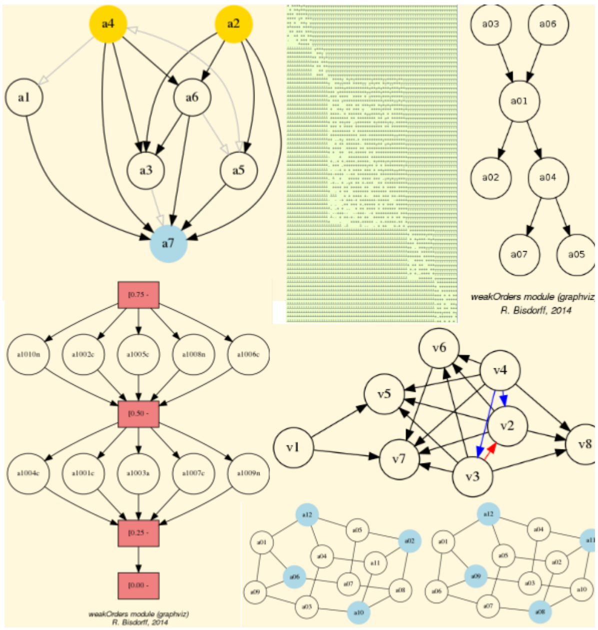
Documentation of the DIGRAPH3 software collection



Tutorials and Advanced Topics

Raymond BISDORFF

Luxembourg, 2020

Last updated : February 19, 2026

This documentation is dedicated to our
late colleague and dear friend
Prof. Marc ROUBENS.

More documents are freely available here

B. Digraph3 Advanced Topics

HTML Version

In this part of the **Digraph3** *documentation*, we provide an insight in computational enhancements one may get when working in a *bipolar-valued epistemic logic* framework, like - easily coping with *missing data* and uncertain criterion *significance weights*, - computing valued *ordinal correlations* between bipolar-valued outranking digraphs, - compting digraph *kernels* and solving bipolar-valued *kernel equation systems*, - testing for stability and confidence of outranking statements when facing uncertain performance criteria significance weights or decision objectives' importance weights and, - applying bipolar-valued *base 3 Bachet numbers* for ranking multicriteria incommensurable performance records.

Contents

1 Enhancing the outranking based MCDA approach **2**

1.1 On confident outrankings with uncertain criteria significance weights . . 2
1.2 On stable outrankings with ordinal criteria significance weights 11
1.3 On unopposed outrankings with multiple decision objectives 22

2 Enhancing social choice procedures **27**

2.1 *Condorcet*'s critical perspective on the simple plurality voting rule 27
2.2 Two-stage elections with multipartisan primary selection 36
2.3 Tempering plurality tyranny effects with bipolar approval voting 45
2.4 Selecting the winner of a primary election: a critical commentary 59

3 Theoretical and computational advancements **63**

3.1 Coping with missing data and indeterminateness 63
3.2 Ordinal correlation equals bipolar-valued relational equivalence 69
3.3 Applications of bipolar-valued base 3 encoded Bachet numbers 79
3.4 On computing digraph kernels 99
3.5 Computing with bipolar-valued sets 119
3.6 Bipolar-valued kernel membership characteristic vectors 124
3.7 On characterizing bipolar-valued outranking digraphs 132
3.8 Consensus quality of the bipolar-valued outranking relation 140

4 Appendix **150**

References **150**

1 Enhancing the outranking based MCDA approach

"*The goal of our research was to design a resolution method* [..] *that is easy to put into practice, that requires as few and reliable hypotheses as possible, and that meets the needs* [of the decision maker]." – B. Roy et al.[13]

- *On confident outrankings with uncertain criteria significance weights* (page 2)
- *On stable outrankings with ordinal criteria significance weights* (page 11)
- *On unopposed outrankings with multiple decision objectives* (page 22)

1.1 On confident outrankings with uncertain criteria significance weights

- *Modelling uncertain criteria significance weights* (page 3)
- *Bipolar-valued likelihood of ''at least as good as " situations* (page 4)
- *Confidence level of outranking situations* (page 6)

When modelling preferences following the outranking approach, the signs of the majority margins do sharply distribute validation and invalidation of pairwise outranking situations. How can we be confident in the resulting outranking digraph, when we acknowledge the usual imprecise knowledge of criteria significance weights coupled with small majority margins?

To answer this question, one usually requires *qualified* majority margins for confirming outranking situations. But how to choose such a qualifying majority level: two third, three fourth of the significance weights ?

In this tutorial we propose to link the qualifying significance majority with a required alpha%-confidence level. We model therefore the significance weights as random variables following more or less widespread distributions around an average significance value that corresponds to the given deterministic weight. As the bipolar-valued random credibility of an outranking statement hence results from the simple sum of positive or negative independent random variables, we may apply the Central Limit Theorem (CLT) for computing the *bipolar likelihood* that the expected majority margin will indeed be positive, respectively negative.

[13] [ROY-1966p]

Modelling uncertain criteria significance weights

Let us consider the significance weights of a family F of m criteria to be **independent random variables** *Wj*, distributing the potential significance weights of each criterion j = 1, . . . , m around a mean value *E(Wj)* with variance *V(Wj)*.

Choosing a specific stochastic model of uncertainty is usually application specific. In the limited scope of this tutorial, we will illustrate the consequence of this design decision on the resulting outranking modelling with four slightly different models for taking into account the uncertainty with which we know the numerical significance weights: *uniform*, *triangular*, and two models of *Beta laws*, one more *widespread* and, the other, more *concentrated*.

When considering, for instance, that the potential range of a significance weight is distributed between 0 and two times its mean value, we obtain the following random variates:

1. A continuous **uniform** distribution on the range 0 to *2E(Wj)*. Thus *Wj* ~ U(0, *2E(Wj)*) and *V(Wj)* = 1/3(*E(Wj)*)^2;
2. A **symmetric beta** distribution with, for instance, parameters *alpha* = 2 and *beta* = 2. Thus, *Wi* ~ Beta(2,2) * *2E(Wj)* and *V(Wj)* = 1/5(*E(Wj)*)^2.
3. A **symmetric triangular** distribution on the same range with mode *E(Wj)*. Thus *Wj* ~ Tr(0, *2E(Wj)*, *E(Wj)*) with *V(Wj)* = 1/6(*E(Wj)*)^2;
4. A **narrower beta** distribution with for instance parameters *alpha* = 4 and *beta* = 4. Thus *Wj* ~ Beta(4,4) * *2E(Wj)* , *V(Wj)* = 1/9(*E(Wj)*)^2.

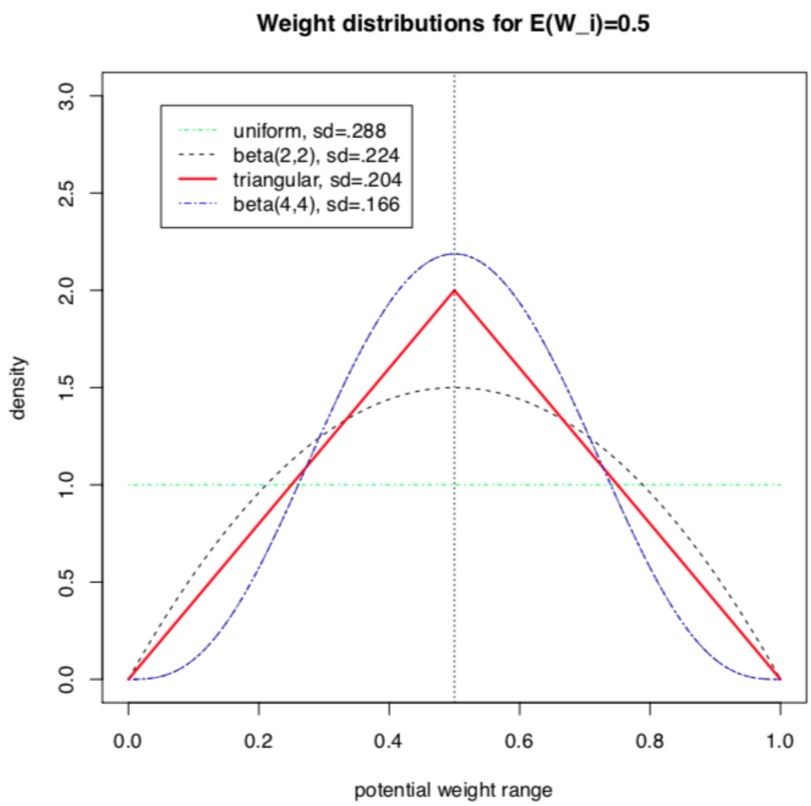


Fig. 1.1: Four models of uncertain significance weights

It is worthwhile noticing that these four uncertainty models all admit the same expected value, *E(Wj)*, however, with a respective variance which goes decreasing from 1/3, to 1/9 of the square of *E(W)* (see Fig. 1.1).

Bipolar-valued likelihood of ''at least as good as " situations

Let $A = \{x,\ y,\ z, \ldots\}$ be a finite set of n potential decision actions, evaluated on $F = \{1, \ldots, m\}$, a *finite* and *coherent* family of m performance criteria. On each criterion j in F, the decision actions are evaluated on a real performance scale [0; *Mj*], supporting an upper-closed indifference threshold *indj* and a lower-closed preference threshold *prj* such that 0 <= *indj* < *prj* <= *Mj*. The marginal performance of object x on criterion j is denoted *xj*. Each criterion j is thus characterising a marginal double threshold order $\geq_j$ on A (see Fig. 1.2):

$$r(x \geq_j y) \;=\; \begin{cases} +1 & \text{if} \quad x_j - y_j \geq -ind_j, \\ -1 & \text{if} \quad x_j - y_j \leq -pr_j, \\ 0 & \text{otherwise.} \end{cases}$$

Semantics of the marginal bipolar-valued characteristic function:

- +1 signifies x is performing at least as good as y on criterion j,
- -1 signifies that x is not performing at least as good as y on criterion j,
- 0 signifies that it is unclear whether, on criterion j, x is performing at least as good as y.

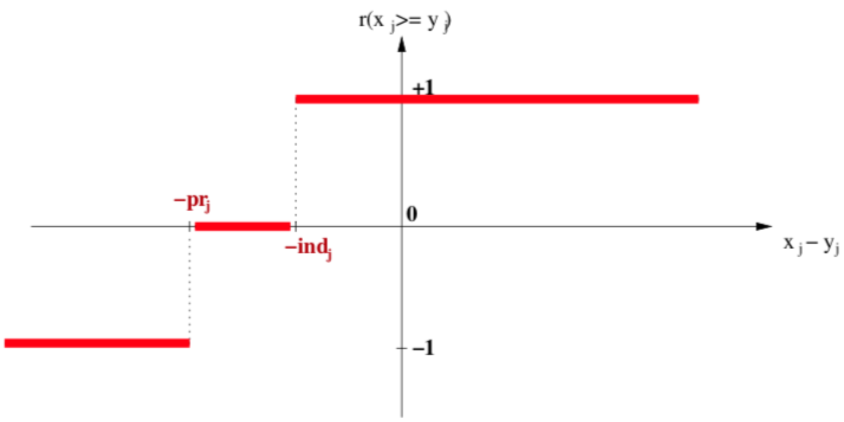


Fig. 1.2: Bipolar-valued outranking characteristic function

Each criterion j in F contributes the random significance *Wj* of his '*at least as good as*' characteristic $r(x \geq_j y)$ to the global characteristic $\tilde{r}(x \geq y)$ in the following way:

$$\tilde{r}(x \geq y) \;=\; \sum_{j \in F} W_j \times r(x \geq_j y))$$

Thus, $\tilde{r}(x \geq y)$ becomes a simple sum of positive or negative independent random variables with known means and variances where $\tilde{r}(x \geq y) > 0$ signifies x is globally performing at least as good as y, $\tilde{r}(x \geq y) < 0$ signifies that x is not globally performing at least as good as y, and $\tilde{r}(x \geq y) = 0$ signifies that it is unclear whether x is globally performing at least as good as y.

From the *Central Limit Theorem* (CLT), we know that such a sum of random variables leads, with m getting large, to a Gaussian distribution Y with

$E(Y) = \sum_{j \in F} \big(E(W_j) \times r(x \geq_j y) \big)$, and

$V(Y) = \sum_{j \in F} \big(V(W_j) \times |r(x \geq_j y)| \big)$.

And the **likelihood of validation**, respectively **invalidation** of an '*at least as good as*' situation, denoted $lh(x \geq y)$, may hence be assessed by the probability *P(Y>0)* = 1.0 - *P(Y<=0)* that Y takes a positive, resp. *P(Y<0)* takes a negative value. In the bipolar-valued case here, we can judiciously make usage of the standard Gaussian **error function** , i.e. the bipolar *2P(Z)* - 1.0 version of the standard Gaussian *P(Z)* probability distribution function:

$$lh(x \geq y) \;=\; -\mathrm{erf}\Big(\frac{1}{\sqrt{2}} \frac{-E(Y)}{\sqrt{V(Y)}}\Big)$$

The range of the bipolar-valued $lh(x \geq y)$ hence becomes [-1.0;+1.0], and $-lh(x \geq y) = lh(x \not\geq y)$, i.e. a **negative likelihood** represents the likelihood of the correspondent **negated** '*at least as good as*' situation. A likelihood of +1.0 (resp. -1.0) means the corresponding preferential situation appears **certainly validated** (resp. **invalidated**).

Example

Let x and y be evaluated wrt 7 equisignificant criteria; Four criteria positively support that x is *as least as good performing* than y and three criteria support that x is *not at least as good* performing than y. Suppose $E(Wj) = w$ for $j = 1,\ldots,7$ and Wj ~ Tr(0, $2w$, w) for $j = 1,\ldots 7$. The expected value of the global '*at least as good as*' characteristic value becomes: $E\big(\tilde{r}(x \geq y)\big) = 4w - 3w = w$ with a variance $V\big(\tilde{r}(x \geq y)\big) = 7\frac{1}{6}w^2$.

If $w = 1$, $E\big(\tilde{r}(x \geq y)\big) = 1$ and $sd\big(\tilde{r}(x \geq y)\big) = 1.08$. By the CLT, the bipolar likelihood of the *at least as good* performing situation becomes: $lh(x \geq y) = 0.66$, which corresponds to a probability of $(0.66 + 1.0)/2 = 83\%$ of being supported by a positive significance majority of criteria.

A *Monte Carlo* simulation with 10 000 runs empirically confirms the effective convergence to a Gaussian (see Fig. 1.3 realised with *gretl*[4]).

[4] The *Gnu Regression, Econometrics and Time-series Library* http://gretl.sourceforge.net/

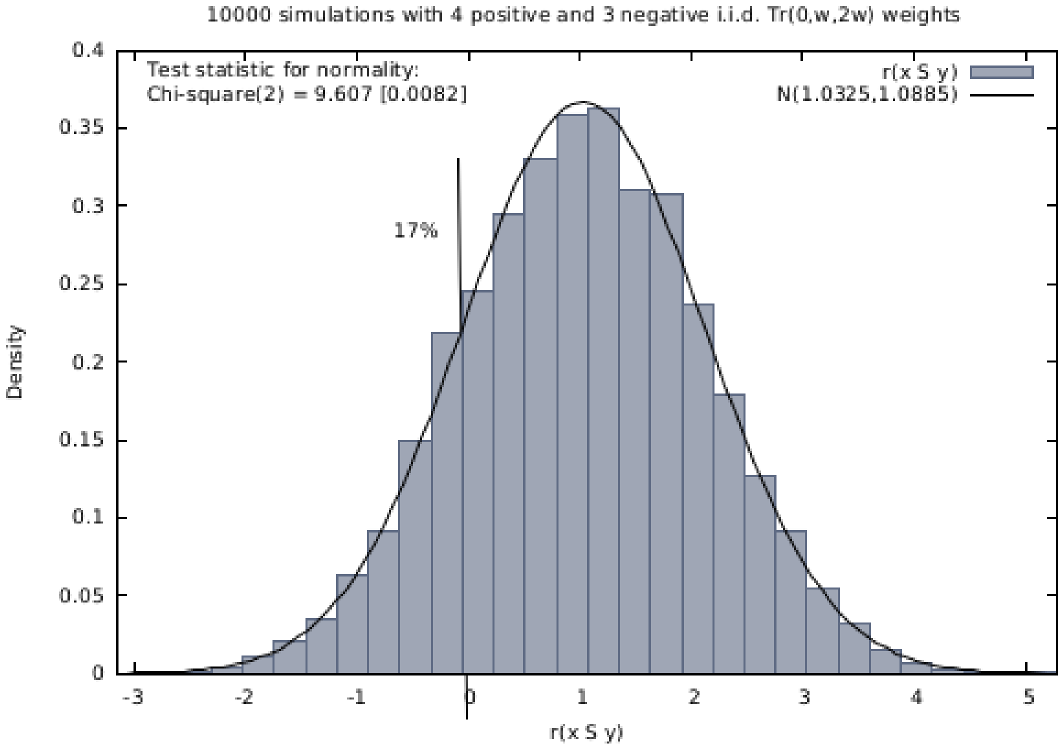


Fig. 1.3: Distribution of 10 000 random outranking characteristic values

Indeed, $\tilde{r}(x \geq y) \rightsquigarrow Y = \mathcal{N}(1.03, 1.089)$, with an empirical probability of observing a negative majority margin of about 17%.

Confidence level of outranking situations

Now, following the classical outranking approach (see [BIS-2013p]), we may say, from an epistemic perspective, that decision action x **outranks** decision action y at *confidence* level *alpha* %, when

1. an expected majority of criteria validates, at confidence level *alpha* % or higher, a global '*at least as good as*' situation between x and y, and
2. no considerably less performing is observed on a discordant criterion.

Dually, decision action x **does not outrank** decision action y at confidence level *alpha* %, when

1. an expected majority of criteria at confidence level *alpha* % or higher, invalidates a global '*at least as good as*' situation between x and y, and
2. no considerably better performing situation is observed on a concordant criterion.

Time for a coded example

Let us consider the following random performance tableau.

```
>>> from randomPerfTabs import RandomPerformanceTableau
>>> t = RandomPerformanceTableau(
```

(continues on next page)

(continued from previous page)

```
...          numberOfActions=7,
...          numberOfCriteria=7,seed=100)

>>> t.showPerformanceTableau(Transposed=True)
 *----  performance tableau -----*
 criteria | weights |   'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'
 ---------|-------------------------------------------------------------
    'g1'  |    1    |  15.17  44.51  57.87  58.00  24.22  29.10  96.58
    'g2'  |    1    |  82.29  43.90    NA   35.84  29.12  34.79  62.22
    'g3'  |    1    |  44.23  19.10  27.73  41.46  22.41  21.52  56.90
    'g4'  |    1    |  46.37  16.22  21.53  51.16  77.01  39.35  32.06
    'g5'  |    1    |  47.67  14.81  79.70  67.48    NA   90.72  80.16
    'g6'  |    1    |  69.62  45.49  22.03  33.83  31.83    NA   48.80
    'g7'  |    1    |  82.88  41.66  12.82  21.92  75.74  15.45   6.05
```

For the corresponding confident outranking digraph, we require a confidence level of *alpha* = 90%. The `ConfidentBipolarOutrankingDigraph` class provides such a construction.

```
>>> from outrankingDigraphs import\
...                           ConfidentBipolarOutrankingDigraph

>>> g90 = ConfidentBipolarOutrankingDigraph(t,confidence=90)
>>> print(g90)
 *------- Object instance description ------*
 Instance class        : ConfidentBipolarOutrankingDigraph
 Instance name         : rel_randomperftab_CLT
 # Actions             : 7
 # Criteria            : 7
 Size                  : 15
 Uncertainty model     : triangular(a=0,b=2w)
 Likelihood domain     : [-1.0;+1.0]
 Confidence level      : 0.80 (90.0%)
 Confident credibility: > abs(0.143) (57.1%)
 Determinateness (%)   : 62.07
 Valuation domain      : [-1.00;1.00]
 Attributes            : ['name', 'bipolarConfidenceLevel',
                          'distribution', 'betaParameter', 'actions',
                          'order', 'valuationdomain', 'criteria',
                          'evaluation', 'concordanceRelation',
                          'vetos', 'negativeVetos',
                          'largePerformanceDifferencesCount',
                          'likelihoods', 'confidenceCutLevel',
                          'relation', 'gamma', 'notGamma']
```

The resulting 90% confident expected outranking relation is shown below.

```
1  >>> g90.showRelationTable(LikelihoodDenotation=True)
2   * ---- Outranking Relation Table -----
3   r/(lh) |  'a1'      'a2'    'a3'    'a4'    'a5'    'a6'    'a7'
4   -------|------------------------------------------------------------
5     'a1' | +0.00   +0.71   +0.29   +0.29   +0.29   +0.29   +0.00
6          | ( - )  (+1.00) (+0.95) (+0.95) (+0.95) (+0.95) (+0.65)
7     'a2' | -0.71   +0.00   -0.29   +0.00   +0.00   +0.29   -0.57
8          |(-1.00)  ( - )  (-0.95) (-0.65) (+0.73) (+0.95) (-1.00)
9     'a3' | -0.29   +0.29   +0.00   -0.29   +0.00   +0.00   -0.29
10         |(-0.95) (+0.95)  ( - )  (-0.95) (-0.73) (-0.00) (-0.95)
11    'a4' | +0.00   +0.00   +0.57   +0.00   +0.29   +0.57   -0.43
12         |(-0.00) (+0.65) (+1.00)  ( - )  (+0.95) (+1.00) (-0.99)
13    'a5' | -0.29   +0.00   +0.00   +0.00   +0.00   +0.29   -0.29
14         |(-0.95) (-0.00) (+0.73) (-0.00)  ( - )  (+0.99) (-0.95)
15    'a6' | -0.29   +0.00   +0.00   -0.29   +0.00   +0.00   +0.00
16         |(-0.95) (-0.00) (+0.73) (-0.95) (+0.73)  ( - )  (-0.00)
17    'a7' | +0.00   +0.71   +0.57   +0.43   +0.29   +0.00   +0.00
18         |(-0.65) (+1.00) (+1.00) (+0.99) (+0.95) (-0.00)  ( - )
19  Valuation domain       : [-1.000; +1.000]
20  Uncertainty model      : triangular(a=2.0,b=2.0)
21  Likelihood domain      : [-1.0;+1.0]
22  Confidence level       : 0.80 (90.0%)
23  Confident credibility  : > abs(0.14) (57.1%)
24  Determinateness        : 0.24 (62.1%)
```

The (lh) figures, indicated in the table above, correspond to bipolar likelihoods and the required bipolar confidence level equals $(0.90{+}1.0)/2 = 0.80$ (see Line 22 above). Action '*a1*' thus confidently outranks all other actions, except '*a7*' where the actual likelihood $(+0.65)$ is lower than the required one (0.80) and we furthermore observe a considerable counter-performance on criterion '*g1*'.

Notice also the lack of confidence in the outranking situations we observe between action '*a2*' and actions '*a4*' and '*a5*'. In the deterministic case we would have $r(a2 \geq a4) = -0.143$ and $r(a2 \geq a5) = +0.143$. All outranking situations with a characteristic value lower or equal to abs(0.143), i.e. a majority support of $1.143/2 = 57.1\%$ and less, appear indeed to be *not confident* at level 90% (see Line 23 above).

We may draw the corresponding strict 90%-confident outranking digraph, oriented by its initial and terminal *strict* prekernels (see Fig. 1.4).

```
>>> gcd90 = ~ (-g90)
>>> gcd90.showPreKernels()
 *--- Computing preKernels ---*
 Dominant preKernels :
 ['a1', 'a7']
    independence :  0.0
    dominance    :  0.2857
    absorbency   :  -0.7143
```

(continues on next page)

(continued from previous page)

```
   covering     :  0.800
Absorbent preKernels :
['a2', 'a5', 'a6']
   independence :  0.0
   dominance    :  -0.2857
   absorbency   :  0.2857
   covered      :  0.583
>>> gcd90.exportGraphViz(fileName='confidentOutranking',
...     firstChoice=['a1', 'a7'],lastChoice=['a2', 'a5', 'a6'])

*---- exporting a dot file for GraphViz tools ---------*
Exporting to confidentOutranking.dot
dot -Grankdir=BT -Tpng confidentOutranking.dot -o confidentOutranking.
↪png
```

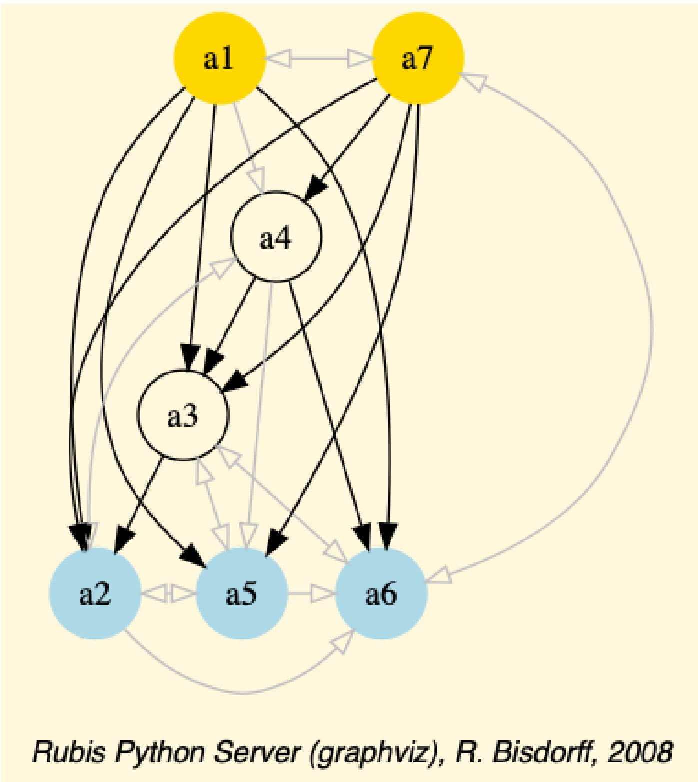


Fig. 1.4: Strict 90%-confident outranking digraph oriented by its prekernels

Now, what becomes this 90%-confident outranking digraph when we require a stronger confidence level of, say 99% ?

```
>>> g99 = ConfidentBipolarOutrankingDigraph(t,confidence=99)
>>> g99.showRelationTable()
* ---- Outranking Relation Table -----
r/(lh) |  'a1'       'a2'    'a3'    'a4'    'a5'    'a6'    'a7'
```

(continues on next page)

(continued from previous page)

```
5  -------|------------------------------------------------------------
6   'a1' |  +0.00   +0.71   +0.00   +0.00   +0.00   +0.00   +0.00
7        |  ( - )  (+1.00) (+0.95) (+0.95) (+0.95) (+0.95) (+0.65)
8   'a2' |  -0.71   +0.00   +0.00   +0.00   +0.00   +0.00   -0.57
9        | (-1.00)  ( - )  (-0.95) (-0.65) (+0.73) (+0.95) (-1.00)
10  'a3' |  +0.00   +0.00   +0.00   +0.00   +0.00   +0.00   +0.00
11       | (-0.95) (+0.95)  ( - )  (-0.95) (-0.73) (-0.00) (-0.95)
12  'a4' |  +0.00   +0.00   +0.57   +0.00   +0.00   +0.57   -0.43
13       | (-0.00) (+0.65) (+1.00)  ( - )  (+0.95) (+1.00) (-0.99)
14  'a5' |  +0.00   +0.00   +0.00   +0.00   +0.00   +0.29   +0.00
15       | (-0.95) (-0.00) (+0.73) (-0.00)  ( - )  (+0.99) (-0.95)
16  'a6' |  +0.00   +0.00   +0.00   +0.00   +0.00   +0.00   +0.00
17       | (-0.95) (-0.00) (+0.73) (-0.95) (+0.73)  ( - )  (-0.00)
18  'a7' |  +0.00   +0.71   +0.57   +0.43   +0.00   +0.00   +0.00
19       | (-0.65) (+1.00) (+1.00) (+0.99) (+0.95) (-0.00)  ( - )
20 Valuation domain       : [-1.000; +1.000]
21 Uncertainty model      : triangular(a=2.0,b=2.0)
22 Likelihood domain      : [-1.0;+1.0]
23 Confidence level       : 0.98 (99.0%)
24 Confident credibility : > abs(0.286) (64.3%)
25 Determinateness        : 0.13 (56.6%)
```

At 99% confidence, the minimal required significance majority support amounts to 64.3% (see Line 24 above). As a result, most outranking situations don't get anymore validated, like the outranking situations between action '*a1*' and actions '*a3*', '*a4*', '*a5*' and '*a6*' (see Line 5 above). The overall epistemic determination of the digraph consequently drops from 62.1% to 56.6% (see Line 25).

Finally, what becomes the previous 90%-confident outranking digraph if the uncertainty concerning the criteria significance weights is modelled with a larger variance, like *uniform* variates (see Line 2 below).

```
1  >>> gu90 = ConfidentBipolarOutrankingDigraph(t,
2  ...            confidence=90,distribution='uniform')
3
4  >>> gu90.showRelationTable()
5   * ---- Outranking Relation Table -----
6   r/(lh) |  'a1'      'a2'    'a3'    'a4'    'a5'    'a6'    'a7'
7  -------|------------------------------------------------------------
8   'a1' |  +0.00   +0.71   +0.29   +0.29   +0.29   +0.29   +0.00
9        |  ( - )  (+1.00) (+0.84) (+0.84) (+0.84) (+0.84) (+0.49)
10  'a2' |  -0.71   +0.00   -0.29   +0.00   +0.00   +0.29   -0.57
11       | (-1.00)  ( - )  (-0.84) (-0.49) (+0.56) (+0.84) (-1.00)
12  'a3' |  -0.29   +0.29   +0.00   -0.29   +0.00   +0.00   -0.29
13       | (-0.84) (+0.84)  ( - )  (-0.84) (-0.56) (-0.00) (-0.84)
14  'a4' |  +0.00   +0.00   +0.57   +0.00   +0.29   +0.57   -0.43
15       | (-0.00) (+0.49) (+1.00)  ( - )  (+0.84) (+1.00) (-0.95)
```

(continues on next page)

(continued from previous page)

```
16    'a5' |  -0.29   +0.00   +0.00   +0.00   +0.00   +0.29   -0.29
17         | (-0.84) (-0.00) (+0.56) (-0.00)  ( - )  (+0.92) (-0.84)
18    'a6' |  -0.29   +0.00   +0.00   -0.29   +0.00   +0.00   +0.00
19         | (-0.84) (-0.00) (+0.56) (-0.84) (+0.56)  ( - )  (-0.00)
20    'a7' |  +0.00   +0.71   +0.57   +0.43   +0.29   +0.00   +0.00
21         | (-0.49) (+1.00) (+1.00) (+0.95) (+0.84) (-0.00)  ( - )
22   Valuation domain   : [-1.000; +1.000]
23   Uncertainty model  : uniform(a=2.0,b=2.0)
24   Likelihood domain  : [-1.0;+1.0]
25   Confidence level   : 0.80 (90.0%)
26   Confident majority : 0.14 (57.1%)
27   Determinateness    : 0.24 (62.1%)
```

Despite lower likelihood values (see the *g90* relation table above), we keep the same confident majority level of 57.1% (see Line 25 above) and, hence, also the same 90%-confident outranking digraph.

Note

For concluding, it is worthwhile noticing again that it is in fact the **neutral** value of our *bipolar-valued epistemic logic* that allows us to easily handle alpha% confidence or not of outranking situations when confronted with uncertain criteria significance weights. Remarkable furthermore is the usage, the standard **Gaussian error function** (erf) provides by delivering *signed likelihood values* immediately concerning either a *positive* relational statement, or when negative, its *negated* version.

Back to *Content Table* (page 1)

1.2 On stable outrankings with ordinal criteria significance weights

- *Cardinal or ordinal criteria significance weights* (page 11)
- *Qualifying the stability of outranking situations* (page 13)
- *Computing the stability denotation of outranking situations* (page 17)
- *Robust bipolar-valued outranking digraphs* (page 19)

Cardinal or ordinal criteria significance weights

The required cardinal significance weights of the performance criteria represent the *Achilles*' heel of the outranking approach. Rarely will indeed a decision maker be cognitively competent for suggesting precise decimal-valued criteria significance weights. More often, the decision problem will involve more or less equally important decision objectives with more or less equi-significant criteria. A random example of such a decision problem may be generated with the `Random3ObjectivesPerformanceTableau` class.

Listing 1.1: Random 3 Objectives Performance Tableau

```
1  >>> from randomPerfTabs import \
2  ...           Random3ObjectivesPerformanceTableau
3  
4  >>> t = Random3ObjectivesPerformanceTableau(
5  ...           numberOfActions=7,
6  ...           numberOfCriteria=9,seed=102)
7  
8  >>> t
9   *------- PerformanceTableau instance description ------*
10  Instance class   : Random3ObjectivesPerformanceTableau
11  Seed             : 102
12  Instance name    : random3ObjectivesPerfTab
13  # Actions        : 7
14  # Objectives     : 3
15  # Criteria       : 9
16  Attributes       : ['name', 'valueDigits', 'BigData', 'OrdinalScales',
17                      'missingDataProbability', 'negativeWeightProbability',
18                      'randomSeed', 'sumWeights', 'valuationPrecision',
19                      'commonScale', 'objectiveSupportingTypes', 'actions',
20                      'objectives', 'criteriaWeightMode', 'criteria',
21                      'evaluation', 'weightPreorder']
22 >>> t.showObjectives()
23  *------ show objectives -------"
24  Eco: Economical aspect
25     ec1 criterion of objective Eco 8
26     ec4 criterion of objective Eco 8
27     ec8 criterion of objective Eco 8
28    Total weight: 24.00 (3 criteria)
29  Soc: Societal aspect
30     so2 criterion of objective Soc 12
31     so7 criterion of objective Soc 12
32    Total weight: 24.00 (2 criteria)
33  Env: Environmental aspect
34     en3 criterion of objective Env 6
35     en5 criterion of objective Env 6
36     en6 criterion of objective Env 6
37     en9 criterion of objective Env 6
38    Total weight: 24.00 (4 criteria)
```

In this example (see Listing 1.1), we face seven decision alternatives that are assessed with respect to three *equally important* decision objectives concerning: first, an *economical* aspect (Line 24) with a coalition of three performance criteria of significance weight

8, secondly, a *societal* aspect (Line 29) with a coalition of two performance criteria of significance weight 12, and thirdly, an *environmental* aspect (Line 33) with a coalition four performance criteria of significance weight 6.

The question we tackle is the following: How *dependent* on the actual values of the significance weights appears the corresponding bipolar-valued outranking digraph ? In the previous section, we assumed that the criteria significance weights were random variables. Here, we shall assume that we know for sure only the preordering of the significance weights. In our example we see indeed three increasing weight equivalence classes (Listing 1.2).

Listing 1.2: Significance weights preorder

```
>>> t.showWeightPreorder()
 ['en3', 'en5', 'en6', 'en9'] (6) <
 ['ec1', 'ec4', 'ec8'] (8) <
 ['so2', 'so7'] (12)
```

How stable appear now the outranking situations when assuming only ordinal significance weights?

Qualifying the stability of outranking situations

Let us construct the normalized bipolar-valued outranking digraph corresponding with the previous 3 Objectives performance tableau *t*.

Listing 1.3: Example Bipolar Outranking Digraph

```
1  >>> from outrankingDigraphs import BipolarOutrankingDigraph
2  >>> g = BipolarOutrankingDigraph(t,Normalized=True)
3  >>> g.showRelationTable()
4   * ---- Relation Table -----
5   r(>=) |  'p1'   'p2'   'p3'   'p4'   'p5'   'p6'   'p7'
6   ------|-------------------------------------------------
7    'p1' | +1.00  -0.42  +0.00  -0.69  +0.39  +0.11  -0.06
8    'p2' | +0.58  +1.00  +0.83  +0.00  +0.58  +0.58  +0.58
9    'p3' | +0.25  -0.33  +1.00  +0.00  +0.50  +1.00  +0.25
10   'p4' | +0.78  +0.00  +0.61  +1.00  +1.00  +1.00  +0.67
11   'p5' | -0.11  -0.50  -0.25  -0.89  +1.00  +0.11  -0.14
12   'p6' | +0.22  -0.42  +0.00  -1.00  +0.17  +1.00  -0.11
13   'p7' | +0.22  -0.50  +0.17  -0.06  +0.78  +0.42  +1.00
```

We notice on the principal diagonal, the *certainly validated* reflexive terms +1.00 (see Listing 1.3 Lines 7-13). Now, we know for sure that *unanimous* outranking situations are completely independent of the significance weights. Similarly, all outranking situations that are supported by a *majority* significance in *each* coalition of equi-significant criteria are also in fact independent of the actual importance we attach to each individual criteria coalition. But we are also able to test (see [BIS-2014p]) if an outranking situation is independent of all the potential significance weights that respect the given *preordering* of the weights. Mind that there are, for sure, always outranking situations that are indeed *dependent* on the very values we allocate to the criteria significance weights.

Such a stability denotation of outranking situations is readily available with the common `showRelationTable()` method.

Listing 1.4: Relation Table with Stability Denotation

```
1  >>> g.showRelationTable(StabilityDenotation=True)
2  * ---- Relation Table -----
3  r/(stab)  |  'p1'  'p2'  'p3'  'p4'  'p5'  'p6'  'p7'
4  ----------|------------------------------------------
5    'p1'    | +1.00 -0.42 +0.00 -0.69 +0.39 +0.11 -0.06
6            |  (+4)  (-2)  (+0)  (-3)  (+2)  (+2)  (-1)
7    'p2'    | +0.58 +1.00 +0.83  0.00 +0.58 +0.58 +0.58
8            |  (+2)  (+4)  (+3)  (+2)  (+2)  (+2)  (+2)
9    'p3'    | +0.25 -0.33 +1.00  0.00 +0.50 +1.00 +0.25
10           |  (+2)  (-2)  (+4)   (0)  (+2)  (+2)  (+1)
11   'p4'    | +0.78  0.00 +0.61 +1.00 +1.00 +1.00 +0.67
12           |  (+3)  (-1)  (+3)  (+4)  (+4)  (+4)  (+2)
13   'p5'    | -0.11 -0.50 -0.25 -0.89 +1.00 +0.11 -0.14
14           |  (-2)  (-2)  (-2)  (-3)  (+4)  (+2)  (-2)
15   'p6'    | +0.22 -0.42  0.00 -1.00 +0.17 +1.00 -0.11
16           |  (+2)  (-2)  (+1)  (-2)  (+2)  (+4)  (-2)
17   'p7'    | +0.22 -0.50 +0.17 -0.06 +0.78 +0.42 +1.00
18           |  (+2)  (-2)  (+1)  (-1)  (+3)  (+2)  (+4)
```

We may thus distinguish the following bipolar-valued stability levels:

- **+4 | -4** : *unanimous* outranking | outranked situation. The pairwise trivial reflexive outrankings, for instance, all show this stability level;
- **+3 | -3** : *validated* outranking | outranked situation in *each* coalition of equi-significant criteria. This is, for instance, the case for the outranking situation observed between alternatives *p1* and *p4* (see Listing 1.4 Lines 6 and 12);
- **+2 | -2** : outranking | outranked situation *validated* with *all* potential significance weights that are *compatible* with the given significance *preorder* (see Listing 1.2. This is case for the comparison of alternatives *p1* and *p2* (see Listing 1.4 Lines 6 and 8);
- **+1 | -1** : *validated* outranking | outranked situation with the given significance weights, a situation we may observe between alternatives *p3* and *p7* (see Listing 1.4 Lines 10 and 16);
- **0** : *indeterminate* relational situation, like the one between alternatives *p1* and *p3* (see Listing 1.4 Lines 6 and 10).

It is worthwhile noticing that, in the one limit case where all performance criteria appear equi-significant, i.e. there is given a single equivalence class containing all the performance criteria, we may only distinguish stability levels +4 and +3 (rep. -4 and -3). Furthermore, when in such a case an outranking (resp. outranked) situation is validated at level +3 (resp. -3), no potential preordering of the criteria significance weights exists that could qualify the same situation as outranked (resp. outranking) at level -2 (resp. +2).

In the other limit case, when all performance criteria admit different significance weights, i.e. the significance weights may be linearly ordered, no stability level +3 or -3 may be observed.

As mentioned above, all *reflexive* comparisons confirm an unanimous outranking situation: all decision alternatives are indeed trivially *as well performing as* themselves. But there appear also two non reflexive unanimous outranking situations: when comparing, for instance, alternative *p4* with alternatives *p5* and *p6* (see Listing 1.4 Lines 14 and 16).

Let us inspect the details of how alternatives *p4* and *p5* compare.

Listing 1.5: Comparing Decision Alternatives *a4* and *a5*

```
>>> g.showPairwiseComparison('p4','p5')
 *------------  pairwise comparison ----*
 Comparing actions : (p4, p5)
 crit. wght.  g(x)  g(y)    diff  | ind   pref    r()        |
 ec1   8.00  85.19  46.75  +38.44 | 5.00  10.00   +8.00      |
 ec4   8.00  72.26   8.96  +63.30 | 5.00  10.00   +8.00      |
 ec8   8.00  44.62  35.91   +8.71 | 5.00  10.00   +8.00      |
 en3   6.00  80.81  31.05  +49.76 | 5.00  10.00   +6.00      |
 en5   6.00  49.69  29.52  +20.17 | 5.00  10.00   +6.00      |
 en6   6.00  66.21  31.22  +34.99 | 5.00  10.00   +6.00      |
 en9   6.00  50.92   9.83  +41.09 | 5.00  10.00   +6.00      |
 so2  12.00  49.05  12.36  +36.69 | 5.00  10.00  +12.00      |
 so7  12.00  55.57  44.92  +10.65 | 5.00  10.00  +12.00      |
 Valuation in range: -72.00 to +72.00; global concordance: +72.00
```

Alternative *p4* is indeed performing unanimously *at least as well as* alternative *p5*: *r(p4 outranks p5) = +1.00* (see Listing 1.4 Line 11).

The converse comparison does not, however, deliver such an unanimous *outranked* situation. This comparison only qualifies at stability level -3 (see Listing 1.4 Line 13 *r(p5 outranks p4) = 0.89*).

Listing 1.6: Comparing Decision Alternatives *p5* and *p4*

```
>>> g.showPairwiseComparison('p5','p4')
 *------------  pairwise comparison ----*
 Comparing actions : (p5, p4)
 crit. wght.  g(x)  g(y)    diff  | ind   pref    r()        |
 ec1   8.00  46.75  85.19  -38.44 | 5.00  10.00   -8.00      |
 ec4   8.00   8.96  72.26  -63.30 | 5.00  10.00   -8.00      |
 ec8   8.00  35.91  44.62   -8.71 | 5.00  10.00   +0.00      |
 en3   6.00  31.05  80.81  -49.76 | 5.00  10.00   -6.00      |
 en5   6.00  29.52  49.69  -20.17 | 5.00  10.00   -6.00      |
 en6   6.00  31.22  66.21  -34.99 | 5.00  10.00   -6.00      |
 en9   6.00   9.83  50.92  -41.09 | 5.00  10.00   -6.00      |
```

(continues on next page)

(continued from previous page)

```
12 so2  12.00  12.36  49.05  -36.69 | 5.00  10.00  -12.00      |
13 so7  12.00  44.92  55.57  -10.65 | 5.00  10.00  -12.00      |
14 Valuation in range: -72.00 to +72.00; global concordance: -64.00
```

Indeed, on criterion *ec8* we observe a small negative performance difference of -8.71 (see Listing 1.6 Line 7) which is effectively below the supposed *preference discrimination threshold* of 10.00. Yet, the outranked situation is supported by a majority of criteria in each decision objective. Hence, the reported preferential situation is completely independent of any chosen significance weights.

Let us now consider a comparison, like the one between alternatives *p2* and *p1*, that is only qualified at stability level +2, resp. -2.

Listing 1.7: Comparing Decision Alternatives *p2* and *p1*

```
>>> g.showPairwiseOutrankings('p2','p1')
*------------  pairwise comparison ----*
Comparing actions : (p2, p1)
crit. wght.  g(x)  g(y)    diff  | ind   pref    r()       |
ec1   8.00  89.77  38.11  +51.66 | 5.00  10.00   +8.00      |
ec4   8.00  86.00  22.65  +63.35 | 5.00  10.00   +8.00      |
ec8   8.00  89.43  77.02  +12.41 | 5.00  10.00   +8.00      |
en3   6.00  20.79  58.16  -37.37 | 5.00  10.00   -6.00      |
en5   6.00  23.83  31.40   -7.57 | 5.00  10.00   +0.00      |
en6   6.00  18.66  11.41   +7.25 | 5.00  10.00   +6.00      |
en9   6.00  26.65  44.37  -17.72 | 5.00  10.00   -6.00      |
so2  12.00  89.12  22.43  +66.69 | 5.00  10.00  +12.00      |
so7  12.00  84.73  28.41  +56.32 | 5.00  10.00  +12.00      |
Valuation in range: -72.00 to +72.00; global concordance: +42.00
*------------  pairwise comparison ----*
Comparing actions : (p1, p2)
crit. wght.  g(x)  g(y)    diff  | ind   pref    r()       |
ec1   8.00  38.11  89.77  -51.66 | 5.00  10.00   -8.00      |
ec4   8.00  22.65  86.00  -63.35 | 5.00  10.00   -8.00      |
ec8   8.00  77.02  89.43  -12.41 | 5.00  10.00   -8.00      |
en3   6.00  58.16  20.79  +37.37 | 5.00  10.00   +6.00      |
en5   6.00  31.40  23.83   +7.57 | 5.00  10.00   +6.00      |
en6   6.00  11.41  18.66   -7.25 | 5.00  10.00   +0.00      |
en9   6.00  44.37  26.65  +17.72 | 5.00  10.00   +6.00      |
so2  12.00  22.43  89.12  -66.69 | 5.00  10.00  -12.00      |
so7  12.00  28.41  84.73  -56.32 | 5.00  10.00  -12.00      |
Valuation in range: -72.00 to +72.00; global concordance: -30.00
```

In both comparisons, the performances observed with respect to the environmental decision objective are not validating with a significant majority the otherwise unanimous outranking, resp. outranked situations. Hence, the stability of the reported preferential situations is in fact dependent on choosing significance weights that are compatible with the given significance weights preorder (see *Significance weights preorder* (page 13)).

Let us finally inspect a comparison that is only qualified at stability level +1, like the one between alternatives *p7* and *p3* (see Listing 1.8).

Listing 1.8: Comparing Decision Alternatives *p7* and *p3*

```
>>> g.showPairwiseOutrankings('p7','p3')
*------------  pairwise comparison ----*
Comparing actions : (p7, p3)
crit. wght.  g(x)  g(y)    diff  | ind   pref    r()         |
ec1   8.00  15.33  80.19  -64.86 | 5.00  10.00   -8.00       |
ec4   8.00  36.31  68.70  -32.39 | 5.00  10.00   -8.00       |
ec8   8.00  38.31  91.94  -53.63 | 5.00  10.00   -8.00       |
en3   6.00  30.70  46.78  -16.08 | 5.00  10.00   -6.00       |
en5   6.00  35.52  27.25   +8.27 | 5.00  10.00   +6.00       |
en6   6.00  69.71   1.65  +68.06 | 5.00  10.00   +6.00       |
en9   6.00  13.10  14.85   -1.75 | 5.00  10.00   +6.00       |
so2  12.00  68.06  58.85   +9.21 | 5.00  10.00  +12.00       |
so7  12.00  58.45  15.49  +42.96 | 5.00  10.00  +12.00       |
Valuation in range: -72.00 to +72.00; global concordance: +12.00
*------------  pairwise comparison ----*
Comparing actions : (p3, p7)
crit. wght.  g(x)  g(y)    diff  | ind   pref    r()         |
ec1   8.00  80.19  15.33  +64.86 | 5.00  10.00   +8.00       |
ec4   8.00  68.70  36.31  +32.39 | 5.00  10.00   +8.00       |
ec8   8.00  91.94  38.31  +53.63 | 5.00  10.00   +8.00       |
en3   6.00  46.78  30.70  +16.08 | 5.00  10.00   +6.00       |
en5   6.00  27.25  35.52   -8.27 | 5.00  10.00   +0.00       |
en6   6.00   1.65  69.71  -68.06 | 5.00  10.00   -6.00       |
en9   6.00  14.85  13.10   +1.75 | 5.00  10.00   +6.00       |
so2  12.00  58.85  68.06   -9.21 | 5.00  10.00   +0.00       |
so7  12.00  15.49  58.45  -42.96 | 5.00  10.00  -12.00       |
Valuation in range: -72.00 to +72.00; global concordance: +18.00
```

In both cases, choosing significance weights that are just compatible with the given weights preorder will not always result in positively validated outranking situations.

Computing the stability denotation of outranking situations

Stability levels 4 and 3 are easy to detect, the case given. Detecting a stability level 2 is far less obvious. Now, it is precisely again the bipolar-valued epistemic characteristic domain that will give us a way to implement an effective test for stability level +2 and -2 (see [BIS-2004_1p], [BIS-2004_2p]).

Let us consider the significance equivalence classes we observe in the given weights preorder. Here we observe three classes: 6, 8, and 12, in increasing order (see Listing 1.2). In the pairwise comparisons shown above these equivalence classes may appear positively or negatively, besides the indeterminate significance of value *0*. We thus get the following ordered bipolar list of significance weights:

W = [-12. -8. -6, 0, 6, 8, 12].

In all the pairwise marginal comparisons shown in the previous Section, we may observe that each one of the nine criteria assigns one precise item out of this list *W*. Let us denote *q[i]* the number of criteria assigning item *W[i]*, and *Q[i]* the cumulative sums of these *q[i]* counts, where i is an index in the range of the length of list *W*.

In the comparison of alternatives *a2* and *a1*, for instance (see Listing 1.7), we observe the following counts:

W[i]	-12	-8	-6	0	6	8	12
q[i]	0	0	2	1	1	3	2
Q[i]	0	0	2	3	4	7	9

Let use denote -q and -Q the reversed versions of the q and the Q lists. We thus obtain the following result.

W[i]	-12	-8	-6	0	6	8	12
-*q[i]*	2	3	1	1	2	0	0
-*Q[i]*	2	5	6	7	9	9	9

Now, a pairwise outranking situation will be qualified at stability level +2, i.e. positively validated with any significance weights that are compatible with the given weights preorder, when for all i, we observe *Q[i]* <= -*Q[i]* and there exists one i such that *Q[i]* < -*Q[i]*. Similarly, a pairwise outranked situation will be qualified at stability level -2, when for all i, we observe *Q[i]* >= -*Q[i]* and there exists one i such that *Q[i]* > -*Q[i]* (see [BIS-2004_2p]).

We may verify, for instance, that the outranking situation observed between *a2* and *a1* does indeed verify this *first order distributional dominance* condition.

W[i]	-12	-8	-6	0	6	8	12
Q[i]	0	0	2	3	4	7	9
-*Q[i]*	2	5	6	7	9	9	9

Notice that outranking situations qualified at stability levels 4 and 3, evidently also verify the stability level 2 test above. The outranking situation between alternatives *a7* and *a3* does not, however, verify this test (see Listing 1.8).

W[i]	-12	-8	-6	0	6	8	12
q[i]	0	3	1	0	3	0	2
Q[i]	0	3	4	4	7	7	9
-*Q[i]*	2	2	5	5	6	9	9

This time, *not* all the *Q[i]* are *lower or equal* than the corresponding -*Q[i]* terms. Hence the outranking situation between *a7* and *a3* is not positively validated with all potential significance weights that are compatible with the given weights preorder.

Using this stability denotation, we may, hence, define the following **robust** version of a bipolar-valued outranking digraph.

Robust bipolar-valued outranking digraphs

We say that decision alternative x **robustly outranks** decision alternative y when

- x positively outranks y at stability level *higher or equal to 2* and we may not observe any *considerable counter-performance* of x on a discordant criterion.

Dually, we say that decision alternative x **does not robustly outrank** decision alternative y when

- x negatively outranks y at stability level *lower or equal to -2* and we may not observe any considerable *better performance* of x on a discordant criterion.

The corresponding *robust* outranking digraph may be computed with the `RobustOutrankingDigraph` class as follows.

Listing 1.9: Robust outranking digraph

```
>>> from outrankingDigraphs import RobustOutrankingDigraph
>>> rg = RobustOutrankingDigraph(t) # same t as before
>>> rg
 *------- Object instance description ------*
 Instance class      : RobustOutrankingDigraph
 Instance name       : robust_random3ObjectivesPerfTab
 # Actions           : 7
 # Criteria          : 9
 Size                : 22
 Determinateness (%) : 68.45
 Valuation domain    : [-1.00;1.00]
 Attributes          : ['name', 'methodData', 'actions', 'order',
                        'criteria', 'evaluation', 'vetos',
                        'valuationdomain', 'cardinalRelation',
                        'ordinalRelation', 'equisignificantRelation',
                        'unanimousRelation', 'relation',
                        'gamma', 'notGamma']
>>> rg.showRelationTable(StabilityDenotation=True)
 * ---- Relation Table -----
 r/(stab) |  'p1'   'p2'   'p3'   'p4'   'p5'   'p6'   'p7'
 ---------|------------------------------------------------------------
   'p1'   | +1.00  -0.42  +0.00  -0.69  +0.39  +0.11  +0.00
          |  (+4)   (-2)   (+0)   (-3)   (+2)   (+2)   (-1)
   'p2'   | +0.58  +1.00  +0.83  +0.00  +0.58  +0.58  +0.58
          |  (+2)   (+4)   (+3)   (+2)   (+2)   (+2)   (+2)
   'p3'   | +0.25  -0.33  +1.00  +0.00  +0.50  +1.00  +0.00
          |  (+2)   (-2)   (+4)   (+0)   (+2)   (+2)   (+1)
   'p4'   | +0.78  +0.00  +0.61  +1.00  +1.00  +1.00  +0.67
```

(continues on next page)

(continued from previous page)

```
29             |  (+3)   (-1)   (+3)   (+4)   (+4)   (+4)   (+2)
30     'p5'    | -0.11  -0.50  -0.25  -0.89  +1.00  +0.11  -0.14
31             |  (-2)   (-2)   (-2)   (-3)   (+4)   (+2)   (-2)
32     'p6'    | +0.22  -0.42  +0.00  -1.00  +0.17  +1.00  -0.11
33             |  (+2)   (-2)   (+1)   (-2)   (+2)   (+4)   (-2)
34     'p7'    | +0.22  -0.50  +0.00  +0.00  +0.78  +0.42  +1.00
35             |  (+2)   (-2)   (+1)   (-1)   (+3)   (+2)   (+4)
```

We may notice that all outranking situations, qualified at stability level +1 or -1, are now put to an *indeterminate* status. In the example here, we actually drop three positive outrankings: between *p3* and *p7*, between *p7* and *p3*, and between *p6* and *p3*, where the last situation is already put to doubt by a veto situation (see Listing 1.9 Lines 22-35). We drop as well three negative outrankings: between *p1* and *p7*, between *p4* and *p2*, and between *p7* and *p4* (see Listing 1.9 Lines 22-35).

Notice by the way that outranking (resp. outranked) situations, although qualified at level +2 or +3 (resp. -2 or -3) may nevertheless be put to doubt by considerable performance differences. We may observe such an outranking situation when comparing, for instance, alternatives *p2* and *p4* (see Listing 1.9 Lines 24-25).

Listing 1.10: Comparing alternatives *p2* and *p4*

```
1  >>> rg.showPairwiseComparison('p2','p4')
2   *------------  pairwise comparison ----*
3   Comparing actions : (p2, p4)
4   crit. wght.  g(x)  g(y)    diff      | ind   pref    r()      |    v    veto
5   --------------------------------------------------------------------
6   ec1    8.00  89.77  85.19  +4.58      | 5.00  10.00   +8.00    |
7   ec4    8.00  86.00  72.26  +13.74     | 5.00  10.00   +8.00    |
8   ec8    8.00  89.43  44.62  +44.81     | 5.00  10.00   +8.00    |
9   en3    6.00  20.79  80.81  -60.02     | 5.00  10.00   -6.00    | 60.00 -1.00
10  en5    6.00  23.83  49.69  -25.86     | 5.00  10.00   -6.00    |
11  en6    6.00  18.66  66.21  -47.55     | 5.00  10.00   -6.00    |
12  en9    6.00  26.65  50.92  -24.27     | 5.00  10.00   -6.00    |
13  so2    12.00  89.12  49.05  +40.07    | 5.00  10.00  +12.00    |
14  so7    12.00  84.73  55.57  +29.16    | 5.00  10.00  +12.00    |
15  Valuation in range: -72.00 to +72.00; global concordance: +24.00
```

Despite being robust, the apparent positive outranking situation between alternatives *p2* and *p4* is indeed put to doubt by a considerable counter-performance (-60.02) of *p2* on criterion *en3*, a negative difference which exceeds slightly the assumed veto discrimination threshold $v = 60.00$ (see Listing 1.10 Line 9).

We may finally compare in Fig. 1.5 the *standard* and the *robust* version of the corresponding strict outranking digraphs, both oriented by their respective identical initial and terminal prekernels.

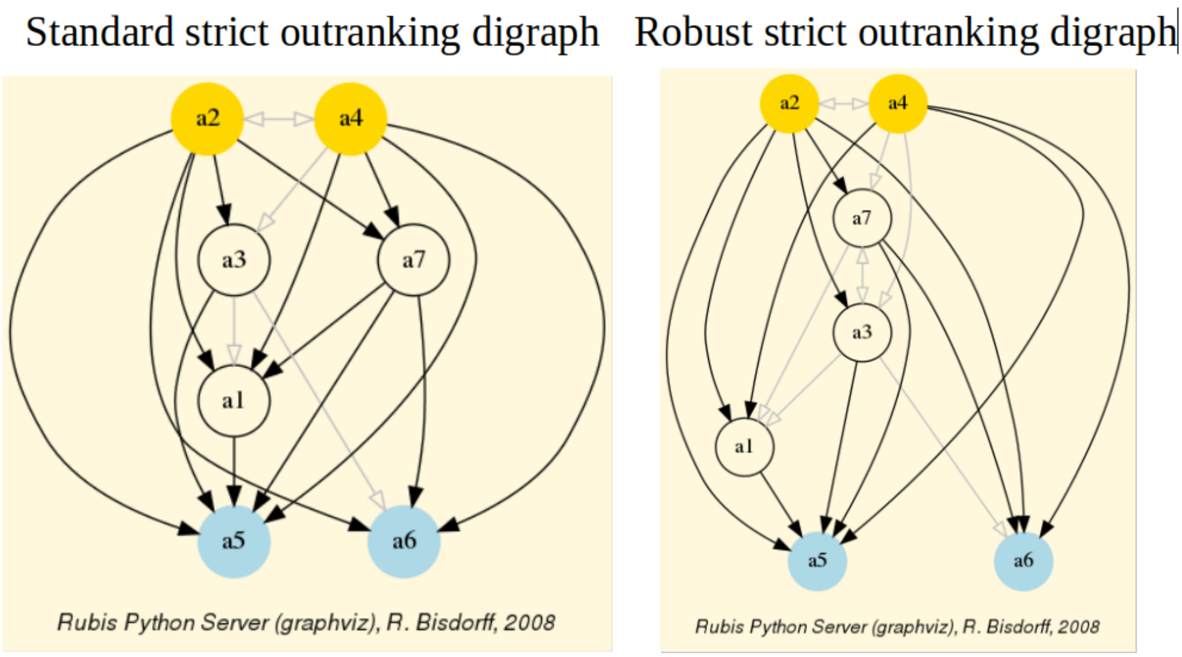


Fig. 1.5: Standard versus robust strict outranking digraphs oriented by their initial and terminal prekernels

The robust version drops two strict outranking situations: between *p4* and *p7* and between *p7* and *p1*. The remaining 14 strict outranking (resp. outranked) situations are now all verified at a stability level of +2 and more (resp. -2 and less). They are, hence, only depending on potential significance weights that must respect the given significance preorder (see Listing 1.2).

To appreciate the apparent orientation of the standard and robust strict outranking digraphs shown in Fig. 1.5, let us have a final heat map view on the underlying performance tableau ordered by the *NetFlows* ranking rule.

```
>>> t.showHTMLPerformanceHeatmap(Correlations=True,
...                              rankingRule='NetFlows')
```

Heatmap of Performance Tableau 'random3ObjectivesPerfTab'

criteria	ec4	so2	ec1	ec8	en9	en3	so7	en6	en5
weights	+8.00	+12.00	+8.00	+8.00	+6.00	+6.00	+12.00	+6.00	+6.00
tau(*)	+0.55	+0.52	+0.45	+0.38	+0.31	+0.29	+0.24	+0.05	+0.05
p4	72.26	49.05	85.19	44.62	50.92	80.81	55.57	66.21	49.69
p2	86.00	89.12	89.77	89.43	26.65	20.79	84.73	18.66	23.83
p3	68.70	58.85	80.19	91.94	14.85	46.78	15.49	1.65	27.25
p7	36.31	68.06	15.33	38.31	13.10	30.70	58.45	69.71	35.52
p1	22.65	22.43	38.11	77.02	44.37	58.16	28.41	11.41	31.40
p6	75.76	48.59	21.06	29.63	32.96	12.54	26.40	36.04	43.09
p5	8.96	12.36	46.75	35.91	9.83	31.05	44.92	31.22	29.52

Color legend:

quantile	14.29%	28.57%	42.86%	57.14%	71.43%	85.71%	100.00%

(*) tau: *Ordinal (Kendall) correlation between marginal criterion and global ranking relation*
Outranking model: **standard**, *Ranking rule*: **NetFlows**
Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.942**
Mean marginal correlation (a) : **+0.338**

Fig. 1.6: Heat map of the random 3 objectives performance tableau ordered by the *NetFlows* ranking rule

As the initial prekernel is here validated at stability level +2, recommending alternatives *p4*, as well as *p2*, as potential first choices, appears well justified. Alternative *a4* represents indeed an overall *best compromise choice* between all decision objectives, whereas alternative *p2* gives an unanimous best choice with respect to two out of three decision objectives. Up to the decision maker to make his final choice.

For concluding, let us mention that it is precisely again our bipolar-valued *logical characteristic framework* that provides us here with a **first order distributional dominance** test for effectively qualifying the stability level 2 *robustness* of an outranking digraph when facing performance tableaux with criteria of only ordinal-valued significance weights. A real world application of our stability analysis with such a kind of performance tableau may be consulted in [BIS-2015p].

Back to *Content Table* (page 1)

1.3 On unopposed outrankings with multiple decision objectives

- *Characterising unopposed multiobjective outranking situations* (page 23)
- *Computing unopposed multiobjective choice recommendations* (page 26)

When facing a performance tableau involving multiple decision objectives, the robustness level +/-**3**, introduced in the previous Section, may lead to distinguishing what we call **unopposed** outranking situations, like the one shown between alternative *p4* and *p1* ($r(p4 \succsim p1) = +0.78$, see Listing 1.4 Line11), namely preferential situations that are more or less validated or invalidated by all the decision objectives.

Characterising unopposed multiobjective outranking situations

Formally, we say that decision alternative x **outranks** decision alternative y **unopposed** when

- x positively outranks y on one or more decision objective without x being positively outranked by y on any decision objective.

Dually, we say that decision alternative x **does not outrank** decision alternative y **unopposed** when

- x is positively outranked by y on one or more decision objective without x outranking y on any decision objective.

Let us reconsider, for instance, the previous performance tableau with three decision objectives (see Listing 1.1):

Listing 1.11: Performance tableau with three decision objectives

```
1  >>> from randomPerfTabs import\
2  ...             Random3ObjectivesPerformanceTableau
3
4  >>> t = Random3ObjectivesPerformanceTableau(
5  ...             numberOfActions=7,
6  ...             numberOfCriteria=9,seed=102)
7
8  >>> t.showObjectives()
9   *------ show objectives -------"
10  Eco: Economical aspect
11   ec1 criterion of objective Eco 8
12   ec4 criterion of objective Eco 8
13   ec8 criterion of objective Eco 8
14  Total weight: 24.00 (3 criteria)
15  Soc: Societal aspect
16   so2 criterion of objective Soc 12
17   so7 criterion of objective Soc 12
18  Total weight: 24.00 (2 criteria)
19  Env: Environmental aspect
20   en3 criterion of objective Env 6
21   en5 criterion of objective Env 6
22   en6 criterion of objective Env 6
23   en9 criterion of objective Env 6
24  Total weight: 24.00 (4 criteria)
```

We notice in this example three decision objectives of equal importance (see Listing 1.11 Lines 10,15,19). What will be the outranking situations that are positively (resp. negatively) validated for each one of the decision objectives taken individually ?

We may obtain such *unopposed multiobjective* outranking situations by operating an **epistemic o-average fusion** (see the `~digraphsTools.symmetricAverage` method) of the marginal outranking digraphs restricted to the coalition of criteria supporting each one of the decision objectives (see Listing 1.12 below).

Listing 1.12: Computing unopposed outranking situations

```
1  >>> from outrankingDigraphs import BipolarOutrankingDigraph
2  >>> geco = BipolarOutrankingDigraph(t,objectivesSubset=['Eco'])
3  >>> gsoc = BipolarOutrankingDigraph(t,objectivesSubset=['Soc'])
4  >>> genv = BipolarOutrankingDigraph(t,objectivesSubset=['Env'])
5  >>> from digraphs import FusionLDigraph
6  >>> objectiveWeights = \
7  ...   [t.objectives[obj]['weight'] for obj in t.objectives]
8  
9  >>> uopg = FusionLDigraph([geco,gsoc,genv],
10 ...                 operator='o-average',
11 ...                 weights=objectiveWeights)
12 
13 >>> uopg.showRelationTable(ReflexiveTerms=False)
14 * ---- Relation Table -----
15  r   |  'p1'   'p2'   'p3'   'p4'   'p5'   'p6'   'p7'
16 -----|------------------------------------------------------------
17 'p1' |    -   +0.00  +0.00  -0.69  +0.39  +0.11  +0.00
18 'p2' | +0.00    -    +0.83  +0.00  +0.00  +0.00  +0.00
19 'p3' | +0.00  -0.33    -    +0.00  +0.50  +0.00  +0.00
20 'p4' | +0.78  +0.00  +0.61    -    +1.00  +1.00  +0.67
21 'p5' | -0.11  +0.00  +0.00  -0.89    -    +0.11  +0.00
22 'p6' | +0.00  +0.00  +0.00  -0.44  +0.17    -    +0.00
23 'p7' | +0.00  +0.00  +0.00  +0.00  +0.78  +0.42    -
24 Valuation domain: [-1.000; 1.000]
```

Positive (resp. negative) $r(x \succsim y)$ characteristic values, like $r(p1 \succsim p5) = 0.39$ (see Listing 1.12 Line 17), show hence only outranking situations being validated (resp. invalidated) by one or more decision objectives without being invalidated (resp. validated) by any other decision objective.

For easily computing this kind of *unopposed multiobjective* outranking digraphs, the `outrankingDigraphs module` conveniently provides a corresponding `UnOpposedBipolarOutrankingDigraph` constructor.

Listing 1.13: Unopposed outranking digraph constructor

```
>>> from outrankingDigraphs import\
...         UnOpposedBipolarOutrankingDigraph

>>> uopg = UnOpposedBipolarOutrankingDigraph(t)
>>> uopg
 *------- Object instance description ------*
 Instance class       : UnOpposedBipolarOutrankingDigraph
 Instance name        : unopposed_outrankings
```

(continues on next page)

(continued from previous page)

```
9  # Actions              : 7
10 # Criteria             : 9
11 Size                   : 13
12 Oppositeness (%)       : 43.48
13 Determinateness (%)    : 61.71
14 Valuation domain       : [-1.00;1.00]
15 Attributes             : ['name', 'actions', 'valuationdomain', 'objectives',
16                             'criteria', 'methodData', 'evaluation', 'order',
17                             'runTimes', 'relation', 'marginalRelationsRelations',
18                             'gamma', 'notGamma']
19 >>> uopg.computeOppositeness(InPercents=True)
20  {'standardSize': 23, 'unopposedSize': 13,
21   'oppositeness': 43.47826086956522}
```

The resulting *unopposed* outranking digraph keeps in fact 13 (see Listing 1.13 Lines 12-13) out of the 23 positively validated *standard* outranking situations, leading to a degree of **oppositeness** -preferential disagreement between decision objectives- of $(1.0 - 13/23) = 0.4348$.

We may now, for instance, verify the unopposed status of the outranking situation observed between alternatives *p1* and *p5*.

Listing 1.14: Example of unopposed multiobjective outranking situation

```
>>> uopg.showPairwiseComparison('p1','p5')
 *------------  pairwise comparison ----*
 Comparing actions : (p1, p5)
 crit. wght.  g(x)  g(y)    diff      | ind   pref    r()      |
 ec1   8.00  38.11  46.75  -8.64      | 5.00  10.00   +0.00    |
 ec4   8.00  22.65  8.96  +13.69      | 5.00  10.00   +8.00    |
 ec8   8.00  77.02  35.91  +41.11     | 5.00  10.00   +8.00    |
 en3   6.00  58.16  31.05  +27.11     | 5.00  10.00   +6.00    |
 en5   6.00  31.40  29.52  +1.88      | 5.00  10.00   +6.00    |
 en6   6.00  11.41  31.22  -19.81     | 5.00  10.00   -6.00    |
 en9   6.00  44.37  9.83  +34.54      | 5.00  10.00   +6.00    |
 so2   12.00  22.43  12.36  +10.07    | 5.00  10.00   +12.00   |
 so7   12.00  28.41  44.92  -16.51    | 5.00  10.00   -12.00   |
  Valuation in range: -72.00 to +72.00; global concordance: +28.00
```

In Listing 1.14 we see that alternative *p1* does indeed positively outrank alternative *p5* from the economic perspective $(r(p1 \succsim_{Eco} p5) = +16/24)$ as well as from the environmental perspective $(r(p1 \succsim_{Env} p5) = +12/24)$. Whereas, from the societal perspective, both alternatives appear incomparable $(r(p1 \succsim_{Soc} p5) = 0/24)$.

When fixed proportional criteria significance weights per objective are given, these outranking situations appear hence **stable** with respect to all possible importance weights we could allocate to the decision objectives.

This gives way for computing multiobjective choice recommendations.

Computing unopposed multiobjective choice recommendations

Indeed, best choice recommendations, computed from an *unopposed multiobjective* outranking digraph, will in fact deliver **efficient** choice recommendations.

Listing 1.15: Efficient multiobjective choice recommendation

```
1  >>> uopg.showBestChoiceRecommendation()
2   Best choice recommendation(s) (BCR)
3    (in decreasing order of determinateness)
4   Credibility domain: [-1.00,1.00]
5    === >> potential first choice(s)
6    choice              : ['p2', 'p4', 'p7']
7     independence        : 0.00
8     dominance           : 0.33
9     absorbency          : 0.00
10    covering (%)        : 33.33
11    determinateness (%) : 50.00
12   === >> potential last choice(s)
13   choice              : ['p3', 'p5', 'p6', 'p7']
14    independence        : 0.00
15    dominance           : -0.61
16    absorbency          : 0.11
17    covered (%)         : 33.33
18    determinateness (%) : 50.00
```

Our previous *robust best choice recommendation* (*p2* and *p4*, see Fig. 1.5) remains, in this example here, **stable**. We recover indeed the best choice recommendation ['p2', 'p4', 'p7'] (see Listing 1.15 Line 6). Yet, notice that decision alternative *p7* appears to be at the same time a potential *first* as well as a potential *last* choice recommendation (see Line 13), a consequence of *p7* being completely *incomparable* to the other decision alternatives when restricting the comparability to only unopposed strict outranking situations.

We may visualize this kind of **efficient** choice recommendation in Fig. 1.7 below.

```
>>> (~(-uopg)).exportGraphViz(fileName = 'unopDigraph',
...                 firstChoice = ['p2', 'p4'],
...                 lastChoice = ['p3', 'p5', 'p6'])
 *---- exporting a dot file for GraphViz tools ---------*
  Exporting to unopDigraph.dot
  dot -Grankdir=BT -Tpng unopDigraph.dot -o unopDigraph.png
```

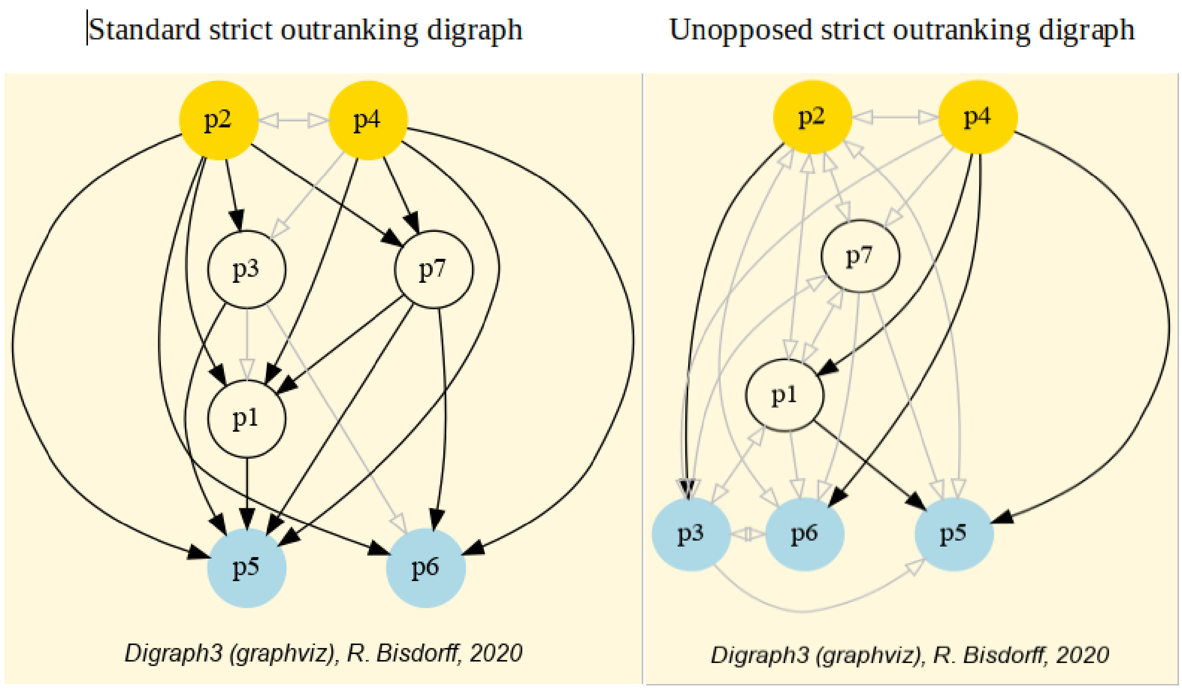


Fig. 1.7: Standard versus *unopposed* strict outranking digraphs oriented by first and last choice recommendations

In order to make now an eventual best unique choice, a decision maker will necessarily have to weight, in a second stage of the decision aiding process, the relative importance of the individual decision objectives (see tutorial on computing a best choice recommendation).

Back to *Content Table* (page 1)

---

2 Enhancing social choice procedures

- *Condorcet's critical perspective on the simple plurality voting rule* (page 27)
- *Two-stage elections with multipartisan primary selection* (page 36)
- *Tempering plurality tyranny effects with bipolar approval voting* (page 45)
- *Selecting the winner of a primary election: a critical commentary* (page 59)

2.1 *Condorcet*'s critical perspective on the simple plurality voting rule

> "*In order to meet both essential conditions for making* [social] *choices –the probability to obtain a decision & the one that the decision may be correct– it is required* [...], *in case of decisions on complicated questions, to thouroughly*

develop the system of simple propositions that make them up, that every potential opinion is well explained, that the opinion of each voter is collected on each one of the propositions that make up each question & not only on the global result." – J.-A. N. Condorcet (1785)[12]

- *Bipolar approval voting of motions* (page 28)
- *Who wins the election?* (page 30)
- *Resolving circular social preferences* (page 32)
- *The* Borda *rank analysis method* (page 35)

In his seminal 1785 critical perspective on simple plurality voting rules for solving social choice problems, *Condorcet* developed several case studies for supporting his analysis. A first case concerns the decision to be taken by a Committee on two motions ([CON-1785p] P. xlvij).

Bipolar approval voting of motions

Suppose that an Assembly of 33 voters has to decide on two motions A and B. 11 voters are in favour of both, 10 voters support A and reject B, 3 voters reject A and support B, and 9 voters reject both. Following naively a simple plurality rule, the decision of the Assembly would be to accept both motion A and motion B, as a plurality of 11 voters apparently supports them both. Is this the correct social decision?

To investigate the question, we model the given preference data in the format of a `BipolarApprovalVotingProfile` object. The corresponding content, shown in Listing 2.1, is contained in a file named *condorcet1.py* to be found in the *examples* directory of the Digraph3 resources.

Listing 2.1: Bipolar approval-disapproval voting profile

```
# BipolarApprovalVotingProfile:
# Condorcet 1785, p. lviij
from collections import OrderedDict
candidates = OrderedDict([
('A', {'name': 'A'}),
('B', {'name': 'B'}) ])
voters = OrderedDict([
('v1', {'weight':11}),
('v2', {'weight':10}),
('v3', {'weight': 3}),
('v4', {'weight': 9}) ])
```

(continues on next page)

[12] "*Pour réunir les deux conditions essentielles à toute décision* [publique], *la probabilité d'avoir une décision, & celle que la décision obtenue sera vraie, il faut* [. . .] *dans le cas des décisions sur des questions compliquées, faire en sorte que le système des propositions simples qui les forment soit rigoureusement développé, que chaque avis possible soit bien exposé, que la voix de chaque Votant soit prise sur chacune des propositions qui forment cet avis, & non sur le résultat seul.*" [CON-1785p] P. lxix

(continued from previous page)

```
approvalBallot = {
'v1': {'A':  1,'B':  1},
'v2': {'A':  1,'B': -1},
'v3': {'A': -1,'B':  1},
'v4': {'A': -1,'B': -1} }
```

We can inspect this data with the `BipolarApprovalVotingProfile` class, as shown in Listing 2.2 Line 3 below.

Listing 2.2: Bipolar approval-disapproval voting profile

```
1  >>> from votingProfiles import\
2  ...                      BipolarApprovalVotingProfile
3  >>> v1 = BipolarApprovalVotingProfile('condorcet1')
4  >>> v1
5   *------- VotingProfile instance description ------*
6    Instance class    : BipolarApprovalVotingProfile
7    Instance name     : condorcet1
8    Candidates        : 2
9    Voters            : 4
10   Attributes        : ['name', 'candidates', 'voters',
11         'approvalBallot', 'netApprovalScores', 'ballot']
12  >>> v1.showApprovalResults()
13   Approval results
14    Candidate: A obtains 21 votes
15    Candidate: B obtains 14 votes
16   Total approval votes: 35
17  >>> v1.showDisapprovalResults()
18   Disapproval results
19    Candidate: A obtains 12 votes
20    Candidate: B obtains 19 votes
21   Total disapproval votes: 31
22  >>> v1.showNetApprovalScores()
23   Net Approval Scores
24    Candidate: A obtains 9 net approvals
25    Candidate: B obtains -5 net approvals
```

Actually, a majority of 60% supports motion A (21/35, see Line 14) whereas a majority of 54% rejects motion B (19/35, see Line 20). The simple plurality rule violates thus clearly the voters actual preferences. The *correct* decision —accepting A and rejecting B as promoted by *Condorcet*– is indeed correctly modelled by the net approval scores obtained by both motions (see Lines 24-25).

A second example of incorrect simple plurality rule results, developed by *Condorcet* in 1785, concerns uninominal general elections ([CON-1785p] P. lviij)

Who wins the election?

Suppose an Assembly of 60 voters has to select a winner among three potential candidates A, B, and C. 23 voters vote for A, 19 for B and 18 for C. Suppose furthermore that the 23 voters voting for A prefer C over B, the 19 voters voting for B prefer C over A and among the 18 voters voting for C, 16 prefer B over A and only 2 prefer A over B.

We may organize this data in the format of the following `LinearVotingProfile` object.

Listing 2.3: Linear voting profile

```
from collections import OrderedDict
candidates = OrderedDict([
('A', {'name': 'Candidate A'}),
('B', {'name': 'Candidate B'}),
('C', {'name': 'Candidate C'}) ])
voters = OrderedDict([
('v1', {'weight':23}),
('v2', {'weight':19}),
('v3', {'weight':16}),
('v4', {'weight':2}) ])
linearBallot = {
'v1': ['A','C','B'],
'v2': ['B','C','A'],
'v3': ['C','B','A'],
'v4': ['C','A','B'] }
```

With an uninominal plurality rule, it is candidate A who is elected. Is this decision correctly reflecting the actual preference of the Assembly ?

The linear voting profile shown in Listing 2.3 is contained in a file named *condorcet2.py* provided in the *examples* directory of the Digraph3 resources. With the `LinearVotingProfile` class, this file may be inspected as follows.

Listing 2.4: Computing the winner

```
>>> from votingProfiles import\
...                     LinearVotingProfile
>>> v2 = LinearVotingProfile('condorcet2')
>>> v2.showLinearBallots()
  voters           marginal
  (weight)    candidates rankings
   v1(23):    ['A', 'C', 'B']
   v2(19):    ['B', 'C', 'A']
   v3(16):    ['C', 'B', 'A']
   v4( 2):    ['C', 'A', 'B']
  Nbr of voters:  60.0
>>> v2.computeUninominalVotes()
 {'A': 23, 'B': 19, 'C': 18}
>>> v2.computeSimpleMajorityWinner()
```

(continues on next page)

(continued from previous page)

```
15  ['A']
16 >>> v2.computeInstantRunoffWinner(Comments=True)
17  Total number of votes =  60.000
18  Half of the Votes =  30.00
19   ==> stage =  1
20      remaining candidates ['A', 'B', 'C']
21      uninominal votes {'A': 23, 'B': 19, 'C': 18}
22      minimal number of votes =  18
23      maximal number of votes =  23
24      candidate to remove =  C
25      remaining candidates =  ['A', 'B']
26   ==> stage =  2
27      remaining candidates ['A', 'B']
28      uninominal votes {'A': 25, 'B': 35}
29      minimal number of votes =  25
30      maximal number of votes =  35
31      candidate B obtains an absolute majority
32  ['B']
```

In ordinary elections, only the votes for first-ranked candidates are communicated and counted, so that candidate A with a plurality of 23 votes would actually win the election. As A does not obtain an absolute majority of votes (23/60 38.3%), it is often common practice to organise a runoff voting. In this case, candidate C with the lowest uninominal votes will be eliminated in the first stage (see Line 24). If the voters do not change their preferences in between the election stages, candidate B eventually wins against A with a 58.3% (35/60) majority of votes (see Line 31). Is candidate B now a more convincing winner than candidate A ?

Disposing supposedly here of a complete linear voting profile, *Condorcet*, in order to answer this question, recommends to compute an election result for all 6 pairwise comparisons of the candidates. This may be done with the `MajorityMarginsDigraph` class constructor as shown in Listing 2.5.

Listing 2.5: Computing the Condorcet winner

```
>>> from votingProfiles import\
...                 MajorityMarginsDigraph
>>> mm = MajorityMarginsDigraph(v2)
>>> mm.showMajorityMargins()
 * ---- Relation Table -----
   S   |  'A'  'B'   'C'
 ------|-----------------
  'A'  |    0  -10   -14
  'B'  |  +10    0   -22
  'C'  |  +14  +22     0
 Valuation domain: [-60;+60]
>>> mm.computeCondorcetWinners()
```

(continues on next page)

(continued from previous page)

```
13  ['C']
```

In a pairwise competition, candidate C beats both candidate A with a majority of 61.5% (37/60) as well as candidate B with a majority of 68.3% (41/60). Candidate C represents in fact the absolute majority supported candidate. C is what we call now a *Condorcet Winner* (see Lines 10 and 13 above).

Yet, is *Condorcet*'s approach always a decisive social choice rule?

Resolving circular social preferences

Let us this time suppose that the 23 voters voting for A prefer B over C, that the 19 voters voting for B prefer C over A, and that the 18 voters voting for C actually prefer A over B.

This resulting linear voting profile, as shown in Listing 2.6, is contained in a file named *condorcet3.py* provided in the *examples* directory of the Digraph3 resources and may be inspected as follows.

Listing 2.6: A circular linear voting profile

```
1   >>> from votingProfiles import\
2   ...                     LinearVotingProfile
3   >>> v3 = LinearVotingProfile('condorcet3')
4   >>> v3.showLinearBallots()
5     voters           marginal
6     (weight)    candidates rankings
7      v1(23):     ['A', 'B', 'C']
8      v2(19):     ['B', 'C', 'A']
9      v3(18):     ['C', 'A', 'B']
10    Nbr of voters:  60.0
11  >>> v3.computeSimpleMajorityWinner()
12   ['A']
13  >>> v3.computeInstantRunoffWinner()
14   ['A']
15  >>> m3 = MajorityMarginsDigraph(v3)
16  >>> m3.showMajorityMargins()
17   *---- Relation Table -----
18      S   |  'A'  'B'  'C'
19    ------|----------------
20     'A'  |   0   +24  -22
21     'B'  |  -24   0   +14
22     'C'  |  +22  -14   0
23    Valuation domain: [-60;+60]
```

We may notice in Listing 2.6 Lines 7-9 that we thus circularly swap in each linear ranking the first with the last candidate. This time, the majority margins do not show anymore a *Condorcet* winner (see Lines 20-22) and the plurality supported social preferences appear to be circular as illustrated in Fig. 2.1:

```
>>> m3.exportGraphViz('circularPreference')
 *---- exporting a dot file for GraphViz tools ---------*
  Exporting to circularPreference.dot
  dot -Grankdir=BT -Tpng circularPreference.dot\
               -o circularPreference.png
```

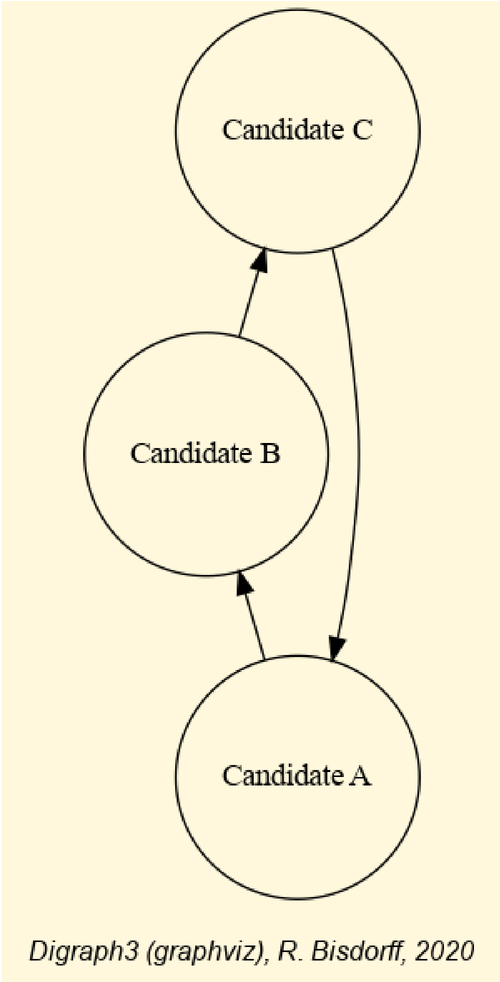


Fig. 2.1: Circular majority margins

Condorcet did recognize this potential failure of the decisiveness of his approach and proposed, in order to effectively solve such a circular decision problem, a kind of prudent *RankedPairs* rule where a potential majority margins circuit is broken up at its weakest margin. In this example, the weakest positive majority margin in the apparent circuit $-C > A > B > C-$ is the last one, characterising $B > C$ (+14, see Listing 2.6 Line 21).

We may use the `RankedPairsRanking` class from the `linearOrders` module to apply such a rule to our majority margins digraph *m3* (see Listing 2.7).

Listing 2.7: Prudent ranked pairs rule based ranking

```
>>> from linearOrders import RankedPairsRanking
>>> rp = RankedPairsRanking(m3,Comments=True)
```

(continues on next page)

(continued from previous page)

```
3  Starting the ranked pairs rule with the following partial order:
4  * ---- Relation Table -----
5    S   |  'A'   'B'   'C'
6  ------|------------------
7   'A' |  0.00  0.00  0.00
8   'B' |  0.00  0.00  0.00
9   'C' |  0.00  0.00  0.00
10 Valuation domain: [-1.00;1.00]
11 (Decimal('48.0'), ('A', 'B'), 'A', 'B')
12 next pair:  ('A', 'B') 24.0
13 added: (A,B) characteristic: 24.00 (1.0)
14 added: (B,A) characteristic: -24.00 (-1.0)
15 (Decimal('44.0'), ('C', 'A'), 'C', 'A')
16 next pair:  ('C', 'A') 22.0
17 added: (C,A) characteristic: 22.00 (1.0)
18 added: (A,C) characteristic: -22.00 (-1.0)
19 (Decimal('28.0'), ('B', 'C'), 'B', 'C')
20 next pair:  ('B', 'C') 14.0
21 Circuit detected !!
22 (Decimal('-28.0'), ('C', 'B'), 'C', 'B')
23 next pair:  ('C', 'B') -14.0
24 added: (C,B) characteristic: -14.00 (1.0)
25 added: (B,C) characteristic: 14.00 (-1.0)
26 (Decimal('-44.0'), ('A', 'C'), 'A', 'C')
27 (Decimal('-48.0'), ('B', 'A'), 'B', 'A')
28 Ranked Pairs Ranking =  ['C', 'A', 'B']
```

The *RankedPairs* rule drops indeed the $B > C$ majority margin in favour of the converse $C > B$ situation (Lines 20-23) and delivers hence the linear ranking $C > A > B$ (Line 28). And, it is eventually candidate C –neither the uninominal simple plurality candidate nor the instant runoff winner (see Listing 2.6 Lines 11-14)– who is, despite the apparent circular social preference, still winning this sample election game.

Condorcet's last example concerns the *Borda* rule. The Chevalier *Jean-Charles de Borda*, geometer and French navy officer, contemporary colleague of *Condorcet* in the French "Academie des Sciences" correctly contested already in 1784 the actual decisiveness of *Condorcet*'s pairwise majority margins approach when facing circular social preferences. He proposed instead the now famous *rank analysis* method named after him[17].

[17] *Borda* (1733-1799) was an early and most active promoter of the introduction of an universal *metric* measurement system. He even elaborated a metric angle measurement system but eventually failed to convince his fellow geometers. See https://fr.wikipedia.org/wiki/Jean-Charles_de_Borda and [BRI-2008p]

The *Borda* rank analysis method

To defend his pairwise voting approach, *Condorcet* showed with a simple example that the *rank analysis* method may give a *Borda* winner who eliminates a candidate who is in fact supported by an absolute majority of voters[18]. He proposed therefore the following example of a linear voting profile, stored in a file named *condorcet4.py* available in the *examples* directory of the *Digraph3* resources.

```
1  >>> from votingProfiles import LinearVotingProfile
2  >>> lv = LinearVotingProfile('condorcet4')
3  >>> lv.showLinearBallots()
4    voters           marginal
5   (weight)     candidates rankings
6    v1(30):      ['A', 'B', 'C']
7    v2(1):       ['A', 'C', 'B']
8    v3(10):      ['C', 'A', 'B']
9    v4(29):      ['B', 'A', 'C']
10   v5(10):      ['B', 'C', 'A']
11   v6(1):       ['C', 'B', 'A']
12   # voters:  81.0
13 >>> lv.computeUninominalVotes()
14  {'A': 31, 'B': 39, 'C': 11}
```

In this example, the simple uninominal plurality winner, with a plurality of 39 votes, is Candidate *B* (see last Line above). When we apply now *Borda*'s rank analysis method we will indeed confirm this Candidate *B* with the smallest *Borda* score –$(39 \times 1) + (31 \times 2) + (11 \times 3) = 134$– as the actual *Borda* winner (see Line 6 below).

```
1 >>> lv.showRankAnalysisTable()
2  *----  Borda rank analysis tableau -----*
3   candi- | alternative-to-rank |       Borda
4   dates  |   1      2      3   | score   average
5   -------|-------------------------------------
6     'B'  |  39     31     11   |  134     1.65
7     'A'  |  31     39     11   |  142     1.75
8     'C'  |  11     11     59   |  210     2.59
```

However, if we compute the corresponding majority margins digraph, we get the following result.

```
>>> from votingProfiles import MajorityMarginsDigraph
>>> mm = MajorityMarginsDigraph(lv)
>>> mm.showRelationTable()
 * ---- Relation Table -----
   S    |  'A'  'B'  'C'
 ------|-----------------
  'A'   |    0   +1  +39
```

(continues on next page)

[18] [CON-1785p] P. clxxvij

(continued from previous page)

```
8   'B'  |  -1    0  +57
9   'C'  | -39  -57    0
10  Valuation domain: [-81;+81]
```

With solely positive pairwise majority margins, Candidate *A* beats in fact both the other two candidates with an absolute majority of votes (see Line 7 above) and gives the *Condorcet* winner. Candidate *A* is hence in this example a more convincing election winner than the one that would result from *Borda*'s rank analysis method and from the uninominal plurality rule.

Could different integer weights allocated to each rank position avoid such a failure of *Borda*'s method? No, as convincingly shown by Condorcet with the help of this example. Indeed, Candidate *A* is 8 times more often than Candidate *B* in the second rank position (39 - 31), whereas Candidate *B* is 8 times more often than Candidate *A* in the first rank position (39 - 31). On the third rank position they both obtain the same score 11 (see Lines 6-7 in the rank analysis table above). As the weight of a first rank must in any case be srictly lower than the weight of a second rank, there does not exist in this example any possible weighing of the rank positions that would make Candidate *A* win over Candidate *B*.

Condorcet did nonetheless aknowledge in his 1785 essay the actual merits of *Borda* and his rank analysis approach which he qualifies as *ingenious* and easy to put into practice[19].

Note

Mind that nearly 250 years after *Condorcet*, most of our modern election systems are still relying either on uninominal plurality rules like the UK Parliament elections or on multi-stage runoff rules like the two stage French presidential elections, which, as convincingly shown by *Condorcet* already in 1785, risk very often to do not deliver correct democratic decisions. No wonder that many of our modern democracies show difficulties to make well accepted social choices.

Back to *Content Table* (page 1)

2.2 Two-stage elections with multipartisan primary selection

- *Converting voting profiles into performance tableaux* (page 37)
- *Multipartisan primary selection of eligible candidates* (page 39)

[19] " … *j'ai cru devoir citer* [Borda], *1. parce qu'il est le premier qui ait observé que la méthode commune* [simple pluralité uninominale] *de faire des élections étoit défectueuse; 2. parce que celle qu'il a proposé d'y substituer est très ingénieuse, quelle seroit très-simple dans la pratique* … " [CON-1785p] P. clxxiX

- *Secondary election winner determination* (page 41)
- *Multipartisan preferences in divisive politics* (page 42)

In a *social choice* context, where decision objectives would match different political parties, *efficient multiobjective choice recommendations* represent in fact **multipartisan social choices** that could judiciously deliver the primary selection in a two stage election system.

To compute such efficient social choice recommendations we need to, first, convert a given linear voting profile (with polls) into a corresponding performance tableau.

Converting voting profiles into performance tableaux

We shall illustrate this point with a voting profile we discuss in the tutorial on generating random linear voting profiles.

Listing 2.8: Example of a 3 parties voting profile

```
>>> from votingProfiles import RandomLinearVotingProfile
>>> lvp = RandomLinearVotingProfile(numberOfCandidates=15,
...                                 numberOfVoters=1000,
...                                 WithPolls=True,
...                                 partyRepartition=0.5,
...                                 other=0.1,
...                                 seed=0.9189670954954139)

>>> lvp
 *------- VotingProfile instance description ------*
 Instance class   : RandomLinearVotingProfile
 Instance name    : randLinearProfile
 # Candidates     : 15
 # Voters         : 1000
 Attributes       : ['name', 'seed', 'candidates',
                     'voters', 'WithPolls', 'RandomWeights',
                     'sumWeights', 'poll1', 'poll2',
                     'other', partyRepartition,
                     'linearBallot', 'ballot']
>>> lvp.showRandomPolls()
 Random repartition of voters
  Party_1 supporters : 460 (46.0%)
  Party_2 supporters : 436 (43.6%)
  Other voters       : 104 (10.4%)
 *---------------- random polls ---------------
  Party_1(46.0%) | Party_2(43.6%)|  expected
 -----------------------------------------------
   a06 : 19.91%  | a11 : 22.94%  | a06 : 15.00%
   a07 : 14.27%  | a08 : 15.65%  | a11 : 13.08%
```

(continues on next page)

(continued from previous page)

```
30     a03 : 10.02%  | a04 : 15.07%  | a08 : 09.01%
31     a13 : 08.39%  | a06 : 13.40%  | a07 : 08.79%
32     a15 : 08.39%  | a03 : 06.49%  | a03 : 07.44%
33     a11 : 06.70%  | a09 : 05.63%  | a04 : 07.11%
34     a01 : 06.17%  | a07 : 05.10%  | a01 : 05.06%
35     a12 : 04.81%  | a01 : 05.09%  | a13 : 05.04%
36     a08 : 04.75%  | a12 : 03.43%  | a15 : 04.23%
37     a10 : 04.66%  | a13 : 02.71%  | a12 : 03.71%
38     a14 : 04.42%  | a14 : 02.70%  | a14 : 03.21%
39     a05 : 04.01%  | a15 : 00.86%  | a09 : 03.10%
40     a09 : 01.40%  | a10 : 00.44%  | a10 : 02.34%
41     a04 : 01.18%  | a05 : 00.29%  | a05 : 01.97%
42     a02 : 00.90%  | a02 : 00.21%  | a02 : 00.51%
```

In this example (see `linearVotingProfileWithPolls` Lines 18-), we obtained 460 Party_1 supporters (46%), 436 Party_2 supporters (43.6%) and 104 other voters (10.4%). Favorite candidates of *Party_1* supporters, with more than 10%, appeared to be *a06* (19.91%), *a07* (14.27%) and *a03* (10.02%). Whereas for *Party_2* supporters, favorite candidates appeared to be *a11* (22.94%), followed by *a08* (15.65%), *a04* (15.07%) and *a06* (13.4%).

We may convert this linear voting profile into a PerformanceTableau object where each party corresponds to a decision objective.

Listing 2.9: Converting a voting profile into a performance tableau

```
>>> lvp.save2PerfTab('votingPerfTab')
>>> from perfTabs import PerformanceTableau
>>> vpt = PerformanceTableau('votingPerfTab')
>>> vpt
 *------- PerformanceTableau instance description ------*
  Instance class    : PerformanceTableau
  Instance name     : votingPerfTab
  # Actions         : 15
  # Objectives      : 3
  # Criteria        : 1000
  Attributes        : ['name', 'actions', 'objectives',
                       'criteria', 'weightPreorder', 'evaluation']
>>> vpt.objectives
OrderedDict([
 ('party0', {'name': 'other', 'weight': Decimal('104'),
  'criteria': ['v0003', 'v0008', 'v0011', ... ']}),
 ('party1', {'name': 'party 1', 'weight': Decimal('460'),
  'criteria': ['v0002', 'v0006', 'v0007', ...]}),
 ('party2', {'name': 'party 2', 'weight': Decimal('436'),
   'criteria': ['v0001', 'v0004', 'v0005', ... ]})
```

(continues on next page)

(continued from previous page)

```
])
```

In Listing 2.9 we first store the linear voting in a `PerformanceTableau` format (see Line 1). In Line 3, we reload this performance tableau data. The three parties of the linear voting profile represent three decision objectives and the voters are distributed as performance criteria according to the party they support.

Multipartisan primary selection of eligible candidates

In order to make now a **primary multipartisan selection** of potential election winners, we compute the corresponding *unopposed multiobjective outranking* digraph.

Listing 2.10: Computing unopposed multiobjective outranking situations

```
>>> from outrankingDigraphs import \
...       UnOpposedBipolarOutrankingDigraph

>>> uog = UnOpposedBipolarOutrankingDigraph(vpt)
>>> uog
 *------- Object instance description ------*
  Instance class      : UnOpposedBipolarOutrankingDigraph
  Instance name       : unopposed_outrankings
  # Actions           : 15
  # Criteria          : 1000
  Size                : 34
  Oppositeness (%)    : 67.31
  Determinateness (%) : 57.61
  Valuation domain    : [-1.00;1.00]
  Attributes          : ['name', 'actions', 'valuationdomain',
                         'objectives', 'criteria', 'methodData',
                         'evaluation', 'order', 'runTimes', '
                        relation', 'marginalRelationsRelations',
                         'gamma', 'notGamma']
```

From the potential 105 pairwise outranking situations, we keep 34 positively validated outranking situations, leading to a degree of *oppositeness* between political parties of 67.31%.

We may visualize the corresponding bipolar-valued relation table by orienting the list of candidates with the help of the *initial* and the *terminal prekernels*.

Listing 2.11: Visualizing the unopposed outranking relation

```
>>> uog.showPreKernels()
 *--- Computing preKernels ---*
 Dominant preKernels :
```

(continues on next page)

(continued from previous page)

```
['a11', 'a06', 'a13', 'a15']
    independence :  0.0
    dominance    :  0.18
    absorbency   :  -0.66
    covering     :  0.43
Absorbent preKernels :
['a02', 'a04', 'a14', 'a03']
    independence :  0.0
    dominance    :  0.0
    absorbency   :  0.37
    covered      :  0.46
>>> orientedCandidatesList = ['a06','a11','a13','a15',
...          'a01','a05','a07','a08','a09','a10','a12',
...          'a02','a03','a04','a14']

>>> uog.showHTMLRelationTable(
...     actionsList=orientedCandidatesList,
...     tableTitle='Unopposed three-partisan outrankings')
```

Multipartisan outranking situations

r(x S y)	a06	a11	a13	a15	a01	a05	a07	a08	a09	a10	a12	a02	a03	a04	a14
a06	–	0.00	0.00	0.00	0.44	0.00	0.25	0.00	0.00	0.00	0.56	0.86	0.29	0.00	0.57
a11	0.00	–	0.00	0.00	0.00	0.55	0.00	0.18	0.59	0.51	0.39	0.80	0.00	0.42	0.47
a13	0.00	0.00	–	0.00	0.00	0.52	-0.27	0.00	0.00	0.00	0.00	0.77	0.00	0.00	0.16
a15	0.00	0.00	0.00	–	0.00	0.39	0.00	0.00	0.00	0.00	0.00	0.66	0.00	0.00	0.00
a01	-0.44	0.00	0.00	0.00	–	0.00	0.00	0.00	0.00	0.00	0.00	0.77	0.00	0.00	0.20
a05	0.00	-0.55	-0.52	-0.39	0.00	–	0.00	-0.47	0.00	-0.12	0.00	0.37	0.00	0.00	0.00
a07	-0.25	0.00	0.27	0.00	0.00	0.00	–	0.00	0.00	0.00	0.30	0.83	0.00	0.00	0.38
a08	0.00	-0.18	0.00	0.00	0.00	0.47	0.00	–	0.00	0.00	0.00	0.77	0.00	0.29	0.00
a09	0.00	-0.59	0.00	0.00	0.00	0.00	0.00	0.00	–	0.00	0.00	0.55	0.00	0.00	0.00
a10	0.00	-0.51	0.00	0.00	0.00	0.12	0.00	0.00	0.00	–	0.00	0.50	0.00	0.00	0.00
a12	-0.56	-0.39	0.00	0.00	0.00	0.00	-0.30	0.00	0.00	0.00	–	0.72	0.00	0.00	0.10
a02	-0.86	-0.80	-0.77	-0.66	-0.77	-0.37	-0.83	-0.77	-0.55	-0.50	-0.72	–	0.00	0.00	0.00
a03	-0.29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	–	0.00	0.00
a04	0.00	-0.42	0.00	0.00	0.00	0.00	0.00	-0.29	0.00	0.00	0.00	0.00	0.00	–	0.00
a14	-0.57	-0.47	-0.16	0.00	-0.20	0.00	-0.38	0.00	0.00	0.00	-0.10	0.00	0.00	0.00	–

Valuation domain: [-1.00; +1.00]

Fig. 2.2: Relation table of multipartisan outranking digraph

In Fig. 2.2, we may notice that the dominating outranking prekernel **['a06', 'a11', 'a13', 'a15']** gathers in fact a **multipartisan selection** of potential election winners. It is worthwhile noticing that in Fig. 2.2 the majority margins obtained from a linear voting profile do verify the zero-sum rule $\big(r(x \succsim y) + r(y \succsim x) \; = \; 0.0\big)$. To each positive outranking situation corresponds indeed an equivalent negative converse situation and the resulting outranking and strict outranking digraphs are the same.

Secondary election winner determination

When restricting now, in a secondary election stage, the set of eligible candidates to this dominating prekernel, we may compute the actual best social choice.

Listing 2.12: Secondary election winner recommendation

```
>>> from outrankingDigraphs import BipolarOutrankingDigraph
>>> g2 = BipolarOutrankingDigraph(vpt,
...                 actionsSubset=['a06','a11','a13','a15'])

>>> g2.showRelationTable(ReflexiveTerms=False)
 * ---- Relation Table -----
   r    | 'a06'  'a11'  'a13'  'a15'
 .------|-------------------------------
  'a06' |   -    +0.10  +0.48  +0.52
  'a11' | -0.10    -    +0.27  +0.29
  'a13' | -0.48  -0.27    -    +0.19
  'a15' | -0.52  -0.29  -0.19    -
 Valuation domain: [-1.000; 1.000]
>>> g2.computeCondorcetWinners()
 ['a06']
>>> g2.computeCopelandRanking()
 ['a06', 'a11', 'a13', 'a15']
```

Candidate *a06* appears clearly to be the winner of this election. Notice by the way that the restricted pairwise outranking relation shown in Listing 2.12 represents a linear ordering of the preselected candidates.

We may eventually check the quality of this best choice by noticing that candidate *a06* represents indeed the *simple majority* winner, the *instant-run-off* winner, the *Borda*, as well as the *Condorcet winner* of the initially given linear voting profile *lvp* (see Listing 2.8).

Listing 2.13: Secondary election winner recommendation verification

```
>>> lvp.computeSimpleMajorityWinner()
 ['a06']
>>> lvp.computeInstantRunoffWinner()
 ['a06']
>>> lvp.computeBordaWinners()
 ['a06']
>>> from votingProfiles import MajorityMarginsDigraph
>>> cd = MajorityMarginsDigraph(lvp)
```

(continues on next page)

(continued from previous page)

```
>>> cd.computeCondorcetWinners()
 ['a06']
```

In our example voting profile here, the multipartisan primary selection stage appears quite effective in reducing the number of eligible candidates to four out of a set of 15 candidates without btw rejecting the actual winning candidate.

Multipartisan preferences in divisive politics

However, in a very **divisive two major party system**, like in the US, where preferences of the supporters of one party appear to be very opposite to the preferences of the supporters of the other major party, the multipartisan outranking digraph will become nearly indeterminate.

In Listing 2.14 below we generate such a divisive kind of linear voting profile with the help of the *DivisivePolitics* flag[5] (see Lines 4 and 13-19). When now converting the voting profile into a performance tableau (Lines 20-21), we may compute the corresponding unopposed outranking digraph.

Listing 2.14: A divisive two-party example of a random linear voting profile

```
1  >>> from votingProfiles import RandomLinearVotingProfile
2  >>> lvp = RandomLinearVotingProfile(
3  ...        numberOfCandidates=7,numberOfVoters=500,
4  ...        WithPolls=True, partyRepartition=0.4,other=0.2,
5  ...        DivisivePolitics=True, seed=1)
6
7  >>> lvp.showRandomPolls()
8    Random repartition of voters
9     Party_1 supporters : 240 (48.00%)
10    Party_2 supporters : 160 (32.00%)
11    Other voters       : 100 (20.00%)
12   *---------------- random polls ---------------
13   Party_1(48.0%) | Party_2(32.0%)|   expected
14   ----------------------------------------------
15    a2 : 30.84%  |  a1 : 30.84%  |  a2 : 15.56%
16    a3 : 23.67%  |  a4 : 23.67%  |  a3 : 12.91%
17    a7 : 17.29%  |  a6 : 17.29%  |  a7 : 11.43%
18    a5 : 11.22%  |  a5 : 11.22%  |  a1 : 11.00%
19    a6 : 09.79%  |  a7 : 09.79%  |  a6 : 10.23%
20    a4 : 04.83%  |  a3 : 04.83%  |  a4 : 09.89%
21    a1 : 02.37%  |  a2 : 02.37%  |  a5 : 08.98%
22 >>> lvp.save2PerfTab('divisiveExample')
23 >>> dvp = PerformanceTableau('divisiveExample')
```

(continues on next page)

[5] The `RandomLinearVotingProfile` constructor provides a *DivisivePolitics* flag (*False* by default) for generating random linear voting profiles based on a divisive polls strucure

(continued from previous page)

```
24 >>> from outrankingDigraphs import \
25 ...        UnOpposedBipolarOutrankingDigraph
26
27 >>> uodg = UnOpposedBipolarOutrankingDigraph(dvp)
28 >>> uodg
29  *------- Object instance description ------*
30   Instance class       : UnOpposedBipolarOutrankingDigraph
31   Instance name        : unopposed_outrankings
32   # Actions            : 7
33   # Criteria           : 500
34   Size                 : 0
35   Oppositeness (%)     : 100.00
36   Determinateness (%) : 50.00
37   Valuation domain     : [-1.00;1.00]
```

With an oppositeness degree of 100.0% (see Listing 2.14 Lines 33-34), the preferential disagreement between the political parties is complete, and the unopposed outranking digraph *uodg* becomes completely **indeterminate** as shown in the relation table below.

```
 >>> uodg.showRelationTable(ReflexiveTerms=False)
 * ---- Relation Table -----
  r    |  'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'
 -----|-------------------------------------------------
 'a1' |     -   +0.00  +0.00  +0.00  +0.00  +0.00  +0.00
 'a2' |  +0.00    -    +0.00  +0.00  +0.00  +0.00  +0.00
 'a3' |  +0.00  +0.00    -    +0.00  +0.00  +0.00  +0.00
 'a4' |  +0.00  +0.00  +0.00    -    +0.00  +0.00  +0.00
 'a5' |  +0.00  +0.00  +0.00  +0.00    -    +0.00  +0.00
 'a6' |  +0.00  +0.00  +0.00  +0.00  +0.00    -    +0.00
 'a7' |  +0.00  +0.00  +0.00  +0.00  +0.00  +0.00    -
 Valuation domain: [-1.000; 1.000]
```

As a consequence, a **multipartisan primary selection**, computed with a `showBestChoiceRecommendation()` method, will keep the complete initial set of eligible candidates and, hence, becomes **ineffective** (see Listing 2.15 Line 6).

Listing 2.15: Example of ineffective primary multipartisan selection

```
1 >>> uodg.showBestChoiceRecommendation()
2  Best choice recommendation(s) (BCR)
3   (in decreasing order of determinateness)
4  Credibility domain: [-1.00,1.00]
5  === >> ambiguous choice(s)
6  choice               : ['a1','a2','a3','a4','a5','a6','a7']
7  independence         : 0.00
8  dominance            : 1.00
```

(continues on next page)

(continued from previous page)

```
absorbency           : 1.00
covered (%)          : 100.00
determinateness (%) : 50.00
 - most credible action(s) = { }
```

With such kind of divisive voting profile, there may not always exist an obvious winner. In Listing 2.16 below, we see, for instance, that the *simple majority* winnner is *a2* (Line 2), whereas the *instant-run-off* winner is *a6* (Line 4).

Listing 2.16: Example of secondary selection

```
1  >>> lvp.computeSimpleMajorityWinner()
2   ['a2']
3  >>> lvp.computeInstantRunoffWinner()
4   ['a6']
5  >>> from votingProfiles import MajorityMarginsDigraph
6  >>> cg = MajorityMarginsDigraph(lvp)
7  >>> cg.showRelationTable(ReflexiveTerms=False)
8   * ---- Relation Table -----
9     S   |  'a1' 'a2' 'a3' 'a4' 'a5' 'a6' 'a7'
10  ------|-----------------------------------
11   'a1' |   -   -68  -90  -46  -68  -88  -84
12   'a2' |  +68   -   -32  +80  +46   -6  -24
13   'a3' |  +90  +32   -   +58  +46   +4   +8
14   'a4' |   +4  -80  -58   -   -16  -68  -72
15   'a5' |  +68  -46  -46  +16   -   -26  -64
16   'a6' |  +88   +6   -4  +68  "26   -    -2
17   'a7' |  +84  +24   -8  +72  "64   "2   -
18  Valuation domain: [-500;+500]
19 >>> cg.computeCondorcetWinners()
20  ['a3']
21 >>> lvp.computeBordaWinners()
22  ['a3','a7']
23 >>> cg.computeCopelandRanking()
24  ['a3', 'a7', 'a6', 'a2', 'a5', 'a4', 'a1']
```

But in our example here, we are lucky. When constructing with the *pairwise majority margins* digraph (Line 6), a *Condorcet winner*, namely *a3* becomes apparent (Lines 13,20), which is also one of the two *Borda* winners (Line 22). More interesting even is to notice that the apparent majority margins digraph models in fact a linear ranking *['a3', 'a7', 'a6', 'a2', 'a5', 'a4', 'a1']* of all the eligible candidates, as shown with a Copeland ranking rule (Line 24).

We may eventually visualize in Listing 2.17 this linear ranking with a graphviz drawing where we drop all transitive arcs (Line 1) and orient the drawing with *Condorcet* winner *a3* and loser *a1* (Lines 2).

Listing 2.17: Drawing the linear ordering

```
>>> cg.closeTransitive(Reverse=True)
>>> cg.exportGraphViz('divGraph',firstChoice=['a3'],lastChoice=['a1'])
 *---- exporting a dot file for GraphViz tools ---------*
  Exporting to divGraph.dot
  dot -Grankdir=BT -Tpng divGraph.dot -o divGraph.png
```

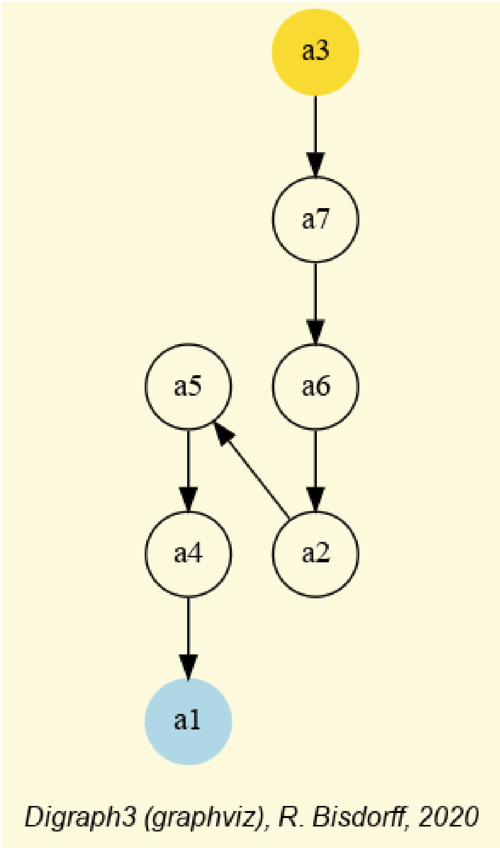


Fig. 2.3: Linear ordering of the eligible candidates

Back to *Content Table* (page 1)

2.3 Tempering plurality tyranny effects with bipolar approval voting

> *The choice of a voting procedure shapes the democracy in which we live.*
>
> —Baujard A., Gavrel F., Igersheim H., Laslier J.-F. and Lebon I. [BAU-2013p].

- *Bipolar approval voting systems* (page 46)
- *Pairwise comparison of bipolar approval votes* (page 49)
- *Three-valued evaluative voting system* (page 51)
- *Favouring multipartisan candidates* (page 54)

Bipolar approval voting systems

In the `votingProfiles` module we provide a `BipolarApprovalVotingProfile` class for handling voting results where, for each eligible candidate c, the voters are invited to **approve** (+1), **disapprove** (-1), or **ignore** (0) the statement that *candidate C should win the election*.

File bpApVotingProfile.py contains such a bipolar approval voting profile concerning 100 voters and 15 eligible candidates. We may inspect its content as follows.

```
1  >>> from votingProfiles import BipolarApprovalVotingProfile
2  >>> bavp = BipolarApprovalVotingProfile('bpApVotingProfile')
3  >>> bavp
4   *------- VotingProfile instance description ------*
5    Instance class    : BipolarApprovalVotingProfile
6    Instance name     : bpApVotingProfile
7    # Candidates      : 15
8    # Voters          : 100
9    Attributes        : ['name', 'candidates', 'voters',
10                        'approvalBallot', 'netApprovalScores',
11                        'ballot']
```

Beside the *bavp.candidates* and *bavp.voters* attributes, we discover in Line 10 above the *bavp.approvalBallot* attribute which gathers bipolar approval votes. Its content is the following.

Listing 2.18: Inspecting a bipolar approval ballot

```
>>> bavp.approvalBallot
 {'v001':
   {'a01': Decimal('0'),
    ...
    'a04': Decimal('1'),
    ...
    'a15': Decimal('0')
   },
  'v002':
    {'a01': Decimal('-1'),
     'a02': Decimal('0'),
     ...
     'a15': Decimal('1')
```

(continues on next page)

(continued from previous page)

```
14      },
15     ...
16    v100':
17    {'a01': Decimal('0'),
18     'a02': Decimal('1'),
19     ...
20     'a15': Decimal('1')
21    }
22  }
```

Let us denote A_v the set of candidates approved by voter v. In Listing 2.18 we hence record in fact the bipolar-valued truth characteristic values $r(c \in A_v)$ of the statements that candidate c **is approved** by voter v. In Line 5, we observe for instance that voter *v001* **positively approves** candidate *a04*. And, in Line 10, we see that voter *v002* **negatively approves**, i.e. **positively disapproves** candidate *a01*. We may now consult how many approvals or disapprovals each candidate receives.

```
>>> bavp.showApprovalResults()
 Approval results
  Candidate: a12 obtains 34 votes
  Candidate: a05 obtains 30 votes
  Candidate: a03 obtains 28 votes
  Candidate: a14 obtains 27 votes
  Candidate: a11 obtains 27 votes
  Candidate: a04 obtains 27 votes
  Candidate: a01 obtains 27 votes
  Candidate: a13 obtains 24 votes
  Candidate: a07 obtains 24 votes
  Candidate: a15 obtains 23 votes
  Candidate: a02 obtains 23 votes
  Candidate: a09 obtains 22 votes
  Candidate: a08 obtains 22 votes
  Candidate: a10 obtains 21 votes
  Candidate: a06 obtains 21 votes
 Total approval votes: 380
 Approval proportion: 380/1500 = 0.25
>>> bavp.showDisapprovalResults()
 Disapproval results
  Candidate: a12 obtains 16 votes
  Candidate: a03 obtains 22 votes
  Candidate: a09 obtains 23 votes
  Candidate: a04 obtains 24 votes
  Candidate: a06 obtains 24 votes
  Candidate: a13 obtains 24 votes
  Candidate: a11 obtains 25 votes
  Candidate: a02 obtains 26 votes
```

(continues on next page)

(continued from previous page)

```
30    Candidate: a07 obtains 26 votes
31    Candidate: a08 obtains 26 votes
32    Candidate: a05 obtains 27 votes
33    Candidate: a10 obtains 27 votes
34    Candidate: a14 obtains 27 votes
35    Candidate: a15 obtains 27 votes
36    Candidate: a01 obtains 32 votes
37   Total disapproval votes: 376
38   Disapproval proportion: 376/1500 = 0.25
```

In Lines 3 and 22 above, we may see that, of all potential candidates, it is Candidate *a12* who receives the highest number of approval votes (34) and the lowest number of disapproval votes (16). Total number of approval, respectively disapproval, votes approaches more or less a proportion of 25% of the 100*15 = 1500 potential approval votes. About 50% of the latter remain hence ignored.

When operating now, for each candidate *c*, the difference between the number of approval and the number of disapproval votes he receives, we obtain per candidate a corresponding **net approval** score; in fact, the bipolar truth characteristic value of the statement *candidate c should win the election*.

$$\mathrm{r}(\textit{Candidate c should win the election}) = \sum_v \big(r(c \in A_v)\big)$$

These bipolar characteristic values are stored in the *bavp.netApprovalScores* attribute and may be printed out as follows.

```
1   >>> bavp.showNetApprovalScores()
2    Net Approval Scores
3     Candidate: a12 obtains 18 net approvals
4     Candidate: a03 obtains 6 net approvals
5     Candidate: a05 obtains 3 net approvals
6     Candidate: a04 obtains 3 net approvals
7     Candidate: a11 obtains 2 net approvals
8     Candidate: a14 obtains 0 net approvals
9     Candidate: a13 obtains 0 net approvals
10    Candidate: a09 obtains -1 net approvals
11    Candidate: a07 obtains -2 net approvals
12    Candidate: a06 obtains -3 net approvals
13    Candidate: a02 obtains -3 net approvals
14    Candidate: a15 obtains -4 net approvals
15    Candidate: a08 obtains -4 net approvals
16    Candidate: a01 obtains -5 net approvals
17    Candidate: a10 obtains -6 net approvals
```

We observe in Line 3 above that Candidate *a12*, with a net approval score of 34 - 16 = 18, represents indeed the **best approved** candidate for winning the election. With a net approval score of 28-22 = 6, Candidate *a03* appears **2nd-best approved**. The net approval scores define hence a potentially weak ranking on the set of eligible election candidates, and the winner(s) of the election is(are) determined by the first-ranked candidate(s).

Pairwise comparison of bipolar approval votes

The approval votes of each voter define now on the set of eligible candidates three ordered categories: his approved (+1), his ignored (0) and his disapproved (-1) ones. Within each of these three categories we consider the voter's actual preferences as **not communicated**, i.e. as *missing data*. This gives for each voter a *partially determined strict order* which we find in the *bavp.ballot* attribute.

```
>>> bavp.ballot['v001']['a12']
 {'a02': Decimal('1'), 'a11': Decimal('1'),
  'a14': Decimal('1'), 'a04': Decimal('0'),
  'a06': Decimal('1'), 'a05': Decimal('1'),
  'a12': Decimal('0'), 'a13': Decimal('0'),
  'a15': Decimal('1'), 'a01': Decimal('1'),
  'a08': Decimal('1'), 'a07': Decimal('1'),
  'a09': Decimal('0'), 'a03': Decimal('1'),
  'a10': Decimal('0')}
```

For voter *v001*, for instance, the best approved candidate *a12* is strictly preferred to candidates: *a01*, *a02*, *a03*, *a05*, *a06*, *a07*, *a08*, *a11*, *a14* and *15*. No candidate is preferred to *a12* and the comparison with *a04*, *a09*, *a10* and *a13* is not communicated, hence indeterminate. Mind by the way that the reflexive comparison of *a12* with itself is, as usual, is ignored, i.e. indeterminate. Each voter v defines thus a partially determined transitive strict preference relation denoted $\succ_v$ on the eligible candidates.

For each pair of eligible candidates, we aggregate the previous individual voter's preferences into a truth characteristic of the statement: candidate x is *better approved than* candidate y, denoted $r(x \succ y)$

$r(x \succ y) \;=\; \sum_v \big(r(x \succ_v y) \big)$.

We say that candidate x is *better approved than* Candidate y when $r(x \succ y) \;>\; 0$, i.e. there is a *majority* of voters who *approve* **more** and *disapprove* **less** x than y. Vice-versa, we say that candidate x is *not better approved than* candidate y when $r(x \succ y) \;<\; 0$, i.e. there is a majority of voters who disapprove more and approve less x than y. This computation is achieved with the `MajorityMarginsDigraph` constructor.

```
>>> from votingProfiles import MajorityMarginsDigraph
>>> m = MajorityMarginsDigraph(bavp)
>>> m
 *------- Digraph instance description ------*
  Instance class      : MajorityMarginsDigraph
  Instance name       : rel_bpApVotingProfile
  Digraph Order       : 15
  Digraph Size        : 97
  Valuation domain    : [-100.00;100.00]
  Determinateness (%) : 52.55
  Attributes          : ['name', 'actions', 'criteria',
```

(continues on next page)

(continued from previous page)

```
12                                  'ballot', 'valuationdomain', '
13                                  relation', 'order',
14                                  'gamma', 'notGamma']
```

The resulting digraph m contains 97 positively validated relations (see Line 8 above) and (see Line 9) for all pairs (x, y) of eligible candidates, $r(x \succ y)$ takes value in an valuation range from -100.00 (all voters opposed) to +100.00 (unanimously supported).

We may inspect these pairwise $r(x \succ y)$ values in a browser view.

```
>>> m.showHTMLRelationTable(relationName='r(x > y)')
```

r(x >= y)	a01	a02	a03	a04	a05	a06	a07	a08	a09	a10	a11	a12	a13	a14	a15
a01	–	1	-10	-10	-8	-2	-2	-3	-3	2	-5	-19	-4	-2	2
a02	-1	–	-5	-4	-9	-2	5	1	-7	3	-2	-20	-1	-4	-2
a03	10	5	–	1	4	6	5	10	7	11	0	-5	2	6	9
a04	10	4	-1	–	2	5	7	8	-1	4	0	-7	8	2	8
a05	8	9	-4	-2	–	4	6	7	4	2	1	-17	6	0	3
a06	2	2	-6	-5	-4	–	-1	2	-3	5	-4	-13	-1	-1	2
a07	2	-5	-5	-7	-6	1	–	7	-4	2	-5	-17	-2	-2	4
a08	3	-1	-10	-8	-7	-2	-7	–	-2	2	-2	-16	0	0	-1
a09	3	7	-7	1	-4	3	4	2	–	3	0	-18	-4	0	2
a10	-2	-3	-11	-4	-2	-5	-2	-2	-3	–	-6	-15	-4	-4	2
a11	5	2	0	0	-1	4	5	2	0	6	–	-15	4	0	5
a12	19	20	5	7	17	13	17	16	18	15	15	–	12	13	18
a13	4	1	-2	-8	-6	1	2	0	4	4	-4	-12	–	1	4
a14	2	4	-6	-2	0	1	2	0	0	4	0	-13	-1	–	-1
a15	-2	2	-9	-8	-3	-2	-4	1	-2	-2	-5	-18	-4	1	–

Valuation domain: [-100; +100]

Fig. 2.4: The bipolar-valued pairwise majority margins

It gets easily apparent that candidate *a12* constitutes a *Condorcet* winner, i.e. the candidate who beats all the other candidates and, with the given voting profile *gavp*, should without doubt win the election. This strongly confirms the first-ranked result obtained with the previous net approval scoring.

Let us eventually compute, with the help of the NetFlows ranking rule), a linear ranking of the 15 eligible candidates and compare the result with the net approval scores' ranking.

```
>>> from linearOrders import NetFlowsOrder
>>> nf = NetFlowsOrder(m,Comments=True)
>>> print('NetFlows versus Net Approval Ranking')
>>> print('Candidate\tNetFlows score\tNet Approval score')
```

(continues on next page)

(continued from previous page)

```
>>> for item in nf.netFlows:
...     print( '%9s\t  %+.3f\t %+.1f' %\
...        (item[1], item[0], bavp.netApprovalScores[item[1]]) )

 NetFlows versus Net Approval Ranking
 Candidate   NetFlows score  Net Approval score
       a12     +410.000       +18.0
       a03     +142.000        +6.0
       a04      +98.000        +3.0
       a05      +54.000        +3.0
       a11      +34.000        +2.0
       a09      -16.000        -1.0
       a14      -20.000        +0.0
       a13      -22.000        +0.0
       a06      -50.000        -3.0
       a07      -74.000        -2.0
       a02      -96.000        -3.0
       a08      -102.000       -4.0
       a15      -110.000       -4.0
       a10      -122.000       -6.0
       a01      -126.000       -5.0
```

On the *better approved than* majority margins digraph m, the *NetFlows* rule delivers a ranking that is very similar to the one previously obtained with the corresponding *Net Approval* scores. Only minor inversions do appear, like in the midfield, where candidate *a09* advances before candidates *a13* and *a14* and *a6* and *a07* swap their positions 9 and 10. And, the two last-ranked candidates also swap their positions.

This confirms again the pertinence of the net approval scoring approach for finding the winner in a bipolar approving voting system. Yet, voting by approving (+1), disapproving (-1) or ignoring (0) eligible candidates, may also be seen as a performance evaluation of the eligible candidates on a {-1, 0, 1}-graded ordinal scale.

Three-valued evaluative voting system

Following such an epistemic perspective, we may effectively convert the given `BipolarApprovalVotingProfile` instance into a `PerformanceTableau` instance, so as to get access to a corresponding *outranking* decision aiding approach.

Mind that, contrary to the majority margins of the *better approved than* relation, all voters consider now the approved candidates to be all equivalent (+1). Same is true for the disapproved (-1), respectively the ignored candidates (0). The voter's marginal preferences model this time a complete preorder with three equivalence classes.

From the saved file *AVPerfTab.py* (see Line 1 below), we may construct an outranking relation on the eligible candidates with our standard `BipolarOutrankingDigraph` constructor. The semantics of this outranking relation are the following:

- We say that Candidate x **outranks** Candidate y when there is a majority of voters who consider x **at least as well evaluated as** y.(see Line3 below).

- We say that Candidate x **is not outranked by** Candidate y when there is a majority of voters who consider x **not at least as well evaluated as** y.

```
1  >>> bavp.save2PerfTab(fileName='AVPerfTab',valueDigits=0)
2   *--- Saving as performance tableau in file: <AVPerfTab.py> ---*
3  >>> from outrankingDigraphs import BipolarOutrankingDigraph
4  >>> odg = BipolarOutrankingDigraph('AVPerfTab')
5  >>> odg
6   *------- Object instance description ------*
7     Instance class       : BipolarOutrankingDigraph
8     Instance name        : rel_AVPerfTab
9     # Actions            : 15
10    # Criteria           : 100
11    Size                 : 210
12    Determinateness (%)  : 69.29
13    Valuation domain     : [-1.00;1.00]
14    Attributes           : ['name', 'actions', 'order,
15                            'criteria', 'evaluation', 'NA',
16                            'valuationdomain', 'relation',
17                            'gamma', 'notGamma', ...]
```

The size ($210 = 15*14$) of the resulting outranking digraph *odg*, shown in Line 11 above, reveals that the corresponding *at least as good evaluated as* (outranking) relation models actually a trivial *complete* digraph. All candidates appear to be **equally** *at least as well evaluated* and the *better evaluated than* (strict outranking) *codual* outranking digraph becomes in fact empty. The converted performance tableau does apparently not contain sufficiently discriminatory performance evaluations for supporting any strict preference situations.

Yet, we may nevertheless try to apply again the *NetFlows* ranking rule to this complete outranking digraph *g* and print side by side the corresponding *NetFlows* scores and the previous *Net Approval* scores.

```
>>> from linearOrders import NetFlowsOrder
>>> nf = NetFlowsOrder(odg)
>>> print('NetFlows versus Net Approval Ranking')
>>> print('Candidate\tNetFlows Score\tNet Approval Score')
>>> for item in nf.netFlows:
...     print('%9s\t  %+.3f\t %+.0f' %\
...         (item[1], item[0],bavp.netApprovalScores[item[1]]) )

 NetFlows versus Net Approval Ranking
 Candidate    NetFlows score Net Approval score
       a12      +4.100         +18.0
       a03      +1.420         +6.0
       a04      +0.980         +3.0
       a05      +0.540         +3.0
```

(continues on next page)

(continued from previous page)

```
        a11     +0.340          +2.0
        a09     -0.160          -1.0
        a14     -0.200          +0.0
        a13     -0.220          +0.0
        a06     -0.500          -3.0
        a07     -0.740          -2.0
        a02     -0.960          -3.0
        a08     -1.020          -4.0
        a15     -1.100          -4.0
        a10     -1.220          -6.0
        a01     -1.260          -5.0
```

Despite its apparent poor strict preference discriminating power, we obtain here *NetFlows* scores that are directly proportional (divided by 100) to the scores obtained with the *better approved than* majority margins digraph *m*.

Encouraged by this positive result, we may furthermore try to compute as well a *best choice recommendation*.

```
>>> odg.showBestChoiceRecommendation()
 ***********************
 Best choice recommendation(s) (BCR)
  (in decreasing order of determinateness)
  Credibility domain: [-1.00,1.00]
 === >> ambiguous first choice(s)
  * choice              : ['a01', 'a02', 'a03', 'a04', 'a05',
                           'a06', 'a07', 'a08', 'a09', 'a10',
                           'a11', 'a12', 'a13', 'a14', 'a15']
   independence        : 0.06
   dominance           : 1.00
   absorbency          : 1.00
   covering (%)        : 100.00
   determinateness (%) : 61.13
   - most credible action(s) = {
     'a12': 0.44, 'a03': 0.34, 'a04': 0.30,
     'a14': 0.28, 'a13': 0.24, 'a06': 0.24,
     'a11': 0.20, 'a10': 0.20, 'a07': 0.20,
     'a01': 0.20, 'a08': 0.18, 'a05': 0.18,
     'a15': 0.14, 'a09': 0.14, 'a02': 0.06, }
 === >> ambiguous last choice(s)
  * choice              : ['a01', 'a02', 'a03', 'a04', 'a05',
                           'a06', 'a07', 'a08', 'a09', 'a10',
                           'a11', 'a12', 'a13', 'a14', 'a15']
   independence        : 0.06
   dominance           : 1.00
   absorbency          : 1.00
   covered (%)         : 100.00
```

(continues on next page)

(continued from previous page)

```
29    determinateness (%) : 63.73
30    - most credible action(s) = {
31      'a13': 0.36, 'a06': 0.36, 'a15': 0.34,
32      'a01': 0.34, 'a08': 0.32, 'a07': 0.30,
33      'a02': 0.30, 'a14': 0.28, 'a11': 0.28,
34      'a09': 0.28, 'a04': 0.26, 'a10': 0.24,
35      'a05': 0.20, 'a03': 0.20, 'a12': 0.06, }
```

The outranking digraph *odg* being actually *empty*, we obtain a unique **ambiguous** –first as well as last– choice recommendation which trivially retains all fifteen candidates (see Lines 6-9 above). Yet, the bipolar-valued best choice membership characteristic vector reveals that, among all the fifteen potential winners, it is indeed Candidate *a12* the most credible one with a 72% majority of voters' support (see Line 16, $(0.44 + 1.0)/2 = 0.72$); followed by Candidate *a03* (67%) and Candidate *a04* (65%). Similarly, Candidates *a13* and *a06* represent the most credible losers with a 68% majority voters' support (Line 31).

> **Note**
>
> We observe here empirically that **evaluative** voting systems, using three-valued ordinal performance scales, match closely **bipolar approval** voting systems. The latter voting system models, however, more *faithfully* the very preferential information that is expressed with *approved*, *disapproved* or *ignored* statements. The corresponding evaluation on a three-graded scale, being value (numbers) based, cannot express the fact that in bipolar approval voting systems there is **no preferential information** given concerning the pairwise comparison of all *approved*, respectively *disapproved* or *ignored* candidates.

Let us finally illustrate how bipolar approval voting systems may favour multipartisan supported candidates. We shall therefore compare *bipolar approval* versus *uninominal plurality* election results when considering a highly divisive and partisan political context.

Favouring multipartisan candidates

In modern democracy, politics are largely structured by political parties and activists movements. Let us so consider a bipolar approval voting profile *dvp* where the random voter behaviour is simulated from two pre-electoral polls concerning a political scene with essentially two major competing parties, like the one existing in the US.

```
>>> dvp = RandomBipolarApprovalVotingProfile(\
...                 numberOfCandidates=15,
...                 numberOfVoters=100,
...                 approvalProbability=0.25,
...                 disapprovalProbability=0.25,
...                 WithPolls=True,
...                 partyRepartition=0.5,
```

(continues on next page)

(continued from previous page)

```
...                   other=0.05,
...                   DivisivePolitics=True,
...                   seed=200)

>>> dvp.showRandomPolls()
 Random repartition of voters
  Party_1 supporters : 45 (45.00%)
  Party_2 supporters : 49 (49.00%)
  Other voters       : 6 (06.00%)
 *---------------- random polls ---------------
  Party_1(45.0%) | Party_2(49.0%)|   expected
 -----------------------------------------------
   a05 : 24.10%  |  a07 : 24.10%  |  a07 : 11.87%
   a14 : 23.48%  |  a10 : 23.48%  |  a10 : 11.60%
   a03 : 15.13%  |  a01 : 15.13%  |  a05 : 10.91%
   a12 : 07.55%  |  a04 : 07.55%  |  a14 : 10.67%
   a08 : 07.11%  |  a09 : 07.11%  |  a01 : 07.67%
   a15 : 04.37%  |  a13 : 04.37%  |  a03 : 07.09%
   a11 : 03.99%  |  a02 : 03.99%  |  a04 : 04.55%
   a06 : 03.80%  |  a06 : 03.80%  |  a09 : 04.49%
   a02 : 02.79%  |  a11 : 02.79%  |  a12 : 04.32%
   a13 : 02.63%  |  a15 : 02.63%  |  a08 : 04.30%
   a09 : 02.24%  |  a08 : 02.24%  |  a06 : 03.57%
   a04 : 01.89%  |  a12 : 01.89%  |  a13 : 03.32%
   a01 : 00.57%  |  a03 : 00.57%  |  a15 : 03.25%
   a10 : 00.20%  |  a14 : 00.20%  |  a02 : 03.21%
   a07 : 00.14%  |  a05 : 00.14%  |  a11 : 03.16%
```

The divisive political situation is reflected by the fact that Party_1 and Party_2 supporters show strict reversed preferences. The leading candidates of Party_1 (*a05* and *a14*) are last choices for Party_2 supporters and, Candidates *a07* and *a10*, leading candidates for Party_2 supporters, are similarly the least choices for Party_1 supporters.

No clear winner may be guessed from these pre-election polls. As Party_2 shows however slightly more supporters than Party_1, the expected winner in an uninominal *plurality* or *instant-runoff* voting system will be Candidate *a07*, i,e, the leading candidate of Party_2 (see below).

```
>>> dvp.computeSimpleMajorityWinner()
 ['a07']
>>> dvp.computeInstantRunoffWinner()
 ['a07']
```

Now, in a corresponding bipolar approval voting system, Party_1 supporters will usually approve their leading candidates and disapprove the leading candidates of Party_2. Vice versa, Party_2 supporters will usually approve their leading candidates and disapprove the leading candidates of Party_1. Let us consult the resulting approval votes per candidate.

```
>>> dvp.showApprovalResults()
  Candidate: a07 obtains 30 votes
  Candidate: a10 obtains 28 votes
  Candidate: a05 obtains 28 votes
  Candidate: a01 obtains 28 votes
  Candidate: a03 obtains 26 votes
  Candidate: a02 obtains 26 votes
  Candidate: a12 obtains 25 votes
  Candidate: a14 obtains 24 votes
  Candidate: a13 obtains 24 votes
  Candidate: a09 obtains 21 votes
  Candidate: a04 obtains 21 votes
  Candidate: a08 obtains 19 votes
  Candidate: a06 obtains 17 votes
  Candidate: a15 obtains 15 votes
  Candidate: a11 obtains 12 votes
 Total approval votes: 344
 Approval proportion: 344/1500 = 0.23
```

When considering only the approval votes, we find confirmed above that the leading candidate of Party_2 obtains in this simulation a plurality of approval votes. In uninominal *plurality* or *instant-runoff* voting systems, this candidate wins hence the election, quite to the despair of Party_1 supporters. As a foreseeable consequence, this election result will be more or less aggressively contested which leads to a loss of popular trust in democratic elections and institutions.

If we look however on the corresponding disapprovals, we discover that, not surprisingly, the leading candidates of both parties collect by far the highest number of disapproval votes.

```
>>> dvp.showDisapprovalResults()
  Candidate: a02 obtains 14 votes
  Candidate: a04 obtains 14 votes
  Candidate: a13 obtains 14 votes
  Candidate: a06 obtains 15 votes
  Candidate: a09 obtains 15 votes
  Candidate: a08 obtains 16 votes
  Candidate: a11 obtains 16 votes
  Candidate: a15 obtains 18 votes
  Candidate: a12 obtains 20 votes
  Candidate: a01 obtains 29 votes
  Candidate: a03 obtains 30 votes
  Candidate: a10 obtains 37 votes
  Candidate: a07 obtains 44 votes
  Candidate: a14 obtains 45 votes
  Candidate: a05 obtains 49 votes
 Total disapproval votes: 376
 Disapproval proportion: 376/1500 = 0.25
```

Balancing now approval against disapproval votes will favour the moderate, bipartisan supported, candidates.

```
>>> dvp.showNetApprovalScores()
 Net Approval Scores
  Candidate: a02 obtains 12 net approvals
  Candidate: a13 obtains 10 net approvals
  Candidate: a04 obtains 7 net approvals
  Candidate: a09 obtains 6 net approvals
  Candidate: a12 obtains 5 net approvals
  Candidate: a08 obtains 3 net approvals
  Candidate: a06 obtains 2 net approvals
  Candidate: a01 obtains -1 net approvals
  Candidate: a15 obtains -3 net approvals
  Candidate: a11 obtains -4 net approvals
  Candidate: a03 obtains -4 net approvals
  Candidate: a10 obtains -9 net approvals
  Candidate: a07 obtains -14 net approvals
  Candidate: a14 obtains -21 net approvals
  Candidate: a05 obtains -21 net approvals
```

Candidate *a02*, appearing in the pre-electoral polls in the midfield (in position 7 for Party_2 and in position 9 for Party_1 supporters), shows indeed the highest net approval score. Second highest net approval score obtains Candidate *a13*, in position 6 for Party_2 and in position 10 for Party_1 supporters.

Fig. 2.5, showing the *NetFlows* ranked relation table of the *better approved than* majority margins digraph, confirms below this net approval scoring result.

```
>>> m = MajorityMarginsDigraph(dvp)
>>> m.showHTMLRelationTable(\
...      actionsList=m.computeNetFlowsRanking(),
...     relationName='r(x > y)')
```

r(x > y)	a02	a13	a04	a09	a12	a08	a06	a01	a11	a15	a03	a10	a07	a14	a05
a02	–	6	5	6	9	5	10	12	12	14	11	15	22	22	23
a13	-6	–	2	5	2	5	8	10	14	10	9	13	18	23	20
a04	-5	-2	–	0	2	2	3	7	7	8	11	13	18	21	18
a09	-6	-5	0	–	0	2	5	5	11	9	5	13	16	21	16
a12	-9	-2	-2	0	–	4	6	2	9	6	10	5	10	23	25
a08	-5	-5	-2	-2	-4	–	4	0	8	9	7	5	11	21	20
a06	-10	-8	-3	-5	-6	-4	–	-2	5	5	5	6	13	17	18
a01	-12	-10	-7	-5	-2	0	2	–	1	-1	3	8	11	9	13
a11	-12	-14	-7	-11	-9	-8	-5	-1	–	1	-2	7	14	13	16
a15	-14	-10	-8	-9	-6	-9	-5	1	-1	–	0	3	10	14	14
a03	-11	-9	-11	-5	-10	-7	-5	-3	2	0	–	-3	7	16	16
a10	-15	-13	-13	-13	-5	-5	-6	-8	-7	-3	3	–	3	6	7
a07	-22	-18	-18	-16	-10	-11	-13	-11	-14	-10	-7	-3	–	3	5
a14	-22	-23	-21	-21	-23	-21	-17	-9	-13	-14	-16	-6	-3	–	0
a05	-23	-20	-18	-16	-25	-20	-18	-13	-16	-14	-16	-7	-5	0	–

Valuation domain: [-100; +100]

Fig. 2.5: The pairwise *better approved than* majority margins

Candidate *a02* appears indeed *better approved than* any other candidate (*Condorcet* winner); and, the leading candidates of Party_1, *a05* and *a14*, are *less approved than* any other candidates (weak *Condorcet* losers).

```
>>> m.computeCondorcetWinners()
 ['a02']
>>> m.computeWeakCondorcetLosers()
 ['a05','a14']
```

We see this result furthermore confirmed when computing the corresponding **first**, respectively **last** choice recommendation.

```
>>> m.showBestChoiceRecommendation()
 Best choice recommendation(s) (BCR)
  (in decreasing order of determinateness)
 Credibility domain: [-100.00,100.00]
  === >> potential first choice(s)
 * choice              : ['a02']
   independence        : 100.00
   dominance           : 5.00
   absorbency          : -23.00
   covering (%)        : 100.00
   determinateness (%) : 52.50
   - most credible action(s) = { 'a02': 5.00, }
```

(continues on next page)

(continued from previous page)

```
 === >> potential last choice(s)
* choice              : ['a05', 'a14']
  independence        : 0.00
  dominance           : -23.00
  absorbency          : 5.00
  covered (%)         : 100.00
  determinateness (%) : 50.00
  - most credible action(s) = { }
```

Candidate *a02*, being actually a *Condorcet* winner, gives an initial **dominating kernel** of digraph *m*, whereas Party_1 leading Candidates *a05* and *a14*, both being weak *Condorcet* losers, give together a terminal **dominated** prekernel. They hence represent our **first choice**, respectively, **last choice** recommendations for winning this simulated election.

Let us conclude by predicting that, for leading political candidates in an aggressively divisive political context, the perspective to easily fail election with bipolar approval voting systems, might or will induce a change in the usual way of running electoral campaigns. Political parties and politicians, who avoid aggressive competitive propaganda and instead propose multipartisan collaborative social choices, will be rewarded with better election results than any kind of extremism. It could mean the end of sterile political obstructions and war like electoral battles.

Let's do it.

Note

It is worthwhile noticing the essential structural and computational role, the **zero value** is again playing in bipolar approval voting systems. This epistemic and logical **neutral** term is needed indeed for handling in a consistent and efficient manner **not communicated votes** and/or **indeterminate preferential statements**.

Back to *Content Table* (page 1)

2.4 Selecting the winner of a primary election: a critical commentary

- *The French popular primary presidential election 2022* (page 60)
- *A bipolar approval-disapproval election* (page 61)
- *Ranking the potential presidential candidates* (page 62)

"*A rating is not a vote.*"[9] – Fr. Hollande (2022)

[9] "*Il faut qu'il y ait un vote et pas une note. Les électeurs ne sont pas des juges, ce sont des*

The French popular primary presidential election 2022

Deploring in the forefront of the presidential election 2022 the utmost division in France of the political landscape on the left and ecological border, a group of young activists took the initiative to organize a popular primary election in order to make appear a unique multipartisan candidate[10].

130,000 engaged citizens proposed and promoted, in view of their respective political programs, seven political personalities for this primary presidential election, namely: *Anna Agueb-Porterie*, *Anne Hidalgo*, *Yannick Jadot*, *Pierre Larrouturou*, *Charlotte Marchandise*, *Jean-Luc Mélenchon* and *Christiane Taubira*.

From January 27 to 30 2022, 392 738 voters participated eventually in a primary presidential election by grading on-line these seven candidates on a five-steps suitability scale: *Very Good*, *Good*, *Quite Good*, *Fair* and *Insufficient* for being a potential multipartisan candidate. Below the resulting grades distribution in percents obtained by each personality.

Table 2.1: The popular primary election results (in %)

Personality	Very Good	Good	Quite Good	Fair	Insufficient
A Agueb-Porterie	2.86	7.34	18.19	21.05	50.56
A Hidalgo	6.33	13.36	20.70	23.80	35.81
Y Jadot	21.57	23.11	20.57	15.54	19.21
P Larrouturou	13.37	14.53	19.42	18.11	34.58
Ch Marchandise	3.41	8.93	19.59	21.87	46.20
J-L Mélenchon	20.49	15.33	16.73	18.29	29.16
Ch Taubira	49.41	18.00	11.68	7.91	12.99

It is important to notice in Table 2.1 that almost half of these 392 738 primary voters (49.41%) appear to be *Taubira* supporters.

For naming the winner of this primary election, the organizers used the *Majority Judgment* -a median grade- approach [BAL-2011]. With this decision algorithm, the election result became obvious. Only *Taubira* obtains a *Good* median grade, followed by *Jadot* and *Mélenchon* with *Quite Good* median grades. Hence *Christiane Taubira* was declared being the most suitable multipartisan presidential candidate.

Yet, this median grade approach makes the implicit hypothesis that the distributions of grades obtained by the candidates show indeed a convincing order statistical center. Suppose for instance that a first personality obtains 51% *Very Good* and 49% of *Insufficient* votes. Her median evaluation will be *Very Good*. A second personality obtains 49% of *Very Good* and 51% of *Good* votes. Her median evaluation will be only *Good*, even if the latter overall evaluation is evidently by far better than the first one. The *Majority Judgment* approach does hence not temper simple plurality induced effects. In the results shown in Table 2.1 the large plurality of *Taubira* supporters clearly forces the issue of this primary election.

citoyens" Fr. Hollande (31/01/2022) https://www.bfmtv.com/politique/elections/presidentielle/une-note-n-est-pas-un-vote-francois-hollande-regrette-que-la-primaire-populaire-ne-change-rien_AN-202201310516.html

[10] See https://primairepopulaire.fr/la-primaire/

The set of voters participating in this primary election does evidently not cover exhaustively all the supporters of each one of the seven potential presidential candidates. Hence, they do not represent a coherent family of performance criteria for selecting the most suitable multipartisan candidate.

To avoid such controversial election results, we need to abandon the evaluative judgment perspective and go instead for a bipolar approval-disapproval approach.

A bipolar approval-disapproval election

Let us therefore notice that the ordinal judgment scale used in the *Majority Judgment* approach shows in fact a bipolar structure. On the positive side, we have three levels of more or less *Good* evaluations, namely *Very Good*, *Good* and *Quite Good* grades, and on the negative side, we have the *Insufficient* grade. The *Fair* votes are constrained by the constant total number of 392 738 votes obtained by each candidates and must hence be neglected. They correspond in an epistemic perspective to a kind of abstention.

Thus, two equally significant decision criteria do emerge. The winner of the popular primary election should obtain:

1. a *maximum* of approvals: sum of *Very Good*, *Good* and *Quite Good* votes, and
2. a *minimum* of disapprovals: *Insufficient* votes.

The best suited multipartisan presidential candidate should as a consequence present the highest **net approval score**: total of approval votes minus total of disapproval votes. In Table 2.2 we show the resulting ranking by descending net approval score.

Table 2.2: The bipolar approval-disapproval results (in %)

Personality	Net approval	Approval	Disapproval	Abstention
Ch Taubira	+66.11	79.10	12.99	07.91
Y Jadot	+46.04	65.25	19.21	15.54
J-L Mélenchon	+23.39	52.55	29.16	18.29
P Larrouturou	+12.74	47.32	34.58	18.11
A Hidalgo	+04.57	40.39	35.81	23.80
Ch Marchandise	-14.28	31.92	46.20	21.87
A Agueb-Porterie	-22.16	28.39	50.56	21.05

Without surprise, it is again *Christaine Taubira* who shows the highest net approval score (+66.11%), followed by *Yannick Jadot* (+46.04%). Notice that both *Ch Marchandise* (-14.28%) and *A Agueb-Porterie* (-22.16%) are positively disapproved as potential multipartisan presidential candidates.

It is furthermore remarkable that both the approval votes and the the disapproval votes model the same linear ranking of the seven candidates.

Ranking the potential presidential candidates

To illustrate this point we provide a corresponding `perfTabs.PerformanceTableau` object in file *primPopRes.py* in the examples directory of the *Digraph3* resources.

```
>>> from perfTabs import PerformanceTableau
>>> t = PerformanceTableau('primPopRes')
>>> t
 *--- PerformanceTableau instance description ---*
  Instance class      : PerformanceTableau
  Instance name       : primPopRes
  Actions             : 7
  Objectives          : 0
  Criteria            : 3
  Attributes          : ['name', 'actions', 'objectives',
                         'criteria', 'weightPreorder',
                         'NA', 'evaluation']
```

When showing now the heatmap of the seven candidates approvals, disapprovals and abstentions, we see confirmed in Fig. 2.6 that both approvals and disapprovals scores model indeed the same linear ranking.

```
>>> t.showHTMLPerformanceHeatmap(Correlations=True,
...          ndigits=2,colorLevels=3,
...          pageTitle='Ranked primary election results',
...          WithActionNames=True)
```

Ranked primary election results

criteria	Approvals	Disapprovals	Abstentions
weights	+1.00	-1.00	+0.00
tau(*)	+1.00	+1.00	+0.00
Christiane Taubira (ct)	79.10	12.99	7.91
Yannick Jadot (yj)	65.25	19.21	15.54
Jean-Luc Mélenchon (jlm)	52.55	29.16	18.29
Pierre Larroutourou (pl)	47.32	34.58	18.11
Anne Hidalgo (ah)	40.39	35.81	23.80
Charlotte Marchandise (cm)	31.92	46.20	21.87
Anna Agueb Porterie (aap)	28.39	50.56	21.05

Color legend:

quantile	33.33%	66.66%	100.00%

(*) tau: *Ordinal (Kendall) correlation between marginal criterion and global ranking relation*
Outranking model: **standard**, *Ranking rule*: **NetFlows**
Ordinal (Kendall) correlation between global ranking and global outranking relation: **+1.000**
Mean marginal correlation (a) : **+1.000**
Standard marginal correlation deviation (b) : **+0.000**
Ranking fairness (a) - (b) : **+1.000**

Fig. 2.6: Ranked popular primary election results

Notice that it is in principle possible to allocate a *negative* significance weight to a performance criterion (see row 2 in Fig. 2.6). The constructor of the `outrankingDigraphs.BipolarOutrankingDigraph` class will, the case given, consider that the corresponding criterion supports a negative preference direction[11]. Allocating furthermore a zero significance weight to the abstentions does allow to ignore this figure in the ranking result. The ordinal correlation index becomes irrelevant in this case and is set to zero (see row 3).

It is eventually interesting to notice that the *NetFlows* ranking does precisely match the unique linear ranking modelled by the approval and disapproval votes. This exceptional situation indicates again that the majority of participating voters appear to belong to a very homogeneous political group –essentially *Taubira* supporters– which unfortunately invalidates thus the claim that the winner of this primary election represents actually the best suited multipartisan presidential candidate on the left and ecological border.

Back to *Content Table* (page 1)

---

3 Theoretical and computational advancements

- *Coping with missing data and indeterminateness* (page 63)
- *Ordinal correlation equals bipolar-valued relational equivalence* (page 69)
- *Applications of bipolar-valued base 3 encoded Bachet numbers* (page 79)
- *On computing digraph kernels* (page 99)
- *Computing with bipolar-valued sets* (page 119)
- *Bipolar-valued kernel membership characteristic vectors* (page 124)
- *On characterizing bipolar-valued outranking digraphs* (page 132)
- *Consensus quality of the bipolar-valued outranking relation* (page 140)

3.1 Coping with missing data and indeterminateness

In a stubborn keeping with a two-valued logic, where every argument can only be true or false, there is no place for efficiently taking into account missing data or logical indeterminateness. These cases are seen as problematic and, at best are simply ignored. Worst, in modern data science, missing data get often replaced with *fictive* values, potentially falsifying hence all subsequent computations.

In social choice problems like elections, *abstentions* are, however, frequently observed and represent a social expression that may be significant for revealing non represented social preferences.

[11] Only the standard bipolar-valued outranking model supports negative significance weights and positive evaluations. When using other outranking models, it is necessary to record, the case given, negative evaluations with a positive significance weight

In marketing studies, interviewees will not always respond to all the submitted questions. Again, such abstentions do sometimes contain nevertheless valid information concerning consumer preferences.

A motivating data set

Let us consider such a performance tableau in file graffiti07.py gathering a *Movie Magazine* 's rating of some movies that could actually be seen in town[1] (see Fig. 3.1).

```
>>> from perfTabs import PerformanceTableau
>>> t = PerformanceTableau('graffiti07')
>>> t.showHTMLPerformanceTableau(title='Graffiti Star wars',
...                               ndigits=0)
```

Graffiti Star wars

criteria	AP	AS	CF	CS	DR	FG	GS	JH	JPT	MR	RR	SF	SJ	TD	VT
weight	1.00	1.00	1.00	1.00	1.00	1.00	1.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	3.00
mv_AL	3	-1	2	-1	NA	NA	3	NA	2	NA	NA	NA	2	NA	2
mv_BI	1	-1	1	1	-1	NA	NA	NA	2	NA	2	NA	NA	NA	NA
mv_CM	NA	3	3	2	NA	NA	3	2	3	2	1	2	2	NA	2
mv_DF	NA	4	3	1	2	NA	1	3	2	2	2	3	-1	NA	1
mv_DG	3	2	2	3	3	1	4	3	3	3	NA	NA	-1	NA	3
mv_DI	1	2	NA	1	2	2	NA	1	2	2	NA	1	1	NA	NA
mv_DJ	3	1	3	NA	NA	NA	3	NA	2	2	NA	2	4	3	-1
mv_FC	3	2	2	1	3	NA	1	3	3	3	2	NA	2	1	3
mv_FF	2	3	2	1	2	2	NA	1	2	2	1	2	1	3	1
mv_GG	2	3	3	1	NA	NA	1	-1	NA	-1	2	NA	-1	NA	1
mv_GH	1	NA	1	-1	2	2	1	2	1	3	4	NA	NA	NA	NA
mv_HN	3	1	3	1	3	NA	4	3	2	NA	1	NA	3	3	3
mv_HP	1	NA	3	1	NA	NA	NA	1	1	2	3	NA	1	2	NA
mv_HS	2	4	2	3	2	2	2	3	4	2	3	NA	1	3	NA
mv_JB	3	4	3	NA	3	2	3	2	3	2	NA	2	2	NA	3
mv_MB	NA	2	NA	NA	1	NA	1	2	2	1	2	NA	2	2	NA
mv_NP	NA	1	3	1	NA	3	2	3	3	3	2	3	NA	NA	3
mv_PE	3	4	2	NA	3	1	NA	2	4	2	NA	3	3	NA	3
mv_QS	NA	3	2	NA	4	3	NA	4	4	3	3	4	NA	4	4
mv_RG	2	2	2	2	NA	NA	3	1	2	2	1	NA	NA	3	NA
mv_RR	3	2	NA	1	4	NA	4	3	3	4	3	3	NA	4	2
mv_SM	3	3	2	2	2	2	NA	3	2	2	3	NA	2	2	3
mv_TF	-1	NA	1	1	1	NA	NA	1	2	NA	-1	NA	-1	1	NA
mv_TM	2	1	2	2	NA	NA	2	2	2	3	NA	2	NA	4	NA
mv_TP	2	3	3	1	2	NA	NA	3	2	NA	2	NA	1	NA	2

Fig. 3.1: *Graffiti* magazine's movie ratings from September 2007

15 journalists and movie critics provide here their rating of 25 movies: 5 stars (*masterpiece*), 4 stars (*must be seen*), 3 stars (*excellent*), 2 stars (*good*), 1 star (*could be seen*), -1 star (*I do not like*), -2 (*I hate*), NA (*not seen*).

[1] *Graffiti*, Edition Revue Luxembourg, September 2007, p. 30. You may find the data file *graffiti07.py* (perfTabs.PerformanceTableau Format) in the *examples* directory of the Digraph3 resources

To aggregate all the critics' rating opinions, the *Graffiti* magazine provides for each movie a global score computed as an *average grade*, just ignoring the *not seen* data. These averages are thus not computed on comparable denominators; some critics do indeed use a more or less extended range of grades. The movies not seen by critic *SJ*, for instance, are favored, as this critic is more severe than others in her grading. Dropping the movies that were not seen by all the critics is here not possible either, as no one of the 25 movies was actually seen by all the critics. Providing any value for the missing data will as well always somehow falsify any global value scoring. What to do ?

A better approach is to rank the movies on the basis of pairwise bipolar-valued *at least as well rated as* opinions. Under this epistemic argumentation approach, missing data are naturally treated as opinion abstentions and hence do not falsify the logical computations. Such a ranking (see the tutorial on Ranking with incommensurable performance criteria) of the 25 movies is provided, for instance, by the **heat map** view shown in Fig. 3.2.

```
>>> t.showHTMLPerformanceHeatmap(Correlations=True,
...                              rankingRule='NetFlows',
...                              ndigits=0)
```

Ranking the movies

criteria	JH	JPT	AP	DR	MR	VT	GS	CS	SJ	RR	TD	CF	SF	AS	FG
weights	+3.00	+1.00	+1.00	+1.00	+1.00	+3.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00
tau(*)	+0.50	+0.43	+0.32	+0.26	+0.25	+0.23	+0.16	+0.14	+0.14	+0.13	+0.11	+0.11	+0.10	+0.08	+0.03
mv_QS	4	4	NA	4	3	4	NA	NA	NA	3	4	2	4	3	3
mv_RR	3	3	3	4	4	2	4	1	NA	3	4	NA	3	2	NA
mv_DG	3	3	3	3	3	3	4	3	-1	NA	NA	2	NA	2	1
mv_NP	3	3	NA	NA	3	3	2	1	NA	2	NA	3	3	1	3
mv_HN	3	2	3	3	NA	3	4	1	3	1	3	3	NA	1	NA
mv_HS	3	4	2	2	2	NA	2	3	1	3	3	2	NA	4	2
mv_SM	3	2	3	2	2	3	NA	2	2	3	2	2	NA	3	2
mv_JB	2	3	3	3	2	3	3	NA	2	NA	NA	3	2	4	2
mv_PE	2	4	3	3	2	3	NA	NA	3	NA	NA	2	3	4	1
mv_FC	3	3	3	3	3	3	1	1	2	2	1	2	NA	2	NA
mv_TP	3	2	2	2	NA	2	NA	1	1	2	NA	3	NA	3	NA
mv_CM	2	3	NA	NA	2	2	3	2	2	1	NA	3	2	3	NA
mv_DF	3	2	NA	2	2	1	1	1	-1	2	NA	3	3	4	NA
mv_TM	2	2	2	NA	3	NA	2	2	NA	NA	4	2	2	1	NA
mv_DJ	NA	2	3	NA	2	-1	3	NA	4	NA	3	3	2	1	NA
mv_AL	NA	2	3	NA	NA	2	3	-1	2	NA	NA	2	NA	-1	NA
mv_RG	1	2	2	NA	2	NA	3	2	NA	1	3	2	NA	2	NA
mv_MB	2	2	NA	1	1	NA	1	NA	2	2	2	NA	NA	2	NA
mv_GH	2	1	1	2	3	NA	1	-1	NA	4	NA	1	NA	NA	2
mv_HP	1	1	1	NA	2	NA	NA	1	1	3	2	3	NA	NA	NA
mv_BI	NA	2	1	-1	NA	NA	NA	1	NA	2	NA	1	NA	-1	NA
mv_DI	1	2	1	2	2	NA	NA	1	1	NA	NA	NA	1	2	2
mv_FF	1	2	2	2	2	1	NA	1	1	1	3	2	2	3	2
mv_GG	-1	NA	2	NA	-1	1	1	1	-1	2	NA	3	NA	3	NA
mv_TF	1	2	-1	1	NA	NA	NA	1	-1	-1	1	1	NA	NA	NA

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%

(*) tau: *Ordinal (Kendall) correlation between marginal criterion and global ranking relation*
Outranking model: **standard**, *Ranking rule*: **NetFlows**
Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.780**
Mean marginal correlation (a) : **+0.234**
Standard marginal correlation deviation (b) : **+0.147**
Ranking fairness (a) - (b) : **+0.087**

Fig. 3.2: *Graffiti* magazine's ordered movie ratings from September 2007

There is no doubt that movie *mv_QS*, with 6 '*must be seen*' marks, is correctly best-ranked and the movie *mv_TV* is worst-ranked with five '*don't like*' marks.

Modelling pairwise bipolar-valued rating opinions

Let us explicitly construct the underlying bipolar-valued outranking digraph and consult in Fig. 3.3 the pairwise characteristic values we observe between the two best-ranked movies, namely *mv_QS* and *mv_RR*.

```
>>> from outrankingDigraphs import BipolarOutrankingDigraph
>>> g = BipolarOutrankingDigraph(t)
>>> g.recodeValuation(-19,19) # integer characteristic values
```

(continues on next page)

(continued from previous page)

```
>>> g.showHTMLPairwiseOutrankings('mv_QS','mv_RR')
```

Pairwise Comparison

Comparing actions : (mv_QS,mv_RR)

crit.	wght.	g(x)	g(y)	diff	ind	wp	p	concord	wv	v	polarisation
AP	1.00	NA	3.00					0.00			
AS	1.00	3.00	2.00	+1.00	None	None	None	+1.00			
CF	1.00	2.00	NA					0.00			
CS	1.00	NA	1.00					0.00			
DR	1.00	4.00	4.00	+0.00	None	None	None	+1.00			
FG	1.00	3.00	NA					0.00			
GS	1.00	NA	4.00					0.00			
JH	3.00	4.00	3.00	+1.00	None	None	None	+3.00			
JPT	1.00	4.00	3.00	+1.00	None	None	None	+1.00			
MR	1.00	3.00	4.00	-1.00	None	None	None	-1.00			
RR	1.00	3.00	3.00	+0.00	None	None	None	+1.00			
SF	1.00	4.00	3.00	+1.00	None	None	None	+1.00			
SJ	1.00	NA	NA					0.00			
TD	1.00	4.00	4.00	+0.00	None	None	None	+1.00			
VT	3.00	4.00	2.00	+2.00	None	None	None	+3.00			

Valuation in range: -19.00 to +19.00; global concordance: +11.00

Pairwise Comparison

Comparing actions : (mv_RR,mv_QS)

crit.	wght.	g(x)	g(y)	diff	ind	wp	p	concord	wv	v	polarisation
AP	1.00	3.00	NA					0.00			
AS	1.00	2.00	3.00	-1.00	None	None	None	-1.00			
CF	1.00	NA	2.00					0.00			
CS	1.00	1.00	NA					0.00			
DR	1.00	4.00	4.00	+0.00	None	None	None	+1.00			
FG	1.00	NA	3.00					0.00			
GS	1.00	4.00	NA					0.00			
JH	3.00	3.00	4.00	-1.00	None	None	None	-3.00			
JPT	1.00	3.00	4.00	-1.00	None	None	None	-1.00			
MR	1.00	4.00	3.00	+1.00	None	None	None	+1.00			
RR	1.00	3.00	3.00	+0.00	None	None	None	+1.00			
SF	1.00	3.00	4.00	-1.00	None	None	None	-1.00			
SJ	1.00	NA	NA					0.00			
TD	1.00	4.00	4.00	+0.00	None	None	None	+1.00			
VT	3.00	2.00	4.00	-2.00	None	None	None	-3.00			

Valuation in range: -19.00 to +19.00; global concordance: -5.00

Fig. 3.3: Pairwise comparison of the two best-ranked movies

Six out of the fifteen critics have not seen one or the other of these two movies. Notice the higher significance (3) that is granted to two locally renowned movie critics, namely *JH* and *VT*. Their opinion counts for three times the opinion of the other critics. All nine critics that have seen both movies, except critic *MR*, state that *mv_QS* is rated at least as well as *mv_RR* and the balance of positive against negative opinions amounts to +11, a characteristic value which positively validates the outranking situation with a majority of (11/19 + 1.0) / 2.0 = 79%.

The complete table of pairwise majority margins of global '*at least as well rated as*' opinions, ranked by the same rule as shown in the heat map above (see Fig. 3.2), may be shown in Fig. 3.4.

```
>>> ranking = g.computeNetFlowsRanking()
>>> g.showHTMLRelationTable(actionsList=ranking, ndigits=0,
...     tableTitle='Bipolar characteristic values of\
...  "rated at least as good as" situations')
```

Bipolar characteristic values of "rated at least as good as" situations

r(x S y)	mv_QS	mv_RR	mv_DG	mv_NP	mv_HN	mv_HS	mv_SM	mv_JB	mv_PE	mv_FC	mv_TP	mv_CM	mv_DF	mv_TM	mv_DJ	mv_AL	mv_RG	mv_MB	mv_GH	mv_HP	mv_BI	mv_DI	mv_FF	mv_GG	mv_TF
mv_QS	–	11	12	11	10	9	14	9	11	13	9	10	9	9	7	6	9	9	7	6	5	9	15	8	8
mv_RR	-5	–	5	7	8	6	4	5	2	9	10	9	12	9	10	8	9	10	8	9	6	10	13	10	9
mv_DG	-8	9	–	9	10	5	9	7	8	13	7	7	10	10	6	8	10	7	9	5	6	7	9	9	9
mv_NP	-7	5	7	–	10	3	7	9	8	11	9	8	12	8	7	6	4	6	8	6	5	7	12	10	7
mv_HN	-10	4	8	8	–	5	9	9	8	10	10	7	10	6	8	10	7	6	8	8	5	7	13	9	11
mv_HS	-3	2	5	5	3	–	10	2	5	6	9	5	10	7	0	1	10	9	8	9	7	11	14	9	11
mv_SM	-8	6	7	5	7	6	–	6	6	10	12	9	10	6	4	9	9	10	7	9	7	11	15	11	11
mv_JB	-9	-1	5	1	3	4	8	–	9	6	6	13	6	8	9	9	9	9	8	8	5	11	15	12	8
mv_PE	-7	2	6	0	4	3	6	11	–	5	4	10	5	7	6	8	8	8	5	6	5	9	13	9	8
mv_FC	-9	5	11	9	8	2	8	8	7	–	10	6	11	5	3	8	6	9	9	5	7	10	12	10	11
mv_TP	-7	4	-1	3	2	3	0	-4	-4	0	–	6	11	6	4	5	7	6	7	7	7	9	14	12	10
mv_CM	-8	-1	-5	-4	-3	-3	-1	5	-2	-2	4	–	3	8	8	9	10	7	5	7	3	9	14	11	8
mv_DF	-9	-2	-2	2	2	2	2	0	1	1	5	1	–	4	6	-1	6	8	6	5	6	8	13	13	9
mv_TM	-3	-7	-6	-2	-2	-1	0	0	1	-1	-2	2	-2	–	2	2	7	6	9	7	5	7	9	5	8
mv_DJ	-7	-8	-4	-3	2	4	2	1	-2	-1	0	0	0	4	–	3	5	4	3	6	4	4	3	1	5
mv_AL	-4	0	-4	-6	-6	1	-1	-3	-4	-2	3	1	3	2	3	–	2	2	5	1	3	1	5	3	3
mv_RG	-7	-7	-6	-4	-1	-2	-1	-5	-4	-2	-3	-2	-2	1	3	4	–	1	0	6	4	8	9	4	9
mv_MB	-9	-8	-5	-4	-4	-9	-4	-1	-2	-1	-2	1	-2	2	-2	2	3	–	2	4	4	4	2	6	8
mv_GH	-5	-8	-7	-6	-8	-4	-3	0	1	-5	-5	1	-2	-1	-3	-3	0	6	–	5	2	5	3	3	5
mv_HP	-4	-5	-5	-2	-4	-3	-3	-4	-4	-3	-1	-3	1	-7	-2	-1	2	-2	-1	–	3	6	5	7	8
mv_BI	-5	-4	-6	-1	-1	-7	-5	-5	-5	-3	-1	-3	0	-3	-2	1	-2	0	2	1	–	1	-1	-1	4
mv_DI	-9	-6	-5	-5	-3	-3	-3	-7	-7	-6	-1	-7	0	-5	0	1	4	0	1	8	5	–	6	4	8
mv_FF	-11	-11	-7	-10	-5	0	-3	-9	-9	-8	-2	-6	1	-1	5	-1	9	0	1	7	5	12	–	9	11
mv_GG	-6	-8	-7	-4	-5	-7	-9	-10	-9	-4	-2	-7	3	-3	1	-3	-2	-2	-1	-3	5	-2	3	–	2
mv_TF	-8	-7	-7	-5	-7	-11	-9	-8	-8	-7	-6	-8	-3	-6	-3	-1	-1	-4	-3	0	2	2	-1	2	–

Valuation domain: [-19.00; +19.00]

Fig. 3.4: Pairwise majority margins of '*at least as well rated as*' rating opinions

Positive characteristic values, validating a global '*at least as well rated as*' opinion are marked in light green (see Fig. 3.4). Whereas negative characteristic values, invalidating such a global opinion, are marked in light red. We may by the way notice that the best-ranked movie *mv_QS* is indeed a *Condorcet* winner, i.e. *better rated than all the other movies* by a 65% majority of critics. This majority may be assessed from the average determinateness of the given bipolar-valued outranking digraph *g*.

```
>>> print( '%.0f%%' % g.computeDeterminateness(InPercents=True) )
65%
```

Notice also the *indeterminate* situation we observe, for instance, when comparing movie *mv_PE* with movie *mv_NP*.

```
>>> g.showHTMLPairwiseComparison('mv_PE','mv_NP')
```

Pairwise Comparison

Comparing actions : (mv_PE,mv_NP)

crit.	wght.	g(x)	g(y)	diff	ind	wp	p	concord	wv	v	polarisation
AP	1.00	3.00	NA					0.00			
AS	1.00	4.00	1.00	+3.00	None	None	None	+1.00			
CF	1.00	2.00	3.00	-1.00	None	None	None	-1.00			
CS	1.00	NA	1.00					0.00			
DR	1.00	3.00	NA					0.00			
FG	1.00	1.00	3.00	-2.00	None	None	None	-1.00			
GS	1.00	NA	2.00					0.00			
JH	3.00	2.00	3.00	-1.00	None	None	None	-3.00			
JPT	1.00	4.00	3.00	+1.00	None	None	None	+1.00			
MR	1.00	2.00	3.00	-1.00	None	None	None	-1.00			
RR	1.00	NA	2.00					0.00			
SF	1.00	3.00	3.00	+0.00	None	None	None	+1.00			
SJ	1.00	3.00	NA					0.00			
TD	1.00	NA	NA					0.00			
VT	3.00	3.00	3.00	+0.00	None	None	None	+3.00			

Valuation in range: -19.00 to +19.00; global concordance: +0.00

Fig. 3.5: Indeterminate pairwise comparison example

Only eight, out of the fifteen critics, have seen both movies and the positive opinions do neatly balance the negative ones. A global statement that *mv_PE* is '*at least as well rated as*' *mv_NP* may in this case hence **neither be validated, nor invalidated**; a preferential situation that cannot be modelled with any scoring approach.

It is fair, however, to eventually mention here that the *Graffiti* magazine's average scoring method is actually showing a very similar ranking. Indeed, average scores usually confirm well all evident pairwise comparisons, yet *enforce* comparability for all less evident ones.

Notice finally the ordinal correlation *tau* values in Fig. 3.2 3rd row. How may we compute these ordinal correlation indexes ?

Back to *Content Table* (page 1)

3.2 Ordinal correlation equals bipolar-valued relational equivalence

- *Kendall's* tau *index* (page 70)
- *Bipolar-valued relational equivalence* (page 71)
- *Fitness of ranking heuristics* (page 73)
- *Illustrating preference divergences* (page 76)
- *Exploring the* better rated *and the* as well as rated *opinions* (page 77)

Kendall's *tau* index

M. G. Kendall ([KEN-1938p]) defined his *ordinal correlation* τ (**tau**) *index* for linear orders of dimension n as a *balancing* of the number $\#Co$ of correctly oriented pairs against the number $\#In$ of incorrectly oriented pairs. The total number of irreflexive pairs being *n(n-1)*, in the case of linear orders, $\#Co + \#In = n(n-1)$. Hence $\tau = \left(\frac{\#Co}{n(n-1)}\right) - \left(\frac{\#In}{n(n-1)}\right)$. In case $\#In$ is zero, $\tau = +1$ (all pairs are *equivalently oriented*); inversely, in case $\#Co$ is zero, $\tau = -1$ (all pairs are *differently oriented*).

Noticing that $\frac{\#Co}{n(n-1)} = 1 - \frac{\#In}{n(n-1)}$, and recalling that the bipolar-valued negation is operated by changing the sign of the characteristic value, *Kendall*'s original *tau* definition implemented in fact the bipolar-valued **negation** of the **non equivalence** of two linear orders:

$$\tau = 1 - 2\frac{\#In}{n(n-1)} = -\left(2\frac{\#In}{n(n-1)} - 1\right) = 2\frac{\#Co}{n(n-1)} - 1,$$

i.e. the **normalized majority margin** of *equivalently oriented* irreflexive pairs.

Let *R1* and *R2* be two random crisp relations defined on a same set of 5 alternatives. We may compute Kendall's *tau* index as follows.

Listing 3.1: Crisp Relational Equivalence Digraph

```
1  >>> from randomDigraphs import RandomDigraph
2  >>> R1 = RandomDigraph(order=5,Bipolar=True)
3  >>> R2 = RandomDigraph(order=5,Bipolar=True)
4  >>> from digraphs import EquivalenceDigraph
5  >>> E = EquivalenceDigraph(R1,R2)
6  >>> E.showRelationTable(ReflexiveTerms=False)
7   * ---- Relation Table -----
8   r(<=>)|  'a1'        'a2'    'a3'    'a4'    'a5'
9   ------|------------------------------------------
10    'a1' |    -        -1.00    1.00   -1.00    1.00
11    'a2' |  -1.00        -     -1.00    1.00   -1.00
12    'a3' |  -1.00      -1.00     -      1.00    1.00
13    'a4' |  -1.00       1.00   -1.00     -      1.00
14    'a5' |  -1.00       1.00   -1.00    1.00     -
15  Valuation domain: [-1.00;1.00]
16 >>> E.correlation
17  {'correlation': -0.1, 'determination': 1.0}
```

In the table of the equivalence relation ($R_1 \Leftrightarrow R_2$) above (see Listing 3.1 Lines 10-14), we observe that the normalized majority margin of equivalent versus non equivalent irreflexive pairs amounts to (9 - 11)/20 = -0.1, i.e. the value of Kendall's *tau* index in this plainly determined crisp case (see Listing 3.1 Line 17).

What happens now with more or less determined and even partially indeterminate relations ? May we proceed in a similar way ?

Bipolar-valued relational equivalence

Let us now consider two randomly bipolar-valued digraphs *R1* and *R2* of order five.

Listing 3.2: Two Random Bipolar-valued Digraphs

```
1  >>> R1 = RandomValuationDigraph(order=5,seed=1)
2  >>> R1.showRelationTable(ReflexiveTerms=False)
3   * ---- Relation Table -----
4    r(R1)|   'a1'        'a2'    'a3'    'a4'    'a5'
5   ------|-----------------------------------------
6    'a1' |    -         -0.66    0.44    0.94   -0.84
7    'a2' |  -0.36         -     -0.70    0.26    0.94
8    'a3' |   0.14        0.20     -      0.66   -0.04
9    'a4' |  -0.48       -0.76    0.24     -     -0.94
10   'a5' |  -0.02        0.10    0.54    0.94     -
11  Valuation domain: [-1.00;1.00]
12 >>> R2 = RandomValuationDigraph(order=5,seed=2)
13 >>> R2.showRelationTable(ReflexiveTerms=False)
14  * ---- Relation Table -----
15   r(R2)|   'a1'        'a2'    'a3'    'a4'    'a5'
16  ------|-----------------------------------------
17   'a1' |    -         -0.86   -0.78   -0.80   -0.08
18   'a2' |  -0.58         -      0.88    0.70   -0.22
19   'a3' |  -0.36        0.54     -     -0.46    0.54
20   'a4' |  -0.92        0.48    0.74     -     -0.60
21   'a5' |   0.10        0.62    0.00    0.84     -
22  Valuation domain: [-1.00;1.00]
```

We may notice in the relation tables shown above that 9 pairs, like *(a1,a2)* or *(a3,a2)* for instance, appear equivalently oriented (see Listing 3.2 Lines 6,17 or 8,19). The `EquivalenceDigraph` class implements this *relational equivalence* relation between digraphs *R1* and *R2* (see Listing 3.3).

Listing 3.3: Bipolar-valued Equivalence Digraph

```
>>> eq = EquivalenceDigraph(R1,R2)
>>> eq.showRelationTable(ReflexiveTerms=False)
 * ---- Relation Table -----
 r(<=>)|  'a1'        'a2'    'a3'    'a4'    'a5'
 ------|-----------------------------------------
  'a1' |   -          0.66   -0.44   -0.80    0.08
  'a2' |  0.36         -     -0.70    0.26   -0.22
  'a3' | -0.14        0.20     -     -0.46   -0.04
  'a4' |  0.48       -0.48    0.24     -      0.60
  'a5' | -0.02        0.10    0.00    0.84     -
 Valuation domain: [-1.00;1.00]
```

In our bipolar-valued epistemic logic, logical disjunctions and conjunctions are implemented as *max*, respectively *min* operators. Notice also that the logical equivalence

$(R_1 \Leftrightarrow R_2)$ corresponds to a double implication $(R_1 \Rightarrow R_)) \wedge (R_2 \Rightarrow R_1)$ and that the implication $(R_1 \Rightarrow R_2)$ is logically equivalent to the disjunction $(\neg R_1 \vee R_2)$.

When $r(x\,R_1\,y)$ and $r(x\,R_2\,y)$ denote the bipolar-valued characteristic values of relation *R1*, resp. *R2*, we may hence compute as follows a majority margin $M(R_1 \Leftrightarrow R_2)$ between equivalently and not equivalently oriented irreflexive pairs *(x,y)*.

$$M(R_1 \Leftrightarrow R_2) = \sum_{(x \neq y)} \Big[\min\Big(\max\big(-r(x\,R_1\,y), r(x\,R_2\,y)\big), \max\big(-r(x\,R_2\,y), r(x\,R_1\,y)\big)\Big)\Big].$$

$M(R_1 \Leftrightarrow R_2)$ is thus given by the sum of the non reflexive terms of the relation table of *eq*, the relation equivalence digraph computed above (see Listing 3.3).

In the crisp case, $M(R_1 \Leftrightarrow R_2)$ is now normalized with the maximum number of possible irreflexive pairs, namely *n(n-1)*. In a generalized r-valued case, the maximal possible equivalence majority margin M corresponds to the sum D of the **conjoint determinations** of $(x\,R_1\,y)$ and $(x\,R2\,y)$ (see [BIS-2012p]).

$$D \;=\; \sum_{x \neq y} \min\Big[abs\big(r(x\,R_1\,y)\big), abs\big(r(x\,R_2\,y)\big)\Big]\;.$$

Thus, we obtain in the general r -valued case:

$$\tau(R_1, R_2) \;=\; \frac{M(R_1 \Leftrightarrow R_2)}{D}\;.$$

$\tau(R_1, R_2)$ corresponds thus to a classical ordinal correlation index, but restricted to the **conjointly determined parts** of the given relations *R1* and *R2*. In the limit case of two crisp linear orders, D equals *n(n-1)*, i.e. the number of irreflexive pairs, and we recover hence *Kendall* 's original *tau* index definition.

It is worthwhile noticing that the ordinal correlation index $\tau(R_1, R_2)$ we obtain above corresponds to the ratio of

> $r(R_1 \Leftrightarrow R_2) \;=\; \frac{M(R_1 \Leftrightarrow R_2)}{n(n-1)}$: the normalized majority margin of the pairwise *relational* equivalence statements, also called *valued ordinal correlation*, and
>
> $d \;=\; \frac{D}{n(n-1)}$: the normalized determination of the corresponding pairwise relational equivalence statements, in fact the *determinateness* of the relational equivalence digraph.

We have thus successfully **out-factored** the *determination* effect from the *correlation* effect. With completely determined relations, $\tau(R_1, R_2) \;=\; r(R_1 \Leftrightarrow R_2)$. By convention, we set the ordinal correlation with a *completely indeterminate* relation, i.e. when D = *0*, to the *indeterminate* correlation value 0.0. With *uniformly* chosen random r-valued relations, the **expected** *tau* index is **0.0**, denoting in fact an **indeterminate** correlation. The corresponding expected normalized determination d is about 0.333 (see [BIS-2012p]).

We may verify these relations with help of the corresponding equivalence digraph *eq* (see Listing 3.4).

Listing 3.4: Computing the Ordinal Correlation Index from the Equivalence Digraph

```
>>> eq = EquivalenceDigraph(R1,R2)
>>> M = Decimal('0'); D = Decimal('0')
>>> n2 = eq.order*(eq.order - 1)
>>> for x in eq.actions:
...     for y in eq.actions:
...         if x != y:
...             M += eq.relation[x][y]
...             D += abs(eq.relation[x][y])
>>> print('r(R1<=>R2) = %+.3f, d = %.3f, tau = %+.3f' % (M/n2,D/n2,M/D))

 r(R1<=>R2) = +0.026, d = 0.356, tau = +0.073
```

In general we simply use the `computeOrdinalCorrelation()` method which renders a dictionary with a '*correlation*' (*tau*) and a '*determination*' (*d*) attribute. We may recover *r(<=>)* by multiplying *tau* with *d* (see Listing 3.5 Line 4).

Listing 3.5: Directly Computing the Ordinal Correlation Index

```
1 >>> corrR1R2 = R1.computeOrdinalCorrelation(R2)
2 >>> tau = corrR1R2['correlation']
3 >>> d = corrR1R2['determination']
4 >>> r = tau * d
5 >>> print('tau(R1,R2) = %+.3f, d = %.3f,\
6 ...        r(R1<=>R2) = %+.3f' % (tau, d, r))
7
8  tau(R1,R2) = +0.073, d = 0.356, r(R1<=>R2) = +0.026
```

We provide for convenience a direct `showCorrelation()` method:

```
>>> corrR1R2 = R1.computeOrdinalCorrelation(R2)
>>> R1.showCorrelation(corrR1R2)
 Correlation indexes:
  Extended Kendall tau       : +0.073
  Epistemic determination    :  0.356
  Bipolar-valued equivalence : +0.026
```

We may now illustrate the quality of the global ranking of the movies shown with the heat map in Fig. 3.2.

Fitness of ranking heuristics

We reconsider the bipolar-valued outranking digraph g modelling the pairwise global '*at least as well rated as*' relation among the 25 movies seen in the topic before (see Fig. 3.2).

Listing 3.6: Global Movies Outranking Digraph

```
>>> from perfTabs import PerformanceTableau
>>> t = PerformanceTableau('graffiti07')
>>> from outrankingDigraphs import BipolarOutrankingDigraph
>>> g = BipolarOutrankingDigraph(t,Normalized=True)
 *------- Object instance description ------*
 Instance class   : BipolarOutrankingDigraph
 Instance name    : rel_grafittiPerfTab.xml
 # Actions        : 25
 # Criteria       : 15
 Size             : 390
 Determinateness  : 65%
 Valuation domain : {'min': Decimal('-1.0'),
                     'med': Decimal('0.0'),
                     'max': Decimal('1.0'),}
>>> g.computeCoSize()
 188
```

Out of the 25 x 24 = 600 irreflexive movie pairs, digraph *g* contains 390 positively validated, 188 positively invalidated, and 22 *indeterminate* outranking situations (see the zero-valued cells in Fig. 3.4).

Let us now compute the normalized majority margin *r(<=>)* of the equivalence between the marginal critic's pairwise ratings and the global *NetFlows* ranking shown in the ordered heat map (see Fig. 3.2).

Listing 3.7: Marginal Criterion Correlations with global *NetFlows* Ranking

```
>>> from linearOrders import NetFlowsOrder
>>> nf = NetFlowsOrder(g)
>>> nf.netFlowsRanking
 ['mv_QS', 'mv_RR', 'mv_DG', 'mv_NP', 'mv_HN', 'mv_HS', 'mv_SM',
  'mv_JB', 'mv_PE', 'mv_FC', 'mv_TP', 'mv_CM', 'mv_DF', 'mv_TM',
  'mv_DJ', 'mv_AL', 'mv_RG', 'mv_MB', 'mv_GH', 'mv_HP', 'mv_BI',
  'mv_DI', 'mv_FF', 'mv_GG', 'mv_TF']
>>> for i,item in enumerate(\
...        g.computeMarginalVersusGlobalRankingCorrelations(\
...               nf.netFlowsRanking,ValuedCorrelation=True) ):\
...     print('r(%s<=>nf) = %+.3f' % (item[1],item[0]) )

 r(JH<=>nf)  = +0.500
 r(JPT<=>nf) = +0.430
 r(AP<=>nf)  = +0.323
 r(DR<=>nf)  = +0.263
 r(MR<=>nf)  = +0.247
 r(VT<=>nf)  = +0.227
```

(continues on next page)

(continued from previous page)

```
19  r(GS<=>nf)  = +0.160
20  r(CS<=>nf)  = +0.140
21  r(SJ<=>nf)  = +0.137
22  r(RR<=>nf)  = +0.133
23  r(TD<=>nf)  = +0.110
24  r(CF<=>nf)  = +0.110
25  r(SF<=>nf)  = +0.103
26  r(AS<=>nf)  = +0.080
27  r(FG<=>nf)  = +0.027
```

In Listing 3.7 (see Lines 13-27), we recover above the relational equivalence characteristic values shown in the third row of the table in Fig. 3.2. The global *NetFlows* ranking represents obviously a rather balanced compromise with respect to all movie critics' opinions as there appears no valued negative correlation with anyone of them. The *NetFlows* ranking apparently takes also correctly in account that the journalist *JH*, a locally renowned movie critic, shows a higher significance weight (see Line 13).

The ordinal correlation between the global *NetFlows* ranking and the digraph g may be furthermore computed as follows:

Listing 3.8: Ordinal correlation between *NetFlows* ranking and outranking digraph

```
1  >>> corrgnf = g.computeOrdinalCorrelation(nf)
2  >>> g.showCorrelation(corrgnf)
3   Correlation indexes:
4    Extended Kendall tau       : +0.780
5    Epistemic determination    :  0.300
6    Bipolar-valued equivalence : +0.234
```

We notice in Listing 3.8 Line 4 that the ordinal correlation *tau(g,nf)* index between the *NetFlows* ranking *nf* and the determined part of the outranking digraph g is quite high ($+0.78$). Due to the rather high number of missing data, the r -valued relational equivalence between the *nf* and the g digraph, with a characteristics value of *only* $+0.234$, may be misleading. Yet, $+0.234$ still corresponds to an epistemic majority support of nearly 62% of the movie critics' rating opinions.

It would be interesting to compare similarly the correlations one may obtain with other global ranking heuristics, like the *Copeland* or the *Kohler* ranking rule.

Illustrating preference divergences

The valued relational equivalence index gives us a further measure for studying how **divergent** appear the rating opinions expressed by the movie critics.

```
>>> g = BipolarOutrankingDigraph(t,Normalized=True)
>>> g.showCriteriaCorrelationTable(ValuedCorrelation=True)
Criteria valued ordinal correlation index
      |  AP    AS    CF    CS    DR    FG    GS    JH   JPT    MR    RR    SF    SJ    TD    VT
------|--------------------------------------------------------------------------------------------
   AP | +0.63 +0.04 +0.19 +0.09 +0.22 -0.01 +0.11 +0.23 +0.25 +0.08 +0.02 +0.04 +0.19 +0.04 +0.12
   AS |       +0.77 +0.12 +0.12 +0.04 -0.02 -0.06 +0.02 +0.24 -0.08 +0.07 +0.04 -0.07 -0.01 +0.02
   CF |             +0.77 +0.07 +0.11 +0.03 +0.05 +0.07 +0.10 -0.03 +0.01 +0.00 +0.06 +0.03 -0.04
   CS |                   +0.63 +0.04 -0.02 +0.07 +0.13 +0.25 +0.01 +0.03 +0.00 +0.02 +0.03 +0.07
   DR |                         +0.45 +0.03 +0.07 +0.17 +0.23 +0.16 +0.06 +0.03 +0.10 +0.07 +0.10
   FG |                               +0.15 -0.01 +0.04 +0.01 +0.06 -0.00 +0.02 +0.01 +0.01 +0.02
   GS |                                     +0.40 +0.07 +0.07 +0.09 -0.02 +0.00 +0.06 +0.04 +0.04
   JH |                                           +0.77 +0.28 +0.26 +0.15 +0.12 +0.10 +0.05 +0.14
  JPT |                                                 +0.92 +0.15 +0.06 +0.09 +0.08 +0.08 +0.17
   MR |                                                       +0.63 +0.10 +0.08 +0.03 +0.09 +0.10
   RR |                                                             +0.51 +0.04 +0.01 +0.05 +0.05
   SF |                                                                   +0.18 +0.01 +0.02 +0.05
   SJ |                                                                         +0.51 +0.03 +0.07
   TD |                                                                               +0.26 +0.00
   VT |                                                                                     +0.40
```

Fig. 3.6: Pairwise valued correlation of movie critics

It is remarkable to notice in the criteria correlation matrix (see Fig. 3.6) that, due to the quite numerous missing data, all pairwise valued ordinal correlation indexes *r(x<=>y)* appear to be of low value, except the *diagonal* ones. These reflexive indexes *r(x<=>x)* would trivially all amount to +1.0 in a plainly determined case. Here they indicate a reflexive normalized determination score d, i.e. the *proportion* of pairs of movies each critic did evaluate. Critic JPT (the editor of the Graffiti magazine), for instance, evaluated all but one ($d = 24{*}23/600 = 0.92$), whereas critic FG evaluated only 10 movies among the 25 in discussion ($d = 10{*}9/600 = 0.15$).

To get a picture of the actual *divergence of rating opinions* concerning **jointly seen** pairs of movies, we may develop a *Principal Component Analysis* ([2]) of the corresponding *tau* correlation matrix. The 3D plot of the first 3 principal axes is shown in Fig. 3.7.

```
>>> g.export3DplotOfCriteriaCorrelation(ValuedCorrelation=False)
```

[2] The 3D PCA plot method requires a running *R statistics software* (https://www.r-project.org/) installation and the Calmat matrix calculator (see the calmat directory in the Digraph3 ressources)

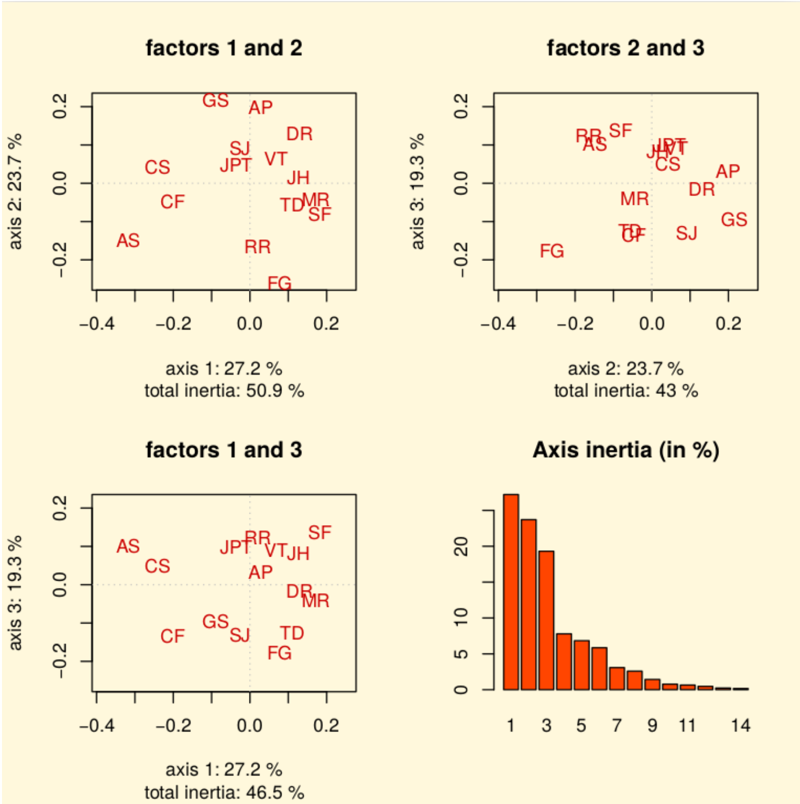


Fig. 3.7: 3D PCA plot of the criteria ordinal correlation matrix

The first 3 principal axes support together about 70% of the total inertia. Most *eccentric* and *opposed* in their respective rating opinions appear, on the first principal axis with 27.2% inertia, the conservative daily press against labour and public press. On the second principal axis with 23.7.7% inertia, it is the people press versus the cultural critical press. And, on the third axis with still 19.3% inertia, the written media appear most opposed to the radio media.

Exploring the *better rated* and the *as well as rated* opinions

In order to furthermore study the quality of a ranking result, it may be interesting to have a separate view on the asymmetrical and symmetrical parts of the '*at least as well rated as*' opinions (see the tutorial on Manipulating Digraph objects).

Let us first have a look at the pairwise asymmetrical part, namely the '*better rated than*' and '*less well rated than*' opinions of the movie critics.

```
>>> from digraphs import AsymmetricPartialDigraph
>>> ag = AsymmetricPartialDigraph(g)
>>> ag.showHTMLRelationTable(actionsList=g.computeNetFlowsRanking(),
↪ndigits=0)
```

Valued Adjacency Matrix

r(x S y)	mv_QS	mv_RR	mv_DG	mv_NP	mv_HN	mv_HS	mv_SM	mv_JB	mv_PE	mv_FC	mv_TP	mv_CM	mv_DF	mv_TM	mv_DJ	mv_AL	mv_RG	mv_MB	mv_GH	mv_HP	mv_BI	mv_DI	mv_FF	mv_GG	mv_TF
mv_QS	–	11	12	11	10	9	14	9	11	13	9	10	9	9	7	6	9	9	7	6	5	9	15	8	8
mv_RR	-5	–	0	0	0	0	0	5	0	0	0	9	12	9	10	8	9	10	8	9	6	10	13	10	9
mv_DG	-8	0	–	0	0	0	0	0	0	0	7	7	10	10	6	8	10	7	9	5	6	7	9	9	9
mv_NP	-7	0	0	–	0	0	0	0	8	0	0	8	0	8	7	6	4	6	8	6	5	7	12	10	7
mv_HN	-10	0	0	0	–	0	0	0	0	0	0	7	0	6	0	10	7	6	8	8	5	7	13	9	11
mv_HS	-3	0	0	0	0	–	0	0	0	0	0	5	0	7	0	0	10	9	8	9	7	11	14	9	11
mv_SM	-8	0	0	0	0	0	–	0	0	0	12	9	0	6	0	9	9	10	7	9	7	11	15	11	11
mv_JB	-9	-1	0	0	0	0	0	–	0	0	6	0	6	8	0	9	9	9	8	8	5	11	15	12	8
mv_PE	-7	0	0	0	0	0	0	0	–	0	4	10	0	0	6	8	8	8	0	6	5	9	13	9	8
mv_FC	-9	0	0	0	0	0	0	0	0	–	10	6	0	5	3	8	6	9	9	5	7	10	12	10	11
mv_TP	-7	0	-1	0	0	0	0	-4	-4	0	–	0	0	6	4	0	7	6	7	7	7	9	14	12	10
mv_CM	-8	-1	-5	-4	-3	-3	-1	0	-2	-2	0	–	0	0	8	0	10	0	0	7	3	9	14	11	8
mv_DF	-9	-2	-2	0	0	0	0	0	0	0	0	0	–	4	6	-1	6	8	6	0	6	8	0	0	9
mv_TM	-3	-7	-6	-2	-2	-1	0	0	0	-1	-2	0	-2	–	0	0	0	0	9	7	5	7	9	5	8
mv_DJ	-7	-8	-4	-3	0	4	0	0	-2	-1	0	0	0	0	–	0	0	4	3	6	4	4	0	0	5
mv_AL	-4	0	-4	-6	-6	0	-1	-3	-4	-2	0	0	3	0	0	–	0	0	5	1	0	0	5	3	3
mv_RG	-7	-7	-6	-4	-1	-2	-1	-5	-4	-2	-3	-2	-2	0	0	0	–	0	0	0	4	0	0	4	9
mv_MB	-9	-8	-5	-4	-4	-9	-4	-1	-2	-1	-2	0	-2	0	-2	0	0	–	0	4	4	4	2	6	8
mv_GH	-5	-8	-7	-6	-8	-4	-3	0	0	-5	-5	0	-2	-1	-3	-3	0	0	–	5	0	0	0	3	5
mv_HP	-4	-5	-5	-2	-4	-3	-3	-4	-4	-3	-1	-3	0	-7	-2	-1	0	-2	-1	–	0	0	0	7	8
mv_BI	-5	-4	-6	-1	-1	-7	-5	-5	-5	-3	-1	-3	0	-3	-2	0	-2	0	0	0	–	0	-1	-1	0
mv_DI	-9	-6	-5	-5	-3	-3	-3	-7	-7	-6	-1	-7	0	-5	0	0	0	0	0	0	0	–	0	4	0
mv_FF	-11	-11	-7	-10	-5	0	-3	-9	-9	-8	-2	-6	0	-1	0	-1	0	0	0	0	5	0	–	0	11
mv_GG	-6	-8	-7	-4	-5	-7	-9	-10	-9	-4	-2	-7	0	-3	0	-3	-2	-2	-1	-3	5	-2	0	–	0
mv_TF	-8	-7	-7	-5	-7	-11	-9	-8	-8	-7	-6	-8	-3	-6	-3	-1	-1	-4	-3	0	0	0	-1	0	–

Valuation domain: [-19.00; +19.00]

Fig. 3.8: Asymmetrical part of graffiti07 digraph

We notice here that the *NetFlows* ranking rule inverts in fact just three '*less well ranked than*' opinions and four '*better ranked than*' ones. A similar look at the symmetric part, the pairwise '*as well rated as*' opinions, suggests a preordered preference structure in several *equivalently rated* classes.

```
>>> from digraphs import SymmetricPartialDigraph
>>> sg = SymmetricPartialDigraph(g)
>>> sg.showHTMLRelationTable(actionsList=g.computeNetFlowsRanking(),
↪ndigits=0)
```

Valued Adjacency Matrix

r(x S y)	mv_QS	mv_RR	mv_DG	mv_NP	mv_HN	mv_HS	mv_SM	mv_JB	mv_PE	mv_FC	mv_TP	mv_CM	mv_DF	mv_TM	mv_DJ	mv_AL	mv_RG	mv_MB	mv_GH	mv_HP	mv_BI	mv_DI	mv_FF	mv_GG	mv_TF
mv_QS	–	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
mv_RR	0	–	5	7	8	6	4	0	2	9	10	0	0	0	0	8	0	0	0	0	0	0	0	0	0
mv_DG	0	9	–	9	10	5	9	7	8	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
mv_NP	0	5	7	–	10	3	7	9	8	11	9	0	12	0	0	0	0	0	0	0	0	0	0	0	0
mv_HN	0	4	8	8	–	5	9	9	8	10	10	0	10	0	8	0	0	0	0	0	0	0	0	0	0
mv_HS	0	2	5	5	3	–	10	2	5	6	9	0	10	0	0	1	0	0	0	0	0	0	14	0	0
mv_SM	0	6	7	5	7	6	–	6	6	10	12	0	10	6	4	0	0	0	0	0	0	0	0	0	0
mv_JB	0	0	5	1	3	4	8	–	9	6	0	13	6	8	9	0	0	0	8	0	0	0	0	0	0
mv_PE	0	2	6	0	4	3	6	11	–	5	0	0	5	7	0	0	0	0	5	0	0	0	0	0	0
mv_FC	0	5	11	9	8	2	8	8	7	–	10	0	11	0	0	0	0	0	0	0	0	0	0	0	0
mv_TP	0	4	0	3	2	3	0	0	0	0	–	6	11	0	4	5	0	0	0	0	0	0	0	0	0
mv_CM	0	0	0	0	0	0	0	5	0	0	4	–	3	8	8	9	0	7	5	0	0	0	0	0	0
mv_DF	0	0	0	2	2	2	2	0	1	1	5	1	–	0	6	0	0	0	0	5	6	8	13	13	0
mv_TM	0	0	0	0	0	0	0	0	1	0	0	2	0	–	2	2	7	6	0	0	0	0	0	0	0
mv_DJ	0	0	0	0	2	4	2	1	0	0	0	0	0	4	–	3	5	0	0	0	0	4	3	1	0
mv_AL	0	0	0	0	0	1	0	0	0	0	3	1	0	2	3	–	2	2	0	0	3	1	0	0	0
mv_RG	0	0	0	0	0	0	0	0	0	0	0	0	0	1	3	4	–	1	0	6	0	8	9	0	0
mv_MB	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	2	3	–	2	0	4	4	2	0	0
mv_GH	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	6	–	0	2	5	3	0	0
mv_HP	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	2	0	0	–	3	6	5	0	8
mv_BI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	2	1	–	1	0	0	4
mv_DI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	4	0	1	8	5	–	6	0	8
mv_FF	0	0	0	0	0	0	0	0	0	0	0	0	1	0	5	0	9	0	1	7	0	12	–	9	0
mv_GG	0	0	0	0	0	0	0	0	0	0	0	0	3	0	1	0	0	0	0	0	0	0	3	–	2
mv_TF	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	2	–

Valuation domain: [-19.00; +19.00]

Fig. 3.9: Symmetrical part of graffiti07 digraph

Such a preordering of the movies may, for instance, be computed with the `computeRankingByChoosing()` method, where we iteratively extract *dominant kernels* -remaining first choices- and *absorbent kernels* -remaining last choices- (see the tutorial on *Computing Digraph Kernels* (page 99)). We operate therefore on the asymmetrical '*better rated than*', i.e. the *codual* ([3]) of the '*at least as well rated as*' opinions (see Listing 3.9 Line 2).

Listing 3.9: Ranking by choosing the Grafitti movies

```
1  >>> from transitiveDigraphs import RankingByChoosingDigraph
2  >>> rbc = RankingByChoosingDigraph(g,CoDual=True)
3  >>> rbc.showRankingByChoosing()
4   Ranking by Choosing and Rejecting
5    1st First Choice ['mv_QS']
6      2nd First Choice ['mv_DG', 'mv_FC', 'mv_HN', 'mv_HS', 'mv_NP',
7                        'mv_PE', 'mv_RR', 'mv_SM']
8        3rd First Choice ['mv_CM', 'mv_JB', 'mv_TM']
9          4th First Choice ['mv_AL', 'mv_TP']
10         4th Last Choice ['mv_AL', 'mv_TP']
11       3rd Last Choice ['mv_GH', 'mv_MB', 'mv_RG']
12     2nd Last Choice ['mv_DF', 'mv_DJ', 'mv_FF', 'mv_GG']
13   1st Last Choice ['mv_BI', 'mv_DI', 'mv_HP', 'mv_TF']
```

Back to *Content Table* (page 1)

3.3 Applications of bipolar-valued base 3 encoded Bachet numbers

> "*The complexity of arithmetic circuitry for balanced ternary arithmetic is not much greater than it is for the binary system, and a given number requires only* $\log_3 2 \approx 63\%$ *as many digit positions for its representation. [...] perhaps the symmetric properties and simple arithmetic of this number system will prove to be quite important someday* [...]" – D. Knuth [KNU-1997]

- *Bipolar-valued base 3 encoded Bachet numbers* (page 80)
- *Examples of Bachet numbers* (page 81)
- *The Bachet ranking rule, a new ranking-by-scoring method* (page 84)
- *Efficiency of the Bachet ranking rule settings* (page 89)
- *Smart sampling of the permutations of the actions list* (page 92)
- *Revealing the transitive part of a bipolar-valued digraph* (page 93)
- *The Bachet choice recommender algorithm* (page 95)
- *Notes* (page 98)

[3] A *kernel* in a digraph g is a *clique* in the dual digraph $-g$.

Bipolar-valued base 3 encoded Bachet numbers

> "*Estant proposee telle quantité qu'on voudra pesant un nombre de livres depuis 1. iusques à 40. inclusivement (sans toutefois admettre les fractions) on demande combien de pois pour le moins il faudrait employer à cet effect.*"[24]
> – Cl. G. Bachet (1622) [BAC-1622p]

Bipolar-valued {-1,0,+1} base 3 encoded integers are due to *Claude Gaspard Bachet de Méziriac* (1581-1638)[20]. The idea is to represent the value of an integer n in a base 3 positional numeration where in each position may appear one of the three symbols **{-1,0,+1}** called hereafter **sbits** for short.

Bachet's positional *sbits* numeration, called nowadays **balanced ternary numeral system**[25] , is simulating a weight balance scale where the number n and the potential negative powers of 3 are put on the right tray and the potential positive powers of 3 are put on the left tray. The equation for $n = 5$ gives for instance $3^2 = (n + 3^1 + 3^0)$. And the *sbits* encoding corresponds hence to the string '+1-1-1'. As, this representation is isomorphic to a base 3 ternary encoding, every positive or negative integer may so be represented with a unique *sbits* string. With four powers of 3, namely $3^3, 3^2, 3^1, 3^0$, one may for instance represent any value in the integer range -40 to +40. *Bachet* showed that this bipolar weighing system relies on the smallest possible number of balance weights -base powers- needed in order to balance the scale for any given integer weight n [BAC-1622p].

The Digraph3 `bachetNumbers` module provides with the `BachetInteger` class an implementation for such *sbits* encoded integers. Instantiating a *Bachet* number may be done either with an integer value or with a vector of sbits (see Listing 3.10 Lines 2, 6, 11 and 15). The class provides the classical arithmetic operators from the standard *int* class, like binary *addition* and unary *negating* and *reversing* as illustrated in Lines 20 and 32-34 below.

Listing 3.10: Working with *Bachet* sbits encoded numbers

```
1 >>> from bachetNumbers import BachetInteger as BachetNumber
2 >>> n1 = BachetNumber(5)
3 >>> n1
4   *------- Bachet number description ------*
5   Instance class : BachetInteger
6   String         : '+1-1-1'
```

(continues on next page)

[24] "*Being proposed to assess with a weight balance a quantity weighing an integer number of pounds between 1 and 40 included, how many balance weights at minimum are needed for being able to do so*".

[20] https://en.wikipedia.org/wiki/Claude_Gaspar_Bachet_de_M%C3%A9ziriac *Claude Gaspar Bachet* Sieur de Méziriac (9 October 1581 – 26 February 1638) was a French mathematician and poet who is known today for his 1622 proof of *Bézout*'s theorem stating the special case of the *Bachet-Bézout* identity for two coprime integers. *Étienne Bézout* actually proved this result in 1779 only for polynomials and *Bézout*'s theorem is misattributed to *Bézout* by *Bourbaki*. The general *Bachet-Bézout* identity is a direct algebraic consequence of *Euclid*'s division algorithm and was known before *Bachet* (see the `arithmetics.bezout()` method). It is furthermore in a Latin Translation by *Bachet* of the Arithmetica of *Diophantus* where *Pierre de Fermat* wrote in 1638 his famous margin note about the missing proof of his last theorem.

[25] https://en.wikipedia.org/wiki/Balanced_ternary

(continued from previous page)

```
  Vector         : [1, -1, -1]
  Length         : 3
  Value          : 5
  Attributes     : ['vector']
>>> n2 = BachetNumber(vector=[1,1,1])
>>> n2
  *------- Bachet number description ------*
  Instance class : BachetInteger
  String         : '+1+1+1'
  Vector         : [1, 1, 1]
  Length         : 3
  Value          : 13
  Attributes     : ['vector']
>>> n3 = n1 + n2
>>> n3
  *------- Bachet number description ------*
  Instance class : BachetInteger
  String         : '+1-100'
  Vector         : [1, -1, 0, 0]
  Length         : 4
  Value          : 18
  Attributes     : ['vector']
>>> print('\'%s\' (%d) + \'%s\' (%d) = \'%s\' (%d)'
...         % (n1, int(n1), n2, int(n2), n3, int(n3) ))
  '+1-1-1' (5) + '+1+1+1' (13) = '+1-100' (18)
>>> n4 = ~n1    # ~n1 = n1.reverse()
>>> n5 = -n2
>>> n6 = n4 + n5        # n6 = ~n1 - n2
>>> print('\'%s\' (%d) + \'%s\' (%d) = \'%s\' (%d)'
...        % ( n4, int(n4), n5, int(n5), n6, int(n6) ) )
  '-1-1+1' (-11) + '-1-1-1' (-13) = '-10+10' (-24)
```

Examples of Bachet numbers

Examples of such *sbits* encoded *Bachet* numbers are immediately provided by the rows and columns of the *self.relation* attribute of a polarised outranking digraph instance (see Listing 3.11 Lines 4-6 and 12-15 below).

Listing 3.11: Examples of sbits encoded numbers

```
1 >>> from outrankingDigraphs import *
2 >>> from linearOrders import *
3 >>> from bachetNumbers import BachetInteger as BachetNumber
4 >>> g = RandomBipolarOutrankingDigraph(numberOfActions=4,seed=1)
5 >>> pg = PolarisedDigraph(g,level=g.valuationdomain['med'],
6 ...                       StrictCut=True,KeepValues=False)
```

(continues on next page)

(continued from previous page)

```
7  >>> pg.recodeValuation(ndigits=0)
8  >>> pg.showRelationTable(ReflexiveTerms=False)
9   * ---- Relation Table -----
10      S    | 'a1' 'a2' 'a3' 'a4'
11    ------|-------------------------
12     'a1' |    -    0    1   -1
13     'a2' |    1    -   -1   -1
14     'a3' |    1    1    -   -1
15     'a4' |    1    1    1    -
16 >>> ra1 = BachetNumber(vector=[0,1,-1])
17 >>> ra2 = BachetNumber(vector=[1,-1,-1])
18 >>> ra3 = BachetNumber(vector=[1,1,-1])
19 >>> ra4 = BachetNumber(vector=[1,1,1])
20 >>> print( int(ra1), int(ra2), int(ra3), int(ra4) )
21   2 5 11 13
22 >>> ca1 = BachetNumber(vector=[1,1,1])
23 >>> ca2 = BachetNumber(vector=[0,1,1])
24 >>> ca3 = BachetNumber(vector=[1,-1,1])
25 >>> ca4 = BachetNumber(vector=[-1,-1,-1])
26 >>> print(int(ca1), int(ca2), int(ca3), int(ca4) )
27   13 4 7 -13
28 >>> print( int(ra1-ca1), int(ra2-ca2),
29 ...        int(ra3-ca3), int(ra4-ca4) )
30   -11 1 4 26
```

The *Bachet* numbers, instantiated by the row vectors without reflexive terms and the column vectors without reflexive terms of the digraph's *self.relation* attribute, model in fact respectively an **outrankingness** measure *rx* and an **outrankedness** measure *cx* (see Lines 16-27).

The sum *rx + (-cx)* of both the **outrankingness** and the **negated outrankedness** measures renders now per decision action x a potential ranking score, similar to *Copeland* or *NetFlows* ranking scores[21].

In our example here we obtain the *Bachet* ranking 'a4' (26) > 'a3' (4) > 'a2' (1) > 'a1' (-11) (see Line 30 above). A ranking result, which is the corresponding optimal *Kemeny* ranking, a ranking maximally correlated with the given outranking digraph g (tau = 0.795, see Lines 4 and 8 below).

```
1 >>> from linearOrders import KemenyRanking
2 >>> ke = KemenyRanking(g)
3 >>> ke.kemenyRanking
4  ['a4', 'a3', 'a2', 'a1']
5 >>> corr = g.computeRankingCorrelation(['a4','a3','a2','a1'])
6 >>> g.showCorrelation(corr)
7   Correlation indexes:
```

(continues on next page)

[21] See the tutorial on ranking with multiple incommensurable criteria

(continued from previous page)

```
   Crisp ordinal correlation  : +0.795
   Epistemic determination    :  0.621
   Bipolar-valued equivalence : +0.493
```

If we reverse however the given ordering of the *actions* dictionary, we may obtain other ranking scores resulting in a different ranking result (see Listing 3.12 below).

Listing 3.12: Importance of the actions ordering

```
>>> pg.showRelationTable(actionsSubset=['a4','a3','a2','a1'],
...                      ReflexiveTerms=False)
 * ---- Relation Table -----
    S   | 'a4' 'a3' 'a2' 'a1'
  ------|---------------------
   'a4' |   -    1    1    1
   'a3' |  -1    -    1    1
   'a2' |  -1   -1    -    1
   'a1' |  -1    1    0    -
  Valuation domain: [-1;+1]
>>> ra4 = BachetNumber(vector=[1,1,1])
>>> ra3 = BachetNumber(vector=[-1,1,1])
>>> ra2 = BachetNumber(vector=[-1,-1,1])
>>> ra1 = BachetNumber(vector=[-1,1,0])
>>> print( int(ra1), int(ra2), int(ra3), int(ra4) )
 -6 -11 -5 13
>>> ca4 = BachetNumber(vector=[-1,-1,-1])
>>> ca3 = BachetNumber(vector=[1,-1,1])
>>> ca2 = BachetNumber(vector=[1,1,0])
>>> ca1 = BachetNumber(vector=[1,1,1])
>>> print( int(ca1), int(ca2), int(ca3), int(ca4) )
 13 12 7 -13
>>> print( int(ra4-ca4), int(ra3-ca3),
...        int(ra2-ca2), int(ra1-ca1) )
 26 -12 -23 -19
```

With the reversed *Bachet* numbers we obtain the ranking ‘a4’ (26) > ‘a3’ (-12) > ‘a1’ (-19) > ‘a2’ (-23). This ranking result is less well correlated (+0.526) with the given outranking digraph, yet corresponds in fact to the actual *Copeland* ranking.

```
>>> from linearOrders import CopelandRanking
>>> cop = CopelandRanking(g)
>>> cop.showScores()
 Copeland scores in descending order
  action      score
    a4         +6
    a3          0
    a1         -3
```

(continues on next page)

(continued from previous page)

```
9     a2          -3
10 >>> cop.copelandRanking
11  ['a4', 'a3', 'a1', 'a2']
12 >>> corr = g.computeRankingCorrelation(cop.copelandRanking)
13 >>> g.showCorrelation(corr)
14  Correlation indexes:
15   Crisp ordinal correlation  : +0.526
16   Epistemic determination    :  0.621
17   Bipolar-valued equivalence : +0.327
18 >>> from transitiveDigraphs import WeakCopelandOrder
19 >>> wcop = WeakCopelandOrder(g)
20 >>> corrw = g.computeordinalCorrelation(wcop)
21 >>> g.showCorrelation(corrw)
22  Correlation indexes:
23   Crisp ordinal correlation  : +0.763
24   Epistemic determination    :  0.537
25   Bipolar-valued equivalence : +0.410
```

For the example outranking digraph here, the *Copeland* ranking rule delivers indeed ranking scores with a tie between actions 'a1' and 'a2', which is by convention resolved by following a lexicographic rule favouring in this case action 'a1'. This ranking is however much less correlated with the given outranking digraph than the optimal *Kemeny* ranking (+0.526 vs +0.795). The corresponding weak Copeland ranking shows nonetheless a high correlation index (+0.763, see Line 23).

The *Bachet* ranking scores of the original and the reversed ordering of the polarised relation table thus lead in our example here to very plausible and convincing ranking results. This hindsight gave the positive stimulus for implementing this new *ranking-by-scoring* rule.

The Bachet ranking rule, a new ranking-by-scoring method

The `linearOrders` module provides now a `PolarisedBachetRanking` class implementing a ranking rule based on the *Bachet* ranking scores modelled by the polarised version of the relation table of a given outranking digraph.

```
>>> from linearOrders import PolarisedBachetRanking
>>> ba = PolarisedBachetRanking(g)
>>> ba
 *------- Digraph instance description ------*
  Instance class      : PolarisedBachetRanking
  Instance name       : rel_randomperftab_best_ranked
  Digraph Order       : 4
  Digraph Size        : 6
  Valuation domain    : [-1.00;1.00]
  Determinateness (%) : 100.00
  Attributes          : ['decBachetScores', 'incBachetScores',
```

(continues on next page)

(continued from previous page)

```
                         'bachetRanking', 'bachetOrder', 'correlation',
                         'name', 'actions', 'order', 'valuationdomain',
                         'relation', 'gamma', 'notGamma', 'runTimes']
>>> ba.bachetRanking
 ['a4', 'a3', 'a2', 'a1']
>>> ba.showScores()
 Bachet scores in descending order
   action     score
     a4        +26
     a3         +4
     a2         +1
     a1        -11
```

The class delivers as usual a ranking (*self.bachetRanking*) and a corresponding ordering result (*self.bachetOrder*) besides the decreasing list (*self.decBachetScores*) and the increasing list of the corresponding *Bachet* ranking scores (*self.incBachetScores*). Due to potential ties observed among *Bachet* scores and the lexicographic resolving of such ties, the decreasing and increasing lists of ranking scores might indeed not always be just the reversed version of one another. The *self.correlation* attribute contains the ordinal correlation index between the given outranking relation and the computed *Bachet* ranking.

Note that, like the *Copeland* and the *NetFlows* ranking rules, the *Bachet* ranking scores are **invariant** under the **codual** transform[22].

Mind however that a base 3 sbits numbering system is **positional**, implying that the *Bachet* ranking scores, as noticed before, depend essentially on the very ordering of the rows and columns of the outranking digraph's *self.relation* attribute. However, when the strict outranking digraph is *transitive*, the *Bachet* ranking scores will consistently model the orderings of all transitive triples independently of the ordering of the rows and columns of the *self.relation* attribute. The polarised *Bachet* ranking scores are hence, like the *Copeland* scores, **Condorcet consistent**, i.e. when the polarised strict outranking digraph models a *transitive* relation, its *Bachet* ranking scores will always be consistent with this strict outranking relation[23].

Our random outranking digraph g, generated above in Listing 3.11 Line 4 is for instance not transitive. Its transitivity degree amounts to 0.833 (see below).

```
>>> print('Transitivity degree: %.4f' % (g.computeTransitivityDegree()))
 Transitivity degree: 0.8333
```

(continues on next page)

[22] To prove the *invariance* of the ordering of *Bachet* ranking scores under the *codual transform*, it is sufficient to notice that the contribution to the *Bachet* scores of any pair of actions, outranking each other and situated respectively in positions p and q in a relation relation table, amounts to $(3^p+3^q)-(3^p+3^q) = 0$. Same zero contribution $(-3^p-3^q)-(-3^p-3^q) = 0$ occurs for any pair positively *not outranking* each other.

[23] To prove the *Condorcet consistency* property of the ordering of *Bachet* ranking scores, it is sufficient to notice that the contributions of a transitive triplet *'ai'* > *'aj'* > *'ak'* to the corresponding *Bachet* ranking scores will respect the actual ordering of the triplet with all positional permutations of [. . . , ai, . . . , aj, . . . ,ak, . . .] in a relation table.

(continued from previous page)

```
>>> pg.showRelationTable(ReflexiveTerms= False)
 * ---- Relation Table -----
   S    | 'a1' 'a2' 'a3' 'a4'
 ------|--------------------
  'a1' |    -    0    1   -1
  'a2' |    1    -   -1   -1
  'a3' |    1    1    -   -1
  'a4' |    1    1    1    -
   Valuation domain: [-1;+1]
```

When we reconsider above digraph g's polarised relation table, we may indeed notice that 'a1' outranks 'a3', 'a3' outranks 'a2', but 'a1' does not outrank 'a2'. When measuring now the quality of *Bachet* rankings obtained by operating the *Bachet* rule on each one of all the potential 24 permutations of the four *g.actions* keys ['a1', 'a2', 'a3', 'a4'], we may observe below in Listing 3.13 four different levels of correlation: +0.7950 (8), +0.6890 (3), +0.6113(3) and +0.5265 (10). The first correlation corresponds to eight ['a4', 'a3', 'a2', 'a1'] ranking results, the second to three ['a4', 'a1', 'a3', 'a2'], the third to three ['a4', 'a2', 'a3', 'a1'], and the last to ten ['a4', 'a3', 'a1', 'a2'] ranking results. In 8 out of 12 cases, the reversed actions list delivers the highest possible correlation +0.7950 (see Lines 6-7 and 24-25).

Listing 3.13: Importance of the ordering of the actions dictionary

```
1  >>> from digraphsTools import all_perms
2  >>> for perm in all_perms(['a1', 'a2', 'a3', 'a4']):
3  ...     ba = PolarisedBachetRanking(g,actionsList=perm,
   ↪BestQualified=False)
4  ...     corr = g.computeRankingCorrelation(ba.bachetRanking)
5  ...     print(perm, ba.bachetRanking, '%.4f' % (corr['correlation']))
6   ['a1', 'a2', 'a3', 'a4'] ['a4', 'a3', 'a2', 'a1'] 0.7950
7    ['a4', 'a3', 'a2', 'a1'] ['a4', 'a3', 'a1', 'a2'] 0.5265
8   ['a2', 'a1', 'a3', 'a4'] ['a4', 'a3', 'a2', 'a1'] 0.7950
9    ['a4', 'a3', 'a1', 'a2'] ['a4', 'a3', 'a1', 'a2'] 0.5265
10  ['a2', 'a3', 'a1', 'a4'] ['a4', 'a3', 'a1', 'a2'] 0.5265
11   ['a4', 'a1', 'a3', 'a2'] ['a4', 'a3', 'a2', 'a1'] 0.7950
12  ['a2', 'a3', 'a4', 'a1'] ['a4', 'a3', 'a1', 'a2'] 0.5265
13   ['a1', 'a4', 'a3', 'a2'] ['a4', 'a2', 'a3', 'a1'] 0.6113
14  ['a1', 'a3', 'a2', 'a4'] ['a4', 'a3', 'a2', 'a1'] 0.7950
15   ['a4', 'a2', 'a3', 'a1'] ['a4', 'a3', 'a1', 'a2'] 0.5265
16  ['a3', 'a1', 'a2', 'a4'] ['a4', 'a3', 'a1', 'a2'] 0.5265
17   ['a4', 'a2', 'a1', 'a3'] ['a4', 'a3', 'a2', 'a1'] 0.7950
18  ['a3', 'a2', 'a1', 'a4'] ['a4', 'a3', 'a1', 'a2'] 0.5265
19   ['a4', 'a1', 'a2', 'a3'] ['a4', 'a3', 'a2', 'a1'] 0.7950
20  ['a3', 'a2', 'a4', 'a1'] ['a4', 'a3', 'a1', 'a2'] 0.5265
21   ['a1', 'a4', 'a2', 'a3'] ['a4', 'a2', 'a3', 'a1'] 0.6113
```

(continues on next page)

(continued from previous page)

```
['a1', 'a3', 'a4', 'a2'] ['a4', 'a2', 'a3', 'a1'] 0.6113
 ['a2', 'a4', 'a3', 'a1'] ['a4', 'a3', 'a1', 'a2'] 0.5265
['a3', 'a1', 'a4', 'a2'] ['a4', 'a3', 'a1', 'a2'] 0.5265
 ['a2', 'a4', 'a1', 'a3'] ['a4', 'a3', 'a1', 'a2'] 0.5265
['a3', 'a4', 'a1', 'a2'] ['a4', 'a1', 'a3', 'a2'] 0.6890
 ['a2', 'a1', 'a4', 'a3'] ['a4', 'a3', 'a2', 'a1'] 0.7950
['a3', 'a4', 'a2', 'a1'] ['a4', 'a1', 'a3', 'a2'] 0.6890
 ['a1', 'a2', 'a4', 'a3'] ['a4', 'a3', 'a2', 'a1'] 0.7950
```

It appears hence to be opportune to compute a first *Bachet* ranking result with the given order of the *self.actions* attribute and a second one with the corresponding reversed ordering. The best correlated of both ranking results is eventually returned. The `PolarisedBachetRanking` class provides therefore the *BestQualified* parameter set by default to *True* (see below Listing 3.14 Lines 5 and 19,21,35). Computing the reversed version of the *Bachet* rule is indeed computationally easy as it just requires to reverse the previously used *Bachet* vectors, a method directly provided by the `BachetInteger` class.

Listing 3.14: Optimising the *Bachet* ranking result I

```
1  >>> from outrankingDigraphs import RandomBipolarOutrankingDigraph
2  >>> g = RandomBipolarOutrankingDigraph(numberOfActions=9,seed=1)
3  >>> from linearOrders import PolarisedBachetRanking
4   *---- solely given ordering of the actions ---*')
5  >>> ba1 = PolarisedBachetRanking(g,BestQualified=False)
6  >>> ba1.showScores()
7    Bachet scores in descending order
8     action     score
9       a2       +6020
10      a8       +3353
11      a9       +3088
12      a3       +2379
13      a6        +476
14      a7        +435
15      a4        +322
16      a5       -1254
17      a1       -5849
18 >>> g.computeRankingCorrelation(ba1.bachetRanking)
19   {'correlation': +0.3936, 'determination': 0.4086}
20  *---- given and reversed ordering of the actions ---*')
21 >>> ba2 = PolarisedBachetRanking(g,BestQualified=True)
22 >>> ba2.showScores()
23   Bachet scores in descending order
24    action     score
25      a2       +6380
26      a9       +2480
```

(continues on next page)

(continued from previous page)

```
      a5       +1830
      a8        -877
      a3       -1399
      a6       -1764
      a7       -2039
      a4       -4410
      a1       -6083
>>> g.computeRankingCorrelation(ba2.bachetRanking)
 {'correlation': +0.6315, 'determination': 0.4086}
```

Yet, when observing a somehow lower transitivity degree, as we may notice in Listing 3.15 Line 2, it is recommended to set the *randomized* parameter (default=0) to a positive integer n. In this case, n random orderings of the decision actions with their reversed versions will be generated in order to compute potentially diverse *Bachet* ranking results. The best correlated ranking will eventually be returned (see Listing 3.15 Lines 3 and 17).

Listing 3.15: Optimising the *Bachet* ranking result II

```
1  >>> print('Transitivity degree: %.4f' % (g.computeTransitivityDegree()))
2   Transitivity degree: 0.6806
3   *---- using 100 random ordering and their reversed versions ---*')
4  >>> ba3 = PolarisedBachetRanking(g,BestQualified=True,randomized=100,seed=5)
5  >>> ba3.showScores()
6    Bachet scores in descending order
7     action      score
8       a2       +6552
9       a5       +4442
10      a9       +2300
11      a6         -84
12      a8        -153
13      a4        -978
14      a3       -4031
15      a7       -4605
16      a1       -6453
17 >>> g.computeRankingCorrelation(ba3.bachetRanking)
18  {'correlation': +0.7585, 'determination': 0.4086}
```

The correlation +0.7585 above corresponds, in the example given here, again to the unique optimal *Kemeny* ranking ['a2', 'a5', 'a9', 'a6', 'a8', 'a4', 'a3', 'a7', 'a1'] (see below).

Listing 3.16: Computing an optimal Kemeny ranking

```
>>> from linearOrders import KemenyRanking
>>> KemenyRanking(g,orderLimit=9)
>>> ke.kemenyRanking
 ['a2', 'a5', 'a9', 'a6', 'a8', 'a4', 'a3', 'a7', 'a1']
```

(continues on next page)

(continued from previous page)

```
5 >>> g.computeRankingCorrelation(ke.kemenyRanking)
6  {'correlation': +0.7585, 'determination': 0.4086}
```

+0.7585 (see Line 4) is hence the highest possible correlation any linear ranking can show with the given outranking digraph *g*.

Efficiency of the Bachet ranking rule settings

We may check the quality of the *optimal Bachet ranking* result with a corresponding performance heat map statistic.

```
>>> g.showHTMLPerformanceHeatmap(Correlations=True,
...              actionsList=ba3.bachetRanking,colorLevels=5)
```

Heatmap of Performance Tableau

criteria	g4	g6	g3	g2	g1	g5	g7
weights	+5.00	+10.00	+2.00	+8.00	+8.00	+2.00	+3.00
tau(*)	+0.35	+0.32	+0.31	+0.19	+0.19	+0.10	-0.25
a2	95.22	97.35	96.90	80.18	37.96	74.31	36.71
a5	91.63	39.34	76.37	31.74	89.33	50.08	73.82
a9	72.36	40.43	67.68	88.12	69.58	18.98	16.37
a6	92.22	85.33	93.92	2.23	38.98	96.72	8.65
a8	62.94	74.37	34.57	0.92	76.72	91.02	10.79
a4	41.62	94.09	52.76	10.22	48.79	NA	77.58
a3	92.65	49.94	72.59	NA	21.00	89.56	88.27
a7	10.00	48.02	55.29	64.95	60.74	50.77	66.38
a1	76.09	28.42	68.65	26.63	47.22	29.64	66.47

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%

(*) tau: *Ordinal (Kendall) correlation between marginal criterion and global ranking relation*
Outranking model: **standard**
Ranking rule: **Bachet randomized 100**
Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.759**
Mean marginal correlation (a) : **+0.213**
Standard marginal correlation deviation (b) : **+0.154**
Ranking fairness (a) - (b) : **+0.060**

Fig. 3.10: *Bachet* randomized 100 rule ranked performance heat map view

In Fig. 3.10 we may observe that the *ba3.bachetRanking* is positively correlated to six out of seven performance criteria with a mean correlation of +0.213 and a positive ranking fairness (+0.06). With four out of seven performance grades in the highest quintile [80% - 100%], action *a2* is convincingly first-ranked. Similarly, action *a1*, with only one performance in the fourth quintile [60%-80%] shows the weakest performances and is last-ranked. It is worthwhile noticing that action *a9* is actually ranked before actions *a6* and *a8* despite an apparent lesser performance profile. This is due to both the considerable negative performance differences (-85.89 and -87.20) observed on criterion *g2* triggering in fact a polarised outranking situation in favour of action *a9*.

When comparing now the ranking results obtained from *single*, *best-qualified* and *randomized=500 Bachet* rule settings with the corresponding *Copeland* rankings obtained from

100 random Cost-Benefit performance tableaux of order 20 and involving 13 performance criteria, we may observe the following correlation statistics.

Polarised Bachet ranking rule single	best qualified	randomized = 500	Copeland ranking rule
Min. : +0.3485	Min. : +0.3485	Min. : +0.6011	Min. : +0.5442
1st Qu.: +0.6317	1st Qu.: +0.6877	1st Qu.: +0.8065	1st Qu.: +0.7943
Median : +0.7114	Median : +0.7393	Median : +0.8486	Median : +0.8338
Mean : +0.6900	Mean : +0.7257	Mean : +0.8377	Mean : +0.8260
3rd Qu.: +0.7586	3rd Qu.: +0.7792	3rd Qu.: +0.8718	3rd Qu.: +0.8617
Max. : +0.8975	Max. : +0.8975	Max. : +0.9363	Max. : +0.9354

The statistical figures confirm the expected noticeable performance enhancement one obtains first with the *BestQualified=True* and secondly even more with the *randomized=500* settings of the *Bachet* ranking rule. With run times less than 8 sec. the latter setting renders in fact ranking results of a correlation quality slightly better than the *Copeland* rule. Yet, computing ranking results just from the given ordering of the *self.actions* dictionary and its reversed ordering may render, with a first quartile correlation of +0.6877 and a median correlation of +0.7393, already satisfactory ranking results in most cases. It is finally remarkable that even the single given actions ordering setting, with a first quartile correlation of +0.6317 and a median correlation of +0.7114, shows already quite acceptable results.

Mind however that the *Bachet* ranking rule, even of comparable complexity $O(n^2)$ as the *Copeland* and *NetFlows* rules, is not scalable to large performance tableaux with hundreds of performance records. The integer value range of *Bachet* numbers gets indeed quickly huge with the order of the given outranking digraph. The `PolarisedBachetRanking` constructor provides therefore an *orderLimit* parameter set by default to 50, which allows to represent integer values in the huge range +- 358948993845926294385124.

```
>>> v = [1 for i in range(50)]
>>> n = BachetNumber(vector=v)
>>> int(n)
 358948993845926294385124
```

In Python, the range of integers is luckily only limited by the available CPU memory and the *orderLimit* parameter may be adjusted to tackle, if required, outranking digraphs of orders > 50. The randomized *Bachet* ranking rule might however need in these cases a smart permutohedron sampling strategy in order to achieve convincingly correlated ranking results.

Smart sampling of the permutations of the actions list

The Condorcet consistency of the polarised Bachet ranking rule guarantees that all transitive outranking triples are correctly scored independently of any given actions ordering. Only the intransitive outranking triples may give diverging ranking results. It is hence opportune to keep transitive triples unchanged in their given initial positions and only permute the first and the last pair in a sample of intransitive triples and keep always the best qualified PolarisedBachetRanking or ValuedBachetRanking result. To explore the usefulness of this sampling approach the `linearOrders` module provides the `BachetRanking` class.

When reconsidering the random outranking digraph g, seen in Listing 3.14, we get the following very convincing ranking result.

Listing 3.17: Smart *Bachet* ranking result

```
1  >>> from outrankingDigraphs import RandomBipolarOutrankingDigraph
2  >>> g = RandomBipolarOutrankingDigraph(numberOfActions=9,seed=1)
3  >>> g.computeTransitivityDegree(Comments=True)
4   Transitivity degree of digraph <rel_randomperftab>:
5     #triples x>y>z: 504, #closed: 343, #open: 161
6     (#closed/#triples) =  0.681
7  >>> from linearOrders import BachetRanking
8  >>> sba = BachetRanking(g,Polarised=True,sampleSize=100)
9  >>> sba.showScores()
10  Bachet scores in descending order
11   action      score
12    a2         +6380
13    a5         +4746
14    a9         +2480
15    a6          -306
16    a8          -877
17    a4         -1494
18    a3         -4315
19    a7         -4955
20    a1         -6083
21 >>> sba.bachetRanking
22  ['a2', 'a5', 'a9', 'a6', 'a8', 'a4', 'a3', 'a7', 'a1']
23 >>> g.showCorrelation(g.computeRankingCorrelation(sba.bachetRanking))
24  Correlation indexes:
25   Crisp ordinal correlation  : +0.759
26   Epistemic determination    :  0.409
27   Bipolar-valued equivalence : +0.310
```

In Listing 3.17 Line 5 we notice that the given random outranking digraph presents 161 intransitive outranking triples. When randomly sampling in Line 8 the permutations of 100 of these intransitive triples, we discover a ranking result that corresponds to the optimal Kemeny ranking seen in Listing 3.16.

Running a MonteCarlo simulation with a sample of 500 random 3 objectives –economic, environmental and societal– performance tableaux of 9 decision actions marked on 13 performance criteria, gives the following ordinal correlations statistics between the corresponding ranking results and the given bipolar-valued outranking digraph g.

Ranking rule	Median	Minimum	Maximum	5% perc.	95% perc.
Polarised Bachet	0.90095	0.52690	0.98390	0.73898	0.96338
Copeland	0.85935	0.40320	0.98520	0.66041	0.95302
Valued Bachet	0.90175	0.56450	0.98390	0.76050	0.96286
NetFlows	0.86325	0.34140	0.98130	0.67623	0.94908
Kemeny	0.91350	0.57230	0.99010	0.79058	0.96945

The correlation figures show that both the smart polarised and the valued Bachet ranking rules, by permuting all potential intransitive outranking triples, come very close to the optimal Kemeny ranking rule (median correlation +0.901 vs +0.914). And in 163/500 (32.6%) polarised cases and in 196/500 (39.2%) valued cases, we even get an optimal Kemeny ranking result.

Notice however in Listing 3.18 below, that the `BachetRanking` constructor is no more invariant under the codual transform.

Listing 3.18: Smart codual *Bachet* ranking result

```
1  >>> gcd = ~(-g)
2  >>> gcd.computeTransitivityDegree(Comments=True)
3   Transitivity degree of digraph <converse-dual-rel_randomperftab>:
4    #triples x>y>z: 26, #closed: 15, #open: 11
5    (#closed/#triples) =  0.577
6  >>> sba = BachetRanking(gcd,Polarised=True,sampleSize=100)
7  >>> g.showCorrelation(g.computeRankingCorrelation(sba.bachetRanking))
8   Correlation indexes:
9    Crisp ordinal correlation  : +0.631
10   Epistemic determination    :  0.409
11   Bipolar-valued equivalence : +0.258
```

As we observe only 11 intransitive triples in the codual outranking digraph *gcd* (see Line 4), the smart sampling leads eventually to a ranking result that is no more the optimal Kemeny ranking (+0.631 versus +0.758, see Line 9). The Condorcet consistency of the ranking of all the transitive triples is however still guaranteed.

Revealing the transitive part of a bipolar-valued digraph

As we have noticed before, the randomized versions of the `PolarisedBachetRanking` and the `ValuedBachetRanking` constructors potentially produce multiple ranking results of unequal correlation quality, yet respecting –due to the Condorcet consistency property– all more or less the transitive part of the given digraph. If we collect now a small subset of the best correlated rankings, we can use the `transitiveDigraphs.RankingsFusionDigraph` class for constructing, by epistemic disjunctive fusion of these selected rankings, a partial *Bachet* ranking result –a transitive asymmetrical digraph with indeterminate reflexive relations– showing actually the potential transitive part of a given polarised outranking digraph.

To explore this remarkable opportunity, a new `PartialBachetRanking` class is provided by the `transitiveDigraphs` module. To illustrate its usefulness, let us reconsider the example outranking digraph g of Listing 3.14.

Listing 3.19: Partial *Bachet* ranking result

```
1  >>> from outrankingDigraphs import RandomBipolarOutrankingDigraph
2  >>> g = RandomBipolarOutrankingDigraph(numberOfActions=9,seed=1)
3  >>> from transitiveDigraphs import PartialBachetRanking
4  >>> pbr = PartialBachetRanking(g,randomized=10,seed=4,maxNbrOfRankings=5)
5  >>> pbr.showTransitiveDigraph()
6   Ranking by Choosing and Rejecting
7    1st ranked ['a2', 'a5', 'a9']
8      2nd ranked ['a3', 'a6', 'a8']
9      2nd last ranked ['a3', 'a6', 'a8'])
10   1st last ranked ['a1', 'a4', 'a7'])
11 >>> g.computeOrdinalCorrelation(pbr)
12  Correlation indexes:
13   Crisp ordinal correlation  : +0.872
14   Epistemic determination    :  0.228
15   Bipolar-valued equivalence : +0.198
```

The nine performance records are grouped into three performance equivalence classes. The resulting *partial ranking* is, except the rankings of alternatives *a3* and *a4*, consistent with the optimal *Kemeny* ranking [‘a2’, ‘a5’, ‘a9’, ‘a6’, ‘a8’, ‘a4’, ‘a3’, ‘a7’, ‘a1’] (see Listing 3.16 Line 4) and is therefore highly correlated (+0.872) with the common determined part of the given outranking digraph g leading to a relational equivalence between both digraphs supported by a criteria significance majority of nearly 60% (see Listing 3.19 Line 13 and 15). In Fig. 3.11 below, is shown its *Hasse* diagram.

```
>>> pbr.exportGraphViz('partialBachet1')
 *---- exporting a dot file for GraphViz tools ---------*
 dot -Grankdir=TB -Tpng partialBachet1.dot -o partialBachet1.png
```

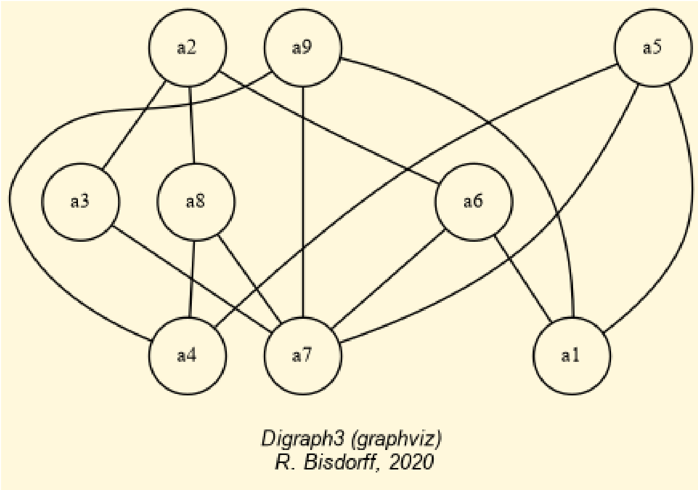


Fig. 3.11: Partial *Bachet* ranking result

A Monte Carlo experiment with the same 500 random Cost-Benefit performance tableaux, reporting the grades obtained by 20 decision actions on 13 criteria already used before, shows with *randomized=100* and *maxNbrOfRankings=5* settings a median partial ranking correlation of +0.936 (min. +0.760) on a common median determination part of 63%% (min. 55%) of the given outranking digraphs. The valued median equivalence of the partial rankings with the given outranking digraphs is hence supported by a median criteria significance majority of 62% (min. 54%).

The `PartialBachetRanking` class provides hence a valuable and effective method for computing *partial rankings* from given bipolar-valued outranking digraphs. The partial *Bachet* ranking digraph represents in fact the directed version of a **comparability** graph, i.e. a Berge or perfect graph .

```
>>> cg = pbr.digraph2Graph(ConjunctiveConversion=False)
>>> cg.isComparabilityGraph()
True
```

We have thus found an algorithm for computing a *transitively orientable* graph, close in the *bipolar-valued relational equivalence* (page 69) sense to the transitive part of a given bipolar-valued outranking digraph. *Partial Bachet* rankings make hence apparent the actual **transitive part** of outranking relations. This interesting finding opens the way to a new design of first- or last-choice recommender algorithms avoiding the necessity to arbitrarily break up potential chordless outranking circuits.

The Bachet choice recommender algorithm

The first and last levels of a topological sort of partial rankings –their **initial and terminal kernels**– give indeed convincing candidates for a *first-choice*, respectively a *last-choice* recommendation (see Listing 3.19 Lines 7-10).

To explore this opportunity, a `showChoiceRecommendation()` method has been added to the `Digraph` class (see Listing 3.20 lines 3-4).

Listing 3.20: *Bachet* choice recommendations

```
1 >>> from outrankingDigraphs import RandomBipolarOutrankingDigraph
2 >>> g = RandomBipolarOutrankingDigraph(numberOfActions=9,seed=1)
3 >>> g.showChoiceRecommendation('Bachet',randomized=100,
4 ...                              maxNbrOfRankings=5,seed=4)
5  *---- Bachet Choice Recommendations ----*
6  Ranking by recursively first and last choosing
7   1st ranked ['a2']
8     2nd ranked ['a5', 'a9']
9     2nd last ranked ['a4', 'a6', 'a8']
10  1st last ranked ['a1', 'a3', 'a7']
11 Quality of partial Bachet ranking
12  Crisp ordinal correlation  : +0.821
13  Epistemic determination    : +0.291
14  Bipolar-valued equivalence : +0.239
15 Execution time: 0.247 seconds
```

Alternative *a2* appears clearly as the first-choice candidate, whereas alternatives *a1*, *a3* and *a7* appear as potential last-choice candidates (see Lines 7 and 10 above). The partial ranking is well ordinally correlated with the outranking digraph g (+0.821) supported by a criteria significance majority of 62%.

In order to provide information about the underlying `PartialBachetRanking` digraph, the corresponding instance is stored in the *g.pbr* attribute. We may thus consult the five Bachet rankings used by the `RankingsFusionDigraph` constructor and provide the best correlated one for ranking the valued adjacency matrix of the outranking digraph g (see Listing 3.21 Lines 2 and 7 below).

Listing 3.21: Details of partial Bachet ranking

```
>>> print(g.pbr.bachetRankings)
[(0.7442, ['a2', 'a5', 'a9', 'a6', 'a4', 'a8', 'a3', 'a7', 'a1']),
 (0.6369, ['a2', 'a5', 'a9', 'a6', 'a8', 'a7', 'a3', 'a4', 'a1']),
 (0.6333, ['a2', 'a5', 'a9', 'a4', 'a6', 'a1', 'a8', 'a3', 'a7']),
 (0.6315, ['a2', 'a9', 'a5', 'a8', 'a3', 'a6', 'a7', 'a4', 'a1']),
 (0.6297, ['a2', 'a5', 'a9', 'a3', 'a6', 'a8', 'a7', 'a4', 'a1'])]
>>> g.showHTMLRelationTable(actionsList=g.pbr.bachetRankings[0][1])
```

Valued Adjacency Matrix

r(x S y)	a2	a5	a9	a6	a4	a8	a3	a7	a1
a2	–	0.42	0.58	0.89	0.58	0.53	0.58	1.00	0.84
a5	-0.16	–	0.58	0.32	0.42	0.37	0.34	1.00	1.00
a9	-0.16	0.39	–	1.00	0.26	1.00	0.26	0.74	0.79
a6	0.16	0.00	0.00	–	0.79	0.58	0.63	1.00	0.42
a4	0.16	-0.26	-0.21	0.58	–	0.26	0.34	0.32	0.42
a8	-0.47	-0.16	0.00	0.37	0.47	–	0.26	0.32	0.18
a3	-0.05	0.37	0.37	-0.05	-0.21	-0.16	–	1.00	0.37
a7	-1.00	0.00	0.26	-1.00	0.08	-0.11	0.00	–	0.68
a1	-0.29	0.08	-0.11	0.13	0.34	-0.05	-0.13	0.00	–

Valuation domain: [-1.00; +1.00]

Fig. 3.12: Ranked adjacency matrix of the outranking digraph

In Fig. 3.12 we see confirmed that alternative *a2* positively outranks all other alternatives and shows hence the very best performance profile. Alternative *a1* and *a7* show the weakest performance profile. This finding is as well confirmed in Listing 3.22 below with the corresponding *Rubis* first and last choice recommendations.

Listing 3.22: *Rubis* choice recommendations

```
>>> g.showChoiceRecommendation('Rubis')
Rubis choice recommendations
***********************
 Credibility domain: [-1.00,1.00]
 === >> potential first choice(s)
```

(continues on next page)

(continued from previous page)

```
6   * choice              : ['a2', 'a4', 'a6']
7     independence        : 0.16
8     dominance           : 0.16
9     absorbency          : -0.42
10    covering (%)        : 50.00
11    determinateness (%) : 63.01
12  - characteristic vector = { 'a2': 0.42, 'a6': 0.16, 'a4': 0.16,
13              'a9': -0.16, 'a5': -0.16, 'a3': -0.16, 'a1': -0.29,
14              'a8': -0.42, 'a7': -0.42, }
15  === >> potential last choice(s)
16  * choice              : ['a1', 'a4', 'a7']
17    independence        : 0.00
18    dominance           : -0.58
19    absorbency          : 0.11
20    covered (%)         : 50.00
21    determinateness (%) : 54.53
22  - characteristic vector = { 'a1': 0.18, 'a4': 0.08,
23               'a7': 0.00, 'a6': 0.00, 'a8': -0.05, 'a5': -0.08,
24               'a9': -0.11, 'a3': -0.13, 'a2': -0.18, }
25   Execution time: 0.033 seconds
26   *****************************
```

Most credible first choice appears indeed to be alternative *a2* with a convincing 71% majority of criteria significance, followed by alternatives *a4* and *a6* with a 58% majority of criteria significance (Line 14). Most credible recommended last choice appears to be alternative *a1* with a 59% majority of criteria significance (Line 22). Notice the ambiguous recommendation of alternative *a4* as potential *first* **and** *last* choice. This explains its appearance in the midfield of the best correlated Bachet ranking here (see Listing 3.21 Line 2).

As the *Bachet* choice recommendation is based on a partial transitive asymmetrical digraph, actually highly correlated with the given outranking digraph *g* (+0.821), a unique initial and a unique terminal prekernel always exist (see Listing 3.20 Lines 7 and 10). Both these properties confer the *Bachet choice recommendation algorithm* a computational advantage over the *Rubis* first choice recommendation algorithm based on initial and terminal prekernels directly extracted from the given strict outranking digraph where we, first, must arbitrarily break, the case given, all chordless outranking circuits (see [BIS-2008p]).

Computing initial and terminal *prekernels* in digraphs is the subject of the next Section.

Notes

Our initial Python implementation of the `BachetInteger` class dates from 2012 when preparing the lectures of a first Semester course on *Discrete Mathematics* for computer scientists. But, it is only in Spring 2025 that we realized how remarkably well *Bachet*'s signed bits weighing design is adapted to our bipolar-valued epistemic logic approach. The *Bachet* ranking rules illustrate here convincingly the benefit one may indeed obtain when computing, not in a binary {0,1} bit world, like today all bit-wise computing devices, but instead in a bipolar-valued {-1,0,+1} world with **balanced ternary Bachet computers**[26].

The power of the **epistemic disjunctive fusion** operator, for instance, is indeed impressive. When two arguments prove the *Truthfulness* of a logical statement, their fusion will be **True**. When two arguments prove the **Falseness** of the statement, their fusion will be **False**, However, when they provide conjointly a proof of *Falseness* **and** a proof of *Truthfulness*, their fusion will be **indeterminate** (zero knowledge). It is worthwhile noticing again the essential computational role this indeterminate **zero** value is taking on in such a *Bachet* computer.

Remarkable is even more the unexpected **Condorcet Consistency** of the *polarised* Bachet ranking scores which allows us to effectively reveal, with the `PartialBachetRanking` constructor, the transitive part of any given bipolar-valued digraph. This consistency property, coupled with a higher discriminatory power than the classic *Copeland* ranking scores, makes the usage of *Bachet* ranking scores very effective for solving inter- and intragroup pairing problems (see the `pairings` module or the tutorial on fairly matching students and internships).

Back to *Content Table* (page 1)

3.4 On computing digraph kernels

- *What is a graph kernel ?* (page 99)
- *Initial and terminal kernels* (page 104)
- *Kernels in lateralized digraphs* (page 109)
- *Computing first and last choice recommendations* (page 112)
- *Tractability* (page 117)

What is a graph kernel ?

We call **choice** in a graph, respectively a digraph, a subset of its vertices, resp. of its nodes or actions. A choice Y is called **internally stable** or **independent** when there exist **no links** (edges) or relations (arcs) between its members. Furthermore, a choice Y is called **externally stable** when for each vertex, node or action x not in Y, there exists at least a member y of Y such that x is linked or related to y. Now, an internally **and** externally stable choice is called a **kernel**.

[26] *Balanced ternary optical* HPC designs have recently gained in China a lot of attention [JIN-2003]. Compared to classical HPC or 'dreamed' quantum computers, balanced ternary based optical computer systems provide indeed many technical and computational advantages. Much less electrical energy required, no water cooling, millions of optical trits may be easily assembled, faithful ultra high speed read and write access, etc.

A first trivial example is immediately given by the maximal independent vertices sets (MISs) of the n-cycle graph (see tutorial on computing isomorphic choices). Indeed, each MIS in the n-cycle graph is by definition independent, i.e. internally stable, and each non selected vertex in the n-cycle graph is in relation with either one or even two members of the MIS. See, for instance, the four non isomorphic MISs of the 12-cycle graph as shown in `MISc12`.

In all graph or symmetric digraph, the *maximality condition* imposed on the internal stability is equivalent to the external stability condition. Indeed, if there would exist a vertex or node not related to any of the elements of a choice, then we may safely add this vertex or node to the given choice without violating its internal stability. All kernels must hence be maximal independent choices. In fact, in a topological sense, they correspond to maximal **holes** in the given graph.

We may illustrate this coincidence between MISs and kernels in graphs and symmetric digraphs with the following random 3-regular graph instance (see Fig. 3.13).

```
>>> from graphs import RandomRegularGraph
>>> g = RandomRegularGraph(order=12,degree=3,seed=100)
>>> g.exportGraphViz('random3RegularGraph')
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to random3RegularGraph.dot
 fdp -Tpng random3RegularGraph.dot -o random3RegularGraph.png
```

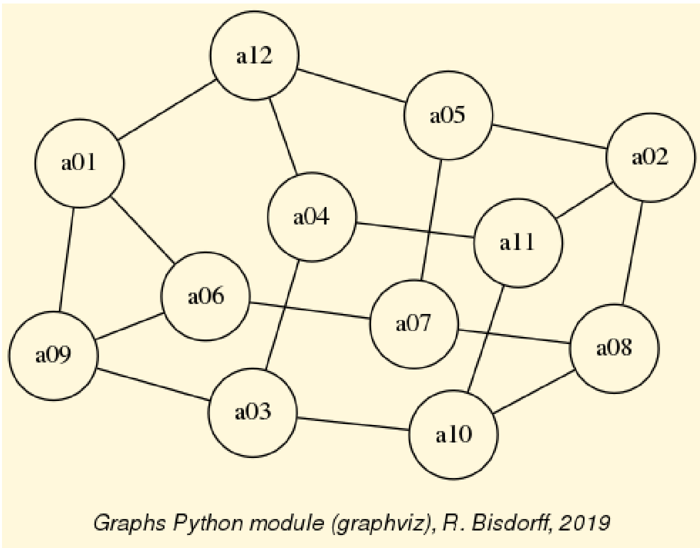


Fig. 3.13: A random 3-regular graph instance

A random MIS in this graph may be computed for instance by using the `MISModel` class.

```
>>> from graphs import MISModel
>>> mg = MISModel(g)
 Iteration:  1
 Running a Gibbs Sampler for 660 step !
 {'a06', 'a02', 'a12', 'a10'}  is maximal !
```

(continues on next page)

(continued from previous page)

```
>>> mg.exportGraphViz('random3RegularGraph_mis')
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to random3RegularGraph-mis.dot
 fdp -Tpng random3RegularGraph-mis.dot -o random3RegularGraph-mis.png
```

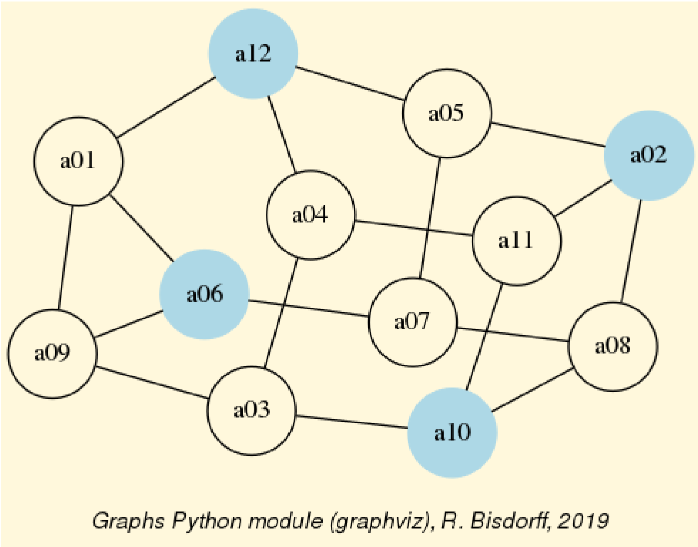


Fig. 3.14: A random MIS colored in the random 3-regular graph

It is easily verified in Fig. 3.14 above, that the computed MIS renders indeed a valid kernel of the given graph. The complete set of kernels of this 3-regular graph instance coincides hence with the set of its MISs.

```
>>> g.showMIS()
 *--- Maximal Independent Sets ---*
 ['a01', 'a02', 'a03', 'a07']
 ['a01', 'a04', 'a05', 'a08']
 ['a04', 'a05', 'a08', 'a09']
 ['a01', 'a04', 'a05', 'a10']
 ['a04', 'a05', 'a09', 'a10']
 ['a02', 'a03', 'a07', 'a12']
 ['a01', 'a03', 'a07', 'a11']
 ['a05', 'a08', 'a09', 'a11']
 ['a03', 'a07', 'a11', 'a12']
 ['a07', 'a09', 'a11', 'a12']
 ['a08', 'a09', 'a11', 'a12']
 ['a04', 'a05', 'a06', 'a08']
 ['a04', 'a05', 'a06', 'a10']
 ['a02', 'a04', 'a06', 'a10']
 ['a02', 'a03', 'a06', 'a12']
 ['a02', 'a06', 'a10', 'a12']
 ['a01', 'a02', 'a04', 'a07', 'a10']
```

(continues on next page)

(continued from previous page)

```
 ['a02', 'a04', 'a07', 'a09', 'a10']
 ['a02', 'a07', 'a09', 'a10', 'a12']
 ['a01', 'a03', 'a05', 'a08', 'a11']
 ['a03', 'a05', 'a06', 'a08', 'a11']
 ['a03', 'a06', 'a08', 'a11', 'a12']
 number of solutions:  22
 cardinality distribution
 card.:  [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
 freq.:  [0, 0, 0, 0, 16, 6, 0, 0, 0, 0, 0, 0, 0]
 execution time: 0.00045 sec.
 Results in self.misset
>>> g.misset
 [frozenset({'a02', 'a01', 'a07', 'a03'}),
  frozenset({'a04', 'a01', 'a08', 'a05'}),
  frozenset({'a09', 'a04', 'a08', 'a05'}),
  ...
  ...
  frozenset({'a06', 'a02', 'a12', 'a10'}),
  frozenset({'a06', 'a11', 'a08', 'a03', 'a05'}),
  frozenset({'a03', 'a06', 'a11', 'a12', 'a08'})]
```

We cannot resist in looking in this 3-regular graph for non isomorphic kernels (MISs, see previous tutorial). To do so we must first, convert the given *graph* instance into a *digraph* instance. Then, compute its automorphism generators, and finally, identify the isomorphic kernel orbits.

```
>>> dg = g.graph2Digraph()
>>> dg.showMIS()
 *---  Maximal independent choices ---*
 ...
  ['a06', 'a02', 'a12', 'a10']
 ...
 number of solutions:  22
 cardinality distribution
 card.:  [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
 freq.:  [0, 0, 0, 0, 16, 6, 0, 0, 0, 0, 0, 0, 0]
 execution time: 0.00080 sec.
 Results in self.misset
>>> dg.automorphismGenerators()
 *----- saving digraph in nauty dre format  -------------*
 ...
  # automorphisms extraction from dre file #
  # Using input file: randomRegularGraph.dre
  echo '<randomRegularGraph.dre -m p >randomRegularGraph.auto x' |
↪dreadnaut
  # permutation = 1['1', '11', '7', '5', '4', '9', '3', '10', '6', '8',
```

(continues on next page)

(continued from previous page)

```
↪'2', '12']
>>> dg.showOrbits(dg.misset)
 *--- Isomorphic reduction of choices
 ...
  current representative:  frozenset({'a09', 'a11', 'a12', 'a08'})
  length   :  4
  number of isomorph choices 2
  isormorph choices
  ['a06', 'a02', 'a12', 'a10']  # <<== the random MIS shown above
  ['a09', 'a11', 'a12', 'a08']
 ...
 *---- Global result ----
  Number of choices:  22
  Number of orbits :  11
  Labelled representatives:
  ...
   ['a09', 'a11', 'a12', 'a08']
  ...
```

In our random 3-regular graph instance (see Fig. 3.13), we may thus find eleven non isomorphic kernels with orbit sizes equal to two. We illustrate below the isomorphic twin of the random MIS example shown in Fig. 3.14 .

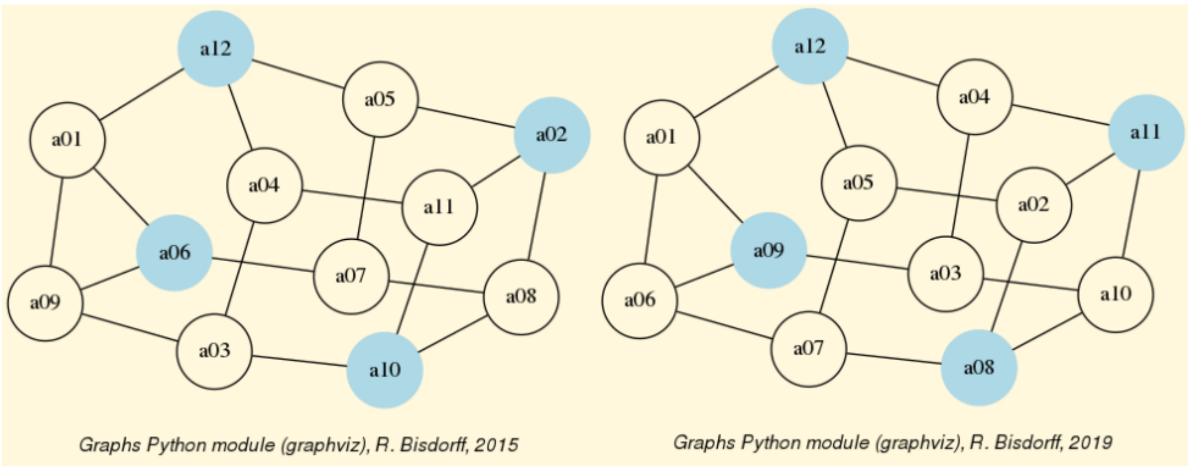


Fig. 3.15: Two isomorphic kernels of the random 3-regular graph instance

All graphs and symmetric digraphs admit MISs, hence also kernels.

It is worthwhile noticing that the **maximal matchings** of a graph correspond bijectively to its line graph's **kernels** (see the `LineGraph` class).

```
>>> from graphs import CycleGraph
>>> c8 = CycleGraph(order=8)
>>> maxMatching = c8.computeMaximumMatching()
```

(continues on next page)

(continued from previous page)

```
>>> c8.exportGraphViz(fileName='maxMatchingcycleGraph',
...                   matching=maxMatching)
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to maxMatchingcyleGraph.dot
 Matching:  {frozenset({'v1', 'v2'}), frozenset({'v5', 'v6'}),
             frozenset({'v3', 'v4'}), frozenset({'v7', 'v8'}) }
 circo -Tpng maxMatchingcyleGraph.dot -o maxMatchingcyleGraph.png
```

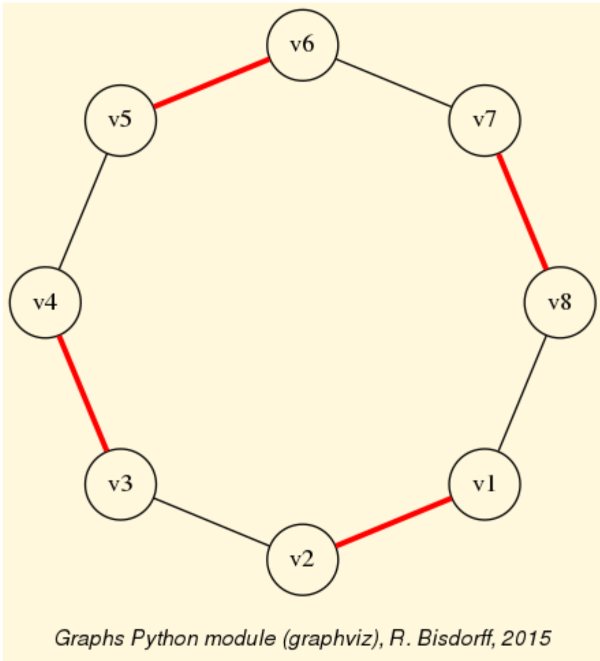


Fig. 3.16: Perfect maximum matching in the 8-cycle graph

In the context of digraphs, i.e. *oriented* graphs, the kernel concept gets much richer and separates from the symmetric MIS concept.

Initial and terminal kernels

In an oriented graph context, the internal stability condition of the kernel concept remains untouched; however, the external stability condition gets indeed split up by the *orientation* into two lateral cases:

1. A **dominant** stability condition, where each non selected node is *dominated* by at least one member of the kernel;
2. An **absorbent** stability condition, where each non selected node is *absorbed* by at least one member of the kernel.

A both *internally* **and** *dominant*, resp. *absorbent stable* choice is called a *dominant* or **initial**, resp. an *absorbent* or **terminal** kernel. From a topological perspective, the initial kernel concept looks from the outside of the digraph into its interior, whereas the terminal kernel looks from the interior of a digraph toward its outside. From an algebraic perspective, the initial kernel is a *prefix* operand, and the terminal kernel is a *postfix* operand in the kernel equation systems (see Digraph3 advanced topic on bipolar-valued kernel membership characteristics).

Furthermore, as the kernel concept involves conjointly a **positive logical refutation** (the *internal stability*) and a **positive logical affirmation** (the *external stability*), it appeared rather quickly necessary in our operational developments to adopt a bipolar characteristic [-1,1] valuation domain, modelling *negation* by change of numerical sign and including explicitly a third **median** logical value (0) expressing logical **indeterminateness** (neither positive, nor negative, see [?] and [?]).

In such a bipolar-valued context, we call **prekernel** a choice which is **externally stable** and for which the **internal stability** condition is **valid or indeterminate**. We say that the independence condition is in this case only **weakly** validated. Notice that all kernels are hence prekernels, but not vice-versa.

In graphs or symmetric digraphs, where there is essentially no apparent ‘ *laterality* ‘, all prekernels are *initial* **and** *terminal* at the same time. They correspond to what we call *holes* in the graph. A *universal* example is given by the **complete** digraph.

```
>>> from digraphs import CompleteDigraph
>>> u = CompleteDigraph(order=5)
>>> u
 *------- Digraph instance description ------*
 Instance class   : CompleteDigraph
 Instance name    : complete
 Digraph Order      : 5
 Digraph Size       : 20
 Valuation domain : [-1.00 ; 1.00]
 --------------------------------
>>> u.showPreKernels()
 *--- Computing preKernels ---*
 Dominant kernels :
 ['1'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 ['2'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 ['3'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 ['4'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 ['5'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 Absorbent kernels :
 ['1'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 ['2'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 ['3'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 ['4'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 ['5'] independence: 1.0; dominance : 1.0; absorbency : 1.0
 *----- statistics -----
 graph name:  complete
 number of solutions
  dominant kernels :  5
  absorbent kernels:  5
 cardinality frequency distributions
```

(continues on next page)

(continued from previous page)

```
cardinality     :  [0, 1, 2, 3, 4, 5]
dominant kernel :  [0, 5, 0, 0, 0, 0]
absorbent kernel:  [0, 5, 0, 0, 0, 0]
Execution time  : 0.00004 sec.
Results in sets: dompreKernels and abspreKernels.
```

In a complete digraph, each single node is indeed both an initial and a terminal prekernel candidate and there is no definite *begin* or *end* of the digraph to be detected. *Laterality* is here entirely *relative* to a specific singleton chosen as reference point of view. The same absence of laterality is apparent in two other universal digraph models, the **empty** and the **indeterminate** digraph.

```
>>> ed = EmptyDigraph(order=5)
>>> ed.showPreKernels()
 *--- Computing preKernels ---*
 Dominant kernel :
 ['1', '2', '3', '4', '5']
    independence :  1.0
    dominance    :  1.0
    absorbency   :  1.0
 Absorbent kernel :
 ['1', '2', '3', '4', '5']
    independence :  1.0
    dominance    :  1.0
    absorbency   :  1.0
 ...
```

In the empty digraph, the whole set of nodes gives indeed at the same time the **unique** *initial* **and** *terminal* prekernel. Similarly, for the **indeterminate** digraph.

```
>>> from digraphs import IndeterminateDigraph
>>> id = IndeterminateDigraph(order=5)
>>> id.showPreKernels()
 *--- Computing preKernels ---*
 Dominant prekernel :
 ['1', '2', '3', '4', '5']
    independence :  0.0   # <<== indeterminate
    dominance    :  1.0
    absorbency   :  1.0
 Absorbent prekernel :
 ['1', '2', '3', '4', '5']
    independence :  0.0   # <<== indeterminate
    dominance    :  1.0
    absorbency   :  1.0
```

Both these results make sense, as in a completely empty or indeterminate digraph, there is no *interior* of the digraph defined, only a *border* which is hence at the same time an initial and terminal prekernel. Notice however, that in the latter indeterminate case, the complete set of nodes verifies only weakly the internal stability condition (see above).

Other common digraph models, although being clearly oriented, may show nevertheless no apparent laterality, like **odd chordless circuits**, i.e. *holes* surrounded by an *oriented cycle* -a circuit- of odd length. They do not admit in fact any initial or terminal prekernel.

```
>>> from digraphs import CirculantDigraph
>>> c5 = CirculantDigraph(order=5,circulants=[1])
>>> c5.showPreKernels()
 *----- statistics -----
 digraph name:  c5
 number of solutions
  dominant prekernels :  0
  absorbent prekernels:  0
```

Chordless circuits of **even** length 2 x k, with $k > 1$, contain however two isomorphic prekernels of cardinality k which qualify conjointly as initial and terminal candidates.

```
>>> c6 = CirculantDigraph(order=6,circulants=[1])
>>> c6.showPreKernels()
 *--- Computing preKernels ---*
 Dominant preKernels :
 ['1', '3', '5'] independence: 1.0, dominance: 1.0, absorbency: 1.0
 ['2', '4', '6'] independence: 1.0, dominance: 1.0, absorbency: 1.0
 Absorbent preKernels :
 ['1', '3', '5'] independence: 1.0, dominance: 1.0, absorbency: 1.0
 ['2', '4', '6'] independence: 1.0, dominance: 1.0, absorbency: 1.0
```

Chordless circuits of even length may thus be indifferently oriented along two opposite directions. Notice by the way that the duals of **all** chordless circuits of *odd* **or** *even* length, i.e. *filled* circuits also called **anti-holes** (see Fig. 3.17), never contain any potential prekernel candidates.

```
>>> dc6 = -c6    # dc6 = DualDigraph(c6)
>>> dc6.showPreKernels()
 *----- statistics -----
 graph name:  dual_c6
 number of solutions
  dominant prekernels :  0
  absorbent prekernels:  0
>>> dc6.exportGraphViz(fileName='dualChordlessCircuit')
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to dualChordlessCircuit.dot
 circo -Tpng dualChordlessCircuit.dot -o dualChordlessCircuit.png
```

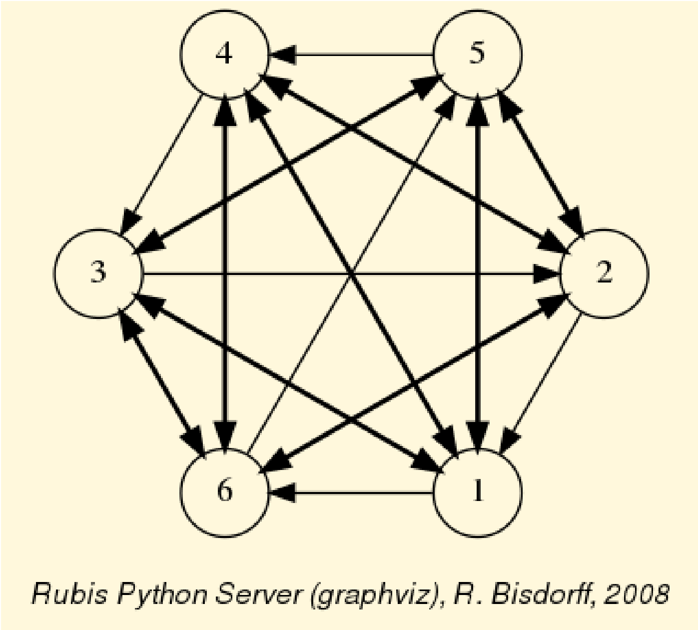


Fig. 3.17: The dual of the chordless 6-circuit

We call **weak**, a *chordless circuit* with *indeterminate inner part*. The `CirculantDigraph` class provides a parameter for constructing such a kind of *weak chordless* circuits.

```
>>> c6 = CirculantDigraph(order=6, circulants=[1],
...                      IndeterminateInnerPart=True)
```

It is worth noticing that the *dual* version of a *weak* circuit corresponds to its *converse* version, i.e. *-c6* = *~c6* (see Fig. 3.18).

```
>>> (-c6).exportGraphViz()
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to dual_c6.dot
 circo -Tpng dual_c6.dot -o dual_c6.png
>>> (~c6).exportGraphViz()
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to converse_c6.dot
 circo -Tpng converse_c6.dot -o converse_c6.png
```

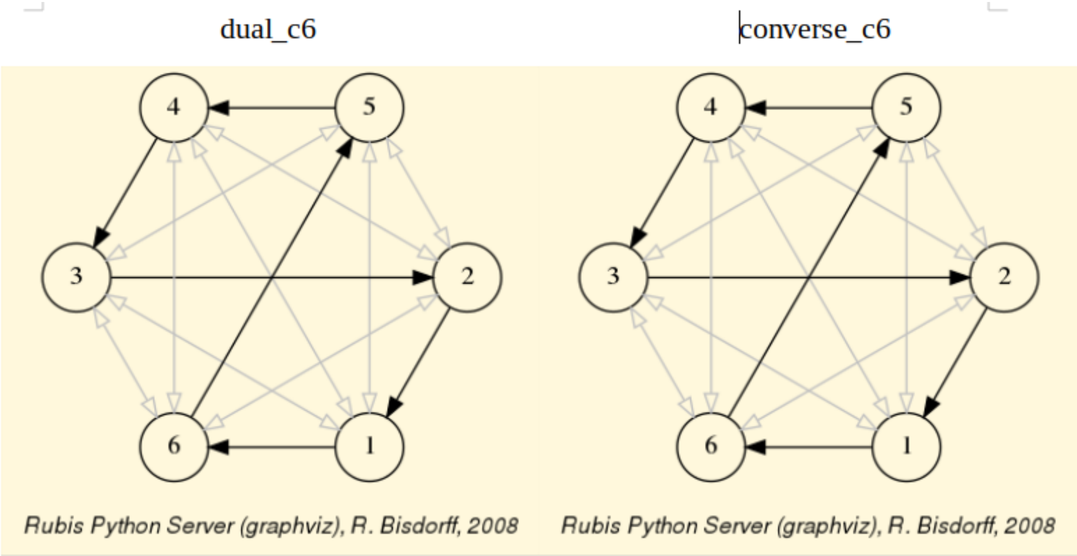


Fig. 3.18: Dual and converse of the weak 6-circuit

It immediately follows that weak chordless circuits are part of the class of digraphs that are **invariant** under the *codual* transform, cn = - (~ cn) = ~ (-cn).

Kernels in lateralized digraphs

Humans do live in an apparent physical space of plain transitive **lateral orientation**, fully empowered in finite geometrical 3D models with **linear orders**, where first, resp. last ranked, nodes deliver unique initial, resp. terminal, kernels. Similarly, in finite **preorders**, the first, resp. last, equivalence classes deliver the unique initial, resp. unique terminal, kernels. More generally, in finite **partial orders**, i.e. asymmetrical and transitive digraphs, topological sort algorithms will easily reveal on the first, resp. last, level all unique initial, resp. terminal, kernels.

In genuine random digraphs, however, we may need to check for each of its MISs, whether *one*, *both*, or *none* of the lateralized external stability conditions may be satisfied. Consider, for instance, the following random digraph instance of order 7 and generated with an arc probability of 30%.

```
>>> from randomDigraphs import RandomDigraph
>>> rd = RandomDigraph(order=7,arcProbability=0.3,seed=5)
>>> rd.exportGraphViz('randomLaterality')
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to randomLaterality.dot
 dot -Grankdir=BT -Tpng randomLaterality.dot -o randomLaterality.png
```

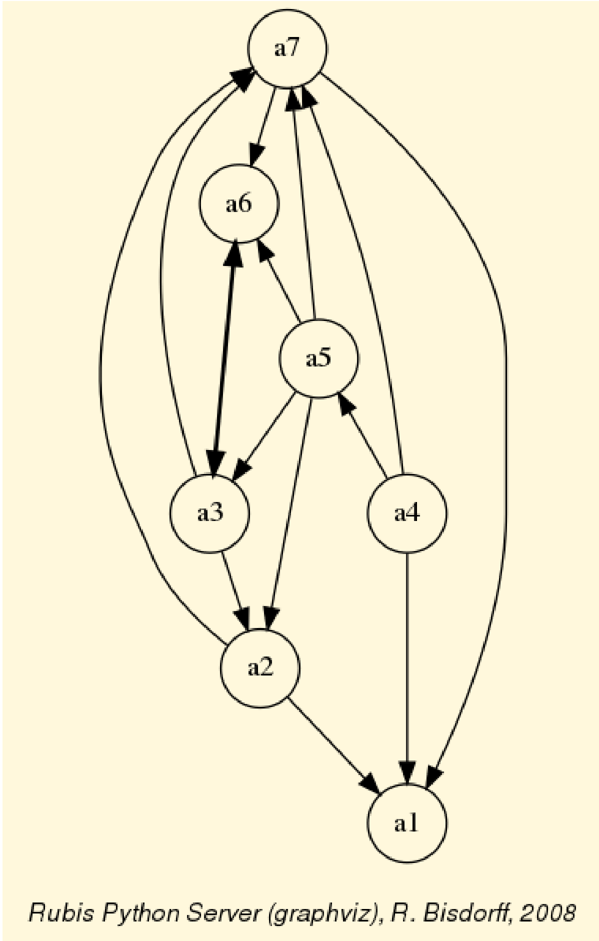


Fig. 3.19: A random digraph instance of order 7 and arc probability 0.3

The random digraph shown in Fig. 3.19 above has no apparent special properties, except from being connected (see Line 3 below).

```
1  >>> rd.showComponents()
2   *--- Connected Components ---*
3   1: ['a1', 'a2', 'a3', 'a4', 'a5', 'a6', 'a7']
4  >>> rd.computeSymmetryDegree(Comments=True,InPercents=True)
5   Symmetry degree (%) of digraph <randomDigraph>:
6    #arcs x>y: 14, #symmetric: 1, #asymmetric: 13
7    #symmetric/#arcs =  7.1
8  >>> rd.computeChordlessCircuits()
9   []  # no chordless circuits detected
10 >>> rd.computeTransitivityDegree(Comments=True,InPercents=True)
11  Transitivity degree (%) of graph <randomDigraph>:
12   #triples x>y>z: 23, #closed: 11, #open: 12
13   #closed/#triples =  47.8
```

The given digraph instance is neither asymmetric (a3 <-> a6) nor symmetric (a2 -> a1, a1 -/> a2) (see Line 6 above); there are no chordless circuits (see Line 9 above); and, the digraph is not transitive (a5 -> a2 -> a1, but a5 -/> a1). More than half of the required transitive closure is missing (see Line 12 above).

Now, we know that its potential prekernels must be among its set of maximal independent choices.

```
1  >>> rd.showMIS()
2   *---  Maximal independent choices ---*
3   ['a2', 'a4', 'a6']
4   ['a6', 'a1']
5   ['a5', 'a1']
6   ['a3', 'a1']
7   ['a4', 'a3']
8   ['a7']
9   ------
10 >>> rd.showPreKernels()
11  *--- Computing preKernels ---*
12  Dominant preKernels :
13  ['a2', 'a4', 'a6']
14     independence :  1.0
15     dominance    :  1.0
16     absorbency   :  -1.0
17     covering     :  0.500
18  ['a4', 'a3']
19     independence :  1.0
20     dominance    :  1.0
21     absorbency   :  -1.0
22     covering     :  0.600   # <<==
23  Absorbent preKernels :
24  ['a3', 'a1']
25     independence :  1.0
26     dominance    :  -1.0
27     absorbency   :  1.0
28     covering     :  0.500
29  ['a6', 'a1']
30     independence :  1.0
31     dominance    :  -1.0
32     absorbency   :  1.0
33     covering     :  0.600   # <<==
34  ...
```

Among the six MISs contained in this random digraph (see above Lines 3-8) we discover two initial and two terminal kernels (Lines 12-34). Notice by the way the covering values (between 0.0 and 1.0) shown by the `digraphs.Digraph.showPreKernels()` method (Lines 17, 22, 28 and 33). The higher this value, the more the corresponding kernel candidate makes apparent the digraph's *laterality*. We may hence redraw the same digraph in Fig. 3.20 by looking into its interior via the *best covering* initial kernel candidate: the dominant choice {'a3','4a'} (coloured in yellow), and looking out of it via the *best covered* terminal kernel candidate: the absorbent choice {'a1','a6'} (coloured in blue).

```
>>> rd.exportGraphViz(fileName='orientedLaterality',
...                   bestChoice=set(['a3', 'a4']),
...                   worstChoice=set(['a1', 'a6']))
*---- exporting a dot file for GraphViz tools ---------*
 Exporting to orientedLaterality.dot
 dot -Grankdir=BT -Tpng orientedLaterality.dot -o orientedLaterality.png
```

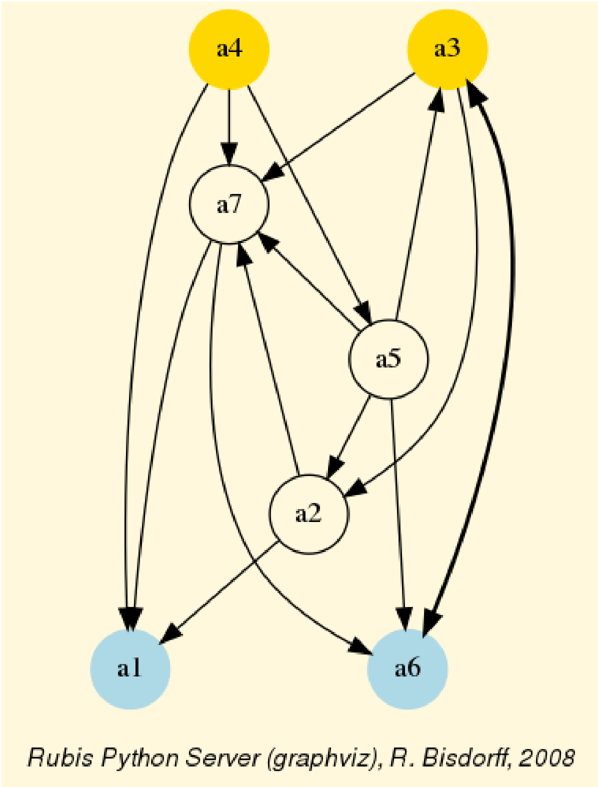


Fig. 3.20: A random digraph oriented by best covering initial and best covered terminal kernel

In algorithmic decision theory, initial and terminal prekernels may provide convincing first, resp. last, choice recommendations (see tutorial on computing a best choice recommendation).

Computing first and last choice recommendations

To illustrate this idea, let us finally compute first and last choice recommendations in the following random bipolar-valued **outranking** digraph.

```
>>> from outrankingDigraphs import RandomBipolarOutrankingDigraph
>>> g = RandomBipolarOutrankingDigraph(seed=5)
>>> g
 *------- Object instance description ------*
 Instance class    : RandomBipolarOutrankingDigraph
 Instance name     : randomOutranking
```

(continues on next page)

(continued from previous page)

```
 # Actions           : 7
 # Criteria          : 7
 Size                : 26
 Determinateness     : 34.275
 Valuation domain : {'min': -100.0, 'med': 0.0, 'max': 100.0}
>>> g.showHTMLPerformanceTableau()
```

Performance table randomOutranking

criterion	g1	g2	g3	g4	g5	g6	g7
a1	64.90	1.31	13.88	98.24	94.10	14.57	31.00
a2	NA	NA	61.75	87.24	69.06	6.51	81.85
a3	11.32	27.95	12.67	28.93	96.66	30.14	48.07
a4	46.91	91.63	0.18	96.15	89.37	60.31	31.58
a5	NA	76.57	87.14	53.92	29.88	0.34	48.12
a6	54.38	15.96	20.95	67.78	36.12	67.79	70.47
a7	57.39	79.71	21.55	20.48	16.60	33.79	5.70

Fig. 3.21: The performance tableau of a random outranking digraph instance

The underlying random performance tableau (see Fig. 3.21) shows the performance grading of 7 potential decision actions with respect to 7 decision criteria supporting each an increasing performance scale from 0 to 100. Notice the missing performance data concerning decision actions ‘a2’ and ‘a5’. The resulting **strict outranking** - i.e. a weighted majority supported - *better than without considerable counter-performance* - digraph is shown in Fig. 3.22 below.

```
>>> gcd = ~(-g)  # Codual: the converse of the negation
>>> gcd.exportGraphViz(fileName='tutOutRanking')
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to tutOutranking.dot
 dot -Grankdir=BT -Tpng tutOutranking.dot -o tutOutranking.png
```

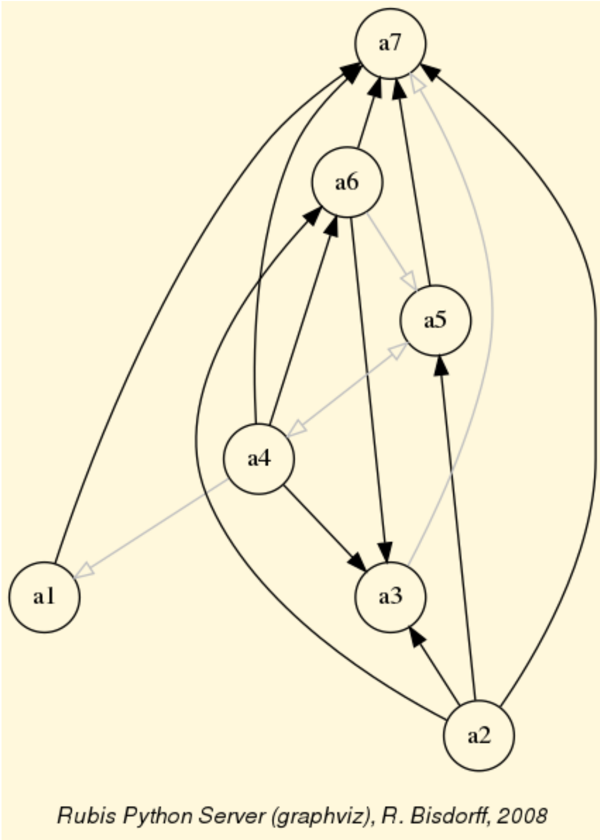


Fig. 3.22: A random strict outranking digraph instance

All decision actions appear strictly better performing than action 'a7'. We call it a **Condorcet loser** and it is an evident terminal prekernel candidate. On the other side, three actions: 'a1', 'a2' and 'a4' are not dominated. They give together an initial prekernel candidate.

```
1  >>> gcd.showPreKernels()
2   *--- Computing preKernels ---*
3   Dominant preKernels :
4   ['a1', 'a2', 'a4']
5      independence :  0.00
6      dominance    :  6.98
7      absorbency   :  -48.84
8      covering     :  0.667
9   Absorbent preKernels :
10  ['a3', 'a7']
11     independence :  0.00
12     dominance    :  -74.42
13     absorbency   :  16.28
14     covered      :  0.800
```

With such unique disjoint initial and terminal prekernels (see Line 4 and 10), the given digraph instance is hence clearly *lateralized*. Indeed, these initial and terminal prekernels of the codual outranking digraph reveal first, resp. last, choice recommendations one may formulate on the basis of a given outranking digraph instance.

```
1 >>> g.showFirstChoiceRecommendation()
2  ***********************
3  First choice recommendation(s) (BCR)
4   (in decreasing order of determinateness)
5  Credibility domain: [-100.00,100.00]
6   === >> potential first choice(s)
7  * choice              : ['a1', 'a2', 'a4']
8    independence        : 0.00
9    dominance           : 6.98
10   absorbency          : -48.84
11   covering (%)        : 66.67
12   determinateness (%) : 57.97
13   - most credible action(s) = { 'a4': 20.93, 'a2': 20.93, }
14  === >> potential last choice(s)
15 * choice              : ['a3', 'a7']
16   independence        : 0.00
17   dominance           : -74.42
18   absorbency          : 16.28
19   covered (%)         : 80.00
20   determinateness (%) : 64.62
21   - most credible action(s) = { 'a7': 48.84, }
```

Notice that solving bipolar-valued kernel equation systems (see *Bipolar-Valued Kernels* (page 124) below) provides furthermore a positive characterization of the most credible decision actions in each respective choice recommendation (see Lines 14 and 23 above). Actions 'a2' and 'a4' are equivalent candidates for a unique best choice, and action 'a7' is clearly confirmed as the last choice.

In Fig. 3.23 below, we orient the drawing of the strict outranking digraph instance with the help of these first and last choice recommendations.

```
>>> gcd.exportGraphViz(fileName='bestWorstOrientation',
...                    bestChoice=['a2','a4'],
...                    worstChoice=['a7'])
 *---- exporting a dot file for GraphViz tools ---------*
 Exporting to bestWorstOrientation.dot
 dot -Grankdir=BT -Tpng bestWorstOrientation.dot -o bestWorstOrientation.
↪png
```

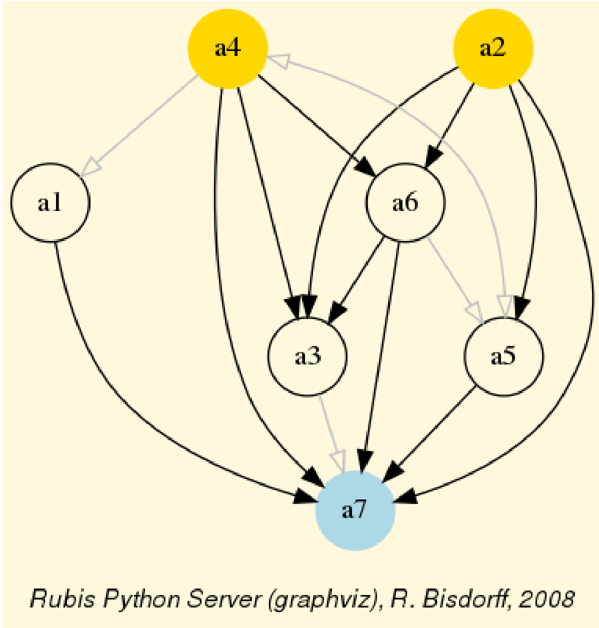


Fig. 3.23: The strict outranking digraph oriented by its first and last choice recommendations

The gray arrows in Fig. 3.23, like the one between actions 'a4' and 'a1', represent indeterminate preferential situations. Action 'a1' appears hence to be rather incomparable to all the other, except action 'a7'. It may be interesting to compare this result with a *Copeland* ranking of the underlying performance tableau (see the tutorial on ranking with uncommensurable criteria).

```
>>> g.showHTMLPerformanceHeatmap(colorLevels=5, ndigits=0,
...                  Correlations=True, rankingRule='Copeland')
```

Heatmap of Performance Tableau 'randomOutranking'

criteria	**g4**	**g7**	**g5**	**g6**	**g1**	**g2**	**g3**
weights	9	10	6	5	4	8	1
tau(*)	+0.64	+0.40	+0.29	+0.17	+0.02	-0.05	-0.10
a4	96	32	89	60	47	92	0
a2	87	82	69	7	NA	NA	62
a6	68	70	36	68	54	16	21
a1	98	31	94	15	65	1	14
a5	54	48	30	0	NA	77	87
a3	29	48	97	30	11	28	13
a7	20	6	17	34	57	80	22

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%

(*) tau: *Ordinal (Kendall) correlation between marginal criterion and global ranking relation*
Ranking rule: **Copeland**
Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.848**

Fig. 3.24: heatmap with Copeland ranking of the performance tableau

In the resulting linear ranking (see Fig. 3.24), action 'a4' is set at first rank, followed by action 'a2'. This makes sense as 'a4' shows three performances in the first quintile, whereas 'a2' is only partially evaluated and shows only two such excellent performances. But 'a4' also shows a very weak performance in the first quintile. Both decision actions, hence, don't show eventually a performance profile that would make apparent a clear preference situation in favour of one or the other. In this sense, the prekernels based best choice recommendations may appear more faithful with respect to the actually definite strict outranking relation than any 'forced' linear ranking result as shown in Fig. 3.24 above.

Tractability

Finally, let us give some hints on the **tractability** of kernel computations. Detecting all (pre)kernels in a digraph is a famously NP-hard computational problem. Checking external stability conditions for an independent choice is equivalent to checking its maximality and may be done in the linear complexity of the order of the digraph. However, checking all independent choices contained in a digraph may get hard already for tiny sparse digraphs of order $n > 20$ (see [?]). Indeed, the worst case is given by an empty or indeterminate digraph where the set of all potential independent choices to check is in fact the power set of the vertices.

```
1  >>> from digraphs import EmptyDigraph
2  >>> e = EmptyDigraph(order=20)
3  >>> e.showMIS()   # by visiting all 2^20 independent choices
4   *---  Maximal independent choices ---*
5   [ '1',  '2',  '3',  '4',  '5',  '6',  '7',  '8',  '9', '10',
6    '11', '12', '13', '14', '15', '16', '17', '18', '19', '20']
7   number of solutions:  1
8   execution time: 1.47640 sec.  # <<== !!!
9  >>> 2**20
10  1048576
```

Now, there exist more efficient specialized algorithms for directly enumerating MISs and dominant or absorbent kernels contained in specific digraph models without visiting all independent choices (see [?]). Alain Hertz provided kindly such a MISs enumeration algorithm for the Digraph3 project (see `showMIS_AH()`). When the number of independent choices is big compared to the actual number of MISs, like in very sparse or empty digraphs, the performance difference may be dramatic (see Line 7 above and Line 15 below).

```
1  >>> e.showMIS_AH()  # by visiting only maximal independent choices
2   *-----------------------------------*
3   * Python implementation of Hertz's  *
4   * algorithm for generating all MISs *
5   * R.B. version 7(6)-25-Apr-2006     *
6   *-----------------------------------*
7   ===>>> Initial solution :
8   [ '1',  '2',  '3',  '4',  '5',  '6',  '7',  '8',  '9', '10',
```

(continues on next page)

(continued from previous page)

```
 '11', '12', '13', '14', '15', '16', '17', '18', '19', '20']
*---- results ----*
[ '1',  '2',  '3',  '4',  '5',  '6',  '7',  '8',  '9', '10',
 '11', '12', '13', '14', '15', '16', '17', '18', '19', '20']
*---- statistics ----*
mis solutions    :  1
execution time   : 0.00026 sec. # <<== !!!
iteration history:  1
```

For more or less dense strict outranking digraphs of modest order, as facing usually in algorithmic decision theory applications, enumerating all independent choices remains however in most cases tractable, especially by using a very efficient Python generator (see `independentChoices()` below).

```python
def independentChoices(self,U):
    """
    Generator for all independent choices with associated
    dominated, absorbed and independent neighborhoods
    of digraph instance self.
    Initiate with U = self.singletons().
    Yields [(independent choice, domnb, absnb, indnb)].
    """
    if U == []:
        yield [(frozenset(),set(),set(),set(self.actions))]
    else:
        x = list(U.pop())
        for S in self.independentChoices(U):
            yield S
            if x[0] <=  S[0][3]:
                Sxgamdom = S[0][1] | x[1]
                Sxgamabs = S[0][2] | x[2]
                Sxindep = S[0][3] &  x[3]
                Sxchoice = S[0][0] | x[0]
                Sx = [(Sxchoice,Sxgamdom,Sxgamabs,Sxindep)]
                yield Sx
```

And, checking maximality of independent choices via the external stability conditions during their enumeration (see `computePreKernels()` below) provides the effective advantage of computing all initial **and** terminal prekernels in a single loop (see Line 10 and [?]).

```python
1 def computePreKernels(self):
2     """
3     computing dominant and absorbent preKernels:
4     Result in self.dompreKernels and self.abspreKernels
5     """
6     actions = set(self.actions)
```

(continues on next page)

(continued from previous page)

```
    n = len(actions)
    dompreKernels = set()
    abspreKernels = set()
    for choice in self.independentChoices(self.singletons()):
        restactions = actions - choice[0][0]
        if restactions <= choice[0][1]:
            dompreKernels.add(choice[0][0])
        if restactions <= choice[0][2]:
            abspreKernels.add(choice[0][0])
    self.dompreKernels = dompreKernels
    self.abspreKernels = abspreKernels
```

Back to *Content Table* (page 1)

3.5 Computing with bipolar-valued sets

- *Bipolar-valued epistemic logic* (page 119)
- *Bipolar-valued propositional calculus* (page 120)
- *Computational bipolar-valued set theory* (page 120)

Bipolar-valued epistemic logic

In bipolar-valued epistemic logic we assemble conjointly some evidence for a statement to be true and other evidence for the same statement to be false. First example is given by the statement P: '*this candidate should win the election*'. There are some voters who support its truthfulness and some voters who support, on the contrary its falseness. Yet, there may also be some voters who don't give any opinion, they abstain from voting. What logical status should be given to the statement P ?

Following *Condorcet* [CON-1785p] we are going to balance the votes in favour against the votes in disfavour. As a result we obtain a bipolar-valued credibility which qualifies statement P as **certainly true** when all voters support its truth, **more or less true** when more voters support its truthfulness than its falseness, **more or less false** when more voters support its falseness than its truthfulness and **certainly false** when all the voters support its falseness. There may however appear a special case when the same number of voters support its truthfulness respectively its falseness, or all voters abstain from voting. In these cases, statement P becomes neither *true* nor *false*, but **indeterminate**. It is a *zero knowledge* situation.

First important consequence is that the logical status of any statement is considered to be indeterminate as long as no evidence is yet collected. Reconsidering for instance proposition P, its logical status is indeterminate as long as no voting has been organized. More important is furthermore the consequence that negating the truthfulness or falseness of a proposition does no more imply its falseness, repective its truthfulness, but its falseness **or** indeterminate state, respecively its truthfulness **or** indeterminate state. Double negation does no more automatically imply truthfulness.

Bipolar-valued propositional calculus

In order to formalize the previous intuitions, we are attaching to any logical proposition or statement P a characteristic function *r(P)* taking value in the decimal interval [-1.0;+1.0] with following semantics:

- $0.0 < r(P) \leq +1.0$ means statement P is **more or less true**;
- $-1.0 \geq r(P) < 0.0$ means statement P is **more or less false**;
- $r(P) = 0.0$ means statement P is **indeterminate**.

Negating a statement is hence operated by changing the sign of its characteristic function: $r(\neg P) = -r(P)$. An important consequence appears. This **negation** $\neg$ operation does not necessarily correspond to taking the complement logical value. Indeed, *not true* does now only imply *false or indeterminate* and *not false* does only imply *true or indeterminate*. In fact, we are working in a balanced ternary logic with **negative** (false), **positive** (true) and **null** (indeterminate) characteristic values. A negative *affirmation* becomes here a positive *refutation* and vice versa [BIS-2004_3p].

Let now P and Q be two bipolar-valued propositions, logical **conjunction** $\wedge$, **disjunction** $\vee$ and **implication** $\Rightarrow$ may be computed as follows:

- $r(P \wedge Q) \;=\; \min\big(r(P), r(Q)\big)$;
- $r(P \vee Q) \;=\; \max\big(r(P), r(Q)\big)$;
- $r(P \Rightarrow Q) \;=\; r\big(\neg(P \wedge \neg Q)\big) \;=\; -r\big(\,\min(r(P), -r(Q)\,)\big)$.

It is worthwhile noticing that the bipolar-valued logical implication is not necessarily transitive. Suppose for instance that a majority of voters validate proposition $P \Rightarrow Q$ and another majority validates the proposition $Q \Rightarrow R$. There is no epistemic reason why the separate evidences of both these statements should necessarily induce the evidence of $P \Rightarrow R$.

Computational bipolar-valued set theory

A bipolar-valued set X, a **bpv-set** for short, consists of a support set E_X of potential elements and a dictionary providing for every element $x \in E_X$ a bipolar-valued membership characteristic function $r(x \in X)$ taking values in the decimal interval [-1.0;+1.0]:

- Element x is more or less included in the set X when $r(x \in X) \;>\; 0.0$;
- Element x is more or less excluded from the set X when $r(x \in X) \;<\; 0.0$;
- When $r(x \in X) \;=\; 0.0$, element x is neither included nor excluded from set X.

We may distinguish three special bpv-sets: the **crisp** bpv-set X where all members of the support E_X are certainly members of X, the **relatively empty** bpv-set X where all members of its support E_X are certainly non-members of X and the special **indeterminate** case where all members of its support E_X are indeterminate members of X, i.e $r(z \in X) \;=\; 0.0,\; \forall\, z \in\, E_X$.

Mind that for any potential element z not included in the support set E_X, we consider its potential X membership characteristic $r(z \in X)$ to be -1.0. The **absolute empty** bpv-set $\emptyset$ consists hence of an empty support set $E_\emptyset \;=\; \emptyset$. And, for any element x it follows that $r(x \in \emptyset) \;=\; -1.0$.

Let X and Y be two bpv-sets. The support for logical bpv-set operations is the union of the support of the arguments. We define the classical **set union** $\cup$ and **intersection** $\cap$ as follows.

For all $z \in E_{X \cup Y}$ the membership characteristic $r\big(\, z \in (X \cup Y)\,\big) \;=$

- $\max\big(\, r(z \in X), r(z \in Y)\,\big)$ when both z in X and z in Y; otherwise
- $r(z \in X)$ when $z \in X$, and
- $r(z \in Y)$ when $z \in Y$.

For all $z \in E_{X \cup Y}$ the membership characteristic $r\big(\, z \in (X \cap Y)\,\big) \;=$

- $\min\big(\, r(z \in X), r(z \in Y)\,\big)$ when both z in X and z in Y;
- -1.0 otherwise.

The **set difference** between two bpv-sets X and Y, denoted X-Y, is the bpv-set of all members of X that are not members of Y. For all $z \in E_{X \cup Y}$, the membership characteristic $r\big(\, z \in (X - Y)\,\big) \;=$

- $\min\big(\, r(z \in X), -r(z \in Y)\,\big)$ when both $z \in E_X \wedge z \in E_Y$;
- $r(z \in X)$ when $z \in E_X$;
- -1.0 otherwise.

The **symmetric difference** of bpv-sets X and Y, denoted X^Y, is the set difference of their union and intersection, $(X \cup Y) \;-\; (X \cap Y)$ or the union of their reciprocal set differences $(X - Y) \,\cup\, (Y - X)$.

The `bipolarValuedSets` Digraph3 module provides, with the `BpvSet` class, a Python implementation of such bpv-sets. In Listing 3.23 we use the `RandomBpvSet` class for generating two random bpv-sets X and Y with a common support of three elements *s1*, *s2* and *s3*.

Listing 3.23: Working with bipolar-valued sets I

```
>>> from bipolarValuedSets import RandomBpvSet
>>> X = RandomBpvSet(numberOfElements=5,elementNamePrefix='s',
...      undeterminateness=0.1,valuationRange=(-1, 1),ndigits=4,seed=1)
>>> X.showMembershipCharacteristics()
 s2:  +0.6949
 s3:  +0.5275
 s5:  +0.0000
```

(continues on next page)

(continued from previous page)

```
s4:  -0.4899
s1:  -0.7313
>>> Y = RandomBpvSet(numberOfElements=3,elementNamePrefix='s',
...     undeterminateness=0.1,valuationRange=(-1, 1),ndigits=4,seed=2)
>>> Y.showMembershipCharacteristics()
s1:  +0.9121
s2:  +0.8957
s3:  -0.8869
```

In Listing 3.24 below we illustrate the bipolar-valued set **union** and **intersection**.

Listing 3.24: Working with bipolar-valued sets II

```
>>> (X|Y).showMembershipCharacteristics()
# Python's set union symbol is |
s1:  +0.9121
s2:  +0.8957
s3:  +0.5275
s5:  +0.0000
s4:  -0.4899
>>> (X&Y).showMembershipCharacteristics()
# Python's set intersection symbol is &
s2:  +0.6949
s1:  -0.7313
s3:  -0.8869
s4:  -1.0000
s5:  -1.0000
```

Finally, in Listing 3.25 below we illustrate the bipolar-valued set **difference** and **symmetric difference**.

Listing 3.25: Working with bipolar-valued sets III

```
>>> (X-Y).showMembershipCharacteristics()
# Python's set difference is -
s3:  +0.5275
s5:  +0.0000
s4:  -0.4899
s2:  -0.8957
s1:  -0.9121
>>> (Y-X).showMembershipCharacteristics()
s1:  +0.7313
s2:  -0.6949
s3:  -0.8869
s4:  -1.0000
s5:  -1.0000
>>> (Y^X).showMembershipCharacteristics()
```

(continues on next page)

(continued from previous page)

```
# Python's symmetrix difference symbol is ^
# X^Y = (X-Y)|(Y-X) or (X|Y)-(X&Y)
 s1:  +0.7313
 s3:  +0.5275
 s5:  +0.0000
 s4:  -0.4899
 s2:  -0.6949
```

The BpvSet class provides furthermore a `isSubset()` method for computing the bipolar-valued subset statement and a `strip()` method which removes potential non-elements from the support of a bpv-set.

```
>>> D = Y - X
>>> D.isSubset(Y)
 Decimal('0.8869')
>>> D1 = D.strip(InSite=False)
>>> D1.showMembershipCharacteristics()
 s1:  +0.7313
 s2:  -0.6949
 s3:  -0.8869
```

Finally, a `polarise()` method is provided for setting all positive and negative membership credibilities of a bpv-set to +1.0, respectively to -1.0 .

```
>>> D2 = D1.polarise(InSite=False)
>>> D2.showMembershipCharacteristics()
 s1:  +1.0000
 s2:  -1.0000
 s3:  -1.0000
```

In the limit case of no indeterminate membership characteristics, we recover this way standard crisp sets and the previous set operations implement in fact a Boolean algebra [BIS-2004_3p].

Note

The *relation* attribute of a bipolar-valued digraph is an example of a bpv-set where the support is given by the oriented pairs of the digraph's nodes. The *edges* attribute of bipolar-valued graphs is a second example of a bpv-set, where the support is given by the non-oriented pairs of the graph's vertices. Further evident examples of bpv-sets are the bipolar-valued prekernel membership charcateristic vectors from the *Rubis* best choice computation [BIS-2006_1p]. This is the topic of the next tutorial.

Back to *Content Table* (page 1)

3.6 Bipolar-valued kernel membership characteristic vectors

- *Kernel equation systems* (page 124)
- *Solving bipolar-valued kernel equation systems* (page 125)
- *Historical notes* (page 131)

Kernel equation systems

Let *G(X,R)* be a crisp irreflexive digraph defined on a finite set X of nodes and where R is the corresponding {-1,+1}-valued adjacency matrix. Let Y be the {-1,+1}-valued membership characteristic (row) vector of a choice in X. When Y satisfies the following equation system

$$Y \circ R \ = \ -Y \ ,$$

where for all x in X,

$$(Y \circ R)(x) \ = \ \max_{y \in X, x \neq y} \big(\min(Y(x), R(x,y)) \big) \ .$$

then Y characterises an **initial** *kernel* ([SCH-1985p]).

When transposing now the membership characteristic vector Y into a column vector Y^t, the following equation system

$$R \circ Y^t \ = \ -Y^t \ ,$$

makes Y^t similarly characterise a **terminal** *kernel*.

Let us verify this result on a tiny random digraph.

```
>>> from digraphs import RandomDigraph
>>> g = RandomDigraph(order=3,seed=1)
>>> g.showRelationTable()
 * ---- Relation Table -----
    R   | 'a1'        'a2'     'a3'
 -------|---------------------
  'a1' |   -1  +1       -1
  'a2' |   -1  -1       +1
  'a3' |   +1  +1       -1
>>> g.showPreKernels()
 *--- Computing preKernels ---*
 Dominant preKernels :
 ['a3']
    independence :  1.0
    dominance    :  1.0
    absorbency   :  -1.0
    covering     :  1.000
 Absorbent preKernels :
```

(continues on next page)

(continued from previous page)

```
['a2']
   independence :  1.0
   dominance    :  -1.0
   absorbency   :  1.0
   covered      :  1.000
```

It is easy to verify that the characteristic vector [-1, -1, +1] satisfies the initial kernel equation system; *a3* gives an *initial* kernel. Similarly, the characteristic vector [-1, +1, -1] verifies indeed the terminal kernel equation system and hence *a2* gives a *terminal* kernel.

We succeeded now in generalizing kernel equation systems to genuine bipolar-valued digraphs ([BIS-2006_1p]). The constructive proof, found by *Marc Pirlot*, is based on the following *fixpoint equation* that may be used for computing bipolar-valued kernel membership vectors,

$$T(Y) \;:=\; -(Y \circ R) = Y,$$

Solving bipolar-valued kernel equation systems

John von Neumann showed indeed that, when a digraph *G(X,R)* is **acyclic** with a **unique initial kernel** K characterised by its membership characteristics vector *Yk*, then the following double bipolar-valued fixpoint equation

$$T^2(Y) \;:=\; -\big(-(Y \circ R) \circ R\big) \;=\; Y\;.$$

will admit a stable high and a stable low fixpoint solution that converge both to *Yk* ([SCH-1985p]).

Inspired by this crisp double fixpoint equation, we observed that for a given bipolar-valued digraph *G(X,R)*, each of its dominant or absorbent prekernels *Ki* in X determines an induced **partial graph** *G(X,R/Ki)* which is *acyclyc* and admits *Ki* as unique kernel (see [BIS-2006_2p]).

Following the *von Neumann* fixpoint algorithm, a similar bipolar-valued extended double fixpoint algorithm, applied to *G(X,R/Ki)*, allows to compute hence the associated bipolar-valued kernel characteristic vectors *Yi* in polynomial complexity.

Algorithm

in : bipolar-valued digraph *G(X,R)*,
out : set { *Y1*, *Y2*, .. } of bipolar-valued kernel membership characteristic vectors.

1. enumerate all initial and terminal crisp prekernels *K1*, *K2*, … in the given bipolar-valued digraph (see the tutorial on *Computing Digraph Kernels* (page 99));
2. for each crisp initial kernel *Ki*:
   a. construct a partially determined subgraph *G(X,R/Ki)* supporting exactly this unique initial kernel *Ki*;

b. Use the double fixpoint equation *T2* with the partially determined adjacency matrix R/Ki for computing a stable low and a stable high fixpoint;

c. Determine the bipolar-valued Ki-membership characteristic vector Yi with an epistemic disjunction of the previous low and high fixpoints;

3. repeat step (2) for each terminal kernel Kj by using the double fixpoint equation *T2* with the transpose of the adjacency matrix R/Kj.

Time for a practical illustration.

Listing 3.26: Random Bipolar-valued Outranking Digraph

```
>>> from outrankingDigraphs import RandomBipolarOutrankingDigraph
>>> g = RandomBipolarOutrankingDigraph(Normalized=True,seed=5)
>>> print(g)
 *------- Object instance description ------*
 Instance class       : RandomBipolarOutrankingDigraph
 Instance name        : rel_randomperftab
 # Actions            : 7
 # Criteria           : 7
 Size                 : 26
 Determinateness (%) : 67.14
 Valuation domain     : [-1.0;1.0]
 Attributes           : ['name', 'actions', 'criteria', 'evaluation',
                         'relation', 'valuationdomain', 'order',
                         'gamma', 'notGamma']
```

The random outranking digraph g, we consider here in Listing 3.26 for illustration, models the pairwise outranking situations between seven decision alternatives evaluated on seven incommensurable performance criteria. We compute its corresponding bipolar-valued prekernels on the associated codual digraph *gcd*.

Listing 3.27: Strict Prekernels

```
>>> gcd = ~(-g) # strict outranking digraph
>>> gcd.showPreKernels()
 *--- Computing prekernels ---*
 Dominant prekernels :
 ['a1', 'a4', 'a2']
    independence :  +0.000
    dominance    :  +0.070
    absorbency   :  -0.488
    covering     :  +0.667
 Absorbent prekernels :
 ['a7', 'a3']
    independence :  +0.000
```

(continues on next page)

(continued from previous page)

```
13      dominance    :  -0.744
14      absorbency   :  +0.163
15      covered      :  +0.800
16   *----- statistics -----
17   graph name:  converse-dual_rel_randomperftab
18   number of solutions
19    dominant kernels :  1
20    absorbent kernels:  1
21   cardinality frequency distributions
22   cardinality     :  [0, 1, 2, 3, 4, 5, 6, 7]
23   dominant kernel :  [0, 0, 0, 1, 0, 0, 0, 0]
24   absorbent kernel:  [0, 0, 1, 0, 0, 0, 0, 0]
25   Execution time  : 0.00022 sec.
```

The codual outranking digraph, modelling a *strict outranking* relation, admits an initial prekernel [*a1*, *a2*, *a4*] and a terminal one [*a3*, *a7*] (see Listing 3.27 Line 5 and 11).

Let us compute the *initial* prekernel restricted adjacency table with the `domkernelrestrict()` method.

```
>>> k1Relation = gcd.domkernelrestrict(['a1','a2','a4'])
>>> gcd.showHTMLRelationTable(
...       actionsList=['a1','a2','a4','a3','a5','a6','a7'],
...       relation=k1Relation,
...       tableTitle='K1 restricted adjacency table')
```

K1 restricted adjacency table

r(x S y)	a1	a2	a4	a3	a5	a6	a7
a1	-	-0.23	-1.00	0.00	0.00	0.00	0.16
a2	-0.21	-	-0.21	0.21	0.44	0.05	0.49
a4	0.00	-0.21	-	0.21	0.00	0.07	0.58
a3	-0.28	-0.21	-0.74	-	0.00	0.00	0.00
a5	-0.26	-0.67	0.00	0.00	-	0.00	0.00
a6	-0.12	-0.49	-0.49	0.00	0.00	-	0.00
a7	-0.51	-0.49	-0.86	0.00	0.00	0.00	-

Valuation domain: [-1.00; +1.00]

Fig. 3.25: Initial kernel [*a1*, *a2*, *a4*] restricted adjacency table

We first notice that this initial prekernel is indeed only *weakly independent*: The outranking situation between *a4* and *a1* appears *indeterminate*. The corresponding initial prekernel membership characteristic vector may be computed with the `computeKernelVector()` method.

Listing 3.28: Fixpoint iterations for initial prekernel ['a1', 'a2', 'a4']

```
>>> gcd.computeKernelVector(['a1','a2','a4'],Initial=True,Comments=True)
 --> Initial prekernel: {'a1', 'a2', 'a4'}
 initial low vector : [-1.00, -1.00, -1.00, -1.00, -1.00, -1.00, -1.00]
 initial high vector: [+1.00, +1.00, +1.00, +1.00, +1.00, +1.00, +1.00]
 1st low vector     : [ 0.00, +0.21, -0.21,  0.00, -0.44, -0.07, -0.58]
 1st high vector    : [+1.00, +1.00, +1.00, +1.00, +1.00, +1.00, +1.00]
 2nd low vector     : [ 0.00, +0.21, -0.21,  0.00, -0.44, -0.07, -0.58]
 2nd high vector    : [ 0.00, +0.21, -0.21, +0.21, -0.21, -0.05, -0.21]
 3rd low vector     : [ 0.00, +0.21, -0.21, +0.21, -0.21, -0.07, -0.21]
 3rd high vector    : [ 0.00, +0.21, -0.21, +0.21, -0.21, -0.05, -0.21]
 4th low vector     : [ 0.00, +0.21, -0.21, +0.21, -0.21, -0.07, -0.21]
 4th high vector    : [ 0.00, +0.21, -0.21, +0.21, -0.21, -0.07, -0.21]
 # iterations       : 4
 low & high fusion  : [ 0.00, +0.21, -0.21, +0.21, -0.21, -0.07, -0.21]
 Choice vector for initial prekernel: {'a1', 'a2', 'a4'}
 a2: +0.21
 a4: +0.21
 a1:  0.00
 a6: -0.07
 a3: -0.21
 a5: -0.21
 a7: -0.21
```

We start the fixpoint computation with an empty set characterisation as first low vector and a complete set X characterising high vector. After each iteration, the low vector is set to the negation of the previous high vector and the high vector is set to the negation of the previous low vector.

A unique stable prekernel characteristic vector *Y1* is here attained at the fourth iteration with positive members *a2*: +0.21 and *a4*: +0.21 (60.5% criteria significance majority); *a1*: 0.00 being an ambiguous potential member. Alternatives *a3*, *a5*, *a6* and *a7* are all negative members, i.e. positive **non members** of this outranking prekernel.

Let us now compute the restricted adjacency table for the outranked, i.e. the *terminal* prekernel [*a3*, *a7*].

```
>>> k2Relation = gcd.abskernelrestrict(['a3','a7'])
>>> gcd.showHTMLRelationTable(
...         actionsList=['a3','a7','a1','a2','a4','a5','a6'],
...         relation=k2Relation,
...         tableTitle='K2 restricted adjacency table')
```

K2 restricted adjacency table

r(x S y)	a3	a7	a1	a2	a4	a5	a6
a3	–	0.00	-0.28	-0.21	-0.74	-0.40	-0.53
a7	-1.00	–	-0.51	-0.49	-0.86	-0.67	-0.63
a1	0.00	0.16	–	0.00	0.00	0.00	0.00
a2	0.21	0.49	0.00	–	0.00	0.00	0.00
a4	0.21	0.58	0.00	0.00	–	0.00	0.00
a5	0.00	0.16	0.00	0.00	0.00	–	0.00
a6	0.30	0.26	0.00	0.00	0.00	0.00	–

Valuation domain: [-1.00; +1.00]

Fig. 3.26: Terminal kernel ['a3','a7'] restricted adjacency table

Again, we notice that this terminal prekernel is indeed only weakly independent. The corresponding bipolar-valued characteristic vector *Y2* may be computed as follows.

```
>>> gcd.computeKernelVector(['a3','a7'],Initial=False,Comments=True)
 --> Terminal prekernel: {'a3', 'a7'}
 initial low vector  : [-1.00, -1.00, -1.00, -1.00, -1.00, -1.00, -1.00]
 initial high vector : [+1.00, +1.00, +1.00, +1.00, +1.00, +1.00, +1.00]
 1st low vector      : [-0.16, -0.49,  0.00, -0.58, -0.16, -0.30, +0.49]
 1st high vector     : [+1.00, +1.00, +1.00, +1.00, +1.00, +1.00, +1.00]
 2nd low vector      : [-0.16, -0.49,  0.00, -0.58, -0.16, -0.30, +0.49]
 2nd high vector     : [-0.16, -0.49,  0.00, -0.49, -0.16, -0.26, +0.49]
 3rd low vector      : [-0.16, -0.49,  0.00, -0.49, -0.16, -0.26, +0.49]
 3rd high vector     : [-0.16, -0.49,  0.00, -0.49, -0.16, -0.26, +0.49]
 # iterations        : 3
 high & low fusion   : [-0.16, -0.49,  0.00, -0.49, -0.16, -0.26, +0.49]
 Choice vector for terminal prekernel: {'a3', 'a7'}
 a7: +0.49
 a3:  0.00
 a1: -0.16
 a5: -0.16
 a6: -0.26
 a2: -0.49
 a4: -0.49
```

A unique stable bipolar-valued high and low fixpoint is attained at the third iteration with *a7* positively confirmed (about 75% criteria significance majority) as member of this terminal prekernel, whereas the membership of *a3* in this prekernel appears indeterminate. All the remaining nodes have *negative* membership characteristic values and are hence positively excluded from this prekernel.

When we reconsider the graphviz drawing of this outranking digraph (see Fig. 52 in the tutorial on *Computing Digraph Kernels* (page 99)),

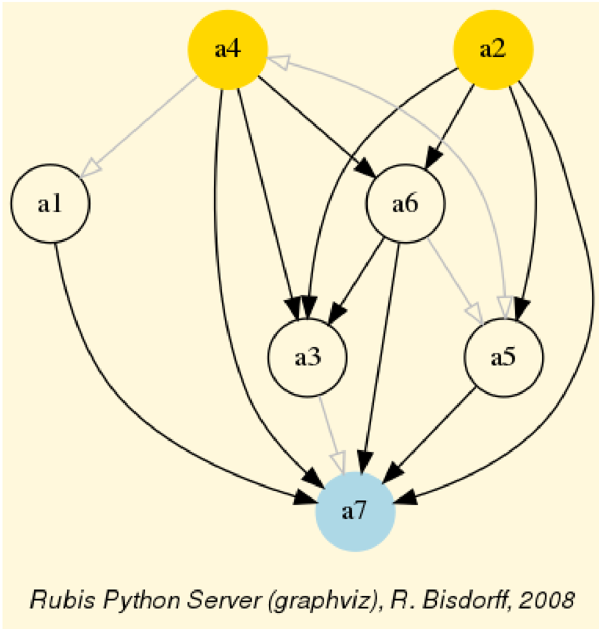


Fig. 3.27: The strict outranking digraph oriented by the positive members of its initial and terminal prekernels

it becomes obvious why alternative *a1* is **neither included nor excluded** from the initial prekernel. Same observation is applicable to alternative *a3* which can **neither be included nor excluded** from the terminal prekernel. It may even happen, in case of more indeterminate outranking situations, that no alternative is positively included or excluded from a weakly independent prekernel; the corresponding bipolar-valued membership characteristic vector being completely indeterminate (see for instance the tutorial on Computing a Best Choice Recommendation).

To illustrate finally why sometimes we need to operate an *epistemic disjunctive fusion* of **unequal** stable low and high membership characteristics vectors (see Step 2.c.), let us consider, for instance, the following crisp 7-*cycle* graph.

```
>>> g = CirculantDigraph(order=7,circulants=[-1,1])
>>> g
 *------- Digraph instance description ------*
 Instance class       : CirculantDigraph
 Instance name        : c7
 Digraph Order        : 7
 Digraph Size         : 14
 Valuation domain     : [-1.00;1.00]
 Determinateness (%)  : 100.00
 Attributes           : ['name', 'order', 'circulants', 'actions',
                         'valuationdomain', 'relation',
                         'gamma', 'notGamma']
```

Digraph *c7* is a symmetric crisp digraph showing, among others, the maximal independent set {‘2’, ‘5’, ‘7’}, i.e. an initial as well as terminal kernel. We may compute the corresponding initial kernel characteristic vector.

```
1  >>> g.computeKernelVector(['2','5','7'],Initial=True,Comments=True)
2   --> Initial kernel: {'2', '5', '7'}
3   initial low vector  : [-1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0]
4   initial high vector : [+1.0, +1.0, +1.0, +1.0, +1.0, +1.0, +1.0]
5   1 st low vector     : [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
6   1 st high vector    : [+1.0, +1.0, +1.0, +1.0, +1.0, +1.0, +1.0]
7   2 nd low vector     : [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
8   2 nd high vector    : [ 0.0, +1.0,  0.0,  0.0, +1.0,  0.0, +1.0]
9   stable low vector   : [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
10  stable high vector  : [ 0.0, +1.0,  0.0,  0.0, +1.0,  0.0, +1.0]
11  #iterations         : 3
12  low & high fusion   : [-1.0, +1.0, -1.0, -1.0, +1.0, -1.0, +1.0]
13  Choice vector for initial prekernel: {'2', '5', '7'}
14  2: +1.00
15  5: +1.00
16  7: +1.00
17  1: -1.00
18  3: -1.00
19  4: -1.00
20  6: -1.00
```

Notice that the stable low vector characterises the **negative membership** part, whereas, the stable high vector characterises the **positive membership** part (see Lines 9-10 above). The bipolar **disjunctive fusion** assembles eventually both stable parts into the correct prekernel characteristic vector (Line 12).

The adjacency matrix of a symmetric digraph staying *unchanged* by the transposition operator, the previous computations, when qualifying the same kernel as a *terminal* instance, will hence produce exactly the same result.

Historical notes

Following the observation that an independent absorbent choice in an acyclic digraph corresponds to the zero values –the kernel– of the associated *Grundy* function, *J. Riguet* [RIG-1948p] introduced the name **noyau** (kernel) for such a choice. Terminal kernels where in the sequel studied by *Claude Berge* [BER-1958p] in the context of Combinatorial Game Theory. Initial kernels –independent and dominating choices– were introduced under the name game solutions by *John von Neumann* [NEU-1944p]. The absorbent version of the crisp kernel equation system was first introduced by *Schmidt G. and Ströhlein Th.* [SCH-1985p] in the context of their thorough exploration of relational algebra.

The fuzzy version of kernel equation systems was first investigated by *Kitainik L.* [KIT-1993p]. Commenting on this work at a meeting in Spring 1995 of the EURO Working Group on Multicriteria Decision Support in Lausanne (Switzerland), *Marc Roubens* feared that solving such fuzzy kernel equation systems could be computationally difficult. Triggered by his pessimistic remark and knowing about kernel equation systems and the *Neumann* fixpoint theorem ([NEU-1944p], [SCH-1985p]), I immediately started to implement in Prolog a solver for the valued version of Equation $T(Y)$, the equation system serving as constraints for a discrete labelling of all possible rational solution vectors. And in Summer 1995, we luckily obtained with a commercial finite domain solver the very first valued initial and terminal kernels from a didactic outranking digraph of order 8, well known in the multiple-criteria decision aiding community. The computation took several seconds on a CRAY 6412 superserver with 12 processors operating in a nowadays ridiculous CPU speed of 90 Mhz. The labelled solution vectors, obtained in the sequel for any outranking digraph with a single initial or terminal kernel, were structured in a way that suggested the converging stages of the *Neumann* fixpoint algorithm and so gave the initial hint for our Algorithm ([BIS-1996p], [BIS-1997p]).

In our present Python3.12 implementation, such a tiny problem is solved in less than a thousandth of a second on a common laptop. And this remains practically the same for any relevant example of outranking digraph observed in a real decision-aiding problem. Several times we wrote in our personal journal that there is certainly now no more potential for any substantial improvement of this computational efficiency; Only to discover, shortly later, that following a new theoretical idea or choosing a more efficient implementation –-using for instance the amazing instrument of iterator generators in Python–-, execution times could well be divided by 20.

This nowadays available computational efficiency confers the bipolar-valued kernel concept a methodological premium for solving first or last choice decision problems on the basis of the bipolar-valued outranking digraph. But it also opens new opportunities for verifying and implementing kernel extraction algorithms for more graph theoretical purposes. New results, like enumerating the non isomorphic maximal independent sets –the kernels– of known difficult graph instances like the n-cycle, could be obtained [ISO-2008p].

Note

It is worthwhile noticing again the essential computational role, the logical **indeterminate value 0.0** is playing in this double fixpoint algorithm. To implement such kind of algorithms without a logical **neutral term** would be like implementing numerical algorithms without a possible usage of the number 0. Infinitely many trivial *impossibility theorems* and *dubious logical results* come up.

Back to *Content Table* (page 1)

3.7 On characterizing bipolar-valued outranking digraphs

- *Necessary properties of the outranking digraph* (page 133)
- *Partial tournaments may be strict outranking digraphs* (page 134)
- *Recognizing bipolar outranking valuations* (page 136)
- *On generating random outranking valuations* (page 139)

Necessary properties of the outranking digraph

Bipolar-valued outranking digraphs verify two necessary properties [BIS-2013p]:

1) They are strongly complete. For all pairs (x, y) of decision actions:

   $\big(r(x \succsim y) + r(y \succsim x)\big) \geqslant 0.0$ and,

2) The construction of the outranking relation verifies the coduality principle. For all pairs (x, y) of decision actions, $r(x \not\succsim y) = r(y \precnsim x)$.

Now, the codual of complete digraphs correspond to the class of asymmetrical digraphs i.e. *partial tournaments*. If, on the one limit, all outranking relations are symmetric, the partial tournament will be empty. On the other hand, if the outranking relation models a linear ranking, the tournament will be complete and transitive.

Let us consider for instance such a partial tournament[6].

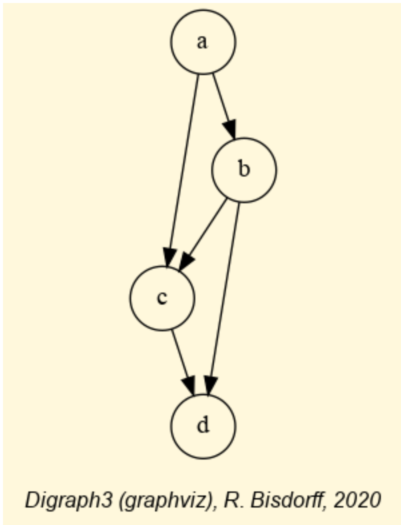


Fig. 3.28: A partial tournament

In Fig. 3.28, only the transitive closure between alternatives a and d is missing. Otherwise, the relation would be modelling a linear ranking from a to d. If this relation is actually supposed to model a *strict* outranking relation then both alternatives a and d positively outrank each other. Is it possible to build a corresponding valid performance tableau which supports epistemically this partial tournament?

It is indeed possible to define such a performance tableau by, first, using a single criterion *g1* of significance weight 2 modelling the apparent linear ranking: $a > b > c > d$. We can, secondly, add a criterion *g2* of significance weight 3 modelling exclusively the missing "*as well evaluated as*" situation between a and d. Both criteria admit without loss of genericity a performance measurement scale of 0 to 100 points with an indifference discrimination threshold of 2.5 and preference discrimination threshold of 5 points. No considerable performance difference discrimination is needed in this example.

[6] The example was proposed in 2005 by *D. Bouyssou* when discussing the necessity or not of a best choice recommendation to be internally stable –pragmatic principle *P3*– [BIS-2008p]

Listing 3.29: A potential performance tableau

```
1  >>> from perfsTab import PerformanceTableau
2  >>> from outrankingDigraphs import BipolarOutrankingDigraph
3  >>> pt = PerformanceTableau('testBouyssou')
4  >>> pt.showPerformanceTableau(ndigits=0)
5   *----  performance tableau ----*
6    Criteria |  'g1'    'g2'
7    Actions  |    2       3
8    ---------|---------------
9      'a'    |   70      70
10     'b'    |   50      NA
11     'c'    |   30      NA
12     'd'    |   10      70
13 >>> g = BipolarOutrankingDigraph(pt)
14 >>> g.showRelationTable()
15  * ---- Relation Table -----
16   r    |  'a'    'b'    'c'    'd'
17  -----|---------------------------
18   'a' | +1.00  +0.40  +0.40  +1.00
19   'b' | -0.40  +1.00  +0.40  +0.40
20   'c' | -0.40  -0.40  +1.00  +0.40
21   'd' | +0.20  -0.40  -0.40  +1.00
22  Valuation domain: [-1.000; 1.000]
```

In Listing 3.29 Lines 9-12 we notice that criterion *g1* models with a majority margin of $2/5 = 0.40$ the requested linear ranking and criterion *g2* warrants with a majority margin of $1/5 = 0.20$ that d is "*at least as well evaluated as*" d (see Lines 18 and 21) leading to the necessary reciprocal outranking situations between a and d.

It becomes apparent with the partial tournament example here that, when the number of criteria is not constrained, we may model in this way compatible pairwise outranking situations independently one of the other.

Partial tournaments may be strict outranking digraphs

In the `randomDigraphs` module we provide the `RandomPartialTournament` class for providing such partial tournament instances.

Listing 3.30: A partial tournament of order 5

```
>>> from randomDigraphs import RandomPartialTournament
>>> rpt = RandomPartialTournament(order=5,seed=998)
>>> rpt.showRelationTable()
 * ---- Relation Table -----
   S    |  'a1'   'a2'   'a3'   'a4'    'a5'
 ------|-----------------------------------
  'a1' |  0.00   1.00   1.00  -1.00    1.00
```

(continues on next page)

(continued from previous page)

```
  'a2' | -1.00   0.00   1.00   1.00    1.00
  'a3' | -1.00  -1.00   0.00   1.00   -1.00
  'a4' |  1.00  -1.00  -1.00   0.00   -1.00
  'a5' | -1.00  -1.00  -1.00   1.00    0.00
 Valuation domain: [-1.00;1.00]
>>> rpt.exportGraphViz()
 *---- exporting a dot file for GraphViz tools ----*
  Exporting to randomPartialTournament.dot
  dot -Grankdir=BT -Tpng randomPartialTournament.dot\
                       -o randomPartialTournament.png
```

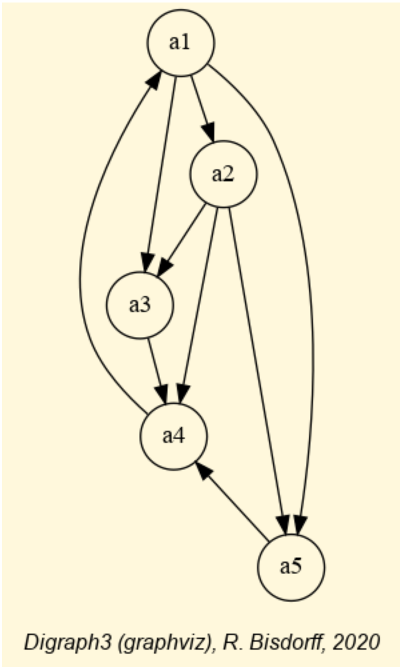


Fig. 3.29: A random partial tournament of order 5

The crisp partial tournament *rpt* shown in Fig. 3.29 corresponds to the potential strict outranking digraph one may obtain with the following multicriteria performance records measured on 10 criteria admitting a 0-100 scale with a 2.5pts indifference and a 5pts preference discrimination thresholds.

```
*----  performance tableau -----*
criteria | weights | 'a1'  'a2'  'a3'  'a4'  'a5'
---------|---------------------------------------
  'g01'  |   1.0   |  60    40    NA    NA    NA
  'g02'  |   1.0   |  60    NA    40    NA    NA
  'g03'  |   1.0   |  40    NA    NA    60    NA
  'g04'  |   1.0   |  60    NA    NA    NA    40
  'g05'  |   1.0   |  NA    60    40    NA    NA
  'g06'  |   1.0   |  NA    60    NA    40    NA
```

(continues on next page)

(continued from previous page)

```
10    'g07'  |   1.0   |  NA    60    NA    NA    40
11    'g08'  |   1.0   |  NA    NA    60    40    NA
12    'g09'  |   1.0   |  NA    NA    50    NA    50
13    'g10'  |   1.0   |  NA    NA    NA    40    60
```

Each one of the ten performance criteria independently models, with a majority margin of $1/10 = 0.10$, one of the 10 links between the five nodes of the tournament *rpt*. Criterion *g01* models for instance the asymmetrical link between *a1* and *a2* (Line 4), criterion *g9* models the symmetric link between *a3* and *a5* (Line 12) and so on. The bipolar-valued strict outranking relation we obtain with this performance tableau is the following:

```
* ---- Relation Table -----
   r   |  'a1'   'a2'   'a3'   'a4'   'a5'
 -----|---------------------------------
 'a1' |    -    +0.10  +0.10  -0.10  +0.10
 'a2' | -0.10    -    +0.10  +0.10  +0.10
 'a3' | -0.10  -0.10    -    +0.10  -0.10
 'a4' | +0.10  -0.10  -0.10    -    -0.10
 'a5' | -0.10  -0.10  -0.10  +0.10    -
 Valuation domain: [-1.000; 1.000]
```

And we recover here exactly the random partial tournament shown in Fig. 3.29.

To all partial tournament we may hence associate a multicriteria performance tableau, making it hence the instance of a potential bipolar-valued strict outranking digraph. Yet, we have not taken care of reproducing the precise characteristic valuation of a given partial tournament. Is it as well possible to always associate a valid performance tableau which produces a strict outranking digraph with exactly the given characteristic valuation?

Recognizing bipolar outranking valuations

From the fact that the epistemic support of a strict outranking –'*better evaluated as*'– situation is a potential sub-part only of the epistemic support of the corresponding outranking –'*at least as well evaluated as*'– situation, it follows that for all irreflexive pairs (x, y), $r(x \succsim y) \geqslant r(x \not\precsim y)$, which induces by the coduality principle the following necessary condition on the valuation of a potential outranking digraph:

$$r(x \succsim y) \geqslant -r(y \succsim x), \quad \forall x \neq y \in X. \tag{3.1}$$

Condition (3.1) strengthens in fact a *weakly completeness* property. Indeed:

$$\big(r(x \succsim y) < 0.0\big) \Rightarrow \big[r(y \succsim x) \geqslant -r(x \succsim y) > 0.0\big]. \tag{3.2}$$

And,

$$\big(r(x \succsim y) = 0.0\big) \;\Rightarrow\; \big(r(y \succsim x) \geqslant 0.0\big). \tag{3.3}$$

The bipolar valuation of a valid outranking digraph is hence necessarily characterised by the following **strong completeness** condition, algebraically equivalent to Condition (3.1):

$$r(x \succsim y) + r(y \succsim x) \;\geqslant\; 0.0, \;\; \forall x \neq y \in X. \tag{3.4}$$

It remains to proof that Condition (3.4) is (or is actually not) also sufficient for characterising the valuation of bipolar-valued outranking digraphs. In other words:

Conjecture

> *For any given bipolar and rational valued digraph verifying* (3.4) *it is possible to construct with an unconstrained number of criteria a valid performance tableau that results in identically valued pairwise outranking situations.*

If the conjecture reveals itself to be true, and we are rather confident that this will indeed be the case, we get a method of complexity $O(n^2)$ for recognizing potential outranking digraph instances with view solely on their relational characteristic valuation (see Listing 3.31 Lines 17-18) [MEY-2008].

Listing 3.31: Recognizing a bipolar outranking valuation

```
1  >>> from randomPerfTabs import RandomPerformanceTableau
2  >>> t = RandomPerformanceTableau(weightDistribution="equiobjectives",
3  ...                              numberOfActions=5,numberOfCriteria=3,
4  ...                              missingDataProbability=0.05,seed=100)
5  >>> from outrankingDigraphs import BipolarOutrankingDigraph
6  >>> g = BipolarOutrankingDigraph(t)
7  >>> g.showRelationTable()
8   * ---- Relation Table -----
9      r   |    'a1'   'a2'   'a3'   'a4'   'a5'
10   ------|-----------------------------------
11    'a1' |   +1.00  -0.33  -0.33  -0.67  -1.00
12    'a2' |   +0.33  +1.00  -0.33  +0.00  +0.33
13    'a3' |   +1.00  +0.33  +1.00  +0.67  +0.33
14    'a4' |   +0.67  +0.00  +0.00  +1.00  +0.67
15    'a5' |   +1.00  -0.33  -0.33  -0.67  +1.00
16    Valuation domain: [-1.000; 1.000]
17  >>> g.isOutrankingDigraph()
18   True
```

Whereas, when we consider in Listing 3.32 a genuine randomly bipolar-valued digraph of order 5, this check will mostly fail.

Listing 3.32: Failing the outranking valuation check

```
1  >>> from randomDigraphs import RandomValuationDigraph
2  >>> rdg = RandomValuationDigraph(order=5)
3  >>> rdg.showRelationTable()
4   * ---- Relation Table -----
5    S    |   'a1'  'a2'   'a3'   'a4'     'a5'
6  ------|-----------------------------------------
7   'a1' |  0.00   0.00  -0.68   0.94    0.06
8   'a2' | -0.14   0.00  -0.44  -0.04    0.84
9   'a3' | -0.14   0.12   0.00  -0.10   -0.62
10  'a4' |  0.40  -0.86   0.98   0.00    0.90
11  'a5' | -0.92   0.18  -0.42   0.14    0.00
12  Valuation domain: [-1.00;1.00]
13 >>> rdg.isOutrankingDigraph(Debug=True)
14  x,y,relation[x][y],relation[y][x]     a1 a2  0.00 -0.14
15  Not a valid outranking valuation
16  x,y,relation[x][y],relation[y][x]     a1 a3 -0.68 -0.14
17  Not a valid outranking valuation
18  x,y,relation[x][y],relation[y][x]     a1 a5  0.06 -0.92
19  Not a valid outranking valuation
20  x,y,relation[x][y],relation[y][x]     a2 a3 -0.44  0.12
21  Not a valid outranking valuation
22  x,y,relation[x][y],relation[y][x]     a2 a4 -0.04 -0.86
23  Not a valid outranking valuation
24  x,y,relation[x][y],relation[y][x]     a3 a5 -0.62 -0.42
25  Not a valid outranking valuation
26  False
```

We observe in Lines 14-25 the absence of any relation between *a1* and *a3*, between *a2* and *a4*, and between *a3* and *a5*. This violates the necessary weak completeness Condition (3.2). The pairs (*a1*, *a2*) and (*a2*, *a3*) furthermore violate Condition (3.3).

A Monte Carlo simulation with randomly bipolar-valued digraphs of order 5 shows that an average proportion of only 0.07% of random instances verify indeed Condition (3.4). With randomly bipolar-valued digraphs of order 6, this proportion drops furthermore to 0.002%. Condition (3.4) is hence a very specific characteristic of bipolar outranking valuations.

Readers challenged by the proof of the sufficiency of Condition (3.4) may find below a bipolar-valued relation verifying (3.4)

```
* ---- Relation Table -----
   r    |  'a1'   'a2'   'a3'   'a4'   'a5'
  -----|---------------------------------
  'a1' | +1.00  +0.60  +0.60  +0.20  +0.20
  'a2' | -0.20  +1.00  +0.00  -0.20  +0.20
  'a3' | -0.40  +0.60  +1.00  +0.20  +0.60
```

(continues on next page)

(continued from previous page)

```
  'a4' | -0.20  +0.20  -0.20  +1.00  +0.20
  'a5' | -0.20  +0.00  -0.20  +0.60  +1.00
Valuation domain: [-1.000; 1.000]
```

Is it possible to construct a corresponding performance tableau giving exactly the shown valuation? *Hint*: the criteria may be equi-significant[7].

Solving the previous problem requires to choose an adequate number of criteria. This raises the following question:

> *What is the minimal number of criteria needed in a performance tableau that corresponds to the valuation of a given bipolar-valued outranking digraph.*

We call this number the **epistemic dimension** of the bipolar-valued outranking digraph. This dimension depends naturally on the potential presence of chordless outranking cycles and indeterminate outranking situations. A crisp linear outranking digraph, for instance, can be modelled with a single performance criterion and is hence of dimension 1. Designing an algorithm for determining *epistemic dimensions* remains an open challenge.

Let us finally mention that the *dual* –the negation– of Condition (3.4) characterizes strict outranking valuations. Indeed, by verifying the coduality principle:

$$-\big(r(x \succsim y) \,+\, r(y \succsim x)\big) \;=\; r(y \not\succsim x) \,+\, r(x \not\succsim y),$$

we obtain the following condition:

$$r(x \not\succsim y) \,+\, r(y \not\succsim x) \;\leqslant\; 0.0, \;\; \forall x \neq y \in X. \tag{3.5}$$

A similar Monte Carlo simulation with randomly bipolar-valued digraphs of order 5 shows that an average proportion of only 0.12% of random instances verify Condition (3.5). With randomly bipolar-valued digraphs of order 6, this proportion drops to 0.006%. Condition (3.5) is hence again a very specific characteristic of bipolar strict outranking valuations.

On generating random outranking valuations

The `RandomOutrankingValuationDigraph` class from the `randomDigraphs` module provides a generator for random outranking valuation digraphs.

[7] A solution is provided under the name *enigmaPT.py* in the *examples* directory of the Digraph3 resources

Listing 3.33: Generating random outranking valuations

```
1 >>> from randomDigraphs import RandomOutrankingValuationDigraph
2 >>> rov = RandomOutrankingValuationDigraph(order=5,
3 ...           weightsSum=10,
4 ...           distribution='uniform',
5 ...           incomparabilityProbability=0.1,
6 ...           polarizationProbability=0.05,
7 ...           seed=1)
8 >>> rov.showRelationTable()
9  * ---- Relation Table -----
10    S    |  'a1'   'a2'   'a3'   'a4'   'a5'
11  ------|----------------------------------
12   'a1' |   10     -2     10      4      4
13   'a2' |   10     10     10      4     10
14   'a3' |  -10    -10     10      0      8
15   'a4' |   -4     -3      0     10      8
16   'a5' |    2    -10      2      3     10
17  Valuation domain: [-10;+10]
18 >>> rov.isOutrankingDigraph()
19  True
```

The generator works like this. For each link between $\{x, y\}$, first a random integer number is uniformly drawn for $r(y, x)$ in the given range $[-weightsSum; +weightsSum]$ (see Listing 3.33 Line 3). Then, $r(x, y)$ is uniformly drawn in the remaining integer interval $[-r(y, x); +weightsSum]$.

In order to favour a gathering around the median zero characteristic value, it is possible to use a *triangular* law instead (see Line 4).

For inserting random considerable performance difference situations, it is possible to define the probabilities of incomparability (default 10%, see Line 5) and/or polarized outranking situations (5%, see Line 6).

The resulting valuation (see Lines 12-16) verifies indeed condition (3.4) (see Lines 18-19).

Back to *Content Table* (page 1)

3.8 Consensus quality of the bipolar-valued outranking relation

- *Circular performance tableaux* (page 141)
- *A difficult decision problem* (page 143)
- *The central CONDORCET point of view* (page 144)

Circular performance tableaux

In order to study the actual consensus quality of a bipolar-valued outranking relation, let us consider a small didactic performance tableau consisting of five decision actions evaluated with respect to five performance criteria of equal significance. On each one of the criteria, we swap first and last ranked evaluations in a circular way (see Lines 8-12 below).

Listing 3.34: Circular performance tableau

```
1  >>> from perfTabs import CircularPerformanceTableau
2  >>> cpt5 = CircularPerformanceTableau(order=5,NoPolarisation=True)
3  >>> cpt5.showPerformanceTableau()
4   *----  performance tableau -----*
5   Criteria |  'g1'   'g2'   'g3'   'g4'   'g5'
6   Actions  |    1      1      1      1      1
7   ---------|-----------------------------------------
8     'a1'   |  0.00  80.00   60.00  40.00  20.00
9     'a2'   | 20.00   0.00   80.00  60.00  40.00
10    'a3'   | 40.00  20.00    0.00  80.00  60.00
11    'a4'   | 60.00  40.00   20.00   0.00  80.00
12    'a5'   | 80.00  60.00   40.00  20.00   0.00
```

In Listing 3.34 Line 2, we do not consider for the moment any considerable performance differences. A performance difference up to 2.5 is considered insignificant, whereas a performance difference of 5.0 and more is attesting a preference situation.

```
>>> cpt5.showCriteria()
 *----  criteria -----*
 g1 RandomPerformanceTableau() instance
  Preference direction: max
  Scale = (0.00, 100.00)
  Weight = 0.200
  Threshold ind : 2.50 + 0.00x ; percentile: 0.00
  Threshold pref : 5.00 + 0.00x ; percentile: 0.00
 g2 RandomPerformanceTableau() instance
  ...
```

All the five decision alternatives show in fact a same performance profile, yet distributed differently on the criteria which are equally significant. The preferential information of such a circular performance tableau does hence not deliver any clue for solving a selection or a ranking decision problem.

Let us inspect the corresponding bipolar-valued outranking digraph.

```
>>> from outrankingDigraphs import BipolarOutrankingDigraph
>>> bodg = BipolarOutrankingDigraph(cpt5)
>>> bodg.exportGraphViz()
 *---- exporting a dot file for GraphViz tools ----*
```

(continues on next page)

(continued from previous page)

```
Exporting to rel_circular-5-PT.dot
dot -Grankdir=BT -Tpng rel_circular-5-PT.dot\
                 -o rel_circular-5-PT.png
```

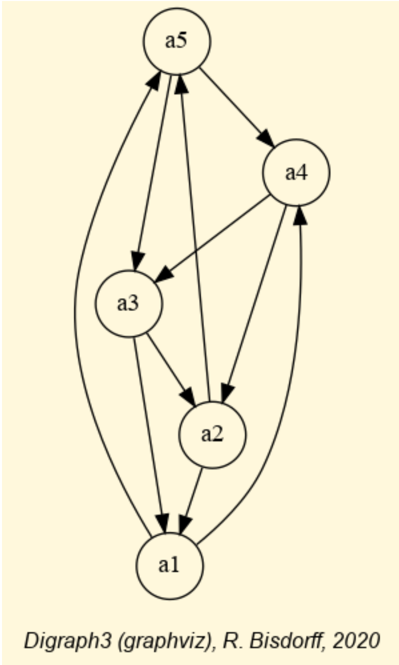


Fig. 3.30: Outranking digraph of circular performance tableau of order 5

In Fig. 3.30 we notice that the outranking digraph models in fact a complete and regular tournament. Each alternative is outranking, respectively outranked by two other alternatives. The outranking relation is not transitive -half of the transitivity arcs are missing- and we observe five equally credible outranking circuits.

```
>>> bodg.computeTransitivityDegree()
 Decimal('0.5')
>>> bodg.computeChordlessCircuits()
>>> bodg.showChordlessCircuits()
 *---- Chordless circuits ----*
 5 circuits.
 1: ['a1', 'a4', 'a3'] , credibility : 0.200
 2: ['a1', 'a4', 'a2'] , credibility : 0.200
 3: ['a1', 'a5', 'a3'] , credibility : 0.200
 4: ['a2', 'a5', 'a3'] , credibility : 0.200
 5: ['a2', 'a5', 'a4'] , credibility : 0.200
```

A difficult decision problem

Due to the regular tournament structure, the *Copeland* scores are the same for each one of the decision alternatives and we end up with a ranking in alphabetic order.

```
>>> from linearOrders import CopelandRanking
>>> cop = CopelandRanking(bodg,Comments=True)
 Copeland scores
  a1 : 0
  a2 : 0
  a3 : 0
  a4 : 0
  a5 : 0
 Copeland Ranking:
  ['a1', 'a2', 'a3', 'a4', 'a5']
```

Same situation appears below with the *NetFlows* scores.

```
>>> nf = NetFlowsOrder(bodg,Comments=True)
 Net Flows :
  a1 : 0.000
  a2 : 0.000
  a3 : 0.000
  a4 : 0.000
  a5 : 0.000
 NetFlows Ranking:
  ['a1', 'a2', 'a3', 'a4', 'a5']
```

Yet, when inspecting in Fig. 3.30 the outranking relation, we may notice that, when ignoring for a moment the upward arcs, an apparent downward ranking ['a5', 'a4', 'a3', 'a2', 'a1'] comes into view. We can try to recover this ranking with the help of the *Kemeny* ranking rule.

```
>>> ke = KemenyRanking(bodg)
>>> ke.maximalRankings
 [['a5', 'a4', 'a3', 'a2', 'a1'],
  ['a4', 'a3', 'a2', 'a1', 'a5'],
  ['a3', 'a2', 'a1', 'a5', 'a4'],
  ['a2', 'a1', 'a5', 'a4', 'a3'],
  ['a1', 'a5', 'a4', 'a3', 'a2']]
```

The *Kemeny* rule delivers indeed five optimal rankings which appear to be the circular versions of the apparent downward ranking ['a5', 'a4', 'a3', 'a2', 'a1'].

The epistemic disjunctive fusion of these five circular rankings gives again an empty relation (see Fig. 3.31 below).

```
>>> from transitiveDigraphs import RankingsFusionDigraph
>>> wke = RankingsFusionDigraph(bodg,ke.maximalRankings)
>>> wke.exportGraphViz()
```

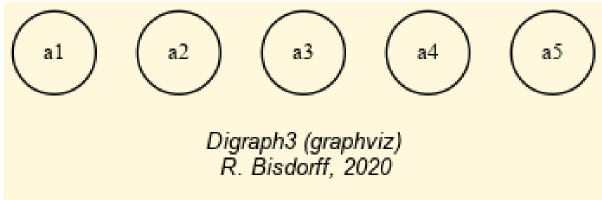


Fig. 3.31: Epistemic fusion of the five optimal Kemeny rankings

All ranking rules based on the bipolar-valued outranking digraph apparently deliver the same result: no effective ranking is possible. When the criteria are supposed to be equally significant, each decision alternative is indeed equally well performing from a multicriteria point of view (see Fig. 3.32).

```
>>> cpt5.showHTMLPerformanceHeatmap(Correlations=False,
...                              rankingRule=None,ndigits=0,
...          pageTitle='The circular performance tableau')
```

The circular performance tableau

criteria	g1	g2	g3	g4	g5
weights	+1.00	+1.00	+1.00	+1.00	+1.00
a1	0	80	60	40	20
a2	20	0	80	60	40
a3	40	20	0	80	60
a4	60	40	20	0	80
a5	80	60	40	20	0

Color legend:

quantile	14.29%	28.57%	42.86%	57.14%	71.43%	85.71%	100.00%

Fig. 3.32: The heatmap of the circular performance tableau

The pairwise outranking relation shown in Fig. 3.30 does hence represent a *faithful consensus* of the preference modelled by each one of the five performance criteria. We can inspect the actual quality of this consensus with the help of the bipolar-valued equivalence index (see the *advanced topic on the ordinal correlation between bipolar-valued digraphs* (page 69)).

The central CONDORCET point of view

The bipolar-valued outranking relation corresponds in fact to the median of the multicriteria points of view, at minimal KENDALL's ordinal correlation distance from all marginal criteria points of view [BAR-1980p].

Listing 3.35: Outranking Consensus quality

```
>>> bodg.computeOutrankingConsensusQuality(Comments=True)
 Consensus quality of global outranking:
  criterion (weight): valued correlation
```

(continues on next page)

(continued from previous page)

```
 -------------------------------------
  g5 (0.200): +0.200
  g4 (0.200): +0.200
  g3 (0.200): +0.200
  g2 (0.200): +0.200
  g1 (0.200): +0.200
 Summary:
  Weighted mean marginal correlation (a): +0.200
  Standard deviation (b)                : +0.000
  Ranking fairness (a)-(b)              : +0.200
```

As all the performance criteria are supposed to be equally significant, the bipolar-valued equivalence index of the outranking relation with each marginal criterion is at constant level +0.200 (see Listing 3.35).

Let us compute the pairwise ordinal correlation indices between each one the five criteria, including the median outranking relation.

```
>>> from digraphs import CriteriaCorrelationDigraph
>>> cc = CriteriaCorrelationDigraph(bodg,WithMedian=True)
>>> cc.showRelationTable()
 * ---- Relation Table -----*
   S   |  'g1'  'g2'  'g3'  'g4'  'g5'  'm'
 ------|-----------------------------------
  'g1' |  1.00  0.20 -0.20 -0.20  0.20  0.20
  'g2' |  0.20  1.00  0.20 -0.20 -0.20  0.20
  'g3' | -0.20  0.20  1.00  0.20 -0.20  0.20
  'g4' | -0.20 -0.20  0.20  1.00  0.20  0.20
  'g5' |  0.20 -0.20 -0.20  0.20  1.00  0.20
   'm' |  0.20  0.20  0.20  0.20  0.20  0.40
   Valuation domain: [-1.00;1.00]
```

We observe the same circular arrangement of the pairwise criteria correlations as the one observed in the circular performance tableau. We may draw a 3D principal plot of this correlation space.

```
>>> cc.exportPrincipalImage(plotFileName='correlation3Dplot')
```

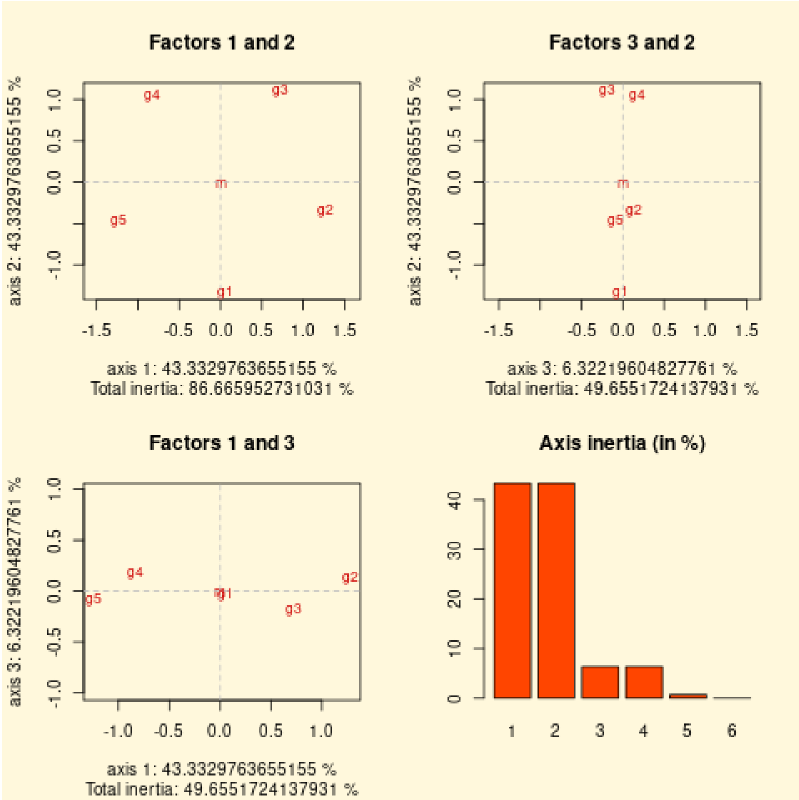


Fig. 3.33: The 3D plot of the principal components of the correlation matrix

In Fig. 3.33 , the median outranking relation **m** is indeed situated exactly in the middle of the regular pentagon of the marginal criteria.

What happens now when we observe imprecise performance evaluations, considerable performance differences, unequal criteria significance weights and missing evaluations? Let us therefore redo the same computations, but with a corresponding random 3-Objectives performance tableau.

Listing 3.36: Outranking consensus quality with 3-objectives tableaux

```
>>> from randomPerfTabs import\
...             Random3ObjectivesPerformanceTableau
>>> pt3Obj = Random3ObjectivesPerformanceTableau(
...             numberOfActions=7,numberOfCriteria=13,
...             missingDataProbability=0.05,seed=1)
>>> pt3Obj.showObjectives()
 Eco: Economical aspect
    ec01 criterion of objective Eco 18
    ec05 criterion of objective Eco 18
    ec09 criterion of objective Eco 18
    ec10 criterion of objective Eco 18
   Total weight: 72.00 (4 criteria)
 Soc: Societal aspect
```

(continues on next page)

(continued from previous page)

```
    so02 criterion of objective Soc 12
    so06 criterion of objective Soc 12
    so07 criterion of objective Soc 12
    so11 criterion of objective Soc 12
    so12 criterion of objective Soc 12
    so13 criterion of objective Soc 12
   Total weight: 72.00 (6 criteria)
 Env: Environmental aspect
    en03 criterion of objective Env 24
    en04 criterion of objective Env 24
    en08 criterion of objective Env 24
   Total weight: 72.00 (3 criteria)
>>> from outrankingDigraphs import\
...         BipolarOutrankingDigraph,
...          CriteriaCorrelationDigraph
>>> g3Obj = BipolarOutrankingDigraph(pt3Obj)
>>> cc3Obj = CriteriaCorrelationDigraph(g3Obj,
...         ValuedCorrelation=True,WithMedian=True)
>>> cc3Obj.saveCSV('critCorrTable.csv')
>>> cc3Obj.exportPrincipalImage(
...         plotFileName='correlation3Dplot-3Obj')
```

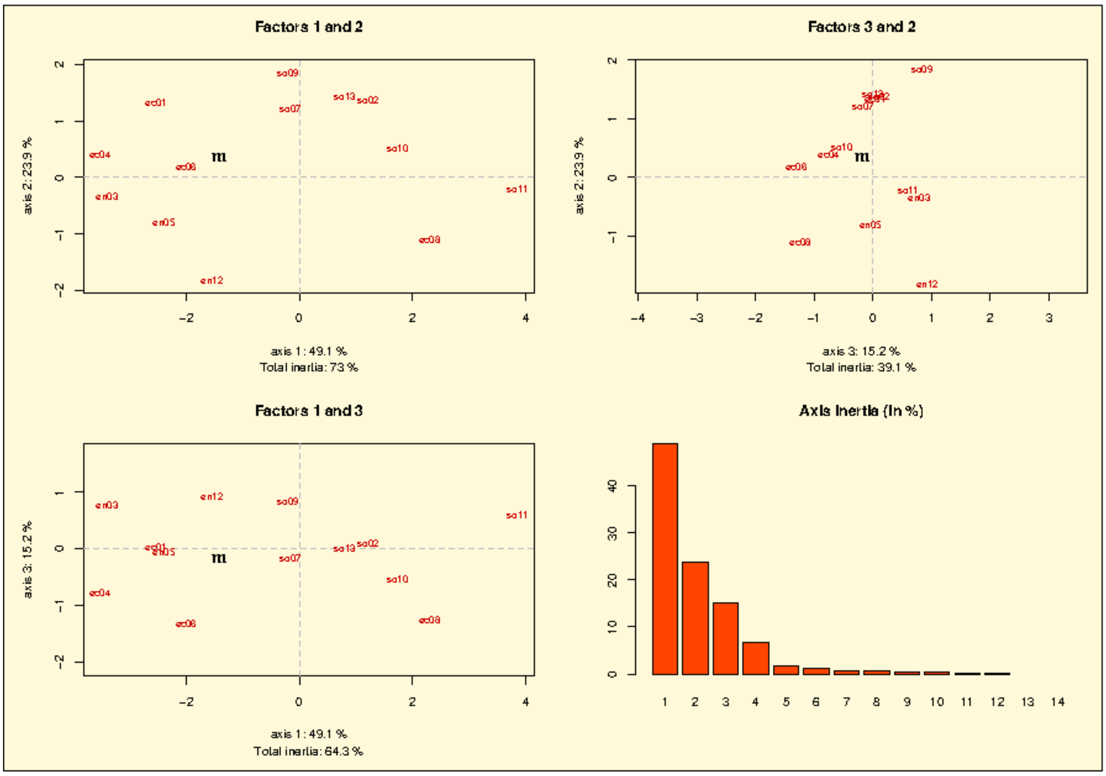


Fig. 3.34: The 3D plot of the principal components of the 3-Objectives correlation matrix

The global outranking relation m remains well situated in the weighted center of the eleven marginal criteria outranking relations. The global outranking relation 'm' is indeed mostly correlated with criteria: ‘ec04’ (+0.333), ‘ec06’ (+0.295), ‘en03’ (+0.243) and ‘ec01’ (+0.232) (see Fig. 3.35).

```
>>> criteriaList = [x for x in cc3Obj.actions]
>>> criteriaList.sort()
>>> cc3Obj.showHTMLRelationTable(actionsList=criteriaList,
...         tableTitle='Valued criteria correlation table',
...         ReflexiveTerms=True,relationName='tau(x,y)',ndigits=3)
```

valued criteria correlation table

tau(x,y)	ec01	ec04	ec06	ec08	en03	en05	en12	m	so02	so07	so09	so10	so11	so13
ec01	0.714	0.500	0.262	-0.357	0.381	0.143	-0.143	0.232	0.095	0.238	0.405	-0.048	-0.381	0.143
ec04	0.500	0.976	0.595	-0.119	0.429	0.262	0.024	0.333	0.024	0.119	0.000	-0.167	-0.548	0.071
ec06	0.262	0.595	0.905	0.143	0.071	0.262	-0.095	0.295	0.024	0.071	-0.071	0.024	-0.381	0.119
ec08	-0.357	-0.119	0.143	0.952	-0.357	-0.071	-0.190	-0.091	-0.024	-0.095	-0.405	0.262	0.286	-0.167
en03	0.381	0.429	0.071	-0.357	0.976	0.405	0.357	0.243	-0.071	-0.048	0.190	-0.262	-0.310	-0.190
en05	0.143	0.262	0.262	-0.071	0.405	0.714	0.238	0.136	-0.238	-0.071	0.024	-0.048	-0.333	-0.214
en12	-0.143	0.024	-0.095	-0.190	0.357	0.238	0.952	0.036	-0.238	-0.286	-0.238	-0.381	-0.048	-0.214
m	0.232	0.333	0.295	-0.091	0.243	0.136	0.036	0.371	0.132	0.077	0.083	-0.030	-0.183	0.112
so02	0.095	0.024	0.024	-0.024	-0.071	-0.238	-0.238	0.132	0.714	0.167	0.262	0.048	0.238	0.500
so07	0.238	0.119	0.071	-0.095	-0.048	-0.071	-0.286	0.077	0.167	0.452	0.262	0.071	-0.119	0.167
so09	0.405	0.000	-0.071	-0.405	0.190	0.024	-0.238	0.083	0.262	0.262	0.976	0.214	0.071	0.190
so10	-0.048	-0.167	0.024	0.262	-0.262	-0.048	-0.381	-0.030	0.048	0.071	0.214	0.714	0.143	-0.024
so11	-0.381	-0.548	-0.381	0.286	-0.310	-0.333	-0.048	-0.183	0.238	-0.119	0.071	0.143	1.000	0.167
so13	0.143	0.071	0.119	-0.167	-0.190	-0.214	-0.214	0.112	0.500	0.167	0.190	-0.024	0.167	0.643

Valuation domain: [-1.00; +1.00]

Fig. 3.35: Bipolar-valued relational equivalence table with included global outranking relation ‘m’

Let us conclude by showing in Listing 3.37 how to draw with the *R* statistics software the dendogram of a hierarchical clustering of the previous relational equivalence table. We use therefore the criteria correlation digraph *cc3Obj* saved in *CSV* format (see Listing 3.36 Line 32).

Listing 3.37: R session for drawing a hierarchical dendogram

```
> x = read.csv('critCorrTable.csv',row.names=1)
> X = as.matrix(x)
> dd = dist(X,method='euclidian')
> hc = hclust(dd)
> plot(hc)
```

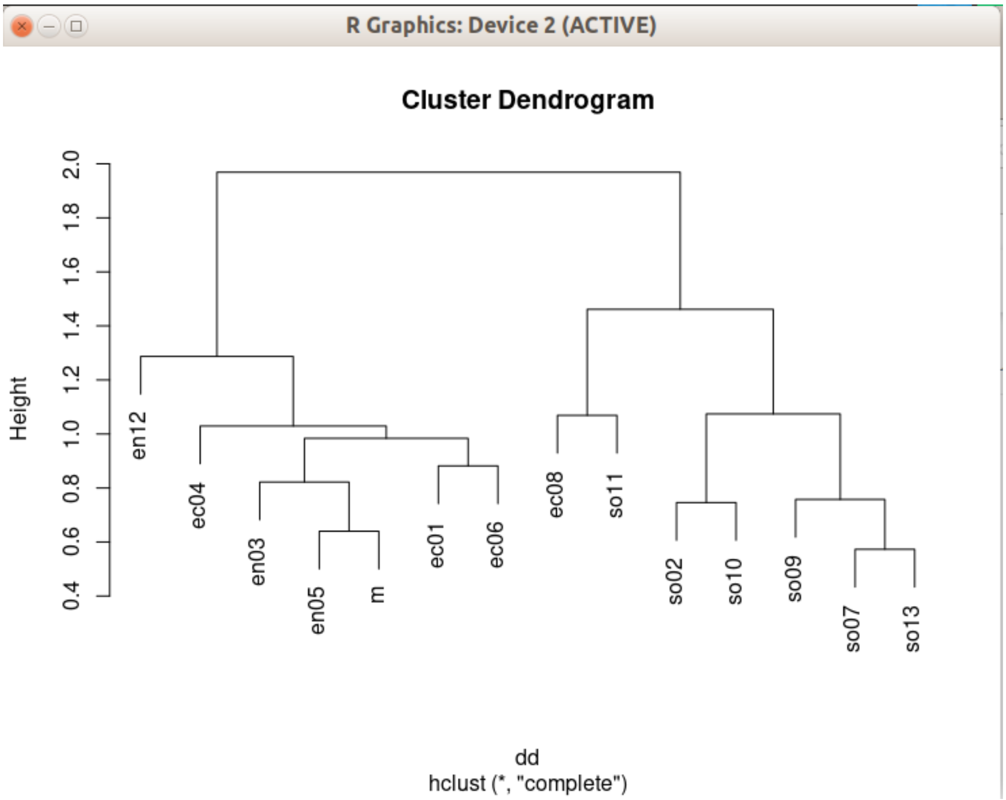


Fig. 3.36: Hierarchical clustering of the criteria correlation table

Fig. 3.36 confirms the actual relational equivalence structure of the marginal criteria outrankings and the global outranking relation. Environmental and economic criteria (left in Fig. 3.34) are opposite to the societal criteria (right in Fig. 3.34). This opposition results in fact from the random generator profile of the given seven decision alternatives as shown in Listing 3.38 below[8].

Listing 3.38: Random generator profile of the decision alternatives

```
>>> pt3Obj.showActions()
 *----- show decision action -----*
 key:  p1
  name: public policy p1 Eco+ Soc- Env+
```

(continues on next page)

[8] See the tutorial on Generating random performance tableaux

(continued from previous page)

```
 profile: {'Eco':'good', 'Soc':'weak', 'Env':'good'}
key:  p2
 name: public policy p2 Eco~ Soc+ Env~
 profile: {'Eco':'fair', 'Soc':'good', 'Env':'fair'}
key:  p3
 name: public policy p3 Eco~ Soc~ Env-
 profile: {'Eco':'fair', 'Soc':'fair', 'Env':'weak'}
key:  p4
 name: public policy p4 Eco~ Soc+ Env+
 profile: {'Eco':'fair', 'Soc':'good', 'Env':'good'}
key:  p5
 name: public policy p5 Eco~ Soc+ Env~
 profile: {'Eco':'fair', 'Soc':'good', 'Env':'fair'}
key:  p6
 name: public policy p6 Eco~ Soc- Env+
 profile: {'Eco':'fair', 'Soc':'weak', 'Env':'good'}
key:  p7
 name: public policy p7 Eco- Soc~ Env~
 profile: {'Eco':'weak', 'Soc':'fair', 'Env':'fair'}
```

Back to *Content Table* (page 1)

---

4 Appendix

References

[BIS-2015p] Bisdorff R. (2015). "The EURO 2004 Best Poster Award: Choosing the Best Poster in a Scientific Conference". Chapter 5 in R. Bisdorff, L. Dias, P. Meyer, V. Mousseau, and M. Pirlot (Eds.), *Evaluation and Decision Models with Multiple Criteria: Case Studies*. Springer-Verlag Berlin Heidelberg, International Handbooks on Information Systems, DOI 10.1007/978-3-662-46816-6_1, pp. 117-166 (downloadable PDF file 754.7 kB (http://hdl.handle.net/10993/23714)).

[BIS-2014p] Bisdorff R., Meyer P. and Veneziano Th. (2014). "Elicitation of criteria weights maximising the stability of pairwise outranking statements". *Journal of Multi-Criteria Decision Analysis* (Wiley) 21: 113-124 (downloadable preprint PDF file 431.4 Kb (http://hdl.handle.net/10993/23701)).

[BIS-2013p] Bisdorff R. (2013) "On Polarizing Outranking Relations with Large Performance Differences" *Journal of Multi-Criteria Decision Analysis* (Wiley) **20**:3-12 (downloadable preprint PDF file 403.5 Kb (http://hdl.handle.net/10993/245)).

[BAU-2013p] Baujard A., Gavrel F., Igersheim H., Laslier J.-F. and Lebon I. (2013). "Approval Voting, Evaluation Voting: An Experiment during the 2012 French

Presidential Election". In *Revue Économique* (Presses de Sciences Po) Volume 64, Issue 2, pp. 345-356 (dowloadable English translation, 652.5 kB).

[BIS-2012p] Bisdorff R. (2012). "On measuring and testing the ordinal correlation between bipolar outranking relations". In Proceedings of DA2PL'2012 *From Multiple Criteria Decision Aid to Preference Learning*, University of Mons 91-100. (downloadable preliminary version PDF file 408.5 kB (http://hdl.handle.net/10993/23909)).

[BAL-2011] Balinski M. and Laraki R. (2011) , Majority Judgment : Measuring, Ranking, and Electing, MIT Press, mars 2011, 1re éd. 448 p. ISBN 978-0-262-01513-4

[BIS-2008p] Bisdorff R., Meyer P. and Roubens M.(2008) "RUBIS: a bipolar-valued outranking method for the choice problem". 4OR, *A Quarterly Journal of Operations Research* Springer-Verlag, Volume 6, Number 2 pp. 143-165. (Online) Electronic version: DOI: 10.1007/s10288-007-0045-5 (downloadable preliminary version PDF file 271.5Kb (http://hdl.handle.net/10993/23716)).

[MEY-2008] Meyer P., Marichal J.-L. and Bisdorff R. (2008). Disagregation of bipolar-valued outranking relations. In Modelling, Computation and Optimization in Information Systems and Management Sciences, H. A. Le Thi, P. Bouvry, and D. Pham (eds), Springer CCIS 14 204-213, ISBN 978-3-540-87476-8 (preliminary version PDF file 129.9Kb (http://hdl.handle.net/10993/14485)

[BIS-2006_1p] Bisdorff R., Pirlot M. and Roubens M. (2006). "Choices and kernels from bipolar valued digraphs". *European Journal of Operational Research*, 175 (2006) 155-170. (Online) Electronic version: DOI:10.1016/j.ejor.2005.05.004 (downloadable preliminary version PDF file 257.3Kb (http://hdl.handle.net/10993/23720)).

[BIS-2006_2p] Bisdorff R. (2006). "On enumerating the kernels in a bipolar-valued digraph". *Annales du Lamsade* 6, Octobre 2006, pp. 1 - 38. Université Paris-Dauphine. ISSN 1762-455X (downloadable version PDF file 532.2 Kb (http://hdl.handle.net/10993/38741)).

[BIS-2004_1p] Bisdorff R. (2004). "Concordant Outranking with multiple criteria of ordinal significance". 4OR, *Quarterly Journal of the Belgian, French and Italian Operations Research Societies*, Springer-Verlag, Issue: Volume 2, Number 4, December 2004, Pages: 293 - 308. [ISSN: 1619-4500 (Paper) 1614-2411 (Online)] Electronic version: DOI: 10.1007/s10288-004-0053-7 (downloadable preliminary version PDF file 137.1Kb (http://hdl.handle.net/10993/23721))

[BIS-2004_2p] Bisdorff R. (2004). Preference aggregation with multiple criteria of ordinal significance. In: D. Bouyssou, M. Janowitz, F. Roberts, and A. Tsouki´s (eds.), Annales du LAMSADE, 3, Octobre 2004, Université Paris-Dauphine, pp. 25-44 [ISSN 1762-455X] (downloadable PDF file 167.6Kb (http://hdl.handle.net/10993/42420)).

[BIS-2004_3p] Bisdorff R. (2004), On a natural fuzzification of Boolean logic, in Erich Peter Klement and Endre Pap (editors), Proceedings of the 25th Linz Seminar on Fuzzy Set Theory, Mathematics of Fuzzy Systems. Bildungszentrum St.

Magdalena, Linz (Austria), February 2004. pp. 20-26 (downloadable PDF file 136.6Kb)

[SCH-1985p] Schmidt G. and Ströhlein Th. (1985), “On kernels of graphs and solutions of games: a synopsis based on relations and fixpoints”. SIAM, J. *Algebraic Discrete Methods*, 6:54–65.

[RIG-1948p] Riguet J. (1948), “Relations binaires, fermetures, correspondances de galois”. *Bull Soc Math France* 76:114–155.

[NEU-1944p] von Neumann J. and Morgenstern O. (1944). *Theory of games and economic behaviour*. Princeton University Press, Pinceton.

[KIT-1993p] Kitainik L. (1993). *Fuzzy decision procedures with binary relations: towards a unified theory*. Kluwer Academic Publisher Boston.

[ISO-2008p] Bisdorff R. and Marichal J. (2008). “Counting non-isomorphic maximal independent sets of the n-cycle graph”. *Journal of Integer Sequences* 11 (Art. 08.5.7):1–16, https://cs.uwaterloo.ca/journals/JIS/VOL11/Marichal/marichal.html

[BIS-1996p] Bisdorff R. (1996). “On computing kernels on fuzzy simple graphs by combinatorial enumeration using a CPL(FD) system”. In: *8th Benelux Workshop on Logic Programming*, Louvain-la-Neuve (BE), 9 September 1996, Université catholique de Louvain, http://hdl.handle.net/10993/46933

[BIS-1997p] Bisdorff R. (1997). “On computing kernels on l-valued simple graphs*. In: *Proceedings of EUFIT’97, 5th European Congress on Fuzzy and Intelligent Technologies*, Aachen, September 8-11, 1997}, pp 97–103.

[BAR-1980p] Barbut M. (1980), “Médianes, Condorcet et Kendall”. *Mathématiques et Sciences Humaines*, 69:9–13.

[BER-1958p] Berge C. (2001), *The theory of graphs*. Dover Publications Inc. 2001. First published in English by Methuen & Co Ltd., London 1962. Translated from a French edition by Dunod, Paris 1958.

[KEN-1938p] Kendall M.G. (1938), “A New Measure of Rank Correlation”. *Biometrica* 30:81–93

[ROY-1966p] Benyaoun S., Roy B. and Sussmann B. (1966), “ELECTRE: une méthode pour guider le choix en présence de points de vue multiples”. *Tech. Rep. 49, SEMA Direction Scientifique Paris.*

[CON-1785p] Condorcet, J.A.N. de Caritat marquis de (1785), *Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix*, Imprimerie royale Paris, https://gallica.bnf.fr/ark:/12148/bpt6k417181/f4.item

[BRI-2008p] Brian E. (2008), “Condorcet and Borda in 1784. Misfits and Documents”, *Journal Electronique d’Histore des Probabilités et de la Statistique*, Vol 4, n°1, Juin/June 2008, https://www.jehps.net/

[BAC-1622p] Claude Gaspard Bachet, sieur de Méziriac, *Problèmes plaisants et délectables...* , 1st ed. Pierre Rigaud & Associates, Lyon, France 1622, pp. 143-146

[JIN-2003] Jin, Y., He, H. and Lü, Y (2003), “Ternary optical computer principle”. *Sci China Ser* F 46:145–150, https://link.springer.com/article/10.1360/03yf9012

[KNU-1997] Knuth, Donald (1997), “The art of Computer Programming Third Edition”. Vol. 2. Addison-Wesley. pp. 207-208. ISBN 0-201-89684-2.