# On the Number of Chordless Circuits in a Digraph

**Theorem 1** *The maximum number of chordless circuits in a digraph on n vertices is at least*

$$\frac{4^{2(\sqrt{n}-2)}}{(\sqrt{n}-2)^4}$$

**Proof.** Define a grid graph with vertex set $V = \{(i,j) : 0 \leq i,j \leq k-1\}$. To define the edges of the graph, embed the graph in the Euclidean plane, with vertex $(i,j)$ at these coordinates. The line $((0,0),(k-1,k-1))$ partitions the (embedding of the) graph into two halves (upper left and lower right). Orient the edges in the digraph such that all edges in the upper left half go up or right, and all edges in the lower right half go down or left. Formally the edge set is defined by: $E = \{((i,j),(i,j+1)) : 0 \leq i < k, 0 \leq j < k-1, i \leq j\} \cup \{((i,j),(i+1,j)) : 0 \leq i < k-1, 0 \leq j < k, i < j\} \cup \{((i,j),(i,j-1)) : 0 \leq i < k, 0 < j < k-1, i \geq j\} \cup \{((i,j),(i-1,j)) : 0 < i < k, 0 \leq j \leq k-1, i > j\}$.

Consider two sets of paths: let $P_1$ denote the set of paths from vertex $(0,1)$ to $(k-2,k-1)$ never crossing the line $((0,1),((k-2,k-1))$ and let $P_2$ denote the set of paths from vertex $(k-1,k-2)$ to $(1,0)$ never crossing the line $((1,0),(k-1,k-2))$. We can construct a chordless circuit by concatenating edge $(0,0),(0,1)$ with any path from $P_1$ followed by edges $((k-2,k-1),(k-1,k-1))$, $((k-1,k-1),(k-1,k-2))$ followed by any path from $P_2$ followed by edge $((1,0),(0,0))$.

It follows that the number $C$ of chordless circuits in this graph is at least $\mid P_1 \mid \cdot \mid P_2 \mid$. For symmetry reasons $\mid P_1 \mid = \mid P_2 \mid$. Thus $C \geq \mid P_1 \mid^2$.

Notice that $\mid P_1 \mid$ is equal to the number of balanced sequences of parentheses of length $2(k-2)$. Since the number of balanced sequences of parenthesis of length $2n$ is the Catalan number $C_n = \frac{1}{n} \cdot \binom{2n}{n}$ we obtain

$$C \geq C_{k-2}^2$$

Using the approximation $C_n > \frac{4^n}{n^2}$ (see wikipedia), we get

$$C > \left(\frac{4^{k-2}}{(k-2)^2}\right)^2 = \frac{4^{2(k-2)}}{(k-2)^4}$$

Since $k = \sqrt{n}$ where $n$ denotes the number of nodes of the above graph we obtain

$$C > \frac{4^{2(\sqrt{n}-2)}}{(\sqrt{n}-2)^4}$$

QED