Chordless circuits

Let $n \geqslant 1$ be an integer. Consider the square with vertices $A = (0,0)$, $B = (0,n)$, $C = (n,n)$ and $D = (n,0)$ and let x_n be the number of paths increasing from A to C above the diagonal AC and then decreasing from C to A below the diagonal AC without touching twice the diagonal at the same point.

Then we have

$$\sum_{k=1}^{n} \sum_{i_1+\cdots+i_k=n} x_{i_1} \cdots x_{i_k} = \left(\frac{(2n)!}{n!(n+1)!}\right)^2$$

The right-hand side represents the number of such paths without the touching constraint. In the left-hand side, we partition the square into smaller squares along the diagonal AC.

Isolating the term $k = 1$ yields x_p in terms of $x_1, \ldots, x_{p-1}$.

The first few elements are: $x_1 = 1$, $x_2 = 3$, $x_3 = 18$, $x_4 = 141$, $x_5 = 1280$,...