THE NUMBER OF CHORDLESS CIRCUITS IN GRID GRAPHS

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For $S \subseteq \omega$, we write $a \prec_S b$ to mean that $a, b \in S$ and $a < b$ and there is no $c \in S$ such that $a < c < b$, that is, it is the covering relation of the natural order of the natural numbers restricted to S.

Define a grid graph $G_k = (V_k, E_k)$ with vertex set $V_k = \{(i, j) : 0 \leq i, j \leq k-1\}$. To define the edge set E_k, embed the graph in the Euclidean plane, with vertex (i, j) at these coordinates. The line $((0, 0), (k-1, k-1))$ partitions (the embedding of) the graph into two halves (upper left and lower right). Orient the edges in the digraph such that all edges in the upper left half go up or right and all edges in the lower right half go down or left. Formally the edge set is defined by

$$\begin{aligned} E_k = {} & \{((i, j), (i, j+1)) : 0 \leq i \leq k-1,\ 0 \leq j < k-1,\ i \leq j\} \cup \\ & \{((i, j), (i+1, j)) : 0 \leq i < k-1,\ 0 \leq j \leq k-1,\ i \leq j\} \cup \\ & \{((i, j), (i, j-1)) : 0 \leq i \leq k-1,\ 0 < j \leq k-1,\ i \geq j\} \cup \\ & \{((i, j), (i-1, j)) : 0 < i \leq k-1,\ 0 \leq j \leq k-1,\ i \geq j\}. \end{aligned}$$

Consider a chordless circuit C that passes through points $(0, 0)$ and $(k-1, k-1)$. It cannot cross the line $((0, 0), (k-1), (k-1))$ in any other points, but it may touch this line from the upper left and the lower right side in several points. It is clear that the circuit does not touch the points $(1, 1)$ and $(k-2, k-2)$, for otherwise a chord is unavoidable. Let

$$\begin{aligned} S_0 &:= \{n \in \{2, \dots, k-3\} \mid C \text{ touches } (n, n) \text{ from the upper left side}\}, \\ S_1 &:= \{n \in \{2, \dots, k-3\} \mid C \text{ touches } (n, n) \text{ from the bottom right side}\}. \end{aligned}$$

We will call the pair (S_0, S_1) the *diagonal system* associated with the chordless circuit C.

We call a pair (A_0, A_1) of subsets of $\{2, \dots, k-3\}$ *admissible,* if $A_0 \cap A_1 = \emptyset$ and for $i \in \{0, 1\}$, the condition $n \in A_i$ implies $n+1,\ n-1 \notin A_{1-i}$. It is easy to verify that a pair (A_0, A_1) of subsets of $\{2, \dots, k-3\}$ is admissible if and only if it is the diagonal system associated with some chordless circuit that passes through the points $(0, 0)$ and $(k-1, k-1)$.

Let (S_0, S_1) be an admissible pair of subsets of $\{0, \dots, k-1\}$. For $i = 0, 1$, let

$$T_i := S_{1-i} \cup (S_{1-i} + 1) = \{n,\ n+1 \mid n \in S_{1-i}\}.$$

For $i = 0, 1$, let $U_i := S_i \cup \{0, k-1\}$, and for all $u \prec_{U_i} v$, let $X_{uv,i}^{(S_0,S_1)} := \{u + 2, \dots, v-1\} \setminus T_i$.

Let C be a chordless circuit whose diagonal system is (S_0, S_1). The upper left part of the circuit C consists of the segments

$$(u, u) \to (u, u+1) \to \cdots \to (v-1, v) \to (v, v)$$

for all $u \prec_{U_0} v$.

The circuit passes through some points of the form $(n-1,n)$ $(n \in \{u+1,\dots,v\})$; it necessarily goes through the points $(u,u+1)$ and $(v-1,v)$ (which might be just a single point), but it does not go through any point $(n-1,n)$ where $n \in T_0$, for otherwise we would have a chord from $(n-1,n)$ to $(n-1,n-1)$ or to (n,n). Let

$$Y := \{n \in X_{uv,0}^{(S_0,S_1)} \mid C \text{ goes through } (n-1,n)\}.$$

Thus the set $Y' := Y \cup \{u+1,v\}$ represents precisely the second coordinates of the points of the form $(n-1,n)$ through which C passes in the segment from (u,u) to (v,v). Thus this segment can be divided further in segments

- $(u,u) \to (u,u+1)$,
- for each $\alpha \prec_{Y\cup\{u+1,v\}} \beta$,
$$(\alpha-1,\alpha) \to (\alpha-1,\alpha+1) \to \cdots \to (\beta-2,\beta) \to (\beta-1,\beta),$$
- $(v-1,v) \to (v,v)$.

The number of all possible directed paths from $(\alpha-1,\alpha+1)$ to $(\beta-2,\beta)$ is $C_{\beta-\alpha-1}$, where C_n denotes the n-th Catalan number. Thus, the number of paths that go from (u,u) to (v,v) through points $(n-1,n)$ for all $n \in Y'$, but not through any other points of the form $(n-1,n)$ is

$$\prod_{\alpha \prec_{Y\cup\{u+1,v\}} \beta} C_{\beta-\alpha-1}.$$

Therefore the number of all possible segments from (u,u) to (v,v) of chordless circuits corresponding to the diagonal system (S_0,S_1) is

$$P_{uv,0}^{(S_0,S_1)} := \sum_{Y \subseteq X_{uv,0}^{(S_0,S_1)}} \prod_{\alpha \prec_{Y\cup\{u+1,v\}} \beta} C_{\beta-\alpha-1}.$$

Thus the number of all possible segments from $(0,0)$ to $(k-1,k-1)$ of chordless circuits corresponding to the diagonal system (S_0,S_1) is

$$N_0^{(S_0,S_1)} := \prod_{u \prec_{U_0} v} P_{uv,0}^{(S_0,S_1)}.$$

A similar argument shows that the number all possible segments from $(k-1,k-1)$ to $(0,0)$ of chordless circuits corresponding to the diagonal system (S_0,S_1) is

$$N_1^{(S_0,S_1)} := \prod_{u \prec_{U_1} v} P_{uv,1}^{(S_0,S_1)},$$

where

$$P_{uv,1}^{(S_0,S_1)} := \sum_{Y \subseteq X_{uv,1}^{(S_0,S_1)}} \prod_{\alpha \prec_{Y\cup\{u+1,v\}} \beta} C_{\beta-\alpha-1}.$$

The total number of chordless circuits corresponding to the diagonal system (S_0,S_1) is $N_0^{(S_0,S_1)} N_1^{(S_0,S_1)}$. Therefore, the total number of chordless circuits passing through points $(0,0)$ and $(k-1,k-1)$ is

$$\begin{aligned} K_k &:= \sum_{(S_0,S_1)} \prod_{i\in\{0,1\}} N_i^{(S_0,S_1)} \\ &= \sum_{(S_0,S_1)} \prod_{i\in\{0,1\}} \prod_{u \prec_{U_i} v} \sum_{Y \subseteq X_{uv,i}^{(S_0,S_1)}} \prod_{\alpha \prec_{Y\cup\{u+1,v\}} \beta} C_{\beta-\alpha-1}, \end{aligned}$$

where the first summation index ranges over the set of all admissible pairs (S_0, S_1) of subsets of $\{2, \dots, k-3\}$.

In addition to the chordless circuits that pass through $(0,0)$ and $(k-1, k-1)$, the grid graph G_k contains several shorter chordless circuits. It is easy to verify that the total number of chordless circuits in G_k is

$$\sum_{j=2}^{k} (k-j+1) \cdot K_j.$$