Disaggregation of bipolar-valued outranking relations and application to the inference of model parameters

Raymond Bisdorff[a], Jean-Luc Marichal[b], Patrick Meyer[c,d]

[a] *University of Luxembourg, Faculty of Sciences, Technology, and Communication, CSC Research Unit, 6 rue Coudenhove-Kalergi, L-1359 Luxembourg*
[b] *University of Luxembourg, Faculty of Sciences, Technology, and Communication, Mathematics Research Unit, 6 rue Coudenhove-Kalergi, L-1359 Luxembourg*
[c] *Institut Télécom; Télécom Bretagne, UMR CNRS 3192 Lab-STICC, Technopôle Brest Iroise CS 83818, F-29238 Brest Cedex 3*
[d] *Université européenne de Bretagne*

Abstract

We show how performances of alternatives, criteria significance weights and preference discrimination thresholds can be reconstructed from a given bipolar-valued outranking relation. We furthermore detail how these theoretical results may be used in real-world decision problems involving multiple criteria to elicit a decision maker's preferences.

Keywords: inverse multiple criteria decision aid, disaggregation, bipolar-valued outranking relation, preference elicitation

1. Introduction

Let $X = \{x, y, z, \ldots\}$ be a set of p alternatives and $N = \{1, \ldots, n\}$ be a set of n criteria. Each alternative of X is evaluated on each of the criteria of N. Let us write $g_i(x)$ for the performance of alternative x on criterion i of N, where $g_i \colon X \to [0,1]$ s.t.:

$$\forall x, y \in X,\ g_i(x) \geq g_i(y) \iff x \text{ is } \textit{at least as good as } y \text{ on criterion } i. \quad (1)$$

With each criterion i of N we associate its *significance weight* represented by a rational number w_i from the unit interval $[0,1]$ such that

$$\sum_{i=1}^{n} w_i = 1$$

.

To represent more accurately a decision maker's (DM's) marginal "*at least as good as*" preferences, the model of Formula (1) can be enriched by discrimination thresholds (weak preference, preference, weak veto, veto; see, e.g., [BMR08]).

Let S be an outranking relation on X. Classically, the proposition "*x outranks y*" (xSy) ($x, y \in X$) is assumed to be validated if a significant majority of criteria validates the fact that x is performing at least as good as y and there is no criterion where y seriously outperforms y [Roy85]. The degree of validation of such outranking statements can be measured by a valuation of the outranking relation $\widetilde{S}\colon X \times X \to [-1, 1]$ which is classically based on concordance and a veto arguments (see Section 2 for further details).

In this paper, given such a valued outranking relation $\widetilde{S}$, we detail how numerical performances of the alternatives, criteria significance weights and preference discrimination thresholds can be reconstructed. We present three definitions of the valued outranking relation, where the first model takes only into account a preference threshold, the second one considers also a weak preference threshold, and finally, the third one adds also two veto thresholds.

From a practical point of view, the determination of the performances of the alternatives on the criteria may be questionable, as in general, in a decision problem, these evaluations are given beforehand. Nevertheless, as we will show later in this paper, from an experimental point of view, the determination of the performances from a given valued outranking relation can be of some interest. Furthermore, as we show in Section 6, particular cases of our developments are applicable in real world Multiple Criteria Decision Aid (MCDA) for the elicitation and the tuning of preferential parameters (see also [BMV09] for further work on such inverse MCDA).

The paper is organized as follows. In Sections 2, 3 and 4 we introduce the three definitions of outranking relations, based on different types of thresholds, and show how it is possible to determine underlying performances of the alternatives, criteria weights, and discrimination thresholds by mathematical programming. Then, in Section 5 we define the concept of *rank* of a valued outranking relation, given a particular model, and illustrate our considerations by two examples. Finally, in Section 6 we present the usefulness of the developments in real-world MCDA, before concluding in Section 7.

2. $\mathcal{M}_1$: Model with a single preference threshold

Starting from Formula (1), this first model enriches the marginal pairwise comparison of two alternatives on each criterion by a preference threshold. The main idea is that a DM considers that for criterion i, x is at least as good as y even if y's performance is *a bit* higher than that of x. Therefore, to characterize such a marginal "*at least as good as*" situation between two alternatives x and y of X, for each criterion i of N, we use the function $C_i : X \times X \to \{-1, 1\}$ defined by:

$$C_i(x,y) = \begin{cases} 1 & \text{if } \; g_i(y) < g_i(x) + p_i\,; \\ -1 & \text{otherwise}\,, \end{cases} \tag{2}$$

where $p_i \in]0,1]$ is a constant preference threshold associated with the i's criterion. According to this marginal concordance index, x is considered as *at least as good as* y for criterion i if $g_i(y) < g_i(x) + p_i$ ($C_i(x,y) = 1$). Else, x is not considered as at least as good as y for criterion i ($C_i(x,y) = -1$). Figure 1 represents this marginal concordance index for a fixed $g_i(x)$.

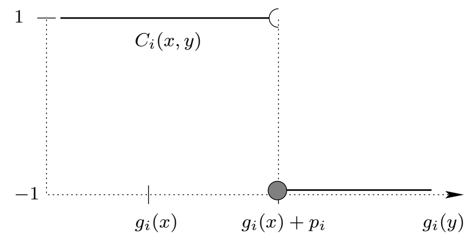

Figure 1: Marginal concordance index for a given $g_i(x)$, under $\mathcal{M}_1$.

We define the overall outranking index $\widetilde{S}$, all pairs of alternatives $(x, y) \in X \times X$, as follows:

$$\widetilde{S}(x, y) = \sum_{i \in N} w_i C_i(x, y). \tag{3}$$

$\widetilde{S}$ represents the *credibility of the validation or non-validation* of an outranking situation observed between each pair of alternatives [BMR08]. The maximum value 1 of $\widetilde{S}$ is reached in the case of *unanimous concordance*, whereas the minimum value -1 is obtained in the case of *unanimous discordance*. $\widetilde{S}$ is called the *bipolar-valued characterization* of the outranking relation S, or, for short, the *bipolar-valued outranking relation*.

Given only the bipolar-valued outranking relation $\widetilde{S}$, in this section we show how values taken by the performance functions $g_i(x)$ $(\forall i \in N, \forall x \in X)$, associated weights w_i $(\forall i \in N)$ and constant preference thresholds p_i $(\forall i \in N)$ can be determined. We therefore formulate a mixed integer linear program whose resolution gives us such performances and related parameters.

In a first attempt we wish to minimize the number of criteria used to rebuild the given valued outranking relation. This first requires that the marginal concordance conditions (2) are written as linear constraints:

$$C_i(x, y) - 1 < g_i(x) - g_i(y) + p_i \leq 2(C_i(x, y) + 1) \quad \forall x \neq y \in X, \forall i \in N, \tag{4}$$

where $C_i(x, y) \in \{-1, 1\}$ for each $x \neq y \in X$. Indeed, $g_i(x) - g_i(y) + p_i > 0$ implies $C_i(x, y) = 1$ whereas $g_i(x) - g_i(y) + p_i \leq 0$ forces $C_i(x, y) = -1$.

We say that a criterion $i \in N$ is *active* if its associated weight $w_i > 0$. This allows us to check whether a criterion is actually needed in the reconstruction of $\widetilde{S}$. To determine whether a criterion is active or not, we use a binary variable $W_i \in \{0, 1\}$ for each $i \in N$ s.t.

$$W_i = \begin{cases} 1 & \text{if } w_i > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The definition of the bipolar-valued outranking relation $\widetilde{S}$ of Formula (3) is not linear with respect to the unknown variables w_i and $C_i(x, y)$ $(\forall x \neq y \in$

$X, \forall i \in N$). In order to linearize it, we replace it by a linear constraint of the type

$$\sum_{i=1}^{n} w_i'(x,y) = \widetilde{S}(x,y) \quad \forall x \neq y \in X,$$

where $w_i'(x,y)$ is a non-negative variable for each $i \in N$, $x \neq y \in X$ s.t.:

$$w_i'(x,y) = \begin{cases} w_i & \text{if } C_i(x,y) = 1; \\ -w_i & \text{otherwise.} \end{cases}$$

This then leads to the following linear constraints:

$$\begin{array}{l} -w_i \leq w_i'(x,y) \leq w_i; \\ w_i + C_i(x,y) - 1 \leq w_i'(x,y); \\ w_i'(x,y) \leq -w_i + C_i(x,y) + 1. \end{array}$$

Indeed, $C_i(x,y) = -1$ implies $w_i'(x,y) = -w_i$, whereas $C_i(x,y) = 1$ forces $w_i'(x,y) = w_i$.

The following mixed integer linear program **MIP1** determines the performances of the alternatives $g_i(x)$ ($\forall x \in X, \forall i \in N$), weights w_i ($\forall i \in N$), as well as the preference thresholds p_i ($\forall i \in N$), given $\widetilde{S}(x,y)$ ($\forall x \neq y \in X$), by minimizing the number of active criteria.

MIP1:

Variables:

$$\begin{array}{ll} g_i(x) \in [0,1] & \forall i \in N, \forall x \in X \\ w_i \in [0,1] & \forall i \in N \\ W_i \in \{0,1\} & \forall i \in N \\ C_i(x,y) \in \{0,1\} & \forall i \in N, \forall x \neq y \in X \\ w_i' \in [-1,1] & \forall i \in N \\ p_i \in]\gamma, 1] & \forall i \in N \end{array}$$

Parameters:

$$\begin{array}{ll} \widetilde{S}(x,y) \in [0,1] & \forall x \neq y \in X \\ \delta \in]0,1[& \\ \gamma \in [\delta, 1[& \end{array}$$

Objective function:

$$\min \quad \sum_{i=1}^{n} W_i$$

Constraints:

$$\begin{array}{llll} \text{s.t.} & \sum_{i=1}^{n} w_i = 1 & & \\ & w_i \leq W_i & \forall i \in N & \\ & -w_i \leq w_i'(x,y) & \forall x \neq y \in X, \forall i \in N & \\ & w_i'(x,y) \leq w_i & \forall x \neq y \in X, \forall i \in N & \\ & w_i + C_i(x,y) - 1 \leq w_i'(x,y) & \forall x \neq y \in X, \forall i \in N & \\ & w_i'(x,y) \leq -w_i + C_i(x,y) + 1 & \forall x \neq y \in X, \forall i \in N & \\ & \sum_{i=1}^{n} w_i'(x,y) = \widetilde{S}(x,y) & \forall x \neq y \in X & \text{(b)} \end{array}$$

$C_i(x,y) - 1 + \delta \leq g_i(x) - g_i(y) + p_i$	$\forall x \neq y \in X, \forall i \in N$
$g_i(x) - g_i(y) + p_i \leq 2(C_i(x,y) + 1)$	$\forall x \neq y \in X, \forall i \in N$
$p_i \geq \gamma$	$\forall i \in N$

Note that the strictly positive separation parameter $0 < \delta < 1$ allows to model the strict inequality of constraints (4). It should be chosen very small to avoid that it produces an infeasible program. The strictly positive parameter γ is merely used to obtain significantly different values for the $g_i(x)$ in order to produce human readable results. From a purely theoretical point of view it can be replaced by δ if readability is not an issue.

The solution of **MIP1**, if it exists, need not be unique. Besides, **MIP1** might be infeasible. Such a situation can arise for at least the following three main reasons:

- The selected maximal number n of criteria is too small.
- The model with a single constant preference threshold for each criterion is too poor to represent the given valuation of the outranking relation.
- The separation parameters δ and γ are chosen inappropriately and do not allow the $g_i(x)$ to take enough distinct values in $[0,1]$.

A less constrained version of **MIP1** can be given by transforming the equality constraint (b) into the inequalities (b') and (b") as shown hereafter in **MIP1relaxed**. The idea is to approach the values taken by $\widetilde{S}$ as closely as possible by minimising the maximal gap between $\widetilde{S}(x,y)$ and $\sum_{i \in N} w_i C_i(x,y)$, for all $x \neq y \in X$, represented by a non-negative variable ε. This may be useful if the given values of $\widetilde{S}(x,y)$ differ by a small amount of a valued outranking relation compatible with model $\mathcal{M}_1$. The variables W_i ($i \in N$) are suppressed and the total number of active criteria is no longer minimised. **MIP1relaxed** then takes the following form:

MIP1relaxed:		
Variables:		
	$\varepsilon \geq 0$	
	$g_i(x) \in [0,1]$	$\forall i \in N, \forall x \in X$
	$w_i \in [0,1]$	$\forall i \in N$
	$C_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
	$w'_i \in [-1,1]$	$\forall i \in N$
	$p_i \in [\gamma,1]$	$\forall i \in N$
Parameters:		
	$\widetilde{S}(x,y) \in [0,1]$	$\forall x \neq y \in X$
	$\delta \in]0,1[$	
	$\gamma \in [\delta,1[$	
Objective function:		
min	ε	
Constraints:		

s.t.	$\sum_{i=1}^{n} w_i = 1$		
	$-w_i \leq w'_i(x,y)$	$\forall x \neq y \in X, \forall i \in N$	
	$w'_i(x,y) \leq w_i$	$\forall x \neq y \in X, \forall i \in N$	
	$w_i + C_i(x,y) - 1 \leq w'_i(x,y)$	$\forall x \neq y \in X, \forall i \in N$	
	$w'_i(x,y) \leq -w_i + C_i(x,y) + 1$	$\forall x \neq y \in X, \forall i \in N$	
	$\sum_{i=1}^{n} w'_i(x,y) \leq \widetilde{S}(x,y) + \varepsilon$	$\forall x \neq y \in X$	(b')
	$\sum_{i=1}^{n} w'_i(x,y) \geq \widetilde{S}(x,y) - \varepsilon$	$\forall x \neq y \in X$	(b")
	$C_i(x,y) - 1 + \delta \leq g_i(x) - g_i(y) + p_i$	$\forall x \neq y \in X, \forall i \in N$	
	$g_i(x) - g_i(y) + p_i \leq 2(C_i(x,y) + 1)$	$\forall x \neq y \in X, \forall i \in N$	
	$p_i \geq \gamma$	$\forall i \in N$	

As for **MIP1**, the solution need not be unique. If the objective function equals 0, then there exist $g_i(x)$ ($\forall i \in N, \forall x \in X$), associated weights w_i and preference thresholds p_i ($\forall i \in N$) generating the overall outranking index $\widetilde{S}$ via Equations (2) and (3). Else there exists no numerically exact reconstruction of the given $\widetilde{S}$ with respect to this model, and the output of **MIP1relaxed** can be considered as an approximation of the given $\widetilde{S}$ by a the constant preference threshold model.

3. $\mathcal{M}_2$: Model with two preference thresholds

Again, starting from Formula (1), this second model enriches the marginal pairwise comparison of two alternatives on each criterion by a weak preference and a preference threshold. The main idea is that a DM considers that, for criterion i, x is still at least as good as y even if y's performance is *a bit* higher than that of x. However, if y's performance is *significantly* higher than that of x, x can no longer be considered as at least as good as y. In this case, the marginal "*at least as good as*" situation between two alternatives x and y of X is characterised by the function $C'_i : X \times X \to \{-1, 0, 1\}$ s.t.:

$$C'_i(x,y) = \begin{cases} 1 & \text{if} \quad g_i(y) < g_i(x) + q_i\,; \\ -1 & \text{if} \quad g_i(y) \geq g_i(x) + p_i\,; \\ 0 & \text{otherwise}\,, \end{cases} \tag{5}$$

where $q_i \in]0, p_i[$ is a constant weak preference threshold associated with the i's preference dimension. If $C'_i(x,y) = 1$ (resp. $C'_i(x,y) = -1$), then x is considered (resp. not considered) as *at least as good as* y for criterion i. If $g_i(x) + q_i \leq g_i(y) < g_i(x) + p_i$ then it cannot be determined whether x is *at least as good as* y or not for criterion i, and $C'_i(x,y) = 0$. Figure 2 represents this marginal concordance index for a fixed $g_i(x)$.

The overall outranking index $\widetilde{S}'$ is defined as follows for all pairs of alternatives $(x,y) \in X \times X$:

$$\widetilde{S}'(x,y) = \sum_{i \in N} w_i C'_i(x,y). \tag{6}$$

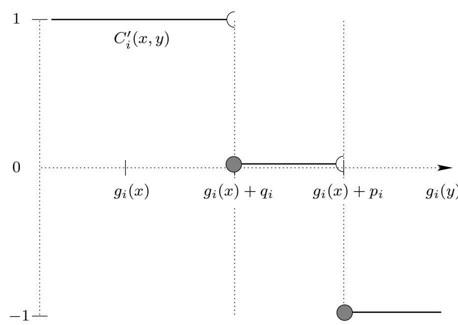

Figure 2: Marginal concordance index for a given $g_i(x)$, under $\mathcal{M}_2$.

According to Equation (6), $\widetilde{S}'$ has its values in $[-1, 1]$. Its maximum value 1 is reached in the case of *unanimous concordance*, its minimum value -1 represents *unanimous discordance*, and the value 0 is obtained if the positive arguments counterbalance the negative arguments for the outranking. The value 0 therefore represents an *indetermined* outranking situation.

The goal of this section is again to determine performances, weights and thresholds, given the bipolar-valued outranking relation $\widetilde{S}'$. In order to represent the three values taken by $C_i'(x,y)$, we use two binary variables $\alpha_i(x,y) \in \{0,1\}$ and $\beta_i(x,y) \in \{0,1\}$ ($\forall i \in N, \forall x \neq y \in X$) s.t.

$$C_i'(x,y) = \alpha_i(x,y) - \beta_i(x,y). \tag{7}$$

Note that $C_i'(x,y) = 1$ if $\alpha_i(x,y) = 1$ and $\beta_i(x,y) = 0$, $C_i'(x,y) = -1$ if $\alpha_i(x,y) = 0$ and $\beta_i(x,y) = 1$, and $C_i'(x,y) = 0$ if $\alpha_i(x,y) = \beta_i(x,y) = 1$ or $\alpha_i(x,y) = \beta_i(x,y) = 0$.

The marginal concordance conditions (5) can then be rewritten as follows as linear constraints ($\forall x \neq y \in X, \forall i \in N$):

$$\left\{ \begin{array}{rcccl} -2(1-\alpha_i(x,y)) & < & g_i(x) - g_i(y) + q_i & \leq & 2\alpha_i(x,y); \\ -2\beta_i(x,y) & < & g_i(x) - g_i(y) + p_i & \leq & 2(1-\beta_i(x,y)). \end{array} \right. \tag{8}$$

Note that, as $p_i > q_i > 0$ ($\forall i \in N$), $g_i(x) - g_i(y) + q_i > 0 \Rightarrow g_i(x) - g_i(y) + p_i > 0$, and $g_i(x) - g_i(y) + p_i < 0 \Rightarrow g_i(x) - g_i(y) + q_i < 0$. Consequently, in constraints (8), $g_i(x) - g_i(y) + q_i > 0$ forces $\alpha_i(x,y) = 1$ and $\beta_i(x,y) = 0$ ($C_i'(x,y) = 1$) whereas $g_i(x) - g_i(y) + p_i < 0$ implies $\beta_i(x,y) = 1$ and $\alpha_i(x,y) = 0$ ($C_i'(x,y) = -1$). Furthermore, $g_i(x) - g_i(y) + q_i < 0$ and $g_i(x) - g_i(y) + p_i > 0$ implies $\alpha_i(x,y) = \beta_i(x,y) = 0$ ($C_i'(x,y) = 0$). Then, $g_i(x) - g_i(y) + q_i = 0 \Rightarrow g_i(x) - g_i(y) + p_i > 0$ forces $\alpha_i(x,y) = \beta_i(x,y) = 0$ and finally $g_i(x) - g_i(y) + p_i = 0 \Rightarrow g_i(x) - g_i(y) + q_i < 0$ implies that $\alpha_i(x,y) = 0$ and $\beta_i(x,y) = 1$ ($C_i'(x,y) = 1$).

It is important to note that constraints (8) linked to the condition $p_i > q_i > 0$ do not allow that $\alpha_i(x,y) = \beta_i(x,y) = 1$ simultanously. Indeed $\alpha_i(x,y) = 1 \Rightarrow g_i(x) - g_i(y) + q_i \geq 0$ and $\beta_i(x,y) = 1 \Rightarrow g_i(x) - g_i(y) + p_i \leq 0$, which is only possible if $p_i = q_i$.

Equation (6) can now be rewritten as follows:

$$\widetilde{S}'(x,y) = \sum_{i\in N} w_i(\alpha_i(x,y) - \beta_i(x,y)) \quad \forall x \neq y \in X,$$

which can be replaced by a linear constraint of the type

$$\sum_{i=1}^{n} w_i''(x,y) = \widetilde{S}'(x,y) \quad \forall x \neq y \in X,$$

where $w_i''(x,y) \in [-1,1]$ for each $i \in N, x \neq y \in X$ s.t.:

$$w_i''(x,y) = \begin{cases} w_i & \text{if } C_i'(x,y) = 1; \\ -w_i & \text{if } C_i'(x,y) = -1; \\ 0 & \text{otherwise.} \end{cases}$$

This then leads to the following linear constraints ($\forall x \neq y \in X, \forall i \in N$):

$$\begin{array}{l} -w_i \leq w_i''(x,y) \leq w_i; \\ w_i + \alpha_i(x,y) - \beta_i(x,y) - 1 \leq w_i''(x,y) \\ w_i''(x,y) \leq -w_i + \alpha_i(x,y) - \beta_i(x,y) + 1 \\ -[\alpha_i(x,y) + \beta_i(x,y)] \leq w_i''(x,y) \leq \alpha_i(x,y) + \beta_i(x,y). \end{array}$$

Indeed, recalling that $\alpha_i(x,y)$ and $\beta_i(x,y)$ cannot simultanously be equal to 1, it is easy to verify that $C_i'(x,y) = 1 \Rightarrow w_i''(x,y) = w_i$, $C_i'(x,y) = -1 \Rightarrow w_i''(x,y) = -w_i$, and $C_i'(x,y) = 0 \Rightarrow w_i''(x,y) = 0$.

These considerations lead to the formulation of the mixed integer program **MIP2**, whose objective is again to minimise a non-negative variable ε representing the maximal gap between $\widetilde{S}'(x,y)$ and $\sum_{i\in N} w_i C_i'(x,y)$, for all $x \neq y \in X$.

MIP2:

Variables:

$$
\begin{array}{ll}
\varepsilon \geq 0 & \\
g_i(x) \in [0,1] & \forall i \in N, \forall x \in X \\
w_i \in [0,1] & \forall i \in N \\
\alpha_i(x,y) \in \{0,1\} & \forall i \in N, \forall x \neq y \in X \\
\beta_i(x,y) \in \{0,1\} & \forall i \in N, \forall x \neq y \in X \\
w''_i(x,y) \in [-1,1] & \forall i \in N, \forall x \neq y \in X \\
q_i \in [\gamma, p_i[& \forall i \in N \\
p_i \in]q_i, 1] & \forall i \in N
\end{array}
$$

Parameters:

$$
\begin{array}{ll}
\widetilde{S}'(x,y) \in [0,1] & \forall x \neq y \in X \\
\delta \in]0,1[& \\
\gamma \in [\delta, 1[&
\end{array}
$$

Objective function:

$$\min \quad \varepsilon$$

Constraints:

$$
\begin{array}{lll}
\text{s.t.} & \sum_{i=1}^{n} w_i = 1 & \\
& -w_i \leq w''_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& w''_i(x,y) \leq w_i & \forall x \neq y \in X, \forall i \in N \\
& w_i + \alpha_i(x,y) - \beta_i(x,y) - 1 \leq w''_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& w''_i(x,y) \leq -w_i + \alpha_i(x,y) - \beta_i(x,y) + 1 & \forall x \neq y \in X, \forall i \in N \\
& -[\alpha_i(x,y) + \beta_i(x,y)] \leq w''_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& w''_i(x,y) \leq \alpha_i(x,y) + \beta_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& \sum_{i=1}^{n} w''_i(x,y) \leq \widetilde{S}'(x,y) + \varepsilon & \forall x \neq y \in X \\
& \sum_{i=1}^{n} w''_i(x,y) \geq \widetilde{S}'(x,y) - \varepsilon & \forall x \neq y \in X \\
& -2(1 - \alpha_i(x,y)) + \delta \leq g_i(x) - g_i(y) + q_i & \forall x \neq y \in X, \forall i \in N \\
& g_i(x) - g_i(y) + q_i \leq 2\alpha_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& -2\beta_i(x,y) + \delta \leq g_i(x) - g_i(y) + p_i & \forall x \neq y \in X, \forall i \in N \\
& g_i(x) - g_i(y) + p_i \leq 2(1 - \beta_i(x,y)) & \forall x \neq y \in X, \forall i \in N \\
& q_i \geq \gamma & \forall i \in N \\
& p_i \geq q_i + \gamma & \forall i \in N
\end{array}
$$

Note that here the strictly positive separation parameter $0 < \gamma < 1$ allows to model the strict difference between the two consecutive thresholds q_i and p_i, $\forall i \in N$. Similar remarks as for **MIP1relaxed** concerning the uniqueness and the characteristics of the solution, as well as the choice of δ and the usefulness of the γ separation parameter apply here.

4. $\mathcal{M}_3$: Model with two preference and two veto thresholds

In this third case, compared to $\mathcal{M}_2$, the outranking relation is enriched by two veto thresholds on the criteria. A veto threshold on a criterion $i \in N$ allows to non-validate an outranking situation between two alternatives if the difference of evaluations on i is too large. A *marginal veto* situation for each criterion i of N is characterised by a veto function $V_i : X \times X \to \{-1, 0, 1\}$ s.t.:

$$V_i(x,y) = \begin{cases} 1 & \text{if} \quad g_i(y) \geq g_i(x) + v_i\,; \\ -1 & \text{if} \quad g_i(y) < g_i(x) + wv_i\,; \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

where $wv_i \in]p_i, v_i[$ (resp. $v_i \in]wv_i, 1]$) is a constant weak veto threshold (resp. veto threshold) associated with the i's preference dimension. If $V_i(x, y) = 1$ (resp. $V_i(x, y) = -1$), then the comparison of x and y for criterion i leads (resp. does not lead) to a veto. If $g_i(x) + wv_i < g_i(y) \leq g_i(x) + v_i$ then it cannot be determined whether we have a veto situation between x and y or not, and $V_i(x, y) = 0$. Figure 3 represents this marginal veto index for a given $g_i(x)$ together with the marginal concordance index C'_i.

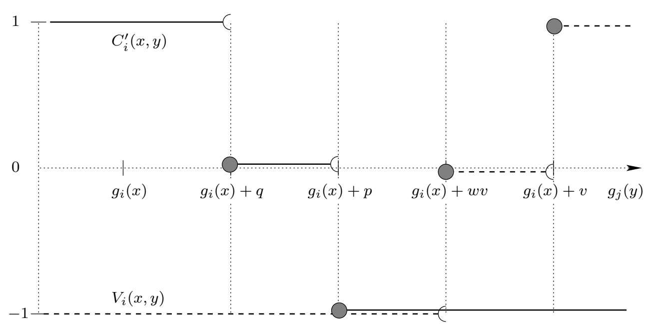

Figure 3: Marginal veto and concordance indexes for a given $g_i(x)$, under $\mathcal{M}_3$.

To take into account these veto effects, the overall outranking index $\widetilde{S}''$ is defined as follows for all pairs of alternatives $(x, y) \in X \times X$:

$$\widetilde{S}''(x,y) = \min\big\{\sum_{i\in N} w_i C'_i(x,y), -V_1(x,y), \ldots, -V_n(x,y)\big\}. \tag{10}$$

The min operator in Formula (10) tranlsates the conjunction between the overall concordance and the negated marginal veto indexes for each criterion. In the case of absence of veto on all the criteria ($V_i = -1\ \forall i \in N$), we have $\widetilde{S}''(x, y) = \widetilde{S}'(x, y)$.

Similarly as in Section 3, the three values taken by the marginal veto function can be represented by means of two binary variables $\alpha'_i(x, y) \in \{0, 1\}$ and $\beta'_i(x, y) \in \{0, 1\}$ ($\forall i \in N, \forall x \neq y \in X$) s.t.

$$V_i(x, y) = \alpha'_i(x, y) - \beta'_i(x, y).$$

Recalling that $wv_i < v_i$, conditions (9) can then be rewritten as follows as linear constraints ($\forall x \neq y \in X, \forall i \in N$):

$$\begin{cases} -2(1-\beta'_i(x,y)) & < & g_i(x) - g_i(y) + wv_i & \leq & 2\beta'_i(x,y); \\ -2\alpha'_i(x,y) & < & g_i(x) - g_i(y) + v_i & \leq & 2(1-\alpha'_i(x,y)). \end{cases} \tag{11}$$

To represent Formula (10) as a set of linear constraints, we need to introduce some further binary variables $z_0(x,y)$ and $z_i(x,y)$ $(\forall x \neq y \in X, \forall i \in N)$ s.t.:

$$\widetilde{S}''(x,y) = \begin{cases} -V_k(x,y) & \text{if } z_k(x,y)=1 \text{ and } z_i(x,y)=0 \; \forall i \in N \cup \{0\} \setminus \{k\}\,; \\ \sum\limits_{i \in N} w_i C'_i(x,y) & \text{if } z_0(x,y)=1 \text{ and } z_i(x,y)=0 \; \forall i \in N\,. \end{cases}$$

This leads to the following linear constraints:

$$\begin{array}{ll} \widetilde{S}''(x,y) \leq \sum\limits_{i \in N} w_i C'_i(x,y) & \forall x \neq y \in X; \\ \widetilde{S}''(x,y) \leq -(\alpha'_i(x,y) - \beta'_i(x,y)) & \forall x \neq y \in X, \forall i \in N; \\ \sum\limits_{i \in N} w_i C'_i(x,y) \leq 2(1 - z_0(x,y)) + \widetilde{S}''(x,y) & \forall x \neq y \in X; \\ -(\alpha'_i(x,y) - \beta'_i(x,y)) \leq 2(1 - z_i(x,y)) + \widetilde{S}''(x,y) & \forall x \neq y \in X, \forall i \in N; \\ \sum\limits_{i=0}^{n} z_i(x,y) = 1 & \forall x \neq y \in X. \end{array} \tag{12}$$

Due to the last condition of Constraints (12), there exists a unique $k \in N \cup \{0\}$ s.t. $z_k = 1$ and $z_i = 0$ for $i \in N \cup \{0\} \setminus \{k\}$. Besides, if $\sum_{i \in N} w_i C'_i(x,y) < -V_i(x,y)$ holds for all $i \in N$, then $z_i(x,y) = 0$ for all $i \in N$ and $z_0(x,y) = 1$ (which implies that $\widetilde{S}''(x,y) = \sum_{i \in N} w_i C'_i(x,y)$). Furthermore, if $\exists k \in N \cup \{0\}$ s.t. $-V_k(x,y) < \sum_{i \in N} w_i C'_i(x,y)$ and $-V_k(x,y) < -V_i(x,y)$ $(\forall i \in N \setminus \{k\})$, then $z_k(x,y) = 1$ (which implies that $\widetilde{S}''(x,y) = -V_k(x,y)$).

Constraints (12) only represent Formula (10) if all criteria have strictly positive weights. Note also that the first and the third condition of Constraints (12) can easily be linearised as in Section 3.

These considerations lead to the formulation of the mixed integer program **MIP3**, whose objective is to minimise a non-negative variable ε representing the maximal gap between $\widetilde{S}''(x,y)$ and $\sum_{i \in N} w_i C'_i(x,y)$, for all $x \neq y \in X$ where the bipolar-valued outranking relation requires no veto. As $\widetilde{S}''(x,y)$ equals -1 or 0 in veto situations, no gap is considered on these values. Remember that all the weights are supposed to be strictly positive.

MIP3:

Variables:

$\varepsilon \geq 0$	
$g_i(x) \in [0,1]$	$\forall i \in N, \forall x \in X$
$w_i \in]0,1]$	$\forall i \in N$
$\alpha_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
$\beta_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
$\alpha'_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
$\beta'_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
$w''_i(x,y) \in [-1,1]$	$\forall i \in N, \forall x \neq y \in X$
$z_i(x,y) \in \{0,1\}$	$\forall i \in N \cup \{0\}, \forall x \neq y \in X$
$q_i \in [\gamma, p_i[$	$\forall i \in N$
$p_i \in]q_i, wv_i[$	$\forall i \in N$

$$
\begin{array}{ll}
wv_i \in]p_i, v_i[& \forall i \in N \\
v_i \in]wv_i, 1] & \forall i \in N
\end{array}
$$

Parameters:

$$
\begin{array}{ll}
\widetilde{S}''(x,y) \in [0,1] & \forall x \neq y \in X \\
\delta \in]0,1[& \\
\gamma \in [\delta, 1[&
\end{array}
$$

Objective function:

$$\min \quad \varepsilon$$

Constraints:

$$
\begin{array}{lll}
\text{s.t.} & \sum_{i=1}^{n} w_i = 1 & \\
& -w_i \leq w''_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& w''_i(x,y) \leq w_i & \forall x \neq y \in X, \forall i \in N \\
& w_i + \alpha_i(x,y) - \beta_i(x,y) - 1 \leq w''_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& w''_i(x,y) \leq -w_i + \alpha_i(x,y) - \beta_i(x,y) + 1 & \forall x \neq y \in X, \forall i \in N \\
& -[\alpha_i(x,y) + \beta_i(x,y)] \leq w''_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& w''_i(x,y) \leq \alpha_i(x,y) + \beta_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& \widetilde{S}''(x,y) - \varepsilon \leq \sum_{i \in N} w''_i & \forall x \neq y \in X \\
& \widetilde{S}''(x,y) \leq -(\alpha'_i(x,y) - \beta'_i(x,y)) & \forall x \neq y \in X, \forall i \in N \\
& \sum_{i \in N} w''_i \leq 2(1 - z_0(x,y)) + \widetilde{S}''(x,y) + \varepsilon & \forall x \neq y \in X \\
& -(\alpha'_i(x,y) - \beta'_i(x,y)) \leq 2(1 - z_i(x,y)) + \widetilde{S}''(x,y) & \forall x \neq y \in X, \forall i \in N \\
& \sum_{i=0}^{n} z_i(x,y) = 1 & \forall x \neq y \in X \\
& -2(1 - \alpha_i(x,y)) + \delta \leq g_i(x) - g_i(y) + q_i & \forall x \neq y \in X, \forall i \in N \\
& g_i(x) - g_i(y) + q_i \leq 2\alpha_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& -2\beta_i(x,y) + \delta_i \leq g_i(x) - g_i(y) + p_i & \forall x \neq y \in X, \forall i \in N \\
& g_i(x) - g_i(y) + p_i \leq 2(1 - \beta_i(x,y)) & \forall x \neq y \in X, \forall i \in N \\
& -2(1 - \beta'_i(x,y)) + \delta \leq g_i(x) - g_i(y) + wv_i & \forall x \neq y \in X, \forall i \in N \\
& g_i(x) - g_i(y) + wv_i \leq 2\beta'_i(x,y) & \forall x \neq y \in X, \forall i \in N \\
& -2\alpha'_i(x,y) + \delta_i \leq g_i(x) - g_i(y) + v_i & \forall x \neq y \in X, \forall i \in N \\
& g_i(x) - g_i(y) + v_i \leq 2(1 - \alpha'_i(x,y)) & \forall x \neq y \in X, \forall i \in N \\
& q_i \geq \gamma & \forall i \in N \\
& p_i \geq q_i + \gamma & \forall i \in N \\
& wv_i \geq p_i + \gamma & \forall i \in N \\
& v_i \geq wv_i + \gamma & \forall i \in N
\end{array}
$$

Similar remarks as for **MIP1relaxed** and **MIP2** concerning the uniqueness and the characteristics of the solution, as well as the choice of δ and the usefulness of the γ separation parameter apply here.

5. Illustrative examples and rank of a bipolar-valued outranking relation

In **MIP1relaxed**, **MIP2** and **MIP3** the number of criteria is not minimised. Nevertheless, the minimal number of criteria necessary to build the given valued outranking relation can easily be obtained. Indeed, starting with $n = 1$, the value of n should be incremented by 1 until the objective function equals 0. This latter value for n then gives the minimal number of criteria necessary to construct the valued outranking relation under the chosen model. This leads us to define the *rank* of a bipolar-valued outranking relation as follows:

Definition 1. *The* rank *of a bipolar-valued outranking relation is given by the minimal number of criteria necessary to construct it via the selected model. The* rank *equals 0 if the bipolar-valued outranking relation is not representable by the selected model.*

Let us now illustrate this concept as well as the previously presented programs on a few examples.

5.1. MIP1 *&* MIP1relaxed

Consider the bipolar-valued outranking relation $\widetilde{S}_1$ of Table 4. Let us determine the performances of the three alternatives a, b and c, the rank of $\widetilde{S}_1$ under $\mathcal{M}_1$, the weights and the thresholds associated with the criteria.

$\widetilde{S}_1$	a	b	c
a	·	0.258	-0.186
b	0.334	·	0.556
c	0.778	0.036	·

Table 4: $\widetilde{S}_1$

	g_1	g_2	g_3	g_4
a	1.000	0.000	0.100	0.000
b	0.500	0.000	0.000	1.000
c	0.000	0.000	0.200	0.100
w_i	0.111	0.296	0.222	0.371
p_i	0.500	1.000	0.100	0.100

Table 5: Performances, weights and thresholds for $\widetilde{S}_1$

Let us fix $\delta = 0.001$, $\gamma = 0.1$ and $n = 5$. Under these conditions, there exists an optimal solution for 4 criteria, which is detailed in Table 5.

Obviously, applying **MIP1relaxed** on $\widetilde{S}_1$ with $n = 4$ generates an optimal solution with $\varepsilon = 0$. Furthermore, for smaller values of n, the optimal solution to **MIP1relaxed** has a strictly positive objective function, which confirms that the rank of $\widetilde{S}_1$ under $\mathcal{M}_1$ equals 4 (the choice of $\gamma = 0.1$ is done to produce human readable results and has no influence on the rank of $\widetilde{S}_1$).

5.2. MIP2 *&* MIP3

The rank of $\widetilde{S}_1$ under $\mathcal{M}_2$ also equals 4. Let us now consider the bipolar-valued outranking relation $\widetilde{S}_2$ of Table 6 and fix again $\delta = 0.001$ and $\gamma = 0.1$.

For $n = 4$, the value of the objective function for the optimal solution of **MIP2** equals 0.593. The weights w_3 and w_4 equal 0. Table 7 summarises the outranking relation associated with its optimal solution determined by solving **MIP2** for $n = 4$. One can easily check that $\widetilde{S}_2$ and $\widetilde{S}_2^*$ differ by at most 0.593. This shows that this outranking relation is not representable by $\mathcal{M}_2$ and at most 4 criteria. We therefore switch to the more general model $\mathcal{M}_3$ with two preference and two veto thresholds.

$\widetilde{S}_2$	a	b	c
a	$\cdot$	0.258	-0.186
b	0.334	$\cdot$	0.556
c	-1.000	0.036	$\cdot$

Table 6: $\widetilde{S}_2$

$\widetilde{S}_2^*$	a	b	c	g_1	g_2
a	$\cdot$	0.407	0.407	0.290	0.000
b	0.296	$\cdot$	1.000	0.100	0.100
c	-0.407	0.407	$\cdot$	0.000	0.01
w_i				0.704	0.296
q_i				0.100	0.100
p_i				0.200	0.200

Table 7: Approximative outranking relation $\widetilde{S}_2^*$ via **MIP2** for $n = 4$, performances and weights

For $n = 4$ the value of the objective function for the optimal solution of **MIP3** equals 0. This means that $\widetilde{S}_2$ can be built from a performance table with 4 criteria via $\mathcal{M}_3$, given the above thresholds. For lower values of n, the objective function for the optimal solution is strictly positive, which shows that the rank of $\widetilde{S}_2$ equals 4 under the selected model. Table 8 shows the performances of the three alternatives and the weights which allow to construct $\widetilde{S}_2$ via model $\mathcal{M}_3$. A veto situation occurs between a and c on criterion g_4 ($\widetilde{S}(c, a) = -1$).

	g_1	g_2	g_3	g_4
a	0.600	0.690	0.000	0.420
b	1.000	0.000	0.200	0.210
c	0.800	0.890	0.100	0.000
w_i	0.186	0.222	0.370	0.222
q_i	0.100	0.100	0.100	0.100
p_i	0.200	0.200	0.210	0.220
wv_i	0.410	0.900	0.310	0.320
v_i	0.510	1.000	0.410	0.420

Table 8: Performances, weights and thresholds to construct $\widetilde{S}_2$ via model $\mathcal{M}_3$

6. Usefulness in MCDA: inference of model parameters

In real-world decision problems involving multiple criteria, the performances $g_i(x)$ are given for each $i \in N$ and each $x \in X$. Nevertheless, the determination of the weights w_i for each $i \in N$ and the thresholds which are present in the three models described in Sections 2, 3 and 4 is often a difficult task. In this section, we focus on $\mathcal{M}_3$ to show how the parameters of the model can be determined from *a priori* knowledge provided by the DM (see also [MD04, DM06] for studies on the disaggregation of ELECTRE-like outranking relations).

As classically done, the asymmetric part of a binary relation $\succeq$ will be denoted by $\succ$ and its symmetric part by $\sim$.

Very generally, in a context of bipolar-valued outranking relations, the *a priori* preferences of the DM could take the form of:

- a partial weak order over the credibilities of the validation of outrankings;
- a partial weak order over the significance of some criteria;
- quantitative intuitions about some credibilities of the validation of outrankings;
- quantitative intuitions about the significance of some criteria;
- quantitative intuitions about some thresholds;
- subsets of criteria important enough for the validation of an outranking situation;
- subsets of criteria not important enough for the validation of an outranking situation;
- etc.

The first two types of information can be rewritten as follows as linear constraints:

- the validation of wSx is strictly more credible than that of ySz can be translated as $\widetilde{S}(w,x) - \widetilde{S}(y,z) \geq \theta$;
- the validation of wSx is similar to that of ySz can be translated as $-\theta \leq \widetilde{S}(w,x) - \widetilde{S}(y,z) \leq \theta$;
- the significance of criterion i is strictly higher than that of j can be translated as $w_i - w_j \geq \theta$;
- the significance of criterion i is similar to that of j can be translated as $-\theta \leq w_i - w_j \leq \theta$;

where $w, x, y, z \in X$, $i, j \in N$ and θ is a non negative separation parameter.

The more quantitative information could be rewritten as follows as linear constraints:

- a quantitative intuition about the credibility of the validation of xSy can be translated as $\eta_{(x,y)} \leq \widetilde{S}(x,y) \leq \theta_{(x,y)}$, where $\eta_{(x,y)} \leq \theta_{(x,y)} \in [-1,1]$ are to be fixed by the DM;
- a quantitative intuition about the significance of criterion i can be translated as $\eta_{w_i} \leq w_i \leq \theta_{w_i}$, where $\eta_{w_i} \leq \theta_{w_i} \in]0,1]$ are to be fixed by the DM;

- a quantitative intuition about the preference threshold p_i of criterion i can be translated as $\eta_{p_i} \leq p_i \leq \theta_{p_i}$, where $\eta_{p_i} \leq \theta_{p_i} \in [0,1]$ are to be fixed by the DM.

Note that in practice a constraint of the form $\widetilde{S}(w,x) - \widetilde{S}(y,z) \geq \delta$ is generally accompanied by the constraint $\widetilde{S}(w,x) \leq 0$ or the constraint $0 \leq \widetilde{S}(y,z)$.

Finally, the information on the sufficiency of a coalition of criteria to validate or non-validate an outranking statement can be written as follows as linear constraints:

- the fact that the subset $M \subset N$ of criteria is sufficient (resp. not sufficient) to validate an outranking statement can be translated as $\sum_{i \in M} w_i \geq \eta_M$ (resp. $\sum_{i \in M} w_i \leq -\eta_M$), where $\eta_M \in]0,1]$ is a parameter of *concordant coalition* which is to be fixed by the DM.

These considerations lead to the following mixed integer program $\mathbf{MIP3} - \mathbf{MCDA}$. Its goal is to determine thresholds and weights of $\mathcal{M}_3$, as well as the bipolar-valued outranking relation $\widetilde{S}$, by considering the *a priori* information provided by the DM.

MIP3-MCDA:

Variables:

$\varepsilon \geq 0$	
$q_i \in [\gamma, p_i[$	$\forall i \in N$
$p_i \in]q_i, wv_i[$	$\forall i \in N$
$wv_i \in]p_i, v_i[$	$\forall i \in N$
$v_i \in]wv_i, 1]$	$\forall i \in N$
$w_i \in]0,1]$	$\forall i \in N$
$\widetilde{S}''(x,y) \in [0,1]$	$\forall x \neq y \in X$
$\alpha_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
$\beta_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
$\alpha'_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
$\beta'_i(x,y) \in \{0,1\}$	$\forall i \in N, \forall x \neq y \in X$
$w''_i(x,y) \in [-1,1]$	$\forall i \in N, \forall x \neq y \in X$
$z_i(x,y) \in \{0,1\}$	$\forall i \in N \cup \{0\}, \forall x \neq y \in X$

Parameters:

$g_i(x) \in [0,1]$	$\forall i \in N, \forall x \in X$
$\delta \in]0,1[$	
$\gamma \in]0,1[$	
$\vdots$	

Objective function:

min ε

Constraints of **MIP3**

$\vdots$

Constraints derived from a priori information (informal):

$\widetilde{S}(w,x) - \widetilde{S}(y,z) \geq \delta$	for some pairs of alternatives
$-\delta \leq \widetilde{S}(w,x) - \widetilde{S}(y,z) \leq \delta$	for some pairs of alternatives
$w_i - w_j \geq \delta$	for some pairs of weights
$-\delta \leq w_i - w_j \leq \delta$	for some pairs of weights
$\eta_{(x,y)} \leq \widetilde{S}(x,y) \leq \theta_{(x,y)}$	for some pairs of alternatives
$\eta_{w_i} \leq w_i \leq \theta_{w_i}$	for some weights
$\eta_{p_i} \leq p_i \leq \theta_{p_i}$	for some thresholds and some weights
$\sum\limits_{i \in M} w_i \geq \eta_M$	for some subsets M of weights
$\sum\limits_{i \in M} w_i \leq -\eta_M$	for some subsets M of weights

Note that there might be no feasible solution to **MIP3-MCDA**. This might occur for example, if some of the constraints are not compatible with the given data or the chosen model.

Let us now illustrate these considerations by a short example. Consider as a starting point the performances of Table 8 for the three alternatives a, b and c on four criteria. Let us call $\widetilde{S}_3$ the unknown outranking relation in this example. The weights on the criteria are also unknown as well as the thresholds on the criteria. Furthermore, the *a priori* preferences of the DM are summarised in Table 10.

$\widetilde{S}_3$	a	b	c
a	$\cdot$	$\in]0, 0.5]$	$\in [-0.5, 0[$
b	$\in]0, 0.5]$	$\cdot$	$\in]0.5, 1]$
c	$= -1$	$\in [-0.1, 0.1]$	$\cdot$

Table 10: A priori information from the DM on $\widetilde{S}_3$

By applying **MIP3-MCDA** on this example, we obtain the results represented in Tables 11 and 12. They are clearly in accordance with the constraints imposed by the DM.

$\widetilde{S}_3$	a	b	c
a	$\cdot$	0.195	-0.010
b	0.500	$\cdot$	0.795
c	-1.000	-0.1	$\cdot$

Table 11: $\widetilde{S}_3$

	g_1	g_2	g_3	g_4
w_i	0.403	0.103	0.347	0.148
q_i	0.100	0.100	0.210	0.100
p_i	0.200	0.200	0.310	0.210
wv_i	0.410	0.900	0.410	0.310
v_i	0.510	1.000	0.510	0.420

Table 12: Model parameters for $\widetilde{S}_3$ via $\mathcal{M}_3$

We can observe that $\widetilde{S}_3(c,a) = -1$ results from a veto situation on criterion 2.

7. Conclusion

In this article we have presented how to determine performances of alternatives on multiple criteria, together with significance weights and preference thresholds, given a bipolar-valued outranking relation. We have also shown that these results can be used to determine the minimal number of criteria necessary to reconstruct a bipolar-valued outranking relation as well as in real-world MCDA when it comes to eliciting the preferential parameters from a DM.

The mathematical programs have all been implemented in the in the GNU MathProg an the AMPL algebraic modeling languages. The examples have been solved on a standard desktop computer with Glpsol, the solver of the GNU Linear Programming Kit (GLPK), and ILOG CPLEX 12.2.

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