strain and stress

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Surface wave

1.1 Introduction

Many texts give expressions for displacements in plane-layered media due point forces and moment tensors. These solutions are useful in regional moment tensor studies. However there are occasions when one in interested in the stresses and strains generated by a seismic source. Ground motions of a large earthquake may be such to change the stresses acting on neighboring faults in a way to cause movement. The newly introduced Distributed Acoustic Sensors (DAS) systems measure the strain in a fiber optic cable with great spatial detail. Interpreting earthquake data requires codes for predicting the observed strain.

In a cylindrical coordinate system for isotropic or transverse isotropic media, 15 Green's functions must be computed. If one is interested in strain, then the partial derivatives of the displacement with respect to the z an r coordinates will require an additional 30 functions to be computed. The partials with respect to azimuth will not require any significant computational effort.

When the epicentral distance is large compared to the wavelength, a superposition of surface-wave models can provide a reasonable approximation to the exact solution by modeling the larger signals following S. in addition the modification of existing code was relatively simple, and the generation of strain and stress signals is speedy.

1.2 Wave equation solutions in cylindrical coordinates

Expressions for the wave equation are available for other curvilinear coordinate systems (Love, 1944; Aki and Richards, 2002). In particular, for a cylindrical coordinate system with coordinates (r, ϕ, z), the equations of motion for $\mathbf{u} = (u_r, u_\phi, u_z)$

are

$$\rho\frac{\partial^2 u_z}{\partial t^2} = \frac{\partial\sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\phi z}}{\partial\phi} + \frac{\partial\sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + F_z$$
$$\rho\frac{\partial^2 u_r}{\partial t^2} = \frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\phi}}{\partial\phi} + \frac{\partial\sigma_{rz}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\phi\phi}}{r} + F_r$$
$$\rho\frac{\partial^2 u_\phi}{\partial t^2} = \frac{\partial\sigma_{r\phi}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{\partial\sigma_{\phi z}}{\partial z} + \frac{2\sigma_{r\phi}}{r} + F_\phi$$

The local coordinate system is assumed to be such that z is positive downward.

In an isotropic medium, the stresses are related to displacements in a cylindrical coordinate system through the relations (Hughes and Gaylord, 1964)

$$\begin{aligned} \sigma_{rz} &= 2\mu e_{rz} & \sigma_{zz} &= \lambda\Delta + 2\mu e_{zz} \\ \sigma_{\phi z} &= 2\mu e_{\phi z} & \sigma_{rr} &= \lambda\Delta + 2\mu e_{rr} \\ \sigma_{rz} &= 2\mu e_{rz} & \sigma_{\phi\phi} &= \lambda\Delta + 2\mu e_{\phi\phi} \end{aligned} \tag{1.2.1}$$

where the strains are defined as

$$\begin{aligned} e_{rr} &= \frac{\partial u_r}{\partial r} & e_{r\phi} &= \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \\ e_{\phi\phi} &= \frac{1}{r}\left(\frac{\partial u_\phi}{\partial\phi} + u_r\right) & e_{rz} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \\ e_{zz} &= \frac{\partial u_z}{\partial z} & e_{\phi z} &= \frac{1}{2}\left(\frac{\partial u_\phi}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\phi}\right) \end{aligned} \tag{1.2.2}$$

and the dilatation Δ is given by

$$\Delta = \nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z}.$$

Solutions of the wave equation in cylindrical coordinates for a point source can be written as follows:

$$
\begin{aligned}
u_z(r,z,h,\omega) &= (F_1 \cos\phi \ + \ F_2 \sin\phi) ZHF \ + \ F_3 ZVF \\
&+ \ M_{11} \left[\frac{ZSS}{2} \cos(2\phi) \ - \ \frac{ZDD}{6} \ + \ \frac{ZEX}{3} \right] \\
&+ \ M_{22} \left[\frac{-ZSS}{2} \cos(2\phi) \ - \ \frac{ZDD}{6} \ + \ \frac{ZEX}{3} \right] \\
&+ \ M_{33} \left[\frac{ZDD}{3} \ + \ \frac{ZEX}{3} \right] \\
&+ \ M_{12} \left[ZSS \ \sin(2\phi) \right] \\
&+ \ M_{13} \left[ZDS \ \cos(\phi) \right] \\
&+ \ M_{23} \left[ZDS \ \sin(\phi) \right] \\
u_r(r,z,h,\omega) &= (F_1 \cos\phi \ + \ F_2 \sin\phi) RHF \ + \ F_3 RVF \\
&+ \ M_{11} \left[\frac{RSS}{2} \cos(2\phi) \ - \ \frac{RDD}{6} \ + \ \frac{REX}{3} \right] \\
&+ \ M_{22} \left[\frac{-RSS}{2} \cos(2\phi) \ - \ \frac{RDD}{6} \ + \ \frac{REX}{3} \right] \\
&+ \ M_{33} \left[\frac{RDD}{3} \ + \ \frac{REX}{3} \right] \\
&+ \ M_{12} \left[RSS \ \sin(2\phi) \right] \\
&+ \ M_{13} \left[RDS \ \cos(\phi) \right] \\
&+ \ M_{23} \left[RDS \ \sin(\phi) \right] \\
u_\phi(r,z,h,\omega) &= (-F_1 \sin\phi + F_2 \cos\phi) THF \\
&+ \ M_{11} \left[\frac{TSS}{2} \sin(2\phi) \right] \\
&+ \ M_{22} \left[\frac{-TSS}{2} \sin(2\phi) \right] \\
&+ \ M_{12} \left[-TSS \ \cos(2\phi) \right] \\
&+ \ M_{13} \left[TDS \ \sin(\phi) \right] \\
&+ \ M_{23} \left[-TDS \ \cos(\phi) \right] \ .
\end{aligned}
\tag{1.2.3}
$$

This expression assumes that the moment tensor, M_{ij}, is symmetric. The functions F_1, F_2 and F_3 are the medium response to a point force, while the other functions are the response to specific moment tensor expressions. The terminology used for these basic force and moment tensor solutions is simple. The leading Z, R or T, indicates the component of motion. The SS indicates that the solution is due to a strike-slip source, with only $M_{12} \neq 0$ or with $M_{11} = -M_{22}$ with other elements zero. The DS solution is associated with a vertical dip-slip source with only $M_{13} \neq 0$ or $M_{23} \neq 0$. The EX solution is for an isotropic center of expansion source with

$M_{11} = M_{22} = M_{33}$. The *DD* solution does not correspond to a fault source, but can be understood as that part of a vertical or radial displacements for a 45° dip-sip source (e.g., $M_{22} = -M_{33}$) observed at an azimuth of 45°. The *DD* component is multiplied by the terms $2M_{33} - M_{11} - M_{22}$ which is known as a compensated linear vector dipole.

The form of the solution of the wave equation in cylindrical coordinates is

$$
\begin{aligned}
u_z &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \sum_{n=-\infty}^{\infty} \int_0^{\infty} U_z^{(n)} J_n(kr) k e^{in\phi} dk\, d\omega \\
u_r &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \left[U_r^{(n)} \frac{dJ_n(kr)}{dr} + U_\phi^{(n)} \frac{in}{r} J_n(kr) \right] e^{in\phi}\, dk\, d\omega \\
u_\phi &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \sum_{n=-\infty}^{\infty} \int_0^{\infty} \left[-U_\phi^{(n)} \frac{dJ_n(kr)}{dr} + U_r^{(n)} \frac{in}{r} J_n(kr) \right] e^{in\phi} dk\, d\omega
\end{aligned}
$$

If one defines the transformed stresses as

$$
\begin{aligned}
T_r &= \mu \left(\frac{dU_r^{(n)}}{dz} + kU_z^{(n)} \right) \\
T_z &= (\lambda + 2\mu) \frac{dU_z^{(n)}}{dz} - k\lambda U_r^{(n)} = \lambda \Delta^n + 2\mu \frac{dU_z^{(n)}}{dz} \\
T_\phi &= \mu \frac{dU_\phi^{(n)}}{dz}
\end{aligned}
$$

then the following ordinary differential equations must be solved:

$$
\frac{d}{dz}\begin{bmatrix} U_r \\ U_z \\ T_z \\ T_r \end{bmatrix} = \begin{bmatrix} 0 & -k & 0 & \frac{1}{\mu} \\ \frac{k\lambda}{\lambda+2\mu} & 0 & \frac{1}{\lambda+2\mu} & 0 \\ 0 & -\rho\omega^2 & 0 & k \\ -\rho\omega^2 + \frac{4k^2\mu(\lambda+\mu)}{\lambda+2\mu} & 0 & \frac{-k\lambda}{\lambda+2\mu} & 0 \end{bmatrix} \cdot \begin{bmatrix} U_r \\ U_z \\ T_z \\ T_r \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ f_z \\ f_r \end{bmatrix}
$$

and

$$
\frac{d}{dz}\begin{bmatrix} U_\phi \\ T_\phi \end{bmatrix} = \begin{bmatrix} 0 & 1/\mu \\ \mu k^2 - \rho\omega^2 & 0 \end{bmatrix} \begin{bmatrix} U_\phi \\ T_\phi \end{bmatrix} - \begin{bmatrix} 0 \\ f_\phi \end{bmatrix}
$$

Because none of the terms within the square matrices involve derivatives with respect to z, and since the medium parameters vary continuously or piecewise continuously with the z-coordinate, we immediately see that the parameters U_r, U_z, U_ϕ, T_r, T_z and T_ϕ must be continuous at depths where the force terms are zero. Discontinuities in these parameters will occur when crossing the source layer.

1.3 Surface wave solution

The required integrals for the Fourier transform of the displacements are of the form

$$
\begin{aligned}
u_z(r,z,\omega) &= \int_0^\infty U_z(k,z,\omega)\, J_n(kr) k dk \\
u_r(r,z,\omega) &= \int_0^\infty U_r(k,z,\omega)\, J_{n-1}(kr) k dk \\
&\quad - \frac{n}{r}\int_0^\infty \left[U_r(k,z,\omega) \,+\, U_\phi(k,z,\omega)\right] J_n(kr) dk \\
u_\phi(r,z,\omega) &= \int_0^\infty U_\phi(k,z,\omega)\, J_{n-1}(kr) k dk \\
&\quad - \frac{n}{r}\int_0^\infty \left[U_r(k,z,\omega) \,+\, U_\phi(k,z,\omega)\right] J_n(kr) dk
\end{aligned}
$$

For $n > 0$, the terms having the factor (n/r) represent near-field terms. These are composed of both P-SV and SH solutions of the ordinary differential equations. Usually the surface-wave solution considers only the far-field terms. The importance of the near-field terms can be seen by rewriting

$$\frac{n}{r}\int_0^\infty G(k) J_n(kr) dk$$

as

$$\frac{n}{kr}\int_0^\infty G(k) J_n(kr) k dk$$

thus showing that that second integral has little influence when $kr \gg 1$, or simply when $r \gg \lambda$, where λ is the wavelength.

For problems that lead to poles in the medium response, the pole contribution of an integral of the form

$$\int_0^\infty F(k,\omega) J_n(kr) dk$$

is

$$-\pi i Res[F(k_j,\omega) H_n^{(2)}(k_j r)$$

1.4 Green's functions

This section summarizes the pole contribution for an observation point at coordinates (r, z) for a point source acting at a depth h at the origin by giving the Fourier transform of the far-field displacement for the basic sources. In addition only the leading term of the asymptotic expansion of $H_n^{(2)}(k_j r)$ is used. If there is more than one mode at a given frequency, the result would be a summation of the terms on the right.

$$ZDD(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}\left(2\frac{dU_z(h)}{dz} + kU_r(h)\right) U_z(z)e^{i(\omega t-kr-\frac{1}{4}\pi)} \tag{1.4.1}$$

$$RDD(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}\left(2\frac{dU_z(h)}{dz} + kU_r(h)\right) U_r(z)e^{i(\omega t-kr-\frac{3}{4}\pi)} \tag{1.4.2}$$

$$ZDS(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}\left(\frac{dU_r(h)}{dz} + kU_z(h)\right) U_z(z)e^{i(\omega t-kr+\frac{1}{4}\pi)} \tag{1.4.3}$$

$$RDS(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}\left(\frac{dU_r(h)}{dz} + kU_z(h)\right) U_r(z)e^{i(\omega t-kr-\frac{1}{4}\pi)} \tag{1.4.4}$$

$$TDS(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_L}{\sqrt{kr}}\frac{dU_\phi(h)}{dz}U_\phi(z)e^{i(\omega t-kr+\frac{3}{4}\pi)} \tag{1.4.5}$$

$$ZSS(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}kU_r(h)\ U_z(z)e^{i(\omega t-kr+\frac{3}{4}\pi)} \tag{1.4.6}$$

$$RSS(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}kU_r(h)\ U_r(z)e^{i(\omega t-kr+\frac{1}{4}\pi)} \tag{1.4.7}$$

$$TSS(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_L}{\sqrt{kr}}kU_\phi(h)U_\phi(z)e^{i(\omega t-kr-\frac{3}{4}\pi)} \tag{1.4.8}$$

$$ZEX(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}\left(\frac{dU_z(h)}{dz} - kU_r(h)\right) U_z(z)e^{i(\omega t-kr-\frac{1}{4}\pi)} \tag{1.4.9}$$

$$REX(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}\left(\frac{dU_z(h)}{dz} - kU_r(h)\right) U_r(z)e^{i(\omega t-kr-\frac{3}{4}\pi)} \tag{1.4.10}$$

$$ZVF(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}U_z(h)U_z(z)e^{i(\omega t-kr-\frac{1}{4}\pi)} \tag{1.4.11}$$

$$RVF(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}kU_z(h)U_r(z)e^{i(\omega t-kr-\frac{3}{4}\pi)} \tag{1.4.12}$$

$$ZHF(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}U_r(h)U_z(z)e^{i(\omega t-kr+\frac{1}{4}\pi)} \tag{1.4.13}$$

$$RHF(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}U_r(h)U_r(z)e^{i(\omega t-kr-\frac{1}{4}\pi)} \tag{1.4.14}$$

$$THF(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_L}{\sqrt{kr}}U_\phi(h)U_\phi(z)e^{i(\omega t-kr+\frac{3}{4}\pi)} \tag{1.4.15}$$

$$\tag{1.4.16}$$

In these expressions, the terms involving eigenfunctions or their vertical derivative at the source depth, h, represent the excitation of a given Green's function. The motion at the receiver depth is controlled by the value of the respective eigenfunction evaluated at z.

The computation of strain requires the partial derivatives given in (1.2.2). Thus the $\partial/\partial r$, $\partial/\partial \phi$ and $\partial/\partial z$ are required.

The $\partial/\partial r$ terms are approximated by multiplying by $-ik$ in (1.4.1), e.g.,

$$\frac{\partial ZDD}{dr}(r,z,h,\omega) = \frac{1}{\sqrt{2\pi}}\frac{A_R}{\sqrt{kr}}\left(2\frac{dU_z(h)}{dz} + kU_r(h)\right) U_z(z)(-ik)$$

and the $\partial u/\partial r$ is composed of combinations of these terms.

The $\partial/\partial\phi$ is obtained directly from partials of (1.2.3), e.g.,

$$\begin{aligned}
\frac{\partial u_z}{\partial\phi}(r,z,h,\omega) &= (-F_1 \sin\phi \;+\; F_2 \cos\phi)\, ZHF \\
&+\; M_{11} \;\; -ZSS \;\; \sin(2\phi) \\
&+\; M_{22}\, ZSS \;\; \sin(2\phi) \\
&+\; M_{12}\, [2ZSS \;\; \cos(2\phi)] \\
&+\; M_{13}\, [-ZDS \;\; \sin(\phi)] \\
&+\; M_{23}\, [ZDS \;\; \cos(\phi)]
\end{aligned}$$

The computation of the $\partial/\partial z$ uses the T_z , T_r and T_ϕ from the differential equation. Thus, $\partial ZDD/\partial z$ is obtained by replacing the $U_z(z)$ on the right side by

$$\frac{dU_z}{dz}(k,z,\omega) = \frac{1}{\lambda + 2\mu}\Big[T_z + k\lambda U_r\Big]$$

Likewise the partials of the radial displacements require

$$\frac{dU_r}{dz}(k,z,\omega) = \frac{1}{\mu}T_r - kU_z$$

and of the transverse displacements require

$$\frac{dU_\phi}{dz}(k,z,\omega) = \frac{1}{\mu}T_\phi$$

1.5 Conversion of cylindrical strain to cartesian

The choice of using a cylindrical coordinate system to describe wave propagation was made for computational efficiency, since the cartesian displacement at any point can be obtained from the cylindrical through a simple coordinate system rotation. Consider the coordinate system shown in Figure 1.1. The displacements

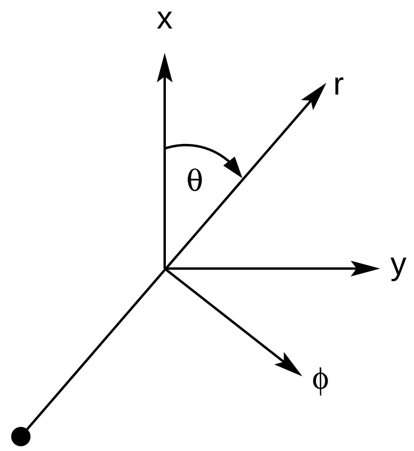

Figure 1.1 Transformation between (r, ϕ, z) coordinate system to an (x, y, z) coordinate system. The z-coordinate is down into the figure. The ϕ component of motion is the transverse component. Often the (x, y) axes are aligned north and east, respectively. In the case of DAS systems, one might align the x-axis with the direction of the fiber.

in the (x, y, z) coordinate system are related to those in the (r, ϕ, z) system through the transformation

$$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_r \\ u_\phi \\ u_z \end{bmatrix}$$

Bower (2010) showed how to relate stresses in a cylindrical coordinate system to those in a Cartesian system through the operation

$$\begin{bmatrix} \sigma_{rr} & \sigma_{r\phi} & \sigma_{rz} \\ \sigma_{r\phi} & \sigma_{\phi\phi} & \sigma_{\phi z} \\ \sigma_{rz} & \sigma_{\phi z} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

These equations can be rearranged to yield

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_{rr} & \sigma_{r\phi} & \sigma_{rz} \\ \sigma_{r\phi} & \sigma_{\phi\phi} & \sigma_{\phi z} \\ \sigma_{rz} & \sigma_{\phi z} & \sigma_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A similar transformation is used to relate cartesian strains to cylindrical strains.

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{xy} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{xz} & \epsilon_{yz} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \epsilon_{rr} & \epsilon_{r\phi} & \epsilon_{rz} \\ \epsilon_{r\phi} & \epsilon_{\phi\phi} & \epsilon_{\phi z} \\ \epsilon_{rz} & \epsilon_{\phi z} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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