

Since $E_1 \subseteq E_2$ and $E_2 \subseteq E_1$, we get that $E_1 = E_2$.

This reiterates that multiplicity doesn't matter.

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- Set Operations
    - Basic Operators
    - Power Set
    - Cartesian Product
    - Partitions

Set Operations

- We have $+, -, \times, \div, \ldots$ operators for numbers.
- We have $\vee, \wedge, \neg, \rightarrow \ldots$ operators for propositions.

Set Operation	Symbol	Idea	Logic
Union of A and B	$A \cup B$	in A or B	$\vee$
Intersection of A and B	$A \cap B$	in A *and* B	$\wedge$
Complement of A	$\overline{A}$	not in A	$\neg$
Difference of A and B	$A \setminus B$	in A and not in B	$A \wedge \neg B$
Symm. difference of A and B	$A \mathit{O} B$	in A or B, not both	$\oplus$
Subsets: A is subset of B	$A \subseteq B$	members: $A \rightarrow B$	$\rightarrow$

Set Union

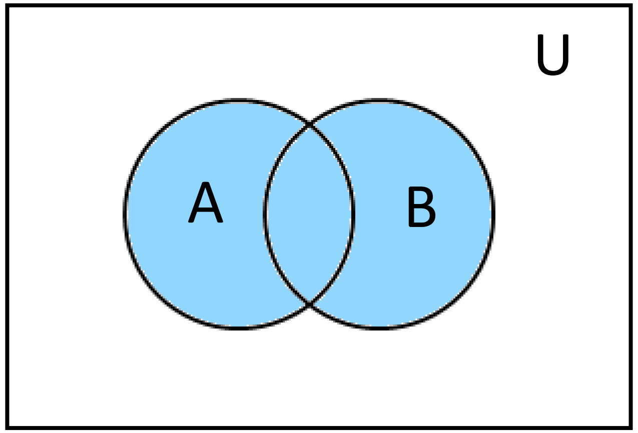
<u>Definition</u>

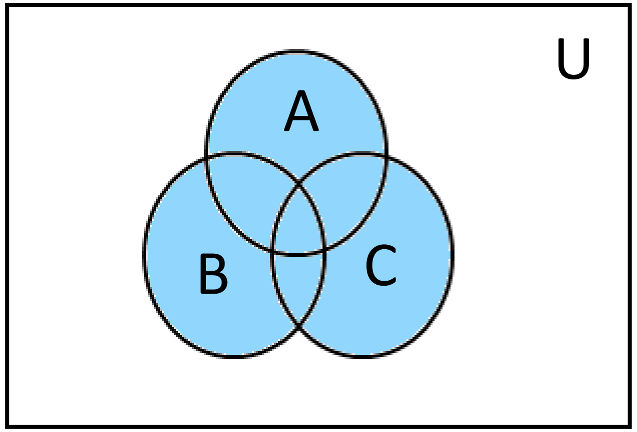
The **union of** two sets A and B is the set that contains exactly all elements that are in either A or B (or in both).

- Denoted by $A \cup B$.
- Formally, $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

Venn Diagrams illustrate results of set operation(s):

$A \cup B$ is shaded





$A \cup B \cup C$ is shaded

Set Intersection

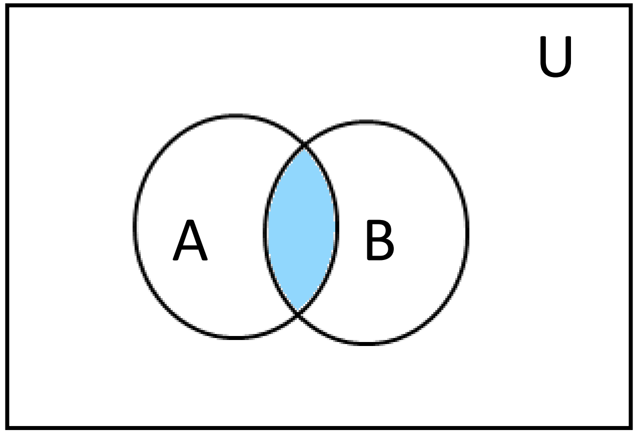
Definition

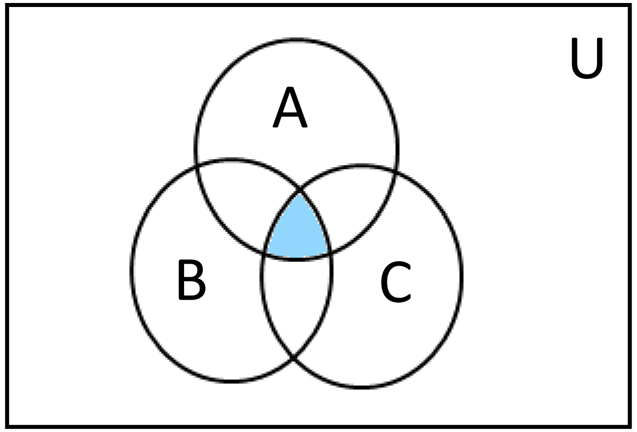
The **intersection of** two sets A and B is the set that contains exactly all the elements that are in both A and B.

- Denoted by $A \cap B$.
- Formally, $A \cap B = \{x \mid x \in A$ and $x \in B\}$.

Venn Diagrams:

$A \cap B$ is shaded





$A \cap B \cap C$ is shaded

Set Intersection

Definition

The **intersection of** two sets A and B is the set that contains exactly all the elements that are in both A and B.

- Denoted by $A \cap B$.
- Formally, $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

Definition

Two sets are called **disjoint** if their intersection is the empty set.

Principle of inclusion-exclusion

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Set Complement

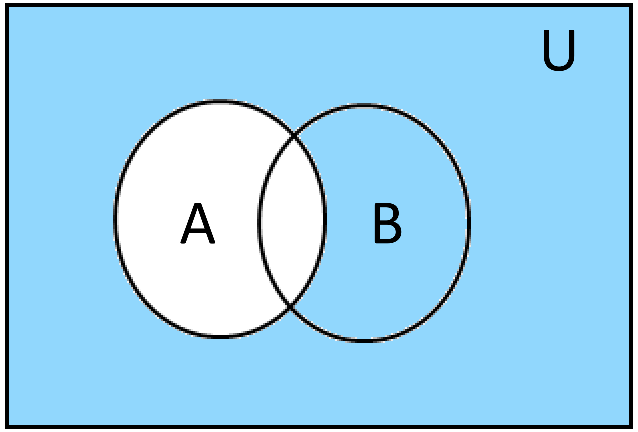
<u>Definition</u>

The **complement of** set A, is the set that contains exactly all the elements that are not in A.

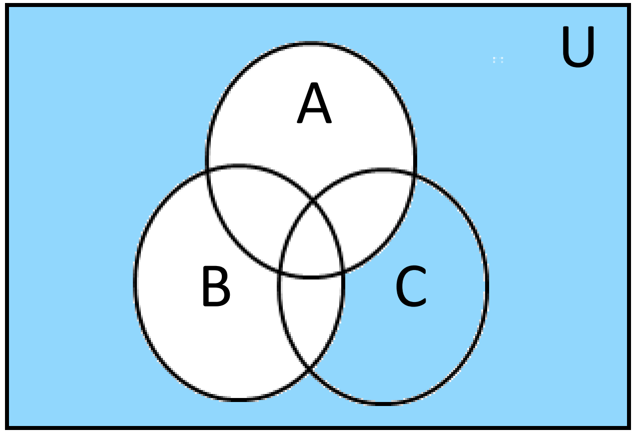
- Denoted by $\overline{A}$.
- Formally, $\overline{A} = \{x \mid x \notin A\}$.

Venn Diagrams:

$\overline{A}$



$\overline{A \cup B}$



Set Difference

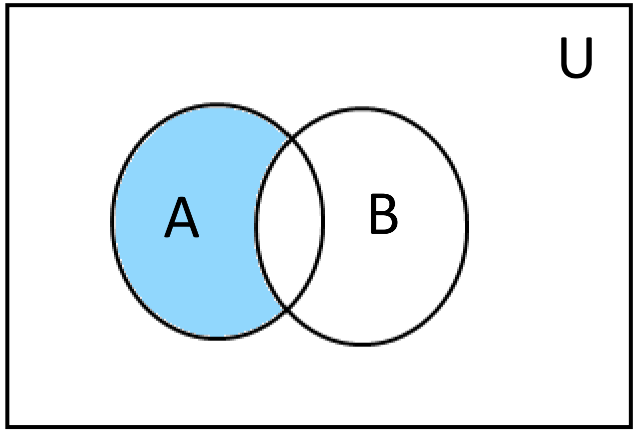
<u>Definition</u>

The **difference of** set A and set B is the set that contains exactly all elements in A but not in B.

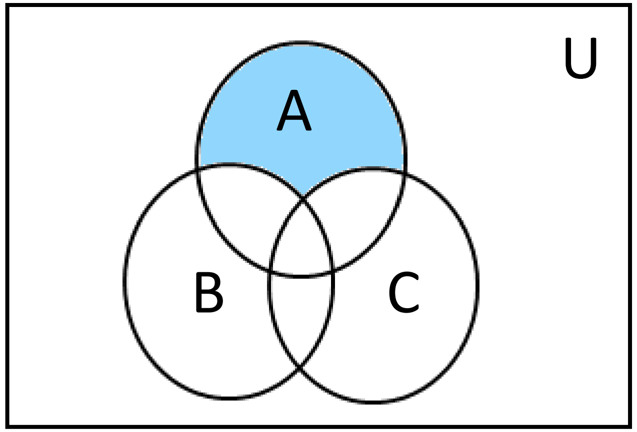
- Denoted by $A - B$ (or $A \setminus B$).
- Formally, $A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$.

Venn Diagrams:

$A - B$



$A - (B \cup C)$



Symmetric Difference

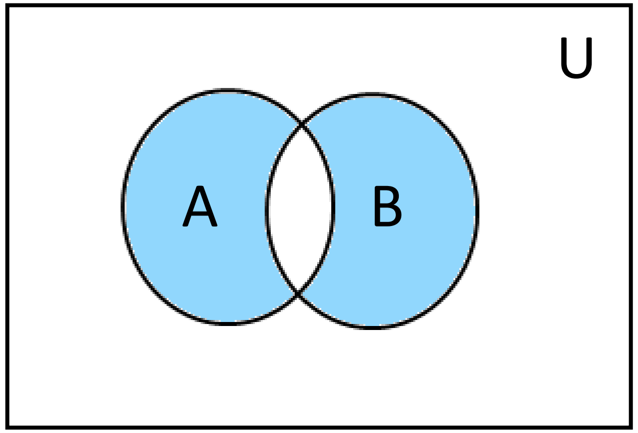
Definition

The **symmetric difference of** set A and set B is the set containing those elements in exactly one of A and B.

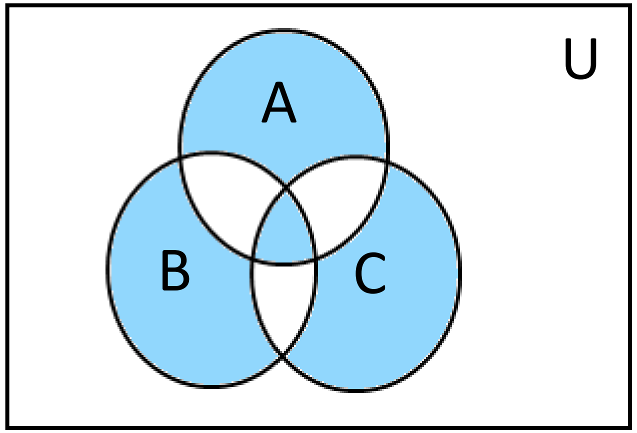
- Denoted by $A \oplus B$ (or A O B).
- Formally, $A \oplus B = (A - B) \cup (B - A)$.

Venn Diagrams:

$A \oplus B$



$A \oplus B \oplus C$



Those values in an odd number of sets, i.e., {x | x ∈ A $\oplus$ x ∈ B $\oplus$ x ∈ C is true}.

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    - Examples
    - Generalized Set Operators
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Operator examples

- Example 1:

Let the universe be $\mathbb{Z}^+$. Write the contents of A in roster form where:

A = $(\{x \mid x \text{ is even}\} - \{x \mid x \text{ is a multiple of } 3\}) \cap \{y \mid y \leq 10\}$

- Example 2:

Let the universe be the 7 colors in a rainbow. Write the contents of C and D in roster form where:

C = $(\{c \mid c \text{ is 6 letters}\} \cup \{c \mid c \text{ has odd length}\}) \oplus \{\text{Red, Blue, Yellow}\}$ and $D = \overline{C}$.

More Practice

Consider the universe $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

- $A \cap B = \{1, 2, 3, 4, 5\}$
- $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $\overline{A} = \{6, 7, 8, 9, 10\}$
- $\overline{B} = \{9, 10\}$
- $A - B = \emptyset$
- $B - A = \{6, 7, 8\}$
- $A \oplus B = (A - B) \cup (B - A) = \{6, 7, 8\}$

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Generalized Set Operators

We can simplify notation for operating on n sets.

- Generalized Union
  - Denoted by: $A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{i=1}^{n} A_i$
  - Formally: $\bigcup_{i=1}^{n} A_i = \{s \mid s \epsilon A_1 \; or \; s \epsilon A_2 \; or \ldots or \; s \epsilon A_n\}$

- Generalized Intersection
  - Denoted by: $A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{i=1}^{n} A_i$
  - Formally: $\bigcap_{i=1}^{n} A_i = \{s \mid s \epsilon A_1 \; and \; s \epsilon A_2 \; and \ldots and \; s \epsilon A_n\}$

Generalized Set Operators

Let $A_i = \{1, 2, . . . , i\}$ for all positive integers i. Then compute:

a) $\bigcup_{i=1}^{n} A_i$

b) $\bigcap_{i=1}^{n} A_i$

Generalized Set Operators

Let $B_i = \{i+1, i+2, \ldots, 2i\}$ for all positive integers i. Compute:

a) $\bigcup_{i=1}^{n} B_i$

b) $\bigcap_{i=1}^{n} B_i$

Let $C_i = \{i\}$ for all positive integers i. Compute:

a) $\bigcup_{i=1}^{n} C_i$

b) $\bigcap_{i=1}^{n} C_i$

c) Prove that $(\bigcup_{i=1}^{n} A_i) = (\bigcup_{i=1}^{n} C_i)$

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Power Set

<u>Definition</u>

The **power set** of set A is the set of all subsets of A.

- Denoted by $\mathcal{P}(A)$.

- In general, $|\mathcal{P}(A)| = 2^{|A|}$.
- For any set A, we always have:
  - $\emptyset \in \mathcal{P}(A)$ (include 0 elements from A)
  - $A \in \mathcal{P}(A)$ (include every element from A)
- Example
- Power set of the set {0 ,1 ,2}:

  $\mathcal{P}(\{0, 1, 2\}) = \{ \emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1, 2\}, \{0, 1, 2\}\}$

Power Set

True or False?

A = {a, {a}, {a, b}, b, {c}, d}

1. $a \in A$
2. $\{b\} \subseteq A$
3. $c \in A$
4. $\{a, d\} \in A$
5. $\{a, b\} \in A$
6. $\{b, d\} \subseteq A$
7. $\{b, d\} \in \mathcal{P}(A)$

$\mathcal{P}(\emptyset) = ?$

$\mathcal{P}(\{\emptyset\}) = ?$

$\mathcal{P}(\{a\}) = ?$

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Imposing Order on Elements

- How can we impose order on elements?
- Sometimes order is important.
  - Rankings, letters in words, etc.

<u>Definition</u>

An **ordered** n**-tuple** $(a_1, a_2, \ldots, a_n)$ has a_1 as its first element, a_2 as its second element, . . ., a_n as its n^{th} element.

- Order is important. Suppose $a_1 \neq a_2$,
  - $(a_1, a_2) \neq (a_2, a_1)$ (*comparing tuples*), but
  - $\{a_1, a_2\} = \{a_2, a_1\}$ (*comparing sets*).

Imposing Order on Elements: Cartesian Product

René Descartes

Definition

The **Cartesian product of two sets** A_1, A_2 is defined as the set of ordered tuples (a_1, a_2) where $a_1 \in A_1, a_2 \in A_2$.

- Denoted by $A_1 \times A_2$.
- Formally, $A_1 \times A_2 = \{(a_1, a_2) \mid a_1 \in A_1, a_2 \in A_2\}$.
- Say "A cross B" to mean $A \times B$.

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How do we compute the Cartesian Product?

Example: $\{1, 2\} \times \{a, b, c\} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$

Consider $A = \{1, 2\}$ and $B = \{a, b, c\}$.

	a	b	c
1	(1, a)	(1, b)	(1, c)
2	(2, a)	(2, b)	(2, c)

$A \times B$ is the set of all elements in our table:

- $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.
- Usually maintain the order of table the rows.

How do we compute the Cartesian Product?

Consider $A = \{1, 2\}$ and $B = \mathbb{Z}^+$.

	1	2	3	. . .
1	$(1, 1)$	$(1, 2)$	$(1, 3)$	
2	$(2, 1)$	$(2, 2)$	$(2, 3)$	

- $A \times B = \{(x, y) \mid x \in \{1, 2\}, y \in \mathbb{Z}^+\}$.
  - Note: $(1, 2)$ and $(2, 1)$ are unique elements in $A \times B$.

<u>Example:</u>

$\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$

- The set of point coordinates in the 2D plane.

Cartesian Product Generalized

- Generalized Cartesian Product: Formally: (for $n \geq 2$):

$A_1 \times A_2 \times \cdot \cdot \cdot \times A_n = \{(a_1, a_2, . . . , a_n) \mid a_1 \in A_1, a_2 \in A_2, . . . , a_n \in A_n\}.$

- Cartesian Power
  - For any integer $n \geq 0$: $A^n = \begin{cases} \{(\,)\} & if\ n = 0 \\ A & if\ n = 1 \\ \underbrace{A \times A \times \ldots \times A}_{\mathrm{n}} & if\ n > 1 \end{cases}$

- Formally:

$$A^n = A \times A \times \cdot \cdot \cdot \times A = \{(a_1, a_2, . . . , a_n) \mid a_1, a_2, . . . , a_n \in A\}.$$

Cartesian Product

A = {x | x is odd integer and x < 10}

B = {y | y is even integer and y < 8}

$A \times B =$?

C = {1, 2}

$C \times C =$?

$|C \times C| =$?

D = {0, 1}

$D^3 =$?

E = {a}

$(C \times E) \times D =$?

Outline

- Set Basics
- Set Equality and Subsets
- Set Operations
  - Basic Operators
  - Power Set
  - Cartesian Product
    - Strings
  - Partitions

Strings

- How can we represent English words?
- Consider: $A = \{a, b, \ldots, z\}$.
  - (c, a, t), (d, o, g) $\in A^3$.
  - (f, r, o, g), (b, i, r, d) $\in A^4$.
- A is the English alphabet.
- Shorthand tuples as “words”:
  - cat, dog $\in A^3$.
  - frog, bird $\in A^4$.

Strings: In Programming

```
s ='cat’
t ='frog'
```

Internally stored as a **null terminated** sequence of characters:

```
s =('c','a','t','\0')
t =('f','r','o','g','\0')
```

```
print(len(s))#prints 3
print(len(t))#prints 4
print(t[1])#prints 'r’
print(s[1:3])#prints 'at’
print(s + t)#prints 'catfrog’
print(s in t)#prints False
print('ro' in t)#prints True
```

Strings

Definition

An **alphabet** is a nonempty finite set of **symbols**.

- A **string** is a finite sequence of symbols from an alphabet written consecutively.
  - Shorthand for tuple from Cartesian power of an alphabet.
- The number of characters in a string is called the **length of the string**.
  - The length of a string s is denoted by $|s|$.

Example

The alphabet $\{0, 1\}$ is used to form **binary strings**.

E.g., 0001 and 110 are **strings over the alphabet** $\{0, 1\}$.

$0001 \in \{0, 1\}^4$. $\quad |0001| = 4$.

$110 \in \{0, 1\}^3$. $\quad |110| = 3$.

Strings

Q: What is the shortest string over any alphabet?

- The smallest Cartesian power is 0.
  - E.g., $\{a, b\}^0 = \{()\}$.
- How can we write the sequence of characters within ()?
  - We let λ denote the **empty string**.
  - Then $\{a, b\}^0 = \{\lambda\}$.
  - $|\lambda| = 0$.
  - E.g.,

```
s = ''#s is the empty string.
print(len(s))#prints 0.
```

Strings

Q: What is the shortest string over any alphabet?

- The smallest Cartesian power is 0.
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- How can we write the sequence of characters within ()?
  - We let λ denote the **empty string**.
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  - E.g.,

```
s =''#s is the empty string.
print(len(s))#prints 0.
```

- Empty string can be formed over any alphabet Σ.
  - Take 0 characters from Σ.
- Many interesting operations: concatenation, substring, prefix, etc.

Strings: Concatenation

The **concatenation** of two strings *s* and *t* is formed by taking all symbols in *s* followed by all symbols in *t*.

- Concatenation of *s* and *t* is denoted by *st*.
- Formally, if $s = s_1s_2 \ldots s_m$ and $t = t_1t_2 \ldots t_n$, then

  $$st = s_1s_2 \ldots s_mt_1t_2 \ldots t_n.$$
- Similar `s + t` in Python.
- Example:
  - s = cat, t = dog, then st = catdog.
  - $s\lambda = \lambda s$ = cat.

Strings: Substrings

t is a **substring** of s if all characters of t appear consecutively within s.

- Similar to `t == s[i:i+n]` in Python.
- A **prefix** of s is a substring that begins at the first character of s.
- A **proper substring** of s is a substring that is not equal to s.

Strings: Substrings

Let s = racecar, t = car, u = race, v = rar then:

- s is a substring of s and s is a prefix of s.
  - s is not a **proper substring** of s.
- t is a substring of s.
- u is a substring of s and is a prefix of s.
- v is not a substring of s.

Outline

- Set Basics
- Set Equality and Subsets
- Set Operations
  - Basic Operators
  - Power Set
  - Cartesian Product
  - Partitions

Pairwise Disjoint Sets

Definition

Two sets A and B are **disjoint** iff $A \cap B = \emptyset$.

- We can extend this to multiple sets.

Definition

A sequence of sets $A_1, A_2, \ldots, A_n$ are **pairwise disjoint** if for any $i, j \in \{1, 2, \ldots, n\}$, where $i \neq j$, we have $A_i \cap A_j = \emptyset$.

Symbolically, we write: $\forall i, j \in \{1, 2, \ldots, n\} : [(i \neq j) \rightarrow (A_i \cap A_j = \emptyset)]$

Pairwise Disjoint Sets Example

Consider the following sets:

- A_1 = $\{$cat, dog$\}$
- A_2 = $\{$apple, banana$\}$
- A_3 = $\{$carrot, celery$\}$

- Are A_1, A_2, A_3 pairwise disjoint?
- Each set is disjoint from the other so A_1, A_2, A_3 are pairwise disjoint.

Pairwise Disjoint Sets Example

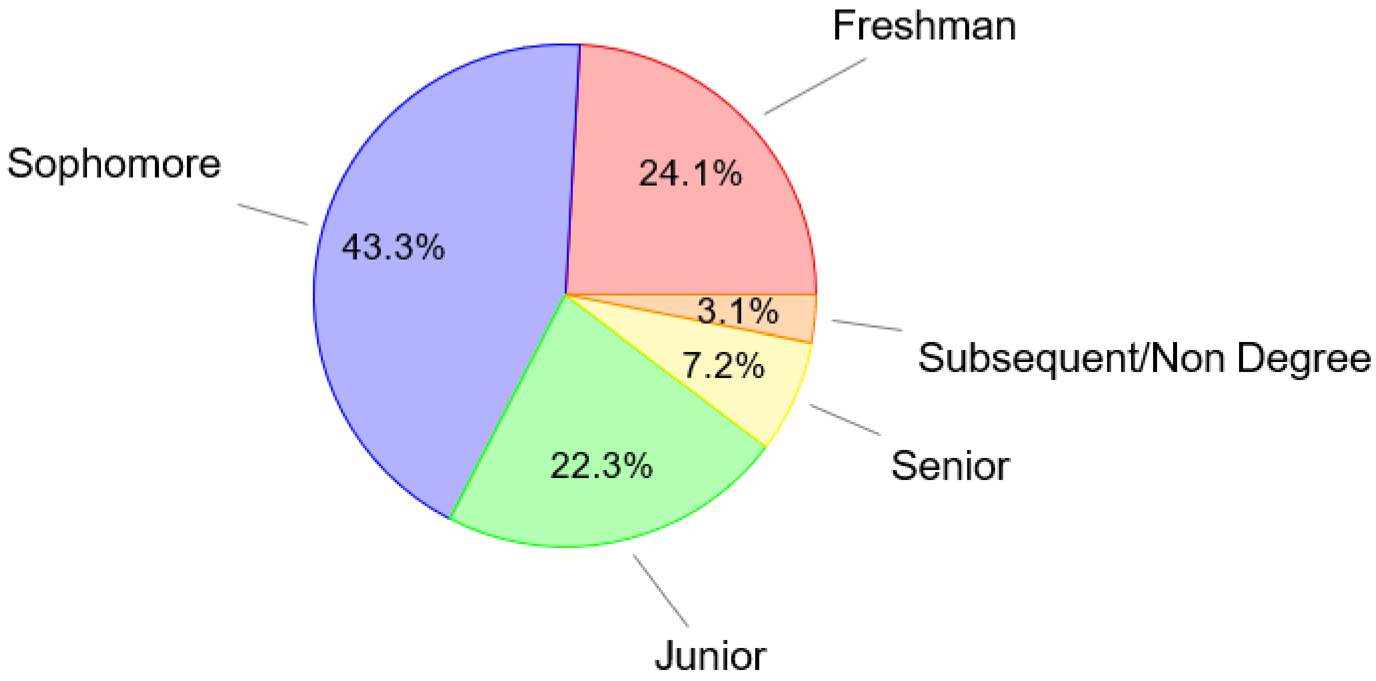
- Consider the following sets:
  - $B_1 = \{x \mid x \in \mathbb{Z}^+,\ x \text{ is even}\}$
  - $B_2 = \{x \mid x \text{ is prime}\}$
  - $B_3 = \mathbb{Z} - \mathbb{Z}^+$
- Are B_1, B_2, B_3 pairwise disjoint?
  - B_1 is disjoint from B_3.
  - B_2 is disjoint from B_3.
  - $B_1 \cap B_2 = \{2\} \neq \emptyset$, so B_1 and B_2 are not disjoint.
  - So B_1, B_2, B_3 are not pairwise disjoint.

Partitions

<u>Definition</u>

A **partition** of a non-empty set A is a list of one or more non-empty subsets of A such that each element of A appears in exactly one of the subsets.

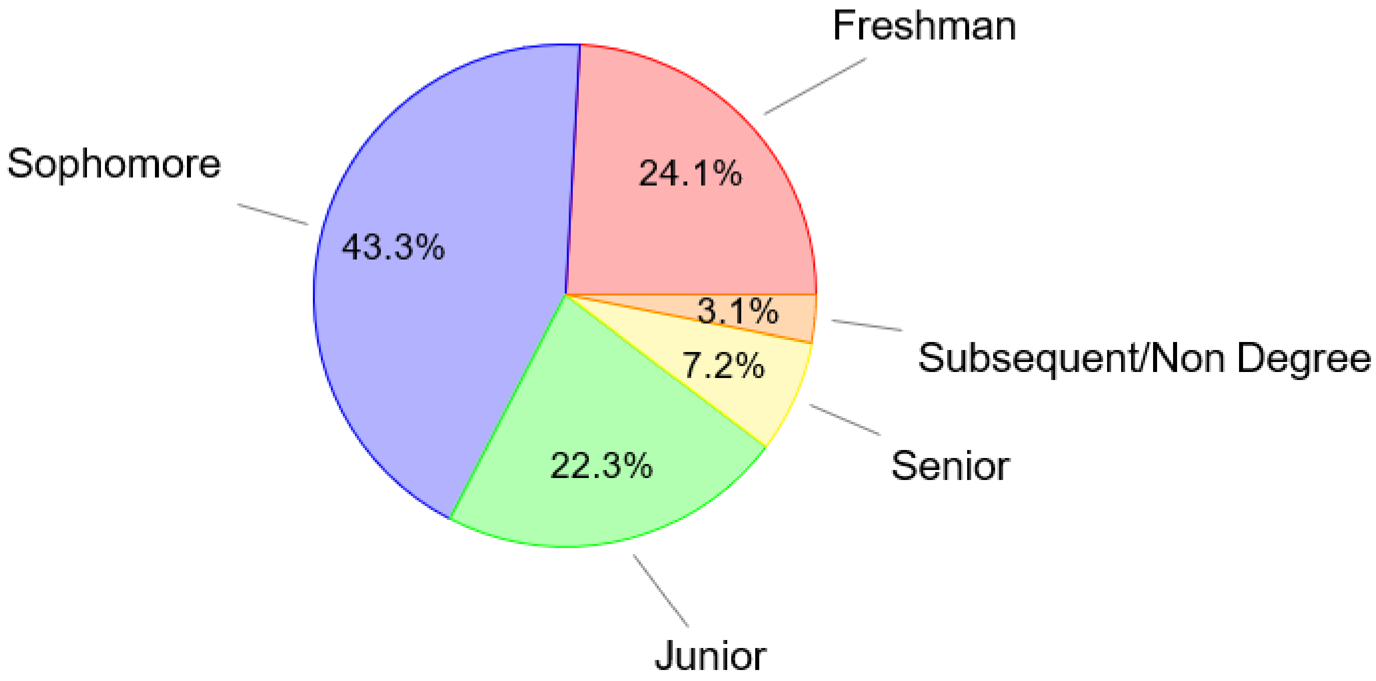
- Consider partitioning students based on their academic level:



Partitions

Formally, a partition of A is a list of sets $A_1, A_2, \ldots, A_k$ ($k \geq 1$) s.t.:

- $\forall i \in [1, k] : A_i \neq \emptyset$ (non-empty sets)
- $\forall i \in [1, k] : A_i \subseteq A$ (subsets of A)
- $\forall i, j \in [1, k] : (i \neq j) \to A_i \cap A_j = \emptyset$ (pairwise disjoint sets)
- $A = \bigcup_{i=1}^{k} A_i$ (contain all elements of A)



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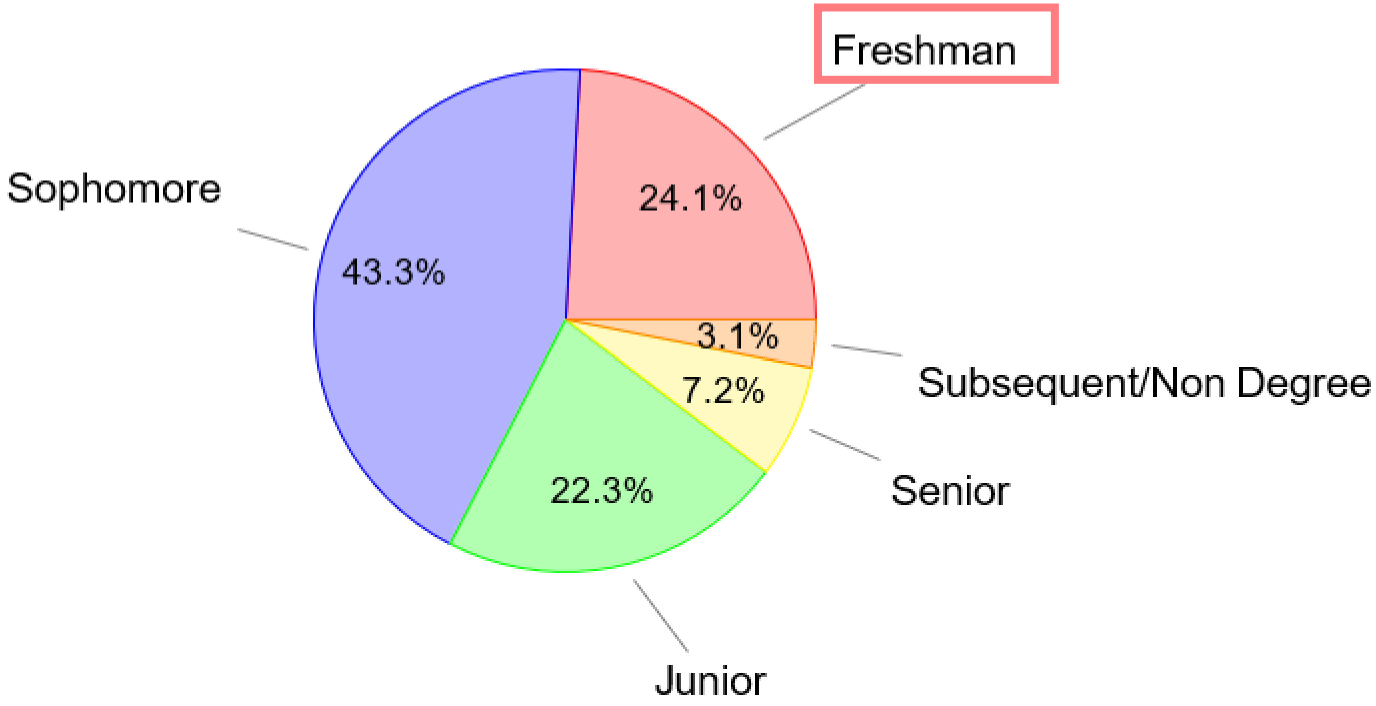

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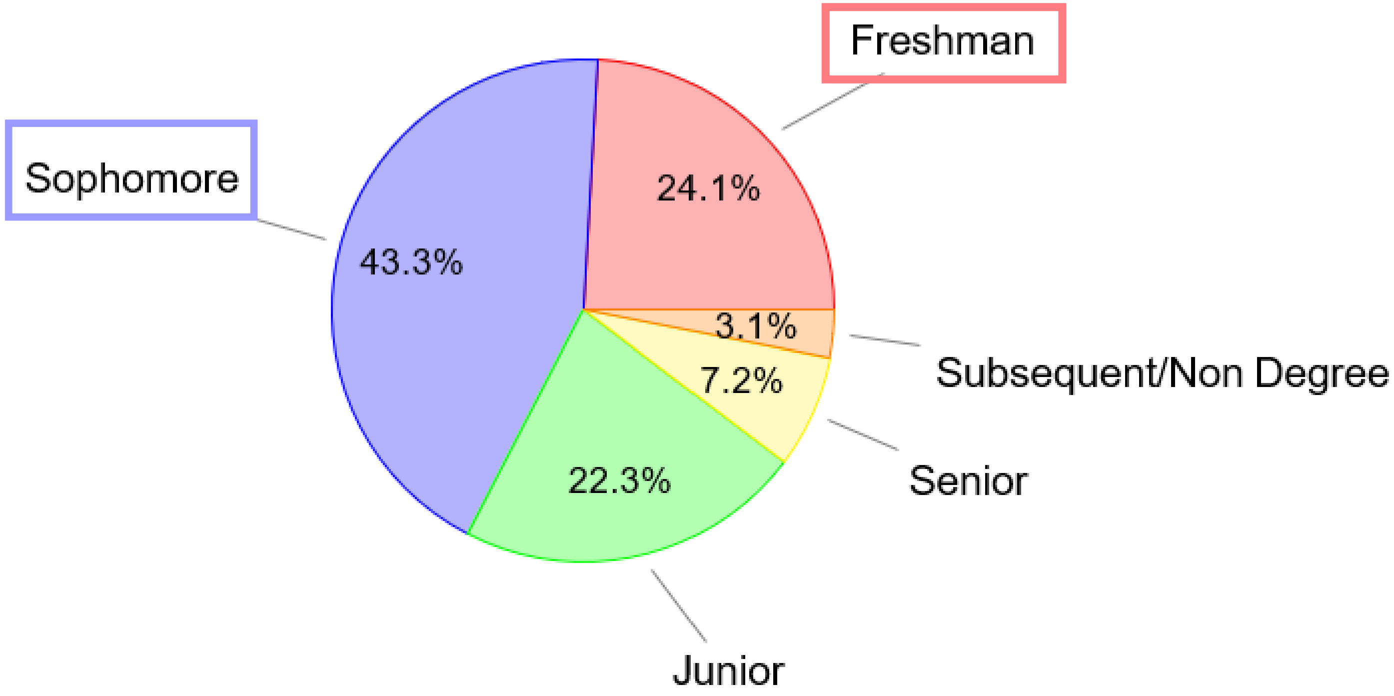

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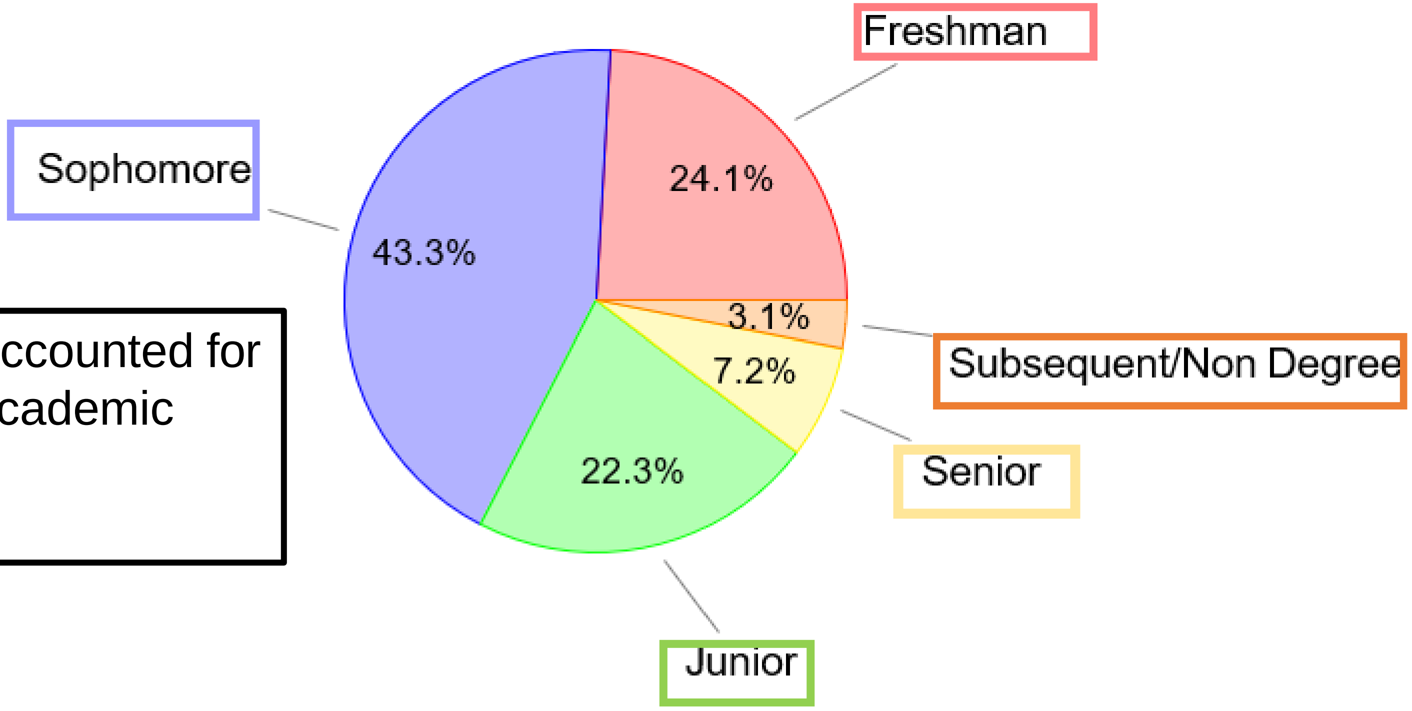
- $\boxed{A = \bigcup_{i=1}^{k} A_i \text{ (contain all elements of } A\text{)}}$

All students are accounted for by one of these academic levels.



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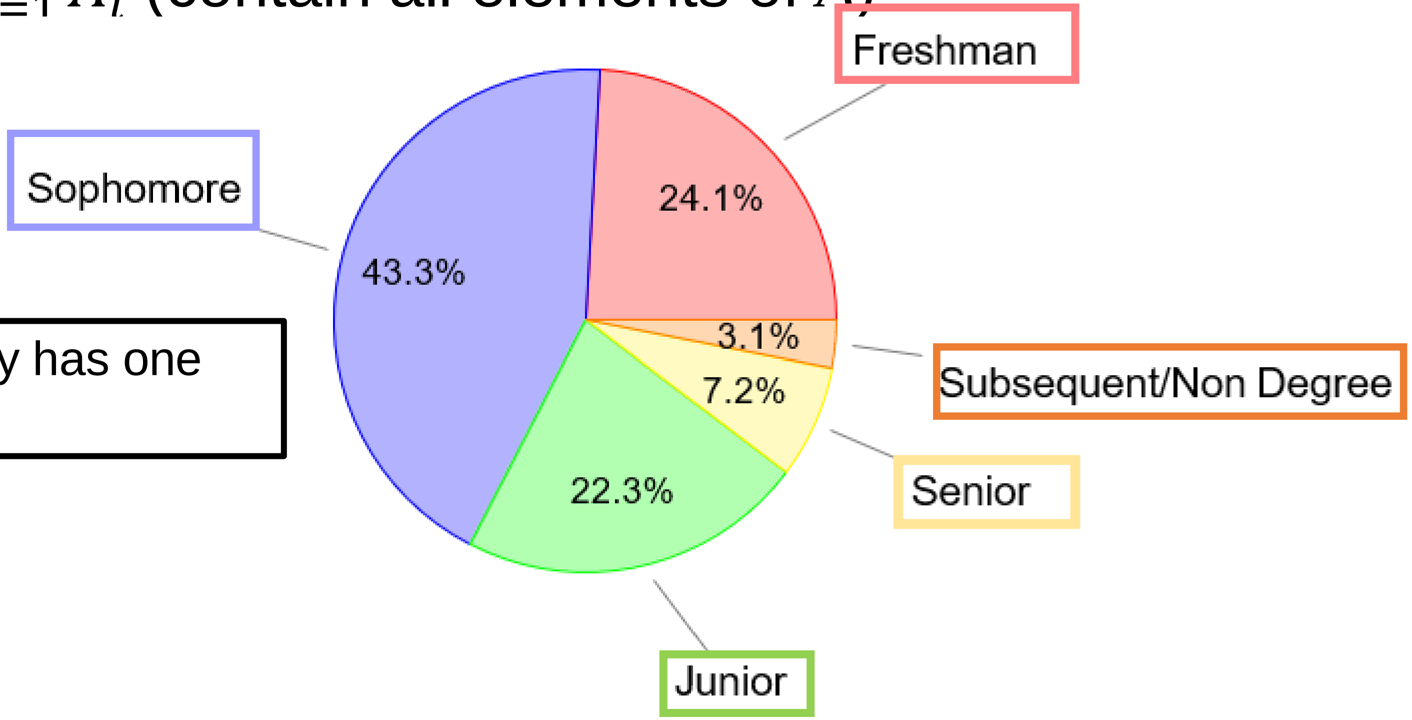
- $A = \bigcup_{i=1}^{k} A_i$ (contain all elements of A)

Each student only has one academic level.



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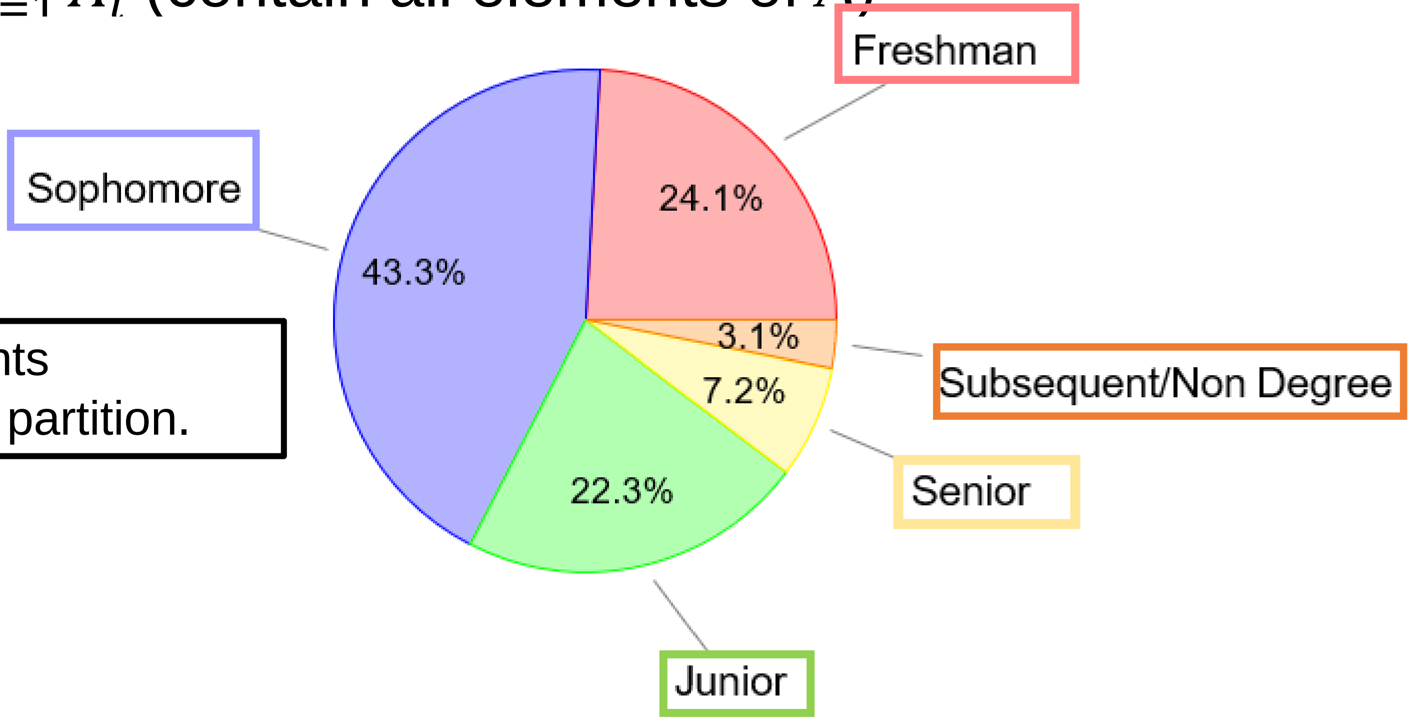


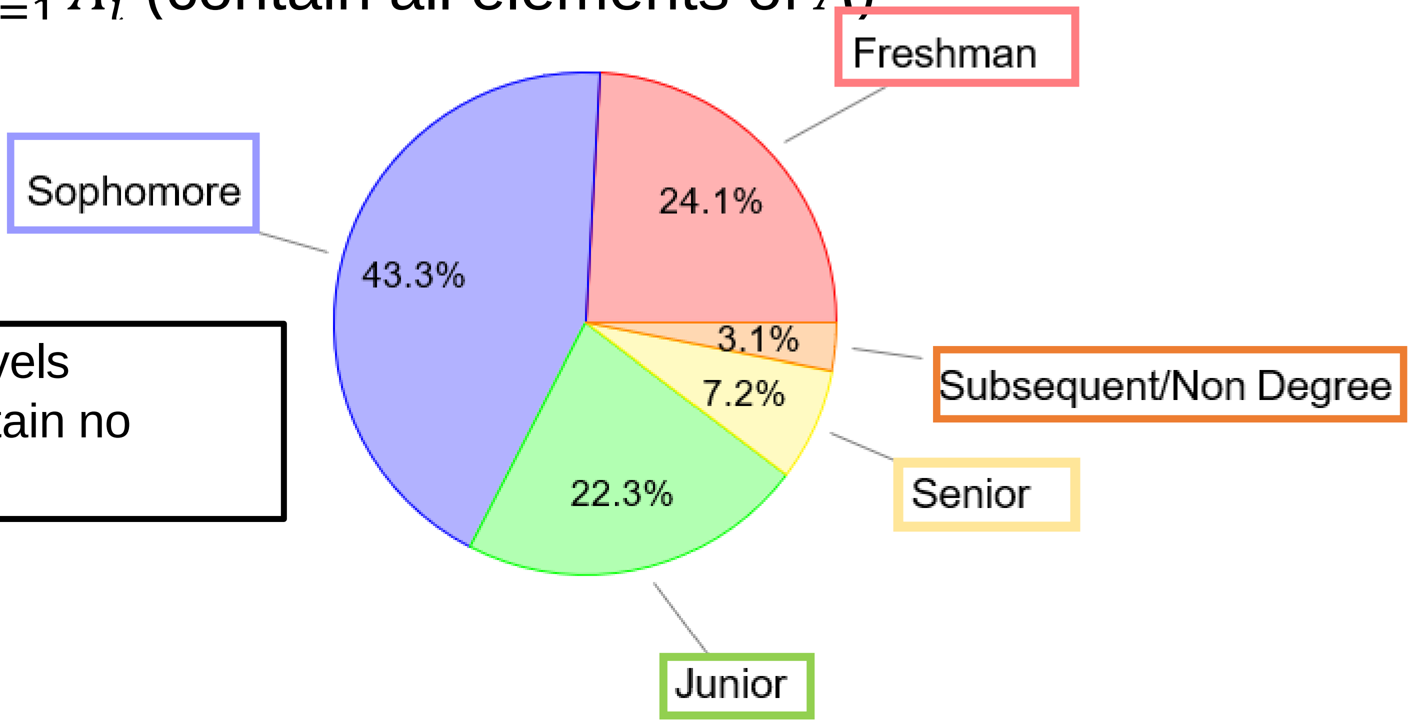
No outside students accounted for by partition.

Partitions

Formally, a partition of A is a list of sets $A_1, A_2, \ldots, A_k$ $(k \geq 1)$ s.t.:

- $\boxed{\forall i \in [1, k] : A_i \neq \emptyset \text{(non-empty sets)}}$
- $\forall i \in [1, k] : A_i \subseteq A$ (subsets of A)
- $\forall i, j \in [1, k] : (i \neq j) \rightarrow A_i \cap A_j = \emptyset$ (pairwise disjoint sets)
- $A = \bigcup_{i=1}^{k} A_i$ (contain all elements of A)

No extraneous levels included that contain no students.



Partitions Example

- Suppose we have the set

COLORS = $\{$red, orange, yellow, green, blue, indigo, violet$\}$

and consider the sets:

- C_1 = $\{$red, orange, yellow$\}$
- C_2 = $\{$blue, violet, green$\}$
- C_3 = $\{$indigo$\}$

Exercise:

- Do C_1, C_2, C_3 partition COLORS?

Partitions Example

- Consider:
  - $O = \{x \mid x \in \mathbb{Z},\ x$ is odd$\}$
  - $E = \{x \mid x \in \mathbb{Z},\ x$ is even$\}$

- Do O and E partition $\mathbb{N}$? No, $-1 \in O \cup E$ but $-1 \neq \mathbb{N}$.
- Do O and E partition $\mathbb{Z}$? Yes, $O \cup E = \mathbb{Z}$.
- Do O and E partition $\mathbb{Q}$? No, $\frac{1}{2} \notin O \cup E$, but $\frac{1}{2} \in \mathbb{Q}$.

Partitions Example

**Exercise

- Define the following sets:
  - $A = \{1, 2, 6\}$
  - $B = \{2, 3, 4\}$
  - $C = \{5\}$
  - $D = \{x \in \mathbb{Z} : 1 \leq x \leq 6\}$

- Do A, B, and C form a partition of D? If not, which condition of a partition is not satisfied?