On a bipolar foundation of the outranking concept

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ABSTRACT In this paper we introduce a bipolarly extended veto principle which allows us to extend the definition of the classic outranking relation in such a way that the identity between the asymmetric part and bipolar codual of the latter outranking relation is given.

KEYWORDS. Multiple criteria decision aid; Outranking; Bipolar veto principle.

1. INTRODUCTION

Recently, Pirlot and Bouyssou (2009) have reported that a strict (asymmetric) outranking relation defined similarly to the classic outranking relation (Roy and Bouyssou, 1993) is in general not identical to its codual relation, that is the converse of its negation. This hiatus is problematic as the asymmetric part of an outranking relation is commonly identified in the decision aid practice as representing in fact its codual relation.

In this contribution we explore this problem in the context of our bipolar credibility calculus (Bisdorff, 2000, 2002, 2006). In a first section, following the hint of Pirlot and Bouyssou, we illustrate formally this unsound hiatus between the asymmetric part and the codual in the case of the classic outranking concept. In a second we introduce a bipolarly extended veto principle which allows us to extend the definition of the classic outranking relation in such a way that the identity between the asymmetric part and the bipolar codual of the outranking relation is indeed given.

2. THE CLASSIC OUTRANKING CONCEPT

Let $A = \{x, y, z, ...\}$ be a finite set of potential decision alternatives and let $F = \{1, ..., n\}$ be a coherent finite family of $n > 1$ criteria (Roy and Bouysoou, 1993). The alternatives are evaluated on each criterion i in F on a real performance scale $[0; M_i]$ supporting coherent indifference (q_i) and preference (p_i) discrimination thresholds such that $0 \leqslant q_i < p_i \leqslant M_i$ (Roy and Bouysoou, 1993). The performance of alternative x on criterion i is denoted x_i .

2.1. Overall preference concordance

In order to characterize between any two alternatives x and y of A a local "*at least as good as*" situation (Roy and Bouysoou, 1993; Bisdorff, 2002), with each criterion i is associated a double threshold order $\geqslant_i$ whose bipolar characteristic representation $r(x \geqslant_i y)$ takes value:

$$r(x \geqslant_i y) = \begin{cases} +1 & \text{if} \quad x_i + q_i \geqslant y_i\,; \\ -1 & \text{if} \quad x_i + p_i \leqslant y_i\,; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we associate with each criterion i in F a *rational significance* weight w_i which represents the contribution of i to the *overall warrant or not* of the *at least as good as* preference situation between all pairs of alternatives. Let W be the set of relative significance weights associated with F such that $W = \{w_i \mid i \text{ in } F\}$, with $0 < w_i < 1$ and $\sum_{i \text{ in } F} w_i = 1$.

The bipolar-valued characteristic representation r of the overall "*at least as good as*" relation (Bisdorff, 2000, 20002), denoted $\geqslant$, aggregating all the *partial at least as good as* situations $\geq_i$ for i in F, is given by:

$$r(x \geqslant y) = \sum_{i \text{ in } F} [w_i \cdot r(x \geqslant_i y)] \qquad (1)$$

For each criterion i in F, we can similarly characterize a local "*better than*" situation between any two alternatives x and y of A with a double threshold order $>_i$ and whose bipolar numerical representation $r\,(x >_i y)$ takes value:

$$\begin{array}{rcl} +1 & \text{if} & x_i - p_i \geqslant y_i\,; \\ -1 & \text{if} & x_i - q_i \leqslant y_i\,; \\ 0 & & \text{otherwise.} \end{array}$$

Again, the overall "*better than*" is characterized by:

$$r\,(x > y) \;=\; \textstyle\sum_{i \text{ in } F} [\, w_i \cdot r\,(x >_i y)\,] \qquad (2)$$

Lemma 2.1 For each criterion i, the codual $(\not\geqslant_i)^{-1}$ of the local "*as good as*" relation $\geqslant_i$ on A is identical to the local "*better than*" relation $>_i$ on A .

Proof: Indeed, for all x, y in A, $r\,(\, y \not\geqslant_i x\,)$ equals:

$$\begin{array}{rcl} -1\,, & \text{if} & y_i + q_i \;\geqslant\; x_i\,; \\ +1\,, & \text{if} & y_i + p_i \;\leqslant\; x_i\,; \\ 0\,, & & \text{otherwise.} \end{array} \quad \square$$

The lemma leads immediately to the following proposition.

Proposition 2.1

The overall "*better than*" relation $>$ on A is the codual, i.e. the converse of the negation, of the overall "*at least as good*" relation $\geqslant$ on A .

Proof: Following lemma (2.1), formula (1) gives the same result for $(\not\geqslant)^{-1}$ than formula (2) gives for $>$. ❑

2.2 The classic veto principle

In order to characterize a local *veto* situation (Roy and Bouyssou, 1993) between any two alternatives x and y of A we may associate with each performance scale $[\,0; M_i\,]$ a *veto* (v_i) discrimination threshold such that $p_i < v_i \leqslant M_i + \varepsilon$ for all i in F (see [8]).

Definition 2.2 We may thus define on each criterion i a single threshold order denoted $\ll_i$ which represents a "*seriously less performing than*" situation (Bisdorff, 2008) and whose numerical representation $r\,(x \ll_i y)$ takes value:

$$\begin{array}{rcl} +1\,, & \text{if} & x_i + v_i \leqslant y_i\,; \\ -1\,, & & \text{otherwise.} \end{array}$$

The characteristic representation of a (global) "*veto*" situation is now given by the overall disjunction of local "*seriously less performing than*" situations:

$$r\,(x \ll y) = r\,\big(\textstyle\bigvee_{i \text{ in } F} (x \ll_i y)\big) \;=\; \max_{i \text{ in } F} [\, r\,(x \ll_i y)\,]. \qquad (3)$$

We are now ready to define the classic outranking relation.

2.3 The classic outranking relation

Definition 2.3 An alternative x "*outranks*" an alternative y, denoted (x $\succcurlyeq$ y), when:

1. a *significant majority* of criteria validates the fact that x is performing at least as good as y, i.e. $x \geqslant y$,
2. and there is no veto raised against this validation, i.e. $\neg(x \ll y)$.

The corresponding bipolar numerical representation gives:

$$r\,(x \succcurlyeq y) \;=\; r\,[\,(x \geqslant y) \wedge \neg(x \ll y)\,] \;=\; \min[\; r\,(x \geqslant y),\; -r\,(x \ll y)\,] \qquad (4)$$

Proposition 2.3 (Pirlot & Bouyssou 2009)

Let $\succcurlyeq$ be a classic outranking relation.

1. The asymmetric part $\succnsim$ of the classic outranking relation, i.e. (x $\succcurlyeq$ y) and $\neg$(y $\succcurlyeq$ x), is in general not identical to its codual relation $\neg(\precsim)$.

2. The absence of any veto situation is a sufficient and necessary condition for making $\succnsim$ identical to

$\neg(\preccurlyeq)$.

Proof:

(1) $r(\neg(y \succcurlyeq x)) = \max[\, r(\neg(y \succcurlyeq x)),\ r(y \gg x)] = \max[\, r(x > y),\ r(y \gg x)\,]$ whereas $r(x \succsim y) = \min[\, r(x \succcurlyeq y), r(\neg(y \succcurlyeq x))\,] \leq r(\neg(y \succcurlyeq x))$. The strict inequality appears when $r(y \gg x) = 1$.

(2) $v_i = M_i + \varepsilon$ implies that $r(x \succcurlyeq y) = r(x \geqslant y)$ and the claimed identity follows from Corrolary 2.1. Conversely, suppose that $v_i < M_i + \varepsilon$ and there exist a strong veto situation ($r(x \ll_i y) = 1$) on some criterion i in F. In this case $\min[\, r(x \succcurlyeq y), r(\neg(y \succcurlyeq x))\,] = \min[\, -1, r(\neg(y \succcurlyeq x))] = -1 < r((y \succcurlyeq x)) = 1$. ❑

As recently reported by Pirlot and Bouyssou (2009), this hiatus between the asymmetric definition and the codual raises a serious concern with respect to the logical soundness of the classic outranking definition. Only the absence of any veto mechanism can guarantee this somehow necessary property from the point of view of the intended semantics of the outranking concept. But this is vanishing the very interest of the outranking concept itself.

3. OUTRANKING WITH BIPOLAR VETO

3.1 The bipolar veto concept

The veto is in fact the concept that we have to put into a bipolar setting in order to overcome the previously mentioned hiatus.

Definition 3.1.a We may thus redefine on each criterion i a single threshold order denoted $\lll_i$ which represents a "*seriously less performing than*" situation and whose bipolar numerical representation $r(x \lll_i y)$ takes value:

$$\begin{array}{rl} +1\,, & \text{if} \quad x_i + v_i \leqslant y_i\,; \\ -1\,, & \text{if} \quad x_i - v_i \geqslant y_i\,; \\ 0\,, & \text{otherwise.} \end{array}$$

Similarly we may define on criterion i a single threshold order denoted $\ggg_i$ representing a "*seriously better performing than*" situation and whose bipolar numerical representation $r(x \ggg_i y)$ takes value:

$$\begin{array}{rl} +1\,, & \text{if} \quad x_i - v_i \geqslant y_i\,; \\ -1\,, & \text{if} \quad x_i + v_i \leqslant y_i\,; \\ 0\,, & \text{otherwise.} \end{array}$$

It is worthwhile noticing that the bipolar negation is thus symmetricaly opposing "*seriously better preforming than*" to "*seriously less performing than*" local veto situations. In case $v_i = M_i + \varepsilon$ again, the criterion i does not supports neither $\lll_i$ nor $\ggg_i$ situations.

Definition 3.1.b The bipolar characteristic representation of a (global) "*veto*" situation is now given by the aggregated determination of all local "*seriously less performing than*" and "*seriously better performing than*" situations:

$$r(x \lll y) = \bigoplus_{i \text{ in } F} r(x \lll_i y)\,. \quad (5)$$

$$r(x \ggg y) .= \bigoplus_{i \text{ in } F} r(x \ggg_i y)\,. \quad (6)$$

where $\oplus$ represents the bipolar sharpening operator (see Grabisch et. al 2009, Bisdorff 1997) defined as follows: $r \oplus r'$ equals $\max(r, r')$ if $r \geqslant 0$ and $r' \geqslant 0$; $\min(r,r')$ if $r \leqslant 0$ and $r' \leqslant 0$; and, 0 otherwise.

We may thus observe that $r(x \lll y) = 1$ iff there exists i in F such that $r(x \lll_i y) = 1$ and there does not exist any j in F such that $r(x \ggg_j y) = 1$. Or conversely, $r(x \ggg y) = 1$ iff there exists i in F such that $r(x \ggg_i y) = 1$ and there does not exist any j in F such that $r(x \lll_j y) = 1$.

Lemma 3.1

The bipolar codual $(\neg\lll)^{-1}$ of the global "*seriously less performing than*" relation $\lll$ on A is identical to the global "*seriously better performing than*" relation $\ggg$ on A .

Proof: On each criterion i, the bipolar codual $(\neg\lll_i)^{-1}$ of the local "*seriously less performing than*" relation $\lll_i$ on A is identical to the local "*seriously better performing than*" relation $\ggg_i$ on A . As the bipolar sharpening operator $\oplus$ is auto-dual, it follows that the codual of the relation $\lll$ is therefore the relation $\ggg$ on A . ❑

We may now define an outranking concept which is coherent with our bipolar approach.

3.2 The bipolar outranking relation

Definition 3.2. Let x and y be two decision alternatives. From a bipolar point of view, we say that :

1. x "*outranks*" y, denoted $x \succsim y$, if a significant majority of criteria validates a global outranking situation between x and y and no serious counter-performance is observed on a discordant criterion,
2. x "*does not outrank*" y, denoted $\neg(x \succsim y)$, if a significant majority of criteria invalidates a global outranking situation between x and y and no serious better performing situation is observed on a concordant criterion.

In terms of our bipolar numeric representation r we obtain the following formal definition:

$$r\,(x \succsim y) \;=\; \big[\, r\,(x \geqslant y) \;\oplus\; -r\,(x \lll y)\,\big] \qquad (5)$$

If $v_i = M_i + \varepsilon$ for all i in F, i.e. in the absence of any vetoes, we recover the previous case where $r\,(x \gtrsim y) = r\,(x \succcurlyeq y) = r\,(x \geqslant y)$. If we observe a seriously better performing situation, $r\,(x \ggg y) = 1$, and $r\,(x \geqslant y) \geqslant 0$, we obtain $r\,(y \gtrsim x) = 1$. Conversely, if we observe a seriuosly less performing situation, $r\,(x \lll y) = 1$, and $r\,(x \geqslant y) \leqslant 0$, we obtain $r\,(y \gtrsim x) = -1$. Otherwise, we observe an indeterminate situation. In this latter case, the apparent preferential information appears contradictory and hence no positive or negative validation conclusion may be drawn from the bipolar information aggregation.

3.3 The bipolar codual of the outranking relation

Let us finally show that the bipolar codual of the outranking relation with the bipolarly extended veto principle is indeed identical with the strict bipolar outranking relation.

Let $(\neg\succsim)^{-1}$ denote the codual of the bipolar outranking relation, i.e. the converse of the symmetric negation of $\succsim$. If we define the strict bipolar outranking relation, denoted $\succnsim$, as follows:

$$r\,(x \succnsim y) \;=\; \big[\, r\,(x > y) \;\oplus\; -r\,(x \lll y)\,\big] \qquad (6)$$

we obtain the following result:

Proposition 3.3

$$r\,(\neg(x \precsim y)) \;=\; r\,(x \succnsim y) \;\text{ forall } (x, y) \text{ in } A^2 \;.$$

Proof:

$$\begin{aligned} r\,(\neg(x \precsim y)) \quad &= \quad -\{\, \big[\, r\,(x \leqslant y) \;\oplus\; -r\,(x \ggg y)\,\big]\,\} \\ &= \quad \big[\, -r\,(x \leqslant y) \;\oplus\; r\,(x \ggg y)\,\big] \\ &= \quad \big[\;\; r\,(x > y) \;\oplus\; -r\,(x \lll y)\,\big] \quad \text{(Proposition 2.1, Lemma 3.1).} \;❑ \end{aligned}$$

4. CONCLUSION

In this paper we have introduced a new bipolar veto principle which allows us to construct an extended bipolar outranking relation guaranteeing the formal identity of the corresponding strict outranking relation, i.e. its asymmetric part, with its bipolar codual relation. Contrary to the classic unipolar outranking relation, taking into account only invalidating causes (via the classic veto principle) and where therefore incomparability situations potentially capture the difficulty to compensate outstanding performances with serious counter-performances, here we rely on the neutral value of the bipolar characteristic calculus for expressing our doubts concerning the effective compensation of such contrasted performances.

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