Performance Analysis of Log-optimal Portfolio Strategies with Transaction Costs [1]

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July 6, 2010

[1]The authors thank László Györfi for his careful reading and useful advices. The paper has greatly benefited and improved by the many helpful suggestions and comments from the two anonymous reviewers. This work is connected to the scientific program of the "Development of quality-oriented and harmonized R+D+I strategy and functional model at BME" project. This project is supported by the New Hungary Development Plan (Project ID: TÁMOP-4.2.1/B-09/1/KMR-2010-0002).

Abstract

In the paper we introduce an empirical approximation of the log-optimal investment strategy that guarantees an almost optimal growth rate of investments. The proposed strategy also considers the effects of portfolio rearrangement costs on growth optimality and advises a suboptimal portfolio for discrete investment periods. We do not assume any parametric structure for the market process, only first order Markov property. The model introduced is based on kernel-based agents' (experts') approximation on the maximum theoretical growth rate with transaction costs. Although the optimal solution is theoretically a complex Bellman programming problem, our sub-optimal empirical result seems to be attractive on Dow Jones 30 shares. The paper presents a performance analysis where the return of the empirical log-optimal portfolio is compared to passive portfolio counterparts compiled from similar components using the CAPM, the three-factor model and the four-factor model. The proposed methods in the presence of transaction costs gain a significant positive abnormal return as compared with the preceding equilibrium models, and is even a survivorship bias free setup.

1 Introduction

Several recent contributions (see Györfi *et al.* 2006,2007) have shown that an empirical, log-optimal investment strategy results in higher return than the return of the stock in the portfolio with the highest return. Both the mathematical proof and the empirical investigation among other boundary conditions suppose that there is no transaction cost. This supposition is acceptable for model formulation as one can see in the case of eg. Sharpe (1964), or Merton (1969,1973), while the cost of transactions on the capital market is incomparably smaller than on any other markets investigated in economics. However, the trading strategies which use large numbers of bids and asks can result in very large values. The log-optimal portfolio strategy rebalances the portfolio in each period, thus neglecting the cost accompanied with the trades, and can result in large errors in the calculated returns. In the paper similarly to Magill and Constantinides (1976), or Constantinides (1997), we introduce proportional transaction costs when revisiting the log-optimal portfolio strategy. In the theory of log-optimal strategies the goal of the investor is to maximize his wealth in the long run without knowing the underlying distribution generating the stock prices. Under this assumption the asymptotic rate of growth has a well-defined maximum which can be achieved in the full knowledge of the underlying distribution generated by the stock returns. Without transaction costs there exist empirical log-optimal portfolio strategies which are able to achieve the theoretical maximum growth rate (see Györfi *et al.* 2006,2008). We present a model using kernel-based agents (experts) which approximates the maximum theoretical growth rate with transaction costs. Although the optimal solution theoretically is a complex Bellman programming problem, our empirical result seems to be attractive.

The proposed investment strategy considers the effects of portfolio rearrangement costs on growth optimality and advises a suboptimal portfolio for discrete investment periods. We do not assume any parametric structure for the market process, only first order Markov property. Although, the method proposed gives only a suboptimal solution, our novel approach is able to capture risk factors that the classical equilibrium models do not cover, even in the presence of proportional transaction costs.

Papers formulating growth optimal investment with transaction costs in discrete time setting are seldom. Cover and Iyengar (2000) formulated the problem of horse race markets, where in every market period one of the horses (assets) pays off and all the others pay nothing, with proportional transaction costs using long run expected average reward criteria. There are results for more general markets; Bobryk and Stettner (1999) considered the case of portfolio selection with consumption, when there are two assets: a bank account and a stock. Furthermore long run expected discounted reward and i.i.d asset returns were assumed. In the case of discrete time, the most far reaching study was Schäfer's (2002) who considered the maximization of the long run expected growth rate with several assets and proportional transaction costs, when the asset returns follow stationary Markov process.

The performance measure of any active or passive portfolio strategy, or simply mutual funds or any other assets, has a well defined way in the world of equilibrium asset pricing. It is not enough to state that a strategy resulted in a highest high return, but one must consider the risk of the return as well. In the case of the empirical studies in log-optimal portfolio strategy, these results and comparisons are still missing.

This paper unambiguously answers the question of the success using the log-optimal portfolio strategy. In our empirical investigation using the components of Dow Jones Industrial Average Index, from which a daily rebalanced log-optimal portfolio is compiled, a 15-year-period is investigated in an efficient capital market. For performance measurement purposes we use the simple Capital Asset Pricing Model (Sharpe 1964), the Fama and French (1993) three-factor model, and the Carhart (1997) four-factor model, where we are searching for significant Jensen alphas. Obviously, in the empirical study we use the log-optimal portfolio strategy introduced previously with transaction costs. We perform tests on three distinct portfolios: (i) a portfolio containing the components of the DJIA observed in December 2005 which is a survivorship biased setup; (ii) a portfolio containing the components of the DJIA in January 1991, which is a survivorship bias free setup; and (iii) a portfolio which tracks the actual components of the DJIA. Each test portfolio contains 30 shares. In addition we compare our results on the log-optimal portfolio strategy to the buy-and-hold (passive) strategy. In the results one can see that this novel trading strategy results in significant positive alphas and significantly higher ones than that of the buy-and-hold strategy gains.

These results strengthen our belief that the log-optimal portfolio strategy yields higher returns than the equilibrium ones, which can be explained from three directions. One would say that the equilibrium models used to explain the return of our dynamic portfolio contains one or more missing explanatory variables, this causes our results to show higher returns than the equilibrium. The other attempt to explain the abnormal return is that the sample (stocks and/or the period) used in our experiment is not adequate, and we are talking only about an anomaly which disappears if another dataset is used. This explanation seems to be weak because other papers (eg. Györfi *et al.* 2006,2008,2007) using different stocks in different time intervals give similar results to ours, however, these studies neglect the transaction cost. The third argument stems from the theoretical difference of the one-period equilibrium models and multi-period investment strategies.

2 Log-optimal strategies with transaction costs

2.1 Trading model

The model of stock market investigated in the section is the one considered, among others, by Breiman (1961), Algoet and Cover (1988). Consider a market of d assets. A *market vector* $\mathbf{x} = (x^{(1)}, \dots x^{(d)})^T \in \mathbb{R}_+^d$ is a vector of d nonnegative numbers representing price relatives for a given trading period. That is, the j-th component $x^{(j)} \geqslant 0$ of $\mathbf{x}$ expresses the ratio of the consecutive closing prices of asset j. In other words, $x^{(j)}$ is the factor by which capital invested in the j-th asset grows during the trading period.

The investor is allowed to diversify his capital at the beginning of each trading period according to a portfolio vector $\mathbf{b} = (b^{(1)}, \dots b^{(d)})^T$. The j-th component $b^{(j)}$ of $\mathbf{b}$ denotes the proportion of the investor's capital invested in asset j. Throughout the paper we assume that the portfolio vector $\mathbf{b}$ has nonnegative components with $\sum_{j=1}^{d} b^{(j)} = 1$. The fact that $\sum_{j=1}^{d} b^{(j)} = 1$ means that the investment strategy is self financing and the consumption of capital is excluded. The non-negativity of the components of $\mathbf{b}$ means that short selling shares on margin are not permitted. Let S_0 denote the investor's initial capital. Then at the end of the trading period the investor's wealth becomes

$$S_1 = S_0 \sum_{j=1}^{d} b_1^{(j)} x_1^{(j)} = S_0 \langle \mathbf{b}_1 , \mathbf{x}_1 \rangle \ ,$$

where $\langle \cdot , \cdot \rangle$ denotes inner product.

The evolution of the market in time is represented by a sequence of market vectors $\mathbf{x}_1, \mathbf{x}_2, \dots \in \mathbb{R}_+^d$, where the j-th component $x_i^{(j)}$ of $\mathbf{x}_i$ denotes the amount obtained after investing a unit capital in the j-th asset on the i-th trading period. For $j \leqslant i$ we abbreviate by $\mathbf{x}_j^i$ the array of market vectors $(\mathbf{x}_j, \dots, \mathbf{x}_i)$ and denote by Δ_d the simplex of all vectors $\mathbf{b} \in \mathbb{R}_+^d$ with nonnegative components summing up to one. An *investment strategy* is a sequence $\mathbf{B}$ of functions

$$\mathbf{b}_i : \left(\mathbb{R}_+^d\right)^{i-1} \to \Delta_d \ , \qquad i = 1, 2, \dots$$

so that $\mathbf{b}_i(\mathbf{x}_1^{i-1})$ denotes the portfolio vector chosen by the investor in the i-th trading period, upon observing the past behavior of the market. We write $\mathbf{b}(\mathbf{x}_1^{i-1}) = \mathbf{b}_i(\mathbf{x}_1^{i-1})$ to ease the notation. Therefore we get by induction that

$$S_n = S_0 \prod_{i=1}^{n} \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) , \mathbf{x}_i \right\rangle = S_0 e^{\sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) , \mathbf{x}_i \right\rangle} = S_0 e^{n W_n(\mathbf{B})} \ ,$$

where $W_n(\mathbf{B})$ denotes the *average growth rate* of the investment strategy $\mathbf{B} = \{b_n\}_{n=1}^{\infty}$:

$$W_n(\mathbf{B}) = \frac{1}{n} \ln S_n = \frac{1}{n} \sum_{i=1}^{n} \ln \left\langle \mathbf{b}(\mathbf{x}_1^{i-1}) , \mathbf{x}_i \right\rangle \ .$$

Remark 2.1. Without transaction costs, the fundamental limits, determined in Algoet and Cover (1988) reveal that on stationary and ergodic markets the so-called *log-optimum portfolio* $\mathbf{B}^* = \{\mathbf{b}^*(\cdot)\}$ is the best possible choice. More precisely, in trading period n let $\mathbf{b}^*(\cdot)$ be such that

$$\mathbf{b}_n^*(\mathbf{X}_1^{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbb{E} \left\{ \ln \left\langle \mathbf{b}(\mathbf{X}_1^{n-1}) , \mathbf{X}_n \right\rangle \middle| \mathbf{X}_1^{n-1} \right\} . \tag{2.1}$$

If $S_n^* = S_n(\mathbf{B}^*)$ denotes the capital achieved by a log-optimum portfolio strategy $\mathbf{B}^*$, after n trading periods, then for any other investment strategy $\mathbf{B}$ with capital $S_n = S_n(\mathbf{B})$ and for any stationary and ergodic return process $\{\mathbf{X}_n\}_{-\infty}^{\infty}$,

$$\liminf_{n\to\infty} \frac{1}{n} \ln \frac{S_n^*}{S_n} \geqslant 0 \quad \text{almost surely}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \ln S_n^* = W^* \quad \text{almost surely,}$$

where

$$W^* = \mathbb{E}\left\{ \max_{\mathbf{b}(\cdot)} \mathbb{E}\left\{ \ln \left\langle \mathbf{b}(\mathbf{X}_{-\infty}^{-1}), \mathbf{X}_0 \right\rangle \middle| \mathbf{X}_{-\infty}^{-1} \right\} \right\}$$

is the maximum possible growth rate of any investment strategy. Moreover, Györfi and Schäfer (2003) and Györfi *et al.* (2006,2007) constructed empirical (data dependent) log-optimum strategies for unknown distributions. Note that for first order Markovian return process

$$\mathbf{b}_n^*(\mathbf{X}_1^{n-1}) = \mathbf{b}_n^*(\mathbf{X}_{n-1}) = \arg\max_{\mathbf{b}(\cdot)} \mathbb{E}\{\ln \langle \mathbf{b}(\mathbf{X}_{n-1}), \mathbf{X}_n \rangle | \mathbf{X}_{n-1}\}.$$

To make the analysis feasible, some simplifying assumptions are used that need to be taken into account in the usual model of log-optimal portfolio theory. Assume that

- the assets are arbitrarily divisible,
- the assets are available in unlimited quantities at the current price at any given trading period,
- the behavior of the market is not affected by the actions of the investor using the strategy under investigation.

2.2 Investment with transaction costs

In this section the formulation of the transaction cost problem profited from the formulation in Iyengar and Cover (2000), Schäfer (2002), Györfi and Vajda (2008) and Stettner (2009). Let S_n denote the wealth at the end of market day n, $n = 0, 1, 2, \cdots$, where without loss of generality let the investor's initial capital S_0 be 1 dollar. At the beginning of a new market day $n+1$, the investor sets up his new portfolio, i.e. purchases/sells shares according to the actual portfolio vector $\mathbf{b}_{n+1}$. During this rearrangement, he has to pay transaction costs, therefore at the beginning of a new market day $n+1$ the net wealth N_n in the portfolio $\mathbf{b}_{n+1}$ is not larger than the gross S_n. Using the above notations the gross wealth S_n at the end of market day n is

$$S_n = N_{n-1} \langle \mathbf{b}_n , \mathbf{x}_n \rangle . \tag{2.2}$$

The rate of proportional transaction costs (commission factors) levied on one asset are denoted by $0 < c_s < 1$ and $0 < c_p < 1$, i.e., the sale of 1 dollar worth of asset i nets only $1 - c_s$ dollars, and similarly we take into account the purchase of an asset in a way that the purchase of 1 dollar's worth of asset i costs an extra c_p dollars. We consider the special case when the rate of costs is constant over the assets. In the currant approach the investor's wealth is always invested in securities, therefore the sale of assets is always followed by purchasing others.

Let's calculate the transaction cost to be paid when selecting the portfolio $\mathbf{b}_{n+1}$. Before rearranging the capitals, at the j-th asset there is $b_n^{(j)} x_n^{(j)} N_{n-1}$ dollars, while after rearranging we need $b_{n+1}^{(j)} N_n$ dollars. If $b_n^{(j)} x_n^{(j)} N_{n-1} > b_{n+1}^{(j)} N_n$ then we have to sell and the transaction cost at the j-th asset is

$$c_s \left(b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right),$$

otherwise no transaction is made.

Let x^+ denote the positive part of x. Thus, the sum of costs on the sale of appropriate assets is

$$\sum_{j=1}^{d} c_s \left(b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right)^+ . \tag{2.3}$$

The money stemming from the sale is used for augmenting the investor's portfolio with new assets. In this case the income from the sale decreased by transaction cost covers the value of the shares to purchase and the collateral cost, i.e.

$$\begin{aligned}
&\sum_{j=1}^{d} \left\{ \left(b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right)^+ - c_s \left(b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right)^+ \right\} \\
&= \sum_{j=1}^{d} \left\{ \left(b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right)^+ + c_p \left(b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right)^+ \right\} .
\end{aligned}$$

In equivalent form we have that

$$\begin{aligned}
&(1 - c_s) \sum_{j=1}^{d} \left(b_n^{(j)} x_n^{(j)} N_{n-1} - b_{n+1}^{(j)} N_n \right)^+ \\
&= (1 + c_p) \sum_{j=1}^{d} \left(b_{n+1}^{(j)} N_n - b_n^{(j)} x_n^{(j)} N_{n-1} \right)^+ .
\end{aligned} \tag{2.4}$$

Using the identity

$$(a - b)^+ = a - b + (b - a)^+$$

(2.4) has an equivalent form,

$$
\begin{aligned}
&(1-c_s)\sum_{j=1}^{d}\left(b_n^{(j)}x_n^{(j)}N_{n-1}-b_{n+1}^{(j)}N_n\right)^+ \\
&=(1+c_p)\left\{\sum_{j=1}^{d}\left(b_{n+1}^{(j)}N_n-b_n^{(j)}x_n^{(j)}N_{n-1}\right)+\sum_{j=1}^{d}\left(b_n^{(j)}x_n^{(j)}N_{n-1}-b_{n+1}^{(j)}N_n\right)^+\right\}\ .
\end{aligned}
$$

Since, $\sum_{j=1}^{d}\left(b_{n+1}^{(j)}N_n\right)=N_n$ and $\sum_{j=1}^{d}\left(b_n^{(j)}x_n^{(j)}N_{n-1}\right)=S_n$, therefore

$$
\frac{c_p+c_s}{1+c_p}\sum_{j=1}^{d}\left(b_n^{(j)}x_n^{(j)}N_{n-1}-b_{n+1}^{(j)}N_n\right)^+=S_n-N_n\ . \tag{2.5}
$$

Dividing both sides by S_n and introducing ratio

$$
w_n=\frac{N_n}{S_n},
$$

where $0<w_n\leqslant 1$. Using (2.2) we get

$$
\frac{c_p+c_s}{1+c_p}\sum_{j=1}^{d}\left(\frac{b_n^{(j)}x_n^{(j)}}{\langle \mathbf{b}_n\ ,\ \mathbf{x}_n\rangle}-b_{n+1}^{(j)}w_n\right)^+=1-w_n\ . \tag{2.6}
$$

Remark 2.2. Equation (2.6) is used in the sequel. Examining this cost equation, it turns out, that for arbitrary portfolio vectors $\mathbf{b}_n$, $\mathbf{b}_{n+1}$, and return vector $\mathbf{x}_n$ there exists a unique cost factor $w_n\in[0,1)$. The value of cost factor w_n at day n is determined by portfolio vectors $\mathbf{b}_n$ and $\mathbf{b}_{n+1}$ as well as by return vector $\mathbf{x}_n$, i.e.

$$
w_n=w(\mathbf{b}_n,\mathbf{b}_{n+1},\mathbf{x}_n),
$$

for some function w. That is, the tuning of w_n is allowed at the end of trading period n, after $\mathbf{x}_n$ is observed and $\mathbf{b}_n$ is calculated at the end of the trading period $n-1$. $\mathbf{b}_{n+1}$ is a target variable which is tuned in parallel with w_n.

If we want to rearrange our portfolio substantially, then our net wealth decreases more considerably, however, it remains positive. Note also, that the cost does not restrict the set of new portfolio vectors, i.e., the optimization algorithm searches for optimal vector $\mathbf{b}_{n+1}$ within the whole simplex Δ_d.

The maximum value of w_n is 1 when no transaction is done. It reaches its minimum if the difference between the gross S_n and the net N_n reaches its maximum, in case the sum in (2.5) is equal to 1. Thus, the value of the cost factor ranges between

$$
\frac{1-c_s}{1+c_p}\leqslant w_n\leqslant 1.
$$

Assume a general condition that $c = c_p = c_s$, so

$$\frac{1-c}{1+c} \leqslant w_n \leqslant 1 \ .$$

in the most cases.

Since w_n cannot be formulated in explicit form, it can only be calculated numerically by using equation (2.6).

Starting with an initial wealth $S_0 = 1$ and $w_0 = 1$, wealth S_n at the closing time of the n-th market day becomes

$$S_n = N_{n-1}\langle \mathbf{b}_n \, , \, \mathbf{x}_n \rangle = w_{n-1}S_{n-1}\langle \mathbf{b}_n \, , \, \mathbf{x}_n \rangle = \prod_{i=1}^{n} [w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \, \langle \mathbf{b}_i \, , \, \mathbf{x}_i \rangle].$$

Introduce the notation

$$g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}, \mathbf{x}_i) = \ln(w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \, \langle \mathbf{b}_i \, , \, \mathbf{x}_i \rangle),$$

then the average growth rate becomes

$$\begin{aligned} \frac{1}{n} \ln S_n &= \frac{1}{n} \sum_{i=1}^{n} \ln(w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}) \, \langle \mathbf{b}_i \, , \, \mathbf{x}_i \rangle) \\ &= \frac{1}{n} \sum_{i=1}^{n} g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{x}_{i-1}, \mathbf{x}_i). \end{aligned} \tag{2.7}$$

Our aim is to maximize this average growth rate.

In the sequel $\mathbf{x}_i$ will be a random variable and is denoted by $\mathbf{X}_i$. Let's use the decomposition

$$\frac{1}{n} \ln S_n = I_n + J_n, \tag{2.8}$$

where

$$I_n = \frac{1}{n} \sum_{i=1}^{n} (g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}, \mathbf{X}_i) - \mathbb{E}\{g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}, \mathbf{X}_i) | \mathbf{X}_1^{i-1}\})$$

and

$$J_n = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\{g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}, \mathbf{X}_i) | \mathbf{X}_1^{i-1}\}.$$

I_n is an average of martingale differences. Under mild conditions on the support of the distribution of $\mathbf{X}$, $g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}, \mathbf{X}_i)$ is bounded, therefore I_n is an average of bounded martingale differences, which converges to 0 almost surely, since according to the Chow Theorem (cf. Theorem 3.3.1 in Stout 1974)

$$\sum_{i=1}^{\infty} \frac{\mathbb{E}\{(g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}, \mathbf{X}_i) - \mathbb{E}\{g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}, \mathbf{X}_i) | \mathbf{X}_1^{i-1}\})^2\}}{i^2} < \infty$$

implies that

$$I_n \to 0$$

almost surely. Thus, the asymptotic maximization of the average growth rate $\frac{1}{n} \ln S_n$ is equivalent to the maximization of J_n.

If the market process $\{\mathbf{X}_i\}$ is a *homogeneous and first order Markov process* then, for appropriate portfolio selection $\{\mathbf{b}_i\}$, we have that

$$\begin{aligned}
& \mathbb{E}\{g(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}, \mathbf{X}_i) | \mathbf{X}_1^{i-1}\} \\
= \; & \mathbb{E}\{\ln(w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}) \langle \mathbf{b}_i \, , \, \mathbf{X}_i \rangle) | \mathbf{X}_1^{i-1}\} \\
= \; & \ln w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}) + \mathbb{E}\{\ln \langle \mathbf{b}_i \, , \, \mathbf{X}_i \rangle | \mathbf{X}_1^{i-1}\} \\
= \; & \ln w(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}) + \mathbb{E}\{\ln \langle \mathbf{b}_i \, , \, \mathbf{X}_i \rangle | \mathbf{b}_i, \mathbf{X}_{i-1}\} \\
\stackrel{\text{def}}{=} \; & \nu(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}),
\end{aligned}$$

therefore the maximization of the average growth rate $\frac{1}{n} \ln S_n$ is asymptotically equivalent to the maximization of

$$J_n = \frac{1}{n} \sum_{i=1}^{n} \nu(\mathbf{b}_{i-1}, \mathbf{b}_i, \mathbf{X}_{i-1}). \tag{2.9}$$

The first order Markov property implies that in the above equations and in the empirical method presented in Section 2.4, the conditional expected value of the portfolio return is based only on the immediately preceding day's return. Time-homogeneity implies that $\mathbb{E}\{\ln \langle \mathbf{b}_i \, , \, \mathbf{X}_i \rangle | \mathbf{b}_i, \mathbf{X}_{i-1}\}$ is independent from i. For a more detailed description of Markov property's appliance in empirical methods see Section 2.4.

2.3 Suboptimal portfolio selection algorithm

In the paper assuming homogeneous and first order Markov process we introduce Györfi and Vajda's (2008) suboptimal solution by a one-step optimization as follows: put $\mathbf{b}_1 = \{1/d, \ldots, 1/d\}$ and for $n > 1$,

$$\begin{aligned}
\mathbf{b}_n^*(\mathbf{X}_{n-1}) &= \arg\max_{\mathbf{b}' \in \Delta_d} \nu(\mathbf{b}_{n-1}(\mathbf{X}_{n-2}), \mathbf{b}', \mathbf{X}_{n-1}) \\
&= \arg\max_{\mathbf{b}' \in \Delta_d} \{\ln w(\mathbf{b}_{n-1}(\mathbf{X}_{n-2}), \mathbf{b}', \mathbf{X}_{n-1}) \\
&+ \mathbb{E}\{\ln \langle \mathbf{b}' \, , \, \mathbf{X}_n \rangle | \mathbf{b}', \mathbf{X}_{n-1}\}\}.
\end{aligned} \tag{2.10}$$

This solution is suboptimal only because the model proposed only gives an optimal portfolio vector regarding the next trading day as opposed to a long run optimization, which in each day considers the optimality regarding the infinite sequence of the following trading periods. If the distribution of the return process is known, Györfi and Vajda (2008) proved that under a homogeneous and first order Markov process there exists a theoretically optimal solution in the presence of proportional transaction costs, although an efficient algorithm is available only for the suboptimal approximation .

In the next section, the performance of the empirical approximation for the suboptimal model is investigated on U.S. stock exchange data series.

2.4 Empirical portfolio selection

In order to construct $\mathbf{b}^*$, one has to know the conditional distribution of $\mathbf{X}_n$ given $\mathbf{X}_1^{n-1}$. The classical Markowitz mean-variance approach (Markowitz 1952) to portfolio optimization for single period investment selects portfolio $\mathbf{b}$ by performance $\mathbb{E}\{\langle \mathbf{b}, \mathbf{X}_n \rangle\}$ and risk $\mathbf{Var}\{\langle \mathbf{b}, \mathbf{X}_n \rangle\}$ in a way that only the first and second moments of $\mathbf{X}_n$ are used in the calculations (cf. Francis 1980). Similarly, if the process $\{\mathbf{X}_n\}$ is log-normally distributed, then again only the first and second moments are needed in the derivations (cf. Schäfer 2002). Ottucsák and Vajda (2007) investigated the connection of the Markowitz mean-variance approach and the log-optimal choice. They find that the log-optimal portfolios may be described as special mean-variance based portfolios with time varying risk aversion factor.

The empirical kernel-based strategies introduced below, estimate $\mathbf{b}^*$ based on previous market data. Although, they work with no knowledge of the theoretical conditional distribution of the market process, they perform well on financial time series. We investigate their empirical behavior in the next section.

The presented empirical methods are based on the previously referred homogeneous and first order Markov property. The first order property implies that the introduced kernel estimators only use returns from the preceding trading days to give forecasts on future returns. The time-homogeneity allows us to base our estimations onto observations from the past, since due to the homogeneity, the state transitions' probability is time independent.

For empirical estimation let's define an infinite array of experts $\mathbf{B}^{(\ell)} = \{\mathbf{b}^{(\ell)}(\cdot)\}$, where ℓ is positive integer. For fixed positive integers ℓ, choose the radius $r_\ell > 0$ in such a way

$$\lim_{\ell \to \infty} r_\ell = 0.$$

Then, for $n > 1$, define the expert $\mathbf{b}^{(\ell)}$ as follows. Let $P_n^{(\ell)}$ be the locations of matches:

$$P_n^{(\ell)} = \{i < n : \|\mathbf{x}_{i-1} - \mathbf{x}_{n-1}\| \leqslant r_\ell\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Put the kernel-based suboptimal (cf. Section 2.3) portfolio determined by expert $\mathbf{b}^{(\ell)}$ for trading day n as follows

$$\mathbf{b}_n^{(\ell)} = \arg\max_{\mathbf{b}' \in \Delta_d} \sum_{\left\{i \in P_n^{(\ell)}\right\}} \ln\{w(\mathbf{b}_{n-1}, \mathbf{b}', \mathbf{x}_{n-1}) \langle \mathbf{b}', \mathbf{x}_i \rangle\} \tag{2.11}$$

if the product is non-void, and $\mathbf{b}_0 = (1/d, \ldots, 1/d)$ otherwise. Thus, $\mathbf{b}_n^{(\ell)}$ seeks market vectors similar to $\mathbf{x}_{n-1}$, then determine the fix portfolio which maximizes the return on market vectors following the previously selected similar ones. The similarity is measured by Euclidean norm.

These experts are mixed as follows: let $\{q_\ell\}$ be a probability distribution over the set of positive integers ℓ such that for all ℓ, $q_\ell > 0$. If $S_n(\mathbf{B}^{(\ell)})$ is the capital accumulated by the elementary strategy $\mathbf{B}^{(\ell)}$ after n periods when starting with an initial capital $S_0 = 1$, then, after period n, the investor's capital becomes

$$S_n = \sum_{\ell} q_\ell S_n(\mathbf{B}^{(\ell)}). \tag{2.12}$$

Note, that (2.12) is a non-parametric estimation of (2.10).

All the proposed algorithms use an infinite array of experts. In practice we take a finite array of size L. In all cases select $L = 10$. Choose the uniform distribution $\{q_\ell\} = 1/(L)$ over the experts in use, and the radius

$$r_l^2 = 0.0001 \cdot d \cdot \ell, \tag{2.13}$$

($\ell = 1, \ldots, L$ and d denotes the number of assets in the portfolio).

3 Empirical results

The goal of our empirical study is to investigate, whether or not, the conventional risk factors, such as *market risk premium*, *momentum factor* (cf. Jegadeesh and Titman 1993, Carhart 1997) and the two *Fama-French factors* (cf. Fama and French 1993) are able to explain the scatter of log-optimal returns through equilibrium models. We have not intended to judge the equilibrium models themselves, however we are interested in measuring the abnormal return compared to the equilibrium ones. Additionally, to evaluate the results of the log-optimal portfolios, we also examine the performance of some passive strategies with the same components. In the comparison we cannot neglect the effect of transaction cost for the log-optimal strategy, since this factor highly affects the performance of any active portfolio strategy compared to the passive buy-and-hold counterpart.

3.1 Data and Methodology

To empirically test the log-optimal method, the return data of the Dow Jones Industrial Average (DJIA) components is applied for a 15-year-long period from January, 1991 through December, 2008. We perform tests on three distinct portfolios: (i) a portfolio containing the components of the DJIA observed in December, 2005; (ii) a portfolio containing the components in January, 1991; and (iii) a portfolio which tracks the actual components. Each test portfolio contains 30 shares.

The source of the asset returns is the database of *The Center for Research in Security Prices* (CRSP). The risk free rate is the yield of the 1 month U.S. Treasury bill collected also from the CRSP database. The CRSP value weighted return index including distributions, is made up of all New York Stock Exchange (NYSE), American Stock Exchange (AMEX) and NASDAQ shares served as market portfolio. The list of the DJIA components is obtained from Dow Jones' official webpage. The proportional transaction cost is set to 0.1% of the traded volume ($c = 0.001$) both for sale and purchase. We shorted of fix cost factor. The applied empirical strategy is defined as the mixture of 10 log-optimal experts, distributing the initial wealth among the experts according to the uniform distribution $q_\ell = \frac{1}{10}$, for all ℓ. The r radius is defined in (2.13).

For testing purposes we built four linear models and investigate the coefficients of common factors for monthly log-optimal returns. Although, the introduced empirical log-optimal strategy provides the facility for daily portfolio rearrangement, in order to eliminate the harmful effect of autocorrelation and heteroscedasticity on linear regressions, monthly returns were formed of the daily log-optimal returns, which means 180 data points for the 15-year-long period. The monthly market premiums, momentum factors and Fama-French factors were collected from CRSP database, too.

We built the most common equilibrium models. In order to estimate the regression coefficients, 4 models are applied, sequentially, the classical *Capital Asset Pricing Model* (CAPM) (cf. Sharpe 1964, Linter 1965, Mossin 1966), the CAPM amended with momentum factor (cf. Carhart 1997), the Fama-French three-factor model (cf. Fama and French 1993) and a four-factor model (cf. Carhart 1997). More precisely, one can estimate the log-optimal premiums by the following ways sequentially:

$$r_l^t - r_f^t = \alpha_l + \beta_l(r_m^t - r_f^t) + \varepsilon_l^t \tag{3.1}$$

$$r_l^t - r_f^t = \alpha_l + \beta_l(r_m^t - r_f^t) + m_l MOM^t + \varepsilon_l^t \tag{3.2}$$

$$r_l^t - r_f^t = \alpha_l + \beta_l(r_m^t - r_f^t) + s_l SMB^t + h_l HML^t + \varepsilon_l^t \tag{3.3}$$

$$r_l^t - r_f^t = \alpha_l + \beta_l(r_m^t - r_f^t) + s_l SMB^t + h_l HML^t + m_l MOM^t + \varepsilon_l^t, \tag{3.4}$$

where l, t, r_f, $(r_m^t - r_f^t)$, ε stand for share l, time, risk free rate, market premium and estimation residuals, respectively. According to Fama and French (1993), SMB (small-minus-big) measures the average return difference between small and large capitalization assets, while HML (high-minus-low) is the average return difference between high and low book-to-market equity (B/M) companies. MOM is the momentum factor (cf. Carhart 1997), which shows the average excess return of the past winners above the return of past loser securities. The regression coefficients α, β, s, h and m were estimated based on equation (3.1), (3.2), (3.3) and (3.4). For better understanding of the performance measure, the four estimations are performed on three distinct passive portfolios as well, which contain the December, 2005, the January, 1991 and the actual DJIA components, respectively. The results are compared to the results of the log-optimal portfolio.

The analysis of regression coefficients statistically investigate the question, as to whether or not an empirical log-optimal strategy has an average return that can be expected from any investment at the risk level of the proposed empirical strategy. In other words, we investigate if the log-optimal strategy also has good empirical properties beyond the traditional models, besides the attractive theoretical accomplishments. Otherwise, this type of log-optimal investments can be empirically rejected.

3.2 Portfolios of December, 2005 DJIA Components

We investigate investments containing stocks which formed the DJIA in December 2005. Since the 15-year long period starts in January 1991, these investigations are not free of survivorship bias. We present the bias free results in the next subsections.

To get a benchmark when evaluate the performance of the log-optimal investment, we also evaluate the performance of a buy-and-hold strategy which is a passively managed alternative of our active method, hence, it does not infer transaction costs. Since the original DJIA is not total-return index, it cannot be used as benchmark. The passively managed portfolio allocates the capital according to the components' price in January 1991, which corresponds to the DJIA's weighting method. The strategy's wealth is the weighted sum of the aggregated capital of each individual asset. We assume costless dividend reinvestment to the issuer firm's stock. The log-optimal strategy applies a dynamic, self-financing and short sale free allocation of the capital regarding Section 2.4.

Table 1 shows the coefficients of the most common equilibrium models and their test statistics. Estimated coefficients are differentiated by a hat from their theoretical values. We measured R^2-s near to 0.55 each case for the log-optimal strategy, which refers to the fact that the classical models are not able to explain log-optimal premiums sufficiently. However the explaining ability of the introduced models cannot be rejected due to significant F-statistics. The higher R^2-s for the passive strategy (above 0.78 each case) suggest that the equilibrium models are more powerful in explaining the excess returns of the passive investment. Since the market proxy is also a passively managed portfolio, containing the majority of the U.S. stocks, it is able to capture the movements of a passive investment rather than an active strategy which may invest into only one asset (and often calls for a lower degree of diversification than the passive counterpart). While $\widehat{\beta}$ is the coefficient of market premium $(r_m^t - r_f^t)$, its near to 1 value implies that –ceteris paribus– the investigated premiums move together with the market. This fact is not surprising if one takes into account that the portfolios are formed of recent Dow Jones 30 components, which are all large cap companies in the majority of the investigated period. When momentum factor is applied, we measure slightly but mostly significant negative $\widehat{m}$-s, which means that the log-optimal premiums, and less the passive investment's premiums, move counterside the returns of the prior 1 year winners in average. The impact of the firm size and the B/M equity ratio is stronger for the passive investment as we measure significant negative loadings on the SMB and the HML factor. Although, for the log-optimal strategy coefficients $\widehat{s}$ and $\widehat{h}$ are not significant anyway, according to model selection measurement *adjusted* R^2, the use of SMB and HML factors increases the power of our models. Economic considerations also support this concept because these factors are able to capture the negative return effects of large capitalization, low B/M companies. The portfolios consist only of shares of large capitalization companies, while 23 companies out of the 30 has low book-to-market equity ratio. This is implied by negative $\widehat{s}$ and $\widehat{h}$. Except for the CAPM, each model shows significant excess returns above the equilibrium value for the log-optimal strategy. That

Table 1: **Regression coefficients of portfolios formed of Dow Jones Industrial Average Components in December 2005.** The table contains coefficients and test statistics regarding the validity of the introduced coefficients and equilibrium models. Coefficients and statistics are presented for both the log-optimal strategy and a passive buy-and-hold strategy constructed by the same stocks. Estimated coefficients are differentiated by a hat from their theoretical values. In each model the expected premiums are represented, which are the equilibrium values that the investigated strategies should achieve on average if the market equilibrium holds in the long-term. The last section contains the measurements on the proposed empirical log-optimal strategy and the passive investment, including average monthly and annual risk premiums and standard deviations.

							CAPM						
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.55	0.78	**Coeff**	0.55	0.25	1.09	1.05						
F-stat	214.4	631.5	**t-stat**	1.77	1.42	14.64	25.13						
p-val	0.00	0.00	**p-val**	0.08	0.16	0.00	0.00						
adj $\mathbf{R^2}$	0.54	0.78											
							CAPM+Mom						
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.56	0.79	**Coeff**	0.69	0.33	1.07	1.04					-0.13	-0.08
F-stat	111.5	325.0	**t-stat**	2.18	1.85	14.18	24.61					-2.12	-2.20
p-val	0.00	0.00	**p-val**	0.03	0.07	0.00	0.00					0.04	0.03
adj $\mathbf{R^2}$	0.55	0.78											
							Three-Factor Model						
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.56	0.84	**Coeff**	0.67	0.47	1.07	1.03	-0.17	-0.36	-0.11	-0.21		
F-stat	73.2	306.1	**t-stat**	2.05	2.98	12.36	24.33	-1.84	-8.03	-1.00	-3.84		
p-val	0.00	0.00	**p-val**	0.04	0.00	0.00	0.00	0.07	0.00	0.32	0.00		
adj $\mathbf{R^2}$	0.55	0.84											
							Four-Factor Model						
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.56	0.84	**Coeff**	0.81	0.53	1.03	1.01	-0.15	-0.35	-0.14	-0.22	-0.13	-0.05
F-stat	56.8	232.5	**t-stat**	2.45	3.29	11.64	23.37	-1.60	-7.80	-1.21	-4.01	-1.98	-1.65
p-val	0.00	0.00	**p-val**	0.02	0.00	0.00	0.00	0.11	0.00	0.23	0.00	0.05	0.10
adj $\mathbf{R^2}$	0.55	0.84											

	Log-opt				**Passive**		
Avg Monthly prem	1.56%	**Std Dev**	6.53%	**Avg Monthly prem**	1.21%	**Std Dev**	5.01%
Avg Annual prem	18.72%	**Std Dev**	21.29%	**Avg Annual prem**	14.52%	**Std Dev**	22.91%

is, the log-optimal premiums are higher than they should be at such risk levels. A surprising fact is that while the foretokens and the magnitudes of the coefficients reflect the properties of the majority of shares making up the log-optimal portfolio, the models are unable to explain the abnormal excess returns. In the case of passive strategy, the $\widehat{\alpha}$-s are significant only for the three-factor and four-factor model. Besides, each of the $\widehat{\alpha}$-s are positive, the log-optimal strategy outperforms its passively managed counterpart regarding all equilibrium models.

The equilibrium models' $\widehat{\alpha}$ measure the abnormal return above the risk adjusted equilibrium value. Besides the $\widehat{\alpha}$-s, we also present the absolute (not risk adjusted) average monthly and annual returns and standard deviations. The results are similar, since the log-optimal method achieves much higher average return than its passive counterpart, albeit with higher monthly deviation, which is close to being equal in the annual level.

3.3 Portfolios of January, 1991 DJIA Components

In this subsection we perform similar tests on the log-optimal and a passively managed portfolio built up according to the same methodology as in the previous subsection, except for the fact that the portfolios are formed of stocks which made up the Industrial Average in January 1991. The investigated period also holds from January 1991 through December 2008. That is, the arrangement of this analysis is free of survivorship bias. The passive portfolio is price weighted at portfolio launch again.

Table 2 presents the coefficients of the same common equilibrium models and their test statistics as in the pervious subsection. The most important difference between the previous and current results that the $\widehat{\alpha}$-s are lower by 0.24% on average for the log-optimal strategy, however, the difference is even higher, 0.50% regarding the strongest (by adjusted R^2) four-factor model. Although we also measure differences for the passive strategy, only one model (CAPM+Mom) has a significant $\widehat{\alpha}$ parameter, so the figures do not refer to the existence of abnormal returns. The superiority of the portfolio formed of Industrial Average components in 2005 is due to the survivorship bias.
However, the log-optimal portfolio also beats its passive counterpart regarding both the $\widehat{\alpha}$-s, and the absolute values of the average monthly and annual returns. The notable difference in the average return helps the log-optimal strategy to achieve more than twice of the passive strategy's final wealth during the 15 years, although it results in higher standard deviation.

Except for the $\widehat{\alpha}$-s, all coefficients are significant, however, the R^2-s are lower than in the case of the previous portfolio, especially for the log-optimal strategy. Since a notable part of the stocks has lost some of their earlier capitalization or has not been traded yet, the equilibrium models are even weaker in explaining the log-optimal premiums. Regarding the CAPM and the CAPM with momentum factor, the portfolios' $\widehat{\beta}$-s are close to 0.8 while the three-factor and the four-factor models measure $\widehat{\beta}$-s around 1. Contrary to the former analysis, the portfolios now have significantly positive loading on the HML factor. That is, these portfolios invest rather to stocks with high book-to-market equity ratio.

3.4 Portfolios of Actual DJIA Components

The portfolios investigated here are the blends of the previous investments as they always hold stocks from the set of the actual DJIA components. This means that the passive portfolio is not a pure buy-and-hold strategy since the portfolio is rebalanced when change

Table 2: **Regression coefficients of portfolios formed of Dow Jones Industrial Average Components in January 1991.** The table contains coefficients and test statistics regarding the validity of the introduced coefficients and equilibrium models. Coefficients and statistics are presented for both the log-optimal strategy and a passive buy-and-hold strategy constructed by the same stocks. Estimated coefficients are differentiated by a hat from their theoretical values. In each model the expected premiums are represented, which are the equilibrium values that the investigated strategies should achieve on average if the market equilibrium holds in the long-term. The last section contains the measurements on the proposed empirical log-optimal strategy and the passive investment, including average monthly and annual risk premiums and standard deviations.

								CAPM					
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.40	0.65	**Coeff**	0.56	0.09	0.86	0.82						
F-stat	118.8	330.9	**t-stat**	1.68	0.49	10.90	18.19						
p-val	0.00	0.00	**p-val**	0.09	0.62	0.00	0.00						
adj $\mathbf{R^2}$	0.40	0.65											
								CAPM+Mom					
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.45	0.69	**Coeff**	0.82	0.27	0.80	0.78					-0.26	-0.18
F-stat	72.3	199.3	**t-stat**	2.52	1.51	10.46	18.16					-3.99	-4.93
p-val	0.00	0.00	**p-val**	0.01	0.13	0.00	0.00					0.00	0.00
adj $\mathbf{R^2}$	0.44	0.69											
								Three-Factor Model					
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.51	0.81	**Coeff**	0.09	-0.12	1.13	1.01	-0.01	-0.25	0.62	0.34		
F-stat	61.6	247.5	**t-stat**	0.28	-0.82	13.55	25.85	-0.14	-6.20	5.66	6.57		
p-val	0.00	0.00	**p-val**	0.78	0.41	0.00	0.00	0.89	0.00	0.00	0.00		
adj $\mathbf{R^2}$	0.50	0.81											
								Four-Factor Model					
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.54	0.82	**Coeff**	0.31	0.00	1.07	0.97	0.02	-0.24	0.58	0.31	-0.20	-0.11
F-stat	51.4	204.4	**t-stat**	0.99	0.02	12.75	25.19	0.26	-5.91	5.43	6.38	-3.25	-3.89
p-val	0.00	0.00	**p-val**	0.33	0.99	0.00	0.00	0.80	0.00	0.00	0.00	0.00	0.00
adj $\mathbf{R^2}$	0.53	0.82											

Log-opt				**Passive**			
Avg Monthly prem	1.13%	**Std Dev**	5.62%	**Avg Monthly prem**	0.79%	**Std Dev**	4.20%
Avg Annual prem	13.56%	**Std Dev**	26.81%	**Avg Annual prem**	9.48%	**Std Dev**	15.96%

occurs in the components list. At the break points the passive portfolio reallocates the capital according to the components' price. Since in this case the portfolio rebalancing is very rare, the transaction costs are negligible[1]. Between two breakpoints the strategy works as the previously introduced passive methods. After the breakpoints the log-optimal strategy is simply restricted to another set of 30 stocks. Although these types

[1]Four changes occurred during the 15-year long period in the DJIA.

of investments repetitively exclude the stocks which lost some of their earlier appeal, they do not suffer from that kind of survivorship bias that worsens the judgement of the first investments since tracking the DJIA is a causal strategy. We summarize the results of the four regressions in Table 3. Through the lens of this fact it is more substantial that, except the three-factor model, the $\widehat{\alpha}$-s are significantly positive for the log-optimal strategy, while they do not differ significantly from zero for the passive investment[2]. It is also worthy to note that the R^2-s are the highest in this case as the strategies track a set of the most important stocks in each subperiod. The $\widehat{\beta}$-s are notably higher for the log-optimal strategy, which is also mirrored in the higher values of the standard deviation. In contrast to the other portfolios, the $\widehat{h}$-s are non-significant for the log-optimal strategy, while they are significantly positive for the passive investment. Both the log-optimal and the passive strategy have negative loading on the momentum factor, although $\widehat{m}$ is not significant for the log-optimal strategy regarding the strongest four-factor model.

In summary, it is proven that no investment strategy has a higher asymptotic growth rate than log-optimal strategies on stationery and ergodic markets without transaction cost. However, now we also get empirical evidence that regarding the common equilibrium models they are able to achieve surpassing returns manifested in positive $\widehat{\alpha}$-s, even in the presence of transaction costs.

4 Extensions and Concluding Remarks

There are several directions in which the model presented in this paper can be extended. On the one hand, one can take into account a riskless investment instrument which gives the possibility to rescue our investment when all the expert forecasts negative price change for the next period. For this reason, the model can be extended with short selling possibility as well. If either the risk-free investment possibility or the short selling option is solved with finding the bull or the bear periods of the market, the estimation of the experts can be more accurate. On the other hand, the model supposes that the demanded security is always available in the market on the closing price, this condition can be solved using a trades and quotes database.

In the paper we abstract from many realistic features of the market in order to highlight the implications of an empirical log-optimal portfolio strategy. Our conclusion is that the log-optimal portfolio strategy shows some kind of market inefficiency, in the sense that the applied kernel-based experts are able to give better advice than pure random selection by which positive abnormal return can be gained. The main explanations for this fact stem from three directions: the dataset used in our analyses, the joint hypothesis which is the equilibrium model used to explain the return, or the difference between the single-period, multi-period approach. The first two explanations have extensive literature, while the last argument is not so researched. The goal of the one-period portfolio theory (Markowitz 1952) and the rival equilibrium models (like CAPM, APT,

[2] The three-factor model also refers to positive $\widehat{\alpha}$ at 0.1 significance level.

Table 3: **Regression coefficients of portfolios formed of actual DJIA Components.** The table contains coefficients and test statistics regarding the validity of the introduced coefficients and equilibrium models. Coefficients and statistics are presented for both the log-optimal strategy and a passive buy-and-hold strategy constructed by the same stocks. Estimated coefficients are differentiated by a hat from their theoretical values. In each model the expected premiums are represented, which are the equilibrium values that the investigated strategies should achieve on average if the market equilibrium holds in the long-term. The last section contains the measurements on the proposed empirical log-optimal strategy and the passive investment, including average monthly and annual risk premiums and standard deviations.

								CAPM					
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.62	0.78	**Coeff**	0.53	0.03	1.12	0.93						
F-stat	287.6	622.3	**t-stat**	1.92	0.16	16.96	24.95						
p-val	0.00	0.00	**p-val**	0.06	0.87	0.00	0.00						
adj $\mathbf{R^2}$	0.62	0.78											
								CAPM+Mom					
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.63	0.81	**Coeff**	0.68	0.19	1.09	0.89					-0.14	-0.16
F-stat	151.6	376.7	**t-stat**	2.43	1.28	16.50	25.50					-2.57	-5.48
p-val	0.00	0.00	**p-val**	0.02	0.20	0.00	0.00					0.01	0.00
adj $\mathbf{R^2}$	0.63	0.81											
								Three-Factor Model					
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.65	0.85	**Coeff**	0.47	-0.14	1.22	1.07	-0.21	-0.17	0.13	0.26		
F-stat	108.3	341.0	**t-stat**	1.67	-1.06	16.35	30.14	-2.72	-4.57	1.37	5.50		
p-val	0.00	0.00	**p-val**	0.10	0.29	0.00	0.00	0.01	0.00	0.17	0.00		
adj $\mathbf{R^2}$	0.64	0.85											
								Four-Factor Model					
	Log-opt	**Passive**		$\widehat{\alpha}_L$	$\widehat{\alpha}_P$	$\widehat{\beta}_L$	$\widehat{\beta}_P$	$\widehat{s}_L$	$\widehat{s}_P$	$\widehat{h}_L$	$\widehat{h}_P$	$\widehat{m}_L$	$\widehat{m}_P$
$\mathbf{R^2}$	0.66	0.87	**Coeff**	0.58	-0.01	1.18	1.03	-0.19	-0.15	0.11	0.23	-0.10	-0.11
F-stat	83.19	289.06	**t-stat**	2.05	-0.10	15.56	29.80	-2.48	-4.21	1.17	5.29	-1.84	-4.52
p-val	0.00	0.00	**p-val**	0.04	0.92	0.00	0.00	0.01	0.00	0.24	0.00	0.07	0.00
adj $\mathbf{R^2}$	0.65	0.87											

	Log-opt				**Passive**		
Avg Monthly prem	1.14%	**Std Dev**	5.93%	**Avg Monthly prem**	0.87%	**Std Dev**	4.35%
Avg Annual prem	13.68%	**Std Dev**	22.69%	**Avg Annual prem**	10.44%	**Std Dev**	16.24%

Fama-French, or Carhart model), is the optimization of the asset allocation in order to achieve optimal trade-off between expected one-period return and risk. This supposes a world where the investors optimize their consumptions and investment strategies for that given one-period. However, most of the mean-variance analysis handles only static models contrary to the expected utility models, whose literature is rich in multi-period models, supposing an individual with longer interval than simply one-period thinking. In the multi-period models the investors are allowed to rebalance their portfolios in each trading period, therefore their investments may be characterized in different ways in one and multiple periods due to the multiplicative effect of consecutive reinvestments. In our experiment the approximation uses an infinite approach, while the 15 years would dictate a multi-period investment model. However, we use a daily rebalancing strategy and the number of periods is more than 3700 which behaves almost as it does in the infinite model.

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