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# Algorithmic Decision Making with Python Programming Resources

A computational journey from multiple criteria performance records to decision algorithms via bipolar-valued outranking digraphs

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*This book is dedicated to my colleague and  
dear friend, the late Prof. Marc ROUBENS.*



## Preface

The reader will find in this monograph a series of tutorials and advanced topics originally written over the last ten years as documentation parts for the DIGRAPH3 collection of Python modules. These programming resources – like the `outrankingDigraphs` module – were essentially used for the preparation and illustration of the Algorithmic Decision Theory course taught at the University of Luxembourg from 2010 to 2020. Some resources, like the `randomNumbers` module, served for preparing and illustrating the Lectures of a Computational Statistics Course. Curious readers will also discover some resources –the `arithmetics` module– used for preparing and illustrating a first Semester Course on Discrete Mathematics.

The DIGRAPH3 Python programming resources are useful in the field of Algorithmic Decision Theory and more specifically for the outranking approach of Multiple Criteria Decision Aid (MCDA). In this latter scientific field, we address essentially three kinds of usage.

1. First, we present algorithms and illustrate computing tools for solving either, a multiple criteria best choice selection or, a dual worst choice rejection problem.
2. We also tackle the problem of how to list a set of items with multiple incommensurable performance criteria either, from the best to the worst (ranking problem), or from the worst to the best (ordering problem).
3. Finally, we present order-statistical algorithms for relative or absolute quantiles-rating of multiple criteria performance records.

It is necessary to mention that the DIGRAPH3 resources do not provide a professional Python software library. The collection of Python modules, I describe in this book, was not built following any professional software development methodology. The design of classes and methods was kept as simple and elementary as was opportune for the author. Sophisticated and cryptic overloading of classes, methods and variables is more or less avoided all over. A simple copy, paste and ad hoc customisation development strategy was generally preferred. As a consequence, the DIGRAPH3 modules keep a large part of independence. Furthermore, the development of the DIGRAPH3 modules being spread over two decades, the programming

style did evolve with growing experience and the changes and enhancement coming up with the ongoing new releases of the standard Python3 libraries. The required backward compatibility necessarily introduced so with time some notation and programming technique changes.

The purpose of this book is to present in a single monograph the many scientific enhancements the author has contributed over the past two decades to the outranking approach-based multiple criteria decision aid field - and that are either left unpublished or published in very specialised media only and difficult to access.

This monograph should provide the reader with a self-contained series of tutorials which explain how to solve multiple criteria selection, as well as ranking or rating decision problems. If successful in this aim, the curious reader will effectively install the DIGRAPH3 programming resources on their laptop and try out and redo for themselves the proposed computations.

The material in this book is valuable for master students and doctoral candidates in Computer Science, Mathematics, Engineering Sciences or Computational Management Sciences taking a course on Algorithmic Decision Theory, Multiple Criteria Decision Aid or Decision Analysis. Some experience in computer programming, in particular with Python, will assist the reader, but it is not a prerequisite. The many coding examples shown throughout the text are purposely kept elementary from a programming point of view.

Chapters presenting algorithms for ranking multiple criteria performance records from best to worst –especially when facing big performance tableaux– may be of interest to designers of web recommender systems.

Similarly, the relative and absolute quantiles-rating algorithms discussed and illustrated will be of practical interest for private or public performance auditors.

Finally, the monograph does not provide any mathematical developments or proofs. Those readers interested in the mathematical background of our decision algorithms are invited to consult the references provided at chapter level. Full texts of most of these references may be downloaded from the <https://orbi.lu.uni.lu/> repository of the University of Luxembourg.

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*Raymond Bisdorff*



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# Introduction

## The editing strategy

The reader will find in the five parts of this monograph several series of tutorials and advanced topics that present and illustrate computational methods and tools mainly useful in the field of Multiple Criteria Decision Aid and Decision Analysis. These methods and tools were designed and implemented in Python3 by the author over the last decade in order to support the preparation and illustration of a Master Course on Algorithmic Decision Theory taught at the University of Luxembourg from 2010 to 2020.

Each chapter illustrates a specific preference modelling aspect, like building a best choice recommendation, ranking a set of potential decision alternatives, or computing the winner of an election. Explicit Python programming examples, purposely kept elementary, are shown in numerous terminal session style listings. A complete list of the numbered listings, shown over all the chapters, is printed in the Appendix.

These programming examples were all checked with the `doctest` module of the standard Python3 library and should work effectively as such either, in a Python3 shell console, or for sure in an `ipython` console. Some chapters will rely on a given data file that is made available in the `examples` directory of the DIGRAPH3 resources.

For easing their reading, most chapters do not provide detailed mathematical developments and proofs. Readers interested in such details are invited to consult the references listed separately at the end of each chapter. Most of these references provide a full text access to the corresponding document on the <https://orbi.lu.uni.lu/> repository of the University of Luxembourg.

Readers interested in the technical aspects of the organisation and implementation of the collection of DIGRAPH3 Python modules are invited to consult the reference manual: <https://digraph3.readthedocs.io/en/latest/techDoc.html>.

## Organisation of the book

The book is organised into five parts.

**Part I** presents three chapters introducing the DIGRAPH3 programming resources and the main formal objects discussed in this book, namely *bipolar-valued digraphs* and, in particular, *outranking digraphs*.

In Chapter 1, the reader will gain contact with the DIGRAPH3 Python resources. First are given the installation instructions and the list of the main DIGRAPH3 Python modules with their purpose. A Python terminal session using the root `digraphs` module eventually illustrates how to generate, save and inspect a random crisp digraph.

Chapter 2 introduces the bipolar-valued digraph model –the root type of all our digraph models–. A randomly bipolar-valued digraph instance is generated. Drawing the digraph, separating its asymmetric and symmetric parts, or its border and inner parts, is illustrated. The initial digraph instance may be reconstructed by epistemic disjunctive fusion from these respective parts. Dual, converse and codual versions, as well as symmetric and transitive closures are generated. Complete, empty and indeterminate digraphs are eventually presented.

Chapter 3 presents the bipolar-valued outranking digraph –the main formal object used and discussed in this monograph–. After illustrating its hybrid type –it is conjointly a multiple criteria performance tableau and a bipolar-valued digraph modelling the outranking situations between the given performance records– pairwise comparisons and the recoding of the digraph characteristic valuation are illustrated. The codual version of the outranking renders the corresponding strict outranking digraph, i.e. its asymmetric part.

**Part II** illustrates in eight methodological chapters multiple criteria performance evaluation models and decision algorithms. These chapters are mostly problem oriented.

Chapter 4 on page 45 presents the RUBIS best choice recommender system. The approach is illustrated with a best office site selection problem. We show how to explore a given performance tableau and compute the corresponding outranking digraph. After presenting the pragmatic principles that govern our best choice recommendation algorithm we solve the best office site choice problem.

Chapter 5 on page 61 illustrates a way of creating a new `PerformanceTableau` instance by editing a given template with 5 decision alternatives, 3 decision objectives and 6 performance criteria. We discuss in detail how to edit the decision alternatives, the decision objectives, the family of performance criteria, and finally, the evaluations of the decision alternatives on the performance criteria.

Chapter 6 on page 73 describes the DIGRAPH3 resources for generating random multiple criteria performance tableaux. These random performance tableaux instances, mainly meant for illustration and training purposes, were serving the preparation and illustration of the Algorithmic Decision Theory Course lectures.

The random generators propose several useful models like a Cost-Benefit tableau, a three Objectives —economic, societal and environmental— tableau, and eventually an academic performance tableau.

Chapter 7 on page 91 is more specifically devoted to handling linear voting profiles and computing the winner of such election results like the simple majority or the instant-run-off winner. By following CONDORCET's recipe, we consider pairwise comparisons of election candidates and balance the number of times the first beats the second against the number of times the second beats the first in order to obtain a majority margins digraph, in fact a bipolar-valued digraph. When the voters express contradictory linear voting profiles one may naturally observe cyclic social preferences without seeing any paradox in this situation. Finally, the chapter presents a more politically realistical random generator for linear voting profiles who takes into account pre-election polls.

Chapter 8 on page 105 introduces several algorithms for solving the ranking problem with a bipolar-valued outranking digraph. The COPELAND, NETFLOWS, KEMENY, SLATER, KOHLER and the RANKEDPAIRS ranking rules are illustrated with a random outranking digraph. The fitness of their respective ranking result is measured with a bipolar-valued version of KENDALL's ordinal correlation index.

Chapter 9 on page 123 applies order statistics for sorting a set  $X$  of  $n$  potential decision alternatives, evaluated on  $m$  incommensurable performance criteria, into  $q$  quantile equivalence classes. The sorting algorithm is based on pairwise outranking characteristics involving the quantile class limits observed on each criterion. Thus we may implement a weak ordering algorithm of complexity  $O(nmq)$ .

Chapter 10 on page 135 addresses the problem of rating multiple criteria performances of a set of potential decision alternatives with respect to performance quantiles learned from historical data gathered from similar decision alternatives observed in the past. We show how to learn performance quantiles from historical performance tableaux. New performance records may now be rated with respect to these learned quantile norms.

Chapter 11 on page 147 tackles the ranking of big multiple criteria performance tableaux with thousands or millions of records. To effectively compute rankings from performance tableaux of these sizes, the chapter proposes a collection of cythonized –C-compiled and optimised– modules that may be run on Linux Debian HPC equipement as available, for instance, at the University of Luxembourg.

**Part III** delivers three realistic algorithmic decision making case studies.

Chapter 12 on page 161 presents a case study concerning the building of a best choice recommendation for Alice, a German student who wants some advice concerning the choice of her future University studies. We present Alice's performance tableau —potential foreign language study programs, her decision objectives, performance criteria and performance grades— and build a best choice recommendation for her. A thorough robustness analysis confirms a very best choice.

In Chapter 13 on page 175 we are resolving with our DIGRAPH3 resources a ranking decision problem based on published data from the Times Higher Education

(THE) World University Rankings 2016 by Computer Science (CS) subject. We first have a look into the THE multiple criteria ranking data with short Python scripts. In a second section, we relax the commensurability hypothesis of the ranking criteria and show how to similarly rank with multiple incommensurable performance criteria of solely ordinal significance. A third section is finally devoted to introduce quality measures for qualifying ranking results.

Chapter 14 on page 195 presents and discusses how to rate with the help of our DIGRAPH3 resources the apparent student enrolment quality of 41 German higher education institutions. The multiple criteria performance tableau, we use, is based on internet published by DER SPIEGEL magazine in 2004 concerning nearly 50,000 students, enrolled in one of fifteen popular academic subjects, like German Studies, Life Sciences, Psychology, Law or Computer Science.

In Chapter 15 on page 213, we propose a series of decision problems of various difficulties which may serve as exercises and exam questions for an Algorithmic Decision Theory or Multiple Criteria Decision Analysis course. They cover selection, ranking and rating decision problems.

**Part IV** presents in five chapters more advanced topics showing some pearls of bipolar-valued epistemic logic.

Starting from a motivating decision problem about how to list, from the best to the worst, a set movies that are star-rated by journalists and movie critics, Chapter 16 on page 221 shows that KENDALL's ordinal correlation index tau may be extended to a relational bipolar-valued equivalence measure of bipolar-valued digraphs. This finding gives way, on the one hand, to measure the fitness and fairness of multiple criteria ranking rules. On the other hand, it provides a tool for illustrating preference divergences between decision objectives and criteria.

We illustrate in Chapter 17 on page 237, first, the concept of graph kernel, i.e. maximal independent set of vertices. In non-symmetric digraphs the kernel concept becomes richer and separates into initial and terminal kernels. In, furthermore, lateralised outranking digraphs, initial and terminal kernels become separate and may deliver suitable first resp. last choice recommendations. After commenting the tractability of kernel computations, we close the chapter with the solving of bipolar-valued kernel equation systems.

In Chapter 18 on page 265 we propose to link a qualifying significance majority for outranking situations with a required  $\alpha\%$ -confidence level. We model therefore the significance weights as random variables following more or less widespread distributions around an average weights value that corresponds to the given deterministic significance weight. As the bipolar-valued random credibility of an outranking situation hence results from the simple sum of positive or negative independent random variables, we may apply the Central Limit Theorem (CLT) for computing the bipolar-valued likelihood that the expected significance majority margin will indeed be positive, respectively negative.

The required cardinal significance weights of the performance criteria represent the 'Achilles' heel of our outranking approach. Rarely will indeed a decision maker be cognitively competent for suggesting precise decimal-valued criteria significance

weights. More often, the decision problem will involve more or less equally important decision objectives with more or less equi-significant criteria per objective. In Chapter 19 on page 275 we study the robustness of the outranking digraph when the criteria significance weights faithfully indicate solely an order of importance. This approach leads furthermore to the concept of unopposed or Pareto efficient multiobjective choices.

In a social choice context, where decision objectives would match different political parties, Pareto efficient choices represent in fact multipartisan social choices. Chapter 20 on page 293 shows that they may judiciously deliver the primary selection in a two stage election system. Our outranking model is based on bipolar approvals-disapprovals of “*as well as performing as*” statements. A similar approach is put into practice with bipolar approval-disapproval voting systems. When converting such approval-disapproval voting ballots into corresponding performance records, one obtains a  $(-1, 0, 1)$ -valued evaluative voting system. We eventually show in this chapter that in bipolar-approval voting systems, the winner tends to be among the more or less multipartisan candidates.

**Part V**, with three chapters, concerns eventually simple undirected graphs and illustrates computational resources for working with such graphs.

Chapter 21 on page 317 introduces bipolar-valued undirected graphs and illustrates several special graph models and algorithms like Q-coloring, MIS and clique enumeration, line graphs and maximal matchings, grid graphs, and n-cycle graphs with their non-isomorphic maximal independent sets of vertices.

Chapter 22 on page 333 specifically addresses working with tree graphs and graph forests. We illustrate how to generate and recognise random tree graphs and how to compute the centres of a tree and draw a rooted and oriented tree. Finally, algorithms for computing spanning trees and forests are presented.

Chapter 23 on page 345 eventually presents some famous classes of perfect graphs, namely comparability, interval, permutation and split graphs. We first present an example of a graph which is at the same time a triangulated, a comparability, a split and a permutation graph. The importance to be an interval is illustrated with BERGE’s mystery story. We discuss furthermore the generation of permutation graphs and close with how to recognise that a given graph is in fact a permutation graph.

## Highlights

Contrary to what is generally thought, it is the preparation of the multiple criteria performance tableau that takes most of the time, not running any decision algorithm. Designing adequate performance grading criteria functions for each decision objective and collecting adequate grades is essential for the success of the decision making. This is a very critical and essential step. Chapter 4 and Chapter 5 illustrate both how to edit a coherent performance tableau. In order to discover examples of

potential performance tableaux, we provide in Chapter 6 on page 73 random generators for several kinds of performance tableau.

Once the performance tableau is ready, starts the thrilling step of discovering the resulting outranking digraph. Are there many chordless outranking circuits? What is the degree of symmetry of the outranking relations? What is the degree of transitivity? If the number of potential decision alternatives is small –less than 20– one may try, in the case of a best choice selection problem to compute initial prekernels in order to find potential best choice decision alternatives? Chapters 4 on page 45, 12 on page 161 and 17 on page 237 are illustrating and discussing this computational problem.

Comparing ranking rules working on bipolar-valued outranking digraphs with multiple performance tableaux of different kinds: Cost-Benefit, 3-Objectives, academic a.-o., has made us confident about the fact that a convincing criterion for judging the quality of a ranking result may not to be found by mathematical properties, like KEMENY optimality or CONDORCET consistency. More useful seams to be the fair balancing of decision objectives and performance criteria. To this respect, the NETFLOWS ranking rule appears to give the fairest and most convincing multiple criteria rankings. Chapter 8 on page 105 on ranking rules, the ranking case study of Chapter 13 on page 175 and Chapter 16 on page 221 illustrate this point.

The bipolar-valued epistemic logic in which our decision algorithms are computing and expressing their decision solutions provides essential assistance for coping with missing data and imprecise performance gradings. An efficient robustness analysis becomes furthermore available for handling, on the one side, uncertain criteria significance weight leading in Chapter 18 to  $\alpha\%$ -confident outranking digraphs. On the other side, we are able to compute in Chapter 19 on page 275 robust outranking digraphs when solely ordinal criteria significance weights are given.

The same kind of robustness analysis allows us in Chapter 20 on page 293 to propose strategies for tempering plurality tyranny effects in social choice problems by favouring multipartisan candidates, like two-stage elections with multipartisan primary selection of candidates or bipolar-valued approval voting.

**Part I**

**Introduction to the DIGRAPH3**

**programming resources**

The first part contains three chapters for introducing the DIGRAPH3 software collection of Python programming resources. The first chapter is devoted to the installation of the DIGRAPH3 Python modules and running a first Python terminal session using the DIGRAPH3 3 resources. The second chapter introduces bipolar-valued digraphs, the root type of all available specialised digraphs. The third chapter finally introduces the main formal objects of this book, namely bipolar-valued outranking digraphs.

# Chapter 1

## Working with the DIGRAPH3 Python resources

**Abstract** The chapter is devoted to a first contact with the DIGRAPH3 Python resources. Following the installation instructions, we list the main Python modules with their purpose and eventually illustrate in a first Python terminal session how to generate, save and inspect a random crisp digraph.

### 1.1 Installing the DIGRAPH3 resources

Using the DIGRAPH3 Python modules is easy<sup>1</sup>. You only need to have installed on your system the Python programming language of version 3 (readily available under Linux and Mac OS). Notice that, from Version 3.3 on, the Python standard decimal module implements very efficiently its Decimal class in C. Now, Decimal numbers are mainly used in the DIGRAPH3 characteristic valuation functions, which makes the recent Python-3.7+ versions much faster (more than twice as fast) when extensive digraph operations are performed. Several download options (easiest under Linux or Mac OS-X) are given. Either, by using a git client from github:

```
...$ git clone https://github.com/rbisdorff/Digraph3
```

or from sourceforge.net:

```
...$ git clone https://git.code.sf.net/p/digraph3/code Digraph3
```

or, with a browser access, download either, from the github link above or, from the sourceforge page the latest distribution zip archive and extract it.

On Linux or Mac OS, ...\$ cd to the extracted Digraph3 directory:

```
.../Digraph3$ make install
```

---

<sup>1</sup> See the DIGRAPH3 documentation site: <https://digraph3.readthedocs.io/en/latest/>, Bisdorff (2021a).

installs (with `sudo` !) the DIGRAPH3 modules in the current running python environment. Python-3.8 (or later) environment is recommended (see the makefile for adapting the `make install` command to your running python environment).

Whereas:

```
.../Digraph3$ make installVenv
```

installs the DIGRAPH3 modules in an activated virtual python environment.

If the *cython* (<https://cython.org>) C-compiled modules for Big Data applications are required, it is necessary to previously install the *Cython* package in the running Python environment:

```
...$ python3 -m pip install cython
```

It is recommended to run a test suite:

```
.../Digraph3$ make tests
```

Test results are stored in the `Digraph3/test` directory.

Notice that the `python3 pytest` package is required:

```
...$ python3 -m pip install pytest
```

A verbose (with `stdout` not captured) `pytest` suite may be run as follows:

```
.../Digraph3$ make verboseTests
```

When the GNU parallel shell tool<sup>2</sup> is installed and multiple cores are detected, the tests may be executed in multiple processing mode:

```
.../Digraph3$ make pTests
```

Individual module `pytest` suites are also provided (see the `makefile`), like the one for the `outrankingDigraphs` module (see Chaptersec:2):

```
.../Digraph3$ make outrankingDigraphsTests
```

Dependencies:

To be fully functional, the DIGRAPH3 resources mainly need:

- The `graphviz` tools ([Gansner and North, 2000](#))<sup>3</sup>, and
- The R statistics resources <sup>4</sup> to be installed.
- When exploring digraph isomorphisms, the `nauty` isomorphism testing program is required ([McKay and Piperno, 2013](#)). On linux you may try: `sudo apt install nauty`. For Mac OS X, corresponding `dmg` installers are available for downloading.
- Two specific methods of the `OutrankingDigraph` class for clustering performance criteria or decision alternatives require furthermore the `calmat` matrix computing resource to be installed (see the `calmat` directory in the DIGRAPH3 resources).

<sup>2</sup> <https://www.gnu.org/software/parallel>

<sup>3</sup> <https://graphviz.org>

<sup>4</sup> <https://www.r-project.org>

## 1.2 Organisation of the DIGRAPH3 Python modules

The main data handling modules of the DIGRAPH3 resources are the following:

1. `digraphs`: Main part of the DIGRAPH3 source code with the root `Digraph` class.
2. `graphs`: Resources for handling undirected graphs with the root `Graph` class and a bridge to the `digraphs` module resources.
3. `perfTabs`: Tools for handling multiple criteria performance tableaux with root `PerformanceTableau` class.
4. `outrankingDigraphs`: Root module for handling outranking digraphs with the abstract root `OutrankingDigraph` class and the main `BipolarOutrankingDigraph` class.<sup>5</sup>
5. `votingProfiles`: Classes and methods for handling voting ballots and computing election results with main `LinearVotingProfile` class.

Various random generators are provided by the following modules:

1. `randomDigraphs` module: Various random digraph models like random crisp digraphs (`RandomDigraph` class) or random bipolar-valued digraphs (`RandomValuationDigraph` class).
2. `randomPerfTabs` module: Various implemented random performance tableau models, like Cost-Benefit tableaux (`RandomCBPerformanceTableau` class) or 3-Objectives tableaux (`Random3ObjectivesPerformanceTableau` class).
3. `randomNumbers` module: Additional random number generators, not available in the standard Python `random.py` library, like a discrete random variable (`DiscreteRandomVariable` class) or a Cauchy random variable (`CauchyRandomVariable` class)

Following modules provide tools for *sorting*, *ranking* and *rating* problems:

1. `sortingDigraphs` module: Tools for solving sorting problems with the root `SortingDigraph` class and the main `QuantilesSortingDigraph` class;
2. `linearOrders` module: Tools for solving linearly ranking or ordering problems with the root `LinearOrder` class;
3. `transitiveDigraphs` module: Additional tools for handling transitive digraphs with root `TransitiveDigraph` class.

Tools for specifically handling Big Data are eventually provided by the following modules:

1. `performanceQuantiles` module: Incremental representation of large performance tableaux via binned cumulated density functions per criteria; Depends on the `randomPerfTabs` module.

---

<sup>5</sup> Notice that the `outrankingDigraph` class defines a hybrid object type, inheriting conjointly from the `Digraph` class and the `PerformanceTableau` class.

2. `sparseOutrankingDigraphs` module: parse implementation design for bipolar-valued outranking digraphs of order  $> 500$ .
3. *Cythonized* modules: C-compiled and optimised Python modules for handling big performance tableaux and bipolar outranking digraphs of order  $> 1000$ .

Readers interested in the technical implementation details are invited to consult the reference manual of the DIGRAPH3 resources, where they will find the documentation and complete source code of all DIGRAPH3 modules, classes and methods (Bisdorff, 2021b).

### 1.3 Starting a DIGRAPH3 terminal session

You may start an interactive Python3 terminal session in the `Digraph3` directory.

```
$HOME/.../Digraph3$ python3
Python 3.9.6 (v3.9.6:db3ff76dal, Jun 28 2021, 11:49:53)
[Clang 6.0 (clang-600.0.57)] on darwin
Type "help", "copyright", "credits" or
"license" for more information.
>>>
```

For exploring the classes and methods provided by the DIGRAPH3 modules <sup>6</sup> enter the Python3 commands following the session prompts marked with `>>>` or `...`; the lines without a prompt are output from the Python3 interpreter. By convention, all class names and boolean parameters start with a capital case; names of other Python objects, like modules, methods, modules all start with a lower case. All Python names and code will be shown in a typewriting font.

**Listing 1.1** Generating a digraph instance

```
1 >>> from randomDigraphs import RandomDigraph
2 >>> dg = RandomDigraph(order=5,arcProbability=0.5, \
3 ...                               seed=101)
4 >>> dg
5 ----- Digraph instance description -----
6 Instance class      : RandomDigraph
7 Instance name       : randomDigraph
8 Digraph Order       : 5
9 Digraph Size        : 12
10 Valuation domain   : [-1.00; 1.00]
11 Determinateness    : 100.000
12 Attributes         : ['actions','valuationdomain',
13                           'relation','order','name',
14                           'gamma','notGamma']
```

In Listing 1.1 we import, for instance, from the `randomDigraphs` module the `RandomDigraph` class in order to generate a random digraph object `dg` of order 5 and arc probability of 50%. The resulting digraph of *order* 5 –number of nodes

---

<sup>6</sup> See the [DIGRAPH3 documentation](#)

called (decision) *actions* – and *size* 12 –number of arcs– is completely determined (see Line 11 in Listing 1.1 on the facing page).

We may save the content of *dg* in a file called *tutorialDigraph.py*.

```
1 >>> dg.save('tutorialDigraph')
2 *---- Saving digraph in file: <tutorialDigraph.py> ----*
```

with the following content:

**Listing 1.2** A stored digraph instance

```
1 from decimal import Decimal
2 from collections import OrderedDict
3 actions = OrderedDict([
4     ('a1', {'shortName': 'a1',
5         'name': 'random decision action'}),
6     ('a2', {'shortName': 'a2',
7         'name': 'random decision action'}),
8     ('a3', {'shortName': 'a3',
9         'name': 'random decision action'}),
10    ('a4', {'shortName': 'a4',
11        'name': 'random decision action'}),
12    ('a5', {'shortName': 'a5',
13        'name': 'random decision action'}),
14])
15 valuationdomain = {
16    'min': Decimal('-1.0'),
17    'med': Decimal('0.0'),
18    'max': Decimal('1.0'),
19    'hasIntegerValuation': True, # representation format
20}
21 relation = {
22    'a1': {'a1':Decimal('-1.0'), 'a2':Decimal('-1.0'),
23            'a3':Decimal('1.0'), 'a4':Decimal('-1.0'),
24            'a5':Decimal('-1.0'),},
25    'a2': {'a1':Decimal('1.0'), 'a2':Decimal('-1.0'),
26            'a3':Decimal('-1.0'), 'a4':Decimal('1.0'),
27            'a5':Decimal('1.0'),},
28    'a3': {'a1':Decimal('1.0'), 'a2':Decimal('-1.0'),
29            'a3':Decimal('-1.0'), 'a4':Decimal('1.0'),
30            'a5':Decimal('-1.0'),},
31    'a4': {'a1':Decimal('1.0'), 'a2':Decimal('1.0'),
32            'a3':Decimal('1.0'), 'a4':Decimal('-1.0'),
33            'a5':Decimal('-1.0'),},
34    'a5': {'a1':Decimal('1.0'), 'a2':Decimal('1.0'),
35            'a3':Decimal('1.0'), 'a4':Decimal('-1.0'),
36            'a5':Decimal('-1.0'),},
37}
```

In the DIGRAPH3 resources, all digraph object instances are of root *Digraph* type and contain at least the following attributes (see Listing 1.1 on the preceding page Lines 11-12):

1. A *name* attribute, holding usually the actual name of the stored instance that was used to create the instance;

2. An ordered dictionary of digraph nodes called `actions` (decision alternatives) with at least a `name` attribute;
3. An `order` attribute containing the number of graph nodes (length of the `actions` dictionary) automatically added by the object constructor;
4. A logical characteristic `valuation` domain dictionary with three decimal entries: the minimum ( $-1.0$ , means certainly false), the median ( $0.0$ , means missing information) and the maximum characteristic value ( $+1.0$ , means certainly true);
5. A double dictionary called `relation` and indexed by an oriented pair of actions (nodes) and carrying a decimal characteristic value in the range of the previous valuation domain;
6. Its associated `gamma` attribute, a dictionary containing the direct successors, respectively predecessors of each action, automatically added by the object constructor;
7. Its associated `notGamma` attribute, a dictionary containing the actions that are not direct successors respectively predecessors of each action, automatically added by the object constructor.

## 1.4 Inspecting a digraph object

Different `show...` methods, like the `showShort()` method, reveal us now that `dg` is a crisp, irreflexive and connected digraph of order five (see Listing 1.3 Lines 1, 16, 26, 37 below).

**Listing 1.3** Random crisp digraph object

```

1 >>> dg.showShort()
2 *----- show short -----*
3 Digraph      : tutorialDigraph
4 Actions       : OrderedDict([
5   ('a1', {'shortName': 'a1', 'name': 'random decision action'}),
6   ('a2', {'shortName': 'a2', 'name': 'random decision action'}),
7   ('a3', {'shortName': 'a3', 'name': 'random decision action'}),
8   ('a4', {'shortName': 'a4', 'name': 'random decision action'}),
9   ('a5', {'shortName': 'a5', 'name': 'random decision action'})
10  ])
11 Valuation domain :
12  'min': Decimal('-1.0'),
13  'max': Decimal('1.0'),
14  'med': Decimal('0.0'), 'hasIntegerValuation': True
15  }
16 >>> dg.showRelationTable()
17 * ----- Relation Table -----
18   S   |  'a1'  'a2'  'a3'  'a4'  'a5'
19  -----|-----
20  'a1' |    -1    -1     1    -1    -1
21  'a2' |     1    -1    -1     1     1
22  'a3' |     1    -1    -1     1    -1

```

```

23 'a4' | 1 1 1 -1 -1
24 'a5' | 1 1 1 -1 -1
25 Valuation domain: [-1;+1]
26 >>> dg.showComponents()
27 *--- Connected Components ---*
28 1: ['a1', 'a2', 'a3', 'a4', 'a5']
29 >>> dg.showNeighborhoods()
30 Neighborhoods:
31     Gamma :
32 'a1': in => {'a2', 'a4', 'a3', 'a5'}, out => {'a3'}
33 'a2': in => {'a5', 'a4'}, out => {'a1', 'a4', 'a5'}
34 'a3': in => {'a1', 'a4', 'a5'}, out => {'a1', 'a4'}
35 'a4': in => {'a2', 'a3'}, out => {'a1', 'a3', 'a2'}
36 'a5': in => {'a2'}, out => {'a1', 'a3', 'a2'}
37     Not Gamma :
38 'a1': in => set(), out => {'a2', 'a4', 'a5'}
39 'a2': in => {'a1', 'a3'}, out => {'a3'}
40 'a3': in => {'a2'}, out => {'a2', 'a5'}
41 'a4': in => {'a1', 'a5'}, out => {'a5'}
42 'a5': in => {'a1', 'a4', 'a3'}, out => {'a4'}

```

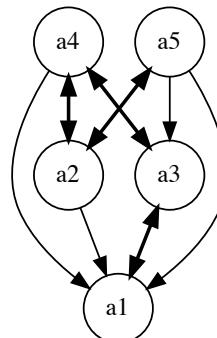
The `exportGraphViz()` method generates in the current working directory a `tutorialDigraph.dot` file and a `tutorialdigraph.png` picture of the tutorial digraph `dg` (see Fig. 1.1), if the `graphviz` tools are installed on your system (Gansner and North, 2000).

```

1 >>> dg.exportGraphViz('tutorialDigraph')
2 *--- exporting a dot file do GraphViz tools -----
3 Exporting to tutorialDigraph.dot
4 dot -Grankdir=BT -Tpng tutorialDigraph.dot -o
    tutorialDigraph.png

```

**Fig. 1.1** The tutorial crisp digraph. The `exportGraphViz()` method is depending on drawing tools from <https://graphviz.org>. On Linux Ubuntu or Debian you may try `'sudo apt-get install graphviz'` to install them. For Mac OSX There are ready `dmg` installers available



*Digraph3 (graphviz), R. Bisdorff, 2020*

Further methods are provided for inspecting this random Digraph object `dg`, like the following `showStatistics()` method.

**Listing 1.4** Inspecting a Digraph object

```

1 >>> dg.showStatistics()
2 *----- general statistics -----*
3 for digraph : <tutorialDigraph.py>
4 order : 5 nodes
5 size : 12 arcs
6 undetermined : 0 arcs
7 determinateness (%) : 100.0
8 arc density : 0.60
9 double arc density : 0.40
10 single arc density : 0.40
11 absence density : 0.20
12 strict single arc density: 0.40
13 strict absence density : 0.20
14 nbr. of components : 1
15 nbr. of strong components : 1
16 transitivity degree (%) : 60.0
17 : [0, 1, 2, 3, 4, 5]
18 outdegrees distribution : [0, 1, 1, 3, 0, 0]
19 indegrees distribution : [0, 1, 2, 1, 1, 0]
20 mean outdegree : 2.40
21 mean indegree : 2.40
22 : [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
23 symmetric degrees dist. : [0, 0, 0, 0, 1, 4, 0, 0, 0, 0, 0]
24 mean symmetric degree : 4.80
25 outdegrees concentration index : 0.1667
26 indegrees concentration index : 0.2333
27 symdegrees concentration index : 0.0333
28 : [0, 1, 2, 3, 4, 'inf']
29 neighbourhood depths distribution: [0, 1, 4, 0, 0, 0]
30 mean neighbourhood depth : 1.80
31 digraph diameter : 2
32 agglomeration distribution :
33 a1 : 58.33
34 a2 : 33.33
35 a3 : 33.33
36 a4 : 50.00
37 a5 : 50.00
38 agglomeration coefficient : 45.00

```

The preceding `show...()` methods usually rely upon corresponding `compute` methods, like: `computeSize()`, `computeDeterminateness()`, or `computeTransitivityDegree()`.

**Listing 1.5** Various `compute...()` methods.

```

1 >>> dg.computeSize()
2 12
3 >>> dg.computeDeterminateness(InPercents=True)
4 Decimal('100.00')
5 >>> dg.computeTransitivityDegree(InPercents=True)
6 Decimal('60.00')

```

Mind that `show...` methods output their results in the *Python* console. We provide also some `showHTML...` methods which output their results in a system browser's tab or window.

```
1 >>> dg.showHTMLRelationMap(relationName='r(x,y)', \
2 ...                                rankingRule=None)
```

**Fig. 1.2** Browsing the relation map of the tutorial digraph.  $+$  indicates a certainly valid and  $-$  indicates a certainly invalid relation. Here we find confirmed again that our random digraph instance `dg`, is indeed a crisp, i.e. 100% determined irreflexive digraph instance

## Relation Map

### Ranking rule: Alphabetic

r(x,y)	a1	a2	a3	a4	a5
a1	-	-	+	-	-
a2	+	-	-	+	+
a3	+	-	-	+	-
a4	+	+	+	-	-
a5	+	+	+	-	-

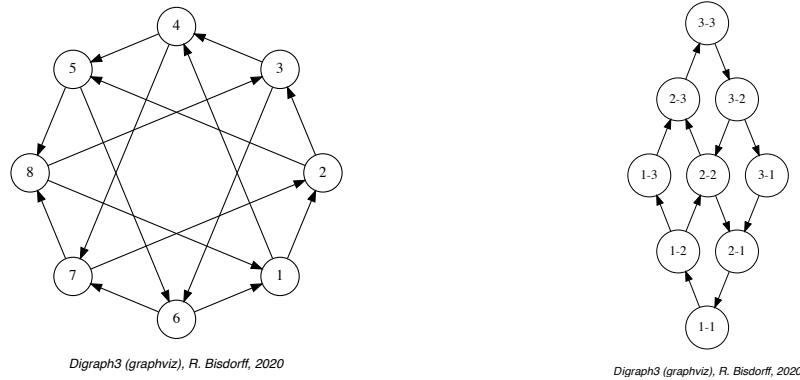
Some special types of digraph instances, like the `CirculantDigraph` or the `GridDigraph` classes, are readily available (see Fig. 1.3 on the following page).

```
1 >>> from digraphs import CirculantDigraph, GridDigraph
2 >>> c8 = CirculantDigraph(order=8, circulants=[1, 3])
3 >>> c8.exportGraphViz('c8')
4     ----- exporting a dot file for GraphViz tools -----
5     Exporting to c8.dot
6     circo -Tpng c8.dot -o c8.png
7 >>> grid3 = GridDigraph(n=3, m=3, \
8     ...                                hasMedianSplitOrientation=True)
9 >>> grid3.exportGraphViz('grid3')
10    ----- exporting a dot file for GraphViz tools -----
11    Exporting to grid3.dot
12    dot -Grankdir=BT -Tpng grid3.dot -o grid3.png
```

The next Chapter 2 on page 19 will introduce the fondamental *bipolar-valued digraph* model which is *root object type* to all the digraph models implemented in the DIGRAPH3 modules (Bisdorff, 2021b).

## References

Bisdorff R (2021a) Documentation of the Digraph3 collection of Python modules for Algorithmic Decision Theory. URL <https://digraph3.readthedocs.io/en/latest/> 9



**Fig. 1.3** The circulant [1,3] digraph and the 3x3 grid digraph

Bisdorff R (2021b) Technical documentation of the Digraph3 collection of Python modules. URL <https://digraph3.readthedocs.io/en/latest/techDoc.html> 12, 17

Gansner E, North S (2000) An open graph visualization system and its applications to software engineering. SOFTWARE - PRACTICE AND EXPERIENCE 30(11):1203–1233, URL <https://graphviz.org/documentation/> 10, 15

McKay B, Piperno A (2013) Practical Graph Isomorphism, II. J Symbolic Computation 60:94–112, URL <https://www.cs.sunysb.edu/~algorith/implement/nauty/implement.shtml> 10

## Chapter 2

# Working with bipolar-valued digraphs

**Abstract** We introduce bipolar-valued digraphs, the fundamental root type of all the specialised digraphs implemented in the DIGRAPH3 modules. With the help of a randomly valued digraph, we illustrate some basic digraph manipulation methods, like drawing the digraph, dividing the digraph into its asymmetric and symmetric parts, separating the border from the inner part, computing associated dual, converse and codual digraphs, and operating symmetric and transitive closures.

### 2.1 Random bipolar-valued digraphs

We are starting the tutorial with generating a uniformly random  $[-1.0; +1.0]$ -valued digraph of order 7, denoted `rdg` and modelling, for instance, a binary relation  $S(x, y)$  defined on the set of nodes of `rdg`. For this purpose, the DIGRAPH3 resources provide in `randomDigraphs` module a specific `RandomValuationDigraph` constructor (Bisdorff, 2021).

**Listing 2.1** Random bipolar-valued digraph instance

```
1 >>> from randomDigraphs import RandomValuationDigraph
2 >>> rdg = RandomValuationDigraph(order=7)
3 >>> rdg.save('tutRandValDigraph')
4 >>> from digraphs import Digraph
5 >>> rdg = Digraph('tutRandValDigraph')
6 >>> rdg
7     *----- Digraph instance description -----*
8     Instance class      : Digraph
9     Instance name       : tutRandValDigraph
10    Digraph Order       : 7
11    Digraph Size        : 22
12    Valuation domain   : [-1.00;1.00]
13    Determinateness (%) : 75.24
14    Attributes          : ['name','actions','order',
15                           'valuationdomain','relation',
16                           'gamma','notGamma']
```

With the `save()` method (see Listing 2.1 Line 3) we may keep a backup version for future use of `rdg` which will be stored in a file called `tutRandValDigraph.py` in the current working directory.

The genuine `Digraph` class constructor may restore the `rdg` object from the stored file (Line 4-5). We may easily inspect the content of `rdg` (Line 6). The digraph size 22 indicates the number of positively valued arcs. The valuation domain is uniformly distributed in the interval  $[-1.0; 1.0]$  and the mean absolute arc valuation is  $(0.7524 \times 2) - 1.0 = 0.5048$  (Line 13).

As mentioned in the previous Chapter 1, all objects of `Digraph` type contain at least the list of attributes shown here in Lines 14-16: –a name (string), –a dictionary of `actions` (digraph nodes), –an `order` (integer) attribute containing the number of actions, –a `valuationdomain` dictionary, –a double dictionary `relation` representing the adjacency table of the digraph relation, –a `gamma` and –a `notGamma` dictionary containing the direct neighbourhood of each action.

The `Digraph` class provides some generic `show...` methods for exploring the content of a given `Digraph` object, like the `showRelationTable()`, the `showComponents()` and the `showNeighborhoods()` methods.

**Listing 2.2** Example of random valuation digraph

```

1  >>> rdg.showRelationTable()
2  * ---- Relation Table ----
3  r(xSy) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
4  -----|-----
5  '1'   |  0.00 -0.48  0.70  0.86  0.30  0.38  0.44
6  '2'   | -0.22  0.00 -0.38  0.50  0.80 -0.54  0.02
7  '3'   | -0.42  0.08  0.00  0.70 -0.56  0.84 -1.00
8  '4'   |  0.44 -0.40 -0.62  0.00  0.04  0.66  0.76
9  '5'   |  0.32 -0.48 -0.46  0.64  0.00 -0.22 -0.52
10 '6'  | -0.84  0.00 -0.40 -0.96 -0.18  0.00 -0.22
11 '7'  |  0.88  0.72  0.82  0.52 -0.84  0.04  0.00
12 >>> rdg.showComponents()
13 *--- Connected Components ---*
14 1: ['1', '2', '3', '4', '5', '6', '7']
15 >>> rdg.showNeighborhoods()
16 *---- Neighborhoods -----*
17     Gamma:
18 '1': in => {'5', '7', '4'}, out => {'5', '7', '6', '3', '4'}
19 '2': in => {'7', '3'}, out => {'5', '7', '4'}
20 '3': in => {'7', '1'}, out => {'6', '2', '4'}
21 '4': in => {'5', '7', '1', '2', '3'}, out => {'5', '7', '1', '6'}
22 '5': in => {'1', '2', '4'}, out => {'1', '4'}
23 '6': in => {'7', '1', '3', '4'}, out => set()
24 '7': in => {'1', '2', '4'}, out => {'1', '2', '3', '4', '6'}
25     Not Gamma:
26 '1': in => {'6', '2', '3'}, out => {'2'}
27 '2': in => {'5', '1', '4'}, out => {'1', '6', '3'}
28 '3': in => {'5', '6', '2', '4'}, out => {'5', '7', '1'}
29 '4': in => {'6'}, out => {'2', '3'}
30 '5': in => {'7', '6', '3'}, out => {'7', '6', '2', '3'}
31 '6': in => {'5', '2'}, out => {'5', '7', '1', '3', '4'}
32 '7': in => {'5', '6', '3'}, out => {'5'}
```

Mind that some `DiGraph` class methods will ignore the *reflexive* links by considering that they are *indeterminate*, i.e. the characteristic value  $r(xSx)$  for all action  $x$  is set to the *median*, i.e. *indeterminate* value 0.0 in this case (see Listing 2.2 Lines 5-11 and Bisdorff (2004)).

## 2.2 Graphviz drawings

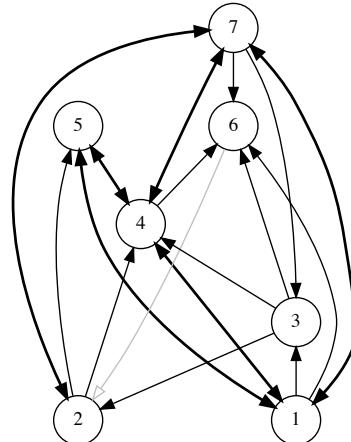
We may have an even better insight into the `DiGraph` object `rdg` by looking at a **graphviz** drawing<sup>1</sup>.

```

1 >>> rdg.exportGraphViz('tutRandValDigraph')
2 *---- exporting a dot file for GraphViz tools ----*
3 Exporting to tutRandValDigraph.dot
4 dot -Grankdir=BT -Tpng tutRandValDigraph.dot\
5                               -o tutRandValDigraph.png

```

**Fig. 2.1** The tutorial random valuation digraph. Double links are drawn in bold black with an arrowhead at each end, whereas single asymmetric links are drawn in black with an arrowhead showing the direction of the link. Notice the undetermined relational situation ( $r(6S2) = 0.00$ ) observed between nodes '6' and '2'. The corresponding link is marked in gray with an open arrowhead in the drawing



*DiGraph3 (graphviz), R. Bisdorff, 2020*

---

<sup>1</sup> The `exportGraphViz()` method is depending on drawing tools from the `graphviz` software (<https://graphviz.org/>). On Linux Ubuntu or Debian you may try `sudo apt-get install graphviz` to install them. There are ready `dmg` installers for Mac OSX.

### 2.3 Asymmetric and symmetric parts

We may now extract both the *symmetric* as well as the *asymmetric* part of digraph rdg with the help of two corresponding constructors (see Listing 2.3).

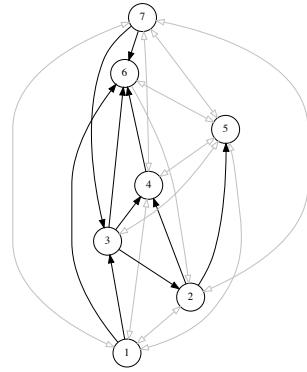
**Listing 2.3** Computing asymmetric and symmetric Parts

```

1 >>> from digraphs import AsymmetricPartialDigraph, \
2     ...                               SymmetricPartialDigraph
3 >>> asymDg = AsymmetricPartialDigraph(rdg)
4 >>> asymDg.exportGraphViz()
5 >>> symDg = SymmetricPartialDigraph(rdg)
6 >>> symDg.exportGraphViz()

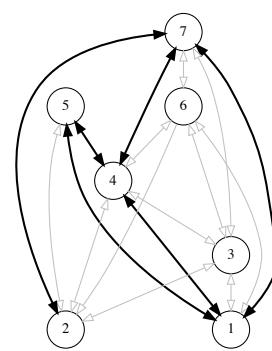
```

Asymmetric Part



Digraph3 (graphviz), R. Bischoff, 2020

Symmetric Part



Digraph3 (graphviz), R. Bischoff, 2020

**Fig. 2.2** Asymmetric and symmetric part of the tutorial random valuation digraph

The constructor of the partial objects `asymDg` and `symDg` puts to the indeterminate characteristic value all non-asymmetric, respectively non-symmetric links between nodes (see Fig. 2.2).

Here below, for illustration the source code of the `relation` constructor of the `AsymmetricPartialDigraph` class.

**Listing 2.4** Computing the asymmetric part of a bipolar-valued relation

```

1 def __constructRelation(self):
2     actions = self.actions
3     Min = self.valuationdomain['min']
4     Max = self.valuationdomain['max']
5     Med = self.valuationdomain['med']
6     relationIn = self.relation
7     relationOut = {}
8     for a in actions:
9         relationOut[a] = {}
10        for b in actions:
11            if a != b:

```

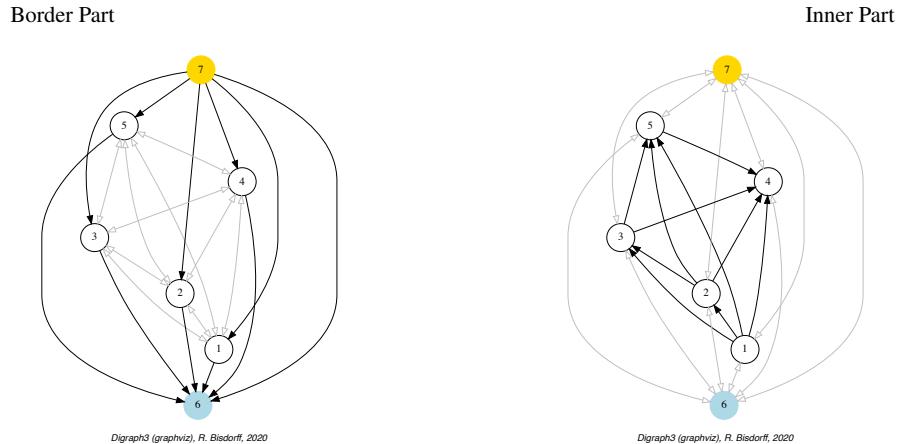
```

12         if relationIn[a][b] >= Med and relationIn[b][a] <= Med:
13             relationOut[a][b] = relationIn[a][b]
14         elif relationIn[a][b] <= Med and relationIn[b][a] >= Med:
15             relationOut[a][b] = relationIn[a][b]
16         else:
17             relationOut[a][b] = Med
18         else:
19             relationOut[a][b] = Med
20
21     return relationOut

```

## 2.4 Border and inner parts

We may also extract the border –the part of a digraph induced by the union of its initial and terminal prekernels (see Chapter 17)– as well as, the inner part –the complement of the border– with the help of two corresponding class constructors: `GraphBorder` and `GraphInner` (see Fig. 2.3).



**Fig. 2.3** *Border and inner part of a linear order oriented by terminal and initial kernels.*

Let us illustrate the digraph border and inner parts on a linear ordering obtained from the tutorial random valuation digraph `rdg` with the NETFLOWS ranking rule (see Section 8.3).

**Listing 2.5** Border and inner part of a linear order

```

1 >>> from digraphs import GraphBorder, GraphInner
2 >>> from linearOrders import NetFlowsOrder
3 >>> nf = NetFlowsOrder(rdg)
4 >>> nf.netFlowsOrder
5   ['6', '4', '5', '3', '2', '1', '7']
6 >>> bnf = GraphBorder(nf)

```

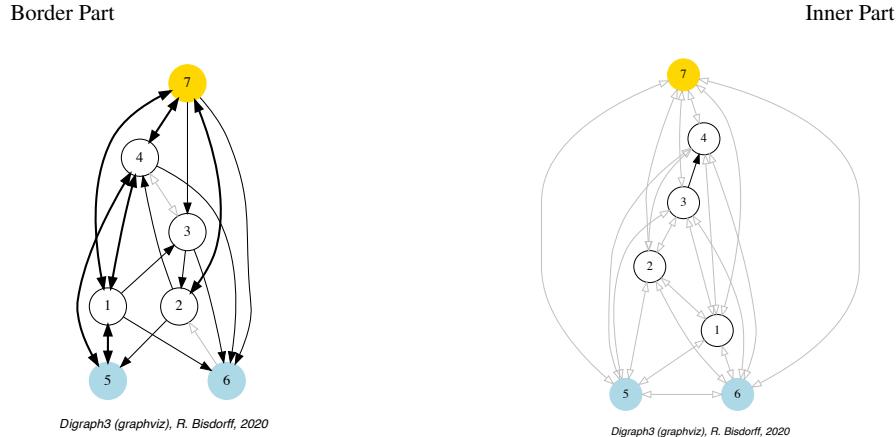
```

7 >>> bnf.exportGraphViz(worstChoice=['6'],bestChoice=['7'])
8 >>> inf = GraphInner(nf)
9 >>> inf.exportGraphViz(worstChoice=['6'],bestChoice=['7'])

```

We may orient the `graphviz` drawings in Fig. 2.3 with the terminal node 6 (`worstChoice` parameter) and initial node 7 (`bestChoice` parameter) (see Listing 2.5 Lines 7 and 9).

The constructor of the partial digraphs `bnf` and `inf` (see Lines 3 and 6) puts to the *indeterminate* characteristic value all links not in the *border*, respectively *not* in the *inner* part (see Fig. 2.3). Being much *denser* than a linear order, the actual inner part of our tutorial random valuation digraph `rdg` is reduced to a single arc between nodes 3 and 4 (see Fig. 2.4). Indeed, a complete digraph on the limit has no inner part (privacy!) at all, whereas empty and indeterminate digraphs admit both, an empty border and an empty inner part.



**Fig. 2.4** Border and inner part of the tutorial random valuation digraph `rdg`

## 2.5 Fusion by epistemic disjunction

We may recover object `rdg` from both partial objects `asymDg` and `symDg`, or as well from the border `bg` and the inner part `ig`, with a *bipolar fusion* constructor, also called *epistemic disjunction*, available via the `FusionDigraph` class.

### Listing 2.6 Epistemic fusion of partial digraphs

```

1 >>> from digraphs import FusionDigraph
2 >>> fusDg = FusionDigraph(asymDg,symDg,operator='o-max')
3 >>> # fusDg = FusionDigraph(bg,ig,operator='o-max')

```

```

4 >>> fusDg.showRelationTable()
5   * ----- Relation Table -----
6   r(xSy) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
7   -----|-----
8   '1'   |  0.00 -0.48  0.70  0.86  0.30  0.38  0.44
9   '2'   | -0.22  0.00 -0.38  0.50  0.80 -0.54  0.02
10  '3'   | -0.42  0.08  0.00  0.70 -0.56  0.84 -1.00
11  '4'   |  0.44 -0.40 -0.62  0.00  0.04  0.66  0.76
12  '5'   |  0.32 -0.48 -0.46  0.64  0.00 -0.22 -0.52
13  '6'   | -0.84  0.00 -0.40 -0.96 -0.18  0.00 -0.22
14  '7'   |  0.88  0.72  0.82  0.52 -0.84  0.04  0.00

```

The epistemic fusion operator  $\circ\text{-max}$  (see Listing 2.6 Line 2) is defined as follows:

**Definition 2.1 (Disjunctive epistemic fusion operator  $\circ\text{-max}$ ).**

Let  $r$  and  $r'$  characterise two bipolar-valued epistemic situations:

- $\circ\text{-max}(r, r') = \max(r, r')$  when both  $r$  and  $r'$  are more or less valid or indeterminate;
- $\circ\text{-max}(r, r') = \min(r, r')$  when both  $r$  and  $r'$  are more or less invalid or indeterminate;
- $\circ\text{-max}(r, r') = 0.0$ , i.e. indeterminate otherwise.

## 2.6 Dual, converse and codual digraphs

We may as readily compute the *dual* (negated relation <sup>2</sup>), the *converse* (transposed relation) and the *codual* (transposed and negated relation) of the digraph instance `rdg`.

**Listing 2.7** Computing associated dual, converse and codual digraphs

```

1 >>> from digraphs import \
2 ...           DualDigraph, ConverseDigraph, CoDualDigraph
3 >>> # dual of rdg
4 >>> ddg = DualDigraph(rdg)
5 >>> ddg.showRelationTable()
6   -r(xSy) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
7   -----|-----
8   '1'   |  0.00  0.48 -0.70 -0.86 -0.30 -0.38 -0.44
9   '2'   |  0.22  0.00  0.38 -0.50  0.80  0.54 -0.02
10  '3'   |  0.42  0.08  0.00 -0.70  0.56 -0.84  1.00
11  '4'   | -0.44  0.40  0.62  0.00 -0.04 -0.66 -0.76
12  '5'   | -0.32  0.48  0.46 -0.64  0.00  0.22  0.52
13  '6'   |  0.84  0.00  0.40  0.96  0.18  0.00  0.22
14  '7'   |  0.88 -0.72 -0.82 -0.52  0.84 -0.04  0.00
15 >>> # converse of rdg

```

<sup>2</sup> Not to be confused with the dual graph of a plane graph  $g$  that has a vertex for each face of  $g$ . Here we mean the *less than* (strict converse) relation corresponding to a *greater or equal* relation, or the *less than or equal* relation corresponding to a (strict) *better than* relation.

```

16 >>> cdg = ConverseDigraph(rdg)
17 >>> cdg.showRelationTable()
18     * ---- Relation Table ----
19     r(ySx) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
20     -----|-----
21     '1'   | 0.00 -0.22 -0.42  0.44  0.32 -0.84  0.88
22     '2'   | -0.48  0.00  0.08 -0.40 -0.48  0.00  0.72
23     '3'   | 0.70 -0.38  0.00 -0.62 -0.46 -0.40  0.82
24     '4'   | 0.86  0.50  0.70  0.00  0.64 -0.96  0.52
25     '5'   | 0.30  0.80 -0.56  0.04  0.00 -0.18 -0.84
26     '6'   | 0.38 -0.54  0.84  0.66 -0.22  0.00  0.04
27     '7'   | 0.44  0.02 -1.00  0.76 -0.52 -0.22  0.00
28 >>> # codual of rdg
29 >>> cddg = CoDualDigraph(rdg)
30 >>> cddg.showRelationTable()
31     * ---- Relation Table ----
32     -r(ySx) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
33     -----|-----
34     '1'   | 0.00  0.22  0.42 -0.44 -0.32  0.84 -0.88
35     '2'   | 0.48  0.00 -0.08  0.40  0.48  0.00 -0.72
36     '3'   | -0.70  0.38  0.00  0.62  0.46  0.40 -0.82
37     '4'   | -0.86 -0.50 -0.70  0.00 -0.64  0.96 -0.52
38     '5'   | -0.30 -0.80  0.56 -0.04  0.00  0.18  0.84
39     '6'   | -0.38  0.54 -0.84 -0.66  0.22  0.00 -0.04
40     '7'   | -0.44 -0.02  1.00 -0.76  0.52  0.22  0.00

```

Computing the *dual*, respectively the *converse* of a digraph, may also be done with prefixing the `__neg__` (`-`) or the `__invert__` (`~`) operator. The *codual* of a `Digraph` object may, hence, as well be computed with a *composition* (in either order) of both operations.

**Listing 2.8** Computing the dual, the converse and the codual of a digraph

```

1 >>> ddg = -rdg    # dual of rdg
2 >>> cdg = ~rdg    # converse of rdg
3 >>> cddg = ~(~rdg) # = -(~(rdg)) codual of rdg
4 >>> (-(~rdg)).showRelationTable()
5     * ---- Relation Table ----
6     -r(ySx) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
7     -----|-----
8     '1'   | 0.00  0.22  0.42 -0.44 -0.32  0.84 -0.88
9     '2'   | 0.48  0.00 -0.08  0.40  0.48  0.00 -0.72
10    '3'   | -0.70  0.38  0.00  0.62  0.46  0.40 -0.82
11    '4'   | -0.86 -0.50 -0.70  0.00 -0.64  0.96 -0.52
12    '5'   | -0.30 -0.80  0.56 -0.04  0.00  0.18  0.84
13    '6'   | -0.38  0.54 -0.84 -0.66  0.22  0.00 -0.04
14    '7'   | -0.44 -0.02  1.00 -0.76  0.52  0.22  0.00

```

## 2.7 Symmetric and transitive closures

Symmetric and transitive closures, by default in-site constructors, are also available (see Fig. 2.5). Note that it is a good idea, before going ahead with these in-site operations, who irreversibly modify the original `rdg` object, to previously make a backup version of `rdg`. The simplest storage method, always provided by the generic `Digraph.save()` method, writes out in a named file the python content of the `Digraph` object in string representation (see Section 1.3).

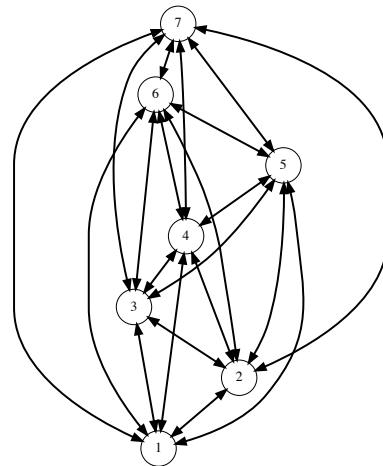
**Listing 2.9** Symmetric and transitive closures

```

1 >>> rdg.save('tutRandValDigraph')
2 >>> rdg.closeSymmetric(InSite=True)
3 >>> rdg.closeTransitive(InSite=True)
4 >>> rdg.exportGraphViz('strongComponents')

```

**Fig. 2.5** Symmetric and transitive closure of the tutorial random valuation digraph `rdg`.



*Digraph3 (graphviz), R. Bisдорff, 2020*

The `closeSymmetric()` method (see Listing 2.9 Line 2), of complexity  $O(n^2)$  where  $n$  denotes the digraph's order, changes, on the one hand, all single pairwise links it may detect into double links by operating a disjunction of the pairwise relations. On the other hand, the `closeTransitive()` method (see Line 3), implements the Roy-Warshall transitive closure algorithm of complexity  $O(n^3)$  (Roy (1959) and Warshall (1962)).

The same `closeTransitive()` with a `Reverse = True` flag may be readily used for eliminating all transitive arcs from a transitive digraph instance. We make usage of this feature when drawing Hasse diagrams of `TransitiveDigraph` objects.

## 2.8 Strong components

As the original digraph `rdg` was connected (see above the result of the `showShort()` command), both the symmetric and the transitive closures operated together, will necessarily produce a single strong component, i.e. a **complete** digraph. We may sometimes wish to collapse all strong components in a given digraph and construct the so *collapsed* digraph. Using the `StrongComponentsCollapsedDigraph` constructor here will render a single hyper-node gathering all the original nodes (see Line 7 below).

**Listing 2.10** Computing the strong components in a digraph

```

1 >>> from digraphs import StrongComponentsCollapsedDigraph
2 >>> sc = StrongComponentsCollapsedDigraph(rdg)
3 >>> sc.showAll()
4     *---- show detail ----*
5     Digraph           : tutRandValDigraph_Scc
6     *---- Actions ----*
7     ['_7_1_2_6_5_3_4_']
8     *---- Relation Table ----
9     S      | 'Scc_1'
10    -----|-----
11    'Scc_1' | 0.00
12     *---- strong Components ----*
13     short      content
14     'Scc_1'     '_7_1_2_6_5_3_4_'
15     *---- Neighborhoods ----*
16     Gamma       :
17     'frozenset({'7','1','2','6','5','3','4'})':
18           in => set(), out => set()
19     Not Gamma  :
20     'frozenset({'7','1','2','6','5','3','4'})':
21           in => set(), out => set()

```

## 2.9 CSV storage

Sometimes it is required to exchange the graph valuation data in CSV format with a statistical package like **R**<sup>3</sup>. For this purpose it is possible to export the digraph data into a CSV file. The valuation domain is hereby normalised by default to the range  $[-1.0, 1.0]$  and the diagonal is put by default to the minimal value  $-1.0$ .

```

1 >>> rdg = Digraph('tutRandValDigraph')
2 >>> rdg.saveCSV('tutRandValDigraph')
3     # content of file tutRandValDigraph.csv
4     "d", "1", "2", "3", "4", "5", "6", "7"
5     "1", -1.0, 0.48, -0.7, -0.86, -0.3, -0.38, -0.44
6     "2", 0.22, -1.0, 0.38, -0.5, -0.8, 0.54, -0.02

```

<sup>3</sup> <https://www.r-project.org/>

```

7  "3", 0.42, -0.08, -1.0, -0.7, 0.56, -0.84, 1.0
8  "4", -0.44, 0.4, 0.62, -1.0, -0.04, -0.66, -0.76
9  "5", -0.32, 0.48, 0.46, -0.64, -1.0, 0.22, 0.52
10 "6", 0.84, 0.0, 0.4, 0.96, 0.18, -1.0, 0.22
11 "7", -0.88, -0.72, -0.82, -0.52, 0.84, -0.04, -1.0

```

It is possible to reload a `Digraph` instance from its previously saved CSV file content.

```

1 >>> from digraphs import CSVDigraph
2 >>> rdgcsv = CSVDigraph('tutRandValDigraph')
3 >>> rdgcsv.showRelationTable(ReflexiveTerms=False)
4 * ---- Relation Table ----
5   r(xSy) |  '1'  '2'  '3'  '4'  '5'  '6'  '7'
6  -----|-----
7  '1'  |   -  -0.48  0.70  0.86  0.30  0.38  0.44
8  '2'  | -0.22   -  -0.38  0.50  0.80 -0.54  0.02
9  '3'  | -0.42  0.08   -  0.70 -0.56  0.84 -1.00
10 '4'  |  0.44 -0.40 -0.62   -  0.04  0.66  0.76
11 '5'  |  0.32 -0.48 -0.46  0.64   - -0.22 -0.52
12 '6'  | -0.84  0.00 -0.40 -0.96 -0.18   - -0.22
13 '7'  |  0.88  0.72  0.82  0.52 -0.84  0.04   -

```

It is as well possible to show a coloured version of the valued relation table in a system browser window tab (see Fig. 2.5).

```

1 >>> rdgcsv.showHTMLRelationTable(tableTitle="Tutorial random
digraph")

```

**Fig. 2.6** The valued relation table shown in a browser window. Positive arcs are shown in green and negative arcs in red. Indeterminate –zero-valued– links, like the reflexive diagonal ones or the link between node '6' and node '2', are shown in gray.

## Tutorial random digraph

r(x S y)	1	2	3	4	5	6	7
1	0.00	-0.48	0.70	0.86	0.30	0.38	0.44
2	-0.22	0.00	-0.38	0.50	0.80	-0.54	0.02
3	-0.42	0.08	0.00	0.70	-0.56	0.84	-1.00
4	0.44	-0.40	-0.62	0.00	0.04	0.66	0.76
5	0.32	-0.48	-0.46	0.64	0.00	-0.22	-0.52
6	-0.84	0.00	-0.40	-0.96	-0.18	0.00	-0.22
7	0.88	0.72	0.82	0.52	-0.84	0.04	0.00

## 2.10 Complete, empty and indeterminate digraphs

Let us finally mention some special universal classes of digraphs that are readily available in the `digraphs` module<sup>4</sup>, like:

<sup>4</sup> See Bisdorff, 2021

- the `CompleteDigraph`,
- the `EmptyDigraph` and
- the `IndeterminateDigraph` class,

who put all characteristic values respectively to the *maximum*, the *minimum* or the *median* indeterminate characteristic value.

**Listing 2.11** Complete, empty and indeterminate digraphs

```

1  >>> from digraphs import CompleteDigraph,EmptyDigraph,\ 
2  ...           IndeterminateDigraph
3  >>> # the empty digraph
4  >>> e = EmptyDigraph(order=5)
5  >>> e.showRelationTable()
6  * ---- Relation Table ----
7  S | '1' '2' '3' '4' '5'
8  ---- | -----
9  '1' | -1.00 -1.00 -1.00 -1.00 -1.00
10 '2' | -1.00 -1.00 -1.00 -1.00 -1.00
11 '3' | -1.00 -1.00 -1.00 -1.00 -1.00
12 '4' | -1.00 -1.00 -1.00 -1.00 -1.00
13 '5' | -1.00 -1.00 -1.00 -1.00 -1.00
14 >>> e.showNeighborhoods()
15 Neighborhoods:
16     Gamma :
17     '1': in => set(), out => set()
18     '2': in => set(), out => set()
19     '5': in => set(), out => set()
20     '3': in => set(), out => set()
21     '4': in => set(), out => set()
22     Not Gamma :
23     '1': in => {'2','4','5','3'}, out => {'2','4','5','3'}
24     '2': in => {'1','4','5','3'}, out => {'1','4','5','3'}
25     '5': in => {'1','2','4','3'}, out => {'1','2','4','3'}
26     '3': in => {'1','2','4','5'}, out => {'1','2','4','5'}
27     '4': in => {'1','2','5','3'}, out => {'1','2','5','3'}
28 >>> # the indeterminate digraph
29 >>> i = IndeterminateDigraph()
30 * ---- Relation Table ----
31 S | '1' '2' '3' '4' '5'
32 ---- | -----
33 '1' | 0.00 0.00 0.00 0.00 0.00
34 '2' | 0.00 0.00 0.00 0.00 0.00
35 '3' | 0.00 0.00 0.00 0.00 0.00
36 '4' | 0.00 0.00 0.00 0.00 0.00
37 '5' | 0.00 0.00 0.00 0.00 0.00
38 >>> i.showNeighborhoods()
39 Neighborhoods:
40     Gamma :
41     '1': in => set(), out => set()
42     '2': in => set(), out => set()
43     '5': in => set(), out => set()
44     '3': in => set(), out => set()
45     '4': in => set(), out => set()
46     Not Gamma :
```

```
47 '1': in => set(), out => set()
48 '2': in => set(), out => set()
49 '5': in => set(), out => set()
50 '3': in => set(), out => set()
51 '4': in => set(), out => set()
```

Mind the subtle difference between the neighbourhoods of an *empty* and the neighbourhoods of an *indeterminate* digraph instance. In the first kind, the neighbourhoods are known to be completely *empty* (see Listing 2.11 Lines 22-27) whereas, in the latter, *nothing is known* about the actual neighbourhoods of the nodes (see Lines 46-51). These two cases illustrate why in the case of *bipolar-valued* digraphs, we may sometimes need both a `gamma` **and** a `notGamma` attribute.

In the following Chapter 3 we introduce the main formal object of this book, namely *bipolar-valued outranking* digraphs.

## References

- Bisdorff R (2004) On a natural fuzzification of boolean logic. In: Klement EP, Pap E (eds) Linz Seminar on Fuzzy Set Theory, Mathematics of Fuzzy Systems, Bildungszentrum St. Magdalena, Linz (Austria), pp 20–26, URL <http://hdl.handle.net/10993/23721> 21
- Bisdorff R (2021) Documentation of the Digraph3 collection of Python modules for Algorithmic Decision Theory. URL <https://digraph3.readthedocs.io/en/latest/> 19, 29
- Roy B (1959) Transitivité et connexité. C R Acad Sci Paris 249:216–218 27
- Warshall S (1962) A Theorem on Boolean Matrices. J ACM 9:11–12 27

## Chapter 3

# Working with outranking digraphs

*“The rule for the combination of independent concurrent arguments takes a very simple form when expressed in terms of the intensity of belief ... It is this: Take the sum of all the feelings of belief which would be produced separately by all the arguments pro, subtract from that the similar sum for arguments con, and the remainder is the feeling of belief which ought to have the whole. This is a proceeding which men often resort to, under the name of balancing reasons.”* – C.S. Peirce, The probability of induction (1878)

**Abstract** In this chapter, we introduce the main formal object of this book, namely the bipolar-valued outranking digraph. With a randomly generated multiple criteria performance tableau, we construct the corresponding bipolar-valued outranking relation from pairwise comparisons. The resulting bipolar-valued outranking characteristics may be recoded. Finally, the codual outranking digraph gives us the associated strict outranking relation.

### 3.1 The hybrid outranking digraph model

In the `outrankingDigraphs` module, the `BipolarOutrankingDigraph` class provides our standard *outranking digraph* constructor. Such an instance represents a *hybrid* object of both the `PerformanceTableau` type **and** the `OutrankingDigraph` type.

A given `BipolarOutrankingDigraph` object contains hence always at least the following attributes:

1. `actions`: An ordered dictionary describing the potential decision actions or alternatives with `name` and `comment` attributes,
2. `objectives`: A possibly empty ordered dictionary of decision objectives with `name` and `comment` attributes, describing the multiple preference dimensions involved in the decision problem,

3. **criteria**: An ordered dictionary of performance criteria, i.e. *preferentially independent* and *non-redundant* decimal-valued functions used for measuring the performance of each potential decision action with respect to a decision objective,
4. **evaluation**: A double dictionary gathering performance grades for each decision action or alternative on each criterion function.
5. **valuationdomain**: A dictionary with three entries: the minimum ( $-1.0$ , *certainly outranked*), the median ( $0.0$ , *indeterminate*) and the maximum characteristic value ( $+1.0$ , *certainly outranking*),
6. **relation**: A double dictionary defined on the Cartesian product of the set of decision alternatives capturing the credibility of the pairwise *outranking situation* computed on the basis of the performance differences observed between couples of decision alternatives on the given family of criteria functions.

Let us consider, for instance, a random bipolar-valued outranking digraph with seven decision actions denoted  $a_1, a_2, \dots, a_7$ . We need therefore, first, to generate a corresponding random performance tableaux.

**Listing 3.1** Generating a random performance tableau.

```

1 >>> from perfTabs import RandomPerformanceTableau
2 >>> pt = RandomPerformanceTableau(numberOfActions=7, \
3 ...                                     seed=100)
4 >>> pt
5      *----- PerformanceTableau instance description -----*
6      Instance class      : RandomPerformanceTableau
7      Seed                 : 100
8      Instance name       : randomperftab
9      Actions              : 7
10     Criteria             : 7
11     NaN proportion (%) : 6.1
12 >>> pt.showActions()
13      *----- show digraphs actions -----*
14      key: a1
15      name:      action 1
16      comment:   RandomPerformanceTableau() generated.
17      key: a2
18      name:      action 2
19      comment:   RandomPerformanceTableau() generated.
20      ...
21      ...
22      key: a7
23      name:      action 7
24      comment:   RandomPerformanceTableau() generated.

```

In this example we, furthermore, consider a family of seven *equisignificant* cardinal *criteria functions*  $g_1, g_2, \dots, g_7$ , measuring the performance of each alternative on a rational scale from 0.0 (worst) to 100.00 (best). In order to capture the grading procedure's potential *uncertainty* and *imprecision*, each criterion function  $g_1$  to  $g_7$  admits three performance *discrimination thresholds* of 2.5, 5.0 and 80.0 pts for warranting respectively any *indifference*, *preference* or *considerable performance difference* situation.

**Listing 3.2** Inspecting the performance criteria.

```

1 >>> pt.showCriteria()
2     *---- criteria ----*
3     g1 'RandomPerformanceTableau() instance'
4         Scale = [0.0, 100.0]
5         Weight = 1.0
6         Threshold ind : 2.50 + 0.00x ; percentile: 4.76
7         Threshold pref : 5.00 + 0.00x ; percentile: 9.52
8         Threshold veto : 80.00 + 0.00x ; percentile: 95.24
9     g2 'RandomPerformanceTableau() instance'
10        Scale = [0.0, 100.0]
11        Weight = 1.0
12        Threshold ind : 2.50 + 0.00x ; percentile: 6.67
13        Threshold pref : 5.00 + 0.00x ; percentile: 6.67
14        Threshold veto : 80.00 + 0.00x ; percentile: 100.00
15        ...
16        ...
17     g7 'RandomPerformanceTableau() instance'
18        Scale = [0.0, 100.0]
19        Weight = 1.0
20        Threshold ind : 2.50 + 0.00x ; percentile: 0.00
21        Threshold pref : 5.00 + 0.00x ; percentile: 4.76
22        Threshold veto : 80.00 + 0.00x ; percentile: 100.00

```

On criteria function `g1` (see Lines Listing 3.2 6-8 above) we observe, for instance, about 5% of *indifference* situations, about 90% of *preference* situations and about 5% of *considerable* performance difference situations.

The individual *performance evaluation* of all decision alternative on each criterion are gathered in a *performance table*.

**Listing 3.3** Inspecting the performance table

```

1 >>> pt.showPerformanceTableau()
2     *---- performance tableau ----*
3     criteria | 'a1'  'a2'  'a3'  'a4'  'a5'  'a6'  'a7'
4     -----|-----
5     'g1'    | 15.2  44.5  57.9  58.0  24.2  29.1  96.6
6     'g2'    | 82.3   43.9   NA    35.8  29.1  34.8  62.2
7     'g3'    | 44.2   19.1   27.7  41.5  22.4  21.5  56.9
8     'g4'    | 46.4   16.2   21.5  51.2  77.0  39.4  32.1
9     'g5'    | 47.7   14.8   79.7  67.5  NA    90.7  80.2
10    'g6'   | 69.6   45.5   22.0  33.8  31.8  NA    48.8
11    'g7'   | 82.9   41.7   12.8  21.9  75.7  15.4  6.0

```

It is noteworthy to mention the three *missing data* (NA) cases: action `a3` is missing, for instance, a grade on criterion `g2` (see Listing 3.3 Line 6 above).

## 3.2 The bipolar-valued outranking digraph

Given the previous random performance tableau `pt`, the `BipolarOutrankingDigraph` constructor computes the corresponding *bipolar-valued outranking digraph*.

**Listing 3.4** Example of random bipolar-valued outranking digraph.

```

1 >>> from outrankingDigraphs import \
2     ...                                BipolarOutrankingDigraph
3 >>> odg = BipolarOutrankingDigraph(pt)
4 >>> odg
5     *----- Object instance description -----*
6     Instance class      : BipolarOutrankingDigraph
7     Instance name       : rel_randomperftab
8     Actions             : 7
9     Criteria            : 7
10    Size                : 20
11    Determinateness (%) : 63.27
12    Valuation domain    : [-1.00;1.00]
13    Attributes          : [
14        'name', 'actions',
15        'criteria', 'evaluation', 'NA',
16        'valuationdomain', 'relation',
17        'order', 'gamma', 'notGamma', ...
18    ]

```

The resulting digraph contains 20 positive (valid) outranking relations. And, the mean majority criteria significance support of all the pairwise outranking situations is 63.3% (see Listing 3.4 Lines 9-10).

We may inspect the complete  $[-1.0, +1.0]$ -valued adjacency table as follows:

**Listing 3.5** Inspecting the valued adjacency table.

```

1 >>> odg.showRelationTable()
2     *----- Relation Table -----
3     r(x,y) |  'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'
4     -----|-----
5     'a1' | +1.00  +0.71  +0.29  +0.29  +0.29  +0.29  +0.00
6     'a2' | -0.71  +1.00  -0.29  -0.14  +0.14  +0.29  -0.57
7     'a3' | -0.29  +0.29  +1.00  -0.29  -0.14  +0.00  -0.29
8     'a4' | +0.00  +0.14  +0.57  +1.00  +0.29  +0.57  -0.43
9     'a5' | -0.29  +0.00  +0.14  +0.00  +1.00  +0.29  -0.29
10    'a6' | -0.29  +0.00  +0.14  -0.29  +0.14  +1.00  +0.00
11    'a7' | +0.00  +0.71  +0.57  +0.43  +0.29  +0.00  +1.00
12    Valuation domain: [-1.0; 1.0]

```

The `BipolarOutrankingDigraph` class constructor computes from the given performance tableau `pt` the characteristic value  $r(x \succsim y)$  of a *pairwise outranking relation*  $x \succsim y$  in a default *valuation domain*  $[-1.0, +1.0]$  with the *median* value 0.0 acting as *indeterminate* characteristic value<sup>1</sup>.

---

<sup>1</sup> See Definition 4.1 (Bisdorff, 2013)

**Definition 3.1 (Semantics of the bipolar-valued characteristic function  $r$ ).** The semantics of  $r(x \succsim y)$  are the following:

- When  $r(x \succsim y) > 0.0$ , it is more *True* than *False* that  $x$  outranks  $y$ , i.e. alternative  $x$  is at least as well performing than alternative  $y$  on a weighted majority of criteria **and** there is no considerable negative performance difference observed in disfavour of  $x$ ,
- When  $r(x \succsim y) < 0.0$ , it is more *False* than *True* that  $x$  outranks  $y$ , i.e. alternative  $x$  is **not** at least as well performing on a weighted majority of criteria than alternative  $y$  **and** there is no considerable positive performance difference observed in favour of  $x$ ,
- When  $r(x \succsim y) = 0.0$ , it is **indeterminate** whether  $x$  outranks  $y$  or not.

### 3.3 Pairwise comparisons

From above given semantics, we notice in Listing 3.5 Line 5 above that  $a1$  outranks  $a2$  ( $r(a1 \succsim a2) > 0.0$ ), but not  $a7$  ( $r(a1 \succsim a7) = 0.0$ ). In order to comprehend the characteristic values shown in the relation table above, we may for instance inspect the details of the pairwise multiple criteria comparison between alternatives  $a1$  and  $a2$ .

**Listing 3.6** Inspecting a pairwise multiple criteria comparison

```

1 >>> odg.showPairwiseComparison('a1','a2')
2 *----- pairwise comparison -----
3 Comparing actions : (a1, a2)
4 crit. wght. g(x) g(y) diff | ind pref r()
5 -----
6 g1 1.00 15.17 44.51 -29.34 | 2.50 5.00 -1.00
7 g2 1.00 82.29 43.90 +38.39 | 2.50 5.00 +1.00
8 g3 1.00 44.23 19.10 +25.13 | 2.50 5.00 +1.00
9 g4 1.00 46.37 16.22 +30.15 | 2.50 5.00 +1.00
10 g5 1.00 47.67 14.81 +32.86 | 2.50 5.00 +1.00
11 g6 1.00 69.62 45.49 +24.13 | 2.50 5.00 +1.00
12 g7 1.00 82.88 41.66 +41.22 | 2.50 5.00 +1.00
13 -----
14 Valuation in range: -7.00 to +7.00; r(x,y): +5/7 = +0.71

```

The outranking characteristic value  $r(a1 \succsim a2)$  represents the *majority margin* resulting from the difference between the significance weights of the criteria in favour and the significance weights of the criteria in disfavour of the statement that alternative  $a1$  is “*at least as well performing as*” alternative  $a2$ . No considerable performance difference being observed, no polarising situation is triggered in this pairwise comparison.

Such a situation is, however, observed when we pairwise compare the performances of alternatives  $a1$  and  $a7$ .

**Listing 3.7** Pairwise comparison with considerable performance difference

```

1 >>> odg.showPairwiseOutrankings('a1','a7')

```

```

2  *----- pairwise comparison -----*
3  Comparing actions : (a1, a7)
4  crit. wght. g(a1) g(a7) diff | ind pref r() | v veto
5  -----
6  g1 1.00 15.17 96.58 -81.41 | 2.50 5.00 -1.00 | 80.00 -1.00
7  g2 1.00 82.29 62.22 +20.07 | 2.50 5.00 +1.00 |
8  g3 1.00 44.23 56.90 -12.67 | 2.50 5.00 -1.00 |
9  g4 1.00 46.37 32.06 +14.31 | 2.50 5.00 +1.00 |
10 g5 1.00 47.67 80.16 -32.49 | 2.50 5.00 -1.00 |
11 g6 1.00 69.62 48.80 +20.82 | 2.50 5.00 +1.00 |
12 g7 1.00 82.88 6.05 +76.83 | 2.50 5.00 +1.00 |
13
14 Valuation in range: -7.00 to +7.00; r(x,y)= +1/7 => 0.0
15 *----- pairwise comparison -----*
16 Comparing actions : (a1, a7)
17 crit. wght. g(a7) g(a1) diff | ind pref r() | v veto
18 -----
19 g1 1.00 96.58 15.17 +81.41 | 2.50 5.00 +1.00 | 80.00 +1.00
20 g2 1.00 62.22 82.29 -20.07 | 2.50 5.00 -1.00 |
21 g3 1.00 56.90 44.23 +12.67 | 2.50 5.00 +1.00 |
22 g4 1.00 32.06 46.37 -14.31 | 2.50 5.00 -1.00 |
23 g5 1.00 80.16 47.67 +32.49 | 2.50 5.00 +1.00 |
24 g6 1.00 48.80 69.62 -20.82 | 2.50 5.00 -1.00 |
25 g7 1.00 6.05 82.88 -76.83 | 2.50 5.00 -1.00 |
26
27 Valuation in range: -7.00 to +7.00; r(x,y)= -1/7 => 0.0

```

This time, we observe a  $(1/7 + 1)/2 = 57.1\%$  majority of criteria significance warranting an “*as well as performing*” situation between alternative  $a_1$  and alternative  $a_7$ . Yet, we also observe a considerable *negative* performance difference on criterion  $g_1$  (see Listing 3.7 Line 6 above and first row in the relation table shown in Listing 3.5). Both contradictory facts trigger eventually in Line 14 an *indeterminate* outranking situation.

The inverse polarisation effect appears when considering in Lines 19-25 the converse performance differences between alternative  $a_7$  and alternative  $a_1$ . The considerable better performing situation on criterion  $g_1$  makes doubtful the otherwise “*not as well performing*” situation (see Line 19 and 27).

Notice that the occurrence in a pairwise comparison of conjointly considerable positive and negative performance differences will also trigger an indeterminate outranking situation. When observing at the same time a positive (resp. negative) “*at least as well performing*” situation and one or more considerable positive (resp. negative) performance difference, the outranking situation gets validated (resp. invalidated) for certain (Bisdorff, 2013).

### 3.4 Recoding the digraph valuation

All outranking digraphs, being of root type `Digraph`, inherit the methods available under this latter class. The characteristic valuation domain of a digraph may, for instance, be recoded with the `recodeValuation()` method to the *integer* range  $[-7, +7]$ , i.e. plus or minus the total significance weights of the family of criteria considered in this example instance `odg`.

**Listing 3.8** Recoding the digraph valuation

```

1 >>> odg.recodeValuation(-37,+37)
2 >>> odg.valuationdomain['hasIntegerValuation'] = True
3 >>> Digraph.showRelationTable(odg,ReflexiveTerms=False)
4   * ----- Relation Table -----
5   r(x,y) | 'a1' 'a2' 'a3' 'a4' 'a5' 'a6' 'a7'
6   -----|-----
7   'a1' | 0 +5 +2 +2 +2 +2 0
8   'a2' | -5 0 -1 -1 +1 +2 -4
9   'a3' | -1 +2 0 -1 -1 0 -1
10  'a4' | 0 +1 +4 0 +2 +4 -3
11  'a5' | -1 0 +1 0 0 +2 -1
12  'a6' | -1 0 +1 -1 +1 0 0
13  'a7' | 0 +5 +4 +3 +2 0 0
14 Valuation domain: [-7;+7]

```

Notice in Listing 3.8 Line 3 above, that the reflexive self comparison characteristic  $r(x \succsim x)$  may be ignored by setting the `ReflexiveTerms` flag to `False`. They are in this case set to the *indeterminate* valuation value 0. Mind that the trivial reflexive terms of outranking relations may be ignored in some of the DIGRAPH3 resources.

### 3.5 The strict outranking digraph

From theory we know that a bipolar-valued outranking digraph is *weakly complete*, i.e. if  $r(x,y) < 0.0$  then  $r(y,x) \geq 0.0$  (Bisdorff, 2013, 2020). From this property follows that a bipolar-valued outranking relation verifies the *coduality principle*: the *dual*<sup>2</sup>–strict negation– $(-)$  of the *converse* (–inverse– $\sim$ ) of the outranking relation corresponds to its *strict outranking* part.

We may visualize the *codual (strict)* outranking digraph with a graphviz drawing<sup>3</sup>.

```

1 >>> cdodg = -(~odg)
2 >>> cdodg.exportGraphViz('codualOdg')
3   *----- exporting a dot file for GraphViz tools -----
4   Exporting to codualOdg.dot
5   dot -Grankdir=BT -Tpng codualOdg.dot -o codualOdg.png

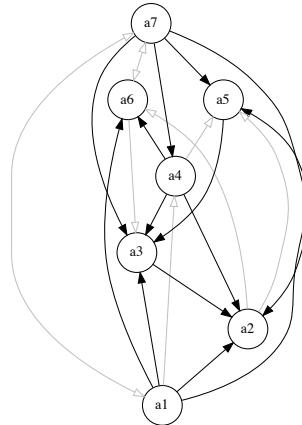
```

Many more tools for exploiting bipolar-valued outranking digraphs are available in the DIGRAPH3 resources (Bisdorff, 2021).

<sup>2</sup> Not to be confused with the dual graph of a plane graph  $g$  that has a vertex for each face of  $g$ . Here we mean the *less than* (strict converse) relation corresponding to a *greater or equal* relation, or the *less than or equal* relation corresponding to a (strict) *better than* relation.

<sup>3</sup> The `exportGraphViz()` method is depending on drawing tools from the graphviz software (<https://graphviz.org/>). On Linux Ubuntu or Debian you may try `sudo apt-get install graphviz` to install them. There are ready `dmg` installers for Mac OSX.

**Fig. 3.1** The codual outranking digraph. It becomes readily clear now from the picture above that both alternatives  $a_1$  and  $a_7$  are *not outranked* by any other alternatives. Hence,  $a_1$  and  $a_7$  appear as *weak* CONDORCET winners and may be recommended as potential *best* decision alternatives in this illustrative preference modelling example.



In the next, methodological Part of the book, we present and discuss operational decision methods and tools, like recommending a best choice, determining the winner of an election, computing linear rankings or quantile ratings with multiple incommensurable criteria. We also introduce the main performance tableau models we are going to use in the sequel.

## Notes

The seminal work on outranking digraphs goes back to the seventies and eighties of the twentieth century when Bernard Roy joined the just starting University Paris-Dauphine and founded there the '*Laboratoire d'Analyse et de Modélisation de Systèmes pour l'Aide à la Décision*' (LAMSADE). The LAMSADE became the major site in the development of the outranking approach to multiple criteria decision aid. (Roy and Bouyssou, 1993).

The ongoing success of the original *outranking* concept stems from the fact that it is rooted in a sound pragmatism. The multiple criteria performance tableau, necessarily associated with a given outranking digraph, is indeed convincingly objective and meaningful (Roy, 1991). And, ideas from social choice theory gave initially the insight that a pairwise voting mechanism à la CONDORCET could provide an order-statistical tool for aggregating a set of preference points of view into what M. Barbut called the *central* CONDORCET point of view (de Caritat, Marquis de Condorcet (1784) and Barbut (1980)); in fact the median of the multiple preference points of view, at minimal absolute KENDALL's ordinal correlation distance from all individual points of view (see Chapter 16).

Considering thus each performance criterion as a subset of unanimous voters and balancing the votes in favour against considerable counter-performances in disfavour gave eventually rise to the concept of *outranking situation*, a distinctive feature of the Multiple Criteria Decision Aid approach (Bisdorff et al, 2015). A modern definition would be : An alternative  $x$  is said to *outrank* alternative  $y$  when – a *significant majority* of criteria confirm that alternative  $x$  has to be considered as *at least as well performing as* an alternative  $y$  (the *concordance argument*); and – no discordant criterion opens to significant doubt the validity of the previous confirmation by revealing a considerable counter-performance of alternative  $x$  compared to  $y$  (the *discordance argument*).

If the concordance argument was always well received, the discordance argument however, very confused in the beginning (Benayoun et al, 1966), could only be handled in an epistemically correct and logically sound way by using a bipolar-valued epistemic logic (see Definition 4.1, Bisdorff (2013)). The outranking situation had consequently to receive an explicit negative definition: An alternative  $x$  is said to *do not outrank* an alternative  $y$  when – a *significant majority* of criteria confirm that alternative  $x$  has to be considered as *not at least as well performing as* alternative  $y$ ; and – no discordant criterion opens to significant doubt the validity of the previous confirmation by revealing a considerable *better* performance of alternative  $x$  compared to  $y$ .

Furthermore, the initial conjunctive aggregation of the concordance and discordance arguments had to be replaced by a disjunctive epistemic fusion operation, polarising in a logically and epistemically sound way the concordance with the discordance argument. This way, outranking digraphs gain two very useful properties from a measure theoretical perspective. They are *weakly complete* ; incomparability situations are no longer attested by the absence of outranking relations, but instead by epistemic indeterminateness. And they verify the *coduality principle*: the negation of the epistemic “*at least as well performing as*” situation corresponds formally to the strict converse epistemic “*less well performing than*” situation.

## References

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**Part II**  
**Evaluation models and decision algorithms**

The second and main Part of the book presents in eight chapters multiple criteria decision algorithms for selecting a best decision alternative, for ranking the potential decision alternative from best to worst, and for relative or absolute rating with the help of bipolar-valued outranking digraphs. We also show how to edit a new multiple criteria performance tableau and present several models of random performance tableau generators. A last chapter is devoted to HPC ranking of big performance tableaux.

# Chapter 4

## Computing a best choice recommendation

“... *The goal of our research was to design a resolution method ... that is easy to put into practice, that requires as few and reliable hypotheses as possible, and that meets the needs [of the decision maker]...*”

—(Benayoun, Roy, and Sussmann, 1966).

**Abstract** In this chapter we present the RUBIS best choice recommender system. Our approach is illustrated with a best office site selection problem. We show how to explore the given performance tableau and compute the corresponding outranking digraph. After presenting the pragmatic principles that govern our best choice recommendation algorithm we solve the best office site choice problem.

### 4.1 What office site to choose?

A SME, specialized in printing and copy services, has to move into new offices, and its CEO has gathered seven *potential new office sites* (see Tab. 4.1).

**Table 4.1** The potential new office sites

ID	Name	Address	Comment
A	Ave	Avenue de la liberté	High standing city center
B	Bon	Bonnevoie	Industrial environment
C	Ces	Cessange	Residential suburb location
D	Dom	Dommeldange	Industrial suburb environment
E	Bel	Esch-Belval	New and ambitious urbanization far from the city
F	Fen	Fentange	Out in the countryside
G	Gar	Avenue de la Gare	Main city shopping street

Three *decision objectives*, in order of decreasing importance, are guiding the CEO's choice:

1. *maximize* the future turnover of the SME,
2. *minimize* the future yearly costs induced by the moving,
3. *maximize* the new working conditions.

The decision consequences to take into account for evaluating the potential new office sites with respect to each of the three objectives are modelled by the *family of performance criteria*<sup>1</sup> shown in Table 4.2.

**Table 4.2** The family of performance criteria

Objective	ID	Name	Weight	Comment
Yearly costs	C	Costs	45	Annual rent, charges, and cleaning
Future turnover	Pr	Proximity	32	Distance from town center
Future turnover	V	Visibility	26	Circulation of potential customers
Future turnover	St	Standing	23	Image and presentation
Working conditions	W	Space	10	Working space
Working conditions	Cf	Comfort	6	Quality of office equipment
Working conditions	P	Parking	3	Available parking facilities

In Table 4.2 we notice that the *Costs* criterion admits the highest significance (45), followed by the *Future turnover* criteria ( $32 + 26 + 23 = 81$ ). The *Working conditions* criteria are the less significant ( $10 + 6 + 3 = 19$ )<sup>2</sup>. It follows that the CEO considers *maximizing the future turnover* the most important objective (81.0), followed by minimizing the future yearly costs (45.0), and less important, *maximizing working conditions* (19.0).

The evaluation of the seven potential new sites on each performance criterion are gathered in a *performance table* shown in Table 4.3 on the next page.

All criteria, except the *Costs* Criterion, admit for grading a qualitative satisfaction scale from 0% (worst) to 100% (best). We may thus notice in Table 4.3 on the facing page that site A (Ave) is the most expensive, but also 100% satisfying the *Proximity* as well as the *Standing* criterion. Whereas site C (Ces) is the cheapest one; providing however no satisfaction at all on both the *Standing* and the *Working Space* criteria.

Concerning yearly costs, we suppose that the CEO is indifferent up to a performance difference of 1000.00€, and he actually prefers a site when there is at least a positive difference of 2500.00€. The grades observed on the six qualitative criteria (measured in percentages of satisfaction) are very subjective and rather imprecise. The CEO is hence *indifferent* up to a satisfaction difference of 10%, and he claims

<sup>1</sup> See Roy, 2000

<sup>2</sup> These criteria weights were supposedly established with a swing weighing MCDA method (Keeney and Raiffa, 1976).

**Table 4.3** Performance evaluations of the potential office sites

Criterion	A	B	C	D	E	F	G
Costs	35.0K€	17.8K€	6.7K€	14.1K€	34.8K€	18.6K€	12.0K€
Proximity	100	20	80	70	40	0	60
Visibility	60	80	70	50	60	0	100
Standing	100	10	0	30	90	70	20
Working space	75	30	0	55	100	0	50
Working comfort	0	100	10	30	60	80	50
Parking	90	30	100	90	70	0	80

a significant *preference* when the satisfaction difference is at least of 20%. Furthermore, a satisfaction difference of 80% represents for him a *considerably large* performance difference, triggering the case given a *polarisation* of the preferential situations (Bisdorff, 2013).

In view of Table 4.3, what is now the office site we may recommend to the CEO as **best choice**?

## 4.2 The given performance tableau

In file `officeChoice.py`, stored in the `examples` directory of the DIGRAPH3 resources, a corresponding, `PerformanceTableau` object, is provided. We may inspect its actual content with the computing resources provided by the `perfTabs` module .

**Listing 4.1** Inspecting the `officeChoice` performance tableau.

```

1 >>> from perfTabs import PerformanceTableau
2 >>> t = PerformanceTableau('officeChoice')
3 >>> t
4 *----- PerformanceTableau instance description -----*
5   Instance class      : PerformanceTableau
6   Instance name       : officeChoice
7   Actions             : 7
8   Objectives          : 3
9   Criteria            : 7
10  NaN proportion (%) : 0.0
11  Attributes          : ['name', 'actions', 'objectives',
12                           'criteria', 'weightPreorder',
13                           'NA', 'evaluation']
14 >>> t.showPerformanceTableau()
15 *---- performance tableau -----*
16 Criteria| 'C'      'Cf'    'P'    'Pr'    'St'    'V'    'W'
17 Weights | 45.00    6.00    3.00   32.00   23.00   26.00  10.00
18 -----| -----

```

19	'Ave'		-35000.00	0.00	90.00	100.00	100.00	60.00	75.00
20	'Bon'		-17800.00	100.00	30.00	20.00	10.00	80.00	30.00
21	'Ces'		-6700.00	10.00	100.00	80.00	0.00	70.00	0.00
22	'Dom'		-14100.00	30.00	90.00	70.00	30.00	50.00	55.00
23	'Bel'		-34800.00	60.00	70.00	40.00	90.00	60.00	100.00
24	'Fen'		-18600.00	80.00	0.00	0.00	70.00	0.00	0.00
25	'Gar'		-12000.00	50.00	80.00	60.00	20.00	100.00	50.00

We thus recover all the input data shown in Section 4.1 on page 45. Notice the *negative* figures on the Costs criterion indicate a negative preference direction: the *lower* the costs, the *better* it is. We may use the `showCriteria()` method for measuring the actual preference discrimination we observe on each criterion.

**Listing 4.2** Inspecting the performance criteria.

```

1 >>> t.showCriteria(IntegerWeights=True)
2 *---- criteria ----*
3 C 'Costs'
4   Scale = (Decimal('0.00'), Decimal('50000.00'))
5   Weight = 45
6   Threshold ind : 1000.00 + 0.00x ; percentile: 9.5
7   Threshold pref : 2500.00 + 0.00x ; percentile: 14.3
8 Cf 'Comfort'
9   Scale = (Decimal('0.00'), Decimal('100.00'))
10  Weight = 6
11  Threshold ind : 10.00 + 0.00x ; percentile: 9.5
12  Threshold pref : 20.00 + 0.00x ; percentile: 28.6
13  Threshold veto : 80.00 + 0.00x ; percentile: 90.5
14 ...

```

On the *Costs* criterion, 9.5% of the performance differences are considered insignificant and 14.3% below the preference discrimination threshold (see List 4.2 lines 6-7). On the qualitative *Comfort* criterion, we observe again 9.5% of insignificant performance differences (line 11). Due to the imprecision in the subjective grading, we notice here 28.6% of performance differences below the preference discrimination threshold (Line 12). Furthermore,  $100.0 - 90.5 = 9.5\%$  of the performance differences are judged *considerably large* (Line 13) and triggering hence a polarisation of the concerned outranking situations (Bisdorff, 2013). Same information is available for all the other criteria.

A colorful comparison of all the performances is shown in Figure 4.1 on the facing page by the *heatmap* statistics, illustrating the respective quantile class of each performance. As the set of potential alternatives is tiny, we choose here a classification into performance quintiles.

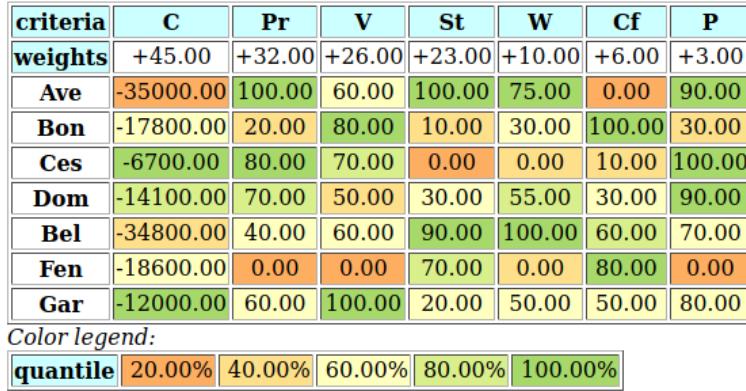
```

1 >>> t.showHTMLPerformanceHeatmap(colorLevels=5, \
2 ...                                         rankingRule=None)

```

Site A (*Ave*) shows extreme and contradictory performances: highest *Costs* and no *Working Comfort* on the one hand, and total satisfaction with respect to *Standing*, *Proximity* and *Parking facilities* on the other hand. Similar, but opposite, situation is given for site C (*Ces*): unsatisfactory *Working Space*, no *Standing* and no *Working Comfort* on the one hand, and lowest *Costs*, best *Proximity* and *Parking facilities*

### Heatmap of Performance Tableau 'officeChoice'



**Fig. 4.1** Unranked heatmap of the office choice performance tableau

on the other hand. Contrary to these contradictory alternatives, we observe two appealing compromise decision alternatives: sites D (*Dom*) and G (*Gar*). Finally, site F (*Fen*) is clearly the less satisfactory alternative of all.

To help now the CEO choosing the best office site, we are going to compute pairwise outranking situations on the set of potential decision alternatives (see [Bisdorff \(2013\)](#)).

### 4.3 Computing the outranking digraph

#### Definition 4.1 (Outranking situation).

For two potential decision alternatives  $x$  and  $y$ :

- “Alternative  $x$  outranks alternative  $y$ ”, denoted  $(x \succ y)$ , is given when there is:
  1. A *majority* of criteria significance concordantly supporting that alternative  $x$  is *at least as well performing as* alternative  $y$ , and
  2. *No considerable* counter-performance observed on any discordant criterion.
- “Alternative  $x$  does not outrank  $y$ ”, denoted  $(x \not\succ y)$ , is given when there is:
  1. A *majority* of criteria concordantly supporting that alternative  $x$  is **not at least as well performing as** alternative  $y$ , and
  2. *No considerable* better performance observed on any discordant criterion.

- Otherwise, the outranking situation between alternatives  $x$  and  $y$  is considered to be *indeterminate*.

The credibility of each pairwise outranking situation, denoted  $r(x \succsim y)$ , is measured in a bipolar significance valuation  $[-1.0, 1.0]$ , where *positive* terms  $r(x \succsim y) > 0.0$  indicate a *validated*, and *negative* terms  $r(x \succsim y) < 0.0$  indicate a *non-validated* outranking situation. The *median* value  $r(x \succsim y) = 0.0$  represents an *indeterminate* situation (see Bisдорff (2004) and Bisдорff (2013)).

For computing such a bipolar-valued binary outranking relation from the given performance tableau  $t$ , we use the `BipolarOutrankingDigraph` constructor from the `outrankingDigraphs` module. The corresponding `showHTMLRelationTable` method shows here the resulting bipolar-valued adjacency matrix in a system browser window (see Fig. 4.2).

**Listing 4.3** Computing a bipolar-valued outranking digraph

```
1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> g = BipolarOutrankingDigraph(t)
3 >>> g.showHTMLRelationTable()
```

**Fig. 4.2** Bipolar-valued adjacency matrix. In the resulting outranking relation we may notice, on the one hand, that Alternative D is *positively outranking* all other potential office sites. On the other hand, alternatives A (the most expensive) and C (the cheapest) are *not outranked* by any other site

## Valued Adjacency Matrix

$r(x \succsim y)$	<b>A</b>	<b>B</b>	<b>C</b>	<b>D</b>	<b>E</b>	<b>F</b>	<b>G</b>
<b>A</b>	-	0.00	1.00	0.30	0.78	0.00	0.00
<b>B</b>	0.00	-	0.00	-0.56	0.00	1.00	-0.60
<b>C</b>	0.00	0.00	-	0.46	0.00	1.00	0.10
<b>D</b>	0.10	0.56	0.02	-	0.46	1.00	0.25
<b>E</b>	0.52	0.00	0.00	-0.10	-	1.00	-0.42
<b>F</b>	0.00	-1.00	-1.00	-1.00	-	-	-1.00
<b>G</b>	0.00	0.92	-0.10	1.00	0.54	1.00	-

Valuation domain: [-1.00; +1.00]

Alternative D gives a CONDORCET winner<sup>3</sup>, whereas alternatives A (the most expensive) and C (the cheapest) give in fact *weak* CONDORCET winners.

```
1 >>> g.computeCondorcetWinners()
2 ['D']
3 >>> g.computeWeakCondorcetWinners()
4 ['A', 'C', 'D']
```

From theory, we know that outranking digraphs are *weakly complete*, i.e. for all  $x$  and  $y$  in  $X$ ,  $r(x \succsim y) < 0.0$  implies that  $r(y \succsim x) \geq 0.0$ . And, they verify the *coduality principle*:  $r(x \not\succsim y) = r(y \not\succsim x)$  (Bisдорff, 2013).

### Definition 4.2 (Strict outranking situation).

For two potential decision alternatives  $x$  and  $y$ :

<sup>3</sup> See Chapter 7 on page 91 on computing the winner of an election.

- “Alternative  $x$  strictly outranks alternative  $y$ ”, denoted  $(x \succ y)$ , is given when there is:
  1. A *majority* of criteria significance concordantly supporting that alternative  $x$  is *better performing than* alternative  $y$ , and
  2. *No considerable* counter-performance observed on any discordant criterion.
- “Alternative  $x$  is not strictly outranking alternative  $y$ ”, denoted  $(x \not\succ y)$ , is given when there is:
  1. A *majority* of criteria concordantly supporting that alternative  $x$  is *not better performing than* alternative  $y$ , and
  2. *No considerable* better performance observed on any discordant criterion.
- Otherwise, the *strict* outranking situation between alternatives  $x$  and  $y$  is considered to be *indeterminate*.

Our best office site choice will now be based on such strict outranking situations.

## 4.4 Designing a best choice recommender system

Traditionally, solving a best-choice problem consists in finding *the* unique best decision alternative. We adopt here instead a modern recommender system’s approach which shows a non empty subset of decision alternatives which contains by construction the potential best alternative(s).

Our five *pragmatic principles* for computing such a *best-choice recommendation* (BCR) are the following:

- P1: *Elimination for well motivated reasons*; each eliminated alternative has to be outranked by at least one alternative in the BCR.
- P2: *Minimal size*; the BCR must be as limited in cardinality as possible.
- P3: *Efficient and informative*; The BCR must not contain a self-contained sub-recommendation.
- P4: *Effectively better*; the BCR must not be ambiguous in the sense that it may not be both a best choice as well as a worst choice recommendation.
- P5: *Maximally determined*; the BCR is, of all potential best-choice recommendations, the most determined one in the sense of the characteristics of the bipolar-valued outranking relation.

Let  $X$  be the set of potential decision alternatives. Let  $Y$  be a non empty subset of  $X$ , called a *choice* in the strict outranking digraph  $G(X, r(\succ))$ . A *weakly independent* and *outranking* (resp. *outranked*) choice is called an *outranking* (resp. *outranked*) strict *prekernel* (see Chapter 17 on page 237 on computing kernels in digraphs).

We may now qualify a BCR  $Y$  in the following terms:

- $Y$  is called *outranking* (resp. *outranked*) when for all not selected alternative  $x$  there exists an alternative  $y \in X$  retained such that  $r(y \succsim x) > 0.0$  (resp.  $r(x \precsim y) > 0.0$ ). Such a choice verifies principle P1.
- $Y$  is called *weakly independent* when for all  $x \neq y$  in  $Y$ , we observe  $r(x \succsim y) \leq 0.0$ . Such a choice verifies principles P3 (*internal stability*).
- $Y$  is conjointly a *strictly outranking* (resp. *strictly outranked*) **and** *weakly independent* choice. This choice now verifies principles P1, P2, P3 and P4.
- To finally verify principle P5, we recommend among all potential strict outranking (resp. strict outranked) prekernels, the *most determined* one, i.e. the one supported with the highest criteria significance. And in this prekernel we eventually recommend the alternatives that are included in this prekernel with highest criteria significance.

Following this best choice algorithm, our potential best choice recommendations are determined by the outranking *prekernels* –*weakly independent* and *outranking* choices– of the *strict outranking* digraph (see [Bisdorff et al \(2008\)](#)). If not reduced to a singleton, the actual *best choice recommendation* will have to be refined in a later stage of the decision process.

Mind that a given outranking digraph  $g$  may not always admit any outranking (resp. outranked) prekernels. This is the case when  $g$  contains chordless odd circuits. The case given, we previously need to break open with the `BrokenCocsDigraph` class all these chordless odd circuits at their weakest link.

```

1 >>> from digraphs import BrokenCocsDigraph
2 >>> bcg = BrokenCocsDigraph(g)
3 >>> bcg.brokenLinks
4   set()
```

Luckily our outranking digraph  $g$  here does not show any chordless outranking circuits.

## 4.5 Computing a best choice recommendation

To compute now our RUBIS best choice recommendation we work on the *codual* outranking digraph, i.e. the converse ( $\sim$ ) of the dual ( $\neg$ )<sup>4</sup>, of the outranking digraph instance  $g$  ([Bisdorff, 2013](#)).<sup>5</sup>:

**Listing 4.4** Computing the best choice recommendation

```

1 >>> g.showBestChoiceRecommendation(\n2 ...           CoDual=True, # default setting\n3 ...           ChoiceVector = True)\n4   * --- Best and worst choice recommendation(s) ---*
```

<sup>4</sup> Not to be confused with the dual graph of a plane graph  $g$  that has a vertex for each face of  $g$ . Here we mean the *less than* (strict converse) relation corresponding to a *greater or equal* relation, or the *less than or equal* relation corresponding to a (strict) *better than* relation.

<sup>5</sup> See Section 2.6 on page 25

```

5      (in decreasing order of determinateness)
6      Credibility domain: [-1.00,1.00]
7      === >> potential best choice(s)
8      * choice           : ['A', 'C', 'D']
9      independence      : 0.00
10     dominance         : 0.10
11     absorbency        : 0.00
12     covering (%)      : 41.67
13     determinateness (%) : 50.59
14     - characteristic vector = {
15         'D': +0.02, 'G': 0.00, 'C': 0.00, 'A': 0.00,
16         'F': -0.02, 'E': -0.02, 'B': -0.02, }
17     === >> potential worst choice(s)
18     * choice           : ['A', 'F']
19     independence      : 0.00
20     dominance         : -0.52
21     absorbency        : 1.00
22     covered (%)       : 50.00
23     determinateness (%) : 50.00
24     - characteristic vector = {
25         'G': 0.00, 'F': 0.00, 'E': 0.00, 'D': 0.00,
26         'C': 0.00, 'B': 0.00, 'A': 0.00, }

```

It is interesting to notice in Listing 4.4 on the preceding page (Line 8) that our *best choice recommendation* consists actually in the previously mentioned set of weak CONDORCET winners: A, C and D (see Fig. 4.2 on page 50). In the corresponding characteristic vector (see Lines 15-16 and Section 17.6 on page 253), representing the bipolar credibility degree with which each alternative may indeed be included in, or excluded from this recommendation, we find that alternative D is the only positively validated one, whereas both extreme alternatives –A (the most expensive) and C (the cheapest)– stay in an indeterminate situation (see Bisdorff et al (2006); Bisdorff (2006)). They may *be or not be* potential best choices. Notice furthermore that compromise alternative G, while not actually being included in this strictly outranking prekernel, shows as well an indeterminate situation with respect to *being or not being* recommended as potential best choice. Alternatives B, E and F are all positively excluded from this best choice recommendation.

To inspect why alternative D is the only positive best choice recommendation, we shall compare now the performances of alternatives D and G in a pairwise perspective.

**Listing 4.5** Inspecting pairwise comparison between alternatives G and D

```

1 >>> g.showPairwiseComparison('G','D')
2 *----- pairwise comparison -----*
3 Comparing actions : ('G', 'D')
4 crit. wght.   g(x)      g(y)      diff. |   ind.      pref.   concord.
5 =====
6 Costs 45.00 -12000.00 -14100.00 +2100.00 | 1000.00 2500.00  +45.00
7 Comf.  6.00      50.00     30.00     +20.00 |   10.00  20.00  +6.00
8 Park.  3.00      80.00     90.00    -10.00 |   10.00  20.00  +3.00
9 Prox. 32.00     60.00     70.00    -10.00 |   10.00  20.00  +32.00
10 Stdg. 23.00     20.00     30.00    -10.00 |   10.00  20.00  +23.00
11 Visi. 26.00     100.00    50.00    +50.00 |   10.00  20.00  +26.00
12 Spac. 10.00     50.00     55.00    -5.00 |   10.00  20.00  +10.00
13 =====

```

```
14 Valuation in range: -145.00 to +145.00; global concordance: +145.00
```

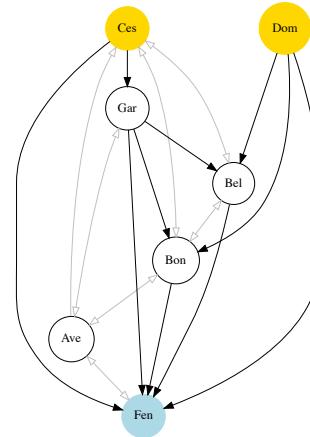
In Listing 4.5 on the preceding page, we notice that, with the given preference discrimination thresholds, alternative G is actually *certainly at least as well performing as* alternative D:  $r(G \succsim D) = +145/145 = +1.0$ .

Yet, we must as well acknowledge in Listing 4.6, that the cheapest alternative C is in fact *strictly outranking* alternative G:  $r(C \succsim G) = +15/145 > 0.0$ , and  $r(G \succsim C) = -15/145 < 0.0$ .

**Listing 4.6** Inspecting pairwise comparison between alternatives C and G

```
1 >>> g.showPairwiseComparison('C', 'G')
2 *----- pairwise comparison -----*
3 Comparing actions : (C,G)/(G,C)
4 crit. wght. g(x) g(y) diff. | ind. pref. (C,G) / (G,C)
5 =====
6 'C' 45.00 -6700.00 -12000.00 +5300.00 | 1000.00 2500.00 +45.00/-45.00
7 'Cf' 6.00 10.00 50.00 -40.00 | 10.00 20.00 -6.00/+6.00
8 'P' 3.00 100.00 80.00 +20.00 | 10.00 20.00 +3.00/-3.00
9 'Pr' 32.00 80.00 60.00 +20.00 | 10.00 20.00 +32.00/-32.00
10 'St' 23.00 0.00 20.00 -20.00 | 10.00 20.00 -23.00/+23.00
11 'V' 26.00 70.00 100.00 -30.00 | 10.00 20.00 -26.00/+26.00
12 'W' 10.00 0.00 50.00 -50.00 | 10.00 20.00 -10.00/+10.00
13
14 Valuation in range: -145 to +145; r(C >= G) / r(G >= c) : +15.00/-15.00
```

**Fig. 4.3** Best office choice recommendation from strict outranking digraph. Notice that site A (Ave) (the most expensive) is appearing *incomparable* to all the other alternatives



Digraph3 (graphviz), R. Bischoff, 2020

We may finally notice in Listing 4.4 on page 52 (Lines 16 and 19) that both alternatives A and F are reported as potential worst choice recommendation. Yet, this potential worst choice recommendation appears to be globally indeterminate. This confirms the *incomparability* status of alternative A (see Fig. 4.3).

```
1 >>> gcd = ~(-g) # codual of g
2 >>> gcd.exportGraphViz(fileName='bestOfficeChoice', \
```

```

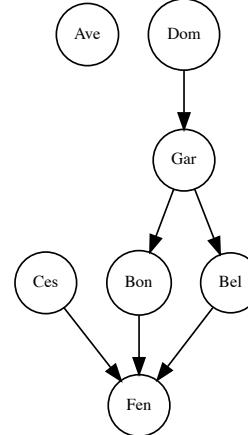
3 ...                                bestChoice=['C','D'], \
4 ...                                worstChoice=['F'])
5 *---- exporting a dot file for GraphViz tools -----*
6 Exporting to bestOfficeChoice.dot
7 dot -Grankdir=BT -Tpng bestOfficeChoice.dot \
8 -o bestOfficeChoice.png

```

## 4.6 Weakly ordering the outranking digraph

To get a global insight in the overall strict outranking situations, we may use the `RankingByChoosingDigraph` constructor imported from the `transitive-Digraphs` module for computing a *ranking-by-choosing* result from the codual, i.e. the strict outranking digraph instance `gcd` (see above). If the computing node supports multiple processor cores, *first* and *last* choosing iterations may be run in parallel (see Line 3 in List. 4.7).

**Fig. 4.4** In this *ranking-by-choosing* method, where we operate the *epistemic fusion* of iterated (strict) first and last choices, compromise alternative `D` is now ranked before compromise alternative `G`. The overall partial ordering result shows again the important fact that the most expensive site `A`, and the cheapest site `C`, due to their contradictory performances, appear both *incomparable* with most of the other alternatives.



Digraph3 (graphviz)  
R. Bisdorff, 2020

**Listing 4.7** Ranking-by-choosing the outranking digraph

```

1 >>> from transitiveDigraphs import \
2 ...                               RankingByChoosingDigraph
3 >>> rbc = RankingByChoosingDigraph(gcd)
4 Threading ... # multiprocessing if 2 cores are available

```

```

5  Exiting computing threads
6 >>> rbc.showRankingByChoosing()
7  Ranking by Choosing and Rejecting
8      1st ranked ['D']
9      2nd ranked ['C', 'G']
10     2nd last ranked ['B', 'C', 'E']
11     1st last ranked ['A', 'F']
12 >>> rbc.exportGraphViz('officeChoiceRanking')
13 *----- exporting a dot file for GraphViz tools -----*
14 Exporting to officeChoiceRanking.dot
15 dot -Grankdir=TB -Tpng officeChoiceRanking.dot \
16                               -o officeChoiceRanking.png

```

The best choice recommendation appears hence depending on the very importance the CEO is attaching to each of the three decision objectives he is considering. In the given setting here, where he considers that *maximizing the future turnover* is the most important objective followed by *minimizing the Costs* and, less important, *maximizing the working conditions*, site D represents actually the *best compromise*. However, if *Costs* do not play much a role, it would be perhaps better to decide to move to the most advantageous site A; or if, on the contrary, *Costs* do matter a lot, moving to the cheapest alternative C could definitely represent a more convincing recommendation.

It might be worth editing the criteria significance weights in the `officeChoice.py` data file in such a way that:

- All three decision objectives are considered *equally important*, and
- All criteria under each objective are considered *equi-significant*.

What will become the best choice recommendation under this working hypothesis?

[6](#)

In the next Chapter [5 on page 61](#) we precisely show how to edit a new performance tableau from a given template file.

## Notes

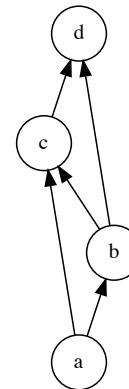
Following a seminar presentation in 2005 at the LAMSADE<sup>7</sup>, where the author promoted the use of kernels of the outranking digraph as suitable candidates for delivering best choice recommendations ([Bisdorff, 2005](#)), a critical discussion started about the methodological requirement for a convincing best choice recommendation to be internally stable (pragmatic principle P3). Denis Bouyssou illustrated his doubts with the example of supposed outranking digraph shown in Figure [4.5 on the facing page](#).

---

<sup>6</sup> See also the notes of Lecture 7 from the MICS Algorithmic Decision Theory course ([Bisdorff, 2020](#)).

<sup>7</sup> Laboratoires d'Analyse et de Modélisation de Systèmes d'Aide à la Décision, Université Paris-Dauphine, UMR 7243 CNRS

**Fig. 4.5** The internal stability of a best choice recommendation in question. The only kernel of this digraph is the pair  $\{a,d\}$ ; yet, it is an ambiguous recommendation, as  $\{a,d\}$  is conjointly an outranking and outranked choice. If the instability of the best choice recommendation is, however, not considered a problem then the choice  $\{a,b\}$  shows the most convincing strict outranking quality and could be considered in priority for recommendation as potential best choice candidates.



*Digraph3 (graphviz), R. Bisdorff, 2020*

His commentary was the following: Adding alternative  $d$  to the set of potential best choice candidates is not convincing as there exists in the given digraph the node  $b$ , which is better than  $d$ . The argument that the incomparability between  $a$  and  $d$  should favour  $d$  as potential best choice is interesting but another hypothesis could be that  $b$  perhaps outranks  $a$ . In this latter case, it seems clear that the actual best choice recommendation should be reduced to node  $b$ , unless one disposes of other information, like a performance tableau and/or the actual computation method of the outranking situations. In any case, one has to be very clear about the available information when judging a best choice procedure.

It became thereafter clear for us all that both the lack of a specific performance tableau as well as the lack of a precisely defined algorithm for computing valid outranking situations does not allow to judge if a given digraph does indeed model a potential outranking relation. In our present bipolar-valued epistemic approach, a valid outranking digraph instance, following from a given performance tableau and the disjunctive epistemic fusion construction of the outranking relation (see Chapter 3 on page 33), will necessarily verify the weak completeness condition and the coduality principle. As a consequence, incomparability situations are now modelled by epistemic indeterminateness not by the actual absence of a relation.

The digraph put forward by D. Bouyssou in the October 2005 discussion is not weakly complete –node  $a$  is not outranking node  $b$  and vice versa– does hence not represent, in our present sense, a valid outranking digraph instance. Yet, it may be a partial tournament and as such it is a strict outranking digraph, i.e. the asymmetric part –the codual– of a valid outranking digraph. In this case, nodes  $a$  and  $d$  –the kernel of the strict outranking digraph– would actually for sure outrank each other and, hence, represent both the natural best choice recommendation. However, in

this codual digraph, node  $a$  becomes a CONDORCET winner –outranking for sure all other nodes– and becomes the evident unique best choice recommendation.

It stays an interesting open mathematical problem to show (or not) that both necessary conditions: –weak completeness and the coduality principle– are also sufficient for qualifying any bipolar-valued digraph as potential instance of an outranking digraph.

## References

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# Chapter 5

## How to create a new performance tableau instance

**Abstract** In this chapter we illustrate a way of creating a new *PerformanceTableau* instance by editing a given template with 5 decision alternatives, 3 decision objectives and 6 performance criteria. We discuss in detail editing the decision alternatives, the decision objectives, the family of performance criteria, and finally, the evaluations of the decision alternatives on the performance criteria.

### 5.1 Editing a template file

For this purpose we provide the following `perfTab_Template.py` file in the `examples` directory of the DIGRAPH3 resources.

**Listing 5.1** `PerformanceTableau` object template

```
1 ######
2 # Digraph3 documentation
3 # Template for creating a new PerformanceTableau instance
4 # (C) R. Bisdorff Mar 2021
5 # Digraph3/examples/perfTab_Template.py
6 #####
7 from decimal import Decimal
8 from collections import OrderedDict
9 #####
10 # edit the decision actions
11 # avoid special characters, like '_', '/' or ':',
12 # in action identifiers and short names
13 actions = OrderedDict([
14     ('a1', {
15         'shortName': 'action1',
16         'name': 'decision alternative a1',
17         'comment': 'some specific features of this alternative',
18         })),
19     ...
20     ...
21 ])
22 #####
23 # edit the decision objectives
24 # adjust the list of performance criteria
25 # and the total weight (sum of the criteria weights)
26 # per objective
```

```

27     objectives = OrderedDict([
28         ('obj1', {
29             'name': 'decision objective obj1',
30             'comment': "some specific features of this objective",
31             'criteria': ['g1', 'g2'],
32             'weight': Decimal('6'),
33         }),
34         ...
35     })
36     #####
37     # edit the performance criteria
38     # adjust the objective reference
39     # Left Decimal of a threshold = constant part and
40     # right Decimal = proportional part of the threshold
41     criteria = OrderedDict([
42         ('g1', {
43             'shortName': 'crit1',
44             'name': "performance criteria 1",
45             'objective': 'obj1',
46             'preferenceDirection': 'max',
47             'comment': 'measurement scale type and unit',
48             'scale': (Decimal('0.0'), Decimal('100.0')),
49             'thresholds': {'ind': (Decimal('2.50'), Decimal('0.0')),
50                             'pref': (Decimal('5.00'), Decimal('0.0')),
51                             'veto': (Decimal('60.00'), Decimal('0.0'))},
52             },
53             'weight': Decimal('3'),
54         }),
55         ...
56     })
57     #####
58     # default missing data symbol = -999
59     NA = Decimal('-999')
60     #####
61     # edit the performance evaluations
62     # criteria to be minimized take negative grades
63     evaluation = {
64         'g1': {
65             'a1': Decimal("41.0"),
66             'a2': Decimal("100.0"),
67             'a3': Decimal("63.0"),
68             'a4': Decimal('23.0'),
69             'a5': NA,
70         },
71         # g2 is of ordinal type and scale 0-10
72         'g2': {
73             'a1': Decimal("4"),
74             'a2': Decimal("10"),
75             'a3': Decimal("6"),
76             'a4': Decimal('2'),
77             'a5': Decimal('9'),
78         },
79         # g3 has preferenceDirection = 'min'
80         'g3': {
81             'a1': Decimal("-52.2"),
82             'a2': NA,
83             'a3': Decimal("-47.3"),
84             'a4': Decimal('35.7'),
85             'a5': Decimal('68.00'),
86         },
87         ...
88     }
89     ...
90 }
91 #####
92 #####

```

The template file, shown in Listing 5.1 on page 61, contains first the instructions to import the required `Decimal` and `OrderedDict` classes (see Lines 7-8). Four main sections are following: the potential decision *actions*, the decision *objectives*, the performance *criteria*, and finally the performance *evaluation*.

## 5.2 Editing the decision alternatives

Decision alternatives are stored in attribute `actions` under the `OrderedDict` format. The `OrderedDict` object keeps this given order when iterating over the decision alternatives<sup>1</sup>.

Required attributes of each decision alternative, besides the object identifier<sup>2</sup>, are: `shortName`, `name` and `comment` (see Lines 15-17). The `shortName` attribute is essentially used when showing the performance tableau or the performance heatmap in a browser view.

The random performance tableau models, introduced in Chapter 6 on page 73, use the `actions` attribute for storing special features of the decision alternatives. The *Cost-Benefit* model, for instance, uses a `type` attribute for distinguishing between *advantageous*, *neutral* and *cheap* alternatives (see Section 6.3 on page 77). The *3-Objectives* model keeps a detailed record of the performance profile per decision objective and the corresponding random generators per performance criteria (see Lines 7- below).

```

1  >>> from randomPerfTabs import \
2  ...                  Random3ObjectivesPerformanceTableau
3  >>> t = Random3ObjectivesPerformanceTableau()
4  >>> t.actions
5  OrderedDict([
6      ('p01', {'shortName': 'p01',
7          'name': 'action p01 Eco~ Soc- Env+',
8          'comment': 'random public policy',
9          'Eco': 'fair',
10         'Soc': 'weak',
11         'Env': 'good',
12         'profile': {'Eco':'fair',
13                     'Soc':'weak',
14                     'Env':'good'},
15         'generators': {'ec01': ('triangular', 50.0, 0.5),
16                         'so02': ('triangular', 30.0, 0.5),
17                         'en03': ('triangular', 70.0, 0.5),
18                         ...
19                         },
20         },
21     ),
22

```

<sup>1</sup> See the `OrderedDict` description in the Python documentation (Python Software Foundation, 2021).

<sup>2</sup> Mind that *graphviz* drawings require node identifier strings without any special characters like `'_'` or `'/'`.

```

22     ...
23     ])

```

The second section of the template file concerns the decision *objectives*.

### 5.3 Editing the decision objectives

The minimal required attributes (see Listing 5.1 on page 61 Lines 27-33) of the ordered decision *objectives* dictionary, besides the individual objective identifiers, are *name*, *comment*, *criteria* (the list of significant performance criteria) and *weight* (the importance of the decision objective). The latter attribute contains the sum of the *significance* weights of the objective's criteria list.

The *objectives* attribute is methodologically useful for specifying the performance criteria significance in building decision recommendations. Mostly, we assume indeed that decision objectives are all equally important and the performance criteria are equi-significant per objective. This is, for instance, the default setting in the random 3-*Objectives* performance tableau model.

**Listing 5.2** Example of decision objectives' description

```

1  >>> # t = Random3ObjectivesPerformanceTableau()
2  >>> t.objectives
3  OrderedDict([
4      ('Eco',
5          {'name': 'Economical aspect',
6              'comment': 'Random3ObjectivesPerformanceTableau generated',
7              'criteria': ['ec01', 'ec06', 'ec09'],
8              'weight': Decimal('48')}),
9      ('Soc',
10         {'name': 'Societal aspect',
11             'comment': 'Random3ObjectivesPerformanceTableau generated',
12             'criteria': ['so02', 'so12'],
13             'weight': Decimal('48')}),
14      ('Env',
15         {'name': 'Environmental aspect',
16             'comment': 'Random3ObjectivesPerformanceTableau generated',
17             'criteria': ['en03', 'en04', 'en05', 'en07',
18                         'en08', 'en10', 'en11', 'en13'],
19             'weight': Decimal('48')})
20  ])

```

The importance weight sums up to 48 for each one of the three example decision objectives shown in Listing 5.2 (Lines 8,13 and 19), so that the significance of each one of the 3 economic criteria is set to 16, of both societal criteria is set to 24, and of each one of the 6 environmental criteria is set to 8.

Mind that the *objectives* attribute is always present in a *PerformanceTableau* object instance, even when empty. In this case, we consider that each performance criterion canonically represents in fact its own decision objective. The criterion

significance equals in this case the corresponding decision objective's importance weight.

The third section of the template file concerns now the *performance criteria*.

## 5.4 Editing the family of performance criteria

In order to assess how well each potential decision alternative is satisfying a given decision objective, we need *performance criteria*, i.e. decimal-valued grading functions gathered in an ordered criteria dictionary. The required attributes (see Listing 5.3), besides the criteria identifiers, are the usual `shortName`, `name` and `comment`. Specific for a criterion are furthermore the `objective` reference, the significance weight, the grading scale (minimum and maximum performance values), the `preferenceDirection` ('max' or 'min') and the performance discrimination thresholds attributes.

**Listing 5.3** Example of performance criteria description

```

1  criteria = OrderedDict([
2      ('g1', {
3          'shortName': 'crit1',
4          'name': "performance criteria 1",
5          'comment': 'measurement scale type and unit',
6          'objective': 'obj1',
7          'weight': Decimal('3'),
8          'scale': (Decimal('0.0'), Decimal('100.0')),
9          'preferenceDirection': 'max',
10         'thresholds': {'ind': (Decimal('2.50'), Decimal('0.0')),
11                         'pref': (Decimal('5.00'), Decimal('0.0')),
12                         'veto': (Decimal('60.00'), Decimal('0.0'))
13                     },
14     }),
15     ...
16     ...
17 ])

```

In our bipolar-valued outranking approach, all performance criteria implement *decimal-valued* grading functions, where preferences are either *increasing* or *decreasing* with measured performances. In order to model a *coherent* performance tableau, the family of decision criteria must satisfy two methodological requirements:

1. *Independance*: Each decision criterion implements a grading that is *functionally independent* of the grading of the other decision criteria, i.e. the performance measured on one of the criteria does not *constrain* in any sense the performance measured on any other criterion.
2. *Non redundancy*: Each performance criterion is only *significant* for a *single* decision objective.

In order to take into account any, usually *unavoidable, imprecision* of the performance grading procedures, we may specify three performance *discrimination*

*thresholds*: an *indifference* ('ind'), a *preference* ('pref') and a *considerable performance difference* ('veto') threshold (see Listing 5.3 Lines 10-12). The left decimal number of a threshold description tuple indicates a *constant part*, whereas the right decimal number indicates a *proportional part*.

On the template performance criterion *g1*, shown in Listing 5.3, we observe for instance a grading scale from 0.0 to 100.0 with a constant *indifference* threshold of 2.5, a constant *preference* threshold of 5.0, and a constant *considerable performance difference* threshold of 60.0. The latter threshold will trigger, the case given, a *polarisation* of the outranking statement (Bisdorff, 2013).

In a random *Cost-Benefit* performance tableau model we may obtain by default the following content:

**Listing 5.4** Example of cardinal Costs criterion

```

1  >>> from randomPerfTabs import \
2  ...           RandomCBPerformanceTableau
3  >>> tcb = RandomCBPerformanceTableau()
4  >>> tcb.showObjectives()
5  *----- decision objectives -----*
6  C: Costs
7      c1 random cardinal cost criterion 6
8      Total weight: 6.00 (1 criteria)
9      ...
10     ...
11 >>> tcb.criteria
12 OrderedDict([
13     ('c1', {'preferenceDirection': 'min',
14             'scaleType': 'cardinal',
15             'objective': 'C',
16             'shortName': 'c1',
17             'name': 'random cardinal cost criterion',
18             'scale': (0.0, 100.0),
19             'weight': Decimal('6'),
20             'randomMode': ['triangular', 50.0, 0.5],
21             'comment': 'Evaluation generator: triangular law ...',
22             'thresholds': OrderedDict([
23                 ('ind', (Decimal('1.49'), Decimal('0'))),
24                 ('pref', (Decimal('3.7'), Decimal('0'))),
25                 ('veto', (Decimal('67.71'), Decimal('0'))),
26             ]),
27         },
28     ...
29     ...
30 ])

```

Criterion 'c1' appears here (see Listing 5.4) to be a cardinal criterion to be minimized and significant for the *Costs* (C) decision objective. We may use the *showCriteria()* method for printing the corresponding performance discrimination thresholds.

```

1 >>> tcb.showCriteria(IntegerWeights=True)
2 *----- criteria -----*
3   c1 'Costs/random cardinal cost criterion'

```

```

4     Scale = (0.0, 100.0)
5     Weight = 6
6     Threshold ind : 1.49 + 0.00x ; percentile: 5.13
7     Threshold pref : 3.70 + 0.00x ; percentile: 10.26
8     Threshold veto : 67.71 + 0.00x ; percentile: 96.15

```

The *indifference* threshold on this criterion amounts to a constant value of 1.49 (Line 6 above). More or less 5% of the observed performance differences on this criterion appear hence to be *insignificant*. Similarly, with a preference threshold of 3.70, about 90% of the observed performance differences are preferentially *significant* (Line 7). Furthermore,  $100.0 - 96.15 = 3.85\%$  of the observed performance differences appear to be *considerable* (Line 8) and will trigger a *polarisation* of the corresponding outranking statements.

After the performance criteria description, we are ready for recording the actual *performance table*.

## 5.5 Editing the performance table

The individual grades of each decision alternative on each decision criterion are recorded in a double *criterion*  $\times$  *action* dictionary called *evaluation* (see Listing 5.5). As we may encounter missing data cases, we previously define a *missing data* symbol *NA* which is set here to a value disjoint from all the measurement scales, by default *Decimal*(''-999') (Line 2 in Listing ?? on page ??).

**Listing 5.5** Editing performance grades

```

1 #-----
2 NA = Decimal('-999')
3 #-----
4 evaluation = {
5     'g1': {
6         'a1':Decimal("41.0"),
7         'a2':Decimal("100.0"),
8         'a3':Decimal("63.0"),
9         'a4':Decimal('23.0'),
10        'a5': NA,  # missing data
11    },
12    ...
13    ...
14    # g3 has preferenceDirection = 'min'
15    'g3': {
16        'a1':Decimal("-52.2"), # negative grades
17        'a2':NA,
18        'a3':Decimal("-47.3"),
19        'a4':Decimal(''-35.7'),
20        'a5':Decimal(''-68.00'),
21    },
22    ...
23    ...
24 }

```

Notice in Listing 5.5 on the preceding page (Lines 16 and following) that on a criterion with `preferenceDirection = 'min'` all performance grades are recorded as *negative* values.

We may now inspect the eventually recorded complete template performance table.

```

1 >>> from perfTabs import PerformanceTableau
2 >>> t = PerformanceTableau('perfTab_Template')
3 >>> t.showPerformanceTableau(ndigits=1)
4 *---- performance tableau ----*
5   Criteria | 'g1'  'g2'  'g3'  'g4'  'g5'  'g6'
6   Actions  |   3    3    6    2    2    2
7   -----|-----
8   'action1' |  41.0  4.0 -52.2  71.0  63.0  22.5
9   'action2' | 100.0 10.0    NA   89.0  30.7  75.0
10  'action3' |  63.0  6.0 -47.3  55.4  63.5    NA
11  'action4' |  23.0  2.0 -35.7  83.5  37.5  54.9
12  'action5' |    NA   9.0 -68.0  10.0  88.0  75.0

```

We may furthermore compute the associated outranking digraph and check if we observe any polarised outranking situations.

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> g = BipolarOutrankingDigraph(t)
3 >>> g.showPolarisations()
4 *---- Negative polarisations ----*
5   number of negative polarisations : 1
6   1: r(a4 >= a2) = -0.44
7     criterion: g1
8     Considerable performance difference : -77.00
9     Veto discrimination threshold      : -60.00
10    Polarisation: r(a4 >= a2) = -0.44 ==> -1.00
11 *---- Positive polarisations ----*
12   number of positive polarisations : 1
13   1: r(a2 >= a4) = 0.56
14     criterion: g1
15     Considerable performance difference : 77.00
16     Counter-veto threshold           : 60.00
17     Polarisation: r(a2 >= a4) = 0.56 ==> +1.00

```

Indeed, due to the considerable performance difference (77.00) observed on performance criterion `g1`, alternative `a2` *for sure outranks* alternative `a4`, respectively `a4` *for sure does not outrank* `a2`.

## 5.6 Inspecting the template outranking relation

Let us have a look at the outranking relation table.

**Listing 5.6** The template outranking relation

```

1 >>> g.showRelationTable()
2 * ---- Relation Table ----

```

r	'a1'	'a2'	'a3'	'a4'	'a5'
'a1'	+1.00	-0.44	-0.22	-0.11	+0.06
'a2'	+0.44	+1.00	+0.33	+1.00	+0.28
'a3'	+0.67	-0.33	+1.00	+0.00	+0.17
'a4'	+0.11	-1.00	+0.00	+1.00	+0.06
'a5'	-0.06	-0.06	-0.17	-0.06	+1.00

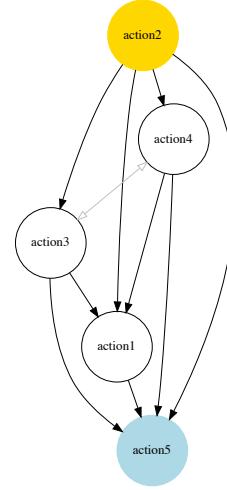
We may notice in the outranking relation table above (see Listing 5.6) that decision alternative  $a_2$  positively *outranks* all the other four alternatives (Line 6). Similarly, alternative  $a_5$  is positively *outranked* by all the other alternatives (see Line 9). We may orient this way the *graphviz* drawing of the template outranking digraph.

```

1 >>> g.exportGraphViz(fileName= 'template', \
2 ...                      bestChoice =['a2'], \
3 ...                      worstChoice=['a5']) \
4 *----- exporting a dot file for GraphViz tools -----* \
5   Exporting to template.dot \
6   dot -Grankdir=BT -Tpng template.dot -o template.png

```

**Fig. 5.1** The template outranking digraph models in fact a *partial order* on the five potential decision alternatives



Digraph3 (graphviz), R. Bisдорff, 2020

In Fig. 5.1, alternatives  $action3$  ( $a_3$ ) and  $action4$  ( $a_4$ ) appear actually *incomparable*. In Listing 5.6 on the facing page their pairwise outranking characteristics show indeed the *indeterminate* value 0.00 (Lines 7-8). We may check their pairwise comparison as follows.

```

1 >>> g.showPairwiseComparison('a3','a4') \
2 *----- pairwise comparison -----* \
3   Comparing actions : ('a3','a4')

```

```

4 crit. wght. g(x) g(y) diff | ind pref r() |
5 -----
6 'g1' 3.00 63.00 23.00 +40.00 | 2.50 5.00 +3.00 |
7 'g2' 3.00 6.00 2.00 +4.00 | 0.00 1.00 +3.00 |
8 'g3' 6.00 -47.30 -35.70 -11.60 | 0.00 10.00 -6.00 |
9 'g4' 2.00 55.40 83.50 -28.10 | 2.09 4.18 -2.00 |
10 'g5' 2.00 63.50 37.50 +26.00 | 0.00 10.00 +2.00 |
11 'g6' NA 54.90
12 Outranking characteristic value: r(a3 >= a4) = +0.00
13 Valuation in range: -18.00 to +18.00

```

The incomparability situation between  $a_3$  and  $a_4$  results here from a perfect balancing of positive (+8) and negative (-8) criteria significance weights.

We may eventually rank the five decision alternatives with a heatmap browser view following the COPELAND ranking rule (see Section 8.2) which consistently reproduces the partial outranking order shown in Fig. 5.1 on the previous page.

```

1 >>> g.showHTMLPerformanceHeatmap(ndigits=1, \
2 ...     colorLevels=5, Correlations=True, \
3 ...     rankingRule='Copeland', \
4 ...     pageTitle= \
5 ...         'Heatmap of the template performance tableau')

```

### Heatmap of the template performance tableau

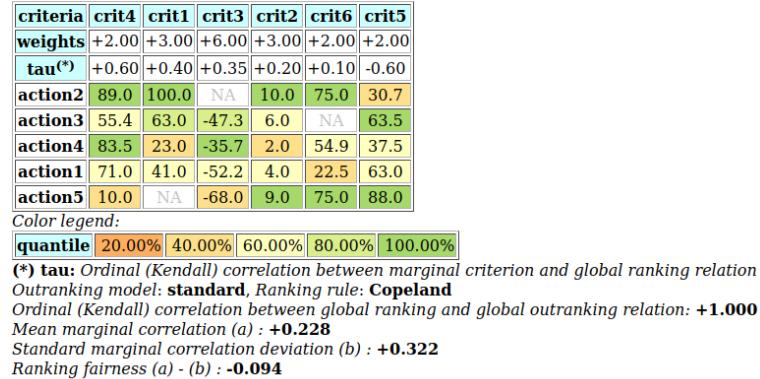


Fig. 5.2 COPELAND ranked heatmap of the template performance tableau

Due to a 11 against 7 *plurality tyranny* effect, the COPELAND ranking rule, essentially based on crisp majority outranking counts, puts here alternative  $action_5$  (a5) last, despite its excellent grades observed on criteria  $g_2$ ,  $g_5$  and  $g_6$ . A slightly *fairer* ranking result may be obtained with the NETFLOWS ranking rule (see Section 8.3 on page 109).

```

1 >>> g.showHTMLPerformanceHeatmap(ndigits=1, \
2 ...     colorLevels=5, Correlations=True, \

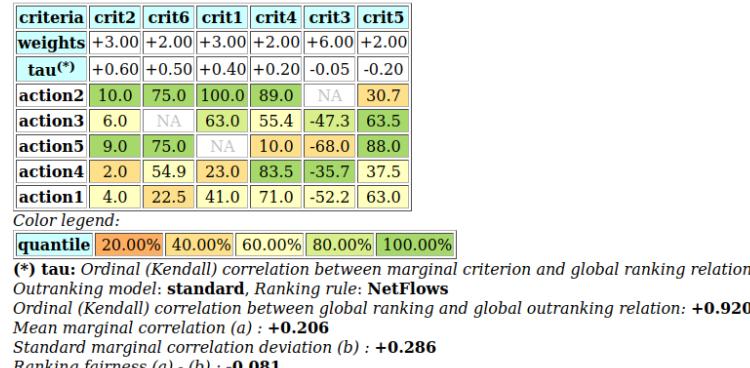
```

```

3 ...     rankingRule='NetFlows', \
4 ...     pageTitle=\
5 ...         'Heatmap of the template performance tableau')

```

### Heatmap of the template performance tableau



**Fig. 5.3** NETFLOWS ranked heatmap of the template performance tableau

It might be opportun to furthermore study the robustness of the apparent outranking situations when assuming only *ordinal* or *uncertain* criteria significance weights (see Sections 19.5 on page 286).

## References

- Bisdorff R (2013) On polarizing outranking relations with large performance differences. *Journal of Multi-Criteria Decision Analysis*, Wiley 20:3–12, URL <http://hdl.handle.net/10993/24566>
- Python Software Foundation (2021) Python documentation. URL <https://docs.python.org/3/> 63

# Chapter 6

## Generating random performance tableaux

**Abstract** The chapter describes the DIGRAPH3 resources for generating random multiple criteria performance tableaux. These random performance tableaux instances, mainly meant for illustration and training purposes, were serving the preparation and illustration of the *Algorithmic Decision Theory* Course lectures. The random generators propose several useful models like a *Cost-Benefit* tableau, a three Objectives –*economic, societal* and *environmental*– tableau, and eventually an *academic* performance tableau.

### 6.1 Introduction

The `randomPerfTabs` module provides several classes for generating random performance tableaux models of different kind, mainly for the purpose of testing implemented methods and tools presented and discussed in the Algorithmic Decision Theory course lectures at the University of Luxembourg. This chapter introduces the most useful models.

The simplest model, called `RandomPerformanceTableau`, generates a set of  $n$  decision actions, a set of  $m$  real-valued performance criteria, ranging by default from 0.0 to 100.0, associated with default discrimination thresholds: 2.5 (ind.), 5.0 (pref.) and 60.0 (veto). The generated performances are  $\text{Beta}(2, 2)$  distributed on each measurement scale.

One of the most useful models, called `RandomCBPerformanceTableau`, proposes a performance tableau involving two decision objectives, named *Costs* (to be minimized) respectively *Benefits* (to be maximized); its purpose being to generate more or less contradictory performances on these two, usually conflicting, objectives. *Low costs* will randomly be coupled with *low benefits*, whereas *high costs* will randomly be coupled with *high benefits*.

Many public policy decision problems involve three often conflicting decision objectives taking into account *economical, societal* as well as *environmental* aspects.

For this type of performance tableau model, we provide a specific model, called `Random3ObjectivesPerformanceTableau`.

Deciding which students, based on the grades obtained in a number of examinations, validate or not their academic studies, is the genuine decision practice of universities and academies. To thoroughly study these kind of decision problems, we provide a corresponding performance tableau model, called `RandomAcademicPerformanceTableau`, which gathers grades obtained by a given number of students in a given number of weighted courses.

In order to study aggregation of election results (see Chapter [7 on page 91](#)) in the context of bipolar-valued outranking digraphs, we provide furthermore a specific performance tableau model called `RandomRankPerformanceTableau` which provides ranks (linearly ordered performances without ties) of a given number of election candidates (decision actions) for a given number of weighted voters (performance criteria).

## 6.2 Random standard performance tableaux

The `RandomPerformanceTableau` class, the simplest of the kind, specializes the generic `PerformanceTableau` class, and takes the following parameters:

- `numberOfActions` := number of decision actions.
- `numberOfCriteria` := number of performance criteria.
- `weightDistribution` :=  
`'random'` (default) | `'fixed'` | `'equisignificant'`:
  - If `'random'`, weights are uniformly selected randomly from the given weight scale;
  - If `'fixed'`, the weightScale must provided a corresponding weights distribution;
  - If `'equisignificant'`, all criterion weights are put to unity.
- `weightScale` := `[Min, Max]` (default = `(1, numberOfCriteria)`).
- `IntegerWeights` := `True` (default) | `False` (normalized to proportions of 1.0).
- `commonScale` := `[a, b]`; common performance measuring scales (default = `[0.0, 100.0]`)
- `commonThresholds` := `[(q0, q1), (p0, p1), (v0, v1)]`; indifference( $q$ ), preference ( $p$ ) and considerable performance difference ( $v$ ) discrimination thresholds. For each threshold type  $x \in \{q, p, v\}$ , the float  $x0$  value represents a *constant percentage* of the common scale and the float  $x1$  value a *proportional value* of the actual performance measure. Default values are `[(2.5, 0.0), (5.0, 0.0), (60.0, 0, 0)]`.

- `commonMode` := common distribution of random performance measurements<sup>1</sup>:
  - ('beta', `None` (default setting),  $(\alpha, \beta)$ ), a beta generator with default  $\alpha = 2$  and  $\beta = 2$  parameters.
  - ('uniform', `None`, `None`), uniformly distributed float values on the given common scales' range [Min, Max];
  - ('normal',  $\mu$ ,  $\sigma$ ), truncated Gaussian distribution, by default  $\mu = (b - a)/2$  and  $\sigma = (b - a)/4$ ;
  - ('triangular', *mode*, *repartition*), generalized triangular distribution with a probability repartition parameter specifying the probability mass accumulated until the mode value. By default, *mode* =  $(b - a)/2$  and *repartition* = 0.5.<sup>2</sup>
- `valueDigits` := integer, precision of performance measurements (2 decimal digits by default).
- `missingDataProbability` :=  $0.0 \leq \text{float} \leq 1.0$ ; probability of missing performance evaluation on a criterion for an alternative (default 0.025).
- `NA` := `Decimal` (default = `-999`); missing data symbol.

**Code example:**

**Listing 6.1** Generating a random performance tableau

```

1 >>> from randomPerfTabs import RandomPerformanceTableau
2 >>> t = RandomPerformanceTableau(numberOfActions=21, \
3 ...                               numberOfCriteria=13, seed=100)
4 >>> t.actions
5   {'a01': {
6     'comment': 'RandomPerformanceTableau() generated.',
7     'name': 'random decision action'
8   },
9   'a02': { ... },
10  ...
11 }
12 >>> t.criteria
13   {'g01': {
14     'thresholds': {
15       'ind' : (Decimal('10.0'), Decimal('0.0')),
16       'veto': (Decimal('80.0'), Decimal('0.0')),
17       'pref': (Decimal('20.0'), Decimal('0.0'))},
18     'scale': [0.0, 100.0],
19     'weight': Decimal('1'),
20     'name': 'RandomPerformanceTableau() instance',
21     'comment': "Arguments: weightDistribution=random;
22                           weightScale=(1, 1);
23                           commonMode=None"
24   },

```

<sup>1</sup> See Lecture 3 of the Computational Statistic Course ([Bisdorff, 2020](#)).

<sup>2</sup> The `randomNumbers` module provides for this purpose the `ExtendedTriangularRandomVariable` class.

```

25     'g02': { ... },
26     ...
27   }
28 >>> t.evaluation
29 {'g01': {'a01': Decimal('15.17'),
30           'a02': Decimal('44.51'),
31           'a03': Decimal('-999'), # missing evaluation
32           ... },
33   ...
34 }
35 >>> t.showHTMLPerformanceTableau()

```

## Performance table randomperftab

criteria	g01	g02	g03	g04	g05	g06	g07	g08	g09	g10	g11	g12	g13
<b>weight</b>	<b>1.00</b>												
<b>a01</b>	15.17	46.37	82.88	41.14	59.94	41.19	58.68	44.73	22.19	64.64	34.93	42.36	17.55
<b>a02</b>	44.51	16.22	41.66	53.58	31.39	65.22	71.96	57.84	78.08	77.37	8.30	63.41	61.55
<b>a03</b>	NA	21.53	12.82	56.93	26.80	48.03	54.35	62.42	94.27	73.57	71.11	21.81	56.90
<b>a04</b>	58.00	51.16	21.92	65.57	59.02	44.77	37.49	58.39	80.79	55.39	46.44	19.57	39.22
<b>a05</b>	24.22	77.01	75.74	83.87	40.85	8.55	85.44	67.34	57.40	39.08	64.83	29.37	96.39
<b>a06</b>	29.10	39.35	15.45	34.99	49.12	11.49	28.44	52.89	64.24	62.92	58.28	32.02	10.25
<b>a07</b>	96.58	32.06	6.05	49.56	NA	66.06	41.64	13.08	38.31	24.82	48.39	57.03	42.91
<b>a08</b>	82.29	47.67	9.96	79.43	29.45	84.17	31.99	90.88	39.58	50.78	61.88	44.40	48.26
<b>a09</b>	43.90	14.81	60.55	42.37	6.72	56.14	34.20	51.54	21.79	79.13	50.95	93.16	81.89
<b>a10</b>	38.75	79.70	27.88	42.39	71.88	66.09	58.33	58.88	17.10	44.25	48.73	30.63	52.73
<b>a11</b>	35.84	67.48	38.81	33.75	26.87	64.10	71.95	62.72	NA	85.80	58.37	49.33	NA
<b>a12</b>	29.12	13.97	67.45	38.60	48.30	11.87	NA	57.76	74.86	26.57	48.80	43.57	7.68
<b>a13</b>	34.79	90.72	38.93	57.38	64.14	97.86	91.16	43.80	33.68	38.98	28.87	63.36	60.03
<b>a14</b>	62.22	80.16	19.26	62.34	60.96	24.72	73.63	71.21	56.43	46.12	26.09	51.43	12.86
<b>a15</b>	44.23	69.62	94.95	34.95	63.46	52.97	98.84	78.74	36.64	65.12	22.46	55.52	68.79
<b>a16</b>	19.10	45.49	65.63	64.96	50.57	55.91	10.02	34.70	29.31	50.15	70.68	62.57	71.09
<b>a17</b>	27.73	22.03	48.00	79.38	23.35	74.03	58.74	59.42	50.95	82.27	49.20	43.27	38.61
<b>a18</b>	41.46	33.83	7.97	75.11	49.00	55.70	64.99	38.47	49.86	17.45	28.08	35.21	67.81
<b>a19</b>	22.41	NA	34.86	49.30	65.18	39.84	81.16	NA	55.99	66.55	55.38	43.08	29.72
<b>a20</b>	21.52	69.98	71.81	43.74	24.53	55.39	52.67	13.67	66.80	57.46	70.81	5.41	76.05
<b>a21</b>	56.90	48.80	31.66	15.31	40.57	58.14	70.19	67.23	61.10	31.04	60.72	22.39	70.38

**Fig. 6.1** Browser view on random performance tableau instance

Best and worst performance on each criterion are marked in *light green*, respectively in *light red*. Notice that missing (NA) evaluation are registered in a performance tableau by default as `Decimal('999')` value (see Listing 6.1 on the previous page Line 24).

### 6.3 Random Cost-Benefit performance tableaux

We provide the `RandomCBPerformanceTableau` class for generating random *Costs* versus *Benefits* organized performance tableaux following the directives below:

- We distinguish three types of decision actions: *cheap*, *neutral* and *expensive* ones with an equal proportion of 1/3. We also distinguish two types of weighted criteria: *Costs* criteria to be *minimized*, and *Benefits* criteria to be *maximized*; in the proportions 1/3 respectively 2/3.
- Random performances on each type of criteria are drawn, either from an ordinal scale [0; 10], or from a cardinal scale [0.0; 100.0], following a parametric triangular law of mode: 30% performance for cheap, 50% for neutral, and 70% performance for expensive decision actions, with constant probability repartition 0.5 on each side of the respective mode.
- Costs criteria use mostly cardinal scales (3/4), whereas Benefits criteria use mostly ordinal scales (2/3).
- The sum of weights of the Costs criteria by default equals the sum weights of the Benefits criteria: `weighDistribution = 'equiobjectives'`.
- On cardinal criteria, both of cost or of benefit type, we observe following constant preference discrimination quantiles: 5% indifferent situations, 90% strict preference situations, and 5% veto situation.

#### Parameters:

- If `numberOfActions == None`, a uniform random number between 10 and 31 of *cheap*, *neutral* or *advantageous* actions (equal 1/3 probability each type) actions is instantiated;
- If `numberOfCriteria == None`, a uniform random number between 5 and 21 of cost or benefit criteria (1/3 respectively 2/3 probability) is instantiated;
- `weightDistribution := 'equisignificant'` (default) | `'equiobjectives'` | `'fixed'` | `'random'`;
- default `weightScale` for `'random'` weight distribution is `1 - numberOfCriteria`;
- All *cardinal* criteria are evaluated with decimals between 0.0 and 100.0 whereas *ordinal* criteria are evaluated with integers between 0 and 10.
- `commonThresholds` is obsolete. Preference discrimination is specified as *percentiles* of concerned performance differences (see below).
- `commonPercentiles := {'ind':5, 'pref':10, 'veto':95}` are expressed in percents (reversed for vetoes), and only concern cardinal criteria.
- `missingDataProbability := 0.0 ≤ float ≤ 1.0`; probability of missing performance evaluation on a criterion for an alternative (default 0.025).
- `NA := Decimal` (default = -999); missing data symbol.

Minimal number of decision actions required is 3 !

### Example Python session:

**Listing 6.2** Generating a random Cost-Benefit performance tableau

```

1  >>> from randomPerfTabs import\
2  ...          RandomCBPerformanceTableau
3  >>> t = RandomCBPerformanceTableau(
4  ...          numberActions=7,\n
5  ...          numberCriteria=5,\n
6  ...          weightDistribution='equiobjectives',\n
7  ...          commonPercentiles={'ind':0.05,\n
8  ...                      'pref':0.10,\n
9  ...                      'veto':0.95},\n
10 ...          seed=100)
11 >>> t.showActions()
12     *----- show decision action -----*
13     key: a1
14     short name: a1c
15     name: random cheap decision action
16     key: a2
17     short name: a2n
18     name: random neutral decision action
19     ...
20     key: a7
21     short name: a7a
22     name: random advantageous decision action
23 >>> t.showCriteria()
24     *----- criteria -----*
25     b1 'random ordinal benefit criterion'
26     Scale = (0, 10)
27     Weight = 3
28     ...
29     c1 'random cardinal cost criterion'
30     Scale = (0.0, 100.0)
31     Weight = 2
32     Threshold ind : 1.76 + 0.00x ; percentile: 9.5
33     Threshold pref : 2.16 + 0.00x ; percentile: 14.3
34     Threshold veto : 73.19 + 0.00x ; percentile: 95.2
35     ...

```

In the example above, we may notice the three types of decision actions (see Listing 6.2 Lines 10-20), as well as the two types (Lines 22-32) of criteria with either an *ordinal* or a *cardinal* performance measuring scale. In the latter case, by default about 5% of the random performance differences will be below the *indifference* and 10% below the *preference* discriminating threshold. About 5% will be considered as *considerably large*. More statistics about the generated performances may be inspected with the `showStatistics()` method.

```

1  >>> t.showStatistics()
2  *----- Performance tableau summary statistics -----*
3  Instance name      : randomCBperftab
4  Actions           : 7
5  Criteria          : 5
6  Criterion name    : b1
7  Criterion weight  : 3

```

```

8      criterion scale      : 0.00 - 10.00
9      mean evaluation     : 5.14
10     standard deviation  : 2.64
11     maximal evaluation  : 8.00
12     quantile Q3 (x_75)  : 8.00
13     median evaluation   : 6.50
14     quantile Q1 (x_25)  : 3.50
15     minimal evaluation  : 1.00
16     mean absolute difference : 2.94
17     standard difference deviation : 3.74
18     ...
19     Criterion name      : c1
20     Criterion weight    : 2
21     criterion scale      : -100.00 - 0.00
22     mean evaluation      : -49.32
23     standard deviation   : 27.59
24     maximal evaluation   : 0.00
25     quantile Q3 (x_75)  : -27.51
26     median evaluation   : -35.98
27     quantile Q1 (x_25)  : -54.02
28     minimal evaluation  : -91.87
29     mean absolute difference : 28.72
30     standard difference deviation : 39.02
31     ...

```

A heatmap view with 5 color levels gives the result shown in Fig. 6.2.

```

1 >>> t.showHTMLPerformanceHeatmap(colorLevels=5, \
2 ...      rankingRule=None, \
3 ...      pageTitle='Random Cost-Benefit Performance Tableau')

```

## RandomCost-Benefit Performance Tableau

criteria	b1	b2	c1	c2	c3
<b>weights</b>	+3.00	+3.00	+2.00	+2.00	+2.00
<b>a1c</b>	2.00	20.66	-28.65	-79.26	-3.19
<b>a2n</b>	3.00	47.34	-15.47	-52.85	-44.47
<b>a3n</b>	3.00	57.44	-81.69	-77.77	-66.35
<b>a4c</b>	2.00	69.06	-77.23	-57.77	-34.80
<b>a5a</b>	7.00	35.23	-57.33	-39.65	-43.45
<b>a6n</b>	9.00	52.77	-39.33	-58.08	-71.82
<b>a7a</b>	5.00	92.47	-79.57	-34.87	-87.54

Color legend:

<b>quantile</b>	20.00%	40.00%	60.00%	80.00%	100.00%
-----------------	--------	--------	--------	--------	---------

**Fig. 6.2** Unordered heatmap of a random Cost-Benefit performance tableau

Such a performance tableau may be stored and re-accessed as follows.

```

1 >>> t.save('temp')
2     ----- saving performance tableau in XMCDA 2.0 format
3     -----
4     File: temp.py saved !
5 >>> from perfTabs import PerformanceTableau
6 >>> t = PerformanceTableau('temp')

```

## 6.4 Random three objectives performance tableaux

We provide the `Random3ObjectivesPerformanceTableau` class for generating random performance tableaux concerning potential public policies evaluated with respect to three preferential decision objectives taking respectively into account  *economical, societal* as well as *environmental* aspects.

Each public policy is qualified randomly as performing *weak* (–), *fair* (~) or *good* (+) on each of the three objectives. Generator directives are the following:

- `numberOfActions` = 20 (default), minimal number required is 3,
- `numberOfCriteria` = 13 (default),
- `weightDistribution` = 'equiobjectives' (default) | 'random' | 'equisignificant',
- `weightScale` = (1, `numberOfCriteria`): only used when random criterion weights are requested,
- `integerWeights` = `True` (default): `False` gives normalized rational weights,
- `commonScale` = (0.0, 100.0),
- `commonThresholds` = [(5.0, 0.0), (10.0, 0.0), (60.0, 0.0)]: Performance discrimination thresholds may be set for 'ind', 'pref' and 'veto' thresholds,
- `commonMode` = ['triangular', 'variable', 0.5]: random number generators of various other types ('uniform', 'beta') are available. If the mode of the 'triangular' distribution is set to 'variable', three modes at 0.3(–), 0.5(~), respectively 0.7(+) of the common scale span are set at random for each coalition and action.
- `valueDigits` = 2 (default): evaluations are encoded as decimals,
- `missingDataProbability` = 0.05 (default): random insertion of missing values with given probability,
- `NA` := `Decimal` (default = –999); missing data symbol,
- `seed` = `None` (default).

### Example Python session:

**Listing 6.3** Generating a random 3 Objectives performance tableau

```

1 >>> from randomPerfTabs import \
2 ...           Random3ObjectivesPerformanceTableau

```

```

3 >>> t = Random3ObjectivesPerformanceTableau(\n
4     ...     numberActions=7,\n
5     ...     numberCriteria=13,\n
6     ...     weightDistribution='equiobjectives',\n
7     ...     seed=120)\n
8 >>> t.showObjectives()\n
9     ----- show objectives -----"\n
10    Eco: Economical aspect\n
11        ec01 criterion of objective Eco 18\n
12        ec05 criterion of objective Eco 18\n
13        ec06 criterion of objective Eco 18\n
14        ec12 criterion of objective Eco 18\n
15        Total weight: 72.00 (4 criteria)\n
16    Soc: Societal aspect\n
17        so02 criterion of objective Soc 24\n
18        so11 criterion of objective Soc 24\n
19        so13 criterion of objective Soc 24\n
20        Total weight: 72.00 (3 criteria)\n
21    Env: Environmental aspect\n
22        en03 criterion of objective Env 12\n
23        en04 criterion of objective Env 12\n
24        en07 criterion of objective Env 12\n
25        en08 criterion of objective Env 12\n
26        en09 criterion of objective Env 12\n
27        en10 criterion of objective Env 12\n
28        Total weight: 72.00 (6 criteria)

```

In Listing 6.3 on the facing page above, we notice that four *equisignificant* criteria (ec01, ec05, ec06, and ec12) evaluate for instance the performance of the public policies from an *economic* point of view (Lines 16-22). Three *equisignificant* criteria do the same from a *societal* (Lines 10-15), and six from an *environmental* point of view (Lines 23-28). The 'equiobjectives' directive (see Line 6) results hence in a balanced total weight (72.00) for each decision objective.

Variable *triangular* modes (0.3, 0.5 or 0.7 of the span of the measure scale) for each objective result in a different performance status for each public policy with respect to the three objectives.

```

1 >>> t.showActions()\n
2     key: p1\n
3         short name: p1\n
4         name:     action p1 Eco- Soc+ Env+\n
5         profile: {'Eco':'weak', 'Soc':'good', 'Env':'good'}\n
6     key: p2\n
7         short name: p2\n
8         name:     action p2 Eco- Soc- Env-\n
9         profile: {'Eco':'weak', 'Soc':'weak', 'Env':'weak'}\n
10    ...
11    key: p7\n
12        short name: p7\n
13        name:     action p7 Eco+ Soc- Env-\n
14        profile: {'Eco':'good', 'Soc':'weak', 'Env':'weak'}

```

Policy p01, for instance, will probably show *good* performances wrt the *societal* and *environmental* aspects, and *weak* performances wrt the *economical* aspect, whereas policy p2 will probably appear among the potential worst choices.

We may inspect in Fig. 6.3 the given random three-objectives performance tableau with the `showHTMLPerformanceTableau()` method.

```
1 >>> t.showHTMLPerformanceTableau()
```

## Performance table random3ObjectivesPerfTab

criteria	ec01	ec05	ec06	ec12	en03	en04	en07	en08	en09	en10	so02	so11	so13
<b>weight</b>	<b>18.00</b>	<b>18.00</b>	<b>18.00</b>	<b>18.00</b>	<b>12.00</b>	<b>12.00</b>	<b>12.00</b>	<b>12.00</b>	<b>12.00</b>	<b>12.00</b>	<b>24.00</b>	<b>24.00</b>	<b>24.00</b>
<b>p1</b>	31.05	26.95	35.91	32.85	68.88	70.52	22.94	50.04	17.06	32.52	20.90	NA	62.37
<b>p2</b>	12.54	11.31	29.63	9.77	43.09	36.04	32.96	54.35	82.25	22.29	75.76	51.42	27.96
<b>p3</b>	30.70	58.45	38.31	24.89	72.36	87.02	30.57	80.17	90.38	86.60	54.51	43.21	6.58
<b>p4</b>	16.18	61.70	60.27	15.10	26.63	66.28	12.84	58.78	86.68	57.92	NA	71.91	70.18
<b>p5</b>	86.00	89.43	44.41	18.32	31.10	74.55	29.89	40.26	26.86	41.91	23.83	57.41	22.41
<b>p6</b>	77.65	94.24	24.75	65.64	3.85	36.15	47.36	81.22	72.54	80.54	27.25	37.04	34.11
<b>p7</b>	72.26	44.62	70.55	29.82	52.69	37.80	50.06	28.27	28.53	29.50	29.81	41.38	26.31

Fig. 6.3 Browser view on the given random three-objectives performance tableau.

Light green cells show best performances and light red worst performances. Policy p01 shows thus the worst performance (17.09/100.00) on the environmental criterion en09, whereas policy p03 shows the best performances on four out of the six *environmental* criteria.

No trivial best choice becomes apparent when looking at the performance tableau shown in Fig. 6.3. Let us therefore compute a RUBIS best choice recommendation (see Chapter 4 on page 45).

**Listing 6.4** What is the public policy to recommend as best choice ?

```
1 >>> from outrankingDigraphs import\
2 ...     BipolarOutrankingDigraph
3 >>> g = BipolarOutrankingDigraph(t)
4 >>> g.showBestChoiceRecommendation()
5 ****
6 Rubis best choice recommendation(s) (BCR)
7 (in decreasing order of determinateness)
8 Credibility domain: [-1.00,1.00]
9 === >> potential best choice(s)
10 * choice           : ['p3', 'p4', 'p5', 'p6']
11 independence      : 0.00
12 dominance        : 0.17
13 absorbency        : -1.00
14 covering (%)      : 41.67
15 determinateness (%) : 52.98
16 - most credible action(s) = { 'p3': 0.17, }
```

```

17     === >> potential_worst_choice(s)
18     * choice           : ['p1', 'p2', 'p5', 'p7']
19     independence      : 0.00
20     dominance         : -0.44
21     absorbency        : 0.19
22     covered (%)       : 41.67
23     determinateness (%) : 50.79
24     - most credible action(s) = { 'p7': 0.06, }

```

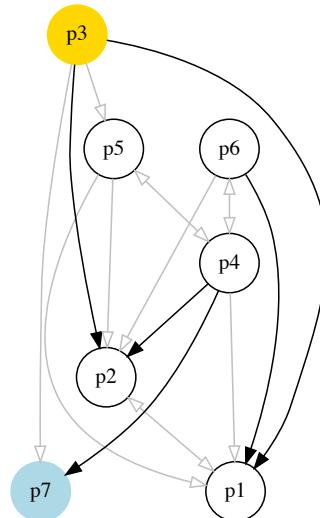
Policy p3 gives the most credible best choice candidate with the support of a 57.5% majority of significance, whereas policy p7 appears to be the less supported. Policy p5 gives an ambiguous best and worst choice candidate. A drawing of the strict outranking digraph oriented by best and worst choices gives a more complete preferential picture (see Fig. 6.4).

```

1 >>> (~(-g)).exportGraphViz(\n2 ...           fileName='3ObjPerfTabBestChoice',\n3 ...           bestChoice=['p3'],worstChoice=['p7'])\n4 *---- exporting a dot file for GraphViz tools ----*\n5   Exporting to 3ObjPerfTabBestChoice.dot\n6   dot -Grankdir=BT -Tpng 3ObjPerfTabBestChoice.dot\\
7   -o 3ObjPerfTabBestChoice.png

```

**Fig. 6.4** The strict outranking digraph oriented by best and worst choices. Policy p5 is indeed incomparable –in a strict outranking sense– to all the other six policies. Policies p1, p2 and p7 appear strictly outranked.



*Diagram3 (graphviz), R. Bisдорff, 2020*

A heatmap view on the COPELAND ranked performance tableau confirms the best choice recommendation (see Section 8.2 on page 108).

```

1 >>> t.showHTMLPerformanceHeatmap(Correlations=True, \
2 ...                               colorLevels=5, ndigits=1, \
3 ...                               rankingRule='Copeland')

```

### Heatmap of Performance Tableau 'random3ObjectivesPerfTab'

criteria	en10	ec05	en09	en04	en08	ec01	ec12	ec06	so11	so02	en07	so13	en03
<b>weights</b>	+12.00	+18.00	+12.00	+12.00	+12.00	+18.00	+18.00	+18.00	+24.00	+24.00	+12.00	+24.00	+12.00
<b>tau<sup>(*)</sup></b>	+0.79	+0.52	+0.45	+0.40	+0.38	+0.14	+0.12	+0.10	+0.02	-0.02	-0.10	-0.10	-0.17
<b>p3</b>	86.6	58.5	90.4	87.0	80.2	30.7	24.9	38.3	43.2	54.5	30.6	6.6	72.4
<b>p4</b>	57.9	61.7	86.7	66.3	58.8	16.2	15.1	60.3	71.9	NA	12.8	70.2	26.6
<b>p6</b>	80.5	94.2	72.5	36.1	81.2	77.7	65.6	24.8	37.0	27.2	47.4	34.1	3.9
<b>p5</b>	41.9	89.4	26.9	74.5	40.3	86.0	18.3	44.4	57.4	23.8	29.9	22.4	31.1
<b>p7</b>	29.5	44.6	28.5	37.8	28.3	72.3	29.8	70.5	41.4	29.8	50.1	26.3	52.7
<b>p1</b>	32.5	26.9	17.1	70.5	50.0	31.1	32.9	35.9	NA	20.9	22.9	62.4	68.9
<b>p2</b>	22.3	11.3	82.2	36.0	54.4	12.5	9.8	29.6	51.4	75.8	33.0	28.0	43.1

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: standard, Ranking rule: Copeland

Ordinal (Kendall) correlation between global ranking and global outranking relation: +0.896

Mean marginal correlation (a) : +0.161

Standard marginal correlation deviation (b) : +0.258

Ranking fairness (a) - (b) : -0.098

**Fig. 6.5** Browser view on the COPELAND ranked performance tableau.

The heatmap view with its COPELAND ranking based on the bipolar outranking digraph confirms policy's p3 as first-ranked. Notice also that the three strict outranked policies do effectively appear in the last positions.

## 6.5 Random academic performance tableaux

The `RandomAcademicPerformanceTableau` class constructor generates performance tableaux with random grades for a given number of students in different courses.

Generator directives:

- `numberOfStudents` := `Integer` (default 10)
- `numberOfCourses` := `Integer` (default 5)
- `weightDistribution` := `'equisignificant'` — `'random'` (default),
- `weightScale` := `1, 1 - numberOfCourses` (default when random)),
- `IntegerWeights` := `Boolean` (`True` = default),
- `commonScale` := `(Integer,integer)` (0,20) (default),
- `ndigits` := `Integer` (default 0),
- `WithTypes` := `Boolean` (default False),
- `commonMode` := `('triangular',xm=14,r=0.25)` (default),
- `commonThresholds` := `'ind':(0,0), 'pref':(1,0)` (default),
- `missingDataProbability` := `0.0` (default),
- `NA` := `Decimal` (default = `-999`); missing data symbol.

When Parameter `WithTypes` is set to `True`, the students are randomly allocated to one of the following four categories *weak* (1/6), *fair* (1/3), *good* (1/3), and *excellent* (1/3) – in the bracketed proportions. In a default 0 – 20 grading range, the random range of a weak student is 0 – 10, of a fair student 4 – 16, of a good student 8 – 20, and of an excellent student 12 – 20. The random grading generator follows in this case a double triangular probability law with *mode* (*xm*) equal to the middle of the random range and median repartition (*r* = 0.5) of probability each side of the mode.

### Listing 6.5 Generating a random academic performance tableau

```

1 >>> from randomPerfTabs import RandomAcademicPerformanceTableau
2 >>> t = RandomAcademicPerformanceTableau(\ 
3 ...           numberOfStudents=11, \
4 ...           numberOfCourses=7, missingDataProbability=0.03, \
5 ...           WithTypes=True, seed=100)
6 >>> t
7 *----- PerformanceTableau instance description -----*
8   Instance class    : RandomAcademicPerformanceTableau
9   Seed              : 100
10  Instance name    : randstudPerf
11  Actions           : 11
12  Criteria          : 7
13  NA proportion(%) : 5.2
14  Attributes        : ['randomSeed', 'name', 'actions',
15                      'criteria', 'evaluation', 'weightPreorder']
16 >>> t.showPerformanceTableau()
17 *---- performance tableau ----*
18   Courses |   'm1'   'm2'   'm3'   'm4'   'm5'   'm6'   'm7'
19   ECTS    |     2      1      3      4      1      1      5

```

```

20   ----- | -----
21   's01f' | 12 13 15 08 16 06 15
22   's02g' | 10 15 20 11 14 15 18
23   's03g' | 14 12 19 11 15 13 11
24   's04f' | 13 15 12 13 13 10 06
25   's05e' | 12 14 13 16 15 12 16
26   's06g' | 17 13 10 14 NA 15 13
27   's07e' | 12 12 12 18 NA 13 17
28   's08f' | 14 12 09 13 13 15 12
29   's09g' | 19 14 15 13 09 13 16
30   's10g' | 10 12 14 17 12 16 09
31   's11w' | 10 10 NA 10 10 NA 08
32 >>> t.weightPreorder
33   [['m2', 'm5', 'm6'], ['m1'], ['m3'], ['m4'], ['m7']]

```

The example tableau, generated for instance above with `missingDataProbability = 0.03`, `WithTypes = True` and `seed = 100` (see Listing 6.5 on the preceding page Lines 2-5), results in a set of two excellent (`s05e` and `s07e`), five good (`s02g`, `s03g`, `s06g`, `s09g` and `s10g`), three fair (`s01f`, `s04f` and `s08f`) and one weak (`s11w`) students. Notice that six students get a grade below the course validating threshold of 10 and we observe four missing grades (NA), two in course `m5` and, one in courses `m3` and `m6` (see Lines 20-30).

A statistical summary of the students' grades obtained in the highest weighted course, namely `m7`, is shown with the `showCourseStatistics()` method.

**Listing 6.6** Student performance summary statistics per course

```

1 >>> t.showCourseStatistics('m7')
2   ----- Summary performance statistics -----
3   Course name      : m7
4   Course weight    : 5
5   Students         : 11
6   Grading scale   : 0.00 - 20.00
7   Missing evaluations : 0
8   Mean evaluation  : 12.82
9   Standard deviation : 3.79
10  Maximal evaluation : 18.00
11  Quantile Q3 (x_75) : 16.25
12  Median evaluation  : 14.00
13  Quantile Q1 (x_25)  : 10.50
14  Minimal evaluation : 6.00
15  Mean absolute difference : 4.30
16  Standard difference deviation : 5.35

```

In Listing 6.6, the Course `m7` evaluation statistics show a mean grade of 12.82 and a median grade of 14. Maximal (resp. minimal) grade is 18 (resp. 6).

With a ranked heatmap view on all the grades we get a global pictures of the performance of all the eleven students. The ranking shown here in Fig. 6.6 on the facing page, produced with the NETFLOWS ranking rule (see Section 8.3 on page 109), reports a high correlation of +0.887 with the corresponding bipolar-valued outranking digraph (see Section 8.3 on page 109).

## Ranking the students

criteria	m7	m4	m2	m3	m1	m6	m5
weights	+5.00	+4.00	+1.00	+3.00	+2.00	+1.00	+1.00
tau <sup>(*)</sup>	+0.73	+0.31	+0.29	+0.20	+0.11	+0.09	+0.07
s07e	17.00	18.00	12.00	12.00	12.00	13.00	NA
s02g	18.00	11.00	15.00	20.00	10.00	15.00	14.00
s09g	16.00	13.00	14.00	15.00	19.00	13.00	9.00
s05e	16.00	16.00	14.00	13.00	12.00	12.00	15.00
s06g	13.00	14.00	13.00	10.00	17.00	15.00	NA
s03g	11.00	11.00	12.00	19.00	14.00	13.00	15.00
s10g	9.00	17.00	12.00	14.00	10.00	16.00	12.00
s01f	15.00	8.00	13.00	15.00	12.00	6.00	16.00
s08f	12.00	13.00	12.00	9.00	14.00	15.00	13.00
s04f	6.00	13.00	15.00	12.00	13.00	10.00	13.00
s11w	8.00	10.00	10.00	NA	10.00	NA	10.00

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: *Ordinal (Kendall) correlation between marginal criterion and global ranking relation*

Outranking model: **standard**, Ranking rule: **NetFlows**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.887**

**Fig. 6.6** Ranking the students in a performance heatmap view.

The NETFLOWS ranking represents also a rather *fair weighted consensus* between the individual courses' marginal rankings as made apparent in Listing 6.7 below with the `showRankingConsensusQuality()` method.

**Listing 6.7** Consensus quality of the students's ranking

```

1  >>> from outrankingDigraphs import\
2  ...                           BipolarOutrankingDigraph
3  >>> g = BipolarOutrankingDigraph(t)
4  >>> t.showRankingConsensusQuality(\n5  ...                           g.computeNetFlowsRanking())
6  Consensus quality of ranking:
7  ['s07', 's02', 's09', 's05', 's06', 's03', 's10',
8  's01', 's08', 's04', 's11']
9  Criterion (weight): correlation
10 -----
11   'm7' (0.294): +0.727
12   'm4' (0.235): +0.309
13   'm2' (0.059): +0.291
14   'm3' (0.176): +0.200
15   'm1' (0.118): +0.109
16   'm6' (0.059): +0.091

```

```
17      'm5' (0.059): +0.073
18  Summary:
19      Weighted mean marginal correlation (a): +0.361
20      Standard deviation (b)                 : +0.248
21      Ranking fairness (a)-(b)               : +0.113
```

The correlation with the marginal course rankings follows precisely the given ECTS weights. Course m7, with the highest relative weight (0.294) is most present (+0.727) in the global NETFLOWS ranking and no course marginal ranking appears negatively correlated with this ranking. The mean weighted correlation is +0.361.

Useful multiple incommensurable criteria ranking rules are presented and discussed in detail in Chapter 8 on page 105. In the next Chapter 7 on page 91, we first show how to compute the winner of an election on the basis of a different kind of performance evaluation models, namely *linear voting profiles* who are discussed in social choice theory.

## References

Bisdorff R (2020) Continuous Random Variables. Computational Statistics Course Lecture 3, University of Luxembourg. URL [http://hdl.handle.net/10993/37870\\_75](http://hdl.handle.net/10993/37870_75)



# Chapter 7

## Who wins the election ?

**Abstract** This chapter is more specifically devoted to handling linear voting profiles and computing the winner of such election results like the simple majority or the instant-run-off winner. By following CONDORCET recipiee, we also consider pairwise comparisons of election candidates and balance the number of times the first beats the second against the number of times the second beats the first. Thus we obtain the majority margins digraph, in fact a bipolar-valued digraph. When the voters express contradictory linear voting profiles we may naturally observe cyclic social preferences without seeing any paradox in this situation. Finally we present a more politically realistical random generator for linear voting profiles who takes into account pre-election polls.

### 7.1 Linear voting profiles

The `votingProfiles` module provides resources for handling election results (Bisdorff, 2020), like the `LinearVotingProfile` class. To illustrates these resources let us consider an election involving a finite set of candidates and finite set of weighted voters, who express their voting preferences in a complete linear ranking (without ties) of the candidates.

The module provides a `RandomLinearVotingProfile` class for generating random instances of the `LinearVotingProfile` class. In the interactive Python session shown in Listing 7.1 for the election of 3 candidates by 5 voters one may obtain the following result:

**Listing 7.1** Example of random linear voting profile

```
1 >>> from votingProfiles import \
2 ...                               RandomLinearVotingProfile
3 >>> lvp = RandomLinearVotingProfile(numberOfVoters=5, \
4 ...                               numberOfCandidates=3, \
5 ...                               RandomWeights=True)
6 >>> lvp.candidates
```

```

7     OrderedDict([ ('c1',{'name':'Candidate 1}),
8                     ('c2',{'name':'Candidate 2'}),
9                     ('c3',{'name':'Candidate 3'}) ])
10    >>> lvp.voters
11    OrderedDict([('v1',{'weight': 2}),
12                  ('v2':{'weight': 3}),
13                  ('v3',{'weight': 1}),
14                  ('v4':{'weight': 5}),
15                  ('v5',{'weight': 4}))])
16    >>> lvp.linearBallot
17    {'v1': ['c2', 'c1', 'c3'],
18     'v2': ['c3', 'c1', 'c2'],
19     'v3': ['c1', 'c3', 'c2'],
20     'v4': ['c1', 'c2', 'c3'],
21     'v5': ['c3', 'c1', 'c2']}

```

The linear voting profile data concerning the eligible candidates and the voters is internally stored in two ordered dictionaries: attribute `candidates` (Lines 8-10) and attribute `voters` (Lines 12-16). Notice that in this random example, the five voters are weighted. The linear ballots are stored in a standard dictionary: attribute `linearBallot` (Lines 17-22). Their linear ballots can be inspected with the `showLinearBallots` method.

```

1 >>> lvp.showLinearBallots()
2     voters(weight)      candidates  rankings
3     v1(2):      ['c2', 'c1', 'c3']
4     v2(3):      ['c3', 'c1', 'c2']
5     v3(1):      ['c1', 'c3', 'c2']
6     v4(5):      ['c1', 'c2', 'c3']
7     v5(4):      ['c3', 'c1', 'c2']
8     nbr. of voters: 15

```

Editing of the linear voting profile may be done by saving first the data into a file, edit this file, and reload it again.

```

1 >>> lvp.save(fileName='tutorialLinearVotingProfile1')
2     *--- Saving linear profile in file:
3             <tutorialLinearVotingProfile1.py> ---*
4 >>> from votingProfiles import LinearVotingProfile
5 >>> lvp = LinearVotingProfile('tutorialLinearVotingProfile1')

```

## 7.2 Computing the winner

The `computeUninominalVotes()` and the `computeSimpleMajorityWinner()` methods compute *uninominal votes*, i.e. how many times a candidate was ranked first, and who is consequently the *simple majority* (plurality) winner(s) in this election.

```

1 >>> lvp.computeUninominalVotes()
2     {'c1': 6, 'c2': 2, 'c3': 7}

```

```

3 >>> lvp.computeSimpleMajorityWinner()
4     ['c3']

```

As we observe no absolute majority (8/15) of votes for any of the three candidates, one may look with the `computeInstantRunoffWinner()` method for the *instant runoff* winner instead (see [Bisdorff \(2020\)](#)).

**Listing 7.2** Example Instant Run Off Winner

```

1 >>> lvp.computeInstantRunoffWinner(Comments=True)
2     Half of the Votes = 7.50
3     ==> stage = 1
4         remaining candidates ['c1', 'c2', 'c3']
5         uninominal votes {'c1': 6, 'c2': 2, 'c3': 7}
6         minimal number of votes = 2
7         maximal number of votes = 7
8         candidate to remove = c2
9         remaining candidates = ['c1', 'c3']
10    ==> stage = 2
11    remaining candidates ['c1', 'c3']
12    uninominal votes {'a1': 8, 'a3': 7}
13    minimal number of votes = 7
14    maximal number of votes = 8
15    candidate a1 obtains an absolute majority
16    Instant run off winner: ['c1']

```

In Listing 7.2 no candidate obtains in stage 1 an absolute majority of votes. Candidate `a2` obtains the minimal number of votes (2/15) and is, hence, eliminated. In stage 2, candidate `c1` eventually obtains an absolute majority of the votes (8/15) and is hence elected.

One may also follow the *Chevalier de Borda*'s advice and, after a **rank analysis** of the linear ballots, compute the **BORDA score** -the average rank- of each candidate and hence determine the **BORDA winner(s)** ([de Borda, 1781](#)).

**Listing 7.3** Example of BORDA rank scores

```

1 >>> lvp.computeRankAnalysis()
2 {'c1': [6, 9, 0], 'c2': [2, 5, 8], 'c3': [7, 1, 7]}
3 >>> v.computeBordaScores()
4 OrderedDict([
5     ('c1', {'BordaScore': 24, 'averageBordaScore': 1.6}),
6     ('c3', {'BordaScore': 30, 'averageBordaScore': 2.0}),
7     ('c2', {'BordaScore': 36, 'averageBordaScore': 2.4}) ])
8 >>> lvp.computeBordaWinners()
9 ['c1']

```

In Listing 7.3, Candidate `c1` obtains the minimal BORDA score, followed by candidate `c3` and finally candidate `c2` (see). The corresponding **BORDA rank analysis table** may be printed out with a corresponding `showRankAnalysisTable()` command .

**Listing 7.4** Rank analysis example

```

1 >>> lvp.showRankAnalysisTable()
2 *---- Borda rank analysis tableau ----*

```

candi-	alternative-to-rank	Borda			
dates	1	2	3	score	average
'c1'	6	9	0	24/15	1.60
'c3'	7	1	7	30/15	2.00
'c2'	2	5	8	36/15	2.40

In our randomly generated election results, we are lucky: Candidate  $c1$  is both the instant runoff winner and the *Borda* winner (see Listings 7.2 on the previous page and 7.3 on the preceding page). However, one could also follow the *Marquis de Condorcet*'s advice, and compute the *majority margins* obtained by voting for each individual pair of candidates (de Caritat, Marquis de Condorcet, 1784).

### 7.3 The majority margins digraph

Candidate  $c1$  is for instance ranked four times before and once behind candidate  $c2$ . Hence the corresponding *majority margin*  $M(a_1, a_2)$  is  $4 - 1 = +3$ . These pairwise *majority margins* define on the set of candidates what we call the *majority margins digraph*. The `MajorityMarginsDigraph` class is provided for handling such kind of digraphs.

**Listing 7.5** Example of *Majority Margins* digraph

```

1 >>> from votingProfiles import MajorityMarginsDigraph
2 >>> mmdg = MajorityMarginsDigraph(lvp,\n3 ...                                IntegerValuation=True)
4 >>> mmdg
5 *----- Digraph instance description -----*
6 Instance class      : MajorityMarginsDigraph
7 Instance name       : rel_randomLinearVotingProfile1
8 Digraph Order       : 3
9 Digraph Size        : 3
10 Valuation domain   : [-15.00;15.00]
11 Determinateness (%) : 55.56
12 Attributes         : ['name', 'actions', 'voters',
13                           'ballot', 'valuationdomain',
14                           'relation', 'order',
15                           'gamma', 'notGamma']
16 >>> mmdg.showAll()
17 *---- show detail -----*
18 Digraph           : rel_randomLinearVotingProfile1
19 *---- Candidates ----*
20 ['c1', 'c2', 'c3']
21 *---- Characteristic valuation domain ----*
22 {'max': Decimal('15.0'),
23  'med': Decimal('0'),
24  'min': Decimal('-15.0'),
25  'hasIntegerValuation': True}
26 *---- majority margins ----*
27 M(x,y) | 'c1'  'c2'  'c3'
```

```

28      -----| -----
29      'c1'  |   0    11    1
30      'c2'  |  -11   0   -1
31      'c3'  |   -1    1    0
32  Valuation domain: [-15;+15]

```

Notice in Listing 7.5 on the preceding page Lines 26-28 that, in the case of linear voting profiles, majority margins always verify a *zero sum property*:  $M(x,y) + M(y,x) = 0$  for all candidates  $x$  and  $y$ . This is not true in general for arbitrary voting profiles. The *majority margins* digraph of linear voting profiles defines in fact a *weak tournament* and belongs, hence, to the class of *self-codual* bipolar-valued digraphs<sup>1</sup>.

Now, a candidate  $x$ , showing a positive majority margin  $M(x,y)$ , is beating candidate  $y$  with an absolute majority in a pairwise voting. Hence, a candidate showing only positive terms in her row in the *majority margins* digraph relation table, beats all other candidates with absolute majority of votes. CONDORCET recommends to declare this candidate (is always unique, why?) the winner of the election. Here we are lucky, it is again candidate  $c1$  who is hence the CONDORCET winner (see Listing 7.5 on the facing page Line 26).

```

1 >>> mmdg.computeCondorcetWinners()
2 ['a1']

```

By seeing the majority margins like a *bipolar-valued characteristic function* of a global preference relation defined on the set of candidates, we may use all operational resources of the generic `DiGraph` class (see Chapter 2 on page 19), and especially its `exportGraphViz()` method<sup>2</sup>, for visualizing an election result.

```

1 >>> mmdg.exportGraphViz(\n2 ...           fileName='tutorialLinearBallots', \
3 ...           graphType='pdf')\n4 *---- exporting a dot file for GraphViz tools -----\n5   Exporting to tutorialLinearBallots.dot\n6   dot -Grankdir=BT -Tpng tutorialLinearBallots.dot \
7   -o tutorialLinearBallots.pdf

```

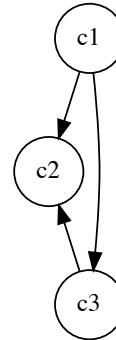
## 7.4 Cyclic social preferences

That a majority margins digraph models a linear ranking of the election candidates represents a very unlikely event. Usually, when aggregating linear ballots, there ap-

<sup>1</sup> The class of self-codual bipolar-valued digraphs consists of all weakly asymmetric digraphs, i.e. digraphs containing only asymmetric and/or indeterminate links. Limit cases consists of, on the one side, full tournaments with indeterminate reflexive links, and, on the other side, fully indeterminate digraphs. In this class, the converse (inverse  $\sim$ ) operator is indeed identical to the dual (negation  $-$ ) one.

<sup>2</sup> The `exportGraphViz()` method is depending on drawing tools from the `graphviz` software (<https://graphviz.org/>).

**Fig. 7.1** Visualizing an election result. In the Figure we notice that the *majority margins* digraph from our example linear voting profile gives a linear order of the candidates:  $c_1 > c_3 > c_2$ , the same actually as given by the BORDA scores (see Listing 7.4 on page 93)



*Digraph3 (graphviz), R. Bisdorff, 2020*

pear cyclic social preferences. Let us consider for instance the following linear voting profile  $v$  and construct the corresponding majority margins digraph.

**Listing 7.6** Example of cyclic social preferences

```

1  >>> v.showLinearBallots()
2  voters(weight): candidates rankings
3      v1(1): ['c1', 'c3', 'c5', 'c2', 'c4']
4      v2(1): ['c1', 'c2', 'c4', 'c3', 'c5']
5      v3(1): ['c5', 'c2', 'c4', 'c3', 'c1']
6      v4(1): ['c3', 'c4', 'c1', 'c5', 'c2']
7      v5(1): ['c4', 'c2', 'c3', 'c5', 'c1']
8      v6(1): ['c2', 'c4', 'c5', 'c1', 'c3']
9      v7(1): ['c5', 'c4', 'c3', 'c1', 'c2']
10     v8(1): ['c2', 'c4', 'c5', 'c1', 'c3']
11     v9(1): ['c5', 'c3', 'c4', 'c1', 'c2']
12 >>> mmdg = MajorityMarginsDigraph(v)
13 >>> mmdg.showRelationTable(ReflexiveTerms=False)
14  * ----- Relation Table -----*
15  M(x,y) | 'c1'  'c2'  'c3'  'c4'  'c5'
16  -----|-----
17  'c1'  |   -    1   -1   -5   -3
18  'c2'  |   -1    -   1    1   -1
19  'c3'  |    1   -1    -   -3   -1
20  'c4'  |    5   -1    3    -   1
21  'c5'  |    3    1    1   -1    -
22  Valuation domain: [-9;+9]
  
```

Now, we cannot find any completely positive row in the relation table (see Listing 7.6 Lines 17 - ). No one of the five candidates is beating all the others with an absolute majority of votes. There is no CONDORCET winner anymore. In fact, when

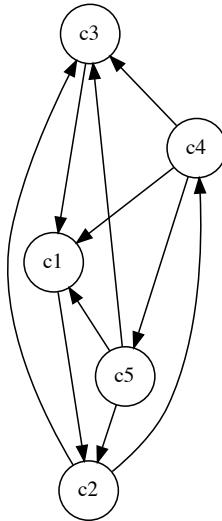
looking at a `graphviz` drawing of this *majority margins* digraph, we may observe **cyclic** preferences, like  $c1 > c2 > c3 > c1$  for instance.

```

1 >>> mmdg.exportGraphViz('cycles',graphType='pdf')
2 *----- exporting a dot file for GraphViz tools -----
3 Exporting to cycles.dot
4 dot -Grankdir=BT -Tpng cycles.dot -o cycles.pdf

```

**Fig. 7.2** Cyclic social preferences.



*Digraph3 (graphviz), R. Bisendorff, 2020*

But, there may be many cycles appearing in a *majority margins* digraph, and, we may detect and enumerate all minimal chordless circuits in a `Digraph` instance with the `computeChordlessCircuits` method.

```

1 >>> mmdg.computeChordlessCircuits()
2 [(['c2', 'c3', 'c1'], frozenset({'c2', 'c3', 'c1'})),
3  (['c2', 'c4', 'c5'], frozenset({'c2', 'c5', 'c4'})),
4  (['c2', 'c4', 'c1'], frozenset({'c2', 'c1', 'c4'}))]

```

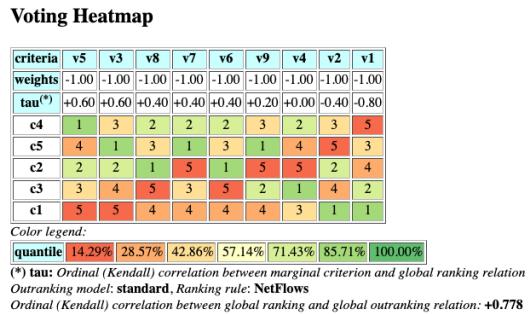
CONDORCET's approach for determining the winner of an election is hence *not decisive* in all circumstances and we need to exploit more sophisticated approaches for finding the winner of the election on the basis of the majority margins of the given linear ballots (see Chapter refsec:8 and [Bisendorff et al \(2008\)](#)).

Many more tools for exploiting voting results are available like the browser heat map view on voting profiles. The number of voters is usually much larger than the

number of candidates. In that case, it is better to generate a transposed *voters*  $\times$  *candidates* view (see Line 2 below)

```
1 >>> v.showHTMLVotingHeatmap(rankingRule='NetFlows',
2 ...                                Transposed=False)
```

**Fig. 7.3** Visualizing a linear voting profile in a heatmap format. Notice that the importance weights of the voters are *negative*, which means that the preference direction of the criteria (in this case the individual voters) is *decreasing*, i.e. goes from lowest (best) rank to highest (worst) rank.



It worthwhile noticing that the compromise *NetFlows* ranking  $a_4 > a_5 > a_2 > a_1 > a_3$ , shown in this heatmap (see Fig. 7.3) results in an optimal *ordinal correlation* index of +0.778 with the pairwise majority voting margins (see Chapter 16 on page 221).

## 7.5 On generating realistic random linear voting profiles

By default, the `RandomLinearVotingProfile` class generates random linear voting profiles where every candidates has the same uniform probabilities to be ranked at a certain position by all the voters. Each voter's random linear ballot is indeed generated via a uniform shuffling of the list of candidates.

In reality, political election data appear quite different. There usually will be different favorite and marginal candidates for each political party. To simulate these aspects into our random generator, we are using two random exponentially distributed polls of the candidates and consider a bipartisan political landscape with a certain random balance (default theoretical party repartition = 0.50) between the two sets of potential party supporters. A certain theoretical proportion (default = 0.1) will not support any party.

Let us generate such a linear voting profile for an election with 1000 voters and 15 candidates.

**Listing 7.7** Generating a linear voting profile with random polls

```
1 >>> from votingProfiles import \
2 ...                               RandomLinearVotingProfile
3 >>> lvp = RandomLinearVotingProfile(\n4 ...                               numberOfCandidates=15,\n5 ...                               numberOfVoters=1000,
```

```

6 ...           WithPolls=True,
7 ...           partyRepartition=0.5,
8 ...           other=0.1,
9 ...           seed=0.9189670954954139)
10 >>> lvp
11 *----- VotingProfile instance description -----*
12 Instance class : RandomLinearVotingProfile
13 Instance name  : randLinearProfile
14 Candidates    : 15
15 Voters        : 1000
16 Attributes     : ['name', 'seed', 'candidates',
17                      'voters', 'RandomWeights',
18                      'sumWeights', 'poll1', 'poll2',
19                      'bipartisan', 'linearBallot',
20                      'ballot']
21 >>> lvp.showRandomPolls()
22 Random repartition of voters
23 Party_1 supporters : 460 (46.0%)
24 Party_2 supporters : 436 (43.6%)
25 Other voters       : 104 (10.4%)
26 *----- random polls -----*
27 Party-1(46.0%) | Party-2(43.6%) | expected
28 -----
29 a06 : 19.91% | a11 : 22.94% | a06 : 15.00%
30 a07 : 14.27% | a08 : 15.65% | a11 : 13.08%
31 a03 : 10.02% | a04 : 15.07% | a08 : 09.01%
32 a13 : 08.39% | a06 : 13.40% | a07 : 08.79%
33 a15 : 08.39% | a03 : 06.49% | a03 : 07.44%
34 a11 : 06.70% | a09 : 05.63% | a04 : 07.11%
35 a01 : 06.17% | a07 : 05.10% | a01 : 05.06%
36 a12 : 04.81% | a01 : 05.09% | a13 : 05.04%
37 a08 : 04.75% | a12 : 03.43% | a15 : 04.23%
38 a10 : 04.66% | a13 : 02.71% | a12 : 03.71%
39 a14 : 04.42% | a14 : 02.70% | a14 : 03.21%
40 a05 : 04.01% | a15 : 00.86% | a09 : 03.10%
41 a09 : 01.40% | a10 : 00.44% | a10 : 02.34%
42 a04 : 01.18% | a05 : 00.29% | a05 : 01.97%
43 a02 : 00.90% | a02 : 00.21% | a02 : 00.51%

```

In this example (see Listing 7.7 on the preceding page Lines 18-), we obtain 460 Party-1 supporters (46%), 436 Party-2 supporters (43.6%) and 104 other voters (10.4%). Favorite candidates of Party-1 supporters, with more than 10%, appear to be  $a_{06}$  (19.91%),  $a_{07}$  (14.27%) and  $a_{03}$  (10.02%). Whereas for Party-2 supporters, favorite candidates appear to be  $a_{11}$  (22.94%), followed by  $a_{08}$  (15.65%),  $a_{04}$  (15.07%) and  $a_{06}$  (13.4%). Being *first* choice for Party-1 supporters and *fourth* choice for Party-2 supporters, this candidate  $a_{06}$  is a natural candidate for clearly winning this election game (see Listing 7.8).

**Listing 7.8** The uninominal and BORDA election winner

```

1 >>> lvp.computeSimpleMajorityWinner()
2 ['a06']
3 >>> lvp.computeInstantRunoffWinner()
4 ['a06']

```

```

5 >>> lvp.computeBordaWinners()
6 ['a06']

```

Is it also a CONDORCET winner ? To verify, we start by creating the corresponding *majority margins* digraph *mmdg* with the help of the `MajorityMarginsDigraph` class. The created digraph instance contains 15 *actions* –the candidates– and 105 *oriented arcs* –the *positive* majority margins– (see Listing 7.9 Lines 7-8).

**Listing 7.9** A majority margins digraph constructed from a linear voting profile

```

1 >>> from votingProfiles import MajorityMarginsDigraph
2 >>> mmdg = MajorityMarginsDigraph(lvp)
3 >>> mmdg
4 *----- Digraph instance description -----*
5 Instance class      : MajorityMarginsDigraph
6 Instance name       : rel_randLinearProfile
7 Digraph Order       : 15
8 Digraph Size        : 104
9 Valuation domain    : [-1000.00;1000.00]
10 Determinateness (%) : 67.08
11 Attributes         : ['name', 'actions', 'voters',
12                           'ballot', 'valuationdomain',
13                           'relation', 'order',
14                           'gamma', 'notGamma']

```

We may visualize the resulting pairwise majority margins by showing the HTML formated version of the *mmdg* relation table in a browser view.

```

1 >>> cdg.showHTMLRelationTable(\n2 ...          tableTitle='Pairwise majority margins',\n3 ...          relationName='M(x>y)')

```

**Fig. 7.4** Browsing the majority margins. *Light green* cells contain the positive majority margins, whereas *light red* cells contain the negative majority margins.

**Pairwise majority margins**

Margins	a01	a02	a03	a04	a05	a06	a07	a08	a09	a10	a11	a12	a13	a14	a15
a01	-	768	138	101	479	436	398	140	238	440	268	148	50	292	218
a02	-768	-	396	-484	364	858	828	-772	546	-496	800	-722	-768	-696	-658
a03	138	768	-	160	598	-266	-80	-	372	522	158	280	210	56	38
a04	101	-396	-484	-160	-	384	288	-288	-	372	522	158	280	210	56
a05	479	364	858	-184	-	-	-730	-640	-472	-234	116	550	-442	-532	-376
a06	436	858	828	370	730	-	248	234	574	692	102	556	482	566	520
a07	180	828	80	180	640	-248	-	-	0	358	602	-94	304	261	384
a08	8	-72	-8	-	-	472	-242	-	0	-	436	396	176	276	134
a09	-238	640	-238	-160	-	-74	-258	-476	-	-	-	-	176	276	134
a10	440	406	522	136	116	492	602	396	-116	-	-310	-310	-442	304	266
a11	268	800	158	420	593	-102	94	176	594	510	-	388	261	474	292
a12	148	722	280	-16	442	-500	-260	-276	120	300	-308	-	92	100	148
a13	202	696	360	-56	376	-566	-384	-298	90	304	-474	-100	-158	-	68
a14	-202	696	360	-56	376	-566	-384	-298	90	304	-474	-100	-158	-	68
a15	-218	658	338	-30	388	-520	-420	-244	14	266	-292	-148	-180	-68	-

Valuation domain [-1000: +1000]

A complete *light green* row reveals a CONDORCET winner, whereas a complete *light green* column reveals a CONDORCET loser. We recover again candidate a06 as *Condorcet* winner<sup>3</sup>.), whereas the obvious *Condorcet* loser is here candidate a02, the candidate with the lowest support in both parties (see Listing 7.7 on page 98 Line 40).

<sup>3</sup> The concept of CONDORCET winner –a generalization of absolute majority winners– proposed by CONDORCET in 1785, is an early historical example of initial digraph kernel (see Chapter 17 on page 237)

With a same *bipolar-first ranked* and *last ranked* candidate- selection procedure, we may *weakly rank* the candidates (with possible ties) by iterating these *first ranked* and *last ranked* choices among the remaining candidates (Bisdorff, 1999).

Before showing the *ranking-by-choosing* result, we have to compute the iterated bipolar selection procedure (see Listing 7.10 Line 2).

**Listing 7.10** Ranking by iterating choosing the *first* and *last* remaining candidates

```

1 >>> cdg.showRankingByChoosing()
2 Error: You must first run
3     self.computeRankingByChoosing(CoDual=False(default) | True) !
4 >>> cdg.computeRankingByChoosing()
5 >>> cdg.showRankingByChoosing()
6 Ranking by Choosing and Rejecting
7     1st first ranked ['a06']
8     2nd first ranked ['a11']
9     3rd first ranked ['a07', 'a08']
10    4th first ranked ['a03']
11    5th first ranked ['a01']
12    6th first ranked ['a13']
13    7th first ranked ['a04']
14    7th last ranked ['a12']
15    6th last ranked ['a14']
16    5th last ranked ['a15']
17    4th last ranked ['a09']
18    3rd last ranked ['a10']
19    2nd last ranked ['a05']
20    1st last ranked ['a02']

```

The first selection concerns a06 (first) and a02 (last), followed by a11 (first) opposed to a05 (last), and so on, until there remains at iteration step 7 a last pair of candidates, namely [a04, a12] (see Lines 13-14).

Notice furthermore the first ranked candidates at iteration step 3 (see Line 9), namely the pair (a07, a08). Both candidates represent indeed conjointly the *first ranked* choice. We obtain here hence a *weak ranking*, i.e. a ranking with a tie.

Let us mention that the *instant-run-off* procedure, we used before (see Listing 7.8 on page 99 Line 3), when operated with a `Comments=True` parameter setting, will deliver a more or less similar *reversed linear ordering-by-rejecting* result, namely [a02, a10, a14, a05, a09, a13, a12, a15, a04, a01, a08, a03, a07, a11, a06], ordered from the *last* to the *first* choice.

Remarkable about both these *ranking-by-choosing* or *ordering-by-rejecting* results is the fact that the random voting behaviour, simulated here with the help of two discrete random variables <sup>4</sup>, defined respectively by the two party polls, is rendering a ranking that is more or less in accordance with the simulated balance of the polls: Party-1 supporters : 460; Party-2 supporters: 436 (see Listing 7.7 on page 98 Lines 26-40 third column). Despite a random voting behaviour per voter, the given polls apparently show a *very strong incidence* on the eventual election result. In order to avoid any manipulation of the election outcome, public media are therefore in

<sup>4</sup> Discrete random variables with a given empirical probability law (here the polls) are provided in the `randomNumbers` module by the `DiscreteRandomVariable` class.

some countries not allowed to publish polls during the last weeks before a general election.

Mind that the specific *ranking-by-choosing* procedure, we use here on the *majority margins* digraph, operates the selection procedure by extracting at each step *initial* and *terminal* kernels, i.e. NP-hard operational problems (see Chapter 17 on page 237 and Bisdorff (1999)); A technique that does not allow in general to tackle voting profiles with) much more than 30 candidates.

Next Chapter 8 on page 105 on ranking methods provides more adequate and efficient techniques for ranking from pairwise majority margins when a larger number of potential candidates is given.

## References

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# Chapter 8

## Ranking with multiple incommensurable criteria

”... whether we are deciding between buying different commodity baskets, or making choices about what to do on a holiday, or deciding for whom to vote for in an election, we are inescapably involved in evaluating alternatives with non-commensurable aspects.”

—Amartya Sen, *Idea of Justice*, Allen Lane London, 2009.

**Abstract** The DIGRAPH3 python resources provide several algorithms for solving the ranking problem with a bipolar-valued outranking digraph. The COPELAND, NETFLOWS, KEMENY, SLATER, KOHLER, and the RANKEDPAIRS ranking rules are presented and illustrated with a random outranking digraph.

### 8.1 The ranking problem

We need to rank without ties a set  $X$  of items (usually decision alternatives) that are evaluated on multiple incommensurable performance criteria; yet, for which we may know their pairwise bipolar-valued *strict outranking* characteristics, i.e.  $r(x \succ y)$  for all  $x, y \in X$  (see Section 3.5 and (Bisdorff, 2013)).

Let us consider a didactic outranking digraph  $g$  generated from a random *Cost-Benefit* performance tableau (see Section 6.3) concerning 9 decision alternatives evaluated on 13 performance criteria. We may compute the corresponding *strict outranking digraph* with a codual transform (see Section 2.6).

**Listing 8.1** Random bipolar-valued strict outranking relation characteristics

```
1 >>> from randomPerfTabs import RandomCBPerformanceTableau
2 >>> t = RandomCBPerformanceTableau(numberOfActions=9, \
3 ...           numberOfCriteria=13, seed=200)
4 >>> from outrankingDigraphs import BipolarOutrankingDigraph
5 >>> g = BipolarOutrankingDigraph(t)
6 >>> gcd = ~(-g) # codual digraph
7 >>> gcd.showRelationTable(ReflexiveTerms=False)
```

```

8  * ---- Relation Table ----
9  r(>) | 'a1' 'a2' 'a3' 'a4' 'a5' 'a6' 'a7' 'a8' 'a9'
10 -----|-----
11 'a1' |   -  0.00 +0.10 -1.00 -0.13 -0.57 -0.23 +0.10 +0.00
12 'a2' | -1.00   -  0.00 +0.00 -0.37 -0.42 -0.28 -0.32 -0.12
13 'a3' | -0.10  0.00   - -0.17 -0.35 -0.30 -0.17 -0.17 +0.00
14 'a4' |  0.00  0.00 -0.42   - -0.40 -0.20 -0.60 -0.27 -0.30
15 'a5' | +0.13 +0.22 +0.10 +0.40   - +0.03 +0.40 -0.03 -0.07
16 'a6' | -0.07 -0.22 +0.20 +0.20 -0.37   - +0.10 -0.03 -0.07
17 'a7' | -0.20 +0.28 -0.03 -0.07 -0.40 -0.10   - +0.27 +1.00
18 'a8' | -0.10 -0.02 -0.23 -0.13 -0.37 +0.03 -0.27   - +0.03
19 'a9' |  0.00 +0.12 -1.00 -0.13 -0.03 -0.03 -1.00 -0.03   -

```

Some ranking rules will work on the associated CONDORCET digraph, i.e. the corresponding *strict median cut* polarised strict outranking digraph.

**Listing 8.2** Median cut polarised strict outranking relation characteristics

```

1 >>> ccd = PolarisedOutrankingDigraph(gcd,\_
2 ...                               level=g.valuationdomain['med'],\
3 ...                               KeepValues=False, StrictCut=True)
4 >>> ccd.showRelationTable(ReflexiveTerms=False,\_
5 ...                               IntegerValues=True)
6 *---- Relation Table ----
7 r(>)_med | 'a1' 'a2' 'a3' 'a4' 'a5' 'a6' 'a7' 'a8' 'a9'
8 -----|-----
9  'a1' |   -  0  +1 -1 -1 -1 -1 +1  0
10 'a2' | -1   -  +0  0 -1 -1 -1 -1 -1
11 'a3' | -1  0   - -1 -1 -1 -1 -1  0
12 'a4' |  0  0 -1   - -1 -1 -1 -1 -1
13 'a5' | +1  +1 +1 +1   - +1 +1 -1 -1
14 'a6' | -1 -1 +1 +1 -1   - +1 -1 -1
15 'a7' | -1 +1 -1 -1 -1 -1   - +1 +1
16 'a8' | -1 -1 -1 -1 -1 +1 -1   - +1
17 'a9' |  0  +1 -1 -1 -1 -1 -1 -1   -

```

Unfortunately, such crisp median-cut CONDORCET digraphs, associated with a given strict outranking digraph, only exceptionally present a linear ordering. Usually, pairwise majority comparisons do not even render a *complete* or, at least, a *transitive* partial order. There may even frequently appear *cyclic* outranking situations (see Section 7.4 on page 95).

To discover how *difficult* this ranking problem can get here, we have a look in Fig. 8.1 on the facing page at the corresponding strict outranking digraph *graphviz* drawing<sup>1</sup>.

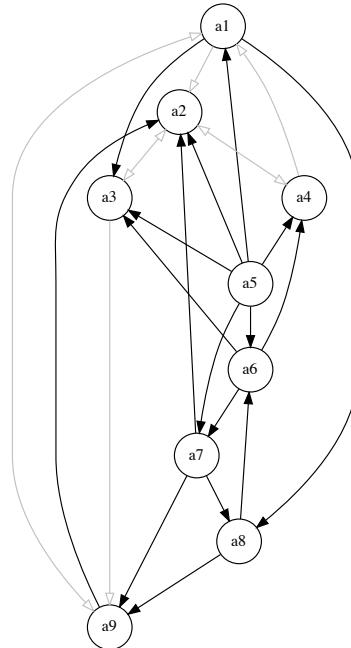
```

1 >>> gcd.exportGraphViz(fileName='rankingTutorial')
2 *---- exporting a dot file for GraphViz tools -----*
3 Exporting to rankingTutorial.dot
4 dot -Grankdir=BT -Tpng rankingTutorial.dot\
5                               -o rankingTutorial.png

```

<sup>1</sup> The `exportGraphViz()` method is depending on drawing tools from *graphviz* software (<https://graphviz.org/>).

**Fig. 8.1** The strict outranking relation  $\succ$  shown here is, for instance, *not transitive*: alternative  $a_8$  outranks alternative  $a_6$  and alternative  $a_6$  outranks  $a_4$ , however,  $a_8$  does not outrank  $a_4$ . Furthermore, alternatives  $a_6$ ,  $a_7$  and  $a_8$  show a cyclic outranking relation



Digraph3 (graphviz), R. Bisdorff, 2020

We may compute the *transitivity degree* of the outranking digraph shown in Fig. 8.1, i.e. the ratio of the difference between the number of outranking arcs and the number of transitive arcs over the difference of the number of arcs of the transitive closure minus the transitive arcs of the digraph gcd.

```

1 >>> gcd.computeTransitivityDegree(Comments=True)
2 Transitivity degree of graph <codual_rel_randomCBperftab>
3   triples x>y>z: 78, closed: 38, open: 40
4   closed/triples = 0.487

```

With only 49% of the required transitive arcs, the strict outranking relation here is hence very far from being transitive; a serious problem when a linear ordering of the decision alternatives is looked for.

Let us furthermore see if there are any cyclic outrankings.

```

1 >>> gcd.computeChordlessCircuits()
2 >>> gcd.showChordlessCircuits()
3   1 circuit(s).
4   *---- Chordless circuits ----*
5   1: ['a6', 'a7', 'a8'], credibility : 0.033

```

There is one chordless circuit detected in the given strict outranking digraph  $gcd$ , namely alternative  $a_6$  outranks alternative  $a_7$ , the latter outranks  $a_8$ , and  $a_8$  outranks again alternative  $a_6$  (see Fig. 8.1 on the preceding page). Any potential linear ordering of these three alternatives will, in fact, always contradict somehow the given outranking relation.

Now, several heuristic ranking rules have been proposed for constructing a linear ordering which is closest in some specific sense to a given outranking relation. The DIGRAPH3 resources provide some of the most common of these ranking rules, like COPELAND's, KEMENY's, SLATER's, KOHLER's, and the RANKEDPAIRS ranking rule.

## 8.2 The COPELAND ranking

**Definition 8.1.** COPELAND's ranking rule computes for each alternative a score resulting from the sum of the differences between the crisp *strict outranking* characteristics  $r(x \succ y)_{>0}$  and the crisp *strict outranked* characteristics  $r(y \succ x)_{>0}$  for all pairs of alternatives where  $y$  is different from  $x$ . The alternatives are ranked in decreasing order of these COPELAND scores; ties, the case given, being resolved with a lexicographical rule applied to the identifiers of the alternatives (Copeland, 1951).

COPELAND's rule, the most intuitive one as it works well for any strict outranking relation which models a linear order on the *median cut* strict outranking digraph  $ccd$ .

**Listing 8.3** Computing a COPELAND Ranking

```

1 >>> from linearOrders import CopelandRanking
2 >>> cop = CopelandRanking(gcd, Comments=True)
3 Copeland decreasing scores
4     a5 : +12
5     a1 : +2
6     a6 : +2
7     a7 : +2
8     a8 : 0
9     a4 : -3
10    a9 : -3
11    a3 : -5
12    a2 : -7
13 Copeland Ranking:
14 ['a5','a1','a6','a7','a8','a4','a9','a3','a2']
```

Alternative  $a_5$  obtains here the best COPELAND score (+12), followed by alternatives  $a_1$ ,  $a_6$  and  $a_7$  with same score (+2); following the lexicographic rule,  $a_1$  is hence ranked before  $a_6$  and  $a_6$  before  $a_7$ . Same situation is observed for  $a_4$  and  $a_9$  with a score of  $-3$  (see Listing 8.3 Lines 4-12).

COPELAND's ranking rule is in fact *invariant* under the codual transform (see Section 2.6 on page 25) and renders a same linear order indifferently from digraphs  $g$  or  $gcd$ . The resulting ranking (see Listing 8.3 Line 14) is rather corre-

lated (+0.463) with the given pairwise outranking relation in the ordinal KENDALL sense<sup>2</sup>.

**Listing 8.4** Checking the quality of the COPELAND ranking

```

1 >>> corr = g.computeRankingCorrelation(cop.copelandRanking)
2 >>> g.showCorrelation(corr)
3 Correlation indexes:
4   Crisp ordinal correlation : +0.463
5   Valued equivalence       : +0.107
6   Epistemic determination  :  0.230

```

With an epistemic determination level of 0.230 (see above), the *extended* KENDALL  $\tau$  index is in fact computed on 61.5%(100.0x(1.0 + 0.23)/2) of the pairwise strict outranking comparisons. Furthermore, the bipolar-valued *relational equivalence* characteristics between the strict outranking relation and the COPELAND ranking equals +0.107, i.e. a *majority* of 55.35% of the criteria significance supports the relational equivalence between the given strict outranking relation and the corresponding COPELAND ranking.

The COPELAND scores deliver actually only a *weak ranking*, i.e. a ranking with potential ties. This weak ranking may be constructed with the `WeakCopelandOrder` class .

**Listing 8.5** Computing a weak COPELAND ranking

```

1 >>> from transitiveDigraphs import WeakCopelandOrder
2 >>> wcop = WeakCopelandOrder(g)
3 >>> wcop.showRankingByChoosing()
4 Ranking by Choosing and Rejecting
5   1st ranked ['a5']
6   2nd ranked ['a1', 'a6', 'a7']
7   3rd ranked ['a8']
8   3rd last ranked ['a4', 'a9']
9   2nd last ranked ['a3']
10  1st last ranked ['a2']

```

We recover in Listing 8.5 above, the ranking with ties delivered by the COPELAND scores (see Listing 8.3 on the facing page). We may draw its corresponding skeleton.

```

1 >>> wcop.exportGraphViz(fileName='weakCopelandRanking')
2 *---- exporting a dot file for GraphViz tools ----*
3 Exporting to weakCopelandRanking.dot
4 dot -Grankdir=TB -Tpng weakCopelandRanking.dot \
      -o weakCopelandRanking.png

```

Let us now consider a similar ranking rule, but working directly on the criteria *significance majority margins*, i.e. the *bipolar-valued* outranking relations.

## 8.3 The NETFLOWS ranking

---

<sup>2</sup> See Chapter 16 on page 221 and Bisdorff (2012).

**Fig. 8.2** Drawing of the weak COPELAND ranking. The graph show the skeleton of the preorder produced by the corresponding ties of the COPELAND scores

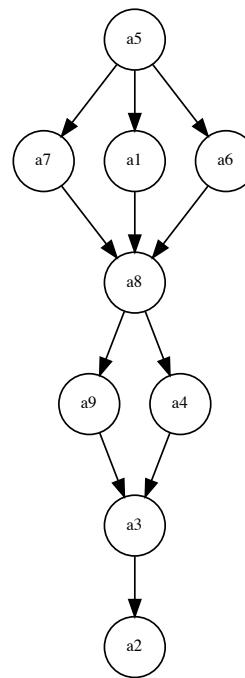


Diagram3 (graphviz)  
R. Bisdorff, 2020

**Definition 8.2.** The bipolar-valued version of the COPELAND ranking rule, we call NETFLOWS<sup>3</sup>, computes for each alternative  $x$  a *net flow* score, i.e. the sum of the differences between the *strict outranking* characteristics  $r(x \succ y)$  and the *strict outranked* characteristics  $r(y \succ x)$  for all pairs of alternatives where  $y$  is different from  $x$ .

**Listing 8.6** Computing a NETFLOWS ranking

```

1 >>> from linearOrders import NetFlowsRanking
2 >>> nf = NetFlowsRanking(gcd,Comments=True)
3     Net Flows :
4         a5 : +3.600
5         a7 : +2.800
6         a6 : +1.300
7         a3 : +0.033
  
```

<sup>3</sup> This ranking rule is also known under the name PROMETHEE ranking rule (Brans and Vincke, 1985).

```

8      a1 : -0.400
9      a8 : -0.567
10     a4 : -1.283
11     a9 : -2.600
12     a2 : -2.883
13 NetFlows Ranking:
14     ['a5','a7','a6','a3','a1','a8','a4','a9','a2']
15 >>> cop.copelandRanking # comparing both
16     ['a5','a1','a6','a7','a8','a4','a9','a3','a2']

```

In our example here, the NETFLOWS scores actually deliver a ranking *without ties* which is rather different from the one delivered by COPELAND's rule (see Listing 8.6 on the preceding page Line 16). It may happen, however, that we obtain, as with the COPELAND scores above, only a ranking with ties, which may then be resolved similarly by following a lexicographic rule applied to the identifiers of the decision alternatives. In such cases, it is possible to construct again a *weak ranking* with the corresponding WeakNetFlowsOrder class.

It is worthwhile noticing again, that similar to the COPELAND ranking rule seen before, the NETFLOWS ranking rule is also *invariant* under the codual transform (see Secion 2.6 on page 25) and delivers again the same ranking result indifferently from digraphs  $g$  or  $gcd$  (see Listing 8.6 on the preceding page Line 14).

The NETFLOWS ranking result appears to be slightly better correlated (+0.638) with the given outranking relation than its crisp cousin, the COPELAND ranking (see Lines 4-6 below).

**Listing 8.7** Checking the quality of the NETFLOWS Ranking

```

1 >>> corr = gcd.computeOrdinalCorrelation(nf)
2 >>> gcd.showCorrelation(corr)
3 Correlation indexes:
4   Extended Kendall tau      : +0.638
5   Epistemic determination   : 0.230
6   Bipolar-valued equivalence : +0.147

```

Indeed, the extended KENDALL  $\tau$  index of +0.638 leads to a bipolar-valued *relational equivalence* characteristics of +0.147, i.e. a majority of 57.35% of the criteria significance supports the relational equivalence between the given outranking digraphs  $g$  or  $gcd$  and the corresponding NETFLOWS ranking (see Chapter 16 on page 221). This lesser ranking performance of the COPELAND rule stems in this example essentially from the *weakness* of the actual ranking result and our subsequent *arbitrary* lexicographic resolution of the many ties given by the COPELAND scores (see Fig. 8.2 on the preceding page).

To appreciate now the more or less correlation of both the COPELAND and the NETFLOWS rankings with the underlying pairwise outranking relation, it is useful to consider KEMENY's and SLATER's '*optimal*' ranking rules.

## 8.4 KEMENY rankings

A KEMENY ranking is a linear ranking without ties which is *closest*, in the sense of the ordinal KENDALL distance (see Chapter 16 on page 221 and Bisdorff (2012)), to the given valued outranking digraphs  $g$  or  $gcd$  (Kemeny, 1959). This rule is also *invariant* under the codual transform.

**Listing 8.8** Computing a KEMENY ranking

```

1 >>> from linearOrders import KemenyRanking
2 >>> ke = KemenyRanking(gcd, orderLimit=9)
3 >>> # default orderLimit is 7
4 >>> ke.showRanking()
5  ['a5','a6','a7','a3','a9','a4','a1','a8','a2']
6 >>> corr = gcd.computeOrdinalCorrelation(ke)
7 >>> gcd.showCorrelation(corr)
8  Correlation indexes:
9    Extended Kendall tau      : +0.779
10   Epistemic determination   :  0.230
11   Bipolar-valued equivalence : +0.179

```

So,  $+0.779$  represents the *highest possible* ordinal correlation –*fitness*– any potential linear ranking can achieve with the given pairwise outranking digraph (see Listing 8.8 Lines 7-10).

A KEMENY ranking may not be unique. In our example here, we obtain in fact two such KEMENY rankings with a same *maximal* KEMENY index of 12.9.

**Listing 8.9** Optimal KEMENY rankings

```

1  [[['a5','a6','a7','a3','a8','a9','a4','a1','a2'],
2   ['a5','a6','a7','a3','a9','a4','a1','a8','a2']]]
3 >>> ke.maxKemenyIndex
4 Decimal('12.9166667')

```

We may visualize the partial order defined by the epistemic disjunction (see Section 2.5 on page 24) of both optimal KEMENY rankings by using the `RankingsFusion` class.

**Listing 8.10** Computing the epistemic disjunction of all optimal KEMENY rankings

```

1 >>> from transitiveDigraphs import RankingsFusion
2 >>> wke = RankingsFusion(ke, ke.maximalRankings)
3 >>> wke.exportGraphViz(fileName='tutorialKemeny')
4  ----- exporting a dot file for GraphViz tools -----
5  Exporting to tutorialKemeny.dot
6  dot -Grankdir=TB -Tpng tutorialKemeny.dot -o tutorialKemeny.png

```

To retain now a specific representative among all the potential rankings with maximal KEMENY index, we will choose, with the help of the `showRankingConsensusQuality()` method, the one proposing the best performance criteria consensus.

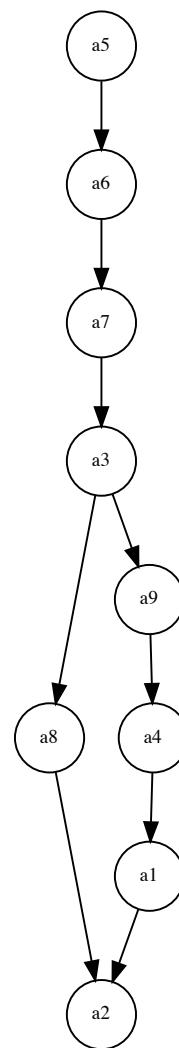
**Listing 8.11** Computing the consensus quality of a ranking

```

1 >>> g.showRankingConsensusQuality(ke.maximalRankings[0])

```

**Fig. 8.3** Epistemic disjunction of optimal KEMENY rankings. It is interesting to notice that both KEMENY rankings only differ in their respective positioning of alternative  $a_8$ ; either before or after alternatives  $a_9$ ,  $a_4$  and  $a_1$ .



Digraph3 (graphviz)  
R. Bisdorff, 2020

```

2  Consensus quality of ranking:
3  ['a5','a6','a7','a3','a8','a9','a4','a1','a2']
4  criterion (weight): correlation

```

```

5  -----
6      b09 (0.050) : +0.361
7      b04 (0.050) : +0.333
8      b08 (0.050) : +0.292
9      b01 (0.050) : +0.264
10     c01 (0.167) : +0.250
11     b03 (0.050) : +0.222
12     b07 (0.050) : +0.194
13     b05 (0.050) : +0.167
14     c02 (0.167) : +0.000
15     b10 (0.050) : +0.000
16     b02 (0.050) : -0.042
17     b06 (0.050) : -0.097
18     c03 (0.167) : -0.167
19 Summary:
20     Weighted mean marginal correlation (a): +0.099
21     Standard deviation (b) : +0.177
22     Ranking fairness (a)-(b) : -0.079
23 >>> g.showRankingConsensusQuality(ke.maximalRankings[1])
24 Consensus quality of ranking:
25 ['a5','a6','a7','a3','a9','a4','a1','a8','a2']
26 criterion (weight): correlation
27 -----
28      b09 (0.050) : +0.306
29      b08 (0.050) : +0.236
30      c01 (0.167) : +0.194
31      b07 (0.050) : +0.194
32      c02 (0.167) : +0.167
33      b04 (0.050) : +0.167
34      b03 (0.050) : +0.167
35      b01 (0.050) : +0.153
36      b05 (0.050) : +0.056
37      b02 (0.050) : +0.014
38      b06 (0.050) : -0.042
39      c03 (0.167) : -0.111
40      b10 (0.050) : -0.111
41 Summary:
42     Weighted mean marginal correlation (a): +0.099
43     Standard deviation (b) : +0.132
44     Ranking fairness (a)-(b) : -0.033

```

Both KEMENY rankings show the same *weighted mean marginal correlation* (+0.099, see Listing 8.11 on page 112 Lines 19-22, 42-44) with all thirteen performance criteria. However, the second ranking shows a slightly lower *standard deviation* (+0.132 versus +0.177), resulting in a slightly *fairer* ranking result (-0.033 versus -0.079).

When several rankings with maximal KEMENY index are given, the Kemeny-Ranking class constructor instantiates the ranking with *highest* mean marginal correlation and, in case of ties, with *lowest* weighted standard deviation. Here we obtain ranking: [a5, a6, a7, a3, a9, a4, a1, a8, a2] (see Line 4 in Listing 8.11 on page 112 above).

## 8.5 SLATER rankings

The SLATER ranking rule is identical to KEMENY's, but it is working, instead, on the *CONDORCET–median cut polarised–* digraph  $ccd$  (Slater, 1961). SLATER's rule is also *invariant* under the codual transform and delivers again indifferently on  $g$  or  $gcd$  the following results:

**Listing 8.12** Computing a SLATER ranking

```

1 >>> from linearOrders import SlaterRanking
2 >>> sl = SlaterRanking(gcd,orderLimit=9)
3 >>> sl.slaterRanking
4   ['a5','a6','a4','a1','a3','a7','a8','a9','a2']
5 >>> corr = gcd.computeOrderCorrelation(sl.slaterRanking)
6 >>> sl.showCorrelation(corr)
7   Correlation indexes:
8     Extended Kendall tau      : +0.676
9     Epistemic determination   :  0.230
10    Bipolar-valued equivalence : +0.156
11 >>> len(sl.maximalRankings)
12    7

```

We notice in Listing 8.12 Line 8 that the first SLATER ranking is a rather good fit (+0.676), slightly better apparently than the NETFLOWS ranking result (+0.638). However, there are in fact 7 such potentially optimal SLATER rankings (see Line 12). The corresponding epistemic disjunction gives the partial ordering shown in Fig 8.4 on the following page:

**Listing 8.13** Computing the epistemic disjunction of optimal SLATER rankings

```

1 >>> slw = RankingsFusion(sl,sl.maximalRankings)
2 >>> slw.exportGraphViz(fileName='tutorialSlater')
3   *---- exporting a dot file for GraphViz tools ----*
4   Exporting to tutorialSlater.dot
5   dot -Grankdir=TB -Tpng tutorialSlater.dot\
6           -o tutorialSlater.png

```

KEMENY's and SLATER's ranking rules are furthermore computationally *difficult* problems and effective ranking results are only computable for tiny outranking digraphs (< 20 objects).

More efficient ranking heuristics, like the COPELAND and the NETFLOWS rules, are therefore needed in practice. Let us finally, after these *ranking-by-scoring* strategies, also present two popular *ranking-by-choosing* strategies.

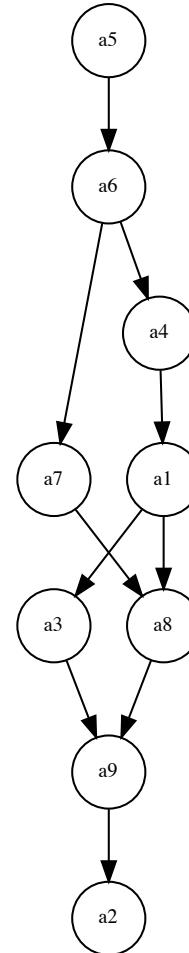
## 8.6 KOHLER's ranking-by-choosing rule

KOHLER's *ranking-by-choosing* rule<sup>4</sup> is formulated like this:

---

<sup>4</sup> (Kohler, 1978)

**Fig. 8.4** Epistemic disjunction of optimal SLATER rankings. What precise SLATER ranking result should we hence adopt?



*DiGraph3 (graphviz)  
R. Bisдорff, 2020*

**Definition 8.3 (KOHLER's *ranking-by-choosing* rule).**

At step  $i$  ( $i$  goes from 1 to  $n$ ) do the following:

1. Compute for each row of the bipolar-valued *strict* outranking relation table (see Listing 8.1 on page 105) the smallest value;

2. Select the row where this minimum is maximal. Ties are resolved in lexicographic order;
3. Put the selected decision alternative at rank  $i$ ;
4. Delete the corresponding row and column from the relation table and restart until the table is empty.

**Listing 8.14** Computing a KOHLER ranking

```

1 >>> from linearOrders import KohlerRanking
2 >>> kocd = KohlerRanking(gcd)
3 >>> kocd.showRanking()
4   ['a5','a7','a6','a3','a9','a8','a4','a1','a2']
5 >>> corr = gcd.computeOrdinalCorrelation(kocd)
6 >>> gcd.showCorrelation(corr)
7   Correlation indexes:
8     Extended Kendall tau      : +0.747
9     Epistemic determination   :  0.230
10    Bipolar-valued equivalence : +0.172

```

With this *min-max* lexicographic ranking-by-choosing strategy, we find a correlation result (+0.747) that is until now clearly the nearest to an optimal KEMENY ranking (see Listing 8.9 on page 112). Only two adjacent pairs: (a6, a7) and (a8, a9) are actually inverted here. Notice that KOHLER's ranking rule, contrary to the previously mentioned rules, is **not** invariant under the codual transform and requires to work on the strict outranking digraph  $g$  for a better correlation result.

```

1 >>> ko = KohlerRanking(g)
2 >>> corr = g.computeOrdinalCorrelation(ko)
3 >>> g.showCorrelation(corr)
4   Correlation indexes:
5     Crisp ordinal correlation  : +0.483
6     Epistemic determination    :  0.230
7     Bipolar-valued equivalence : +0.111

```

But KOHLER's ranking has a *dual* version, the prudent Arrow-Raynaud ordering-by-choosing rule, where a corresponding *max-min* strategy, when used on the *non-strict* outranking digraph  $g$ , for ordering from *last* to *first* produces a similar ranking result (Arrow and Raynaud, 1986).

Noticing that the NETFLOWS score of an alternative  $x$  represents in fact a bipolar-valued characteristic of the assertion “alternative  $x$  is ranked first”, we may enhance KOHLER's rule by replacing the simple *min-max* strategy with an *iterated* maximal NETFLOWS score selection.

#### Definition 8.4 (The iterated NETFLOWS ranking-by-choosing rule).

At step  $i$  ( $i$  goes from 1 to  $n$ ) do the following:

1. Compute for each row of the bipolar-valued outranking relation table (see Listing 8.1 on page 105) the corresponding NETFLOWS score;
2. Select the row where this score is *maximal*, ties being resolved by lexicographic order;
3. Put the corresponding decision alternative at rank  $i$ ;

4. Delete the corresponding row and column from the relation table and restart until the table is empty.

The `IteratedNetFlowsRanking` class from the `linearOrders` module computes this ranking result.

**Listing 8.15** Ranking-by-choosing with iterated maximal NETFLOWS scores

```

1  >>> from linearOrders import IteratedNetFlowsRanking
2  >>> inf = IteratedNetFlowsRanking(g)
3  >>> inf
4  *----- Digraph instance description -----*
5  Instance class      : IteratedNetFlowsRanking
6  Instance name       : rel_randomCBperftab_ranked
7  Digraph Order       : 9
8  Digraph Size        : 36
9  Valuation domain    : [-1.00;1.00]
10 Determinateness (%) : 100.00
11 Attributes          : ['valuedRanks', 'valuedOrdering',
12                         'iteratedNetFlowsRanking',
13                         'iteratedNetFlowsOrdering',
14                         'name', 'actions', 'order',
15                         'valuationdomain', 'relation',
16                         'gamma', 'notGamma']
17 >>> inf.iteratedNetFlowsRanking
18  ['a5','a7','a6','a3','a4','a1','a8','a9','a2']
19 >>> corr = g.computeRankingCorrelation(\n20 ...
21  ...           inf.iteratedNetFlowsRanking)
22 >>> g.showCorrelation(corr)
22 Correlation indexes:
23  Crisp ordinal correlation : +0.743
24  Epistemic determination   : 0.230
25  Bipolar-valued equivalence : +0.171

```

Like KOHLER's rule, the iterated NETFLOWS's rule has also a dual *ordering-by-choosing* version, where instead of choosing at each step  $i$  the row with maximal NETFLOWS score, we choose the row with the *minimal* NETFLOWS score. Both the ranking and ordering result are computed by the `IteratedNetFlowsRanking` class constructor (see Lines 12 and 13 in Listing 8.15).

```

1  >>> inf.iteratedNetFlowsOrdering
2  ['a2','a9','a1','a4','a3','a8','a7','a6','a5']
3  >>> corr = g.computeOrderCorrelation(\n4 ...
4  ...           inf.iteratedNetFlowsOrdering)
5  >>> g.showCorrelation(corr)
6  Correlation indexes:
7  Crisp ordinal correlation : +0.751
8  Epistemic determination   : 0.230
9  Bipolar-valued equivalence : +0.173

```

The iterated NETFLOWS ranking and its dual, the iterated NETFLOWS ordering, do not usually deliver both the same result. With our example outranking digraph  $g$  for instance, it is the *ordering-by-choosing* result who obtains a slightly better

correlation with the given outranking digraph (+0.751), a result that is also slightly better than KOHLER's original result (+0.747, see Listing 8.14 on page 117 Line 8).

With different *ranking-by-choosing* and *ordering-by-choosing* results, it may be useful to *fuse* now, similar to what we have done before with KEMENY's and SLATER's optimal rankings, both, the iterated NETFLOWS ranking and ordering into a partial ranking. But we are hence back to the practical problem of what linear ranking should we eventually retain?

Let us finally mention another interesting *ranking-by-choosing* approach.

## 8.7 The RANKEDPAIRS ranking rule

Tideman's ranking-by-choosing heuristic, the RANKEDPAIRS rule, working best this time on the non strict outranking digraph  $g$ , is based on a *prudent incremental* construction of linear orders that avoids on the fly any cycling outrankings (Tideman, 1987). The ranking rule may be formulated as follows:

**Definition 8.5 (The RANKEDPAIRS ranking rule).**

1. Rank the ordered pairs  $(x, y)$  of alternatives in decreasing order of  $r(x \succsim y) + r(y \not\succsim x)$ ;
2. Consider the pairs in that order (ties are resolved by a lexicographic rule):
  - if the next pair does not create a *circuit* with the pairs already blocked, block this pair;
  - if the next pair creates a *circuit* with the already blocked pairs, skip it.

With our didactic outranking digraph  $g$ , we get the following result.

**Listing 8.16** Computing a RANKEDPAIRS ranking

```

1 >>> from linearOrders import RankedPairsRanking
2 >>> rp = RankedPairsRanking(g)
3 >>> rp.showRanking()
4   ['a5','a6','a7','a3','a8','a9','a4','a1','a2']
```

The RANKEDPAIRS rule renders in our example here luckily one of the two optimal KEMENY ranking, as we may verify below.

```

1 >>> ke.maximalRankings
2   [['a5','a6','a7','a3','a8','a9','a4','a1','a2'],
3    ['a5','a6','a7','a3','a9','a4','a1','a8','a2']]
4 >>> corr = g.computeOrdinalCorrelation(rp)
5 >>> g.showCorrelation(corr)
6   Correlation indexes:
7     Extended Kendall tau      : +0.779
8     Epistemic determination   : 0.230
9     Bipolar-valued equivalence : +0.179
```

Similar to KOHLER's rule, the RANKEDPAIRS rule has also a prudent dual version, the *Dias-Lamboray ordering-by-choosing* rule, which produces, when working

this time on the codual strict outranking digraph gcd, a similar ranking result (see [Dias and Lamboray \(2010\)](#)).

Besides of not providing a unique linear ranking, the *ranking-by-choosing* rules, as well as their duals, the *ordering-by-choosing* rules, are unfortunately not scalable to outranking digraphs of larger orders ( $> 100$ ). For such bigger outranking digraphs, with several hundred or thousands of alternatives, only the COPELAND and the NETFLOWS *ranking-by-scoring* rules, with a polynomial complexity of  $\mathcal{O}(n^2)$ , where  $n$  is the order of the outranking digraph, remain in fact tractable. Furthermore, as computing the COPELAND and NETFLOWS scores may be done separately per alternative, the latter ranking rules can right away be processed in parallel when multiprocessing resources are available.

If the necessity to write a list of items in a linear sequence renders the ranking decision problem very important. Concerning however the representation of preferences, a relative rating of such items into performance quantiles classes may be more expressive and faithful. This is the subject of the following Chapter [10 on page 135](#).

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# Chapter 9

## Rating by sorting into relative performance quantiles

**Abstract** In this Chapter, we apply order statistics for sorting a set  $X$  of  $n$  potential decision alternatives, evaluated on  $m$  incommensurable performance criteria, into  $q$  quantile equivalence classes. The sorting algorithm is based on pairwise outranking characteristics involving the quantile class limits observed on each criterion. Thus we may implement a weak ordering algorithm of complexity  $O(nmq)$ .

### 9.1 Quantile sorting on a single performance criterion

A single criterion sorting category  $K$  is a (usually) lower-closed interval  $[m_k; M_k[$  on a real-valued performance measurement scale, with  $m_k \leq M_k$ . If  $x$  is a measured performance on this scale, we may distinguish three sorting situations.

1.  $x < m_k$  and  $x < M_k$ : The performance  $x$  is lower than category  $K$ .
2.  $x \geq m_k$  and  $x < M_k$ : The performance  $x$  belongs to category  $K$ .
3.  $x > m_k$  and  $x \geq M_k$ : The performance  $x$  is higher than category  $K$ .

As the relation  $<$  is the dual of  $\geq$  ( $\geq$ ), it will be sufficient to check that  $x \geq m_k$  as well as  $x \not\geq M_k$  are true for  $x$  to be considered a member of category  $K$ .

Upper-closed categories (in a more mathematical integration style) may as well be considered. In this case it is sufficient to check that  $m_k \not\geq x$  as well as  $M_k \geq x$  are true for  $x$  to be considered a member of category  $K$ . It is worthwhile noticing that a category  $K$  such that  $m_k = M_k$  is hence always empty by definition. In order to be able to properly sort over the complete range of values to be sorted, we will need to use a special, two-sided closed last, respectively first, category.

Let  $K = K_1, \dots, K_q$  be a non trivial partition of the criterion's performance measurement scale into  $q \geq 2$  ordered categories  $K_k$  – i.e. lower-closed intervals  $[m_k; M_k[$  – such that  $m_k < M_k$ ,  $M_k = m_{k+1}$  for  $k = 0, \dots, q-1$  and  $M_q = \infty$ . And, let  $A = \{a_1, a_2, a_3, \dots\}$  be a finite set of not all equal performance measures observed on the scale in question.

**Property:** For all performance measure  $x \in A$  there exists now a unique  $k$  such that  $x \in K_k$ . If we assimilate, like in descriptive statistics, all the measures gathered

in a category  $K_k$  to the central value of the category – i.e.  $(m_k + M_k)/2$  – the sorting result will hence define a weak order (complete preorder) on  $A$ .

Let  $Q = \{Q_0, Q_1, \dots, Q_q\}$  denote the set of  $q+1$  increasing order-statistical quantiles –like quartiles or deciles– we may compute from the ordered set  $A$  of performance measures observed on a performance scale. If  $Q_0 = \min(X)$ , we may, with the following intervals:  $[Q_0; Q_1[$ ,  $[Q_1; Q_2[$ , ...,  $[Q_{q-1}; \infty[$ , hence define a set of  $q$  lower-closed sorting categories. And, in the case of upper-closed categories, if  $Q_q = \max(X)$ , we would obtain the intervals  $]-\infty; Q_1]$ ,  $]Q_1; Q_2]$ , ...,  $]Q_{q-1}; Q_q]$ . The corresponding sorting of  $A$  will result, in both cases, in a repartition of all measures  $x$  into the  $q$  quantile categories  $K_k$  for  $k = 1, \dots, q$ .

**Example:** Let  $A = \{a_7 = 7.03, a_{15} = 9.45, a_{11} = 20.35, a_{16} = 25.94, a_{10} = 31.44, a_9 = 34.48, a_{12} = 34.50, a_{13} = 35.61, a_{14} = 36.54, a_{19} = 42.83, a_5 = 50.04, a_2 = 59.85, a_{17} = 61.35, a_{18} = 61.61, a_3 = 76.91, a_6 = 91.39, a_1 = 91.79, a_4 = 96.52, a_8 = 96.56, a_{20} = 98.42\}$  be a set of 20 increasing performance measures observed on a given criterion. The lower-closed category limits we obtain with quartiles ( $q = 4$ ) are:  $Q_0 = 7.03 = a_7$ ,  $Q_1 = 34.485$ ,  $Q_2 = 54.945$  (median performance), and  $Q_3 = 91.69$ . And the sorting into these four categories defines on  $A$  a complete preorder with the following four equivalence classes:  $K_1 = \{a_7, a_{10}, a_{11}, a_{10}, a_{15}, a_{16}\}$ ,  $K_2 = \{a_5, a_9, a_{13}, a_{14}, a_{19}\}$ ,  $K_3 = \{a_2, a_3, a_6, a_{17}, a_{18}\}$ , and  $K_4 = \{a_1, a_4, a_8, a_{20}\}$ .

## 9.2 Quantiles sorting with multiple performance criteria

Let us now suppose that we are given a performance tableau with a set  $X$  of  $n$  decision alternatives evaluated on a coherent family of  $m$  performance criteria associated with the corresponding outranking relation  $\succsim$  defined on  $X$ . We denote  $x_j$  the performance of alternative  $x$  observed on criterion  $j$ .

Suppose furthermore that we want to sort the decision alternatives into  $q$  upper-closed quantile equivalence classes. We therefore consider a series :  $k = k/q$  for  $k = 0, \dots, q$  of  $q+1$  equally spaced quantiles, like quartiles: 0, 0.25, 0.5, 0.75, 1; quintiles: 0, 0.2, 0.4, 0.6, 0.8, 1; or deciles: 0, 0.1, 0.2, ..., 0.9, 1, for instance.

The upper-closed  $\mathbf{q}^k$  class corresponds to the  $m$  quantile intervals  $[q_j(p_{k-1}); q_j(p_k)]$  observed on each criterion  $j$ , where  $k = 2, \dots, q$ ,  $q_j(p_q) = \max_X(x_j)$ , and the first class gathers all performances below or equal to  $Q_j(p_1)$ .

The lower-closed  $\mathbf{q}_k$  class corresponds to the  $m$  quantile intervals  $[q_j(p_{k-1}); q_j(p_k)]$  observed on each criterion  $j$ , where  $k = 1, \dots, q-1$ ,  $q_j(p_0) = \min_X(x_j)$ , and the last class gathers all performances above or equal to  $Q_j(p_{q-1})$ .

We call **q-tiles** a complete series of  $k = 1, \dots, q$  upper-closed  $\mathbf{q}^k$ , respectively lower-closed  $\mathbf{q}_k$ , multiple criteria quantile classes.

**Property:** With the help of the bipolar-valued characteristic of the outranking relation  $r(\succsim)$  we may compute the bipolar-valued characteristic of the assertion:  $x$  belongs to upper-closed  $q$ -tiles class  $\mathbf{q}^k$  class, resp. lower-closed class  $\mathbf{q}_k$ , as follows (Bisdorff, 2016) :

$$r(x \in \mathbf{q}^k) = \min [-r(\mathbf{q}(p_{q-1}) \succsim x), r(\mathbf{q}(p_q) \succsim x)] \quad (9.1)$$

$$r(x \in \mathbf{q}_k) = \min [r(x \succsim \mathbf{q}(p_{q-1})), -r(x \succsim \mathbf{q}(p_q))] \quad (9.2)$$

The outranking relation  $\succsim$  verifying the coduality principle,  $-r(\mathbf{q}(p_{q-1}) \succsim x) = r(\mathbf{q}(p_{q-1}) \prec x)$ , resp.  $-r(x \succsim \mathbf{q}(p_q)) = r(x \prec \mathbf{q}(p_q))$ .

We may compute, for instance, a five-tiling of a given random performance tableau with the help of the `QuantilesSortingDigraph` class provided by the `sortingDigraphs` module.

**Listing 9.1** Computing a quintiles sorting result

```

1 >>> from randomPerfTabs import RandomPerformanceTableau
2 >>> t = RandomPerformanceTableau(numberOfActions=50, seed=5)
3 >>> from sortingDigraphs import QuantilesSortingDigraph
4 >>> qs = QuantilesSortingDigraph(t, limitingQuantiles=5)
5 >>> qs
6     *----- Object instance description -----*
7     Instance class  : QuantilesSortingDigraph
8     Instance name   : sorting_with_5-tile_limits
9     # Actions       : 50
10    # Criteria      : 7
11    # Categories    : 5
12    Lowerclosed     : False
13    Size            : 841
14    Valuation domain : [-1.00;1.00]
15    Determinateness (%) : 81.39
16    Attributes      : ['actions','actionsOrig',
17                          'criteria','evaluation','runTimes','name',
18                          'limitingQuantiles','LowerClosed',
19                          'categories','criteriaCategoryLimits',
20                          'profiles','profileLimits','hasNoVeto',
21                          'valuationdomain','nbrThreads','relation',
22                          'categoryContent','order','gamma','notGamma']
23    *----- Constructor run times (in sec.) -----*
24    # Threads       : 1
25    Total time     : 0.03120
26    Data input     : 0.00300
27    Compute profiles : 0.00075
28    Compute relation : 0.02581
29    Weak Ordering   : 0.00052
30 >>> qs.showCriteriaQuantileLimits()
31     Quantile Class Limits (q = 5)
32     Upper-closed classes
33     crit.      0.20      0.40      0.60      0.80      1.00
34     *-----*
35     g1        31.35    41.09    58.53    71.91    98.08
36     g2        27.81    39.19    49.87    61.66    96.18
37     g3        25.10    34.78    49.45    63.97    92.59
38     g4        24.61    37.91    53.91    71.02    89.84
39     g5        26.94    36.43    52.16    72.52    96.25
40     g6        23.94    44.06    54.92    67.34    95.97
41     g7        30.94    47.40    55.46    69.04    97.10
42 >>> qs.showSorting()
```

```

43     *---- Sorting results in descending order ----*
44     ]0.80 - 1.00]: ['a22']
45     ]0.60 - 0.80]: ['a03','a07','a08','a11','a14','a17',
46             'a19','a20','a29','a32','a33','a37',
47             'a39','a41','a42','a49']
48     ]0.40 - 0.60]: ['a01','a02','a04','a05','a06','a08',
49             'a09','a16','a17','a18','a19','a21',
50             'a24','a27','a28','a30','a31','a35',
51             'a36','a40','a43','a46','a47','a48',
52             'a49','a50']
53     ]0.20 - 0.40]: ['a04','a10','a12','a13','a15','a23',
54             'a25','a26','a34','a38','a43','a44',
55             'a45','a49']
56     ] < - 0.20]: ['a44']

```

With the `showCriteriaQuantileLimits()` and the `showSorting()` methods, we may inspect the quintiles sorting result. Most of the decision alternatives (26) are gathered in the median quintile  $]0.40 - 0.60]$  class, whereas the highest quintile  $]0.80 - 1.00]$  and the lowest quintile  $] < - 0.20]$  classes gather each one a unique decision alternative (a22, resp. a44) (see Listing 9.1 on the previous page Lines 43-).

We may inspect the details of the corresponding sorting characteristics with the `showSortingCharacteristics()` method.

**Listing 9.2** Bipolar-valued sorting characteristics (extract)

```

1 >>> qs.valuationdomain
2     {'min': Decimal('-1.0'), 'med': Decimal('0'),
3      'max': Decimal('1.0')}
4 >>> qs.showSortingCharacteristics()
5     x in q^k           r(q^k-1 < x)  r(q^k >= x)  r(x in q^k)
6     a22 in ]< - 0.20]  1.00          -0.86        -0.86
7     a22 in ]0.20 - 0.40] 0.86          -0.71        -0.71
8     a22 in ]0.40 - 0.60] 0.71          -0.71        -0.71
9     a22 in ]0.60 - 0.80] 0.71          -0.14        -0.14
10    a22 in ]0.80 - 1.00] 0.14          1.00        0.14
11    ...
12    ...
13    a44 in ]< - 0.20]  1.00          0.00        0.00
14    a44 in ]0.20 - 0.40] 0.00          0.57        0.00
15    a44 in ]0.40 - 0.60] -0.57         0.86        -0.57
16    a44 in ]0.60 - 0.80] -0.86         0.86        -0.86
17    a44 in ]0.80 - 1.00] -0.86         0.86        -0.86
18    ...
19    ...
20    a49 in ]< - 0.20]  1.00          -0.43        -0.43
21    a49 in ]0.20 - 0.40] 0.43          0.00        0.00
22    a49 in ]0.40 - 0.60] 0.00          0.00        0.00
23    a49 in ]0.60 - 0.80] 0.00          0.57        0.00
24    a49 in ]0.80 - 1.00] -0.57         0.86        -0.57

```

Alternative a22 verifies indeed positively both sorting conditions only for the highest quintile  $[0.80 - 1.00]$  class (see Listing 9.2 Lines 10). Whereas alternatives a44

and  $a49$ , for instance, weakly verify both sorting conditions each one for two, resp. three, adjacent quintile classes (see Lines 13-14 and 21-23).

Quantiles sorting results indeed always verify the following **Properties**:

1. **Coherence**: Each object is sorted into a non-empty subset of *adjacent q-tiles* classes. An alternative that would *miss* evaluations on all the criteria will be sorted conjointly in all *q-tiled* classes.
2. **Uniqueness**: When  $r(x \in \mathbf{q}^k) \neq 0.0$  for  $k = 1, \dots, q$ , the performance  $x$  is sorted into *exactly one single q-tiled class*.
3. **Separability**: Computing the sorting result for performance  $x$  is *independent* from the computing of the other performances' sorting results. This property gives access to efficient parallel processing of class membership characteristics.

The *q-tiles* sorting result leaves us hence with more or less *overlapping* ordered quantile equivalence classes. For constructing now a linearly ranked *q-tiles* partition of  $X$ , we may apply three strategies:

1. **Average** (default): In decreasing lexicographic order of the average of the lower and upper quantile limits and the upper quantile class limit;
2. **Optimistic**: In decreasing lexicographic order of the upper and lower quantile class limits;
3. **Pessimistic**: In decreasing lexicographic order of the lower and upper quantile class limits;

**Listing 9.3** Weakly ranking the quintiles sorting result

```

1 >>> qs.showQuantileOrdering(strategy='average')
2     ]0.80-1.00] : ['a22']
3     ]0.60-0.80] : ['a03','a07','a11','a14','a20','a29',
4                     'a32','a33','a37','a39','a41','a42']
5     ]0.40-0.80] : ['a08','a17','a19']
6     ]0.20-0.80] : ['a49']
7     ]0.40-0.60] : ['a01','a02','a05','a06','a09','a16',
8                     'a18','a21','a24','a27','a28','a30',
9                     'a31','a35','a36','a40','a46','a47',
10                    'a48','a50']
11    ]0.20-0.60] : ['a04','a43']
12    ]0.20-0.40] : ['a10','a12','a13','a15','a23','a25',
13                    'a26','a34','a38','a45']
14    ] < -0.40] : ['a44']
```

Following, for instance, the *average* ranking strategy, we find confirmed in the weak ranking shown in Listing 9.3, that alternative  $a49$  is indeed sorted into three adjacent quintiles classes, namely  $]0.20 - 0.80]$  (see Line 6) and precedes the  $]0.40 - 0.60]$  class, of same average of lower and upper limits.

The `QuantilesSortingDigraph` constructor gives hence a linearly ordered decomposition of the corresponding bipolar-valued outranking digraph. This decomposition leads us to a new **sparse pre-ranked** outranking digraph model.

### 9.3 The sparse pre-ranked outranking digraph model

We may notice that a given outranking digraph -the association of a set of decision alternatives and an outranking relation- is, following the methodological requirements of the outranking approach, necessarily associated with a corresponding performance tableau. And, we may use this underlying performance tableau for linearly decomposing the set of potential decision alternatives into ordered quantiles equivalence classes by using the quantiles sorting technique seen in the previous Section.

In the coding example shown in Listing 9.4 below, we generate for instance, first (Lines 2-3), a simple performance tableau of 75 decision alternatives and, secondly (Lines 4-6), we construct the corresponding `PreRankedOutrankingDigraph` instance called `prg`. Notice by the way the `BigData` flag (Line 3) used here for generating a parsimoniously commented performance tableau.

**Listing 9.4** Computing a *pre-ranked* sparse outranking digraph

```

1 >>> from randomPerfTabs import RandomPerformanceTableau
2 >>> tp = RandomPerformanceTableau(numberOfActions=75, \
3 ...                                         BigData=True, seed=100)
4 >>> from sparseOutrankingDigraphs import \
5 ...             PreRankedOutrankingDigraph
6 >>> prg = PreRankedOutrankingDigraph(Tp, quantiles=5)
7 >>> prg
8     ----- Object instance description -----
9     Instance class      : PreRankedOutrankingDigraph
10    Instance name       : randomperftab_pr
11    Actions             : 75
12    Criteria            : 7
13    Sorting by          : 5-Tiling
14    Ordering strategy   : average
15    Components          : 9
16    Minimal order       : 1
17    Maximal order        : 25
18    Average order        : 8.3
19    fill rate           : 20.432%
20    Attributes          :
21        ['actions','criteria','evaluation','NA','name',
22        'order','runTimes','dimension','sortingParameters',
23        'valuationdomain','profiles','categories','sorting',
24        'decomposition','nbrComponents','components',
25        'fillRate','minimalComponentSize','maximalComponentSize',
26        ... ]

```

The ordering of the 5-tiling result is following the *average* lower and upper quintile limits strategy (see Listing 9.4 Line 14). We obtain here 9 ordered components of minimal order 1 and maximal order 25. The corresponding *pre-ranked decomposition* may be visualized as follows.

**Listing 9.5** The quantiles decomposition of a pre-ranked outranking digraph

```

1 >>> prg.showDecomposition()

```

```

2      *--- quantiles decomposition in decreasing order---*
3      c1. ]0.80-1.00] : [5, 42, 43, 47]
4      c2. ]0.60-1.00] : [73]
5      c3. ]0.60-0.80] : [1, 4, 13, 14, 22, 32, 34, 35, 40,
6                      41, 45, 61, 62, 65, 68, 70, 75]
7      c4. ]0.40-0.80] : [2, 54]
8      c5. ]0.40-0.60] : [3, 6, 7, 10, 15, 18, 19, 21, 23, 24,
9                      27, 30, 36, 37, 48, 51, 52, 56, 58,
10                     63, 67, 69, 71, 72, 74]
11     c6. ]0.20-0.60] : [8, 11, 25, 28, 64, 66]
12     c7. ]0.20-0.40] : [12, 16, 17, 20, 26, 31, 33, 38, 39,
13                     44, 46, 49, 50, 53, 55]
14     c8. ] <-0.40] : [9, 29, 60]
15     c9. ] <-0.20] : [57, 59]

```

The highest quintile class (]80% – 100%]) contains decision alternatives 5, 42, 43 and 47. Lowest quintile class (] – 20%]) gathers alternatives 57 and 59 (see Listing 9.5 on the facing page Lines 3 and 15). We may inspect the resulting sparse outranking relation map as follows in a browser view.

```
1      >>> prg.showHTMLRelationMap()
```

In Fig. 9.1 on the next page we easily recognize the 9 linearly ordered quantile equivalence classes. *Green* and *light-green* show positive **outranking** situations, whereas positive **outranked** situations are shown in *red* and *light-red*. Indeterminate situations appear in white. In each one of the 9 quantile equivalence classes we recover in fact the corresponding bipolar-valued outranking *sub-relation*, which leads to an actual **fill-rate** of 20.4% (see Listing 9.4 on the facing page Line 19).

We may now check how faithful the sparse model represents the complete outranking relation.

```

1      >>> g = BipolarOutrankingDigraph(tp)
2      >>> corr = prg.computeOrdinalCorrelation(g)
3      >>> g.showCorrelation(corr)
4      Correlation indexes:
5          Crisp ordinal correlation : +0.863
6          Epistemic determination : 0.315
7          Bipolar-valued equivalence : +0.272

```

The ordinal correlation index between the standard and the sparse outranking relations is quite high (+0.863) and their bipolar-valued equivalence is supported by a mean criteria significance majority of  $(1.0 + 0.272)/2 = 64\%$ .

It is worthwhile noticing in Listing 9.4 on the preceding page (Lines 20- ) that sparse pre-ranked outranking digraphs do not contain a *relation* attribute. The access to pairwise outranking characteristic values is here provided via a corresponding *relation()* function.

**Listing 9.6** Access to the binary relation characteristics

```

1  def relation(self,x,y):
2      """
3          Dynamic construction of the global
4          outranking characteristic function  $r(x,y)$ .

```

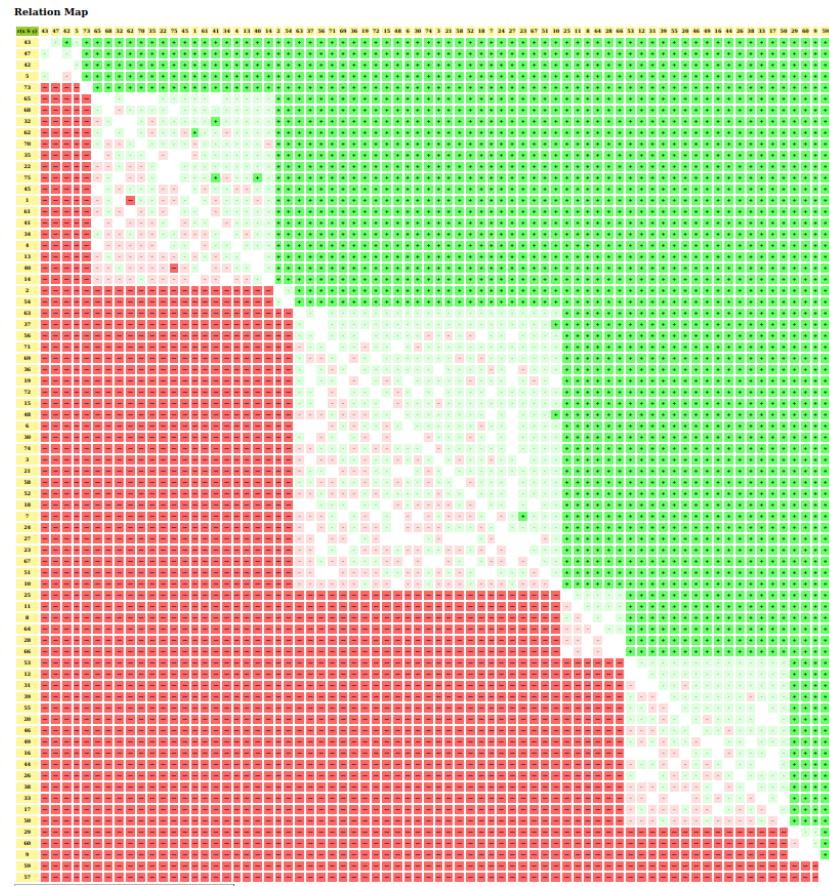


Fig. 9.1 The relation map of a sparse outranking digraph

```

5
6
7
8
9
10
11
12
13
14
15
16
17
18

```

```

    """
    Min = self.valuationdomain['min']
    Med = self.valuationdomain['med']
    Max = self.valuationdomain['max']
    if x == y:
        return Med
    cx = self.actions[x]['component']
    cy = self.actions[y]['component']
    if cx == cy:
        return self.components[cx]['subGraph'].relation[x][y]
    elif self.components[cx]['rank'] >
        self.components[cy]['rank']:
        return Min
    else:
        return Max

```

In Listing 9.6 on page 129 Lines 9-10, all reflexive situations are set to the *indeterminate* value. When two decision alternatives belong to a same component –quantile equivalence class– we access the relation attribute of the corresponding outranking sub-digraph (Lines 13-14). Otherwise we just check the respective ranks of the components (Lines 15-18).

## 9.4 Ranking pre-ranked sparse outranking digraphs

Each one of these nine ordered components may now be locally ranked by using a suitable ranking rule. Best operational results, both in run times and quality, are more or less equally obtained with the COPELAND and the NETFLOWS rules<sup>1</sup>. The eventually obtained linear ordering (from the worst to best) is stored in a `prg.boostedOrder` attribute. A reversed linear order (from the best to the worst) is stored in a `prg.boostedRanking` attribute.

```

1 >>> prg.boostedRanking
2 [43, 47, 42, 5, 73, 65, 68, 32, 62, 70, 35, 22, 75, 45, 1,
3 61, 41, 34, 4, 13, 40, 14, 2, 54, 63, 37, 56, 71, 69, 36,
4 19, 72, 15, 48, 6, 30, 74, 3, 21, 58, 52, 18, 7, 24, 27,
5 23, 67, 51, 10, 25, 11, 8, 64, 28, 66, 53, 12, 31, 39, 55,
6 20, 46, 49, 16, 44, 26, 38, 33, 17, 50, 29, 60, 9, 59, 57]
```

Alternative 43 appears *first ranked*, whereas alternative 57 is *last ranked* (see Line 2 and 6 above).

The quality of this ranking result may be assessed by computing its ordinal correlation with the standard outranking relation.

```

1 >>> corr = g.computeRankingCorrelation(prg.boostedRanking)
2 >>> g.showCorrelation(corr)
3 Correlation indexes:
4   Crisp ordinal correlation : +0.807
5   Epistemic determination   : 0.315
6   Bipolar-valued equivalence : +0.254
```

We may also verify below that the COPELAND ranking obtained from the standard outranking digraph is highly correlated (+0.822) with the one obtained from the sparse outranking digraph.

```

1 >>> from linearOrders import CopelandOrder
2 >>> cop = CopelandOrder(g)
3 >>> print(cop.computeRankingCorrelation(prg.boostedRanking))
4   {'correlation': 0.822, 'determination': 1.0}
```

Noticing the computational efficiency of the quantiles sorting construction, coupled with the separability property of the quantile class membership characteristics computation, we will make usage of the `PreRankedOutrankingDigraph`

---

<sup>1</sup> See Sections 8.2 on page 108 and 8.3 on page 109

constructor in Chapter [11 on page 147](#) for HPC ranking big and even huge performance tableaux. Before we shall address in the next Chapter [10 on page 135](#) the problem of absolute rating into learned quantile performance norms.

## References

- Bisdorff R (2016) Computing linear rankings from trillions of pairwise outranking situations. In: Busa-Fekete R, Hüllermeier E, Mousseau V, Pfannschmidt K (eds) From Multiple Criteria Decision Aid to Preference Learning , DAP'2016, University of Paderborn (Germany), pp 1–16, URL <http://hdl.handle.net/10993/28613 124>



# Chapter 10

## Rating by ranking with learned performance quantile norms

**Abstract** We address in Chapter 10 the problem of rating multiple criteria performances of a set of potential decision alternatives with respect to performance quantiles learned from historical performance data gathered from similar decision alternatives observed in the past. We show how to learn performance quantiles from historical performance tableaux. New performance records may now be rated with respect to these quantile norms.

### 10.1 The absolute rating problem

To illustrate the *absolute rating* problem we face, consider for a moment that, in a given decision making problem, we observe, for instance, in Table 10.1 below, the multi-criteria performances of two potential decision alternatives, named a1001 and a1010, graded on 7 *incommensurable* performance criteria: 2 *Costs* criteria c1 and c2 (to *minimize*) and 5 *Benefits* criteria b1 to b5 (to *maximize*). The perfor-

**Table 10.1** Multi-criteria performances of two potential decision alternatives

Criterion (weight)	b1 (2)	b2 (2)	b3 (2)	b4 (2)	b5 (2)	c1 (5)	c2 (5)
a1001	37.0	2	2	61.0	31.0	-4	-40.0
a1010	32.0	9	6	55.0	51.0	-4	-35.0

mances on *Benefits* criteria b1, b4 and b5 are measured on a cardinal scale from 0.0 (worst) to 100.0 (best) whereas, the performances on the *Benefits* criteria b2 and b3 and on the *Costs* criterion c1 are measured on an ordinal scale from 0 (worst) to 10 (best), respectively -10 (worst) to 0 (best). The performances on the *Costs* criterion c2 are eventually measured on a cardinal *negative* scale from -100.00 (worst) to 0.0 (best). The two *Costs* criteria are equi-significant (weight 5). Similarly, the

five Benefits criteria are also equi-significant (weight 2). The *importance* (sum of significance weights:  $2 \times 5 = 10$ ) of the *Costs* criteria is hence *equivalent* to the *importance* (sum of significance weights:  $5 \times 2 = 10$ ) of the *Benefits* criteria.

The non trivial decision problem we now face here, is to decide, how the previous two multiple criteria performances of  $a1001$ , respectively  $a1010$ , may be rated (*excellent* ? *good* ?, or *fair* ?; perhaps even, *weak* ? or *very weak* ?) in an *order statistical sense*, when compared with similar multi-criteria performances one has already encountered in the past.

To solve this *absolute* rating problem, first, we need to estimate multi-criteria *performance quantiles* from historical records<sup>1</sup>.

## 10.2 Incremental learning of historical performance quantiles

Suppose that we see flying in random multiple criteria performances from a given model of random performance tableau (see Chapter 5 on page 61). The question we address here is to estimate empirical performance quantiles on the basis of so far observed performance vectors. For this task, we are inspired by Chambers et al (2006), who present an efficient algorithm for incrementally updating a quantile-binned cumulative distribution function (CDF) with newly observed CDFs<sup>2</sup>.

The `PerformanceQuantiles` class, using the `IncrementalQuantiles-Estimator` class from the `randomNumbers` module, implements such a performance quantiles estimation based on a given performance tableau.

Its main attributes are:

- Ordered `objectives` and a `criteria` dictionaries from a valid performance tableau instance;
- A list `quantileFrequencies` of quantile frequencies like:
  - `quartiles` [0.0, 0.25, 0.5, 0.75, 1.0],
  - `quintiles` [0.0, 0.2, 0.4, 0.6, 0.8, 1.0] or
  - `deciles` [0.0, 0.1, 0.2, ...1.0] for instance;
- An ordered dictionary `limitingQuantiles` of so far estimated *lower* (default) or *upper* quantile class limits for each frequency per criterion;
- An ordered dictionary `historySizes` for keeping track of the number of evaluations seen so far per criterion. Missing data, the case given, make these sizes vary from criterion to criterion.

Below, we show an example Python session concerning 900 decision alternatives randomly generated with a *Cost-Benefit* performance tableau model (see Section 5.3 on page 64) from which are also drawn the performances of alternatives  $a1001$  and  $a1010$  shown in Table 10.1 on the preceding page above.

---

<sup>1</sup> See Lecture 5 of the Computational Statistics Course (Bisdorff, 2020).

<sup>2</sup> We have adapted a C++ implementation published by (Press, Teukolsky, Vetterling, and Flannery, 2007, Chapter 5).

**Listing 10.1** Computing performance quantiles from a given performance tableau

```

1 >>> from performanceQuantiles import PerformanceQuantiles
2 >>> from randomPerfTabs import RandomCBPerformanceTableau
3 >>> nbrActions=900
4 >>> nbrCrit = 7
5 >>> seed = 100
6 >>> pt = RandomCBPerformanceTableau(numberOfActions=nbrActions,\n7 ...                               numberOfCriteria=nbrCrit,seed=seed)
8 >>> pq = PerformanceQuantiles(pt,\n9 ...                               numberOfBins = 'quartiles',\n10 ...                             LowerClosed=True)
11 >>> pq
12 *----- PerformanceQuantiles instance description -----*
13     Instance class      : PerformanceQuantiles
14     Instance name       : 4-tiled_performances
15     Objectives          : 2
16     Criteria            : 7
17     Quantiles           : 4
18     History sizes       : {'c1': 887,'b1': 888,'b2': 891,'b3': 895,
19 ...                               'b4': 892,'c2': 893,'b5': 887}
20     Attributes          : ['perfTabType','valueDigits',
21 ...                               'actionsTypeStatistics',
22 ...                               'objectives', 'BigData',
23 ...                               'missingDataProbability',
24 ...                               'criteria', 'LowerClosed', 'name',
25 ...                               'quantilesFrequencies', 'historySizes',
26 ...                               'limitingQuantiles', 'cdf']
```

The `PerformanceQuantiles` class parameter `numberOfBins` (see Listing 10.1 on the preceding page Line 9 above), choosing the wished number of quantile frequencies, may be either *quartiles* (4 bins), *quintiles* (5 bins), *deciles* (10 bins), *duo-deciles* (20 bins) or any other integer number of quantile bins. The quantile bins may be either *lower closed* (default) or *upper-closed*.

Inspecting the estimated quartile limits may be operated with the `showLimitingQuantiles()` method.

**Listing 10.2** Printing out the estimated quartile limits

```

1 >>> pq.showLimitingQuantiles(ByObjectives=True)
2 *----- Historical performance quantiles -----*
3 Costs
4 criteria | weights | '0.00' '0.25' '0.50' '0.75' '1.00'
5 -----
6   'c1' | 5 | -10   -7   -5   -3   0
7   'c2' | 5 | -96.37 -70.65 -50.10 -30.00 -1.43
8 Benefits
9 criteria | weights | '0.00' '0.25' '0.50' '0.75' '1.00'
10 -----
11  'b1' | 2 | 1.99  29.82 49.44 70.73 99.83
12  'b2' | 2 | 0      3      5      7      10
13  'b3' | 2 | 0      3      5      7      10
14  'b4' | 2 | 3.27  30.10 50.82 70.89 98.05
15  'b5' | 2 | 0.85  29.08 48.55 69.98 97.56
```

Both objectives are *equi-important*; the sum of weights (10) of the *Costs* criteria balance the sum of weights (10) of the *Benefits* criteria (see Listing 10.2 on the preceding page column 2). The preference direction of the *Costs* criteria *c1* and *c2* is *negative*; the lesser the costs, the better it is, whereas all the *Benefits* criteria *b1* to *b5* show *positive* preference directions, i.e. the higher the benefits, the better it is. The columns entitled '0.00', resp. '1.00' show the *quartile*  $Q_0$ , resp.  $Q_4$ , i.e. the *worst*, resp. *best* performance observed so far on each criterion. Column '0.50' shows the *median* ( $Q_2$ ) performance observed on the criteria.

New decision alternatives with random multiple criteria performance vectors from the same random performance tableau model as *pt* (see Listing 10.1 on page 136) may now be generated with a generic *RandomPerformanceGenerator* class, provided in the *randomPerfTabs* module.<sup>3</sup>

**Listing 10.3** Generating 100 new random decision alternatives of the same model

```
1 >>> from randomPerfTabs import RandomPerformanceGenerator
2 >>> rpg = RandomPerformanceGenerator(tp, seed=seed)
3 >>> newTab = rpg.randomPerformanceTableau(100)
```

The so far estimated historical quantile limits may be updated with this newly arriving 100 data records:

```
1 >>> # Updating the quartile norms shown above
2 >>> pq.updateQuantiles(newTab, historySize=None)
```

Parameter *historySize* of the *updateQuantiles()* method (Line 2 above) allows to *balance* the *new* evaluations against the *historical* ones.

With *historySize* = *None* (the default setting), the balance in the example above is 900/1000 (90%, the weight of historical data) against 100/1000 (10%, the weight of the new incoming observations). Putting *historySize* = 0, for instance, will ignore all historical data (0/100 against 100/100) and restart building the quantile estimation with solely the new incoming data. The updated quantile limits may be shown in a browser view (see Fig. 10.1 on the next page).

```
1 >>> # showing the updated quantile limits in a browser view
2 >>> pq.showHTMLLimitingQuantiles(Transposed=True)
```

### 10.3 Rating-by-ranking new performances with quantile norms

For *rating* a newly given set of decision alternatives with the help of empirical performance quantiles estimated from historical data, we provide in the *sortingDigraphs* module the *LearnedQuantilesRatingDigraph* class, a specialisation of the *QuantilesSortingDigraph* class (see Chapter 9 on page 123). The absolute

---

<sup>3</sup> The *RandomPerformanceGenerator* class works for the *standard* performance tableau model (see Section 5.2 on page 63), the *Cost-Benefit* model (see Section 5.3 on page 64), and the 3-objectives model (see Section 5.4 on page 65).

**Fig. 10.1** Showing updated quartiles limits per criterion. The 0.25 column shows the first quartile ( $Q_1$ ) limits, the 0.50 column shows the second quartile ( $Q_2$ ) limits and the 0.75 column shows the third quartile ( $Q_3$ ) limits. Column 0.00 (resp. 1.00) shows the minimum (resp. maximum) performance on each criterion

## Performance quantiles

Sampling sizes between 986 and 995.

criterion	0.00	0.25	0.50	0.75	1.00
<b>b1</b>	1.99	28.77	49.63	75.27	99.83
<b>b2</b>	0.00	2.94	4.92	6.72	10.00
<b>b3</b>	0.00	2.90	4.86	8.01	10.00
<b>b4</b>	3.27	35.91	58.58	72.00	98.05
<b>b5</b>	0.85	32.84	48.09	69.75	99.00
<b>c1</b>	-10.00	-7.35	-5.39	-3.38	0.00
<b>c2</b>	-96.37	-72.22	-52.27	-33.99	-1.43

rating result is computed by *ranking* the new performance records together with the learned quantile limits.

The constructor requires a valid `PerformanceQuantiles` instance and, by default, uses COPELAND's or the NETFLOWS ranking rule, whichever fits best in an ordinal correlation sense with the underlying outranking digraph.

It is important to notice that the `LearnedQuantilesRatingDigraph` class, contrary to the generic `OutrankingDigraph` class, does not inherit from the generic `PerformanceTableau` class, but instead from the `PerformanceQuantiles` class.

The `actions` attribute in such a `LearnedQuantilesRatingDigraph` class instance contains not only the newly given decision alternatives, but also the historical quantile limits (attribute `profiles`) obtained from a given `PerformanceQuantiles` class instance, i.e. estimated quantile bins' performance limits from historical performance data.

We reconsider now the `PerformanceQuantiles` object instance `pq` as computed in the previous section. Let `newActions` be a list of 10 new decision alternatives generated with the same random performance tableau generator `rpg` and including, for our didactical purpose, the two decision alternatives `a1001` and `a1010` mentioned at the beginning.

**Listing 10.4** Computing the absolute rating of 10 new decision alternatives

```

1 >>> from sortingDigraphs import LearnedQuantilesRatingDigraph
2 >>> newActions = rpg.randomActions(10)
3 >>> lqr = LearnedQuantilesRatingDigraph(pq, newActions, \
4 ...                                         rankingRule='best')
5 >>> lqr
6     ----- Object instance description
7     Instance class      : LearnedQuantilesRatingDigraph
8     Instance name       : normedRatingDigraph
9     Criteria            : 7
10    Quantile profiles   : 4
11    Lower-closed bins   : True

```

```

12  New actions      : 10
13  Size            : 93
14  Determinateness (%) : 76.1
15  Ranking rule     : Copeland
16  Ordinal correlation : +0.95
17  Attributes: ['runTimes','objectives','criteria',
18    'LowerClosed','quantilesFrequencies',
19    'limitingQuantiles','historySizes','cdf','name',
20    'newActions','evaluation','categories',
21    'criteriaCategoryLimits','profiles','profileLimits',
22    'hasNoVeto','actions','completeRelation','relation',
23    'concordanceRelation','valuationdomain','order',
24    'gamma','notGamma','rankingRule','rankingCorrelation',
25    'rankingScores','actionsRanking','ratingCategories',
26    'ratingRelation','relationOrig']
27  *---- Constructor run times (in sec.)
28  Threads          : 1
29  Total time       : 0.02218
30  Data input        : 0.000134
31  Quantile classes : 0.000008
32  Compute profiles : 0.000021
33  Compute relation : 0.01869
34  Compute rating   : 0.00186
35  Compute sorting   : 0.00000

```

Data input to the `LearnedQuantilesRatingDigraph` class constructor are a valid `PerformanceQuantiles` object *pq* and a `newActions` list of newly generated decision alternatives with the same random generator *rpg* (see Listing 10.3 on page 138 Line 2-3).

Let us have a look at the digraph's nodes, here called `newActions`.

**Listing 10.5** Performance tableau of the new incoming decision alternatives

```

1 >>> lqr.showPerformanceTableau(actionsSubset=lqr.newActions)
2 *---- performance tableau ----*
3 criteria | a1001 a1002 a1003 a1004 a1005 a1006 a1007 a1008 a1009 a1010
4 -----
5 'b1' | 37.0 27.0 24.0 16.0 42.0 33.0 39.0 64.0 42.0 32.0
6 'b2' | 2.0 5.0 8.0 3.0 3.0 3.0 6.0 5.0 4.0 9.0
7 'b3' | 2.0 4.0 2.0 1.0 6.0 3.0 2.0 6.0 6.0 6.0
8 'b4' | 61.0 54.0 74.0 25.0 28.0 20.0 20.0 49.0 44.0 55.0
9 'b5' | 31.0 63.0 61.0 48.0 30.0 39.0 16.0 96.0 57.0 51.0
10 'c1' | -4.0 -6.0 -8.0 -5.0 -1.0 -5.0 -1.0 -6.0 -6.0 -4.0
11 'c2' | -40.0 -23.0 -37.0 -37.0 -24.0 -27.0 -73.0 -43.0 -94.0 -35.0

```

Among the 10 new incoming decision alternatives (see Listing 10.5), we recognize alternatives `a1001` (see column 2) and `a1010` (see last column) we have shown in Table 10.1 on page 135.

The `actions` dictionary of a `LearnedQuantilesRatingDigraph` class instance includes, besides the 10 performance evaluations of the ten new alternatives, also the closed lower limits of the four quartile classes:  $m1 = [0.0 - [$ ,  $m2 = [0.25 - [$ ,  $m3 = [0.5 - [$ ,  $m4 = [0.75 - [$ . We find these limits in the `profiles` attribute (see Listing 10.6 below).

**Listing 10.6** Showing the limiting profiles of the rating quantiles

```

1 >>> lqr.showPerformanceTableau(actionsSubset=lqr.profiles)

```

```

2      *---- Quartiles limit profiles ----*
3      criteria | 'm1'   'm2'   'm3'   'm4'
4      -----|-----
5      'b1'   | 2.0    28.8   49.6   75.3
6      'b2'   | 0.0    2.9    4.9    6.7
7      'b3'   | 0.0    2.9    4.9    8.0
8      'b4'   | 3.3    35.9   58.6   72.0
9      'b5'   | 0.8    32.8   48.1   69.7
10     'c1'   | -10.0  -7.4   -5.4   -3.4
11     'c2'   | -96.4  -72.2  -52.3  -34.0

```

The main run time is spent by the class constructor in computing a bipolar-valued outranking relation on the extended actions set including both the new alternatives as well as the quartile class limits (see Listing 10.4 on page 139 Lines 23-29).<sup>4</sup>

The actual rating procedure will rely on a complete ranking of the new decision alternatives as well as the quartile class limits obtained from the corresponding bipolar-valued outranking digraph. Two efficient and scalable ranking rules, the COPELAND and its valued version, the NETFLOWS rule may be used for this purpose. The `rankingRule` parameter allows to choose one of both. With `rankingRule='best'` (see Listing 10.4 on page 139 Line 3) the `LearnedQuantilesRatingDigraph` constructor will choose the ranking rule that results in the highest ordinal correlation with the given outranking relation (see Chapter 22 on page 333 and (Bisdorff, 2012)).

In this rating example, the COPELAND rule appears to be the more appropriate ranking rule.

**Listing 10.7** COPELAND ranking of new alternatives and historical quartile limits

```

1 >>> lqr.rankingRule
2   'Copeland'
3 >>> lqr.actionsRanking
4   ['m4', 'a1005', 'a1010', 'a1002', 'a1008', 'a1006', 'a1001',
5   'a1003', 'm3', 'a1007', 'a1004', 'a1009', 'm2', 'm1']
6 >>> lqr.showCorrelation(lqr.rankingCorrelation)
7   Correlation indexes:
8   Crisp ordinal correlation : +0.945
9   Epistemic determination   :  0.522
10  Bipolar-valued equivalence : +0.493

```

We achieve here (see Listing 10.7) a linear ranking without ties (from best to worst) of the digraph's actions set, i.e. including the new decision alternatives as well as the quartile limits  $m1$  to  $m4$ , which is very close in an ordinal sense ( $\tau = 0.945$ ) to the underlying strict outranking relation.

The eventual rating procedure is based in this example on the *lower* quartile limits, such that we may collect the quartile classes' contents in increasing order of the *quartiles*.

```

1 >>> lqr.ratingCategories
2 OrderedDict([

```

<sup>4</sup> In case of large volumes, i.e. many new decision alternatives and centile classes for instance, a multi-threading version may be used when multiple processing cores are available.

```

3 ('m2', ['a1007','a1004','a1009']),
4 ('m3', ['a1005','a1010','a1002','a1008',
5   'a1006','a1001','a1003'])
6 ]

```

We notice above that no new decision alternative is actually rated into the lowest  $[0.0 - 0.25[$ , respectively highest  $[0.75 - [$  quartile classes. Indeed, the rating result is shown with the `showQuantilesRating()` method:

**Listing 10.8** Absolute quartiles rating result

```

1 >>> lqr.showQuantilesRating()
2 ----- Quartiles rating result -----
3 [0.50 - 0.75[ ['a1005', 'a1010', 'a1002', 'a1008',
4   'a1006', 'a1001', 'a1003']
5 [0.25 - 0.50[ ['a1007', 'a1004', 'a1009']

```

The same result may also be shown in a browser view with the `showHTMLRatingHeatmap()` method using a specialised rating heatmap format:

```

1 >>> lqr.showHTMLRatingHeatmap(
2 ...           pageTitle='Heatmap of Quartiles Rating', \
3 ...           Correlations=True, colorLevels=5)

```

**Fig. 10.2** Heatmap of absolute quartiles ranking. The quartile equivalence classes appear lower-closed. No alternative is rated into the Q1 class ( $[0.00 - 0.25[$ ) and no alternative is rated into the Q4 class ( $[0.75 - 1.00]$ )

### Heatmap of Quartiles Rating

Ranking rule: Copeland; Ranking correlation: 0.938

criteria	c2	b3	c1	b4	b1	b2	b5
weights	5	2	5	2	2	2	2
tau <sup>(*)</sup>	+0.64	+0.54	+0.43	+0.37	+0.37	+0.35	+0.34
[0.75 -	-30.00	7.00	-3.00	70.89	70.73	7.00	69.98
<b>a1005c</b>	-24.00	6.00	-1.00	28.00	42.00	3.00	30.00
<b>a1010n</b>	-35.00	6.00	-4.00	55.00	32.00	9.00	51.00
<b>a1002c</b>	-23.00	4.00	-6.00	54.00	27.00	5.00	63.00
<b>a1008n</b>	-43.00	6.00	-6.00	49.00	64.00	5.00	96.00
<b>a1006c</b>	-27.00	3.00	-5.00	20.00	33.00	3.00	39.00
<b>a1001c</b>	-40.00	2.00	-4.00	61.00	37.00	2.00	31.00
<b>a1003a</b>	-37.00	2.00	-8.00	74.00	24.00	8.00	61.00
[0.50 -	-50.10	5.00	-5.00	50.82	49.44	5.00	48.55
<b>a1007c</b>	-73.00	2.00	-1.00	20.00	39.00	6.00	16.00
<b>a1004c</b>	-37.00	1.00	-5.00	25.00	16.00	3.00	48.00
<b>a1009n</b>	-94.00	6.00	-6.00	44.00	42.00	4.00	57.00
[0.25 -	-70.65	3.00	-7.00	30.10	29.82	3.00	29.08
[0.00 -	-96.37	0.00	-10.00	3.27	1.99	0.00	0.85

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

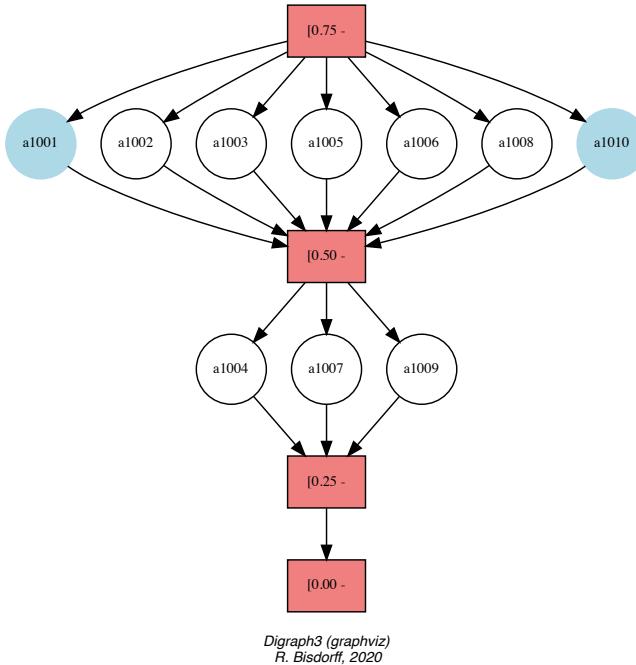
(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation.

Using furthermore a specialised version of the `exportGraphViz()` method allows drawing the same rating result in a Hasse diagram format (see Fig. 10.3 on the next page).

```

1 >>> lqr.exportRatingGraphViz('quartileRatingDigraph')
2 *---- exporting a dot file for GraphViz tools -----*
3 Exporting to quartileRatingDigraph.dot
4 dot -Grankdir=TB -Tpng quartileRatingDigraph.dot -o
    quartileRatingDigraph.png

```



**Fig. 10.3** Absolute quartiles rating digraph

We may now answer the *absolute rating* problem stated at the beginning. Decision alternatives  $a1001$  and  $a1010$  (see Table 10.1 on page 135) are both rated into the same third quartile  $Q3$  class (see Fig. 10.3), even if the COPELAND ranking, obtained from the underlying strict outranking digraph (see Fig. 10.2 on the facing page), suggests that alternative  $a1010$  is effectively *better performing than* alternative  $a1001$ .

A *preciser* rating result may indeed be achieved when using *deciles* instead of *quartiles* for estimating the historical marginal cumulative distribution functions.

**Listing 10.9** Absolute deciles rating result

```

1 >>> pq1 = PerformanceQuantiles(pt, numberOfBins = 'deciles', \
2 ...                                LowerClosed=True)

```

```

3 >>> pq1.updateQuantiles(newTab, historySize=None)
4 >>> lqr1 =
5     LearnedQuantilesRatingDigraph(pq1, newActions, rankingRule='best')
6 >>> lqr1.showQuantilesRating()
7     ----- Deciles rating result -----
8     [0.60 - 0.70[ ['a1005', 'a1010', 'a1008', 'a1002']
9     [0.50 - 0.60[ ['a1006', 'a1001', 'a1003']
10    [0.40 - 0.50[ ['a1007', 'a1004']
11    [0.30 - 0.40[ ['a1009']

```

Compared with the quartiles rating result, we notice in Listing 10.9 on the previous page that the seven alternatives (a1001, a1002, a1003, a1005, a1006, a1008 and a1010), rated before into the third quartile class  $[0.50 - 0.75[$ , are now divided up: alternatives a1002, a1005, a1008 and a1010 attain now the 7th decile class  $[0.60 - 0.70[$ , whereas alternatives a1001, a1003 and a1006 attain only the 6th decile class  $[0.50 - 0.60[$ . Of the three Q2  $[0.25 - 0.50[$  rated alternatives (a1004, a1007 and a1009), alternatives a1004 and a1007 are now rated into the 5th decile class  $[0.40 - 0.50[$  and a1009 is lowest rated into the 4th decile class  $[0.30 - 0.40[$ .

A browser heatmap view may more conveniently illustrate this refined rating result (see Fig. 10.4 on the facing page).

```

1 >>> lqr1.showHTMLRatingHeatmap(\n2 ...     pageTitle='Heatmap of the deciles rating', \
3 ...     colorLevels=5, Correlations=True)

```

To avoid having to recompute performance deciles from historical data when wishing to refine a rating result, it is useful, depending on the actual size of the historical data, to initially compute performance quantiles with a relatively high number of bins, for instance *dodeciles* or even *centiles*. It is then possible to interpolate on the fly *quartiles* or *deciles* for instance, when constructing the rating digraph.

#### Listing 10.10 From deciles interpolated quartiles rating result

```

1 >>> lqr2 = LearnedQuantilesRatingDigraph(pq1, newActions,
2 ...             quantiles='quartiles')
3 >>> lqr2.showQuantilesRating()
4     ----- Deciles rating result -----
5     [0.50 - 0.75[ ['a1005', 'a1010', 'a1002', 'a1008',
6     'a1006', 'a1001', 'a1003']
7     [0.25 - 0.50[ ['a1004', 'a1007', 'a1009']

```

With the *quantiles* parameter (see Listing 10.10 Line 2), we may recover by interpolation the same quartiles rating as obtained directly with historical performance quartiles (see Listing 10.8 on page 142). Mind that a correct interpolation of quantiles from a given cumulative distribution function requires more or less uniform distributions of observations in each bin.

More generally, in the case of industrial production monitoring problems, for instance, where large volumes of historical performance data may be available, it may be of interest to estimate even more precisely the marginal cumulative distribution functions, especially when *tail* rating results, i.e. distinguishing *very best*, or *very weak* multiple criteria performances, become a critical issue. Similarly, the

**Fig. 10.4** Heatmap of absolute deciles rating. Decision alternatives a1001 and a1010 are now, as expected, rated in the 6<sup>th</sup> decile (D6), respectively in the 7<sup>th</sup> decile (D7)

### Heatmap of Deciles rating

Ranking rule: **NetFlows**; Ranking correlation: **0.960**

criteria	c2	b3	c1	b1	b5	b2	b4
<b>weights</b>	5	2	5	2	2	2	2
<b>tau(*)</b>	0.67	0.65	0.58	0.57	0.53	0.53	0.48
[0.90 -]	-20.32	7.73	-2.53	86.83	82.16	7.66	82.04
[0.80 -]	-29.70	7.26	-3.35	79.30	75.15	6.64	74.66
[0.70 -]	-37.97	6.67	-4.14	70.95	60.20	5.88	69.76
<b>a1005c</b>	-24.00	6.00	-1.00	42.00	30.00	3.00	28.00
<b>a1010n</b>	-35.00	6.00	-4.00	32.00	51.00	9.00	55.00
<b>a1008n</b>	-43.00	6.00	-6.00	64.00	96.00	5.00	49.00
<b>a1002c</b>	-23.00	4.00	-6.00	27.00	63.00	5.00	54.00
[0.60 -]	-44.23	5.92	-5.04	60.56	56.01	5.37	62.23
<b>a1006c</b>	-27.00	3.00	-5.00	33.00	39.00	3.00	20.00
<b>a1001c</b>	-40.00	2.00	-4.00	37.00	31.00	2.00	61.00
<b>a1003a</b>	-37.00	2.00	-8.00	24.00	61.00	8.00	74.00
[0.50 -]	-52.22	4.64	-6.02	49.56	48.07	4.83	58.45
<b>a1007c</b>	-73.00	2.00	-1.00	39.00	16.00	6.00	20.00
<b>a1004c</b>	-37.00	1.00	-5.00	16.00	48.00	3.00	25.00
[0.40 -]	-60.50	3.84	-6.69	39.61	40.16	4.25	49.82
<b>a1009n</b>	-94.00	6.00	-6.00	42.00	57.00	4.00	44.00
[0.30 -]	-67.14	3.12	-7.32	30.85	34.33	3.30	40.89
[0.20 -]	-77.07	2.55	-7.94	23.84	29.57	2.27	30.45
[0.10 -]	-83.04	1.99	-8.48	16.64	16.91	1.58	24.78
[0.00 -]	-96.37	0.00	-10.00	1.99	0.85	0.00	3.27

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation.

historySize parameter may be used for monitoring on the fly *unstable* random multiple criteria performance data.

## References

- Bisdorff R (2012) On measuring and testing the ordinal correlation between bipolar outranking relations. In: Mousseau V, Pirlot M (eds) DAP'2012 From Multiple Criteria Decision Aid to Preference Learning, University of Mons (Belgium), pp 91–100, URL <http://hdl.handle.net/10993/23909> 141
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# Chapter 11

## HPC ranking of big performance tableaux

**Abstract** The sparse outranking digraph, introduced in Chapter 9 is suitable for tackling the ranking of big multiple criteria performance tableaux with thousands or millions of records. To effectively compute rankings from performance tableaux of these sizes, we propose in this chapter a collection of C-compiled and optimised DIGRAPH3 modules that may be run on HPC equipment as available, for instance at the University of Luxembourg.

### 11.1 C-compiled Python modules

The DIGRAPH3 collection provides cythonized <sup>1</sup>, i.e. C-compiled and optimised versions of the main python modules for tackling multiple criteria decision problems facing very large sets of decision alternatives ( $> 10000$ ). Such problems appear usually with a combinatorial organisation of the potential decision alternatives, as is frequently the case in bioinformatics for instance. If HPC facilities with nodes supporting numerous cores ( $> 20$ ) and big RAM ( $> 50\text{GB}$ ) are available, ranking up to several millions of alternatives becomes effectively tractable (see (Bisdorff, 2016)).

Four cythonized DIGRAPH3 modules, prefixed with the letter `c` and taking a `.pyx` extension, are provided with their corresponding setup tools in the `cython` directory of the DIGRAPH3 resources, namely:

```
cRandPerfTabs.pyx,  
cIntegerOutrankingDigraphs.pyx,  
cIntegerSortingDigraphs.pyx, and  
cSparseIntegerOutrankingDigraphs.pyx.
```

Their automatic compilation and installation, alongside the standard DIGRAPH3 python3 modules, requires the `cython` compiler:

```
...$ python3 -m pip install cython
```

---

<sup>1</sup> CYTHON C-extension for Python, <https://cython.org/>

and a C compiler:

```
...$ sudo apt install gcc on Ubuntu, for instance.
```

These cythonized modules, specifically designed for being run on HPC clusters (see <https://hpc.uni.lu>), require the Unix *forking* start method of subprocesses (see start methods of the [Python multiprocessing library](#)) and therefore, due to forking problems on Mac OS platforms, may only operate safely on Linux platforms.

## 11.2 Big Data performance tableaux

In order to efficiently type the C variables, the `cRandPerfTabs` module provides the root `cPerformanceTableau` class, but, with *integer* action keys, *float* performance evaluations, *integer* criteria weights and *float* discrimination thresholds. And, to limit as much as possible memory occupation of class instances, all the usual verbose comments are dropped from the description of the actions and `criteria` dictionaries. A random instance may be generated with the `cRandomPerformanceTableau` class:

**Listing 11.1** Big data performance tableau format

```

1 >>> from cRandPerfTabs import cRandomPerformanceTableau
2 >>> t = cRandomPerformanceTableau(numberOfActions=4,\n3 ...                                numberOfCriteria=2)
4 >>> t
5     ----- PerformanceTableau instance description -----
6     Instance class      : cRandomPerformanceTableau
7     Seed                : None
8     Instance name       : cRandomperftab
9     Actions             : 4
10    Criteria            : 2
11    Attributes          : ['randomSeed', 'name', 'actions',
12                           'criteria', 'evaluation', 'weightPreorder']
13 >>> t.actions
14     OrderedDict([(1, {'name': '#1'}), (2, {'name': '#2'}),
15                  (3, {'name': '#3'}), (4, {'name': '#4'})])
16 >>> t.criteria
17     OrderedDict([
18         ('g1', {'name': 'RandomPerformanceTableau() instance',
19                  'thresholds': {'ind': (10.0, 0.0),
20                               'pref': (20.0, 0.0),
21                               'veto': (80.0, 0.0)},
22                  'scale': (0.0, 100.0),
23                  'weight': 1,
24                  'preferenceDirection': 'max'}),
25         ('g2', {'name': 'RandomPerformanceTableau() instance',
26                  'thresholds': {'ind': (10.0, 0.0),
27                               'pref': (20.0, 0.0),
28                               'veto': (80.0, 0.0)},
29                  'scale': (0.0, 100.0),
30                  'weight': 1,
31                  'preferenceDirection': 'max'}))
32 >>> t.evaluation
33     {'g1': {1: 35.17, 2: 56.4, 3: 1.94, 4: 5.51},
34      'g2': {1: 95.12, 2: 90.54, 3: 51.84, 4: 15.42}}
35 >>> t.showPerformanceTableau()
36     Criteria |  'g1'    'g2'
37     Actions  |    1      1
38     -----|-----

```

```

39      '#1'  |  91.18  90.42
40      '#2'  |  66.82  41.31
41      '#3'  |  35.76  28.86
42      '#4'  |  7.78   37.64

```

Several methods for converting from the Big Data model to the standard model and vice versa are provided by the `cPerformanceTableau` class.

```

1 >>> t1 = t.convert2Standard()
2 >>> t1.convertWeight2Decimal()
3 >>> t1.convertEvaluation2Decimal()
4 >>> t1
5      *----- PerformanceTableau instance description -----*
6      Instance class      : PerformanceTableau
7      Seed                 : None
8      Instance name       : std_cRandomperftab
9      Actions              : 4
10     Criteria             : 2
11     Attributes          : ['name', 'actions', 'criteria',
12                           'weightPreorder', 'evaluation',
13                           'randomSeed']

```

## 11.3 C-implemented integer-valued outranking digraphs

The C compiled version of the `BipolarOutrankingDigraph` class takes integer  $r$ -characteristic values.

**Listing 11.2** Constructing big bipolar-valued outranking digraphs

```

1 >>> t = cRandomPerformanceTableau(numberOfActions=1000, \
2 ...                                         numberOfCriteria=2)
3 >>> from cIntegerOutrankingDigraphs import \
4 ...                                         IntegerBipolarOutrankingDigraph
5 >>> g = IntegerBipolarOutrankingDigraph(t, \
6 ...                                         Threading=True, nbrCores=4)
7 >>> g
8      *----- Object instance description -----*
9      Instance class      : IntegerBipolarOutrankingDigraph
10     Instance name       : rel_cRandomperftab
11     Actions              : 1000
12     Criteria             : 2
13     Size                 : 465024
14     Determinateness     : 56.877
15     Valuation domain   : {'min': -2, 'med': 0, 'max': 2,
16                           'hasIntegerValuation': True}
17     ---- Constructor run times (in sec.) ----
18     Total time          : 4.23880
19     Data input          : 0.01203
20     Compute relation   : 3.60788
21     Gamma sets          : 0.61889
22     Threads              : 4
23     Attributes          : ['name', 'actions', 'criteria',

```

```

24     'totalWeight', 'valuationdomain',
25     'methodData', 'evaluation',
26     'order', 'runTimes', 'nbrThreads',
27     'relation', 'gamma', 'notGamma']

```

On a classical intel-i7 equipped PC with four single threaded cores, the `IntegerBipolarOutrankingDigraph` class constructor takes about four seconds for computing a *million* pairwise outranking characteristic values (see Listing 11.2 on the previous page). In a similar setting, the standard `BipolarOutrankingDigraph` constructor operates more than two times slower.

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> t1 = t.convert2Standard()
3 >>> g1 = BipolarOutrankingDigraph(t1,\n4 ...                         Threading=True, nbrCores=4)
5 >>> g1
6 *----- Object instance description -----*
7   Instance class   : BipolarOutrankingDigraph
8   Instance name    : rel_std_cRandomperftab
9   Actions          : 1000
10  Criteria         : 2
11  Size             : 465024
12  Determinateness  : 56.817
13  Valuation domain : {'min': Decimal('-1.0'),
14 ...                           'med': Decimal('0.0'),
15 ...                           'max': Decimal('1.0'),
16 ...                           'precision': Decimal('0')}
17  ---- Constructor run times (in sec.) ----
18  Total time       : 8.63340
19  Data input        : 0.01564
20  Compute relation : 7.52787
21  Gamma sets       : 1.08987
22  Threads          : 4

```

By far, most of the run time is in each case needed for computing the individual pairwise outranking characteristic values (3.61sec versus 7.53sec.). Notice also below the memory occupations –about 103MB for `g` and 202MB for `g1`– of both outranking digraph instances.

```

1 >>> from digraphsTools import total_size
2 >>> total_size(g)
3 108662777
4 >>> total_size(g1)
5 212679272
6 >>> total_size(g1.relation)/total_size(g1)
7 0.34
8 >>> total_size(g1.gamma)/total_size(g1)
9 0.45

```

The standard `Decimal` valued `BipolarOutrankingDigraph` instance `g1` thus nearly doubles the memory occupation of the corresponding `IntegerBipolarOutrankingDigraph` `g` instance (see Line 3 and 5 above). 3/4 of this memory occupation is due to the `g1.relation` (34%) and the `g1.gamma` (45%) dictionaries. And these ratios quadratically grow with the digraph order. To limit the

object sizes for really big outranking digraphs, we need to abandon the complete implementation of adjacency tables and gamma functions.

## 11.4 The sparse implementation of big outranking digraphs

The idea is to first decompose the complete outranking relation into an ordered collection of equivalent quantile performance classes. Let us consider for this illustration a random performance tableau with 100 decision alternatives evaluated on 7 criteria.

```

1 >>> from cRandPerfTabs import cRandomPerformanceTableau
2 >>> t = cRandomPerformanceTableau(numberOfActions=100, \
3 ...                                     numberOfCriteria=7, seed=100)

```

The `cSparseIntegerOutrankingDigraphs` module provides the `SparseIntegerOutrankingDigraph` class for sorting the 100 decision alternatives into overlapping quartile classes and rank all 100 with respect to the average quantile limits.

**Listing 11.3** Constructing the sparse integer outranking digraph

```

1 >>> from cSparseIntegerOutrankingDigraphs import \
2 ...                               SparseIntegerOutrankingDigraph
3 >>> sg = SparseIntegerOutrankingDigraph(t, quantiles=4)
4 >>> sg
5     *----- Object instance description -----*
6     Instance class      : SparseIntegerOutrankingDigraph
7     Instance name       : cRandomperftab_mp
8     Actions             : 100
9     Criteria            : 7
10    Sorting by          : 4-Tiling
11    Ordering strategy   : average
12    Ranking rule         : Copeland
13    Components           : 6
14    Minimal order        : 1
15    Maximal order        : 35
16    Average order        : 16.7
17    fill rate            : 24.970%
18    *----- Constructor run times (in sec.) -----
19    Nbr of threads       : 1
20    Total time           : 0.08212
21    QuantilesSorting    : 0.01481
22    Preordering           : 0.00022
23    Decomposing          : 0.06707
24    Ordering              : 0.00000
25    Attributes            : ['runTimes', 'name', 'actions',
26                                'criteria', 'evaluation', 'order', 'dimension',
27                                'sortingParameters', 'nbrOfCPUs',
28                                'valuationdomain', 'profiles', 'categories',
29                                'sorting', 'minimalComponentSize',
30                                'decomposition', 'nbrComponents', 'nd',

```

```

31     'components', 'fillRate',
32     'maximalComponentSize', 'componentRankingRule',
33     'boostedRanking']

```

We obtain in this example here a decomposition into 6 linearly ordered components with a maximal component size of 35 for component  $c3$ .

```

1 >>> sg.showDecomposition()
2     *--- quantiles decomposition in decreasing order---*
3     c1. ]0.75-1.00] : [3,22,24,34,41,44,50,53,56,62,93]
4     c2. ]0.50-1.00] : [7,29,43,58,63,81,96]
5     c3. ]0.50-0.75] : [1,2,5,8,10,11,20,21,25,28,30,33,
6                     35,36,45,48,57,59,61,65,66,68,70,
7                     71,73,76,82,85,89,90,91,92,94,95,97]
8     c4. ]0.25-0.75] : [17,19,26,27,40,46,55,64,69,87,98,100]
9     c5. ]0.25-0.50] : [4,6,9,12,13,14,15,16,18,23,31,32,
10                    37,38,39,42,47,49,51,52,54,60,67,72,
11                    74,75,77,78,80,86,88,99]
12    c6. ]<-0.25]     : [79,83,84]

```

A restricted outranking relation is stored for each component with more than one alternative. The resulting global relation map of the first ranked 75 alternatives looks as follows (see Fig. 11.1 on the facing page):

```

1 >>> sg.showRelationMap(toIndex=75)

```

With a fill rate of 25%, the memory occupation of this sparse outranking digraph `sg` instance takes now only 769kB, compared to the 1.7MB required by a corresponding standard `BipolarOutrankingDigraph` instance (see Listing ?? on page ??).

```

1 >>> print('%.0f kB' % (total_size(sg)/1024) )
2 769kB

```

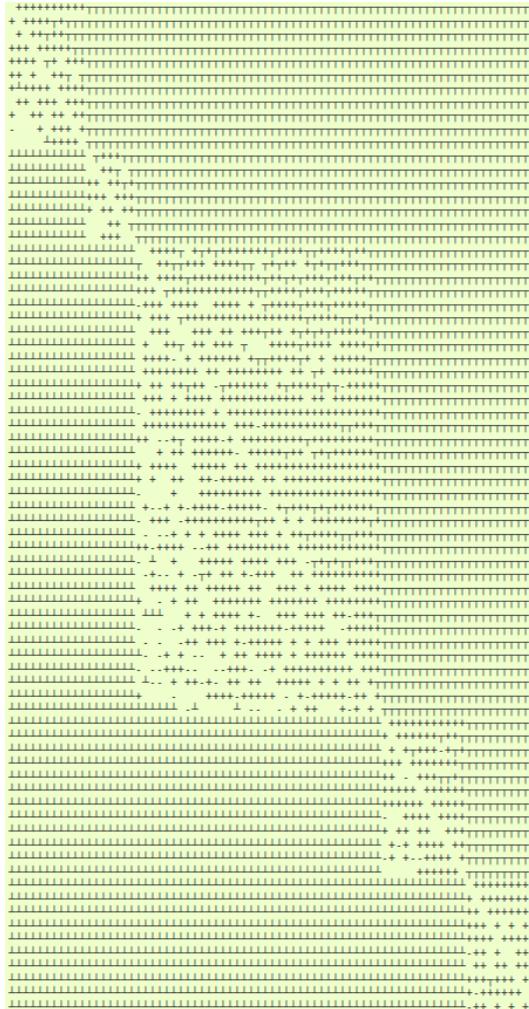
For sparse outranking digraphs, the adjacency table is implemented as a dynamic `relation()` function instead of a double dictionary.

**Listing 11.4** The `relation()` function of a sparse outranking digraph

```

1  def relation(self, int x, int y):
2      """
3          *Parameters*:
4          * x (int action key),
5          * y (int action key).
6          Dynamic construction of the global outranking
7          characteristic function *r(x S y)*.
8          """
9          cdef int Min, Med, Max, rx, ry
10         Min = self.valuationdomain['min']
11         Med = self.valuationdomain['med']
12         Max = self.valuationdomain['max']
13         if x == y:
14             return Med
15         cx = self.actions[x]['component']
16         cy = self.actions[y]['component']
17         #print(self.components)
18         rx = self.components[cx]['rank']
19         ry = self.components[cy]['rank']
20         if rx == ry:
21             try:

```



**Fig. 11.1** Sparse quartiles-  
sorting decomposed outrank-  
ing relation (extract). **Legend:**  
outranking for certain ( $\top$ );  
outranked for certain ( $\perp$ );  
more or less outranking (+);  
more or less outranked (-);  
indeterminate

```

22     rxtg = self.components[cx]['subGraph'].relation
23     return rxtg[x][y]
24 except AttributeError:
25     componentRanking = self.components[cx]['componentRanking']
26     if componentRanking.index(x) < componentRanking.index(y):
27         return Max
28     else:
29         return Min
30 elif rx > ry:
31     return Min
32 else:
33     return Max

```

## 11.5 Quantiles ranking of big performance tableaux

We may now rank the complete set of 100 decision alternatives by locally ranking all these individual components with the COPELAND or the NETFLOWS rule.

```

1  >>> sg.boostedRanking
2  [22, 53, 3, 34, 56, 62, 24, 44, 50, 93, 41, 63, 29, 58,
3  96, 7, 43, 81, 91, 35, 25, 76, 66, 65, 8, 10, 1, 11, 61,
4  30, 48, 45, 68, 5, 89, 57, 59, 85, 82, 73, 33, 94, 70,
5  97, 20, 92, 71, 90, 95, 21, 28, 2, 36, 87, 40, 98, 46, 55,
6  100, 64, 17, 26, 27, 19, 69, 6, 38, 4, 37, 60, 31, 77, 78,
7  47, 99, 18, 12, 80, 54, 88, 39, 9, 72, 86, 42, 13, 23, 67,
8  52, 15, 32, 49, 51, 74, 16, 14, 75, 79, 83, 84]

```

When actually computing linear rankings of a set of alternatives, the local outranking relations are of no practical usage, and we may furthermore reduce the memory occupation of the resulting digraph by:

1. refining the ordering of the quantile classes by taking into account how well an alternative is outranking the lower limit of its quantile class, respectively the upper limit of its quantile class is *not* outranking the alternative;
2. dropping the local outranking digraphs and keeping for each quantile class only a locally ranked list of alternatives.

We provide therefore the `cQuantilesRankingDigraph` class from the `cSparseIntegerOutrankingDigraphs` module.

**Listing 11.5** Ranking the sparse integer outranking digraph

```

1  >>> from cSparseIntegerOutrankingDigraphs import \
2  ...          cQuantilesRankingDigraph
3  >>> qr = cQuantilesRankingDigraph(t, 4)
4  >>> qr
5  *----- Object instance description -----*
6  Instance class      : cQuantilesRankingDigraph
7  Instance name       : cRandomperftab_mp
8  Actions             : 100
9  Criteria            : 7
10 Sorting by          : 4-Tiling
11 Ordering strategy   : optimal
12 Ranking rule         : Copeland
13 Components           : 47
14 Minimal order        : 1
15 Maximal order        : 10
16 Average order        : 2.1
17 fill rate            : 2.566%
18 *----- Constructor run times (in sec.) -----*
19 Nbr of threads        : 1
20 Total time            : 0.03702
21 QuantilesSorting      : 0.01785
22 Preordering           : 0.00022
23 Decomposing           : 0.01892
24 Attributes            : ['runTimes', 'name',
25 'actions', 'order', 'dimension', 'sortingParameters',
26 'nbrOfCPUs', 'valuationdomain', 'profiles', 'categories',

```

```

27     'sorting', 'minimalComponentSize', 'decomposition',
28     'nbrComponents', 'nd', 'components', 'fillRate',
29     'maximalComponentSize', 'componentRankingRule',
      'boostedRanking']

```

With this *optimised* quantile ordering strategy, we obtain now 47 performance equivalence classes (see Listing 11.5 on the facing page Line 13)

**Listing 11.6** The ordered components of the sparse outranking digraph

```

1  >>> qr.components
2      OrderedDict([
3          ('c01', {'rank': 1,
4              'lowQtileLimit': ']0.75',
5              'highQtileLimit': '1.00]',
6              'componentRanking': [53]}),
7          ('c02', {'rank': 2,
8              'lowQtileLimit': ']0.75',
9              'highQtileLimit': '1.00]',
10             'componentRanking': [3, 23, 63, 50]}),
11          ('c03', {'rank': 3,
12              'lowQtileLimit': ']0.75',
13              'highQtileLimit': '1.00]',
14              'componentRanking': [34, 44, 56, 24, 93, 41]}),
15          ...
16          ...
17          ...
18          ('c45', {'rank': 45,
19              'lowQtileLimit': ']0.25',
20              'highQtileLimit': '0.50]',
21              'componentRanking': [49]}),
22          ('c46', {'rank': 46,
23              'lowQtileLimit': ']0.25',
24              'highQtileLimit': '0.50]',
25              'componentRanking': [52, 16, 86]}),
26          ('c47', {'rank': 47,
27              'lowQtileLimit': ']<',
28              'highQtileLimit': '0.25]',
29              'componentRanking': [79, 83, 84]}))
30  >>> print('%.0fkB' % (total_size(qr)/1024) )
31  208kB

```

We observe in Listing 11.6 an even more considerably less voluminous memory occupation 208kB, compared to the 769kB of the SparseIntegerOutrankingDigraph instance. It is opportune, however, to measure the loss of quality of the resulting COPELAND ranking when working with sparse outranking digraphs.

**Listing 11.7** Measuring the loss of quality with respect to the standard outranking digraph

```

1  >>> from cIntegerOutrankingDigraphs import *
2  >>> ig = IntegerBipolarOutrankingDigraph(t)
3  >>> print('Complete outranking : %+.4f' \
4  ...   % (ig.computeOrderCorrelation(\n5  ...       ig.computeCopelandOrder()['correlation'])))
6  Complete outranking : +0.7474

```

```

7
8 >>> print('Sparse 4-tiling : %+.4f' \
9 ...   % (ig.computeOrderCorrelation(\ \
10 ...     list(reversed(sg.boostedRanking)))) \
11 ...     ['correlation'])) \
12 Sparse 4-tiling : +0.7172
13
14 >>> print('Optimzed sparse 4-tiling: %+.4f' \
15 ...   % (ig.computeOrderCorrelation(\ \
16 ...     list(reversed(qr.boostedRanking)))) \
17 ...     ['correlation'])) \
18 Optimzed sparse 4-tiling: +0.7051

```

The best ranking correlation with the pairwise outranking situations (+0.75) is naturally given when we apply the COPELAND rule to the complete outranking digraph (see Listing 10.7 on page 141 Line 6). When we apply the same rule to the sparse 4-tiled outranking digraph, we get a correlation of +0.72 (Line 12), and when applying the COPELAND rule to the optimised 4-tiled digraph, we still obtain a correlation of +0.71 (Line 18). These results actually depend on the number of quantiles we use as well as on the given model of random performance tableau.

In case of `Random3ObjectivesPerformanceTableau` instances, for instance, we would get in a similar setting a complete outranking correlation of +0.86, a sparse 4-tiling correlation of +0.82, and an optimized sparse 4-tiling correlation of +0.81.

## 11.6 HPC quantiles ranking records

Following from the separability property of the  $q$ -tiles sorting of each action into each  $q$ -tiles class, the  $q$ -sorting algorithm may be safely split into as much threads as are multiple processing cores available in parallel. Furthermore, the ranking procedure being local to each diagonal component, these procedures may as well be safely processed in parallel threads on each component restricted outrankingdigraph.

Using the HPC platform of the University of Luxembourg (<https://hpc.uni.lu/>), the following run times for very big ranking problems could be achieved both:

- on Iris -skylake nodes with 28 cores <sup>2</sup>, and
- on the 3TB -bigmem Gaia-183 node with 64 cores <sup>3</sup>,

by running the cythonized Python modules in an Intel compiled virtual Python 3.6.5 environment [GCC Intel(R) 17.0.1 –enable-optimizations c++ gcc 6.3 mode] on Debian 8 Linux.

---

<sup>2</sup> The `iris` cluster: <https://hpc.uni.lu/systems/iris/>

<sup>3</sup> The `gaia` cluster: <https://hpc.uni.lu/systems/gaia/>

**Fig. 11.2** HPC-UL Ranking Performance Records (Spring 2018). On the big memory equipped Gaia-183 node, we were able to rank a million decision alternatives in about 2 minutes. We even could linearly rank up to 6 million decision alternatives in about 40 minutes

$\gtrless^q$ outranking order	relation size	$q$	fill rate	nbr. cores	run time
5 000	$25 \times 10^6$	4	0.005%	28	0.5"
10 000	$1 \times 10^8$	4	0.001%	28	1"
100 000	$1 \times 10^{10}$	5	0.002%	28	10"
1 000 000	$1 \times 10^{12}$	6	0.001%	64	2'
3 000 000	$9 \times 10^{12}$	15	0.004%	64	13'
6 000 000	$36 \times 10^{12}$	15	0.002%	64	41'

## References

- Bisdorff R (2016) Computing linear rankings from trillions of pairwise outranking situations. In: Busa-Fekete R, Hüllermeier E, Mousseau V, Pfannschmidt K (eds) From Multiple Criteria Decision Aid to Preference Learning , DAP'2016, University of Paderborn (Germany), pp 1–16, URL <http://hdl.handle.net/10993/28613 147>

**Part III**  
**Evaluation and decision case studies**

The third part with four chapters proposes decision making case studies illustrating different performance evaluation models and decision problems. Chapter 12 concerns building a best choice recommendation for helping a young student to select a foreign language study program. Chapter 13, inspired by the THE World University Rankings, presents and discusses the mutiple incomensurable criteria ranking of academic Computer Science Departments. Inspired by the 2004 data published by DER SPIEGEL magazine, Chapter 14 showcases a multiple incommensurable criteria based rating of the academic enrolment quality. A last Chapter 15 lists some exercises for students taking a course on Algorithmic Decision Theory.

## Chapter 12

### Alice's best choice: A selection case study

**Abstract** The chapter presents a case study concerning the building of a best choice recommendation for Alice, a German student who wants some advice concerning the choice of her future University studies. We present Alice's performance tableau –potential foreign language study programs, her decision objectives, performance criteria and performance grades– and build a best choice recommendation for her. A thorough robustness analysis confirms a very best choice.

**Fig. 12.1** Alice D., 19 years old German student finishing her secondary studies in Köln (Germany), desires to undertake foreign languages studies.



This case study is inspired by a multiple criteria decision analysis case study published by [Eisenführ et al, 2001](#), pp. 1-17.

## 12.1 The decision problem

Alice will probably receive her "Abitur" with satisfactory and/or good marks and wants to start her further studies thereafter. She would not mind staying in Köln, yet is ready to move elsewhere if necessary. The length of the higher studies do concern her, as she wants to earn her life as soon as possible. Her parents however agree to financially support her study fees as well as her living costs during her studies.

Alice has already identified 10 potential study programs.

**Table 12.1** The potential study programs

ID	Diploma	Institution	City
T-UD	Qualified translator (T)	University (UD)	Düsseldorf
T-FHK	Qualified translator (T)	Higher Technical School (FHK)	Köln
T-FHM	Qualified translator (T)	Higher Technical School (FHM)	München
I-FHK	Graduate interpreter (I)	Higher Technical School (FHK)	Köln
T-USB	Qualified translator (T)	University (USB)	Saarbrücken
I-USB	Graduate interpreter (I)	University (USB)	Saarbrücken
T-UHB	Qualified translator (T)	University (UHB)	Heidelberg
I-UHB	Graduate interpreter (I)	University (UHB)	Heidelberg
S-HKK	Specialized secretary (S)	Chamber of Commerce (HKK)	Köln
C-HKK	Foreign correspondent (C)	Chamber of Commerce (HKK)	Köln

In Table 12.1 we notice that Alice considers three *Graduate Interpreter* studies (8 or 9 Semesters), respectively in Köln, in Saarbrücken or in Heidelberg; and five *Qualified translator* studies (8 or 9 Semesters), respectively in Köln, in Düsseldorf, in Saarbrücken, in Heidelberg or in Munich. She also considers two short (4 Semesters) study programs at the Chamber of Commerce in Köln.

Four **decision objectives** of more or less equal importance are guiding Alice's choice:

1. *maximize* the attractiveness of the study place (GEO),
2. *maximize* the attractiveness of her further studies (LEA),
3. *minimize* her financial dependency on her parents (FIN),
4. *maximize* her professional perspectives (PRA).

The decision consequences Alice wishes to take into account for evaluating the potential study programs with respect to each of the four objectives are modelled by the following *coherent family of criteria* (Roy, 1991; Roy and Bouyssou, 1993). Such a family of performance criteria verifies:

1. *Exhaustiveness*: No argument acceptable to Alice can be put forward to justify a preference in favour of study program  $x$  versus program  $y$  when  $x$  and  $y$  have the same performance level on each of the performance criteria;
2. *Cohesiveness*: Alice recognizes that program  $x$  must be preferred to program  $y$  whenever the performance level of  $x$  is significantly better than that of  $y$  on one

- of the criteria of positive weight, performance levels of  $x$  and  $y$  being the same on each of the other criteria;
3. *Nonredundancy*: One of the above requirements is violated if one of the performance criteria is left out from the family.

**Table 12.2** Alice's family of performance criteria

ID	Name	Comment	Objective	Weight
DH	Proximity	Distance in km to her home (min)	GEO	3
BC	Big City	Number of inhabitants (max)	GEO	3
AS	Studies	Attractiveness of the studies (max)	LEA	6
SF	Fees	Annual study fees (min)	FIN	2
LC	Living	Monthly living costs (min)	FIN	2
SL	Length	Length of the studies (min)	FIN	2
AP	Profession	Attractiveness of the profession (max)	PRA	2
AI	Income	Annual income after studying (max)	PRA	2
PR	Prestige	Occupational prestige (max)	PRA	2

Within each decision objective, the performance criteria are considered to be *equisignificant*. Hence, the four decision objectives get a same *importance weight* of 6 (see Table 12.2 Column 5).

## 12.2 The performance tableau

The actual evaluations of Alice's potential study programs are stored in a file named `AliceChoice.py` of `PerformanceTableau` format<sup>1</sup>.

**Listing 12.1** Alice's performance tableau

```

1 >>> from perfTabs import PerformanceTableau
2 >>> t = PerformanceTableau('AliceChoice')
3 >>> t.showObjectives()
4     *----- decision objectives -----*
5     GEO: Geographical aspect
6     DH Distance to parent's home 3
7     BC Number of inhabitants      3
8     Total weight: 6 (2 criteria)
9     LEA: Learning aspect
10    AS Attractiveness of the study program 6
11    Total weight: 6.00 (1 criteria)

```

<sup>1</sup> The performance tableau `AliceChoice.py` may be found in the `examples` directory of the DIGRAPH3 resources (Bisdorff, 2021).

```

12     FIN: Financial aspect
13         SF Annual registration fees 2
14         LC Monthly living costs 2
15         SL Study time 2
16         Total weight: 6.00 (3 criteria)
17     PRA: Professional aspect
18         AP Attractiveness of the profession 2
19         AI Annual professional income after studying 2
20         OP Occupational Prestige 2
21         Total weight: 6.00 (3 criteria)

```

Details of the performance criteria may be consulted in the browser view below.

```
1 >>> t.showHTMLCriteria()
```

### AliceChoice: Family of Criteria

#	Identifier	Name	Comment	Weight	Scale			Thresholds (ax + b)		
					direction	min	max	indifference	preference	veto
1	AI	Annual professional income after studying	Professional aspect measured in x / 1000 Euros	2.00	max	0.00	50.00	0.00x + 0.00	0.00x + 1.00	
2	AP	Attractiveness of the profession	Professional aspect subjectively measured on a three-level scale: 0 (weak), 1 (fair), 2 (good)	2.00	max	0.00	2.00	0.00x + 0.00	0.00x + 1.00	
3	AS	Attractiveness of the study program	Learning aspect subjectively measured from 0 (weak) to 10 (excellent)	6.00	max	0.00	10.00	0.00x + 0.00	0.00x + 1.00	0.00x + 7.00
4	BC	Number of inhabitants	Geographical aspect: measured in x / 1000	3.00	max	0.00	2000.00	0.01x + 0.00	0.05x + 0.00	
5	DH	Distance to parent's home	Geographical aspect measured in km	3.00	min	0.00	1000.00	0.00x + 0.00	0.00x + 10.00	
6	LC	Monthly living costs	Financial aspect measured in Euros	2.00	min	0.00	1000.00	0.00x + 0.00	0.00x + 100.00	
7	OP	Occupational Prestige	Professional aspect measured in SIOPS points	2.00	max	0.00	100.00	0.00x + 0.00	0.00x + 10.00	
8	SF	Annual registration fees	Financial aspect measured in Euros	2.00	min	400.00	4000.00	0.00x + 0.00	0.00x + 100.00	
9	SL	study time	Financial aspect measured in number of semesters	2.00	min	0.00	10.00	0.00x + 0.00	0.00x + 0.50	

Fig. 12.2 Alice's performance criteria

It is worthwhile noticing in Fig. 12.2 that, on her subjective attractiveness scale of the study programs (criterion AS), Alice considers a performance differences of 7 points to be *considerable* and triggering, the case given, an outranking polarisation (Bisdorff, 2013). Notice also the proportional *indifference* (1%) and *preference* (5%) discrimination thresholds shown on criterion BC (number of inhabitants).

Alice is subjectively evaluating the *Attractiveness* of the studies (criterion AS) on an ordinal scale from *weak* (0) to *excellent* (10). Similarly, she is subjectively evaluating the *Attractiveness* of the respective professions (criterion AP) on a three level ordinal scale from *weak* (0), *fair* (1) to *good* (2). Considering the *Occupational Prestige* (criterion OP), she looked up the SIOPS<sup>2</sup>. All the other evaluation data she found on the internet.

In the *heatmap view* shown in Fig. 12.3 on the facing page, we may now consult Alice's performance gradings.

<sup>2</sup> Standard International Occupational Prestige Scale (Ganzeboom and Treiman, 1996).

```

1 >>> t.showHTMLPerformanceHeatmap(\n
2 ...     colorLevels=5,Correlations=True,ndigits=0)

```

### Heatmap of Performance Tableau 'AliceChoice'

criteria	AS	AP	SF	OP	AI	DH	LC	BC	SL
<b>weights</b>	+6.00	+2.00	+2.00	+2.00	+2.00	+3.00	+2.00	+3.00	+2.00
<b>tau(*)</b>	+0.71	+0.64	+0.36	+0.36	+0.24	+0.03	-0.04	-0.07	-0.24
<b>I-FHK</b>	8	2	-400	62	35	0	0	1015	-8
<b>I-USB</b>	8	2	-400	62	45	-269	-1000	196	-9
<b>T-FHK</b>	5	1	-400	62	35	0	0	1015	-8
<b>I-UHB</b>	8	2	-400	62	45	-275	-1000	140	-9
<b>T-UD</b>	5	1	-400	62	45	-41	-1000	567	-9
<b>T-USB</b>	5	1	-400	62	45	-260	-1000	196	-9
<b>T-FHM</b>	4	1	-400	62	35	-631	-1000	1241	-8
<b>T-UHB</b>	5	1	-400	62	45	-275	-1000	140	-9
<b>C-HKK</b>	2	0	-4000	44	30	0	0	1015	-4
<b>S-HKK</b>	1	0	-4000	44	30	0	0	1015	-4

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Ranking rule: **NetFlows**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.692**

**Fig. 12.3** Heatmap of Alice's performance tableau

Her ten potential study programs are ordered with the **NETFLOWS** ranking rule applied to the corresponding bipolar-valued outranking digraph (see Section 8.3 on page 109). *Graduate interpreter* studies in Köln (I-FHK) or Saarbrücken (I-USB), followed by *Qualified Translator* studies in Köln (T-FHK) appear to be Alice's most preferred alternatives. The least attractive for her appear to be studies at the Chamber of Commerce of Köln (C-HKK, S-HKK). Notice by the way that evaluations on performance criteria to be *minimized*, like *Distance to Home* (criterion DH) or *Study time* (criterion SL), are registered as *negative* values, so that smaller measures are, in this case, preferred to larger ones.

It is finally interesting to observe in Fig. 12.3 (third row) that the *most significant* performance criteria, appear to be for Alice, on the one side, the *Attractiveness* of the study program (criterion AS, tau = +0.72) followed by the *Attractiveness* of the future profession (criterion AP, tau = +0.62). On the other side, *Study times* (criterion SL, tau = -0.24), *Big city* (criterion BC, tau = -0.07) as well as *Monthly living costs* (criterion LC, tau = -0.04) appear to be *not so significant* (see Chapter 16 on page 221).

### 12.3 Building a best choice recommendation

Let us now have a look at the resulting pairwise outranking situations.

**Listing 12.2** Computing Alice's outranking digraph

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> dg = BipolarOutrankingDigraph(t)
3 >>> dg
4     *----- Object instance description -----*
5     Instance class      : BipolarOutrankingDigraph
6     Instance name       : rel_AliceChoice
7     Actions             : 10
8     Criteria            : 9
9     Size                : 67
10    Determinateness (%) : 73.91
11    Valuation domain   : [-1.00;1.00]
12 >>> dg.computeSymmetryDegree(Comments=True)
13     Symmetry degree of graph <rel_AliceChoice> : 0.49

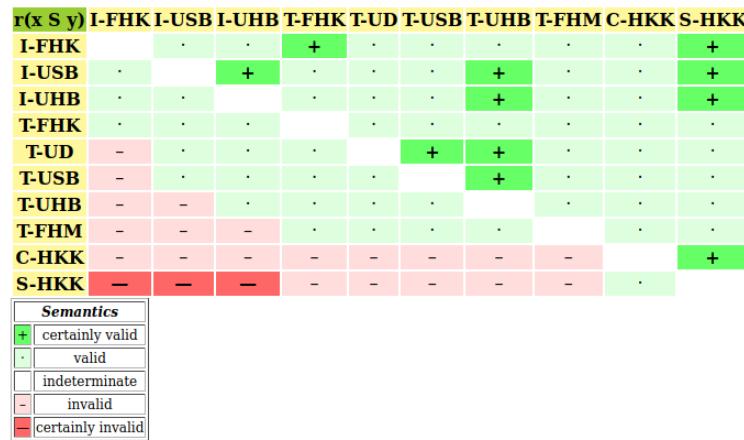
```

From Alice's performance tableau we obtain in the digraph `dg` 67 positively validated pairwise outranking situations, supported by a 74% majority of criteria significance (see Lines 9-10 in Listing 12.2).

Due to the poorly discriminating performance evaluations, nearly half of these outranking situations (see Line 12) are *symmetric* and reveal actually *more or less indifference* situations between the potential study programs. This is well illustrated in the *relation map* of the outranking digraph shown in Fig. 12.4.

## Outranking relation map

### Ranking rule: Copeland



**Fig. 12.4** COPELAND ranked outranking relation map

```

1 >>> dg.showHTMLRelationMap(\n2 ...             tableTitle='Outranking relation map', \n3 ...             rankingRule='Copeland')
```

We have mentioned that Alice considers a performance difference of 7 points on the *Attractiveness of studies* criterion AS to be considerable which triggers, the case given, a potential polarisation of the outranking characteristics. In Fig. 12.4 on the facing page, these polarisations appear in the last column and last row. We may inspect the occurrence of such polarisations with the `showPolarisations()` method.

**Listing 12.3** Polarised outranking situations

```

1 >>> dg.showPolarisations()\n2     *---- Negative polarisations ----*\n3     number of negative polarisations : 3\n4     1: r(S-HKK >= I-FHK) = -0.17\n5         criterion: AS\n6         Considerable performance difference : -7.00\n7         Veto discrimination threshold       : -7.00\n8         Polarisation: r(S-HKK >= I-FHK) = -0.17 ==> -1.00\n9     2: r(S-HKK >= I-USB) = -0.17\n10        criterion: AS\n11        Considerable performance difference : -7.00\n12        Veto discrimination threshold       : -7.00\n13        Polarisation: r(S-HKK >= I-USB) = -0.17 ==> -1.00\n14     3: r(S-HKK >= I-UHB) = -0.17\n15        criterion: AS\n16        Considerable performance difference : -7.00\n17        Veto discrimination threshold       : -7.00\n18        Polarisation: r(S-HKK >= I-UHB) = -0.17 ==> -1.00\n19     *---- Positive polarisations ----*\n20     number of positive polarisations: 3\n21     1: r(I-FHK >= S-HKK) = 0.83\n22         criterion: AS\n23         Considerable performance difference : 7.00\n24         Counter-veto threshold           : 7.00\n25         Polarisation: r(I-FHK >= S-HKK) = 0.83 ==> +1.00\n26     2: r(I-USB >= S-HKK) = 0.17\n27         criterion: AS\n28         Considerable performance difference : 7.00\n29         Counter-veto threshold           : 7.00\n30         Polarisation: r(I-USB >= S-HKK) = 0.17 ==> +1.00\n31     3: r(I-UHB >= S-HKK) = 0.17\n32         criterion: AS\n33         Considerable performance difference : 7.00\n34         Counter-veto threshold           : 7.00\n35         Polarisation: r(I-UHB >= S-HKK) = 0.17 ==> +1.00
```

In Listing 12.3, we see that *considerable* performance differences concerning the *Attractiveness of the studies* (AS criterion) are indeed observed between the *Specialised Secretary* study programm offered in Köln and the *Graduate Interpreter* study programs offered in Köln, Saarbrücken and Heidelberg. They polarise, hence, three *more or less invalid* outranking situations to *certainly invalid* (Lines 8, 13,

18) and corresponding three *more or less valid* converse outranking situations to *certainly valid* ones (Lines 25, 30, 35).

We may finally notice in the relation map, shown in Fig. 12.4 on page 166, that the four best-ranked study programs, I-FHK, I-USB, I-UHB and T-FHK, are in fact CONDORCET winners (see Listing 12.4 Line 2), i.e. they are all four *indifferent* one of the other **and** positively *outranking* all other alternatives, a result confirmed in Listing 12.4 below by our RUBIS best choice recommendation (Bisdorff, Meyer, and Roubens, 2008).

**Listing 12.4** Alice's best choice recommendation

```

1 >>> dg.computeCondorcetWinners()
2   ['I-FHK', 'I-UHB', 'I-USB', 'T-FHK']
3 >>> dg.showBestChoiceRecommendation()
4   Best choice recommendation(s) (BCR)
5     (in decreasing order of determinateness)
6     Credibility domain: [-1.00,1.00]
7   === >> potential best choice(s)
8     choice : ['I-FHK', 'I-UHB', 'I-USB', 'T-FHK']
9     independence : 0.17
10    dominance : 0.08
11    absorbency : -0.83
12    covering (%) : 62.50
13    determinateness (%) : 68.75
14    most credible action(s) = {'I-FHK': 0.75, 'T-FHK': 0.17,
15                                'I-USB': 0.17, 'I-UHB': 0.17}
16   === >> potential worst choice(s)
17     choice : ['C-HKK', 'S-HKK']
18     independence : 0.50
19     dominance : -0.83
20     absorbency : 0.17
21     covered (%) : 100.00
22     determinateness (%) : 58.33
23     most credible action(s) = {'S-HKK': 0.17, 'C-HKK': 0.17}

```

Most credible best choice among the four best-ranked study programs eventually becomes the *Graduate Interpreter* study program at the Technical High School in Köln (see Line 14) supported by a  $(0.75 + 1)/2.0 = 87.5\%(18/24)$  majority of global criteria significance <sup>3</sup>.

In the relation map, shown in Fig. 12.4 on page 166, we see in the left lower corner that the *asymmetric part* of the outranking relation, i.e. the corresponding *strict* outranking relation, is actually *transitive* (see Line 2 below). Hence, a *graphviz* drawing of its *skeleton*, oriented by the previous *best*, respectively *worst* choice, may well illustrate our *best choice recommendation*.

**Listing 12.5** Alice's strict best choice recommendation

```

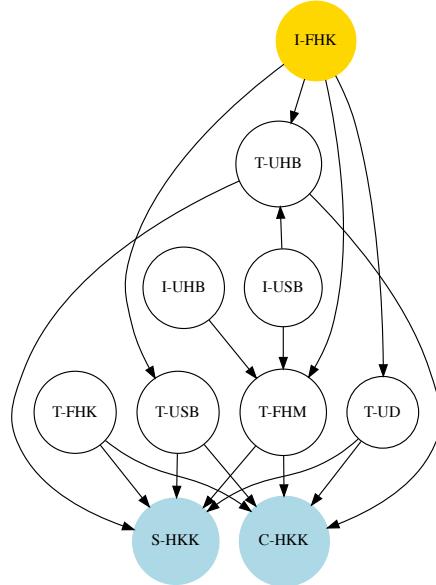
1 >>> dgcd = ~(-dg)
2 >>> dgcd.isTransitive()
3   True
4 >>> dgcd.closeTransitive(Reverse=True, InSite=True)

```

<sup>3</sup> See Section 17.6 on page 253 on solving BERGE kernel equation systems.

```
5 >>> dgcd.exportGraphViz('aliceBestChoice', \
6 ...                         bestChoice=['I-FHK'], \
7 ...                         worstChoice=['S-HKK', 'C-HKK']) \
8 *---- exporting a dot file for GraphViz tools -----* \
9 Exporting to aliceBestChoice.dot \
10 dot -Grankdir=BT -Tpng aliceBestChoice.dot -o \
     aliceBestChoice.png
```

**Fig. 12.5** In Alice's best choice recommendation, we notice that the *Graduate Interpreter* studies come first, followed by the *Qualified Translator* studies. Last come the *Chamber of Commerce*'s specialised studies. This confirms again the high significance that Alice attaches to the *attractiveness* of her further studies and of her future profession (see criteria AS and AP in Fig. 12.3 on page 165)



Let us now, for instance, check the pairwise outranking situations observed between the first and second-ranked alternative, i.e. *Graduate Interpreter* studies in Köln versus *Graduate Interpreter* studies in Saabrücken (see I-FHK and I-USB in Fig. 12.3 on page 165).

```
>>> dg.showHTMLPairwiseOutrankings('I-FHK','I-USB')
```

In a similar way, one may finally compute a *weak ranking* of all the potential study programs with the help of the `RankingByChoosingDigraph` constructor, that computes a bipolar ranking by conjointly *best-choosing* and *last-rejecting* (Bisdorff, 1999).

**Listing 12.6** Weakly ranking by bipolar best-choosing and last-rejecting

```
1 >>> from transitiveDigraphs import\  
2 ... RankingByChoosingDigraph
```

**Fig. 12.6** Comparing the first and second best-ranked study programs. The Köln alternative is performing *at least as well as* the Saarbrücken alternative on all the performance criteria, except the *Annual income* (of significance 2/24). Conversely, the Saarbrücken alternative is clearly *outperformed* from the geographical (0/6) as well as from the financial perspective (2/6)

## Pairwise Comparison

### Comparing actions : (I-FHK,I-USB)

crit	wght.	g(x)	g(y)	diff	ind	pref	concord	v	polarisation
AI	2.00	35.00	+45.00	-10	0.00	1.00	-2.00		
AP	2.00	2.00	+2.00	0	0.00	1.00	+2.00		
AS	6.00	8.00	+8.00	0	0.00	1.00	+6.00		
BC	3.00	1015.00	+196.00	819	10.15	50.75	+3.00		
DH	3.00	0.00	-269.00	269	0.00	10.00	+3.00		
LC	2.00	0.00	-1000.00	1000	0.00	100.00	+2.00		
OP	2.00	62.00	+62.00	0	0.00	10.00	+2.00		
SF	2.00	-400.00	-400.00	0	0.00	100.00	+2.00		
SL	2.00	-8.00	-9.00	1	0.00	0.50	+2.00		

Valuation in range: -24.00 to +24.00; global concordance: +20.00

## Pairwise Comparison

### Comparing actions : (I-USB,I-FHK)

crit	wght.	g(x)	g(y)	diff	ind	pref	concord	v	polarisation
AI	2.00	45.00	+35.00	10	0.00	1.00	+2.00		
AP	2.00	2.00	+2.00	0	0.00	1.00	+2.00		
AS	6.00	8.00	+8.00	0	0.00	1.00	+6.00		
BC	3.00	196.00	+1015.00	-819	10.15	50.75	-3.00		
DH	3.00	-269.00	+0.00	-269	0.00	10.00	-3.00		
LC	2.00	-1000.00	+0.00	-1000	0.00	100.00	-2.00		
OP	2.00	62.00	+62.00	0	0.00	10.00	+2.00		
SF	2.00	-400.00	-400.00	0	0.00	100.00	+2.00		
SL	2.00	-9.00	-8.00	-1	0.00	0.50	-2.00		

Valuation in range: -24.00 to +24.00; global concordance: +4.00

```

3 >>> rbc = RankingByChoosingDigraph(dg)
4 >>> rbc.showRankingByChoosing()
5     Ranking by Choosing and Rejecting
6     1st ranked ['I-FHK']
7     2nd ranked ['I-USB']
8     3rd ranked ['I-UHB']
9     4th ranked ['T-FHK']
10    5th ranked ['T-UD']
11    5th last ranked ['T-UD']
12    4th last ranked ['T-UHB', 'T-USB']
13    3rd last ranked ['T-FHM']
14    2nd last ranked ['C-HKK']
15    1st last ranked ['S-HKK']

```

In Listing 12.6 on the preceding page, we find confirmed that the *Interpreter* studies appear all preferred to the *Translator* studies. Furthermore, the *Interpreter* studies in Saarbrücken appear preferred to the same studies in Heidelberg. The Köln alternative is apparently the preferred one of all the *Translator* studies. And, the *Foreign Correspondent* and the *Specialised Secretary* studies appear second-last and last ranked.

Yet, how *robust* are our findings with respect to potential settings of the decision objectives' importance and the performance criteria significance?

## 12.4 Robustness analysis

Alice considers her four decision objectives as being *more or less* equally important. Here we have, however, allocated *strictly equal* importance weights with *strictly equi-significant* criteria per objective. How robust is our previous best choice recommendation when, now, we would consider the importance of the objectives and, hence, the significance of the respective performance criteria to be *more or less uncertain*?

To answer this question, we will consider the respective criteria significance weights  $w_j$  for  $j = 1, \dots, 9$  to be *triangular random variables* in the range 0 to  $2w_j$  with mode =  $w_j$ . We may compute a corresponding *90%-confident outranking digraph* with the help of the `ConfidentBipolarOutrankingDigraph` constructor (see Chapter 18 on page 265).

**Listing 12.7** Computing the 90% confident outranking digraph

```

1 >>> from outrankingDigraphs import \
2 ...           ConfidentBipolarOutrankingDigraph
3 >>> cdg = ConfidentBipolarOutrankingDigraph(t, \
4 ...           distribution='triangular', confidence=90.0)
5 >>> cdg
6      *----- Object instance description -----*
7      Instance class      : ConfidentBipolarOutrankingDigraph
8      Instance name       : rel_AliceChoice_CLT
9      Actions             : 10
10     Criteria            : 9
11     Size                : 44
12     Valuation domain    : [-1.00;1.00]
13     Uncertainty model   : triangular(a=0,b=2w)
14     Likelihood domain   : [-1.0;+1.0]
15     Confidence level    : 90.0%
16     Confident majority   : 14/24 (58.3%)
17     Determinateness (%) : 68.19

```

Of the original 67 valid outranking situations, we retain 44 outranking situations as being 90%-confident (see Listing 12.7 Line 11). The corresponding 90%-confident *qualified majority* of criteria significance amounts to  $14/24 = 58.3\%$  (Line 16).

Concerning now a 90%-confident best choice recommendation, we are lucky.

**Listing 12.8** Computing the 90%-confident best choice recommendation.

```

1 >>> cdg.computeCondorcetWinners()
2      ['I-FHK']
3 >>> cdg.showBestChoiceRecommendation()
4      ****
5      Best choice recommendation(s) (BCR)
6      (in decreasing order of determinateness)
7      Credibility domain: [-1.00,1.00]
8      ==> potential best choice(s)
9      choice      : ['I-FHK','I-UHB','I-USB',
10        'T-FHK','T-FHM']

```

```

11      independence      : 0.00
12      dominance        : 0.42
13      absorbency       : 0.00
14      covering (%)     : 20.00
15      determinateness (%) : 61.25
16      - most credible action(s) = { 'I-FHK': 0.75, }

```

The *Graduate Interpreter* studies in Köln remain indeed a 90%-confident CONDORCET winner (see Fig. 12.7 on the next page Line 2). Hence, the same study program also remains our 90%-confident most credible best choice supported by a continual 18/24(87.5%) majority of the global criteria significance (see Lines 9-10 and 16 in Listing 12.8 on the preceding page).

When previously comparing the two best-ranked study programs (see Fig. 12.6 on page 170), we have observed that I-FHK actually positively outranks I-USB on all four decision objectives. When admitting equi-significant criteria significances per objective, this outranking situation is hence valid independently of the importance weights Alice may allocate to each of her decision objectives.

The `UnOpposedBipolarOutrankingDigraph` class constructor computes these *unopposed* outranking situations (see Section 19.5 on page 286).

**Listing 12.9** Computing the unopposed outranking situations

```

1 >>> from outrankingDigraphs import \
2 ...                               UnOpposedBipolarOutrankingDigraph
3 >>> uop = UnOpposedBipolarOutrankingDigraph(t)
4 >>> uop
5      ----- Object instance description -----
6      Instance class      : UnOpposedBipolarOutrankingDigraph
7      Instance name       : AliceChoice_unopposed_outrankings
8      Actions             : 10
9      Criteria            : 9
10     Size                : 28
11     Oppositeness (%)   : 58.21
12     Determinateness (%) : 62.94
13     Valuation domain   : [-1.00;1.00]
14 >>> uop.isTransitive()
15     True

```

We keep 28 out the 67 standard outranking situations, which leads to an *oppositeness degree* of  $(1.0 - 28/67) = 58.21\%$  (see Lines 10-11 in Listing 12.9). Remarkable furthermore is that this unopposed outranking digraph `uop` is actually *transitive*, i.e. modelling a *partial ranking* of the study programs (Lines 14-15).

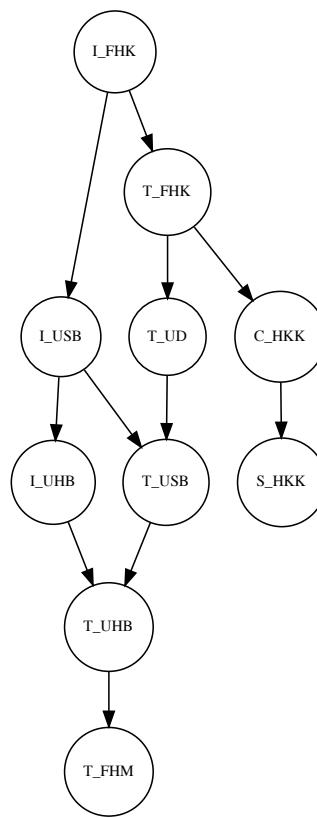
We may hence make use of the `exportGraphViz()` method of the `TransitiveDigraph` class for drawing the corresponding partial ranking.

```

1 >>> from transitiveDigraphs import TransitiveDigraph
2 >>> TransitiveDigraph.exportGraphViz(uop, \
3 ...                   fileName='choiceUnopposed')
4      ----- exporting a dot file for GraphViz tools -----
5      Exporting to choiceUnopposed.dot
6      dot -Grankdir=TB -Tpng choiceUnopposed.dot -o
          choiceUnopposed.png

```

**Fig. 12.7** Unopposed partial ranking of the potential study programs. Again, when *equi-significant* performance criteria are assumed per decision objective, we observe in the Figure here that *I-FHK* remains the stable best choice, *independently* of the actual importance weights that Alice may wish to allocate to her four decision objectives



*Digraph3 (graphviz)  
R. Bisdorff, 2020*

In view of her performance tableau shown in Fig. 12.3 on page 165, *Graduate Interpreter* studies at the Technical High School Köln, thus, represent definitely **Alice's very best choice**.

For further reading about the RUBIS *Best Choice* methodology, one may consult in Bisdorff (2015) the study of a *real decision aid case* about choosing a best poster in a scientific conference.

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## Chapter 13

# The best academic Computer Science Depts: A ranking case study

**Abstract** In this case study, we are solving with our DIGRAPH3 resources a ranking decision problem based on published data from the *Times Higher Education* (THE) *World University Rankings* 2016 by *Computer Science* (CS) subject. Several hundred academic CS Departments, from all over the world, were ranked that year following an overall numerical score based on the weighted average of five performance criteria: *Teaching* (the learning environment, 30%), *Research* (volume, income and reputation, 30%), *Citations* (research influence, 27.5%), *International outlook* (staff, students, and research, 7.5%), and *Industry income* (innovation, 5%). To illustrate our DIGRAPH3 programming resources, we shall first have a look into the THE multiple criteria ranking data with short Python scripts. In a second section, we shall relax the commensurability hypothesis of the ranking criteria and show how to similarly rank with multiple incommensurable performance criteria of solely ordinal significance. A third section is finally devoted to introduce quality measures for qualifying ranking results.

### 13.1 The THE performance tableau

For this decision making case study, we use an extract of the published TIMES HIGHER EDUCATION (THE) World University rankings 2016 by Computer Science (CS) subject, concerning the 75 first-ranked academic Institutions<sup>1</sup>. The multiple criteria performance tableau data, collected from the THE web pages, is stored in a file named `the-cs-2016.py` of PerformanceTableau format<sup>2</sup>.

**Listing 13.1** Performance tableau of the

```
1 >>> from perfTabs import PerformanceTableau
2 >>> cspt = PerformanceTableau('the_cs_2016')
```

---

<sup>1</sup> [THE World University Rankings 2016 by Computer Science subject](#).

<sup>2</sup> The performance tableau `the-cs-2016.py` is available in the `examples` directory of the DIGRAPH3 resources ([Bisdorff, 2021](#)).

```

3 >>> cspt
4 *----- PerformanceTableau instance description -----*
5 Instance class      : PerformanceTableau
6 Instance name       : the-cs-2016
7 Actions             : 75
8 Objectives          : 5
9 Criteria            : 5
10 NaN proportion (%) : 0.0
11 Attributes          : ['name','description','actions',
12                      'objectives','criteria',
13                      'weightPreorder','NA','evaluation']

```

Potential decision alternatives, in our case here, are the 75 THE best-ranked CS Departments in 2016, all of them located at world renowned Institutions, like California Institute of Technology, Swiss Federal Institute of Technology Zurich, Technical University München, University of Oxford or the National University of Singapore (see Listing 13.2 below).

Instead of using prefigured DIGRAPH3 `show...` methods, readily available for inspecting `PerformanceTableau` instances, Listing 13.2 below illustrates how to write small Python scripts for printing out its content.

**Listing 13.2** Printing the CS Departments

```

1 >>> for x in cspt.actions:
2 ...     print('%s:%s (%s)' % \
3 ...           (x,cspt.actions[x]['name'],cspt.actions[x]['comment']) )
4 albt: University of Alberta (CA)
5 anu: Australian National University (AU)
6 ariz: Arizona State University (US)
7 bjup: Beijing University (CN)
8 bro: Brown University (US)
9 calt: California Institute of Technology (US)
10 cbu: Columbia University (US)
11 chku: Chinese University of Hong Kong (HK)
12 cihk: City University of Hong Kong (HK)
13 cir: University of California at Irvine (US)
14 cmel: Carnegie Mellon University (US)
15 cou: Cornell University (US)
16 csb: University of California at Santa Barbara (US)
17 csd: University Of California at San Diego (US)
18 dut: Delft University of Technology (NL)
19 eind: Eindhoven University of Technology (NL)
20 ens: Superior Normal School at Paris (FR)
21 epfl: Swiss Federal Institute of Technology Lausanne (CH)
22 epfr: Polytechnic school of Paris (FR)
23 ethz: Swiss Federal Institute of Technology Zurich (CH)
24 frei: University of Freiburg (DE)
25 git: Georgia Institute of Technology (US)
26 glas: University of Glasgow (UK)
27 hels: University of Helsinki (FI)
28 hkpu: Hong Kong Polytechnic University (CN)
29 hkst: Hong Kong University of Science and Technology (HK)
30 hku: Hong Kong University (HK)
31 humb: Berlin Humboldt University (DE)
32 icl: Imperial College London (UK)
33 indis: Indian Institute of Science (IN)
34 itmo: ITMO University (RU)
35 kcl: King's College London (UK)
36 kist: Korea Adv. Institute of Science and Technology (KR)
37 kit: Karlsruhe Institute of Technology (DE)
38 kth: KTH Royal Institute of Technology (SE)
39 kuj: Kyoto University (JP)

```

```

40 kul: Catholic University Leuven (BE)
41 lms: Lomonosov Moscow State University (RU)
42 man: University of Manchester (UK)
43 mcp: University of Maryland College Park (US)
44 mel: University of Melbourne (AU)
45 mil: Polytechnic University of Milan (IT)
46 mit: Massachusetts Institute of Technology (US)
47 naji: Nanjing University (CN)
48 ntu: Nanyang Technological University of Singapore (SG)
49 ntw: National Taiwan University (TW)
50 nyu: New York University (US)
51 oxf: University of Oxford (UK)
52 pud: Purdue University (US)
53 qut: Queensland University of Technology (AU)
54 rcu: Rice University (US)
55 rwth: RWTH Aachen University (DE)
56 shJi: Shanghai Jiao Tong University (CN)
57 sing: National University of Singapore (SG)
58 sou: University of Southampton (UK)
59 stut: University of Stuttgart (DE)
60 tech: Technion - Israel Institute of Technology (IL)
61 tlavi: Tel Aviv University (IR)
62 tsu: Tsinghua University (CN)
63 tub: Technical University of Berlin (DE)
64 tud: Technical University of Darmstadt (DE)
65 tum: Technical University of Muenchen (DE)
66 ucl: University College London (UK)
67 ued: University of Edinburgh (UK)
68 uiu: University of Illinois at Urbana-Champaign (US)
69 unlu: University of Luxembourg (LU)
70 unsw: University of New South Wales (AU)
71 unt: University of Toronto (CA)
72 uta: University of Texas at Austin (US)
73 utj: University of Tokyo (JP)
74 utw: University of Twente (NL)
75 uwa: University of Waterloo (CA)
76 wash: University of Washington (US)
77 wtu: Vienna University of Technology (AUS)
78 zhej: Zhejiang University (CN)

```

The THE authors base their 2016 ranking on five decision objectives (Times Higher Education, 2016).

**Listing 13.3** The THE ranking objectives

```

1 >>> for obj in cspt.objectives:
2 ...     print('%s: %s (%.1f%%),\n\t%s' % \
3 ...           (obj,cspt.objectives[obj]['name'], \
4 ...            cspt.objectives[obj]['weight'], \
5 ...            cspt.objectives[obj]['comment'])) \
6 ...
7 Teaching: Best learning environment (30.0%)
8 Reputation survey; Staff-to-student ration;
9 Doctorate-to-student ratio;
10 Doctorate-to-academic-staff ratio;
11 Institutional income.
12 Research: Highest volume and reputation (30.0%)
13 Reputation survey;
14 Research income;
15 Research productivity.
16 Citations: Highest research influence (27.5%)
17 Impact.
18 International outlook: Most international staff,
19             students and research (7.5%)
20             Proportions of international students;
21             Proportions of international staff;
22             International collaborations.

```

```

23  Industry income: Best knowledge transfer (5.0%)
24  Volume.

```

With a cumulated importance of 87% (see above), *Teaching*, *Research* and *Citations* represent clearly the major ranking objectives. *International outlook* and *Industry income* are considered of minor importance (12.5%).

THE authors do, unfortunately, not publish the detail of their performance assessments for grading CS Depts with respect to each one of performance criteria per ranking objective.<sup>3</sup> The THE 2016 ranking publication reveals solely a compound assessment on a single performance criteria per ranking objective. The five retained performance criteria may be printed out as follows.

```

1 >>> for g in cspt.criteria:
2 ...     print('%.1f%%' % \
3 ...         (g,cspt.criteria[g]['name'],cspt.criteria[g]['comment'],\
4 ...          cspt.criteria[g]['weight']))
5 ...     gtch: Teaching, The learning environment (30.0%)
6 ...     gres: Research, Volume, income and reputation (30.0%)
7 ...     gcit: Citations, Research influence (27.5%)
8 ...     gint: International outlook, In staff, students and research
9 ...         (7.5%)
9 ...     gind: Industry income, knowledge transfer (5.0%)

```

The largest part (87.5%) of criteria significance is, hence canonically, allocated to the major ranking criteria: *Teaching* (30%), *Research* (30%) and *Citations* (27.5%). The small remaining part (12.5%) goes to *International outlook* (7.5%) and *Industry income* (5%).

In order to render commensurable these performance criteria, the THE authors replace, per criterion, the actual performance grade obtained by each University with the corresponding *quantile* observed in the cumulative distribution of the performance grades obtained by all the surveyed institutions (Times Higher Education, 2016). The THE ranking is eventually determined by an *overall score* per University which corresponds to the weighted average of these five criteria quantiles, as illustrated in Listing 13.4 below. Missing data is

**Listing 13.4** Computing the THE overall scores

```

1 >>> theScores = []
2 >>> for x in cspt.actions:
3 ...     xscore = Decimal('0')
4 ...     for g in cspt.criteria:
5 ...         xscore += cspt.evaluation[g][x] * \
6 ...             (cspt.criteria[g]['weight']/Decimal('100'))
7 ...     theScores.append((xscore,x))

```

In Listing 13.5 on the facing page (Lines 15-16), we may thus notice that, in the 2016 edition of the THE World University rankings by CS subject, the Swiss

---

<sup>3</sup> THE gives some insight on the subject and significance of the actual ranking criteria used for grading along each ranking objective on her website (Times Higher Education, 2016)

Federal Institute of Technology Zürich is first-ranked with an overall score of 92.9; followed by the California Institute of Technology (overall score: 92.4)<sup>4</sup>.

**Listing 13.5** Printing the ranked performance table

```

1 >>> theScores.sort(reverse = True)
2 >>> print('## Univ \tgtch gres gcit gint gind overall')
3 >>> print('-----')
4 >>> i = 1
5 >>> for it in theScores:
6 ...     x = it[1]
7 ...     xScore = it[0]
8 ...     print('%2d: %s' % (i,x), end=' \t')
9 ...     for g in cspt.criteria:
10 ...         print('%.1f' % (cspt.evaluation[g][x]),end=' ')
11 ...         print(' %.1f' % xScore)
12 ...         i += 1
13     ## Univ \tgtch gres gcit gint gind overall
14     -----
15     1: ethz    89.2  97.3  97.1  93.6  64.1   92.9
16     2: calt   91.5  96.0  99.8  59.1  85.9   92.4
17     3: oxf    94.0  92.0  98.8  93.6  44.3   92.2
18     4: mit    87.3  95.4  99.4  73.9  87.5   92.1
19     5: git    87.2  99.7  91.3  63.0  79.5   89.9
20     6: cmel   88.1  92.3  99.4  58.9  71.1   89.4
21     7: icl    90.1  87.5  95.1  94.3  49.9   89.0
22     8: epfl   86.3  91.6  94.8  97.2  42.7   88.9
23     9: tum    87.6  95.1  87.9  52.9  95.1   87.7
24    10: sing   89.9  91.3  83.0  95.3  50.6   86.9
25    11: cou    81.6  94.1  99.7  55.7  45.7   86.6
26    12: ucl    85.5  90.3  87.6  94.7  42.4   86.1
27    13: wash   84.4  88.7  99.3  57.4  41.2   85.6
28    14: hkst   74.3  92.0  96.2  84.4  55.8   85.5
29    15: ntu    76.6  87.7  90.4  92.9  86.9   85.5
30    16: ued    85.7  85.3  89.7  95.0  38.8   85.0
31    17: unt    79.9  84.4  99.6  77.6  38.4   84.4
32    18: uiu    85.0  83.1  99.2  51.4  42.2   83.7
33    19: mcp    79.7  89.3  94.6  29.8  51.7   81.5
34    20: cbu    81.2  78.5  94.7  66.9  45.7   81.3
35    21: tsu    88.1  90.2  76.7  27.1  85.9   80.9
36    22: csd    75.2  81.6  99.8  39.7  59.8   80.5
37    23: uwa    75.3  82.6  91.3  72.9  41.5   80.0
38    24: nyu    71.1  77.4  99.4  78.0  39.8   79.7
39    25: uta    72.6  85.3  99.6  31.6  49.7   79.6
40    26: kit    73.8  85.5  84.4  41.3  76.8   77.9
41    27: bju   83.0  85.3  70.1  30.7  99.4   77.0
42    28: csb    65.6  70.9  94.8  72.9  74.9   76.2
43    29: rwth   77.8  85.0  70.8  43.7  89.4   76.1
44    30: hku    77.0  73.0  77.0  96.8  39.5   75.4
45    31: pud    76.9  84.8  70.8  58.1  56.7   75.2
46    32: kist   79.4  88.2  64.2  31.6  92.8   74.9
47    33: kcl    45.5  94.6  86.3  95.1  38.3   74.8
48    34: chku   64.1  69.3  94.7  75.6  49.9   74.2
49    35: epfr   81.7  60.6  78.1  85.3  62.9   73.7
50    36: dut    64.1  78.3  76.3  69.8  90.1   73.4
51    37: tub    66.2  82.4  71.0  55.4  99.9   73.3
52    38: utj    92.0  91.7  48.7  25.8  49.6   72.9
53    39: cir    68.8  64.6  93.0  65.1  40.4   72.5
54    40: ntw    81.5  79.8  66.6  25.5  67.6   72.0
55    41: anu    47.2  73.0  92.2  90.0  48.1   70.6
56    42: rcu    64.1  53.8  99.4  63.7  46.1   69.8
57    43: mel    56.1  70.2  83.7  83.3  50.4   69.7
58    44: lms    81.5  68.1  61.0  31.1  87.8   68.4

```

<sup>4</sup> The author's own Computer Science Dept at the University of Luxembourg was ranked on position 63 with an overall score of 58.0.

59	45: ens	71.8	40.9	98.7	69.6	43.5	68.3
60	46: wtu	61.8	73.5	73.7	51.9	62.2	67.9
61	47: tech	54.9	71.0	85.1	51.7	40.1	67.1
62	48: bro	58.5	54.9	96.8	52.3	38.6	66.5
63	49: man	63.5	71.9	62.9	84.1	42.1	66.3
64	50: zhej	73.5	70.4	60.7	22.6	75.7	65.3
65	51: frei	54.2	51.6	89.5	49.7	99.9	65.1
66	52: unsw	60.2	58.2	70.5	87.0	44.3	63.6
67	53: kuj	75.4	72.8	49.5	28.3	51.4	62.8
68	54: sou	48.2	60.7	75.5	87.4	43.2	62.1
69	55: shJi	66.9	68.3	62.4	22.8	38.5	61.4
70	56: itmo	58.0	32.0	98.7	39.2	68.7	60.5
71	57: kul	35.2	55.8	92.0	46.0	88.3	60.5
72	58: glas	35.2	52.5	91.2	85.8	39.2	59.8
73	59: utw	38.2	52.8	87.0	69.0	60.0	59.4
74	60: stut	54.2	60.6	61.1	36.3	97.8	58.9
75	61: naji	51.4	76.9	48.8	39.7	74.4	58.6
76	62: tud	46.6	53.6	75.9	53.7	66.5	58.3
77	63: unlu	35.2	44.2	87.4	99.7	54.1	58.0
78	64: qut	45.5	42.6	82.8	75.2	63.0	58.0
79	65: hkpu	46.8	36.5	91.4	73.2	41.5	57.7
80	66: albt	39.2	53.3	69.9	91.9	75.4	57.6
81	67: mil	46.4	64.3	69.2	44.1	38.5	57.5
82	68: heils	48.8	49.6	80.4	50.6	39.5	57.4
83	69: cihk	42.4	44.9	80.1	76.2	67.9	57.3
84	70: tlavu	34.1	57.2	89.0	45.3	38.6	57.2
85	71: indis	56.9	76.1	49.3	20.1	41.5	57.0
86	72: ariz	28.4	61.8	84.3	59.3	42.0	56.8
87	73: kth	44.8	42.0	83.6	71.6	39.2	56.4
88	74: humb	48.4	31.3	94.7	41.5	45.5	55.3
89	75: eind	32.4	48.4	81.5	72.2	45.8	54.4

It is important to notice that a ranking by weighted average scores requires *commensurable ranking criteria* of precise decimal significance and on which precise decimal performance grades are given. It is very unlikely that the THE 2016 performance assessments verify indeed these conditions. Here we show how to relax these methodological requirements –precise commensurable criteria and decimal grades– by following instead a bipolar-valued epistemic logic based ranking methodology (see Chapter 8 on page 105).

## 13.2 Ranking with multiple criteria of ordinal significance

Let us, first, have a critical look in Fig. 13.1 on the facing page at the THE performance criteria.

```
1 >>> cspt.showHTMLCriteria(Sorted=False)
```

Considering a very likely imprecision of the performance grading procedure, followed by some potential violation of uniform distributed quantile classes, we assume here that a performance quantile difference of up to 2.5% is *insignificant*, whereas a difference of 5% and more warrants a *clearly better*, resp. *clearly less good* performance. With quantiles 94%, resp. 87.3%, Oxford's CS teaching environment, for instance, is thus clearly better evaluated than that of the MIT (see Listing 13.5 on the previous page Lines 27-28). We shall furthermore assume that a *considerable* performance quantile difference of 60%, observed on the three major

## the\_cs\_2016: Family of Criteria

#	Identifier	Name	Comment	Weight	Scale			Thresholds (ax + b)		
					direction	min	max	indifference	preference	veto
1	gtch	Teaching	The learning environment	30.00	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	0.00x + 60.00
2	gres	Research	Volume, income and reputation	30.00	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	0.00x + 60.00
3	gcit	Citations	Research influence	27.50	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	0.00x + 60.00
4	gint	International outlook	In staff, students and research	7.50	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	
5	gind	Industry income	Innovation	5.00	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	

**Fig. 13.1** The THE ranking criteria

ranking criteria: *Teaching*, *Research* and *Citations*, will trigger the polarisation of pairwise outranking, respectively outranked situations (Bisdorff, 2013).

The effect these performance discrimination thresholds induce on the outranking modelling can be inspected with the `showCriteria()` method.

### Listing 13.6 Inspecting the performance discrimination thresholds

```

1  >>> cspt.showCriteria()
2  -----
3  ----- criteria -----
4  3  gtch 'Teaching'
5  4  Scale = (Decimal('0.00'), Decimal('100.00'))
6  5  Weight = 0.300
7  6  Threshold ind : 2.50 + 0.00x ; percentile: 8.07
8  7  Threshold pref : 5.00 + 0.00x ; percentile: 15.75
9  8  Threshold veto : 60.00 + 0.00x ; percentile: 99.75
10 gres 'Research'
11 9  Scale = (Decimal('0.00'), Decimal('100.00'))
12 10 Weight = 0.300
13 11 Threshold ind : 2.50 + 0.00x ; percentile: 7.86
14 12 Threshold pref : 5.00 + 0.00x ; percentile: 16.14
15 13 Threshold veto : 60.00 + 0.00x ; percentile: 99.21
16 gcit 'Citations'
17 15 Scale = (Decimal('0.00'), Decimal('100.00'))
18 16 Weight = 0.275
19 17 Threshold ind : 2.50 + 0.00x ; percentile: 11.82
20 18 Threshold pref : 5.00 + 0.00x ; percentile: 22.99
21 19 Threshold veto : 60.00 + 0.00x ; percentile: 100.00
22 gint 'International outlook'
23 20 Scale = (Decimal('0.00'), Decimal('100.00'))
24 21 Weight = 0.075
25 22 Threshold ind : 2.50 + 0.00x ; percentile: 6.45
26 23 Threshold pref : 5.00 + 0.00x ; percentile: 11.75
27 gind 'Industry income'
28 24 Scale = (Decimal('0.00'), Decimal('100.00'))
29 25 Weight = 0.050
30 26 Threshold ind : 2.50 + 0.00x ; percentile: 11.82
31 27 Threshold pref : 5.00 + 0.00x ; percentile: 21.51

```

Between 6% and 12% of the observed quantile differences are, hence, considered to be *insignificant*. Similarly, between 77% and 88% are considered to be *significant*. Less than 1% correspond to *considerable* quantile differences on both the *Teaching* and *Research* criteria; actually triggering an epistemic polarisation effect (Bisdorff, 2013).

Beside the likely imprecise performance discrimination, the *precise decimal significance weights*, as allocated by the THE authors to the five ranking criteria are, as well, quite *questionable*. Criteria significance weights may carry sometimes hidden strategies for rendering the performance evaluations commensurable in view of a numerical computation of the overall ranking scores. The eventual ranking result is, the case given, as much depending on the precise values of the given criteria significance weights as, vice versa, the given precise significance weights are depending on the subjectively expected and accepted ranking results<sup>5</sup>. We will therefore drop these precise decimal weights and, instead, only require a corresponding criteria significance preorder:  $gtch = gres > gcit > gint > gind$ . *Teaching environment* and *Research volume and reputation* are equally considered most significant, followed by *Research influence*. Then comes *International outlook in staff, students and research* and, least significant finally, *Industry income and innovation*.

Both these working hypotheses: –performance discrimination thresholds and – solely ordinal criteria significance, give us way to a ranking methodology based on *robust pairwise outranking* situations (see Chapter 19 on page 275 and (Bisdorff, 2004)):

- We say that CS Dept  $x$  *robustly outranks* CS Dept  $y$  when  $x$  positively outranks  $y$  with **all** significance weight vectors that are compatible with the significance preorder:  $gtch = gres > gcit > gint > gind$ ;
- We say that CS Dept  $x$  is *robustly outranked* by CS Dept  $y$  when  $x$  is positively outranked by  $y$  with **all** significance weight vectors that are compatible with the significance preorder:  $gtch = gres > gcit > gint > gind$ ;
- Otherwise, CS Depts  $x$  and  $y$  are considered to be *incomparable*.

A corresponding digraph constructor is provided by the `RobustOutrankingDigraph` class.

**Listing 13.7** Computing the robust outranking digraph

```

1 >>> from outrankingDigraphs import RobustOutrankingDigraph
2 >>> rdg = RobustOutrankingDigraph(cspt)
3 >>> rdg
4     *----- Object instance description -----*
5     Instance class      : RobustOutrankingDigraph
6     Instance name       : robust_the_cs_2016
7     Actions             : 75
8     Criteria            : 5
9     Size                : 2993
10    Determinateness (%) : 78.16
11    Valuation domain    : [-1.00;1.00]
12 >>> rdg.computeIncomparabilityDegree(Comments=True)

```

<sup>5</sup> In a social choice context, this potential double bind between voting profiles and election result, corresponds to voting manipulation strategies.

```

13 Incomparability degree (%) of digraph <robust_the_cs_2016>:
14   links x<->y y: 2775, incomparable: 102, comparable: 2673
15   (incomparable/links) =  0.037
16 >>> rdg.computeTransitivityDegree(Comments=True)
17   Transitivity degree of digraph <robust_the_cs_2016>:
18   triples x>y>z: 405150, closed: 218489, open: 186661
19   (closed/triples) =  0.539
20 >>> rdg.computeSymmetryDegree(Comments=True)
21   Symmetry degree (%) of digraph <robust_the_cs_2016>:
22   arcs x>y: 2673, symmetric: 320, asymmetric: 2353
23   (symmetric/arcs) =  0.12

```

In the resulting digraph instance `rdg` (see Line 9 in [Listing 13.7 on the preceding page](#)), we observe 2993 such robust pairwise outranking situations validated with a mean significance of 78% (Line 10). Unfortunately, in the case here, they do not deliver any complete linear ranking. The robust outranking digraph `rdg` contains 102 incomparability situations (3.7%, Line 15); nearly half of its transitive closure is missing (46.1%, Line 19) and 12% of the positive outranking situations correspond in fact to symmetric indifference situations (Line 23).

Worse even, the digraph `rdg` admits a really high number of outranking circuits.

#### **Listing 13.8** Inspecting outranking circuits

```

1 >>> rdg.computeChordlessCircuits()
2 >>> rdg.showChordlessCircuits()
3   *---- Chordless circuits ----*
4   145 circuits.
5   1:  ['albt','unlu','ariz','hels'], cred. : 0.300
6   2:  ['albt','tlavu','hels'], cred. : 0.150
7   3:  ['anu', 'man', 'itmo'], cred. : 0.250
8   4:  ['anu', 'zhej', 'rcu'], cred. : 0.250
9   ...
10  ...
11  82:  ['csb','epfr','rwth'], cred. : 0.250
12  83:  ['csb','epfr','pud','nyu'], cred. : 0.250
13  84:  ['csd','kcl','kist'], cred. : 0.250
14  ...
15  ...
16  142:  ['kul','qut','mil'], cred. : 0.250
17  143:  ['lms','rcu','tech'], cred. : 0.300
18  144:  ['mil','stut','qut'], cred. : 0.300
19  145:  ['mil','stut','tud'], cred. : 0.300

```

Among the 145 detected robust outranking circuits reported in [Listing 13.8](#), we notice, for instance, two outranking circuits of length 4 (see circuits 1 and 83).

Let us inspect below the bipolar-valued robust outranking characteristics of the first circuit.

#### **Listing 13.9** Showing the relation table with stability denotation

```

1 >>> rdg.showRelationTable(actionsSubset=\
2 ...           ['albt','unlu','ariz','hels'],\
3 ...           Sorted=False)
4   * ---- Relation Table ----

```

```

5   r/(stab) | 'albt' 'unlu' 'ariz' 'hels'
6   -----|-----
7   'albt' | +1.00 +0.30 +0.00 +0.00
8   | (+4)  (+2)  (-1)  (-1)
9   'unlu' | +0.00 +1.00 +0.40 +0.00
10  | (+0)  (+4)  (+2)  (-1)
11  'ariz' | +0.00 -0.12 +1.00 +0.40
12  | (+1)  (-2)  (+4)  (+2)
13  'hels' | +0.45 +0.00 -0.03 +1.00
14  | (+2)  (+1)  (-2)  (+4)
15  Valuation domain: [-1.0; 1.0]
16  Stability denotation semantics:
17  +4|-4 : unanimous outranking | outranked situation;
18  +2|-2 : outranking | outranked situation validated
19  with all potential significance weights that are
20  compatible with the given significance preorder;
21  +1|-1 : validated outranking | outranked situation
22  with the given significance weights;
23  0   : indeterminate relational situation.

```

In Listing 13.9 on the previous page, we notice that the robust outranking circuit  $[\text{albt}, \text{unlu}, \text{ariz}, \text{hels}]$  will reappear with all potential criteria significance weight vectors that are compatible with given preorder:  $\text{gtch} = \text{gres} > \text{gcit} > \text{gint} > \text{gind}$ . Notice also the  $(\pm 1)$  marked outranking situations, like the one between  $\text{albt}$  and  $\text{ariz}$ . The statement that “*Arizona State University strictly outranks University of Alberta*” is in fact valid with the precise THE criteria significance weights, but not with all potential significance weights vectors that are compatible with the given significance preorder. All these  $(\pm 1)$  marked outranking situations become hence *doubtful* ( $r(x \succsim y) = 0.00$ ) and the corresponding CS Depts, like University of Alberta and Arizona State University, become *incomparable* in a robust outranking sense.

Showing many incomparabilities and indifferences; not being transitive and containing many robust outranking circuits; all these relational characteristics, make that no ranking algorithm, applied to digraph  $\text{rdg}$ , does exist that would produce a *unique* optimal linear ranking result. Methodologically, we are only left with ranking heuristics. In Chapter 8 on page 105 on ranking with multiple criteria we have now seen several potential heuristic ranking rules that can be used for ranking with an outranking digraph; yet, they may deliver potentially more or less diverging results. Considering the order of digraph  $\text{rdg}$  (75) and the largely unequal THE criteria significance weights, we rather opt, in this tutorial, for the NETFLOWS ranking rule <sup>6</sup>. Its complexity in  $O(n^2)$  is indeed quite tractable and, by avoiding potential *tyranny of short majority* effects, the NETFLOWS rule may specifically take the ranking criteria significance weights into a more fairly balanced account.

The NETFLOWS ranking result of the CS Depts can be directly computed with the `computeNetFlowsRanking()` method.

---

<sup>6</sup> The reader might try other ranking rules, like COPELAND’s or KOHLER’s rule. Mind that the latter ranking-by-choosing rule is more complex (see Chapter 8 on page 105).

**Listing 13.10** Computing a robust NETFLOWS ranking

```

1 >>> nfRanking = rdg.computeNetFlowsRanking()
2 >>> nfRanking
3 ['ethz', 'calt', 'mit', 'oxf', 'cmel', 'git', 'epfl',
4  'icl', 'cou', 'tum', 'wash', 'sing', 'hkst', 'ucl',
5  'uiu', 'unt', 'ued', 'ntu', 'mcp', 'csd', 'cbu',
6  'uta', 'tsu', 'nyu', 'uwa', 'csb', 'kit', 'utj',
7  'bju', 'kcl', 'chku', 'kist', 'rwth', 'pud', 'epfr',
8  'hku', 'rcu', 'cir', 'dut', 'ens', 'ntw', 'anu',
9  'tub', 'mel', 'lms', 'bro', 'frei', 'wtu', 'tech',
10 'itmo', 'zhej', 'man', 'kuj', 'kul', 'unsw', 'glas',
11 'utw', 'unlu', 'naji', 'sou', 'hkpu', 'qut', 'humb',
12 'shJi', 'stut', 'tud', 'tlavu', 'cihk', 'albt', 'indis',
13 'ariz', 'kth', 'hels', 'eind', 'mil']

```

We actually obtain in Listing 13.10 on the facing page a very similar ranking result as the one obtained with the THE average scores. The same group of seven Depts: ethz, calt, mit, oxf, cmel, git and epfl, is top-ranked. And a same group of Depts: tlavu, cihk, indis, ariz, kth, hels, eind, and mil appears at the end of the list.

We can print out the difference between the overall scores based THE ranking and the NETFLOWS ranking above with the following short Python script, where we make use of an ordered Python dictionary with net flow scores, stored in the rdg.netFlowsRankingDict attribute by the previous computation.

**Listing 13.11** Comparing the robust NETFLOWS ranking with the THE ranking

```

1 >>> # rdg.netFlowsRankingDict: ordered dictionary with net flow
2 >>> # scores stored in rdg by the computeNetFlowsRanking() method
3 >>> # theScores = [(xScore_1,x_1), (xScore_2,x_2), ... ]
4 >>> # is sorted in decreasing order of xscores_i
5 >>> print(
6 ... ' NetFlows ranking  gtch gres gcit gint gind  THE ranking')
7 >>> for i in range(75):
8 ...     x = nfRanking[i]
9 ...     xScore = rdg.netFlowsRankingDict[x]['netFlow']
10 ...     thexScore, thex = theScores[i]
11 ...     print('%2d: %s (%.2f) ' % (i+1,x,xScore), end=' \t')
12 ...     for g in rdg.criteria:
13 ...         print('%.1f ' % (t.evaluation[g][x]), end=' ')
14 ...         print(' %s (%.2f)' % (thex,thexScore) )
15 NetFlows ranking  gtch gres gcit gint gind  THE ranking
16 1: ethz (116.95) 89.2 97.3 97.1 93.6 64.1  ethz (92.88)
17 2: calt (116.15) 91.5 96.0 99.8 59.1 85.9  calt (92.42)
18 3: mit (112.72) 87.3 95.4 99.4 73.9 87.5  oxf (92.20)
19 4: oxf (112.00) 94.0 92.0 98.8 93.6 44.3  mit (92.06)
20 5: cmel (101.60) 88.1 92.3 99.4 58.9 71.1  git (89.88)
21 6: git (93.40) 87.2 99.7 91.3 63.0 79.5  cmel (89.43)
22 7: epfl (90.88) 86.3 91.6 94.8 97.2 42.7  icl (89.00)
23 8: icl (90.62) 90.1 87.5 95.1 94.3 49.9  epfl (88.86)
24 9: cou (84.60) 81.6 94.1 99.7 55.7 45.7  tum (87.70)
25 10: tum (80.42) 87.6 95.1 87.9 52.9 95.1  sing (86.86)
26 11: wash (76.28) 84.4 88.7 99.3 57.4 41.2  cou (86.59)
27 12: sing (73.05) 89.9 91.3 83.0 95.3 50.6  ucl (86.05)
28 13: hkst (71.05) 74.3 92.0 96.2 84.4 55.8  wash (85.60)
29 14: ucl (66.78) 85.5 90.3 87.6 94.7 42.4  hkst (85.47)
30 15: uiu (64.80) 85.0 83.1 99.2 51.4 42.2  ntu (85.46)
31 16: unt (62.65) 79.9 84.4 99.6 77.6 38.4  ued (85.03)
32 17: ued (58.67) 85.7 85.3 89.7 95.0 38.8  unt (84.42)
33 18: ntu (57.88) 76.6 87.7 90.4 92.9 86.9  uiu (83.67)

```

34	19: mcp (54.08)	79.7	89.3	94.6	29.8	51.7	mcp (81.53)
35	20: csd (46.62)	75.2	81.6	99.8	39.7	59.8	cbu (81.25)
36	21: cbu (44.27)	81.2	78.5	94.7	66.9	45.7	tsu (80.91)
37	22: uta (43.27)	72.6	85.3	99.6	31.6	49.7	csd (80.45)
38	23: tsu (42.42)	88.1	90.2	76.7	27.1	85.9	uwa (80.02)
39	24: nyu (35.30)	71.1	77.4	99.4	78.0	39.8	nyu (79.72)
40	25: uwa (28.88)	75.3	82.6	91.3	72.9	41.5	uta (79.61)
41	26: csb (18.18)	65.6	70.9	94.8	72.9	74.9	kit (77.94)
42	27: kit (16.32)	73.8	85.5	84.4	41.3	76.8	bju (77.04)
43	28: utj (15.95)	92.0	91.7	48.7	25.8	49.6	csb (76.23)
44	29: bju (15.45)	83.0	85.3	70.1	30.7	99.4	rwth (76.06)
45	30: kcl (11.95)	45.5	94.6	86.3	95.1	38.3	hku (75.41)
46	31: chku (9.43)	64.1	69.3	94.7	75.6	49.9	pud (75.17)
47	32: kist (7.30)	79.4	88.2	64.2	31.6	92.8	kist (74.94)
48	33: rwth (5.00)	77.8	85.0	70.8	43.7	89.4	kcl (74.81)
49	34: pud (2.40)	76.9	84.8	70.8	58.1	56.7	chku (74.23)
50	35: epfr (-1.70)	81.7	60.6	78.1	85.3	62.9	epfr (73.71)
51	36: hku (-3.83)	77.0	73.0	77.0	96.8	39.5	dut (73.44)
52	37: rcu (-6.38)	64.1	53.8	99.4	63.7	46.1	tub (73.25)
53	38: cir (-8.20)	68.8	64.6	93.0	65.1	40.4	utj (72.92)
54	39: dut (-8.85)	64.1	78.3	76.3	69.8	90.1	cir (72.50)
55	40: ens (-8.97)	71.8	40.9	98.7	69.6	43.5	ntw (72.00)
56	41: ntw (-11.15)	81.5	79.8	66.6	25.5	67.6	anu (70.57)
57	42: anu (-11.50)	47.2	73.0	92.2	90.0	48.1	rcu (69.79)
58	43: tub (-12.20)	66.2	82.4	71.0	55.4	99.9	mel (69.67)
59	44: mel (-23.98)	56.1	70.2	83.7	83.3	50.4	lms (68.38)
60	45: lms (-25.43)	81.5	68.1	61.0	31.1	87.8	ens (68.35)
61	46: bro (-27.18)	58.5	54.9	96.8	52.3	38.6	wtu (67.86)
62	47: frei (-34.42)	54.2	51.6	89.5	49.7	99.9	tech (67.06)
63	48: wtu (-35.05)	61.8	73.5	73.7	51.9	62.2	bro (66.49)
64	49: tech (-37.95)	54.9	71.0	85.1	51.7	40.1	man (66.33)
65	50: itmo (-38.50)	58.0	32.0	98.7	39.2	68.7	zhej (65.34)
66	51: zhej (-43.70)	73.5	70.4	60.7	22.6	75.7	frei (65.08)
67	52: man (-44.83)	63.5	71.9	62.9	84.1	42.1	unsw (63.65)
68	53: kuj (-47.40)	75.4	72.8	49.5	28.3	51.4	kuj (62.77)
69	54: kul (-49.98)	35.2	55.8	92.0	46.0	88.3	sou (62.15)
70	55: unsw (-54.88)	60.2	58.2	70.5	87.0	44.3	shJi (61.35)
71	56: glas (-56.98)	35.2	52.5	91.2	85.8	39.2	itmo (60.52)
72	57: utw (-59.27)	38.2	52.8	87.0	69.0	60.0	kul (60.47)
73	58: unlu (-60.08)	35.2	44.2	87.4	99.7	54.1	glas (59.78)
74	59: naji (-60.52)	51.4	76.9	48.8	39.7	74.4	utw (59.40)
75	60: sou (-60.83)	48.2	60.7	75.5	87.4	43.2	stut (58.85)
76	61: hkpu (-62.05)	46.8	36.5	91.4	73.2	41.5	naji (58.61)
77	62: qut (-66.17)	45.5	42.6	82.8	75.2	63.0	tud (58.28)
78	63: humb (-68.10)	48.4	31.3	94.7	41.5	45.5	unlu (58.04)
79	64: shJi (-69.72)	66.9	68.3	62.4	22.8	38.5	qut (57.99)
80	65: stut (-69.90)	54.2	60.6	61.1	36.3	97.8	hkpu (57.69)
81	66: tud (-70.83)	46.6	53.6	75.9	53.7	66.5	albt (57.63)
82	67: tlavu (-71.50)	34.1	57.2	89.0	45.3	38.6	mil (57.47)
83	68: cihk (-72.20)	42.4	44.9	80.1	76.2	67.9	hels (57.40)
84	69: albt (-72.33)	39.2	53.3	69.9	91.9	75.4	cihk (57.33)
85	70: indis (-72.53)	56.9	76.1	49.3	20.1	41.5	tlavu (57.19)
86	71: ariz (-75.10)	28.4	61.8	84.3	59.3	42.0	indis (57.04)
87	72: kth (-77.10)	44.8	42.0	83.6	71.6	39.2	ariz (56.79)
88	73: hels (-79.55)	48.8	49.6	80.4	50.6	39.5	kth (56.36)
89	74: eind (-82.85)	32.4	48.4	81.5	72.2	45.8	humb (55.34)
90	75: mil (-83.67)	46.4	64.3	69.2	44.1	38.5	eind (54.36)

The first inversion we observe in Listing 13.12 on the next page (Lines 18-19) concerns Oxford University and the MIT, switching positions 3 and 4. Most inversions are similarly short and concern only switching very close positions in either way. There are some slightly more important inversions concerning, for instance, the Hong Kong University CS Dept, ranked into position 30 in the THE ranking and here in the position 36 (Line 51). The opposite situation may also happen; the Berlin

Humboldt University CS Dept, occupying the 74th position in the THE ranking, advances in the robust NETFLOWS ranking to position 63 (Line 78).

In our bipolar-valued epistemic framework, the NETFLOWS score of any CS Dept  $x$  corresponds to the criteria significance support for the logical statement “ $x$  is first-ranked”. Formally

$$r(x \text{ is first-ranked}) = \sum_{y \neq x} r((x \succsim y) + (y \not\succsim x)) = \sum_{y \neq x} (r(x \succsim y) - r(y \succsim x)). \quad (13.1)$$

Using the robust outranking characteristics of digraph  $\text{rdg}$ , we can thus explicitly compute, for instance, ETH Zürich's NETFLOWS score, denoted  $\text{nf}_X$  below.

```
1 >>> x = 'ethz'
2 >>> nfx = Decimal('0')
3 >>> for y in rdg.actions:
4 ...     if x != y:
5 ...         nfx += (rdg.relation[x][y] \
6 ...                  - rdg.relation[y][x])
7 >>> print(x, nfx)
8 ethz 116.950
```

In Listing 13.12 (Line 16), one may now verify that ETH Zürich obtains indeed the highest NETFLOWS score 116.95, and gives, hence the *most credible* first-ranked CS Dept of the 75 potential candidates.

Yet, how may we now convince the reader, that our pairwise outranking based ranking result here appears more objective and trustworthy, than the classic value theory based THE ranking by average quantile scores?

### 13.3 How to judge the quality of a ranking result?

In a multiple criteria based ranking problem, inspecting pairwise marginal performance differences may give objectivity to global preferential statements. That a CS Dept  $x$  convincingly outranks Dept  $y$  can conveniently be checked with the `showPairwiseOutrankings()` method.. The ETH Zürich CS Dept is, for instance, first ranked before Caltech's Dept in both previous rankings. Lest us check the preferential reasons.

**Listing 13.12** Comparing pairwise criteria performances

```

1 >>> rdg.showPairwiseOutrankings('ethz','calt')
2 *----- pairwise comparisons -----
3 Valuation in range: -100.00 to +100.00
4 Comparing actions : ('ethz', 'calt')
5 crit. wght. g(ethz) g(calt) diff | ind pref r()
6 -----
7 'gcit' 27.50 97.10 99.80 -2.70 | 2.50 5.00 +0.00
8 'gind' 5.00 64.10 85.90 -21.80 | 2.50 5.00 -5.00
9 'gint' 7.50 93.60 59.10 +34.50 | 2.50 5.00 +7.50
10 'gres' 30.00 97.30 96.00 +1.30 | 2.50 5.00 +30.00
11 'gtch' 30.00 89.20 91.50 -2.30 | 2.50 5.00 +30.00
12 -----
13 r(x >= y): +62.50

```

14	crit.	wght.	g(calt)	g(ethz)	diff		ind	pref	r()	
15						-----				-----
16	'gcit'	27.50	99.80	97.10	+2.70		2.50	5.00	+27.50	
17	'gind'	5.00	85.90	64.10	+21.80		2.50	5.00	+5.00	
18	'gint'	7.50	59.10	93.60	-34.50		2.50	5.00	-7.50	
19	'gres'	30.00	96.00	97.30	-1.30		2.50	5.00	+30.00	
20	'gtch'	30.00	91.50	89.20	+2.30		2.50	5.00	+30.00	
21										-----
22										r(y >= x) : +85.00

A significant positive performance difference (+34.50), concerning the *International outlook* criterion (of 7.5% significance), is observed in Listing 13.12 on the previous page in favour of the ETH Zürich Dept (Line 9 above). Similarly, a significant positive performance difference (+21.80), concerning the *Industry income* criterion (of 5% significance), is observed, this time, in favour of the Caltech Dept (Line 17). The former, larger positive, performance difference, observed on a more significant criterion, gives so far a first convincing argument of 12.5% significance for putting ETH Zürich first, before Caltech. Yet, the slightly positive performance difference (+2.70, Line 16) between Caltech and ETH Zürich on the *Citations* criterion (of 27.5% significance) confirms an “*at least as well performing as*” situation in favour of the Caltech Dept.

The inverse negative performance difference (−2.70, Line 7), however, is neither *significant* ( $< -5.00$ ), nor *insignificant* ( $> -2.50$ ), and does hence neither confirm nor infirm a “*not at least as well performing as*” situation in disfavour of ETH Zürich. We observe here a convincing argument of 27.5% significance for ranking Caltech first, before ETH Zürich.

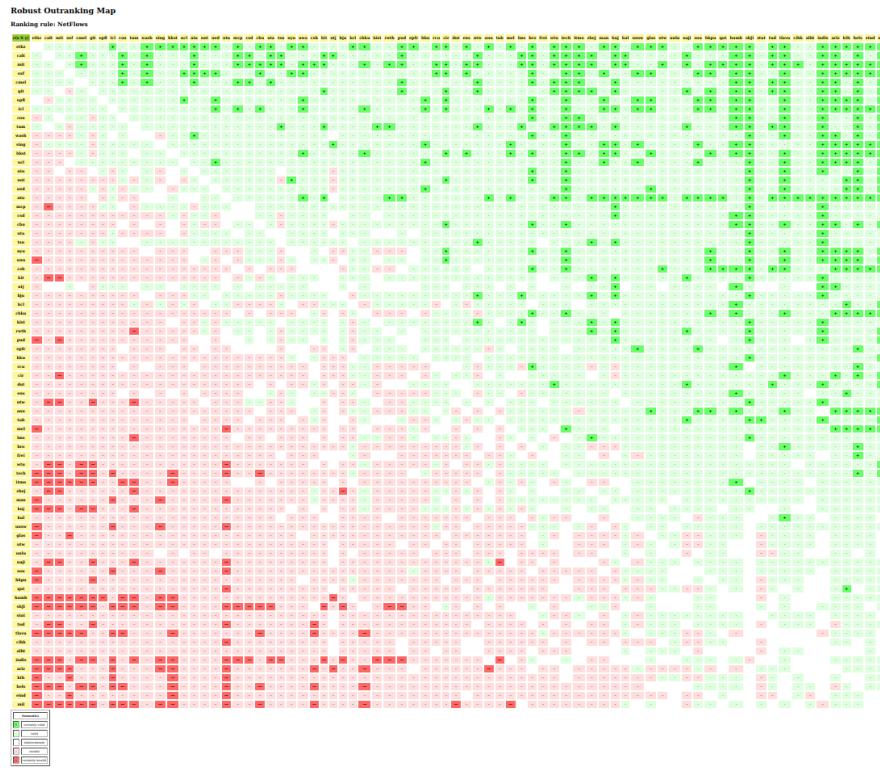
Notice finally, that, on the *Teaching* and *Research* criteria of total significance 60%, both Depts do, with performance differences ( $< |2.50|$ ), perform one as well as the other. As these two major performance criteria together necessarily admid always the highest significance with the imposed significance weight preorder: *gtch* = *gres* > *gcit* > *gint* > *gind*, both outranking situations get in fact globally confirmed at stability level +2 (see Chapter 19 on page 275).

We may well illustrate all such *stable outranking* situations with a browser view of the corresponding robust relation map ordering the CS Depts again with the same NETFLOWS ranking rule.

```
1 >>> rdg.showHTMLRelationMap('
2 ...           tableTitle='Robust Outranking Map',
3 ...           rankingRule='NetFlows')
```

In Fig. 13.2 on the next page, *dark green*, resp. *light green* marked positions show certainly, resp. positively valid outranking situations, whereas *dark red*, resp. *light red* marked positions show certainly, respectively positively valid outranked situations. In the left upper corner one may verify that the five top-ranked Depts ([ethz, calt, oxf, mit, cmel]) are in fact mutually outranking each other and thus are all indifferent one to another. They give even robust CONDORCET winners by robustly outranking all other Depts.

Notice by the way that no certainly valid robust outranking (dark green) and no certainly valid robust outranked situations (dark red) appear in Fig. 13.3 on page 191 below, resp. above the principal diagonal; none of these robust preferential situations



**Fig. 13.2** Relation map of the robust outranking relation

are hence violated by the robust NETFLOWS ranking. The non reflexive *white* positions in the relation map, like the one between the Georgia Institute of Technology and the MIT, mark outranking or outranked situations that are not robust with respect to the given significance weight preorder. They become, hence, doubtful and are set to the indeterminate characteristic value 0.0.

By measuring the ordinal correlation with the underlying pairwise global and marginal robust outranking situations, the quality of the robust NETFLOWS ranking result can be formally evaluated with the `computeRankingCorrelation()` method and the `showCorrelation()` method (see [Chapter 16 on page 221](#)).

**Listing 13.13** Measuring the quality of the NETFLOWS ranking result

```

1 >>> corrnf = rdg.computeRankingCorrelation(nfRanking)
2 >>> rdg.showCorrelation(corrnf)
3   Correlation indexes:
4     Crisp ordinal correlation : +0.901
5     Epistemic determination  :  0.563
6     Bipolar-valued equivalence : +0.507

```

[Listing 13.13](#) Line 4 indicates that the NETFLOWS ranking result is indeed highly ordinally correlated (+0.901, in KENDALL's tau index sense) with the pairwise

global robust outranking relation. Their bipolar-valued *relational equivalence* value (+0.51, Line 6) indicates a more than 75% criteria significance support.

With the `showRankingConsensusQuality()` method, we can also check how the NETFLOWS ranking rule is actually balancing the five ranking objectives.

**Listing 13.14** Measuring the consensus quality of the NETFLOWS ranking result

```

1 >>> rdg.showRankingConsensusQuality(nfRanking)
2   Criterion (weight): correlation
3   -----
4     gtch (0.300): +0.660
5     gres (0.300): +0.638
6     gcit (0.275): +0.370
7     gint (0.075): +0.155
8     gind (0.050): +0.101
9   Summary:
10    Weighted mean marginal correlation (a): +0.508
11    Standard deviation (b) : +0.187
12    Ranking fairness (a)-(b) : +0.321

```

The ordinal correlation indexes with the marginal performance criterion rankings are nearly respecting the given significance weights preorder:  $gtch = gres > gcit > gint > gind$  (see Lines 4-8 above). The mean marginal correlation index is quite high (+0.51). Coupled with a low standard deviation (0.187), we obtain a rather fairly balanced ranking result (Lines 10-12).

We can furthermore inspect with the `showCriteriaCorrelationTable()` method the mutual correlation indexes observed between the individual criterion based marginal robust outranking relations.

**Listing 13.15** Showing the ordinal correlation between the marginal criterion relations

```

1 >>> rdg.showCriteriaCorrelationTable()
2   Criteria ordinal correlation index
3   | gcit      gind      gint      gres      gtch
4   |-----|-----|-----|-----|-----|
5   gcit | +1.00    -0.11    +0.24    +0.13    +0.17
6   gind |           +1.00    -0.18    +0.15    +0.15
7   gint |           |           +1.00    +0.04    -0.00
8   gres |           |           |           +1.00    +0.67
9   gtch |           |           |           |           +1.00

```

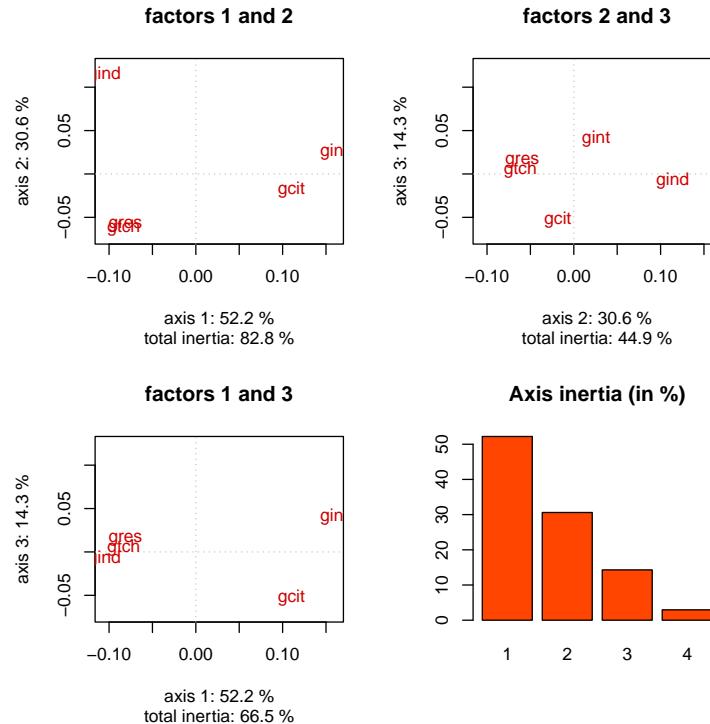
Slightly contradictory (-0.11) appear the *Citations* and *Industrial income* criteria (Line 5 Column 3 in Listing 13.15). Due perhaps to potential confidentiality clauses, it seems perhaps not always possible to publish industrially relevant research results in highly ranked journals. However, criteria *Citations* and *International outlook* show a slightly positive correlation (+0.24, Column 4), whereas the *International outlook* criterion shows no apparent correlation with both the major *Teaching* and *Research* criteria. The latter are however highly correlated (+0.67, Line 9 Column 6).

A Principal Component Analysis may well illustrate the previous findings (Bisdorff, 2020, Lecture 2).

```

1 >>> rdg.export3DplotOfCriteriaCorrelation(graphType='pdf')

```



**Fig. 13.3** 3D PCA plot of the pairwise criteria correlation table.

We notice in Fig. 13.3, first, that more than 80% of the total variance of the previous correlation table is explained by the apparent opposition between the marginal outrankings of criteria: *Teaching*, *Research* and *Industry income* on the left side, and the marginal outrankings of criteria: *Citations* and *International outlook* on the right side. Notice also in the left lower corner the nearly identical positions of the marginal outrankings of the major *Teaching* and *Research* criteria. In the factors 2 and 3 plot, about 30% of the total variance is captured by the opposition between the marginal outrankings of the *Teaching* and *Research* criteria and the marginal outrankings of the *Industrial income* criterion. Finally, in the factors 1 and 3 plot, nearly 15% of the total variance is explained by the opposition between the marginal outrankings of the *International outlook* criterion and the marginal outrankings of the *Citations* criterion.

It is, finally, interesting to similarly assess the ordinal correlation between the THE average scores-based ranking and the robust outranking digraph.

**Listing 13.16** Computing the ordinal quality of the THE ranking

```

1 >>> # theScores = [(xScore_1,x_1), (xScore_2,x_2), ... ]
2 >>> # is sorted in decreasing order of xscores
3 >>> theRanking = [item[1] for item in theScores]

```

```

4 >>> corrthe = rdg.computeRankingCorrelation(theRanking)
5 >>> rdg.showCorrelation(corrthe)
6     Correlation indexes:
7         Crisp ordinal correlation : +0.907
8         Epistemic determination : 0.563
9         Bipolar-valued equivalence : +0.511
10 >>> rdg.showRankingConsensusQuality(theRanking)
11     Criterion (weight): correlation
12     -----
13     gtch (0.300): +0.683
14     gres (0.300): +0.670
15     gcit (0.275): +0.319
16     gint (0.075): +0.161
17     gind (0.050): +0.106
18     Summary:
19     Weighted mean marginal correlation (a): +0.511
20     Standard deviation (b) : +0.210
21     Ranking fairness (a)-(b) : +0.302

```

The THE ranking result is similarly correlated (+0.907, Line 7 in Listing 13.16) with the pairwise global robust outranking situations. By its overall weighted scoring rule, the THE ranking naturally induces marginal criterion correlations that are compatible with the given significance weight preorder (Lines 13-17). Notice that the mean marginal correlation is of a similar value (+0.51, Line 19) as the robust NETFLOWS ranking. Yet, its standard deviation is slightly higher, which leads to a less fairer balancing of the three major ranking criteria.

To conclude, let us emphasize, that, without any commensurability hypothesis and by taking, furthermore, into account, first, the always present more or less imprecision of any performance grading and, secondly, solely ordinal criteria significance weights, we may obtain here with our robust outranking approach a very similar ranking result with a slightly better preference modelling quality. A convincing heatmap view of the 25 first-ranked Institutions may eventually be generated in the default system browser with following command.

```

1 >>> rdg.showHTMLPerformanceHeatmap(
2     ...     WithActionNames=True, \
3     ...     outrankingModel='this', \
4     ...     rankingRule='NetFlows', \
5     ...     ndigits=1, \
6     ...     Correlations=True, \
7     ...     fromIndex=0,toIndex=25)

```

As an exercise, the reader is invited to try out other robust outranking based ranking heuristics. Notice also that we have not challenged in this case study the THE provided criteria significance preorder. It would be very interesting to consider the five ranking objectives as equally important and, consequently, consider the ranking criteria to be equisignificant. Curious to see the ranking results under such settings.

### Heatmap of Performance Tableau 'robust\_the\_cs\_2016'

criteria	gtch	gres	gcit	gint	gind
<b>weights</b>	+30.00	+30.00	+27.50	+7.50	+5.00
<b>tau<sup>(*)</sup></b>	+0.66	+0.64	+0.37	+0.15	+0.10
<b>Swiss Federal Institute of Technology Zürich (ethz)</b>	89.2	97.3	97.1	93.6	64.1
<b>California Institute of Technology (calt)</b>	91.5	96.0	99.8	59.1	85.9
<b>Massachusetts Institute of Technology (mit)</b>	87.3	95.4	99.4	73.9	87.5
<b>University of Oxford (oxf)</b>	94.0	92.0	98.8	93.6	44.3
<b>Carnegie Mellon University (cmel)</b>	88.1	92.3	99.4	58.9	71.1
<b>Georgia Institute of Technology (git)</b>	87.2	99.7	91.3	63.0	79.5
<b>Swiss Federal Institute of Technology Lausanne (epfl)</b>	86.3	91.6	94.8	97.2	42.7
<b>Imperial College London (icl)</b>	90.1	87.5	95.1	94.3	49.9
<b>Cornell University (cou)</b>	81.6	94.1	99.7	55.7	45.7
<b>Technical University of München (tum)</b>	87.6	95.1	87.9	52.9	95.1
<b>University of Washington (wash)</b>	84.4	88.7	99.3	57.4	41.2
<b>National University of Singapore (sing)</b>	89.9	91.3	83.0	95.3	50.6
<b>Hong Kong University of Science and Technology (hkst)</b>	74.3	92.0	96.2	84.4	55.8
<b>University College London (ucl)</b>	85.5	90.3	87.6	94.7	42.4
<b>University of Illinois at Urbana-Champagne (uiu)</b>	85.0	83.1	99.2	51.4	42.2
<b>University of Toronto (unt)</b>	79.9	84.4	99.6	77.6	38.4
<b>University of Edinburgh (ued)</b>	85.7	85.3	89.7	95.0	38.8
<b>Nanyang Technological University of Singapore (ntu)</b>	76.6	87.7	90.4	92.9	86.9
<b>University of Maryland College Park (mcp)</b>	79.7	89.3	94.6	29.8	51.7
<b>University Of California at San Diego (csd)</b>	75.2	81.6	99.8	39.7	59.8
<b>Columbia University (cbu)</b>	81.2	78.5	94.7	66.9	45.7
<b>University of Texas at Austin (uta)</b>	72.6	85.3	99.6	31.6	49.7
<b>Tsinghua University (tsu)</b>	88.1	90.2	76.7	27.1	85.9
<b>New York University (nyu)</b>	71.1	77.4	99.4	78.0	39.8
<b>University of Waterloo (uwa)</b>	75.3	82.6	91.3	72.9	41.5

Color legend:

<b>quantile</b>	14.29%	28.57%	42.86%	57.14%	71.43%	85.71%	100.00%
-----------------	--------	--------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: **this**, Ranking rule: **NetFlows**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.901**

Mean marginal correlation (a) : **+0.508**

Standard marginal correlation deviation (b) : **+0.187**

Ranking fairness (a) - (b) : **+0.321**

**Fig. 13.4** Extract of a heatmap browser view on the NETFLOWS ranking result

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## Chapter 14

# The best students, where do they study? A rating case study

**Abstract** In 2004, the German magazine *Der Spiegel*, with the help of *McKinsey & Company* and *AOL*, conducted an extensive online survey, assessing the apparent quality of German University students. The eventually published results by the *Spiegel* magazine concerned nearly 50,000 students, enroled in one of fifteen popular academic subjects, like *German Studies*, *Life Sciences*, *Psychology*, *Law* or *CS*. Based on this published data Footnote[28], we present and discuss in this chapter, how to *rate* with the help of our DIGRAPH3 software ressources the apparent global *enrolment quality* of these 41 higher education institutions.

## 14.1 The performance tableau

In the 2004 DER SPIEGEL survey, more than 80,000 students, by participating, were questioned on their 'Abitur' and university exams' marks, time of studies and age, grants, awards and publications, IT proficiency, linguistic skills, practical work experience, foreign mobility and civil engagement. Each student received in return a quality score through a specific weighing of the collected data which depended on the subject the student is mainly studying. Publishing only those subject-University combinations, where at least 18 students had correctly filled in the questionnaire, left 41 German Universities where, for at least eight out of the fifteen subjects, an average enrolment quality score could be determined (Friedmann et al, 2004).

Internet published data of the 2004 survey is stored, for our case study here, in a file named `studentenSpiegel04.py` of `PerformanceTableau` format<sup>1</sup>.

**Listing 14.1** The 2004 Spiegel students survey data

```
1 >>> from perfTabs import PerformanceTableau
2 >>> t = PerformanceTableau('studentenSpiegel04')
3 >>> t
```

---

<sup>1</sup> The `performance tableau` `studentenSpiegel04.py` is also available in the examples directory of the DIGRAPH3 software collection (Bisdorff, 2021).

```

4  *----- PerformanceTableau instance description -*
5  Instance class      : PerformanceTableau
6  Instance name       : studentenSpiegel04
7  Actions             : 41 (Universities)
8  Criteria            : 15 (academic subjects)
9  NA proportion (%) : 27.3
10 Attributes          : ['name', 'actions',
11                           'objectives', 'criteria',
12                           'weightPreorder',
13                           'evaluation']
14 >>> t.showHTMLPerformanceHeatmap(ndigits=1,\n15 ...                               rankingRule=None)

```

In Figure 14.1 on the next page, the fifteen popular academic subjects are grouped into topical 'Faculties': - *Humanities*; - *Law, Economics & Management*; - *Life Sciences & Medicine*; - *Natural Sciences & Mathematics*; and - *Technology*. All fifteen subjects are considered equally significant for our rating problem (see Row 2). The recorded average enrolment quality scores appear coloured along a 7-tiling scheme per subject (see last Row).

We may by the way notice that TU Dresden is the only Institution showing enrolment quality scores in all the fifteen academic subjects. Whereas, on the one side, TU Muenchen and Kaiserslautern are only graded in *Sciences* and *Technology* subjects. On the other side, Mannheim, is only graded in *Humanities* and *Law, Economics & Management* studies. Most of the 41 Universities are not graded in *Engineering* studies. We are, hence, facing a large part (27.3%) of irreducible missing data (see Listing 14.1 on the preceding page Line 9).

Details of the enrolment quality criteria (the academic subjects) may be consulted in a browser view (see Fig. 14.2 on page 198 below).

```
1 >>> t.showHTMLCriteria()
```

The evaluation of the individual quality score for a participating student actually depends on his or her mainly enrolled subject<sup>2</sup>. The apparent quality measurement scales thus largely differ indeed from subject to subject (see Fig. 14.2 on page 198), like *Law Studies* (35.0 – 65 – 0) and *Politology* (50.0 – 70.0). The recorded average enrolment quality scores, hence, are in fact *incommensurable* between the subjects.

To take furthermore into account a potential and very likely imprecision of the individual quality scores' computation, we shall assume that, for all subjects, an average enrolment quality score difference of 0.1 is *insignificant*, wheras a difference of 0.5 is sufficient to positively attest a *better* enrolment quality.

The apparent incommensurability and very likely imprecision of the recorded average enrolment quality scores, renders meaningless any global averaging over the subjects per University of the enrolment quality. We shall therefore, similarly to the methodological approach of the *Spiegel* authors, proceed with an *order statistics* based rating-by-ranking approach (see Chapter 10 on page 135 on rating with learned quantile norms).

---

<sup>2</sup> The methodology (in German) guiding the *Spiegel* survey may be consulted in the examples directory of the DIGRAPH3 resources (DER SPIEGEL, 2004).

criteria	germ	pol	psy	soc	law	eco	mgt	bio	med	phys	chem	math	info	elec	mec
weights	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00
aach	53.3	50.8	62.7	51.0	NA	NA	49.6	52.2	49.5	59.1	53.6	58.6	54.6	57.2	54.4
aug	57.9	54.3	NA	54.8	45.6	NA	54.3	NA	NA	62.3	NA	61.2	58.1	NA	NA
berf	54.7	61.4	59.8	55.5	45.7	50.5	52.2	51.6	49.0	61.6	57.4	NA	54.9	NA	NA
berh	57.3	58.5	59.8	59.2	48.8	59.5	55.5	55.0	52.3	61.9	53.2	57.9	55.8	NA	NA
bertu	51.4	NA	57.7	59.1	NA	49.6	54.0	NA	NA	58.9	52.0	56.8	55.4	56.1	54.3
bie	51.4	NA	54.4	55.6	41.9	NA	50.7	49.7	NA	53.9	54.2	56.3	55.8	NA	NA
boc	53.9	NA	55.2	NA	39.1	NA	NA	48.0	49.8	56.8	53.3	57.6	NA	54.2	54.4
bon	54.1	57.3	60.3	56.0	47.2	53.6	NA	50.1	3.0	59.9	53.1	59.4	53.7	NA	NA
brau	53.5	54.0	NA	51.5	NA	NA	53.4	53.1	NA	59.8	50.1	54.7	52.6	54.5	55.2
brem	56.9	55.5	52.5	54.5	40.9	NA	55.4	53.3	NA	59.7	NA	NA	54.1	50.1	NA
chem	54.3	57.1	60.8	53.3	NA	NA	52.7	NA	NA	NA	NA	NA	57.7	57.5	53.6
darm	1.0	59.7	58.6	52.0	NA	NA	NA	1.0	NA	62.5	2.0	59.4	3.0	NA	56.1
dres	55.2	55.9	60.6	56.2	44.0	56.7	54.8	55.3	49.2	59.9	55.8	57.8	56.2	56.1	54.8
dsd	53.5	NA	57.5	48.8	44.9	NA	50.5	47.3	50.5	NA	53.5	NA	NA	NA	NA
duis	50.6	52.5	NA	47.9	NA	NA	47.5	NA	48.0	54.6	52.8	51.6	56.8	53.6	51.9
erl	57.9	55.1	58.7	55.4	42.9	NA	55.6	51.8	49.3	60.3	54.0	60.5	54.6	55.9	55.1
fran	51.7	53.1	58.0	51.5	41.9	53.5	52.0	51.3	51.2	62.1	55.5	57.0	52.4	NA	NA
frei	57.8	60.5	64.1	57.5	50.7	53.3	NA	55.4	54.2	61.6	57.0	60.6	58.1	NA	NA
gie	53.0	59.0	58.0	NA	41.9	NA	51.2	50.4	50.0	57.6	NA	NA	NA	NA	NA
goet	58.7	56.3	59.8	53.5	44.8	53.6	52.6	50.5	48.9	60.4	53.9	63.1	NA	NA	NA
ham	57.0	60.2	57.3	53.6	44.1	52.1	49.8	52.7	49.2	56.4	54.2	54.9	54.7	NA	NA
han	50.4	52.8	NA	49.9	41.2	NA	NA	53.8	NA	57.5	53.6	56.6	58.8	53.5	53.6
hei	61.4	59.5	59.8	52.2	51.1	54.4	NA	55.2	55.5	60.9	56.7	61.3	NA	NA	NA
jena	56.5	55.8	58.5	52.8	45.3	NA	56.2	52.7	51.1	61.6	57.8	NA	57.2	NA	NA
kiel	51.9	52.2	58.4	NA	45.1	50.5	52.8	52.7	49.6	59.7	52.8	NA	54.9	54.2	NA
koel	51.7	57.6	58.9	56.1	46.1	56.1	54.6	51.8	50.7	58.7	54.0	56.5	NA	NA	NA
kons	56.9	65.9	59.1	54.7	48.3	59.0	NA	55.7	NA	59.9	58.0	NA	59.7	NA	NA
ksl	NA	55.9	50.8	NA	59.7	54.6	62.2	56.2	57.5						
leip	57.4	60.4	62.4	59.5	46.3	NA	54.6	56.9	51.5	62.2	57.0	NA	3.0	NA	NA
main	54.2	57.9	56.9	55.7	46.5	50.7	53.1	50.0	49.2	60.8	56.3	54.7	NA	NA	NA
marb	53.6	54.7	57.6	59.8	40.3	NA	55.5	53.2	51.1	62.8	57.8	NA	55.6	NA	NA
mnh	52.2	57.2	61.1	55.0	45.0	57.0	59.7	NA	NA	NA	NA	NA	58.6	NA	NA
mnst	55.4	56.7	62.2	56.3	46.4	54.1	56.3	51.2	52.8	55.2	55.0	56.9	57.7	NA	NA
mu	57.2	60.1	60.9	54.0	47.3	55.8	57.5	50.4	52.6	62.0	58.1	59.6	57.1	NA	NA
reg	54.8	55.4	62.1	NA	46.1	52.3	55.5	54.6	50.9	60.5	55.8	59.2	NA	NA	NA
saar	57.9	NA	56.5	NA	48.1	NA	52.2	NA	49.6	61.2	56.2	NA	57.6	NA	NA
stu	52.5	58.4	NA	NA	NA	NA	NA	56.6	57.1	NA	61.5	55.2	60.6	59.8	60.2
tri	54.1	58.0	58.3	54.9	46.3	52.8	52.8	NA	NA	NA	NA	60.7	52.3	NA	NA
tueb	57.9	57.7	58.4	1.0	46.8	60.8	54.4	53.7	52.1	61.6	57.5	NA	55.2	NA	NA
tum	NA	68.0	53.6	60.1	62.8	58.8	62.6	58.2	58.2						
wrzb	56.9	56.0	59.8	NA	46.4	53.3	52.8	53.0	52.2	60.2	56.6	NA	55.9	NA	NA

Color legend:

quantile	14.29%	28.57%	42.86%	57.14%	71.43%	85.71%	100.00%
----------	--------	--------	--------	--------	--------	--------	---------

Fig. 14.1 Average quality of enrolled students per academic subject

#	Identifier	Name	Comment	Weight	Scale			Thresholds (ax + b)		
					direction	min	max	indifference	preference	veto
1	bio	Life Sciences	Life Sciences & Medicine	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
2	chem	Chemistry	Natural Sciences & Mathematics	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
3	eco	Economics	Law, Economics & Management	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
4	elec	Electrical Engineering	Technology	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
5	germ	German Studies	Humanities	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
6	info	Computer Science	Technology	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
7	law	Law Studies	Law, Economics & Management	1.00	max	35.00	65.00	0.00x + 0.10	0.00x + 0.50	
8	math	Mathematics	Natural Sciences & Mathematics	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
9	mec	Mechanical Engineering	Technology	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
10	med	Medicine	Life Sciences & Medicine	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
11	mgt	Management	Law, Economics & Management	1.00	max	40.00	80.00	0.00x + 0.10	0.00x + 0.50	
12	phys	Physics	Natural Sciences & Mathematics	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
13	pol	Politology	Humanities	1.00	max	50.00	70.00	0.00x + 0.10	0.00x + 0.50	
14	psy	Psychology	Humanities	1.00	max	50.00	70.00	0.00x + 0.10	0.00x + 0.50	
15	soc	Sociology	Humanities	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	

Fig. 14.2 Details of the rating criteria

## 14.2 Rating-by-ranking with lower-closed quantile limits

The *Spiegel* authors opted indeed for a simple 3-tiling of the Universities per valued academic subject, followed by an average BORDA scores based global ranking. Here, our epistemic logic based outranking approach, allows us, with adequate choices of *indifference* (0.1) and *preference* (0.5) discrimination thresholds, to estimate *lower-closed* 9-tiles of the enrolment quality scores per subject and rank conjointly, with the help of the COPELAND ranking rule Footnote[34] applied to a corresponding bipolar-valued outranking digraph, the 41 Universities **and** the lower limits of the estimated 9-tiles limits.

We therefore need in Listing 14.2, first, to import from the `performanceQuantiles` module the `PerformanceQuantiles` class constructor and estimate the lowerclosed 9-tiling of the average enrolment quality scores per academic subject.

Listing 14.2 Computing 9-tiles of the enrolment quality scores per subject

```

1 >>> from performanceQuantiles import PerformanceQuantiles
2 >>> pq = PerformanceQuantiles(t,numberOfBins=9,LowerClosed=True)
3 >>> pq
4     *----- PerformanceQuantiles instance description -----*
5     Instance class      : PerformanceQuantiles
6     Instance name       : 9-tiled_performances
7     Criteria            : 15
8     Quantiles           : 9 (LowerClosed)
9     History sizes      : {
10         'germ': 39, 'pol': 34, 'psy': 34, 'soc': 32,
11         'law': 32, 'eco': 21, 'mgt': 34,
12         'bio': 34, 'med': 28, 'phys': 37, 'chem': 35, 'math': 27,
13         'info': 33, 'elec': 14, 'mec': 13, }
```

The history sizes, reported in Listing 14.2 above, indicate the number of Universities graded in each one of the popular fifteen subjects. *German Studies*, for instance, are graded for 39 out of 41 Universities, whereas *Electrical and Mechan-*

ical Engineering are only graded for 14, respectively 13 Institutions. None of the fifteen subjects are graded in all the 41 Universities <sup>3</sup>.

We may inspect the resulting 9-tiling limits in a browser view.

```
1 >>> pq.showHTMLLimitingQuantiles(Transposed=True,Sorted=False, \
2 ...           ndigits=1,title='9-tiled quality score limits')
```

**Fig. 14.3** 9-tiling quality score limits per academic subject. We see confirmed again the incommensurability between the subjects, we noticed already in the apparent enrolment quality scoring, especially between *Law Studies* (39.1 – 51.1) and *Politology* (50.5 – 65.9). Universities evaluated in *Law studies* but not in *Politology*, like the University of Bielefeld, would see their enrolment quality unfairly weakened when simply averaging the enrolment quality scores over evaluated subjects

## 9-tiled quality score limits

Sampling sizes between 13 and 39.

criterion	0.00	0.11	0.22	0.33	0.44	0.56	0.67	0.78	0.89	1.00
<b>bio</b>	45.0	49.9	50.5	51.4	52.3	53.0	53.5	54.8	55.5	57.1
<b>chem</b>	45.0	52.8	53.5	54.0	54.4	55.6	56.4	57.1	57.8	58.8
<b>eco</b>	49.6	50.6	52.2	53.3	53.5	53.9	55.8	56.8	59.3	60.8
<b>elec</b>	50.1	53.6	54.2	54.9	55.9	56.1	57.3	57.5	59.1	60.2
<b>germ</b>	45.0	51.5	52.4	53.5	54.1	55.1	56.9	57.3	57.9	61.4
<b>info</b>	45.0	52.5	54.6	54.9	55.7	56.2	57.2	58.0	58.7	59.8
<b>law</b>	39.1	41.6	43.0	44.9	45.4	46.1	46.4	47.2	48.5	51.1
<b>math</b>	51.6	54.9	56.6	57.0	57.9	59.4	60.5	60.7	62.2	63.1
<b>mec</b>	51.9	53.6	54.2	54.4	54.7	55.1	55.8	56.4	57.4	57.8
<b>med</b>	45.0	49.0	49.2	49.6	50.2	51.0	51.4	52.3	54.0	60.1
<b>mgt</b>	47.5	50.7	52.2	52.8	53.5	54.6	55.5	55.7	56.8	68.0
<b>phys</b>	53.9	56.9	58.9	59.7	60.0	60.7	61.6	61.8	62.3	62.8
<b>pol</b>	50.8	53.0	54.9	55.8	56.7	57.6	58.3	59.6	60.4	65.9
<b>psy</b>	52.5	56.8	57.7	58.3	58.6	59.7	59.8	60.8	62.2	64.1
<b>soc</b>	45.0	50.5	52.0	53.4	54.5	55.0	55.6	56.2	59.1	59.8

We add, now, these 9-tiling quality score limits to the enrolment quality records of the 41 Universities and rank, by using the COPELAND rule, all these records conjointly together with the help of the `LearnedQuantilesRatingDigraph` class from the `sortingDigraphs` module.

```
1 >>> from sortingDigraphs import \
2 ...           LearnedQuantilesRatingDigraph
3 >>> lqr = LearnedQuantilesRatingDigraph(pq,t, \
4 ...           rankingRule='Copeland')
```

The resulting ranking of the 41 Universities including the lower-closed 9-tiling score limits may be nicely illustrated with the help of a corresponding heatmap view

```
1 >>> lqr.showHTMLRatingHeatmap(colorLevels=7, \
2 ...           Correlations=True, \
3 ...           ndigits=1,rankingRule='Copeland')
```

<sup>3</sup> It would have been interesting to estimate such quantile limits from the individual quality scores of all the nearly 50,000 surveyed students. But this data was not available (Friedmann et al, 2004).

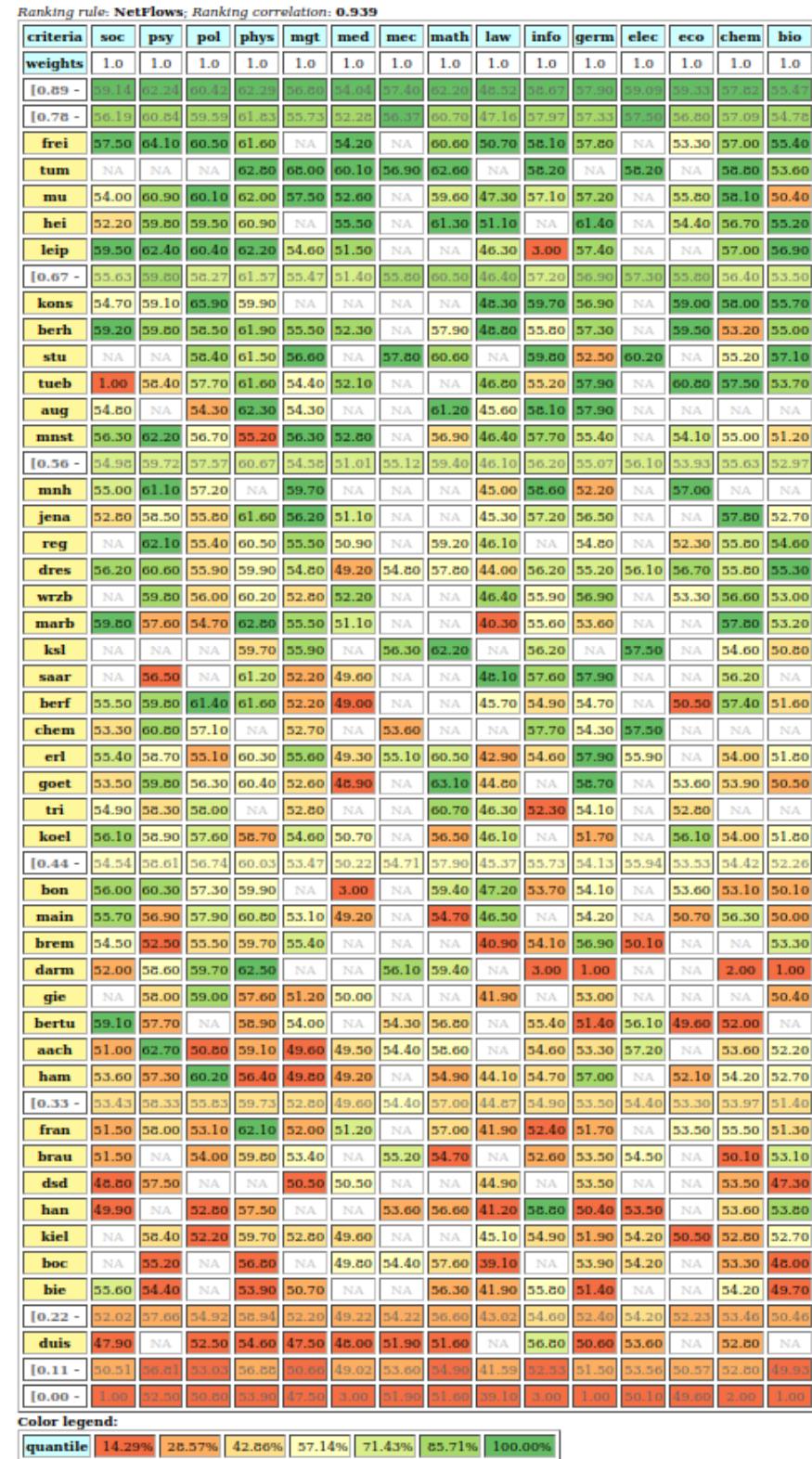


Fig. 14.4 Heatmap view of the 9-tiles rating-by-ranking result

The ordinal correlation (+0.967) of the COPELAND ranking with the underlying bipolar-valued outranking digraph is very high (see Fig. 14.4 on the facing page Row 1). Most correlated subjects with this rating-by-ranking result appear to be *German Studies* (+0.51), *Chemistry* (+0.48), *Management* (+0.47) and *Physics* (+0.46). Both *Electrical* (+0.07) and *Mechanical Engineering* (+0.05) are the less correlated subjects (see Row 3).

From the actual ranking position of the lower 9-tiling limits, we may now immediately deduce the 9-tile enrolment quality equivalence classes. No University reaches the highest 9-tile ([0.89 – ]). In the lowest 9-tile ([0.00 – 0.11]) we find the University Duisburg. The complete rating result may be easily printed out as follows with the `showQuantilesRating()` method:

**Listing 14.3** Showing the 9-tiling of the enrolment quality of the 41 Universities

```

1 >>> lqr.showQuantilesRating()
2      *----- Quantiles rating result -----
3      [0.89 - 1.00] []
4      [0.78 - 0.89[ ['tum','frei','kons','leip','mu','hei']
5      [0.67 - 0.78[ ['stu','berh']
6      [0.56 - 0.67[ ['aug','mnh','tueb','mnst','jena',
7          'reg','saar']
8      [0.44 - 0.56[ ['wrzb','dres','ksl','marb','berf',
9          'chem','koel','erl','tri']
10     [0.33 - 0.44[ ['goet','main','bon','brem']
11     [0.22 - 0.33[ ['fran','ham','kiel','aach',
12          'bertu','brau','darm']
13     [0.11 - 0.22[ ['gie','dsd','bie','boc','han']
14     [0.00 - 0.11[ ['duis']

```

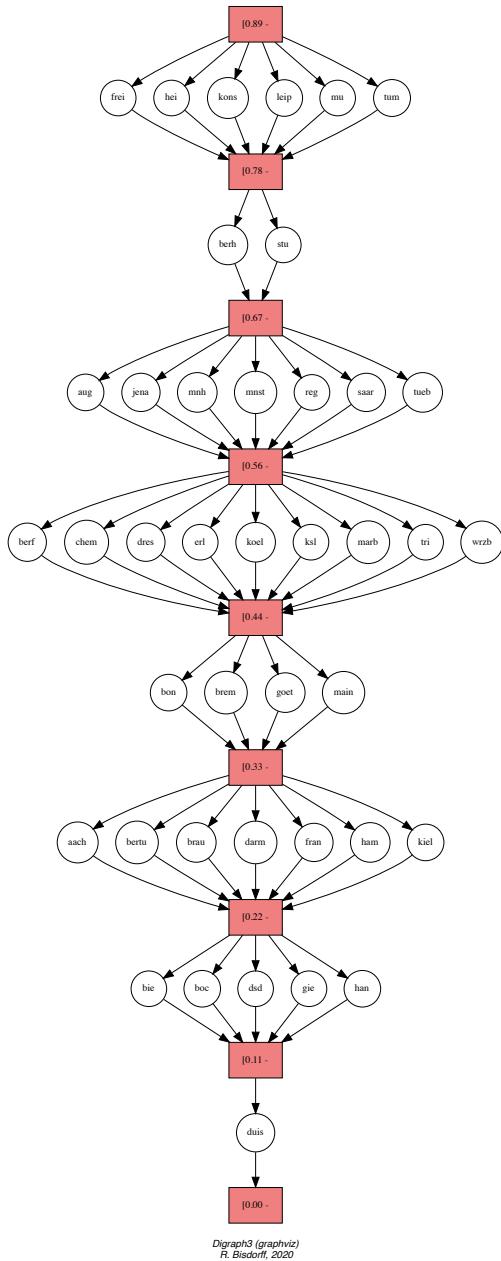
Following Universities: TU München, Freiburg, Konstanz, Leipzig, München as well as Heidelberg, appear best rated in the eighth 9-tile ([0.78 – 0.89[, see Listing 14.3 Line 4). Lowest-rated in the first 9-tile, as mentioned before, appears University Duisburg (Line 14). Midfield, the fifth 9-tile ([0.44 – 0.56[), consists of the Universities Würzburg, TU Dresden, Kaiserslautern, Marburg, FU Berlin, Chemnitz, Köln, Erlangen-Nürnberg and Trier (Lines 8-9).

A corresponding *graphviz* drawing may well illustrate all these enrolment quality equivalence classes.

```

1 >>> lqr.exportRatingByRankingGraphViz(fileName='ratingResult', \
2 ...                                     graphSize='12,12')
3      *----- exporting a dot file for GraphViz tools -----
4      Exporting to ratingResult.dot
5      dot -Grankdir=TB -Tpdf dot -o ratingResult.png

```



**Fig. 14.5** Drawing of the 9-tiles rating-by-ranking result

We have noticed in Chapter 8 on page 105, that there is not a single optimal rule for ranking from a given outranking digraph. The COPELAND rule, for instance, has the advantage of being CONDORCET consistent, i.e. when the outranking digraph models in fact a linear ranking, this ranking will necessarily be the result of the COPELAND \* rule. When this is not the case, and especially when the outranking digraph shows many circuits, all potential ranking rules may give very divergent ranking results, and hence also substantially divergent rating-by-ranking results.

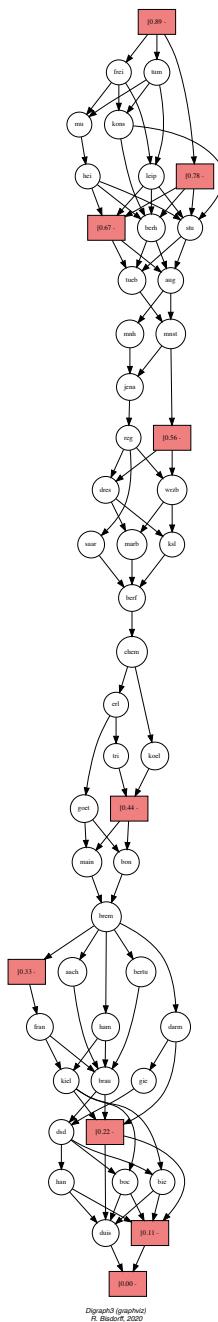
It is, hence, interesting, to verify if the epistemic fusion of the rating-by-ranking results, one may obtain when applying two different ranking rules, like the COPELAND and the NETFLOWS ranking rule, does actually confirm our rating-by-ranking result shown in Fig. 14.5 on the preceding page. For this purpose we make usage of the RankingsFusionDigraph constructor (see Line 9 below).

```

1 >>> lqr = LearnedQuantilesRatingDigraph(\ 
2 ...                               pq,t,rankingRule='Copeland')
3 >>> lqr1 = LearnedQuantilesRatingDigraph(\ 
4 ...                               pq,t,rankingRule='NetFlows')
5 >>> from transitiveDigraphs import\ 
6 ...                               RankingsFusionDigraph
7 >>> rankings = [lqr.actionsRanking, lqr1.actionsRanking]
8 >>> rf = RankingsFusionDigraph(lqr,rankings)
9 >>> rf.exportGraphViz(fileName='fusionResult',\ 
10 ...                           WithRatingDecoration=True,\ 
11 ...                           graphSize='30,30')
12 exporting a dot file for GraphViz tools
13 Exporting to fusionResult.dot
14 dot -Grankdir=TB -Tpng fusionResult.dot\ 
15 ...                           -o fusionResult.png

```

**Fig. 14.6** Disjunctive fusion of the COPELAND and the NETFLOWS rankings (extract).



In Figure 14.6 on the facing page we notice that many Universities appear now rated into several adjacent 9-tiles. The previously best-rated Universities: TU München, Freiburg, München, Leipzig, as well as Heidelberg, for instance, appear now sorted into the 7<sup>th</sup> and 8<sup>th</sup> 9-tile ([0.67 – 0.89]), whereas Konstanz is now, even more imprecisely, rated into the 6<sup>th</sup>, the 7<sup>th</sup> and the 8<sup>th</sup> 9-tile. How confident, hence, is our precise COPELAND rating-by-ranking result? To investigate this question, let us now inspect the outranking digraph on which we actually apply the COPELAND ranking rule.

### 14.3 Inspecting the bipolar-valued outranking digraph

We say that University  $x$  *outranks* (resp. *is outranked by*) University  $y$  in enrolment quality when there exists a majority (resp. only a minority) of valued subjects showing an *at least as good as* average enrolment quality score. To compute these outranking situations, we use the `BipolarOutrankingDigraph` constructor.

**Listing 14.4** Computing 9-tiles of the enrolment quality scores per subject

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> dg = BipolarOutrankingDigraph(t)
3 >>> dg
4     *----- Object instance description -----*
5     Instance class      : BipolarOutrankingDigraph
6     Instance name       : rel_studentenSpiegel04
7     Actions             : 41 (Universities)
8     Criteria            : 15 (subjects)
9     Size                : 828 (outranking situations)
10    Determinateness (%) : 63.67
11    Valuation domain   : [-1.00;1.00]
12 >>> dg.computeTransitivityDegree(Comments=True)
13     Transitivity degree of digraph <rel_studentenSpiegel04>:
14     triples x>y>z: 57837, closed: 30714, open: 27123
15     (closed/triples) = 0.531
16 >>> dg.computeSymmetryDegree(Comments=True)
17     Symmetry degree of digraph <rel_studentenSpiegel04>:
18     arcs x>y: 793, symmetric: 35, asymmetric: 758
19     symmetric/arcs = 0.044

```

The bipolar-valued outranking digraph  $dg$  (see Listing 14.4 Line 2), obtained with the given performance tableau  $t$ , shows 828 positively validated pairwise outranking situations (Line 9). Unfortunately, the transitivity of digraph  $dg$  is far from being satisfied: nearly half of the transitive closure is missing (Line 15). Despite the rather large preference discrimination threshold (0.5) we have assumed (see Fig. 14.2 on page 198), there does not occur many indifference situations (Line 19).

We may furthermore check if there exists any *cyclic* outranking situations.

**Listing 14.5** Enumerating chordless outranking circuits

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph

```

```

2 >>> dg.computeChordlessCircuits()
3 >>> dg.showChordlessCircuits()
4     *---- Chordless circuits ----*
5     93 circuits.
6     1: ['aach','bie','darm','brau'] ,credibility : 0.067
7     2: ['aach','bertu','brau'] ,credibility : 0.200
8     3: ['aach','bertu','brem'] ,credibility : 0.067
9     4: ['aach','bertu','ham'] ,credibility : 0.200
10    5: ['aug','tri','marb'] ,credibility : 0.067
11    6: ['aug','jena','marb'] ,credibility : 0.067
12    7: ['aug','jena','koel'] ,credibility : 0.067
13    ...
14    ...
15    29: ['berh','kons','mu'] ,credibility : 0.133
16    ...
17    ...
18    88: ['main','mnh','marb'] ,credibility : 0.067
19    89: ['marb','saar','wrzb'] ,credibility : 0.067
20    90: ['marb','saar','reg'] ,credibility : 0.067
21    91: ['marb','saar','mnst'] ,credibility : 0.133
22    92: ['marb','saar','tri'] ,credibility : 0.067
23    93: ['mnh','mu','stu'] ,credibility : 0.133

```

Here we observe indeed 93 such outranking circuits, like: Berlin Humboldt > Konstanz > München > Berlin Humboldt, supported by a  $(0.133 + 1.0)/2 = 56.7\%$  majority of subjects Footnote[31] (see circuit 29 above). In the COPELAND ranking result shown in Figure 14.4 on page 200, these Universities appear positioned respectively at ranks 10, 4 and 6. In the NETFLOWS ranking result they would appear respectively at ranks 10, 6 and 5, thus inverting the positions of Konstanz and München. The occurrence in digraph  $dg$  of so many outranking circuits makes thus *doubtful* any *forced* linear ranking, independently of the specific ranking rule we might have applied.

To effectively check the quality of our COPELAND rating-by-ranking result, we shall now compute a direct sorting into 9-tiles of the enrolment quality scores, without using any outranking digraph based ranking rule.

## 14.4 Rating by quantiles sorting

In our case here, the Universities represent the decision actions: *where to study*. We say now that University  $x$  is sorted into the lower-closed 9-tile  $q$  when the performance record of  $x$  positively outranks the lower limit record of 9-tile  $q$  and  $x$  does not positively outrank the upper limit record of 9-tile  $q$ .

**Listing 14.6** Enumerating chordless outranking circuits

```

1 >>> lqr.showActionsSortingResult()
2   Quantiles sorting result per decision action
3   [0.33 - 0.44[: aach with credibility: 0.13 = min(0.13,0.27)
4   [0.56 - 0.89[: aug with credibility: 0.13 = min(0.13,0.27)
5   [0.44 - 0.67[: berf with credibility: 0.13 = min(0.13,0.20)

```

```

6  [0.78 - 0.89[: berh with credibility: 0.13 = min(0.13,0.33)
7  [0.22 - 0.44[: bertu with credibility: 0.20 = min(0.33,0.20)
8  [0.11 - 0.22[: bie with credibility: 0.20 = min(0.33,0.20)
9  [0.22 - 0.33[: boc with credibility: 0.07 = min(0.07,0.07)
10 [0.44 - 0.56[: bon with credibility: 0.13 = min(0.20,0.13)
11 [0.33 - 0.44[: brau with credibility: 0.07 = min(0.07,0.27)
12 [0.33 - 0.44[: brem with credibility: 0.07 = min(0.07,0.07)
13 [0.44 - 0.56[: chem with credibility: 0.07 = min(0.13,0.07)
14 [0.22 - 0.56[: darm with credibility: 0.13 = min(0.13,0.13)
15 [0.56 - 0.67[: dres with credibility: 0.27 = min(0.27,0.47)
16 [0.22 - 0.33[: dsd with credibility: 0.07 = min(0.07,0.07)
17 [0.00 - 0.11[: duis with credibility: 0.33 = min(0.73,0.33)
18 [0.44 - 0.56[: erl with credibility: 0.13 = min(0.27,0.13)
19 [0.22 - 0.44[: fran with credibility: 0.13 = min(0.13,0.33)
20 [0.78 - <[: frei with credibility: 0.53 = min(0.53,1.00)
21 [0.22 - 0.33[: gie with credibility: 0.13 = min(0.13,0.20)
22 [0.33 - 0.44[: goet with credibility: 0.07 = min(0.47,0.07)
23 [0.22 - 0.33[: ham with credibility: 0.07 = min(0.33,0.07)
24 [0.11 - 0.22[: han with credibility: 0.20 = min(0.33,0.20)
25 [0.78 - 0.89[: hei with credibility: 0.13 = min(0.13,0.27)
26 [0.56 - 0.67[: jena with credibility: 0.07 = min(0.13,0.07)
27 [0.33 - 0.44[: kiel with credibility: 0.20 = min(0.20,0.47)
28 [0.44 - 0.56[: koel with credibility: 0.07 = min(0.27,0.07)
29 [0.78 - <[: kons with credibility: 0.20 = min(0.20,1.00)
30 [0.56 - 0.89[: ksl with credibility: 0.13 = min(0.13,0.40)
31 [0.78 - 0.89[: leip with credibility: 0.07 = min(0.20,0.07)
32 [0.44 - 0.56[: main with credibility: 0.07 = min(0.07,0.13)
33 [0.56 - 0.67[: marb with credibility: 0.07 = min(0.07,0.07)
34 [0.56 - 0.89[: mnh with credibility: 0.20 = min(0.20,0.27)
35 [0.56 - 0.67[: mnst with credibility: 0.07 = min(0.20,0.07)
36 [0.78 - 0.89[: mu with credibility: 0.13 = min(0.13,0.47)
37 [0.56 - 0.67[: reg with credibility: 0.20 = min(0.20,0.27)
38 [0.56 - 0.78[: saar with credibility: 0.13 = min(0.13,0.20)
39 [0.78 - 0.89[: stu with credibility: 0.07 = min(0.13,0.07)
40 [0.44 - 0.56[: tri with credibility: 0.07 = min(0.13,0.07)
41 [0.67 - 0.78[: tueb with credibility: 0.13 = min(0.13,0.20)
42 [0.89 - <[: tum with credibility: 0.13 = min(0.13,1.00)
43 [0.56 - 0.67[: wrzb with credibility: 0.07 = min(0.20,0.07)

```

In the 9-tiles sorting result, shown in Listing 14.6 on the facing page, we notice for instance in Lines 3-4 that the RWTH Aachen is precisely rated into the 4th 9-tile ([0.33 – 0.44]), whereas the University Augsburg is less precisely rated conjointly into the 6<sup>th</sup>, the 7<sup>th</sup> and the 8<sup>th</sup> 9-tile ([0.56 – 0.89]). In Line 42, TU München appears best rated into the unique highest 9-tile ([0.89 – <]). All three rating results are supported by a  $(0.07 + 1.0)/2 = 53.5\%$  majority of valuated subjects<sup>4</sup>. With the support of a 76.5% majority of valuated subjects (Line 20), the apparent most confident rating result is the one of University Freiburg (see also Fig. 14.3 on page 199 and Fig. 14.4 on page 200).

We shall now lexicographically sort these individual rating results per University, by *average* rated 9-tile limits and highest-rated upper 9-tile limit, into ordered, but not necessarily disjoint, enrolment quality quantiles.

```

1 >>> lqr.showHTMLQuantilesSorting(strategy='average')

```

With a special *graphviz* drawing of the `LearnedQuantilesRatingDigraph` instance `lqr`, we may, without requiring any specific ordering strategy, as well illustrate our 9-tiles rating-by-sorting result.

<sup>4</sup> Converted by a +1.0 shift and a  $0.5 \times 100$  scale transform from a bipolar-valued credibility of +0.07 in  $[-1.0, +1.0]$  to a majority (in %) support.

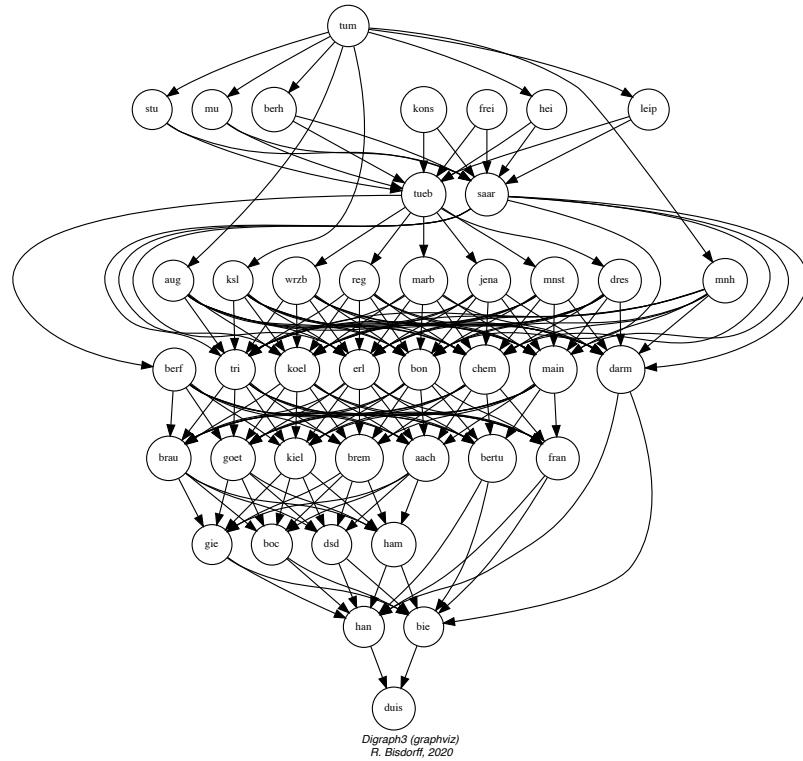
**Fig. 14.7** The ranked 9-tiles rating-by-sorting result. We may notice that the Universities: Augsburg, Kaiserslautern, Mannheim and Tübingen, for instance, show in fact the same average rated 9-tiles score of 0.725; yet, the rated upper 9-tile limit of Tuebingen reaches only 0.78, whereas the one of the other Universities reaches 0.89. Hence, Tuebingen is ranked below Augsburg, Kaiserslautern and Mannheim

quantile limits	Ordering by average quantile class limits
[0.89-<[	['tum']
[0.78-<[	['frei', 'kons']
[0.78-0.89[	['berh', 'hei', 'leip', 'mu', 'stu']
[0.56-0.89[	['aug', 'ksl', 'mnh']
[0.67-0.78[	['tueb']
[0.56-0.78[	['saar']
[0.56-0.67[	['dres', 'jena', 'marb', 'mnst', 'reg', 'wrzb']
[0.44-0.67[	['berf']
[0.44-0.56[	['bon', 'chem', 'erl', 'koel', 'main', 'tri']
[0.22-0.56[	['darm']
[0.33-0.44[	['aach', 'brau', 'brem', 'goet', 'kiel']
[0.22-0.44[	['bertu', 'fran']
[0.22-0.33[	['boc', 'dsd', 'gie', 'ham']
[0.11-0.22[	['bie', 'han']
[0.00-0.11[	['duis']

```

1 >>> lqr.exportRatingBySortingGraphViz(\n2 ...           'nineTilingDrawing', graphSize='12,12')\n3 *---- exporting a dot file for GraphViz tools --*\n4 Exporting to nineTilingDrawing.dot\n5 dot -Grankdir=TB -Tpng nineTilingDrawing.dot\\
6           -o nineTilingDrawing.png

```



**Fig. 14.8** Graphviz drawing of the 9-tiles sorting digraph.

In Figure 14.8 on the previous page we actually see the *skeleton* (transitive closure removed) of a partial order, where an oriented arc is drawn between Universities  $x$  and  $y$  when their 9-tiles sorting results are disjoint and the one of  $x$  is higher rated than the one of  $y$ . The rating for TU München (see Listing 14.6 on page 206 Lines 45), for instance, is disjoint and higher rated than the one of the Universities Freiburg and Konstanz (Lines 23, 32). And, both the ratings of Feiburg and Konstanz are, however, not disjoint from the one, for instance, of the University of Stuttgart (Line 42).

The partial ranking, shown in Figure 14.8 on the previous page, is in fact independent of any ordering strategy: - *average*, - *optimistic* - *pessimistic*, of overlapping 9-tiles sorting results, and confirms that the same Universities as with the previous rating-by-ranking approach, namely TU München, Freiburg, Konstanz, Stuttgart, Berlin Humboldt, Heidelberg and Leipzig appear top-rated. Similarly, the Universities of Duisburg, Bielefeld, Hanover, Bochum, Giessen, Düsseldorf and Hamburg give the lowest-rated group. The midfield here is again consisting of more or less the same Universities as the one observed in the previous rating-by-ranking approach (see Listing 14.3 on page 201).

In the end, both the COPELAND rating-by-ranking, as well as the rating-by-sorting approach give luckily, in our case study here, very similar results. The first approach, with its *forced* linear ranking, determines on the one hand, precise enrolment quality equivalence classes; a result, depending potentially a lot on the actually applied ranking rule. The rating-by-sorting approach, on the other hand, only determines for each University a less precise but prudent rating of its individual enrolment quality, furthermore supported by a known majority of performance criteria significance; a somehow fairer and robuster result, but, much less evident for easily comparing the apparent enrolment quality among Universities. Contradictorily, or sparsely valued Universities, for instance, will appear trivially rated into a large midfield of adjacent 9-tiles.

Let us conclude by saying that we prefer this latter rating-by-sorting approach; perhaps impreciser, due the case given, to missing and contradictory performance data; yet, well grounded in a powerful bipolar-valued logical and epistemic framework.

## References

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- Friedmann J, Hackenbroch V, Hipp D, Klawitter N, Koch J, Lakotta B, Mohr J, Schmitz C, Thimm K, Wüst C (2004) Die Elite von morgen. DER SPIEGEL 48, URL <https://www.spiegel.de/politik/die-elite-von-morgen-a-afd0507a-0002-0001-0000-000037494731> 195, 199



# Chapter 15

## Exercises

**Abstract** In this chapter, we propose a series decision problems of various difficulties which may serve as exercises and exam questions for an Algorithmic Decision Theory or Multiple Criteria Decision Analysis course. They cover *selection*, *ranking* and *rating* decision problems.

### Introduction

The following exercises are marked as follows: *warming up* (§), *home work* (§§), *research work* (§§§). Solutions should be supported both by computational Python code using the DIGRAPH3 programming resources as well as by methodological and algorithmic arguments from the *Algorithmic Decision Theory* Lectures (Bisdorff, 2020, 2021).

### 15.1 Who will receive the best student award? (§)

#### Data

In Table 15.1 on the next page you find the actual grades obtained by four students : *Ariana* (A), *Bruce* (B), *Clare* (C) and *Daniel* (D) in five courses: C1, C2, C3, C4 and C5 weighted by their respective ECTS points.<sup>1</sup> The student grades are measured on an ordinal performance scale from 0 pts (weakest) to 20 pts (highest). Assume that the grading admits a preference discrimination threshold of 1 point. No considerable performance differences are given. The more ECTS points, the more importance a course takes in the curriculum of the students. An award is to be granted to the student showing the *best* result.

---

<sup>1</sup> ECTS stands for *European Credit Transfer System*. It is a unit used to grade diplomas in all participating countries <https://wwwen.uni.lu/studies/ects.credits>

**Table 15.1** Grades obtained by the students

Course ECTS	CF 2	C2 3	C3 4	C4 2	C5 4
Ariana (A)	11	13	9	15	11
Bruce (B)	12	9	13	10	13
Clare (C)	8	11	14	12	14
Daniel (D)	15	10	12	8	13

## Questions

1. Edit a `PerformanceTableau` instance with the data shown above (see Chapter 5).
2. Who would you nominate for the grant?
3. Explain and motivate your selection algorithm.
4. Assume that the grading may actually admit an indifference discrimination threshold of 1 point and a preference discrimination threshold of 2 points. How stable is your grant recommendation with respect to this preference discrimination power of the grading scale?

## 15.2 How to fairly rank movies? (§)

### Data

File `graffiti03.py`<sup>2</sup> contains a performance tableau about the star-rating of movies that could be seen in the city of Luxembourg during Spring 2003 (source: [Graffiti magazine, Luxembourg \(February 2003\)](#)). Its content is shown in Fig. 15.1 on the facing page.

```

1  >>> from perfTabs import PerformanceTableau
2  >>> t = PerformanceTableau('graffiti03')
3  >>> t.showHTMLPerformanceHeatmap(WithActionNames=True, \
4  ...                               pageTitle='Graffiti Star wars',
5  ...                               rankingRule=None, colorLevels=5,
6  ...                               ndigits=0)

```

The critic's opinions are expressed on seven rating levels:  $-2$  (two zeros, *I hate*),  $-1$  (one zero, *I don't like*),  $1$  (one star, *maybe*),  $2$  (two stars, *good*),  $3$  (three stars, *excellent*),  $4$  (four stars, *not to be missed*), and  $5$  (five stars, *a master piece*). Notice the many missing data (NA) when a critic had not seen the respective movie. Mind also that the ratings of two movie critics (`jh` and `vt`) are given a higher significance weight.

<sup>2</sup> The file is provided in the `examples` directory of the DIGRAPH3 ressources ([Bisdorff, 2021](#)).

### Graffiti Star wars

criteria	jh	vt	ap	as	cf	cn	cs	dr	jt	mk	mr	rr	td
weights	+2.00	+2.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00
Ah si j'étais riche (ah)	1	NA	NA	NA	NA	-1	1	NA	1	2	NA	1	3
A walk to remember (aw)	NA	NA	-1	NA	1	NA	2	NA	-1	1	NA	1	NA
Bend it like Beckham (bb)	2	1	2	1	2	2	3	2	2	3	2	3	1
Demonlover (dl)	1	-1	-1	-1	-1	NA	-1	1	1	NA	1	1	1
Gangs of New York (gny)	3	3	2	4	2	4	2	3	4	2	4	3	2
Ghost Ship (gs)	NA	NA	1	NA	-1	NA	1	1	1	NA	-1	-1	-1
El Hija de la Novia (hn)	2	1	3	NA	3	2	2	NA	2	3	2	2	NA
Lantana (la)	3	3	3	2	3	3	2	2	3	3	4	3	NA
Lord of the Rings - The Two Towers (lor)	3	2	2	3	3	NA	3	4	4	1	2	2	2
The Magdalene Sisters (ma)	3	3	NA	NA	NA	3	2	3	3	3	2	2	3
Mr. Deeds (md)	NA	NA	1	1	-1	NA	NA	-1	1	-1	NA	-1	1
Mon Idole (mi)	1	1	NA	-1	NA	1	-1	NA	2	NA	NA	-1	2
the Slaton Sea (sa)	2	NA	NA	NA	NA	NA	2	NA	1	3	1	NA	NA
the santa Clause 2 (sc)	NA	NA	1	NA	1	NA	-1	NA	1	1	NA	1	NA
Sweet home Alabama (sha)	-1	NA	2	-1	1	1	2	2	2	1	1	1	NA
Sweet Sixteen (ss)	3	3	3	3	3	4	2	3	3	3	3	1	3
24 heures de la vie d'une femme (vf)	1	NA	NA	NA	NA	1	NA	NA	1	NA	NA	1	1

Color legend:

quantile 20.00% 40.00% 60.00% 80.00% 100.00%

Fig. 15.1 Graffiti magazine's movie ratings from February 2003

### Questions

1. The GRAFFITI magazine suggests a best rated movie by computing the average number of stars it received, ignoring the missing data and any significance weights of the critics. What movie gets the highest average?
2. By taking into account missing data and varying significance weights, how may one find the best rated movie without computing any average of stars?
3. How would one rank these movies so as to at best respect the weighted rating opinions of each movie critic?
4. In what ranking position would appear a movie not seen by any critic? Confirm computationally your answer by adding such a fictive, *not at all evaluated*, movie to the given performance tableau instance.
5. How robust are the preceding results when the significance weights of the movie critics are considered to be only ordinal grades?

### 15.3 What is your best choice recommendation? (§)

#### Data

A person, who wants to buy a TV set, retains after a first selection, eight potential TV models<sup>3</sup>. To make their best choice these eight models were graded with respect to three decision objectives of equal importance:

1. *Costs* of the set (to be minimised);
2. *Picture and Sound* quality of the TV (to be maximised);
3. *Maintenance contract* quality of the provider (to be maximised).

The *Costs* objective is assessed by the price of the TV set (criterion *Pr* to be minimised). *Picture* quality (criterion *Pq*), *Sound* quality (criterion *Sq*) and *Maintenance contract* quality (criterion *Mq*) are each one assessed on a four levels qualitative performance scale: -1 (*not good*), 0 (*average*), 1 (*good*) and 2 (*very good*).

The actual evaluation data are gathered in Table 15.2.

**Table 15.2** Performance evaluations of the potential TV sets

Criteria Significance	Pr (€) 2	Pq 1	Sq 1	Mq
Model T1	-1300	2	2	0
Model T2	-1200	2	2	1
Model T3	-1150	2	1	1
Model T4	-1000	1	1	-1
Model T5	-950	1	1	0
Model T6	-950	0	1	-1
Model T7	-900	1	0	-1
Model T8	-900	0	0	0

The *Price* criterion *Pr* supports furthermore an indifference threshold of 25.00€ and a preference threshold of 75.00€. No considerable performance differences (veto thresholds) are to be considered.

#### Questions

1. Edit a new `PerformanceTableau` instance with the data shown in Table 15.2 and illustrate its content by best showing objectives, criteria, decision alternatives and performance table. If needed, write adequate python code.
2. What is the best TV set to recommend?
3. Illustrate your best choice recommendation with an adequate `graphviz` drawing.
4. Explain and motivate your selection algorithm.

<sup>3</sup> This didactic decision problem is discussed in [Vincke \(1992, pp.33-35\)](#)

5. Assume that the qualitative criteria: *Picture* quality ( $P_Q$ ), *Sound* quality ( $S_Q$ ), and *Maintenance contract* quality ( $M_Q$ ), are all three considered to be equi-significant and that the significance of the *Price* criterion ( $P_r$ ) equals the significance of these three quality criteria taken together. How stable is your best choice recommendation with respect to thus reviewing the criteria significance weights?

## 15.4 What is the best public policy? (§§)

### Data files

- File `perfTab_1.py`<sup>4</sup> contains a 3-Objectives performance tableau with 100 performance records concerning public policies evaluated with respect to an *economic*, a *societal* and an *environmental* social decision objective.
- File `historicalData_1.py` contains a performance tableau of the same kind with 2000 historical performance records.

### Questions

1. Illustrate the content of the given `perfTab_1.py` performance tableau by best showing *objectives*, *criteria*, *decision alternatives* (public policies) and *performance evaluations*. If needed, write adequate python code.
2. Construct the corresponding bipolar-valued outranking digraph. How *confident* and/or *robust* are the apparent outranking situations?
3. What are apparently the 5 best-ranked decision alternatives in your decision problem from the different decision objectives point of views and from a global fair compromise view? Justify your ranking approach from a methodological point of view.
4. How would you rate your 100 public policies into relative deciles classes?
5. Using the given historical records in file `historicalData_1.py`, how would you rate your 100 public policies into absolute deciles classes?
6. Explain the differences you may observe between the absolute and the previous relative rating result.
7. Select among your 100 potential public policies a shortlist of up to 15 potential best policies, all reaching an absolute performance quantile of at least 66.67%.
8. Based on the previous best policies shortlist (see Question 6), what is your eventual best-choice recommendation? Is it perhaps a best choice unopposed by all three objectives?

---

<sup>4</sup> Files `perfTab_1.py` and `historicalData_1.py` are provided in the `examples` directory of the DIGRAPH3 resources ([Bisdorff, 2021](#)).

## 15.5 A fair diploma validation decision (§§§)

### Data

Use the `RandomAcademicPerformanceTableau` class from the `randomPerfTab` module for generating realistic random students performance tableaux concerning a curriculum of nine ECTS weighted Courses (Bisdorff, 2021). Assume that all the gradings are done on an integer scale from 0 (weakest) to 20 (best). It is known that grading procedures are inevitably somehow imprecise; therefore assume an indifference discrimination threshold of 1 point and a preference discrimination threshold of 2 points. Furthermore, a performance difference of more than 12 points is considerable and will trigger a polarisation situation. To validate eventually their curriculum, the students are required to obtain more or less 10 points in each course.

### Questions

1. Design and implement a fair diploma validation decision rule based on the grades obtained in the nine Courses.
2. Run simulation tests with random students performance tableaux for validating your design and implementation.

### References

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## **Part IV**

### **Advanced topics**

The fourth part gathers five chapters introducing and discussing further going algorithmic developments. In Chapter [16 on the next page](#) KENDALL's ordinal correlation index is consistently extended to bipolar-valued digraphs. Chapter [17 on page 237](#) explains the important concept of digraph kernel and illustrates the DIGRAPH3 algorithmic approach to their computation. In the following Chapter [18 on page 265](#), criteria significance weights are considered to be uncertain and become random variates. This idea opens the way to compute a bipolar-valued likelihood of outranking and outranked situations leading to  $\alpha\%$ -confident outranking digraphs. In Chapter [19 on page 275](#), the criteria significance weights are considered to be only of ordinal type. This working hypothesis induces the need to put into place a specific robustness analysis of the corresponding outranking digraphs. A last Chapter [20 on page 293](#) makes use of the preceding robustness analysis for introducing and discussing ideas, like two-stage elections with multipartisan primary selection or bipolar voting systems, for tempering plurality tyranny effects in social choice problems.

# Chapter 16

## On measuring the fitness of a multiple criteria ranking

**Abstract** Starting from a motivating decision problem about how to list, from the best to the worst, a set movies that are star-rated by journalists and movie critics, the chapter shows that KENDALL's ordinal correlation index tau may be extended to a relational bipolar-valued equivalence measure of bipolar-valued digraphs. This finding gives way, on the one hand, to measure the fitness and fairness of multiple criteria ranking rules. On the other hand, it provides a tool for illustrating preference divergences between decision objectives and criteria.

### 16.1 Listing movies from best star-rated to worst

In a stubborn keeping with a two-valued logic, where every argument can only be true or false, there is no place for efficiently taking into account missing data or logical indeterminateness. These cases are seen as problematic and, at best are simply ignored. Worst, in modern data science, missing data get often replaced with *fictive* values, potentially falsifying hence all subsequent computations.

In social choice problems, voting abstentions are, however, frequently observed and represent a social expression that may be significant for revealing non represented social preferences. And, in marketing studies, interviewees will not always respond to all the submitted questions. Again, such abstentions do sometimes contain nevertheless valid information concerning consumer preferences.

Such a case is given with a list of star-rated movies that could be seen in town in September 2003 (source: *Graffiti Star wars*, Edition Revue Luxembourg, September 2007, p. 30.). The underlying performance tableau data, stored in a file named `graffiti07.py`<sup>1</sup>, is shown below with the `showHTMLPerformanceTableau()` method :

```
1 >>> from outrankingDigraphs import \
2     ...                               PerformanceTableau
```

---

<sup>1</sup> to be found in the `examples` directory of the DIGRAPH3 resources

```

3 >>> gt07 = PerformanceTableau('graffiti07')
4 >>> gt07.showHTMLPerformanceTableau(\n
5 ...           title='Graffiti Star wars', \n
6 ...           ndigits=0)

```

## Graffiti Star wars

criteria	AP	AS	CF	CS	DR	FG	GS	JH	JPT	MR	RR	SF	SJ	TD	VT
weight	1.00	1.00	1.00	1.00	1.00	1.00	1.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	3.00
<b>mv_AL</b>	3	-1	2	-1	NA	NA	3	NA	2	NA	NA	NA	2	NA	2
<b>mv BI</b>	1	-1	1	1	-1	NA	NA	NA	2	NA	2	NA	NA	NA	NA
<b>mv_CM</b>	NA	3	3	2	NA	NA	3	2	3	2	1	2	2	NA	2
<b>mv_DF</b>	NA	4	3	1	2	NA	1	3	2	2	2	3	-1	NA	1
<b>mv_DG</b>	3	2	2	3	3	1	4	3	3	3	NA	NA	-1	NA	3
<b>mv_DI</b>	1	2	NA	1	2	2	NA	1	2	2	NA	1	1	NA	NA
<b>mv_DJ</b>	3	1	3	NA	NA	NA	3	NA	2	2	NA	2	4	3	-1
<b>mv_FC</b>	3	2	2	1	3	NA	1	3	3	3	2	NA	2	1	3
<b>mv_FF</b>	2	3	2	1	2	2	NA	1	2	2	1	2	1	3	1
<b>mv_GG</b>	2	3	3	1	NA	NA	1	-1	NA	-1	2	NA	-1	NA	1
<b>mv_GH</b>	1	NA	1	-1	2	2	1	2	1	3	4	NA	NA	NA	NA
<b>mv_HN</b>	3	1	3	1	3	NA	4	3	2	NA	1	NA	3	3	3
<b>mv_HP</b>	1	NA	3	1	NA	NA	NA	1	1	2	3	NA	1	2	NA
<b>mv_HS</b>	2	4	2	3	2	2	2	3	4	2	3	NA	1	3	NA
<b>mv_JB</b>	3	4	3	NA	3	2	3	2	3	2	NA	2	2	NA	3
<b>mv_MB</b>	NA	2	NA	NA	1	NA	1	2	2	1	2	NA	2	2	NA
<b>mv_NP</b>	NA	1	3	1	NA	3	2	3	3	2	3	NA	NA	3	NA
<b>mv_PE</b>	3	4	2	NA	3	1	NA	2	4	2	NA	3	3	NA	3
<b>mv_QS</b>	NA	3	2	NA	4	3	NA	4	4	3	3	4	NA	4	4
<b>mv_RG</b>	2	2	2	2	NA	NA	3	1	2	2	1	NA	NA	3	NA
<b>mv_RR</b>	3	2	NA	1	4	NA	4	3	3	4	3	3	NA	4	2
<b>mv_SM</b>	3	3	2	2	2	2	NA	3	2	2	3	NA	2	2	3
<b>mv_TF</b>	-1	NA	1	1	1	NA	NA	1	2	NA	-1	NA	-1	1	NA
<b>mv_TM</b>	2	1	2	2	NA	NA	2	2	2	3	NA	2	NA	4	NA
<b>mv_TP</b>	2	3	3	1	2	NA	NA	3	2	NA	2	NA	1	NA	2

Fig. 16.1 Graffiti magazine's movie ratings from September 2007

Figure 16.1 shows the star-ratings of 25 movies by 15 journalists and movie critics: 5 stars (*masterpiece*), 4 stars (*must be seen*), 3 stars (*excellent*), 2 stars (*good*), 1 star (*could be seen*), -1 star (*I do not like*), -2 stars (*I hate*), NA (*not seen*). Notice in the second row the higher significance (3.00) that is granted to two locally renowned movie critics, namely JH and VT. Their opinion counts for three times the opinion of the other critics. With six times a 4 stars (*must be seen*) grade, mv\_QS is best-rated, followed by mv\_RR with four times a 4 stars mark. Fewest stars obtains movie mv\_TF with three times a -1 star (*don't like*) mark and five times a 1 star mark. Notice that many movies, like movie mv\_BI, are only rated by some of the critics.

To aggregate all the critics' star-ratings, the *Graffiti* magazine computes for each movie a global score –the average weighted number of stars it obtained– just ignoring the *not seen* movies. Listing 16.1 illustrates below how to compute these global scores using the data stored in the `gt07` performance tableau. Attribute `gt07.actions`, resp. `gt07.criteria`, contains the description of the 25 movies, resp. the 15 critics. The actual star-ratings are to be found in the `gt07.evaluation` attribute.

**Listing 16.1** Computing the average weighted number of stars per movie

```

1 >>> # gt07 = PerformanceTableau('graffiti07')
2 >>> globalScores = {}
3 >>> for mv in gt07.actions:
4 >>>     globalScores[mv] = Decimal('0')
5 >>>     sumWeights = Decimal('0')
6 >>>     for critic in gt07.criteria:
7 >>>         stars = gt07.evaluation[critic][mv]
8 >>>         if stars != gt07.NA:
9 >>>             weight = gt07.criteria[critic]['weight']
10 >>>             globalScores[mv] += (stars * weight)
11 >>>             sumWeights += weight
12 >>>     globalScores[mv] /= sumWeights
13 >>> graffitiList = [(globalScores[mv],mv) for mv in globalScores]
14 >>> graffitiList.sort(reverse=True)
15 >>> for item in graffitiList:
16 >>>     print('%s: %.2f' % (item[1],item[0]) )
17 mv_QS: 3.60
18 mv_RR: 2.88
19 mv_PE: 2.67
20 mv_JB: 2.62
21 mv_HN: 2.62
22 mv_NP: 2.60
23 mv_HS: 2.60
24 mv_DG: 2.56
25 mv_SM: 2.53
26 mv_FC: 2.41
27 mv_TP: 2.21
28 mv_CM: 2.20
29 mv_TM: 2.17
30 mv_DF: 1.94
31 mv_RG: 1.83
32 mv_MB: 1.73
33 mv_GH: 1.67
34 mv_DJ: 1.67
35 mv_FF: 1.61
36 mv_AL: 1.60
37 mv_HP: 1.55
38 mv_DI: 1.42
39 mv_GG: 0.71
40 mv_BI: 0.71
41 mv_TF: 0.55

```

The global scores ranking confirms in Lines 17-18 both leading movies –`mv_QS` (3.60) and `mv_RR` (2.88) – as well as in Line 41 the weakest rated one –`mv_TF` (0.55). Mind however that these global averages, due to the numerous missing grades, are not computed with commensurable denominators; some critics do indeed use a more or less extended range of stars. The movies not seen for instance by critic `SJ` are favoured, as this critic is severer than others in her rating. Dropping the movies that were not graded by all the critics is not possible either, as none of the 25 movies was actually rated by all the 15 critics. Providing a fictive value for

the many not seen situations, will as well always somehow falsify the global scores. What to do?

A better approach is to rank the movies on the basis of pairwise bipolar-valued “*rated at least as well as*” statements. Under this epistemic argumentation approach, missing grades are naturally treated as opinion abstentions and hence do not falsify the logical computations. Such a NETFLOWS ranking from best-rated to weakest-rated is provided by the **heatmap** browser view generated with the `showHTMLPerformanceHeatmap()` method (see List. 16.2).

**Listing 16.2** Showing the movie from best to worst rated in a heatmap view

```

1 >>> gt07.showHTMLPerformanceHeatmap(\n2 ...     pageTitle='Ranking the movies',\n3 ...     rankingRule='NetFlows',\n4 ...     Correlations=True,\n5 ...     ndigits=0)

```

The NETFLOWS ranking shown in Fig. 16.2 on the next page confirms again that movie `mv_QS`, with 6 *must be seen* marks, is correctly best-ranked and the movie `mv_TF` is worst-ranked with five *don't like* marks.

It is fair, however, to eventually mention here that the *Graffiti* magazine's average stars ranking rule is actually showing a very similar result. Indeed, average scores usually confirm well all evident pairwise comparisons, yet *enforce* comparability for all less evident ones. How to judge now the fitness of a given ranking rule?

This is the purpose of the ordinal correlation *tau* indexes shown in Fig. 16.2 on the facing page 3rd row. Computing these ordinal correlation indexes is the subject of the next Section.

## 16.2 KENDALL's ordinal correlation tau index

M. G. Kendall defined his ordinal correlation  $\tau$  (*tau*) index for linear orders of dimension  $n$  as a balancing of the number  $Co$  of correctly oriented pairs against the number  $In$  of incorrectly oriented pairs (Kendall, 1938). The total number of irreflexive pairs being  $n(n-1)$ , in the case of linear orders,  $Co + In = n(n-1)$ . Hence  $\tau = \left(\frac{Co}{n(n-1)}\right) - \left(\frac{In}{n(n-1)}\right)$ . In case  $In$  is zero,  $\tau = +1$  (all pairs are equivalently oriented); inversely, in case  $Co$  is zero,  $\tau = -1$  (all pairs are differently oriented).

Noticing that  $\frac{Co}{n(n-1)} = 1 - \frac{In}{n(n-1)}$ , and recalling that the bipolar-valued negation is operated by changing the sign of the characteristic value:

$$\tau = 1 - 2 \frac{In}{n(n-1)} = -\left(2 \frac{In}{n(n-1)} - 1\right) = 2 \frac{Co}{n(n-1)} - 1, \quad (16.1)$$

KENDALL's original *tau* definition implemented in fact the bipolar-valued negation of the non equivalence of two linear orders, i.e. the normalized majority margin of equivalently oriented irreflexive pairs.

### Ranking the movies

criteria	JH	JPT	AP	DR	MR	VT	GS	CS	SJ	RR	TD	CF	SF	AS	FG
<b>weights</b>	+3.00	+1.00	+1.00	+1.00	+1.00	+3.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00
<b>tau<sup>(*)</sup></b>	+0.50	+0.43	+0.32	+0.26	+0.25	+0.23	+0.16	+0.14	+0.14	+0.13	+0.11	+0.11	+0.10	+0.08	+0.03
<b>mv_QS</b>	4	4	NA	4	3	4	NA	NA	NA	3	4	2	4	3	3
<b>mv_RR</b>	3	3	3	4	4	2	4	1	NA	3	4	NA	3	2	NA
<b>mv_DG</b>	3	3	3	3	3	3	4	3	-1	NA	NA	2	NA	2	1
<b>mv_NP</b>	3	3	NA	NA	3	3	2	1	NA	2	NA	3	3	1	3
<b>mv_HN</b>	3	2	3	3	NA	3	4	1	3	1	3	3	NA	1	NA
<b>mv_HS</b>	3	4	2	2	2	NA	2	3	1	3	3	2	NA	4	2
<b>mv_SM</b>	3	2	3	2	2	3	NA	2	2	3	3	2	NA	3	2
<b>mv_JB</b>	2	3	3	3	2	3	3	NA	2	NA	NA	3	2	4	2
<b>mv_PE</b>	2	4	3	3	2	3	NA	NA	3	NA	NA	2	3	4	1
<b>mv_FC</b>	3	3	3	3	3	3	1	1	2	2	1	2	NA	2	NA
<b>mv_TP</b>	3	2	2	2	NA	2	NA	1	1	2	NA	3	NA	3	NA
<b>mv_CM</b>	2	3	NA	NA	2	2	3	2	2	1	NA	3	2	3	NA
<b>mv_DF</b>	3	2	NA	2	2	1	1	1	-1	2	NA	3	3	4	NA
<b>mv_TM</b>	2	2	2	NA	3	NA	2	2	NA	NA	4	2	2	1	NA
<b>mv_DJ</b>	NA	2	3	NA	2	-1	3	NA	4	NA	3	3	2	1	NA
<b>mv_AL</b>	NA	2	3	NA	NA	2	3	-1	2	NA	NA	2	NA	-1	NA
<b>mv_RG</b>	1	2	2	NA	2	NA	3	2	NA	1	3	2	NA	2	NA
<b>mv_MB</b>	2	2	NA	1	1	NA	1	NA	2	2	2	NA	NA	2	NA
<b>mv_GH</b>	2	1	1	2	3	NA	1	-1	NA	4	NA	1	NA	NA	2
<b>mv_HP</b>	1	1	1	NA	2	NA	NA	1	1	3	2	3	NA	NA	NA
<b>mv_BI</b>	NA	2	1	-1	NA	NA	NA	1	NA	2	NA	1	NA	-1	NA
<b>mv_DI</b>	1	2	1	2	2	NA	NA	1	1	NA	NA	NA	1	2	2
<b>mv_FF</b>	1	2	2	2	2	1	NA	1	1	1	3	2	2	3	2
<b>mv_GG</b>	-1	NA	2	NA	-1	1	1	1	-1	2	NA	3	NA	3	NA
<b>mv_TF</b>	1	2	-1	1	NA	NA	NA	1	-1	-1	1	1	NA	NA	NA

Color legend:

quantile 20.00% 40.00% 60.00% 80.00% 100.00%

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: standard, Ranking rule: NetFlows

Ordinal (Kendall) correlation between global ranking and global outranking relation: +0.780

Mean marginal correlation (a) : +0.234

Standard marginal correlation deviation (b) : +0.147

Ranking fairness (a) - (b) : +0.087

**Fig. 16.2** Graffiti magazine's movie ratings from September 2007 ranked with the NETFLOWS rule

Let  $r_1$  and  $r_2$  be two random crisp relations defined on a same set of 5 alternatives. We may compute KENDALL's  $\tau$  index as shown in List. 16.3.

**Listing 16.3** Computing a relational equivalence digraph

```

1 >>> from randomDigraphs import RandomDigraph
2 >>> r1 = RandomDigraph(order=5,Bipolar=True)
3 >>> r2 = RandomDigraph(order=5,Bipolar=True)
4 >>> from digraphs import EquivalenceDigraph
5 >>> eqd = EquivalenceDigraph(r1,r2)
6 >>> eqd.showRelationTable(ReflexiveTerms=False)
7   * ----- Relation Table -----
8   r(<=) | 'a1'  'a2'  'a3'  'a4'  'a5'
9   -----|-----
10  'a1' |      -1.00  1.00  -1.00  1.00
11  'a2' | -1.00      -1.00  1.00  -1.00
12  'a3' | -1.00  -1.00      -      1.00  1.00

```

```

13   'a4' | -1.00 1.00 -1.00 - 1.00
14   'a5' | -1.00 1.00 -1.00 1.00 -
15   Valuation domain: [-1.00;1.00]
16 >>> eqd.correlation
17   {'correlation': -0.1, 'determination': 1.0}

```

In the table of the equivalence relation ( $r_1 \Leftrightarrow r_2$ ) above (see List. 16.1 on page 223 Lines 10-14), we observe that the normalized majority margin of equivalent versus non equivalent irreflexive pairs amounts to  $(9 - 11)/20 = -0.1$ , i.e. the value of KENDALL's *tau* index in this plainly determined crisp case (see Line 17).

What happens now with more or less determined and even partially indeterminate relations? May we proceed in a similar way?

### 16.3 Bipolar-valued relational equivalence

Two random bipolar-valued digraphs  $d_1$  and  $d_2$  of order five, generated in List. 16.4, will help exploring this idea.

**Listing 16.4** Two random bipolar-valued digraphs

```

1 >>> from randomDigraphs import RandomValuationDigraph
2 >>> d1 = RandomValuationDigraph(order=5, seed=1)
3 >>> d1.showRelationTable(ReflexiveTerms=False)
4   * ----- Relation Table -----
5   r(d1) | 'a1' 'a2' 'a3' 'a4' 'a5'
6   -----|-----
7   'a1' | - -0.66 0.44 0.94 -0.84
8   'a2' | -0.36 - -0.70 0.26 0.94
9   'a3' | 0.14 0.20 - 0.66 -0.04
10  'a4' | -0.48 -0.76 0.24 - -0.94
11  'a5' | -0.02 0.10 0.54 0.94 -
12  Valuation domain: [-1.00;1.00]
13 >>> d2 = RandomValuationDigraph(order=5, seed=2)
14 >>> d2.showRelationTable(ReflexiveTerms=False)
15   * ----- Relation Table -----
16   r(d2) | 'a1' 'a2' 'a3' 'a4' 'a5'
17   -----|-----
18   'a1' | - -0.86 -0.78 -0.80 -0.08
19   'a2' | -0.58 - 0.88 0.70 -0.22
20   'a3' | -0.36 0.54 - -0.46 0.54
21   'a4' | -0.92 0.48 0.74 - -0.60
22   'a5' | 0.10 0.62 0.00 0.84 -
23  Valuation domain: [-1.00;1.00]

```

In the generated random digraphs  $d_1$  and  $d_2$ , 9 pairs, like  $(a_1, a_2)$  or  $(a_3, a_2)$  for instance, appear equivalently oriented (see Lines 7,18 or 9,20). The *EquivalenceDigraph* class computes this bipolar-valued relational equivalence between digraphs  $d_1$  and  $d_2$  (see List. 16.5).

**Listing 16.5** Bipolar-valued Equivalence Digraph

```

1 >>> from digraphs import EquivalenceDigraph

```

```

2 >>> eqd = EquivalenceDigraph(d1,d2)
3 >>> eqd.showRelationTable(ReflexiveTerms=False)
4   * ----- Relation Table -----
5   r(<=>) |  'a1'  'a2'  'a3'  'a4'  'a5'
6   -----|-----
7   'a1' |      0.66  -0.44 -0.80  0.08
8   'a2' |  0.36      -0.70  0.26 -0.22
9   'a3' | -0.14  0.20      -0.46 -0.04
10  'a4' |  0.48 -0.48  0.24      -  0.60
11  'a5' | -0.02  0.10  0.00  0.84      -
12  Valuation domain: [-1.00;1.00]

```

In our bipolar-valued epistemic logic, logical disjunctions and conjunctions are implemented as `max`, respectively `min` operations. Notice also that the logical equivalence  $(d1 \Leftrightarrow d2)$  corresponds to a double implication  $(d1 \Rightarrow d2) \wedge (d2 \Rightarrow d1)$  and that the implication  $(d1 \Rightarrow d2)$  is logically equivalent to the disjunction  $(\neg d1 \vee d2)$ .

When  $r(x \text{d1} y)$  and  $r(x \text{d2} y)$  denote the bipolar-valued characteristic values of relation  $d1$ , resp.  $d2$ , we may hence compute as follows a majority margin  $M(d1 \Leftrightarrow d2)$  between equivalently and not equivalently oriented irreflexive pairs  $(x,y)$ :

$$M(d1 \Leftrightarrow d2) = \sum_{(x \neq y)} \left[ \min \left( \max(-r(x \text{d1} y), r(x \text{d2} y)), \max(-r(x \text{d2} y), r(x \text{d1} y)) \right) \right]. \quad (16.2)$$

$M(d1 \Leftrightarrow d2)$  is thus given by the sum of the non reflexive terms of the relation table of  $eqd$ , the relational equivalence digraph computed above (see List. 16.5 on the preceding page). In the crisp case,  $M(d1 \Leftrightarrow d2)$  is normalized with the maximum number of possible irreflexive pairs, namely  $n(n-1)$ . In the extended  $r$ -valued case, the maximal possible equivalence majority margin  $M$  corresponds to the sum  $D$  of the conjoint determinations of  $(x \text{d1} y)$  and  $(x \text{d2} y)$  (see Bisdorff (2012)):

$$D = \sum_{x \neq y} \min \left[ \text{abs}(r(x \text{d1} y)), \text{abs}(r(x \text{d2} y)) \right], \quad (16.3)$$

and we obtain hence in the general  $r$ -valued case:

$$\tau(d1, d2) = \frac{M(d1 \Leftrightarrow d2)}{D}. \quad (16.4)$$

$\tau(d1, d2)$  corresponds so to the classical ordinal correlation index, yet restricted to the conjointly determined parts of the given digraphs  $d1$  and  $d2$ . In the limit case of two crisp linear orders,  $D$  equals  $n(n-1)$ , i.e. the number of irreflexive pairs, and we recover KENDALL's original *tau* index definition.

It is worthwhile noticing that the ordinal correlation index  $\tau(d1, d2)$  one obtains above corresponds in fact to the ratio of:

- $r(d1 \Leftrightarrow d2) = \frac{M(d1 \Leftrightarrow d2)}{n(n-1)}$ :  
the normalized majority margin of the pairwise *relational* equivalence statements, also called *valued ordinal correlation*, and
- $d = \frac{D}{n(n-1)}$ :  
the normalized determination of the corresponding pairwise relational equivalence statements, in fact the *determinateness* of the relational equivalence digraph.

The epistemic determination effect is thus successfully *out-factored* from the ordinal correlation effect. With completely determined relations,  $\tau(d1, d2) = r(d1 \Leftrightarrow d2)$ . The ordinal correlation with a completely indeterminate digraph, i.e. when  $D = 0$ , is set by convention to the indeterminate correlation value 0.0. With uniformly chosen random  $r$ -valued digraphs, the expected  $\tau$  index is 0.0, denoting in fact an indeterminate relational equivalence. The corresponding expected normalized determination  $d$  is about 0.333 (see (Bisdorff, 2012)).

We may below verify Eq. ?? on page ?? with help of the equivalence digraph *eqd* computed in List. 16.5 on page 226.

**Listing 16.6** Computing the ordinal correlation index from the equivalence digraph

```

1 >>> # eqd = EquivalenceDigraph(d1, d2)
2 >>> M = Decimal('0'); D = Decimal('0')
3 >>> n2 = eqd.order*(eqd.order - 1)
4 >>> for x in eqd.actions:
5     ...     for y in eqd.actions:
6         ...         M += eqd.relation[x][y]
7     ...         D += abs(eqd.relation[x][y])
8 >>> print('r(r(d1<=>d2)) = %+.3f, d = %.3f, tau = %+.3f' % \
9         ...         (M/n2, D/n2, M/D))
10    r(r(d1<=>d2)) = +0.026, d = 0.356, tau = +0.073

```

The DIGRAPH3 resources directly provide for the preceding computations the `computeOrdinalCorrelation()` method which renders a dictionary with a `correlation` ( $\tau$ ) and a `determination` ( $d$ ) attribute. We may recover  $r(d1 \Leftrightarrow d2)$  by multiplying  $\tau$  with  $d$  (see List. 16.7 Line 4).

**Listing 16.7** Computing the valued ordinal correlation index

```

1 >>> corrd1d2 = d1.computeOrdinalCorrelation(d2)
2 >>> tau = corrd1d2['correlation']
3 >>> d = corrd1d2['determination']
4 >>> r = tau * d
5 >>> print('tau(d1, d2) = %+.3f, d = %.3f, \
6 ...     r(d1<=>d2) = %+.3f' % (tau, d, r))
7     tau(d1, d2) = +0.073, d = 0.356, r(d1<=>d2) = +0.026

```

The DIGRAPH3 resources provide for convenience a direct `showCorrelation()` method:

```

1 >>> d1.showCorrelation(\n
2 ...     d1.computeOrdinalCorrelation(d2) )
3     Correlation indexes:
4     Extended Kendall tau      : +0.073

```

```

5   Epistemic determination    : 0.356
6   Bipolar-valued equivalence : +0.026

```

We are now ready for assessing the quality of the NETFLOWS ranking of the movies shown in the heat map view of Fig. 16.2 on page 225.

## 16.4 Fitness of ranking heuristics

The NETFLOWS ranking of the movies shown in the heatmap view in Fig. 16.2 on page 225 is based on the bipolar-valued outranking digraph modelling the pairwise global “*rated at least as well as*” relation among the 25 movies from the performance tableau instance `gt07`.

**Listing 16.8** The bipolar-valued outranking digraph of the star-rated movies

```

1 >>> bod = BipolarOutrankingDigraph(gt07)
2   *----- Object instance description -----*
3   Instance class    : BipolarOutrankingDigraph
4   Instance name     : rel_grafittiPerfTab.xml
5   Actions          : 25
6   Criteria         : 15
7   Size              : 390
8   Determinateness   : 65%
9   Valuation domain : {'min': Decimal('-1.0'),
10      'med': Decimal('0.0'),
11      'max': Decimal('1.0'),}
12 >>> g.computeCoSize()
13 188

```

Listing 16.8 reveals that the outranking digraph `bod` contains 390 positively validated (Line 7), 188 positively invalidated (Line 13), and 22 indeterminate outranking situations from the potential  $25 \times 24 = 600$  irreflexive movie pairs.

Listing 16.9 illustrates with the `NetFlowsOrder` class from the `linearOrders` module how to compute the global NETFLOWS ranking `nf`, as shown in the ordered heat map of Fig. 16.2 on page 225 and the bipolar-valued relational equivalence of the `nf` ranking with each one the individual critic’s star-ratings.

**Listing 16.9** Computing marginal criterion correlations with global NETFLOWS ranking

```

1 >>> from linearOrders import NetFlowsOrder
2 >>> nf = NetFlowsOrder(bod)
3 >>> nf.netFlowsRanking
4   ['mv_QS', 'mv_RR', 'mv_DG', 'mv_NP', 'mv_HN', 'mv_HS', 'mv_SM',
5   'mv_JB', 'mv_PE', 'mv_FC', 'mv_TP', 'mv_CM', 'mv_DF', 'mv_TM',
6   'mv_DJ', 'mv_AL', 'mv_RG', 'mv_MB', 'mv_GH', 'mv_HP', 'mv_BI',
7   'mv_DI', 'mv_FF', 'mv_GG', 'mv_TF']
8 >>> for i,item in enumerate(\
9 ...           bod.computeMarginalVersusGlobalRankingCorrelations(\ \
10 ...             nf.netFlowsRanking, ValuedCorrelation=True) ):\
11 ...           print('r(%s<=>nf) = %+.3f' % (item[1],item[0]) )
12

```

```

13  r (JH<=>nf)  = +0.500
14  r (JPT<=>nf) = +0.430
15  r (AP<=>nf)  = +0.323
16  r (DR<=>nf)  = +0.263
17  r (MR<=>nf)  = +0.247
18  r (VT<=>nf)  = +0.227
19  r (GS<=>nf)  = +0.160
20  r (CS<=>nf)  = +0.140
21  r (SJ<=>nf)  = +0.137
22  r (RR<=>nf)  = +0.133
23  r (TD<=>nf)  = +0.110
24  r (CF<=>nf)  = +0.110
25  r (SF<=>nf)  = +0.103
26  r (AS<=>nf)  = +0.080
27  r (FG<=>nf)  = +0.027

```

In List. 16.9 on the preceding page (see Lines 13-27), we obtain the relational equivalence characteristic values shown in the third row of the ranked heatmap (see Fig. 16.2 on page 225). The global NETFLOWS ranking `nf` represents obviously a rather balanced compromise with respect to each movie critic's star-ratings, as there appears no negative correlation with anyone of them. The ranking `nf` apparently takes also correctly in account that the journalist *JH*, a locally renowned movie critic, shows a higher significance weight (see Line 13).

The ordinal correlation between the global NETFLOWS ranking and the outranking digraph `bod` may be furthermore computed as illustrated in List. 16.10:

**Listing 16.10** Computing the ordinal correlation between NETFLOWS and global outranking digraph

```

1 >>> corrgnf = bod.computeOrdinalCorrelation(nf)
2 >>> bod.showCorrelation(corrgnf)
3   Correlation indexes:
4     Extended Kendall tau      : +0.780
5     Epistemic determination   : 0.300
6     Bipolar-valued equivalence : +0.234

```

One may notice in Line 4 that the ordinal correlation  $\tau(\text{bod}, \text{nf})$  index between the NETFLOWS ranking `nf` and the determined part of the outranking digraph `bod` is quite high (+0.78). Due to the rather high number of missing data, the  $r$ -valued relational equivalence between the `nf` and the `bod` digraph, with a characteristics value of only +0.234, may be misleading. Yet, +0.234 still corresponds to a 62% majority support of the movie critics' star-ratings.

It would be interesting to compare similarly the correlations one may obtain with other global ranking heuristics, like the COPELAND ranking rule.

## 16.5 Illustrating preference divergences

The bipolar-valued relational equivalence indexes gives us, via the `showCriteriaCorrelationTable(ValuedCorrelation=True)` method, a further

measure for studying how *divergent* may appear the rating opinions expressed by the movie critics.

	AP	AS	CF	CS	DR	FG	GS	JH	JPT	MR	RR	SF	SJ	TD	VT
AP	+0.63	+0.04	+0.19	+0.09	+0.22	-0.01	+0.11	+0.23	+0.25	+0.08	+0.02	+0.04	+0.19	+0.04	+0.12
AS		+0.77	+0.12	+0.12	+0.04	-0.02	-0.06	+0.02	+0.24	-0.08	+0.07	+0.04	-0.07	-0.01	+0.02
CF			+0.77	+0.07	+0.11	+0.03	+0.05	+0.07	+0.10	-0.03	+0.01	+0.00	+0.06	+0.03	-0.04
CS				+0.63	+0.04	-0.02	+0.07	+0.13	+0.25	+0.01	+0.03	+0.00	+0.02	+0.03	+0.07
DR					+0.45	+0.03	+0.07	+0.17	+0.23	+0.16	+0.06	+0.03	+0.10	+0.07	+0.10
FG						+0.15	-0.01	+0.04	+0.01	+0.06	-0.00	+0.02	+0.01	+0.01	+0.02
GS							+0.40	+0.07	+0.07	+0.09	-0.02	+0.00	+0.06	+0.04	+0.04
JH								+0.77	+0.28	+0.26	+0.15	+0.12	+0.10	+0.05	+0.14
JPT									+0.92	+0.15	+0.06	+0.09	+0.08	+0.08	+0.17
MR										+0.63	+0.10	+0.08	+0.03	+0.09	+0.10
RR											+0.51	+0.04	+0.01	+0.05	+0.05
SF												+0.18	+0.01	+0.02	+0.05
SJ													+0.51	+0.03	+0.07
TD														+0.26	+0.00
VT															+0.40

**Fig. 16.3** Pairwise valued correlation of movie critics.

It is remarkable to notice in Fig. 16.3 that, due to the quite numerous missing data, all pairwise valued ordinal correlation indexes  $r(x \leftrightarrow y)$  appear to be of low value, except the diagonal ones. These reflexive indexes  $r(x \leftrightarrow x)$  would trivially all amount to +1.0 in a plainly determined case. Here they indicate a reflexive normalized determination score  $d$ , i.e. the proportion of pairs of movies each critic did evaluate. Critic JPT (the editor of the Graffiti magazine), for instance, evaluated all but one ( $d = 24 \times 23/600 = 0.92$ ), whereas critic FG evaluated only 10 movies among the 25 in discussion ( $d = 10 \times 9/600 = 0.15$ ).

To get a picture of the actual divergence of rating opinions concerning jointly seen pairs of movies, we may develop a Principal Component Analysis of the corresponding  $\tau$  correlation matrix<sup>2</sup>. The 3D plot of the first 3 principal axes is shown in Fig. 16.2 on page 225.

```
1 >>> bod.export3DplotOfCriteriaCorrelation(\n2 ...                         ValuedCorrelation=False)
```

The first 3 principal axes support together about 70% of the total inertia. Most eccentric and opposed in their respective rating opinions appear, on the first principal axis with 27.2% inertia, the conservative daily press against labour and public press. On the second principal axis with 23.7% inertia, it is the people press versus the cultural critical press. And, on the third axis with still 19.3% inertia, the written media appear most opposed to the radio media.

<sup>2</sup> The 3D PCA plot method requires a running R statistics software (<https://www.r-project.org/>) installation and the Calmat matrix calculator (see the calmat directory in the DIGRAPH3 resources)

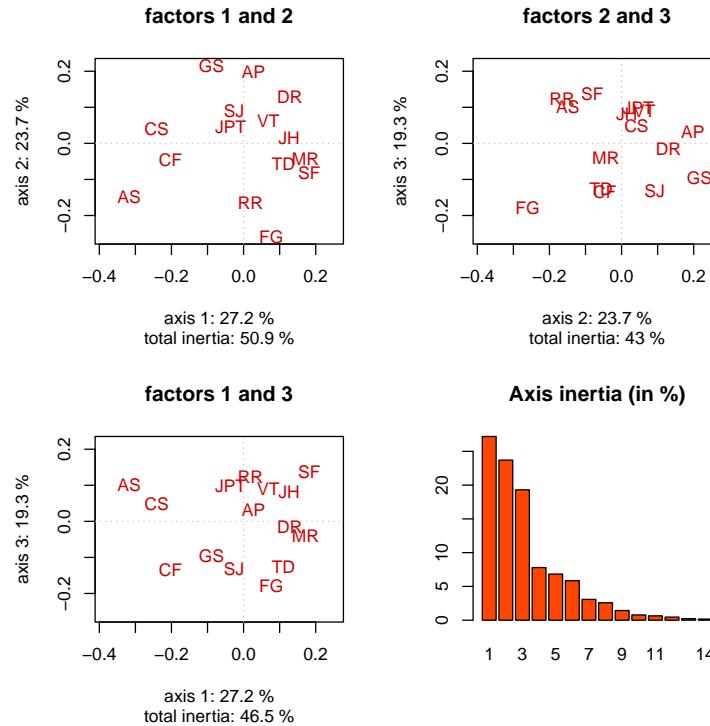


Fig. 16.4 3D PCA plot of the criteria ordinal correlation matrix.

## 16.6 Exploring the “*better rated*” and the “*as well as rated*” opinions

In order to furthermore study the quality of a ranking result, it may be interesting to have a separate view on the asymmetric and symmetric parts of the “*at least as well rated as*” opinions (see Section 2.3 on page 22).

Let us first inspect the pairwise asymmetric part, namely the “*better rated than*” and “*less well rated than*” opinions of the movie critics.

```

1 >>> from digraphs import AsymmetricPartialDigraph
2 >>> ag = AsymmetricPartialDigraph(bod)
3 >>> ag.showHTMLRelationTable(\n4     ...     actionsList=g.computeNetFlowsRanking(), ndigits=0)

```

We notice in Fig. 16.5 on the facing page that the NETFLOWS ranking rule inverts in fact just three “*less well rated than*” opinions and four “*better rated than*” ones. A similar look in Fig. 16.6 on the next page at the symmetric part –the pairwise “*as well rated as*” opinions– suggests a preordered preference structure in several equivalently rated classes.

## Valued Adjacency Matrix

rk	S	Y	mv_QS	mv_RR	mv_DG	mv_NP	mv_HN	mv_HS	mv_SM	mv_JB	mv_PE	mv_FC	mv_TP	mv_CM	mv_DF	mv_TM	mv_DJ	mv_AL	mv_RG	mv_MB	mv_GH	mv_HP	mv_BI	mv_DI	mv_FF	mv_GG	mv_TF
mv_QS	-	11	12	11	10	9	14	9	11	13	9	10	9	9	7	6	9	9	7	6	5	9	15	8	8		
mv_RR	-5	-	0	0	0	0	0	0	5	0	0	0	9	12	9	10	6	8	10	8	9	6	10	13	10	9	
mv_DG	-8	0	-	0	0	0	0	0	0	0	7	7	10	10	6	8	10	7	9	5	6	7	9	9	9		
mv_NP	-7	0	0	-	0	0	0	0	0	8	0	0	8	0	7	6	4	6	8	6	5	7	12	10	7		
mv_HN	-10	0	0	0	0	-	0	0	0	0	0	0	7	0	6	0	10	7	6	8	8	5	7	13	9	11	
mv_HS	-3	0	0	0	0	0	-	0	0	0	0	0	5	0	7	0	0	10	9	8	9	7	11	14	9	11	
mv_SM	-8	0	0	0	0	0	0	-	0	0	0	0	12	9	0	6	0	9	9	10	7	9	7	11	15	11	
mv_JB	-9	-1	0	0	0	0	0	0	-	0	0	0	6	0	6	0	9	9	9	8	8	5	11	15	12	8	
mv_PE	-7	0	0	0	0	0	0	0	0	-	0	4	10	0	0	6	8	8	8	0	6	5	9	13	9	8	
mv_FC	-9	0	0	0	0	0	0	0	0	0	0	-	10	6	0	5	3	8	6	9	9	5	7	10	12	10	11
mv_TP	-7	0	-1	0	0	0	0	-4	-4	0	-	0	0	0	6	4	0	7	6	7	7	7	9	14	12	10	
mv_CM	-8	-1	-5	-4	-3	-3	-1	0	-2	-2	0	-	0	0	0	8	0	10	0	0	0	7	3	9	14	11	8
mv_DF	-9	-2	-2	0	0	0	0	0	0	0	0	0	0	1	-	4	6	6	6	6	0	6	8	0	0	9	
mv_TM	-3	-7	-6	-2	-2	-1	0	0	0	-1	-2	0	-2	0	-	0	0	0	0	0	9	7	5	7	9	5	8
mv_DJ	-7	-6	-4	-3	0	4	0	0	-2	-1	0	0	0	0	-	0	0	4	3	6	4	4	0	0	5	5	
mv_AL	-4	0	-4	-6	-6	0	-1	-3	-4	-2	0	0	3	0	0	-	0	0	5	1	0	0	5	3	3		
mv_RG	-7	-6	-4	-1	-2	-1	-5	-4	-2	-3	0	0	0	-	0	0	0	0	0	4	0	0	4	0	4	9	
mv_MB	-9	-8	-5	-4	-4	-9	-4	-1	-2	-1	-2	0	-	0	-2	0	0	0	0	4	4	4	2	6	8		
mv_GH	-5	-8	-7	-6	-8	-4	-3	0	0	-5	-5	0	-2	-1	-3	-3	0	0	-	5	0	0	0	3	5		
mv_HP	-4	-5	-5	-2	-4	-3	-3	-4	-4	-3	-1	-3	-2	-1	-2	-1	-1	-	0	0	0	0	7	8			
mv_BI	-5	-4	-6	-1	-1	-7	-5	-5	-5	-3	-1	-3	-3	-2	-2	0	-2	0	0	0	0	0	-1	-1	0		
mv_DI	-9	-6	-5	-5	-3	-3	-3	-3	-7	-7	-6	-1	-7	0	-5	0	0	0	0	0	0	0	0	4	0		
mv_FF	-11	-11	-7	-10	-5	0	-3	-9	-9	-8	-2	-6	0	-1	0	-1	0	0	0	0	5	0	-	0	11		
mv_GG	-6	-8	-7	-4	-5	-7	-9	-10	-9	-4	-2	-7	0	-3	0	-3	-2	-2	-1	-3	5	-2	0	-	0		
mv_TF	-8	-7	-7	-5	-7	-7	-11	-9	-8	-7	-6	-8	-3	-6	-3	-1	-1	-4	-3	0	0	0	-1	0	-		

Valuation domain: [-19.00; +19.00]

Fig. 16.5 Asymmetric part of *graffiti07* digraph

```

1 >>> from digraphs import SymmetricPartialDigraph
2 >>> sg = SymmetricPartialDigraph(bod)
3 >>> sg.showHTMLRelationTable(\n4 ...           actionsList=g.computeNetFlowsRanking(), \n5 ...           ndigits=0)

```

## Valued Adjacency Matrix

rk	S	Y	mv_QS	mv_RR	mv_DG	mv_NP	mv_HN	mv_HS	mv_SM	mv_JB	mv_PE	mv_FC	mv_TP	mv_CM	mv_DF	mv_TM	mv_DJ	mv_AL	mv_RG	mv_MB	mv_GH	mv_HP	mv_BI	mv_DI	mv_FF	mv_GG	mv_TF
mv_QS	-	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
mv_RR	0	-	5	7	8	6	4	0	2	9	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
mv_DG	0	9	-	9	10	5	9	7	8	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
mv_NP	0	5	7	10	9	7	9	8	11	9	0	0	12	0	0	0	0	0	0	0	0	0	0	0	0	0	
mv_HN	0	4	8	8	-	5	9	9	6	10	10	0	10	0	8	0	0	0	0	0	0	0	0	0	0	0	
mv_HS	0	2	5	5	3	-	10	2	5	6	9	0	0	10	0	0	1	0	0	0	0	0	0	14	0	0	
mv_SM	0	6	7	5	7	6	-	6	6	10	12	0	0	10	6	4	0	0	0	0	0	0	0	0	0	0	
mv_JB	0	0	5	1	3	4	8	-	9	6	0	13	6	8	9	0	0	0	8	0	0	0	0	0	0	0	
mv_PE	0	2	6	0	4	3	6	11	-	5	0	0	5	7	0	0	0	0	5	0	0	0	0	0	0	0	
mv_FC	0	5	11	9	8	2	8	8	7	-	10	0	11	0	0	0	0	0	0	0	0	0	0	0	0	0	
mv_CM	0	4	0	3	2	3	0	0	0	0	-	6	11	0	4	5	0	0	0	0	0	0	0	0	0	0	
mv_DF	0	0	0	0	0	2	2	2	0	1	5	1	-	0	6	0	0	0	0	5	6	8	13	13	0		
mv_TM	0	0	0	0	0	0	0	0	1	0	0	2	0	-	2	2	7	6	0	0	0	0	0	0	0	0	
mv_DI	0	0	0	0	0	2	4	2	1	0	0	0	0	0	4	-	3	5	0	0	0	0	4	3	1	0	
mv_AL	0	0	0	0	0	0	1	0	0	0	0	3	1	0	2	3	-	2	2	0	0	3	1	0	0	0	
mv_RG	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	3	4	-	1	0	6	0	8	9	0	0	
mv_MB	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	1	2	3	-	2	0	4	4	2	0	0	
mv_GH	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	6	0	2	5	3	0	0	0	
mv_HP	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	2	0	3	6	5	0	8	0	
mv_BI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	2	1	-	1	0	0	4	
mv_DI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	4	0	1	1	8	5	-	6	0	8	
mv_FF	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	5	0	9	0	1	7	0	12	-	9	0	
mv_GG	0	0	0	0	0	0	0	0	0	0	0	0	3	0	1	0	0	0	0	0	0	0	0	3	-	2	
mv_TF	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	2	-	-	

Valuation domain: [-19.00; +19.00]

Fig. 16.6 Symmetric part of *graffiti07* digraph

Such a preordering of the movies may, for instance, be computed with the `computeRankingByChoosing()` method, where we iteratively extract dominant kernels –best remaining choices– and absorbent kernels –worst remaining choices– (see the next Chapter). We operate therefore on the asymmetric “better

*rated than*” opinions, i.e. the codual of the “*at least as well rated as*” opinions <sup>3</sup> (see Line 2 in List. 16.11).

**Listing 16.11** Bipolar ranking-by-choosing the Grafitti movies

```

1 >>> from transitiveDigraphs import RankingByChoosingDigraph
2 >>> rbc = RankingByChoosingDigraph(bod, CoDual=True)
3 >>> rbc.showRankingByChoosing()
4   Ranking by Choosing and Rejecting
5     1st Best Choice ['mv_QS']
6     2nd Best Choice ['mv_DG', 'mv_FC', 'mv_HN', 'mv_HS', 'mv_NP',
7                      'mv_PE', 'mv_RR', 'mv_SM']
8     3rd Best Choice ['mv_CM', 'mv_JB', 'mv_TM']
9     4th Best Choice ['mv_AL', 'mv_TP']
10    4th Worst Choice ['mv_AL', 'mv_TP']
11    3rd Worst Choice ['mv_GH', 'mv_MB', 'mv_RG']
12    2nd Worst Choice ['mv_DF', 'mv_DJ', 'mv_FF', 'mv_GG']
13    1st Worst Choice ['mv_BI', 'mv_DI', 'mv_HP', 'mv_TF']

```

In the next Chapter [17 on page 237](#), we thouroughly discuss the computation of such kernels in bipolar-valued digraphs.

---

<sup>3</sup> A kernel in a digraph  $g$  is a clique in the dual digraph  $-g$ .

**References**

- Bisdorff R (2012) On measuring and testing the ordinal correlation between bipolar outranking relations. In: Mousseau V, Pirlot M (eds) DAP'2012 From Multiple Criteria Decision Aid to Preference Learning, University of Mons (Belgium), pp 91–100, URL <http://hdl.handle.net/10993/23909> 227, 228
- Kendall MG (1938) A new measure of rank correlation. *Biometrika* 30:81–93 224



# Chapter 17

## On computing digraph kernels

**Abstract** We illustrate in this chapter, first, the concept of graph kernel, i.e. maximal independent set of vertices. In non-symmetric digraphs the kernel concept becomes richer and separates into initial and terminal kernels. In, furthermore, lateralised outranking digraphs, initial and terminal kernels become separate and may deliver suitable best resp. worst choice recommendations. After commenting the tractability of kernel computations, we close the chapter with the solving of bipolar-valued kernel equation systems.

### 17.1 What is a graph kernel ?

We call *choice* in a graph, respectively a digraph, a subset of its vertices, resp. of its nodes or actions. A choice  $Y$  is called *internally stable* or *independent* when there exist no links –(edges) or relations (arcs)– between its members. Furthermore, a choice  $Y$  is called *externally stable* when for each vertex, node or action  $x$  not in  $Y$ , there exists at least a member  $y$  of  $Y$  such that  $x$  is linked or related to  $y$ . Now, an internally and externally stable choice is called a *kernel*.

A first trivial example is immediately given by the maximal independent vertices sets (MISs) of the  $n$ -cycle graph (see Fig. 21.5 on page 331). Indeed, each MIS in the  $n$ -cycle graph is by definition independent, i.e. *internally stable*, and each non selected vertex in the  $n$ -cycle graph is in relation with either one or even two members of the MIS.

In all graphs or symmetric digraphs, the *maximality condition* imposed on the internal stability is equivalent to the *external stability* condition. Indeed, if there would exist a vertex or node not related to any of the elements of a choice, we may safely add this vertex or node to the given choice without violating its internal stability. All kernels must hence be maximal independent choices. In fact, in a topological sense, they correspond to maximal *holes* in the given graph.

We may illustrate in Figure 17.1 on the next page this coincidence between MISs and kernels in graphs and symmetric digraphs with a random 3-regular graph in-

stance generated in Listing 17.1. A random MIS in this graph may be computed by using the `MISModel` class (see Lines 5-8 below).

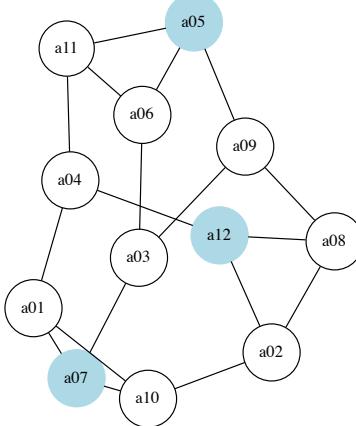
**Listing 17.1** Generating a random 3-regular graph of order 12

```

1 >>> from graphs import RandomRegularGraph
2 >>> r3g12 = RandomRegularGraph(order=12, \
3     degree=3, seed=4)
4 >>> from graphs import MISModel
5 >>> mg = MISModel(r3g12)
6   Iteration: 1
7     Running a Gibbs Sampler for 660 step !
8     {'a05', 'a07', 'a12'} is maximal !
9 >>> mg.exportGraphViz('random3RegularGraph-mis')
10  *---- exporting a dot file for GraphViz tools ---*
11  Exporting to random3RegularGraph-mis.dot
12  fdp -Tpng random3RegularGraph-mis.dot \
13      -o random3RegularGraph-mis.png

```

**Fig. 17.1** A random MIS coloured in the random 3-regular graph `r3g12`. All non blue vertices are covered by a blue vertex



*DiGraph3 (graphviz), R. Bisdorff, 2019*

It is easily verified in Figure 17.1, that the computed MIS renders indeed a valid kernel of the given graph. The complete set of kernels of this 3-regular graph instance coincides hence with the set of its MISs.

**Listing 17.2** Printing out all maximal independent sets of the random 3-regular graph

```

1 >>> g.showMIS()
2  *---- Maximal Independent Sets ---*
3  ['a05', 'a07', 'a12']
4  ['a01', 'a06', 'a08']
5  ['a07', 'a08', 'a11']

```

```

6      ['a01', 'a09', 'a11', 'a12']
7      ['a01', 'a02', 'a09', 'a11']
8      ['a01', 'a06', 'a09', 'a12']
9      ['a01', 'a02', 'a06', 'a09']
10     ['a07', 'a09', 'a11', 'a12']
11     ['a02', 'a07', 'a09', 'a11']
12     ['a06', 'a07', 'a09', 'a12']
13     ['a04', 'a06', 'a07', 'a08']
14     ['a02', 'a04', 'a05', 'a07']
15     ['a04', 'a05', 'a07', 'a08']
16     ['a09', 'a10', 'a11', 'a12']
17     ['a06', 'a09', 'a10', 'a12']
18     ['a04', 'a06', 'a08', 'a10']
19     ['a04', 'a06', 'a09', 'a10']
20     ['a01', 'a03', 'a11', 'a12']
21     ['a01', 'a02', 'a03', 'a11']
22     ['a01', 'a03', 'a08', 'a11']
23     ['a01', 'a03', 'a05', 'a12']
24     ['a01', 'a02', 'a03', 'a05']
25     ['a01', 'a03', 'a05', 'a08']
26     ['a02', 'a03', 'a04', 'a05']
27     ['a03', 'a10', 'a11', 'a12']
28     ['a03', 'a08', 'a10', 'a11']
29     ['a03', 'a05', 'a10', 'a12']
30     ['a02', 'a04', 'a06', 'a07', 'a09']
31     ['a03', 'a04', 'a05', 'a08', 'a10']
32     number of solutions: 29
33     cardinality distribution
34     card.: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
35     freq.: [0, 0, 0, 3, 24, 2, 0, 0, 0, 0, 0, 0, 0]
36     stability number : 5
37     execution time: 0.00056 sec.
38     Results in self.misset
39     >>> g.misset
40     [frozenset({'a05', 'a07', 'a12'}),
41      frozenset({'a11', 'a01', 'a12', 'a09'}),
42      frozenset({'a11', 'a01', 'a02', 'a09'}),
43      ...
44      ...
45      frozenset({'a03', 'a10', 'a08', 'a11'}),
46      frozenset({'a03', 'a10', 'a12', 'a05'}),
47      frozenset({'a03', 'a10', 'a04', 'a05', 'a08'})]

```

All graphs and symmetric digraphs admit MISs, hence also kernels. In the context of digraphs, i.e. *oriented* graphs, the kernel concept gets much richer and separates from the symmetric MIS concept.

## 17.2 Initial and terminal kernels

In an oriented graph context, the internal stability condition of the kernel concept remains untouched; however, the external stability condition gets indeed split up by the orientation into two lateral cases:

1. A *dominant* stability condition, where each non selected node is dominated by at least one member of the kernel;
2. An *absorbent* stability condition, where each non selected node is absorbed by at least one member of the kernel.

A both internally stable **and** dominant, resp. absorbent choice is called a dominant or *initial*, resp. an absorbent or *terminal* kernel. From a topological perspective, the initial kernel concept looks from the outside of the digraph into its interior, whereas the terminal kernel looks from the interior of a digraph toward its outside. From an algebraic perspective, the initial kernel is a prefix operand, and the terminal kernel is a postfix operand in the kernel equation systems (see Sec. 17.6 on page 253).

Furthermore, as the kernel concept involves conjointly a positive logical refutation (the internal stability) and a positive logical affirmation (the external stability), it appeared rather quickly necessary in our operational developments to adopt a bipolar characteristic  $[-1.0, 1.0]$  valuation domain, modelling logical negation by a change of numerical sign and including explicitly a third median value (0.0), expressing logical *indeterminateness* –neither positive, nor negative (Bisdorff, 2000, 2002, 2004).<sup>1</sup>.

In such a bipolar-valued context, we call *prekernel* a choice who is *externally stable* and for which the internal stability condition is *valid or indeterminate*. We say that the independence condition is in this case only *weakly validated*. Notice that all kernels are hence prekernels, but not vice-versa.

In graphs or symmetric digraphs, where there is essentially no apparent *laterality*; all prekernels are initial and terminal at the same time. A universal example is given by the *complete* digraph .

**Listing 17.3** The prekernels of a complete digraph

```

1  >>> from digraphs import CompleteDigraph
2  >>> u = CompleteDigraph(order=5)
3  >>> u
4  *----- Digraph instance description -----*
5  Instance class   : CompleteDigraph
6  Instance name    : complete
7  Digraph Order    : 5
8  Digraph Size     : 20
9  Valuation domain : [-1.00 ; 1.00]
10 -----
11 >>> u.showPreKernels()
12 *--- Computing preKernels ---*
13 Dominant kernels :

```

---

<sup>1</sup> It is D. Bouyssou who first suggested us to adopt bipolar-valued relational characteristics.

```

14      ['1'] ind. : 1.0; dom. : 1.0; abs. : 1.0
15      ['2'] ind. : 1.0; dom. : 1.0; abs. : 1.0
16      ['3'] ind. : 1.0; dom. : 1.0; abs. : 1.0
17      ['4'] ind. : 1.0; dom. : 1.0; abs. : 1.0
18      ['5'] ind. : 1.0; dom. : 1.0; abs. : 1.0
19      Absorbent kernels :
20      ['1'] ind. : 1.0; dom. : 1.0; abs. : 1.0
21      ['2'] ind. : 1.0; dom. : 1.0; abs. : 1.0
22      ['3'] ind. : 1.0; dom. : 1.0; abs. : 1.0
23      ['4'] ind. : 1.0; dom. : 1.0; abs. : 1.0
24      ['5'] ind. : 1.0; dom. : 1.0; abs. : 1.0
25      *----- statistics -----
26      graph name: complete
27      number of solutions
28      dominant kernels : 5
29      absorbent kernels: 5
30      cardinality frequency distributions
31      cardinality : [0, 1, 2, 3, 4, 5]
32      dominant kernel : [0, 5, 0, 0, 0, 0]
33      absorbent kernel: [0, 5, 0, 0, 0, 0]
34      Execution time : 0.00004 sec.
35      Results in sets: dompreKernels
36      and abspreKernels.

```

In a complete digraph, each single node is indeed both an initial and a terminal prekernel candidate and there is no definite begin or end of the digraph to be detected. Laterality is here entirely relative to a specific singleton chosen as reference point of view.

The same absence of laterality is apparent (see List. 17.4) in two other universal digraph models, the *empty* and the *indeterminate* digraph.

**Listing 17.4** The prekernels of the empty or indeterminate digraph

```

1 >>> from digraphs import EmptyDigraph
2 >>> ed = EmptyDigraph(order=5)
3 >>> ed.showPreKernels()
4      *--- Computing preKernels ---*
5      Dominant kernel :
6      ['1', '2', '3', '4', '5']
7          independence : 1.0
8          dominance : 1.0
9          absorbency : 1.0
10     Absorbent kernel :
11     ['1', '2', '3', '4', '5']
12         independence : 1.0
13         dominance : 1.0
14         absorbency : 1.0
15 >>> from digraphs import IndeterminateDigraph
16 >>> id = IndeterminateDigraph(order=5)
17 >>> id.showPreKernels()
18      *--- Computing preKernels ---*
19      Dominant prekernel :
20      ['1', '2', '3', '4', '5']
21          independence : 0.0

```

```

22      dominance   : 1.0
23      absorbency : 1.0
24  Absorbent prekernel :
25  ['1', '2', '3', '4', '5']
26      independence : 0.0
27      dominance   : 1.0
28      absorbency : 1.0

```

In the empty digraph, the whole set of nodes gives indeed at the same time the unique initial and terminal prekernel (see Lines 6 and 11). Similarly, for the *indeterminate* digraph (see Lines 20 and 25).

Both these results make sense, as in a completely empty or indeterminate digraph, there is no *interior* of the digraph defined, only a *border* which is hence at the same time an initial and terminal prekernel (see Section 2.4 on page 23). Notice however, that in the latter indeterminate case, the complete set of nodes verifies only weakly the internal stability condition (see Lines 21 and 26).

Other common digraph models, although being clearly oriented, may show nevertheless no apparent laterality, like *chordless circuits*, i.e. holes surrounded by an oriented cycle -a circuit- of odd length. They do not admit in fact any initial or terminal prekernel.

**Listing 17.5** The prekernels of the 5-circuit digraph

```

1 >>> from digraphs import CirculantDigraph
2 >>> c5 = CirculantDigraph(order=5, circulants=[1])
3 >>> c5.showPreKernels()
4  *----- statistics -----
5  digraph name: c5
6  number of solutions
7  dominant prekernels : 0
8  absorbent prekernels: 0

```

Chordless circuits of *even* length  $2 \times k$ , with  $k > 1$ , contain however two isomorphic prekernels of cardinality  $k$  which qualify conjointly as initial and terminal candidates.

**Listing 17.6** The prekernels of the 6-circuit digraph

```

1 >>> c6 = CirculantDigraph(order=6, circulants=[1])
2 >>> c6.showPreKernels()
3  *--- Computing preKernels ---*
4  Dominant preKernels :
5  ['1', '3', '5'] ind. : 1.0, dom. : 1.0, abs. : 1.0
6  ['2', '4', '6'] ind. : 1.0, dom. : 1.0, abs. : 1.0
7  Absorbent preKernels :
8  ['1', '3', '5'] ind. : 1.0, dom. : 1.0, abs. : 1.0
9  ['2', '4', '6'] ind. : 1.0, dom. : 1.0, abs. : 1.0

```

Chordless circuits of even length may thus be indifferently oriented along two opposite directions. Notice by the way in Fig. 17.2 on the next page that the duals of all chordless circuits of odd or even length, i.e. *filled* circuits also called *anti-holes* (see Fig. 17.3 on page 244), never contain any potential prekernel candidates.

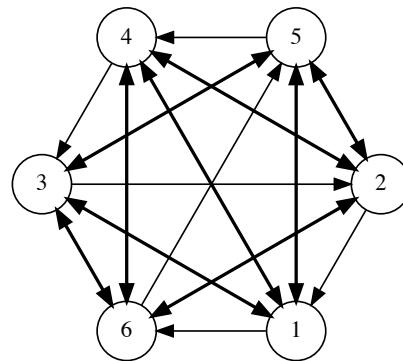
**Listing 17.7** The prekernels of the dual of the 6-circuit digraph

```

1 >>> dc6 = -c6      # dc6 = DualDigraph(c6)
2 >>> dc6.showPreKernels()
3     *----- statistics -----
4     graph name: dual_c6
5     number of solutions
6     dominant prekernels :  0
7     absorbent prekernels:  0
8 >>> dc6.exportGraphViz(fileName='dualChordlessCircuit')
9     *----- exporting a dot file for GraphViz tools -----
10    Exporting to dualChordlessCircuit.dot
11    circo -Tpng dualChordlessCircuit.dot\
12          -o dualChordlessCircuit.png

```

**Fig. 17.2** The dual of the chordless 6-circuit. No initial or terminal prekernel –weakly independent and dominant, resp. absorbent subset of nodes– may be found in this kind of digraphs



*Digraph3 (graphviz), R. Bisdorff, 2020*

We call *weak*, a chordless circuit with indeterminate inner part.

In Listing 17.8 we use the `IndeterminateInnerPart` parameter of the `CirculantDigraph` class for constructing such a weak chordless 6-circuit digraph. It is worth noticing in Fig. 17.3 on the following page that the *dual* version of a weak circuit corresponds to its *converse* version<sup>2</sup>.

**Listing 17.8** The weak 6-circuit digraph

```

1 >>> from digraphs import CirculantDigraph
2 >>> c6 = CirculantDigraph(order=6, circulants=[1], \
3 ...                           IndeterminateInnerPart=True)
4 >>> (-c6).exportGraphViz()

```

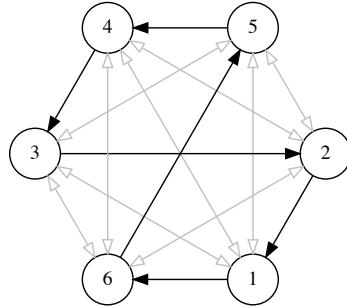
<sup>2</sup> Not to be confused with the dual graph of a plane graph  $g$  that has a vertex for each face of  $g$ . Here we mean the *less than* (strict converse) relation corresponding to a *greater or equal* relation, or the *less than or equal* relation corresponding to a (strict converse) *greater than* relation.

```

5  *---- exporting a dot file for GraphViz tools ----*
6  Exporting to dual_c6.dot
7  circo -Tpng dual_c6.dot -o dual-c6.png
8 >>> (~c6).exportGraphViz()
9  *---- exporting a dot file for GraphViz tools ----*
10 Exporting to converse_c6.dot
11 circo -Tpng converse_c6.dot -o converse-c6.png

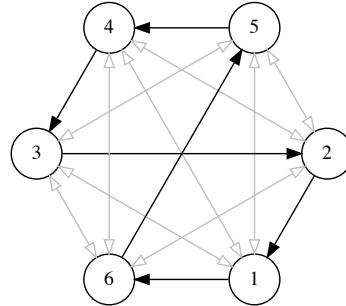
```

Dual of weak 6-circuit



Digraph3 (graphviz), R. Bisdorff, 2020

Converse of weak 6-circuit



Digraph3 (graphviz), R. Bisdorff, 2020

Fig. 17.3 Dual and converse versions of the weak 6-circuit give the same digraph

Weak chordless circuits of length  $n$  are in fact part of the class of digraphs that are invariant under the *codual* transform,  $cn = -(\sim cn) = \sim(-cn)$ <sup>3</sup>. When digraph  $cn$  is a weak chordless  $n$ -circuit,  $cn$ ,  $-cn$ ,  $\sim cn$  and  $\sim(-cn)$  will all admit the same set of prekernels.

### 17.3 Kernels in lateralised digraphs

Humans do live in an apparent physical space of plain transitive lateral orientation, fully empowered in finite geometrical 3D models with linear orders, where first, resp. last ranked, nodes deliver unique initial, resp. terminal, kernels. Similarly, in finite preorders, the first, resp. last, equivalence classes deliver the unique initial,

<sup>3</sup> The class of *self-codual* bipolar-valued digraphs consists of all weakly asymmetric digraphs, i.e. digraphs containing only asymmetric and/or indeterminate links. Limit cases consists of, on the one side, full tournaments with indeterminate reflexive links, and, on the other side, fully indeterminate digraphs. In this class, the converse (inverse  $\sim$ ) operator is indeed identical to the dual (negation  $-$ ) one.

resp. unique terminal, kernels. More generally, in finite partial orders, i.e. asymmetric and transitive digraphs, topological sort algorithms will easily reveal on the first, resp. last, level all unique initial, resp. terminal, kernels.

In genuine random digraphs, however, we may need to check for each of its MISs, whether one, both, or none of the lateralised external stability conditions may be satisfied. Consider, for instance in Listing 17.9, the following random digraph instance of order 7 and generated with an arc probability of 30%.

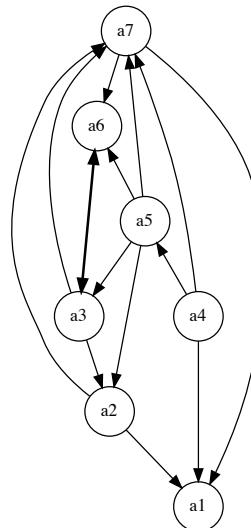
**Listing 17.9** Generating a random digraph `rd` of order 7 and arc probability 0.3

```

1 >>> from randomDigraphs import RandomDigraph
2 >>> rd = RandomDigraph(order=7, arcProbability=0.3, seed=5)
3 >>> rd.exportGraphViz('randomLaterality')
4     *---- exporting a dot file for GraphViz tools ---*
5     Exporting to randomLaterality.dot
6     dot -Grankdir=BT -Tpng randomLaterality.dot\
7             -o randomLaterality.png

```

**Fig. 17.4** A random digraph instance of order 7 and arc probability 0.3. The digraph instance is neither asymmetric ( $a_3 \Leftrightarrow a_6$ ) nor symmetric ( $a_2 \rightarrow a_1$ ,  $a_1 \not\rightarrow a_2$ ) and, the digraph is not transitive ( $a_5 \rightarrow a_2 \rightarrow a_1$ , but  $a_5 \not\rightarrow a_1$ )



*Digraph3 (graphviz), R. Bisdorff, 2020*

The random digraph shown in Figure 17.4 has no apparent special properties, except from being connected (see Line 3 in Listing 17.10).

**Listing 17.10** Inspecting the properties of random digraph `rd`

```

1 >>> rd.showComponents()
2     *--- Connected Components ---*

```

```

3  1: ['a1', 'a2', 'a3', 'a4', 'a5', 'a6', 'a7']
4 >>> rd.computeSymmetryDegree(Comments=True, InPercents=True)
5   Symmetry degree (%) of digraph <randomDigraph>:
6     arcs x>y: 14, symmetric: 1, asymmetric: 13
7     symmetric/arcs =  7.1
8 >>> rd.computeChordlessCircuits()
9   [] # no chordless circuits detected
10 >>> rd.computeTransitivityDegree(Comments=True, InPercents=True)
11   Transitivity degree (%) of graph <randomDigraph>:
12     triples x>y>z: 23, closed: 11, open: 12
13     closed/triples = 47.8

```

There are no chordless circuits (see Line 9 above) and more than half of the required transitive closure is missing (see Line 12 above).

Now, we know that its potential prekernels must be among its set of maximal independent choices.

**Listing 17.11** Inspecting the prekernels of random digraph `rd`

```

1 >>> rd.showMIS()
2   *--- Maximal independent choices ---*
3     ['a2', 'a4', 'a6']
4     ['a6', 'a1']
5     ['a5', 'a1']
6     ['a3', 'a1']
7     ['a4', 'a3']
8     ['a7']
9 >>> rd.showPreKernels()
10  *--- Computing preKernels ---*
11  Dominant preKernels :
12    ['a2', 'a4', 'a6']
13      independence : 1.0
14      dominance   : 1.0
15      absorbency   : -1.0
16      covering     : 0.500
17    ['a4', 'a3']
18      independence : 1.0
19      dominance   : 1.0
20      absorbency   : -1.0
21      covering     : 0.600
22  Absorbent preKernels :
23    ['a3', 'a1']
24      independence : 1.0
25      dominance   : -1.0
26      absorbency   : 1.0
27      covering     : 0.500
28    ['a6', 'a1']
29      independence : 1.0
30      dominance   : -1.0
31      absorbency   : 1.0
32      covering     : 0.600

```

Among the six MISs contained in random digraph `rd` (see Lines 3-8 in Listing 17.11) we discover two initial and two terminal kernels (Lines 12, 17, 23, 28). Notice by the way the covering values (between 0.0 and 1.0) shown by the

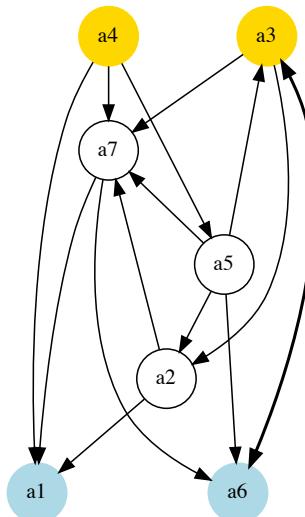
`showPreKernels()` method (Lines 16, 21, 27 and 32). The higher this value, the more the corresponding prekernel candidate makes apparent the digraph's laterality. We may hence redraw the same digraph in Fig. 17.5 by looking into its interior via the best covering initial prekernel candidate: the dominant choice  $\{a_3, a_4\}$  (coloured in yellow), and looking out of it via the best covered terminal prekernel candidate: the absorbent choice  $\{a_1, a_6\}$  (coloured in blue).

```

1 >>> rd.exportGraphViz(fileName='orientedLaterality', \
2 ...                         bestChoice=set(['a3', 'a4']), \
3 ...                         worstChoice=set(['a1', 'a6'])) \
4 *----- exporting a dot file for GraphViz tools -----*
5 Exporting to orientedLaterality.dot
6 dot -Grankdir=BT -Tpng orientedLaterality.dot \
7 -o orientedLaterality.png

```

**Fig. 17.5** A random digraph oriented by best covering initial and best covered terminal prekernels



*Digraph3 (graphviz), R. Bisдорff, 2020*

As all reasonable bipolar-valued outranking digraph usually show some laterality –the marginal criteria preferences being not all totally contradictory– initial and terminal prekernels provide convincing first, resp. last, choice recommendations as illustrated in Chapter 4 on page 45.

## 17.4 Computing first and last choice recommendations

To illustrate this idea, let us finally compute in Listing 17.12 first and last choice recommendations in a random bipolar-valued outranking digraph.

**Listing 17.12** Generating a random bipolar-valued outranking digraph

```

1 >>> from outrankingDigraphs import \
2 ...                               RandomBipolarOutrankingDigraph
3 >>> g = RandomBipolarOutrankingDigraph(seed=5)
4 >>> g
5      *----- Object instance description -----*
6      Instance class      : RandomBipolarOutrankingDigraph
7      Instance name       : randomOutranking
8      Actions             : 7
9      Criteria            : 7
10     Size                : 26
11     Determinateness     : 34.275
12     Valuation domain   : {'min': -100.0, 'med': 0.0, 'max': 100.0}
13 >>> g.showHTMLPerformanceTableau()

```

## Performance table randomOutranking

criterion	g1	g2	g3	g4	g5	g6	g7
a1	64.90	1.31	13.88	98.24	94.10	14.57	31.00
a2	NA	NA	61.75	87.24	69.06	6.51	81.85
a3	11.32	27.95	12.67	28.93	96.66	30.14	48.07
a4	46.91	91.63	0.18	96.15	89.37	60.31	31.58
a5	NA	76.57	87.14	53.92	29.88	0.34	48.12
a6	54.38	15.96	20.95	67.78	36.12	67.79	70.47
a7	57.39	79.71	21.55	20.48	16.60	33.79	5.70

**Fig. 17.6** The performance tableau of a random outranking digraph instance

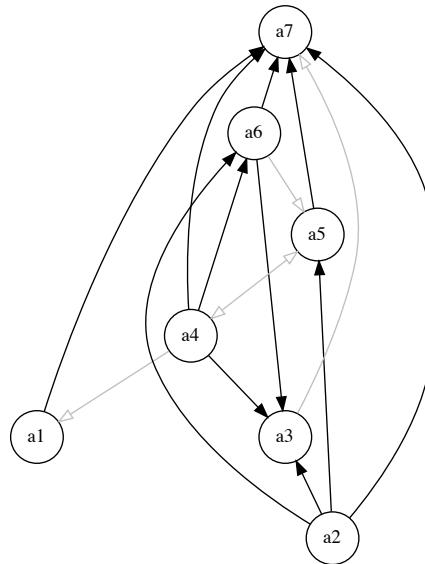
The associated random performance tableau shown in Fig. 17.6 reveals the performance grading of 7 potential decision alternatives with respect to 7 decision criteria supporting each an increasing performance scale from 0.0 to 100.0. Notice the missing performance data concerning decision alternatives a2 and a5. The resulting *strict outranking* - i.e. a weighted majority supported – “*better than without considerable counter-performance*” – digraph is shown in Fig. 17.7 on the next page.

```

1 >>> gcd = ~(-g)  # Codual: the converse of the negation
2 >>> gcd.exportGraphViz(fileName='tutOutRanking')
3      *---- exporting a dot file for GraphViz tools ----*
4      Exporting to tutOutranking.dot
5      dot -Grankdir=BT -Tpng tutOutranking.dot \
6                      -o tutOutranking.png

```

**Fig. 17.7** A random strict outranking digraph instance. All decision alternatives appear strictly better performing than alternative  $a_7$ . We call it a CONDORCET loser and it is an evident terminal prekernel candidate. On the other side, three alternatives:  $a_1$ ,  $a_2$  and  $a_4$  are not dominated. They give together an initial prekernel candidate



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**Listing 17.13** Computing the prekernels of the strict outranking digraph gcd

```

1 >>> gcd.showPreKernels()
2     *--- Computing preKernels ---*
3     Dominant preKernels :
4         ['a1', 'a2', 'a4']
5             independence : 0.00
6             dominance   : 6.98
7             absorbency   : -48.84
8             covering     : 0.667
9     Absorbent preKernels :
10        ['a3', 'a7']
11        independence : 0.00
12        dominance   : -74.42
13        absorbency   : 16.28
14        covered       : 0.800

```

With such unique disjoint initial and terminal prekernels (see Line 4 and 10 in Listing 17.13 on the facing page), the random digraph gcd is hence clearly lateralised. Indeed, these initial and terminal prekernels of the codual outranking digraph reveal *best*, resp. *worst*, choice recommendations one may formulate on the basis of a given outranking digraph instance.

**Listing 17.14** Computing a best and worst choice recommendation from digraph gcd

```

1 >>> g.showBestChoiceRecommendation()

```

```

2 Rubis best choice recommendation(s) (BCR)
3 (in decreasing order of determinateness)
4 Credibility domain: [-100.00,100.00]
5 === >> potential best choice(s)
6 * choice : ['a1', 'a2', 'a4']
7 independence : 0.00
8 dominance : 6.98
9 absorbency : -48.84
10 covering (%) : 66.67
11 determinateness (%) : 57.97
12 - most credible action(s) =
13 {'a4': 20.93, 'a2': 20.93}
14 === >> potential worst choice(s)
15 * choice : ['a3', 'a7']
16 independence : 0.00
17 dominance : -74.42
18 absorbency : 16.28
19 covered (%) : 80.00
20 determinateness (%) : 64.62
21 - most credible action(s) = { 'a7': 48.84, }

```

Notice in Lines 13 and 21 in Listing 17.14 on the previous page that solving the valued BERGE kernel equations provides furthermore a positive characterisation of the most credible decision alternatives in each respective choice recommendation. Alternatives  $a_2$  and  $a_4$  give equivalent candidates for a unique *first* choice, and alternative  $a_7$  is clearly confirmed as the *last* choice.

In Fig. 17.8 on the facing page, we orient the drawing of the strict outranking digraph instance with the help of these best and worst choice recommendations.

```

1 >>> gcd.exportGraphViz(fileName='bestWorstOrientation', \
2 ...                                bestChoice=['a2','a4'], \
3 ...                                worstChoice=['a7'])
4 *---- exporting a dot file for GraphViz tools ----*
5 Exporting to bestWorstOrientation.dot
6 dot -Grankdir=BT -Tpng bestWorstOrientation.dot \
7                                -o bestWorstOrientation.png

```

It may be interesting to compare this result with a COPELAND ranking of the underlying performance tableau (see Section 8.2 on page 108 on ranking with incommensurable criteria).

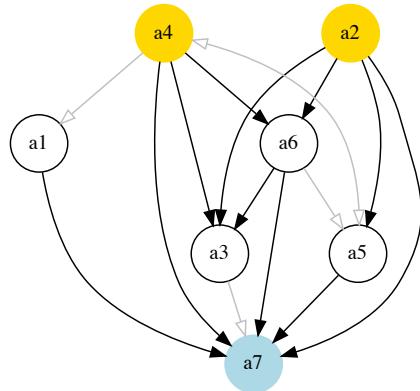
```

1 >>> g.showHTMLPerformanceHeatmap(colorLevels=5, ndigits=0, \
2 ...                                Correlations=True, rankingRule='Copeland')

```

In the resulting linear ranking (see Fig. 17.9 on the facing page), alternative  $a_4$  is set at first rank, followed by alternative  $a_2$ . This makes sense as alternative  $a_4$  shows three performances in the first quintile, whereas  $a_2$  is only partially evaluated and shows only two such excellent performances. But  $a_4$  also shows a very weak performance in the first quintile. Both decision actions, hence, don't show eventually a performance profile that would make apparent a clear preference situation in favour of one or the other. In this sense, the prekernels based best choice recommendations may appear more faithful with respect to the actually definite strict

**Fig. 17.8** The strict outranking digraph oriented by its best and worst choice recommendations. The gray arrows, like the one between alternatives  $a_4$  and  $a_1$ , represent indeterminate preferential situations. Alternative  $a_1$  appears hence to be rather incomparable to all the other, except alternative  $a_7$



*Digraph3 (graphviz), R. Bisdorff, 2020*

**Heatmap of Performance Tableau 'randomOutranking'**

criteria	g4	g7	g5	g6	g1	g2	g3
<b>weights</b>	9	10	6	5	4	8	1
<b>tau<sup>(*)</sup></b>	+0.64	+0.40	+0.29	+0.17	+0.02	-0.05	-0.10
<b>a4</b>	96	32	89	60	47	92	0
<b>a2</b>	87	82	69	7	NA	NA	62
<b>a6</b>	68	70	36	68	54	16	21
<b>a1</b>	98	31	94	15	65	1	14
<b>a5</b>	54	48	30	0	NA	77	87
<b>a3</b>	29	48	97	30	11	28	13
<b>a7</b>	20	6	17	34	57	80	22

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Ranking rule: **Copeland**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.848**

**Fig. 17.9** Heat map with Copeland ranking of the performance tableau

outranking relation than any 'forced' linear ranking result as shown in Fig. 17.9 on the next page.

## 17.5 Tractability of kernel computation

Let us give some hints on the *tractability* of prekernel computations. Detecting all prekernels in a digraph is a famously NP-hard computational problem. Checking external stability conditions for an independent choice is equivalent to checking its

maximality and may be done in the linear complexity of the order of the digraph. However, checking all independent choices contained in a digraph may get hard already for tiny sparse digraphs of order  $n > 20$  (see (Bisdorff, 2006)). Indeed, the worst case is given by an empty or indeterminate digraph where the set of all potential independent choices to check is in fact the power set of the vertices.

```

1 >>> from digraphs import EmptyDigraph
2 >>> e = EmptyDigraph(order=20)
3 >>> e.showMIS()  # by visiting all 2^20 independent choices
4     *--- Maximal independent choices ---*
5     [ '1', '2', '3', '4', '5', '6', '7', '8', '9', '10',
6      '11', '12', '13', '14', '15', '16', '17', '18', '19', '20']
7     number of solutions: 1
8     execution time: 1.47640 sec. # <-----
9 >>> 2**20
10    1048576

```

Now, there exist more efficient specialised algorithms for directly enumerating MISs and dominant or absorbent kernels contained in specific digraph models without visiting all independent choices (see (Bisdorff, 2006)). *Alain Hertz* provided kindly such a MISs enumeration algorithm for the DIGRAPH3 project (the `showMIS_AH()` method). When the number of independent choices is big compared to the actual number of MISs, like in very sparse or empty digraphs, the performance difference may be dramatic (see Line 7 above and Line 15 below).

**Listing 17.15** Enumerating MISs by visiting only maximal independent choices (A. Hertz)

```

1 >>> e.showMIS_AH()
2     # by visiting only maximal independent choices
3     *-----*
4     * Python implementation of Hertz's *
5     * algorithm for generating all MISs *
6     * R.B. version 7(6)-25-Apr-2006 *
7     *-----*
8     =====>>> Initial solution :
9     [ '1', '2', '3', '4', '5', '6', '7', '8', '9', '10', '11',
10      '12', '13', '14', '15', '16', '17', '18', '19', '20']
11     *--- results ---*
12     [ '1', '2', '3', '4', '5', '6', '7', '8', '9', '10', '11',
13      '12', '13', '14', '15', '16', '17', '18', '19', '20']
14     *--- statistics ---*
15     MIS solutions      : 1
16     execution time     : 0.00026 sec. # <-----
17     iteration history: 1

```

For more or less dense strict outranking digraphs of modest order, as facing usually in MCDA applications, enumerating all independent choices remains however in most cases tractable, especially by using a very efficient Python generator with the `independentChoices()` method shown in Listing 17.16.

**Listing 17.16** Generating all independent choices in a digraph

```

1 def independentChoices(self, U):
2     """

```

```

3   Generator for all independent choices with associated
4   dominated, absorbed and independent neighborhoods
5   of digraph instance self.
6   Initiate with U = self.singletons().
7   Yields [(independent choice, domnb, absnb, indnb)].
8   """
9   if U == []:
10      yield [(frozenset(), set(), set(), set(self.actions)) ]
11  else:
12      x = list(U.pop())
13      for S in self.independentChoices(U):
14          yield S
15          if x[0] <= S[0][3]:
16              Sxgamdom = S[0][1] | x[1]
17              Sxgamabs = S[0][2] | x[2]
18              Sxindep = S[0][3] & x[3]
19              Sxchoice = S[0][0] | x[0]
20              Sx = [(Sxchoice, Sxgamdom, Sxgamabs, Sxindep)]
21              yield Sx

```

And, checking maximality of independent choices via the external stability conditions during their enumeration with the `computePreKernels()` method shown in Listing 17.17 provides the effective advantage of computing all initial and terminal prekernels in a single loop (see Line 10 and (Bisdorff, 2006)).

**Listing 17.17** Computing dominant and absorbent preKernels

```

1 def computePreKernels(self):
2     """
3         computing dominant and absorbent preKernels:
4         Result in self.dompreKernels and self.abspreKernels
5         """
6     actions = set(self.actions)
7     n = len(actions)
8     dompreKernels = set()
9     abspreKernels = set()
10    for choice in self.independentChoices(self.singletons()):
11        restactions = actions - choice[0][0]
12        if restactions <= choice[0][1]:
13            dompreKernels.add(choice[0][0])
14        if restactions <= choice[0][2]:
15            abspreKernels.add(choice[0][0])
16        self.dompreKernels = dompreKernels
17        self.abspreKernels = abspreKernels

```

Finally, we use our bipolar-valued epistemic logic framework for computing the credibility that an individual node of the digraph fits with being a member of an initial or terminal prekernel. For this purpose we make use of the kernel equation systems (Berge, 1962; ?).

## 17.6 Solving kernel equation systems

Let  $G(X, R)$  be a crisp irreflexive digraph defined on a finite set  $X$  of nodes and where  $R$  is the corresponding  $\{-1, +1\}$ -valued adjacency matrix. Let  $Y$  be the  $\{-1, +1\}$ -valued membership characteristic (row) vector of a choice in  $X$ .

When  $Y$  satisfies the following equation system:

$$Y \circ R = -Y, \quad (17.1)$$

where for all  $x$  in  $X$ ,

$$(Y \circ R)(x) = \max_{y \in X, x \neq y} (\min(Y(x), R(x, y))), \quad (17.2)$$

then  $Y$  characterises an *initial kernel* (Bisdorff, Pirlot, and Roubens, 2006; Bisdorff, 2006).

When transposing now the membership characteristic vector  $Y$  into a column vector  $Y^t$ , the following equation system:  $R \circ Y^t = -Y^t$ . makes  $Y^t$  similarly characterise a *terminal kernel*.

Let us verify this result in Listing 17.18 on a tiny random digraph.

**Listing 17.18** Verifying the kernel equation system on a tiny random digraph

```

1 >>> from digraphs import RandomDigraph
2 >>> g = RandomDigraph(order=3, seed=1)
3 >>> g.showRelationTable()
4 * ----- Relation Table -----
5   R | 'a1' 'a2' 'a3'
6   -----|-----
7   'a1' | -1   +1   -1
8   'a2' | -1   -1   +1
9   'a3' | +1   +1   -1
10 >>> g.showPreKernels()
11 *---- Computing preKernels ---*
12 Dominant preKernels :
13   ['a3']
14 Absorbent preKernels :
15   ['a2']

```

It is easy to verify by hand that the characteristic vector  $[-1, -1, +1]$  satisfies the initial kernel equation system; node  $a3$  gives an *initial kernel*. Similarly, the characteristic vector  $[-1, +1, -1]$  verifies indeed the terminal kernel equation system and node  $a2$  gives hence a *terminal kernel*.

We succeeded now in generalising BERGE's kernel equation systems to genuine bipolar-valued digraphs (Bisdorff, Pirlot, and Roubens, 2006; Bisdorff, 2006). The constructive proof, found by *Marc Pirlot*, is based on the following *fixpoint equation* that may be used for computing bipolar-valued kernel membership vectors:

$$T(Y) := -(Y \circ R) = Y. \quad (17.3)$$

*John von Neumann* showed indeed that, when a digraph  $G(X, R)$  is acyclic with a unique initial kernel  $K$  characterised by its membership characteristics vector  $Y_k$ , then the following double fixpoint equation:

$$T^2(Y) := -(-(Y \circ R) \circ R) = Y, \quad (17.4)$$

will admit a stable *high* and a stable *low* fixpoint solution that converge both to  $Y_k$  (Schmidt and Ströhlein, 1985).

Inspired by the crisp double fixpoint equation 17.4 on the preceding page, we observed that for a given bipolar-valued digraph  $G(X, R)$ , each one of its dominant or absorbent prekernels  $K_i$  in  $X$  determines an induced partial digraph  $G(X, R/K_i)$  which is acyclic and admits  $K_i$  as unique prekernel (Bisdorff, 2006).

Following the *von Neumann* fixpoint algorithm, a similar bipolar-valued extended double fixpoint algorithm, applied to  $G(X, R/K_i)$ , allows us to compute the associated bipolar-valued kernel characteristic vectors  $Y_i$  in polynomial complexity.

#### Algorithm

*in* : bipolar-valued digraph  $G(X, R)$ ,

*out* : set  $\{Y_1, Y_2, \dots\}$  of bipolar-valued kernel membership characteristic vectors.

1. Enumerate all initial and terminal prekernels  $K_1, K_2, \dots$  in the given bipolar-valued digraph (see List. 17.1 on page 238);
2. For each crisp initial kernel  $K_i$ :
  - a. Construct a partially determined subgraph  $G(X, R/K_i)$  supporting exactly this unique initial kernel  $K_i$ ;
  - b. Use the double fixpoint equation  $T^2$  (Eq. 17.4 on the preceding page) with the partially determined adjacency matrix  $R/K_i$  for computing a stable low and a stable high fixpoint;
  - c. Determine the bipolar-valued  $K_i$ -membership characteristic vector  $Y_i$  with an epistemic disjunction of the previous low and high fixpoints;
3. Repeat step (2) for each terminal kernel  $K_j$  by using the double fixpoint equation  $T^2$  with the transpose of the adjacency matrix  $R/K_j$ .

#### Time for a practical illustration

We reconsider the random digraph  $g$  generated in Listing 17.12 on page 248. Digraph  $g$  models the pairwise outranking situations between seven decision alternatives evaluated on seven incommensurable performance criteria. We recompute its corresponding bipolar-valued prekernels on the associated codual digraph  $gcd$ .

```

1 >>> gcd = ~(-g) # strict outranking digraph
2 >>> gcd.showPreKernels()
3     *--- Computing prekernels ---*
4     Dominant prekernels :
5     ['a1', 'a4', 'a2']
6     independence : +0.000

```

```

7      dominance   : +0.070
8      absorbency  : -0.488
9      covering    : +0.667
10     Absorbent prekernels :
11     ['a7', 'a3']
12     independence : +0.000
13     dominance   : -0.744
14     absorbency  : +0.163
15     covered     : +0.800
16 *----- statistics -----
17     graph name: converse-dual_rel_randomperftab
18     number of solutions
19     dominant kernels : 1
20     absorbent kernels: 1
21     cardinality frequency distributions
22     cardinality   : [0, 1, 2, 3, 4, 5, 6, 7]
23     dominant kernel : [0, 0, 0, 1, 0, 0, 0, 0]
24     absorbent kernel: [0, 0, 1, 0, 0, 0, 0, 0]
25     Execution time : 0.00022 sec.

```

The codual outranking digraph  $gcd$ , modelling a strict outranking relation, admits an initial prekernel  $\{a1, a2, a4\}$  and a terminal one  $\{a3, a7\}$  (see Lines 5 and 11 above).

In Listing 17.19 we now compute, with the `domkernelrestrict()` method, the initial prekernel-restricted adjacency table (see Fig. 17.10).

**Listing 17.19** Computing a dominant prekernel restricted adjacency table

```

1 >>> k1Relation = gcd.domkernelrestrict(['a1','a2','a4'])
2 >>> gcd.showHTMLRelationTable(\n3 ...     actionsList=['a1','a2','a4','a3','a5','a6','a7'],\n4 ...     relation=k1Relation,\n5 ...     tableTitle='K1 restricted adjacency table')

```

**Fig. 17.10** Initial kernel  $\{a1, a2, a4\}$  restricted adjacency table. This initial prekernel is indeed only weakly independent: The outranking situation between alternatives  $a4$  and  $a1$  is indeed *indeterminate*.

## K1 restricted adjacency table

r(x S y)	a1	a2	a4	a3	a5	a6	a7
a1	-	-0.23	-1.00	0.00	0.00	0.00	0.16
a2	-0.21	-	-0.21	0.21	0.44	0.05	0.49
a4	0.00	-0.21	-	0.21	0.00	0.07	0.58
a3	-0.28	-0.21	-0.74	-	0.00	0.00	0.00
a5	-0.26	-0.67	0.00	0.00	-	0.00	0.00
a6	-0.12	-0.49	-0.49	0.00	0.00	-	0.00
a7	-0.51	-0.49	-0.86	0.00	0.00	0.00	-

Valuation domain: [-1.00; +1.00]

The corresponding initial prekernel membership characteristic vector may be computed with the `computeKernelVector()` method.

**Listing 17.20** Fixpoint iterations for initial prekernel  $\{a1, a2, a4\}$

```

1 >>> gcd.computeKernelVector(['a1','a2','a4'], \

```

```

2 ...           Initial=True,Comments=True)
3 --> Initial prekernel: {'a1', 'a2', 'a4'}
4 initial low vector : [-1.00,-1.00,-1.00,-1.00,-1.00,-1.00,-1.00]
5 initial high vector: [+1.00,+1.00,+1.00,+1.00,+1.00,+1.00,+1.00]
6 1st low vector     : [ 0.00,+0.21,-0.21, 0.00,-0.44,-0.07,-0.58]
7 1st high vector   : [+1.00,+1.00,+1.00,+1.00,+1.00,+1.00,+1.00]
8 2nd low vector    : [ 0.00,+0.21,-0.21, 0.00,-0.44,-0.07,-0.58]
9 2nd high vector   : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.05,-0.21]
10 3rd low vector   : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.07,-0.21]
11 3rd high vector  : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.05,-0.21]
12 4th low vector   : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.07,-0.21]
13 4th high vector  : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.07,-0.21]
14 Iterations        : 4
15 low & high fusion : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.07,-0.21]
16 Choice vector for initial prekernel: {'a1', 'a2', 'a4'}
17   'a2': +0.21
18   'a4': +0.21
19   'a1':  0.00
20   'a6': -0.07
21   'a3': -0.21
22   'a5': -0.21
23   'a7': -0.21

```

In Listing 17.20 on the preceding page we start the fixpoint computation with an empty set characterisation as first low vector and a complete set  $X$  characterising high vector. After each iteration, the low vector is set to the negation of the previous high vector and the high vector is set to the negation of the previous low vector.

A unique stable prekernel characteristic vector  $Y_1$  is here attained at the fourth iteration with positive members  $a_2$ : +0.21 and  $a_4$ : +0.21 (60.5% criteria significance majority);  $a_1$ : 0.00 being an ambiguous potential member. Alternatives  $a_3$ ,  $a_5$ ,  $a_6$  and  $a_7$  are all negative members, i.e. positive *non members* of this outranking prekernel.

Let us also compute the restricted adjacency table for the outranked, i.e. the *terminal* prekernel  $\{a_3, a_7\}$ .

```

1 >>> k2Relation = gcd.abskernelrestrict(['a3','a7'])
2 >>> gcd.showHTMLRelationTable(
3 ...     actionsList=['a3','a7','a1','a2','a4','a5','a6'],
4 ...     relation=k2Relation,
5 ...     tableTitle='K2 restricted adjacency table')

```

The corresponding bipolar-valued characteristic vector  $Y_2$  may be computed as follows:

```

1 >>>
2     gcd.computeKernelVector(['a3','a7'],Initial=False,Comments=True)
3 --> Terminal prekernel: {'a3', 'a7'}
4 initial low vector :
5     [-1.00,-1.00,-1.00,-1.00,-1.00,-1.00]
6 initial high vector :
7     [+1.00,+1.00,+1.00,+1.00,+1.00,+1.00]
8 1st low vector     : [-0.16,-0.49,
9     0.00,-0.58,-0.16,-0.30,+0.49]

```

**Fig. 17.11** Terminal prekernel  $\{a_3, a_7\}$  restricted adjacency table. Again, we notice that this terminal prekernel is indeed only weakly independent.

## K2 restricted adjacency table

r(x S y)	a3	a7	a1	a2	a4	a5	a6
<b>a3</b>	-	0.00	-0.28	-0.21	-0.74	-0.40	-0.53
<b>a7</b>	-1.00	-	-0.51	-0.49	-0.86	-0.67	-0.63
<b>a1</b>	0.00	0.16	-	0.00	0.00	0.00	0.00
<b>a2</b>	0.21	0.49	0.00	-	0.00	0.00	0.00
<b>a4</b>	0.21	0.58	0.00	0.00	-	0.00	0.00
<b>a5</b>	0.00	0.16	0.00	0.00	0.00	-	0.00
<b>a6</b>	0.30	0.26	0.00	0.00	0.00	0.00	-

Valuation domain: [-1.00; +1.00]

```

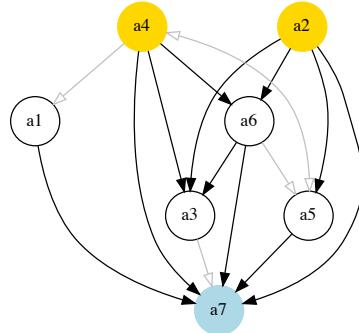
6 1st high vector      :
7   [+1.00,+1.00,+1.00,+1.00,+1.00,+1.00,+1.00]
8 2nd low vector      :
9   0.00,-0.16,-0.49,
10  0.00,-0.58,-0.16,-0.30,+0.49]
11 2nd high vector    :
12   0.00,-0.49,-0.16,-0.26,+0.49]
13 3rd low vector     :
14   0.00,-0.49,-0.16,-0.26,+0.49]
15 3rd high vector    :
16   0.00,-0.49,-0.16,-0.26,+0.49]
17 Iterations          : 3
18 high & low fusion   :
19   0.00,-0.49,-0.16,-0.26,+0.49]
20 Choice vector for terminal prekernel: {'a3','a7'}
21   'a7': +0.49
22   'a3': 0.00
23   'a1': -0.16
24   'a5': -0.16
25   'a6': -0.26
26   'a2': -0.49
27   'a4': -0.49

```

A unique stable bipolar-valued high and low fixpoint is attained at the third iteration with alternative  $a_7$  positively confirmed (about  $(1.0+0.49)/2 = 0.75\%$  criteria significance majority, see Line 14) as member of this terminal prekernel, whereas the membership of alternative  $a_3$  in this prekernel appears indeterminate. All the remaining nodes have negative membership characteristic values and are hence positively excluded from this prekernel.

When we reconsider the graphviz drawing of this outranking digraph in Figure 17.12 on the facing page, it becomes obvious here why alternative  $a_1$  is *neither included nor excluded* from the initial prekernel. Same observation is applicable to alternative  $a_3$  who can *neither be included nor excluded* from the terminal prekernel. It may even happen, in case of more indeterminate outranking situations, that no alternative is positively included or excluded from a weakly independent prekernel; the corresponding bipolar-valued membership characteristic vector being completely indeterminate.

**Fig. 17.12** The strict outranking digraph oriented by its initial and terminal prekernels. Notice the indeterminate outranking situation between alternatives  $a_4$  and  $a_1$ , and the same indeterminate outranking situation between alternatives  $a_3$  and  $a_7$



*Digraph3 (graphviz), R. Bisdorff, 2020*

To illustrate finally why sometimes we need to operate an epistemic disjunctive fusion of unequal stable low and high membership characteristics vectors (see Step 2.c.), let us consider, for instance, the following crisp 7-cycle graph.

```

1 >>> from digraphs import CirculantDigraph
2 >>> c7 = CirculantDigraph(order=7,circulants=[-1,1])
3 >>> c7
4 *----- Digraph instance description -----*
5 Instance class      : CirculantDigraph
6 Instance name       : c7
7 Digraph Order       : 7
8 Digraph Size        : 14
9 Valuation domain    : [-1.00;1.00]
10 Determinateness (%) : 100.00
11 Attributes          : ['name','order','circulants',
12                           'actions','valuationdomain',
13                           'relation','gamma','notGamma']
```

Digraph  $c_7$  is a symmetric crisp digraph showing, among others, the maximal independent set  $\{2', 5', 7'\}$ , i.e. an initial as well as terminal kernel. We may compute the corresponding initial kernel characteristic vector.

```

1 >>> c7.computeKernelVector(['2','5','7'], \
2 ...                           Initial=True,Comments=True)
3 --> Initial kernel: {'2', '5', '7'}
4 initial low vector : [-1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0]
5 initial high vector: [+1.0, +1.0, +1.0, +1.0, +1.0, +1.0, +1.0]
6 1 st low vector   : [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
7 1 st high vector  : [+1.0, +1.0, +1.0, +1.0, +1.0, +1.0, +1.0]
8 2 nd low vector   : [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
9 2 nd high vector  : [ 0.0, +1.0,  0.0,  0.0, +1.0,  0.0, +1.0]
10 stable low vector : [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
11 stable high vector: [ 0.0, +1.0,  0.0,  0.0, +1.0,  0.0, +1.0]
12 Iterations         : 3
```

```

13  low & high fusion  : [-1.0, +1.0, -1.0, -1.0, +1.0, -1.0, +1.0]
14  Choice vector for initial prekernel: {'2', '5', '7'}
15  '7': +1.00
16  '5': +1.00
17  '2': +1.00
18  '6': -1.00
19  '4': -1.00
20  '3': -1.00
21  '1': -1.00

```

Notice that the stable low vector characterises the *negative membership* part, whereas, the stable high vector characterises the *positive membership* part (see Lines 10-11 above). The bipolar epistemic fusion assembles eventually both stable parts into the correct prekernel characteristic vector (Line 13).

The adjacency matrix of a symmetric digraph staying *unchanged* by the transposition operator, the previous computations, when qualifying the same kernel as a terminal instance, will hence produce exactly the same result.

It is worthwhile noticing the essential computational role, the logical indeterminate value 0.0 is playing in this double fixpoint algorithm. To implement such kind of algorithms without a logical *neutral* term would be like implementing numerical algorithms without a possible usage of the number 0. Infinitely many trivial impossibility theorems and dubious logical results come up.

## Notes

Following the observation that an independent absorbent choice in an acyclic digraph corresponds to the kernel of the associated GRUNDY function, [Riguet \(1948\)](#) introduced the name “noyau” (kernel) for such a choice. Terminal kernels where in the sequel studied by [Berge \(1962\)](#) in the context of Combinatorial Game Theory. Initial kernels –independent and dominating choices– were introduced under the name game solutions by [von Neumann and Morgenstern \(1944\)](#).

It is Marc Roubens who in Spring 1995, by commenting a newly published book on *Fuzzy decision theory with binary relations* by [Kitainik \(1993\)](#), suggested that solving fuzzy kernel equation systems would be very difficult. Triggered by his pessimistic remark and knowing about kernel equation systems and the VON NEUMANN fixpoint theorem ([von Neumann and Morgenstern, 1944](#); [Schmidt and Ströhlein, 1985](#)), I immediately started to implement in Prolog a solver for the valued version of such equation systems; the kernel equation system serving as constraints for a discrete labelling of all possible rational solution vectors. And in Summer 1995, we luckily obtained the very first valued outranking kernel from a classical example outranking digraph of order 8 (!) well known in the multiple criteria decision aid community. The computation was done with the help of a commercial finite domain solver. It took several seconds on a CRAY 6412 superserver with 12 processors operating in a nowadays ridiculous CPU speed of 90 Mhz. The labelled solution vectors showing up for a digraph with a single kernel were structured in a

way that suggested the converging stages of the VON NEUMANN fixpoint algorithm and gave the initial idea for eventually computing all kernels of a bipolar-valued outranking digraph (Bisdorff, 1996, 1997; Bisdorff and Roubens, 2004).

In our present Python3.9 implementation, such a tiny problem is solved in less than a thousandth of a second on a common laptop. And this remains practically the same for any relevant example of outranking digraph observed in a real decision aid problem. Several times we wrote in our personal journal that there is certainly now no more potential for any substantial improvement of this computational efficiency; Only to discover, shortly later, that following a new theoretical idea or choosing a more efficient implementation —using for instance the amazing instrument of iterator generators in Python—, execution times could well be divided by 20.

This nowadays available computational efficiency confers the bipolar-valued kernel concept a methodological premium for solving specific decision problems on the basis of the bipolar-valued outranking digraph (see Chapter 4 on page 45 and Chapter 12 on page 161). But it also opens new opportunities for verifying and implementing kernel extraction algorithms for more graph theoretical purposes. New results concerning for instance unlabelled kernels in symmetric graphs could be obtained by (Bisdorff and Marichal, 2008) for enumerating the non isomorphic maximal independent sets —the kernels— of known difficult graph instances like the  $n$ -cycle (see Section 21.7 on page 327).

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# Chapter 18

## On confident outrankings with uncertain criteria significance weights

**Abstract** When modelling preferences following the outranking approach, the signs of the majority margins do sharply distribute validation and invalidation of pairwise outranking situations. How can we be confident in the resulting outranking digraph, when we acknowledge the usual imprecise knowledge of criteria significance weights coupled with small majority margins? To answer this question, one usually requires *qualified* majority margins for confirming outranking situations. But how to choose such a qualifying majority level: two third, three fourth of the significances? In this chapter we propose to link the qualifying significance majority with a required  $\alpha\%$ -confidence level. We model therefore the significance weights as random variables following more or less widespread distributions around an average significance value that corresponds to the given deterministic weight. As the bipolar-valued random credibility of an outranking statement hence results from the simple sum of positive or negative independent random variables, we may apply the Central Limit Theorem (CLT) for computing the bipolar likelihood that the expected majority margin will indeed be positive, respectively negative.

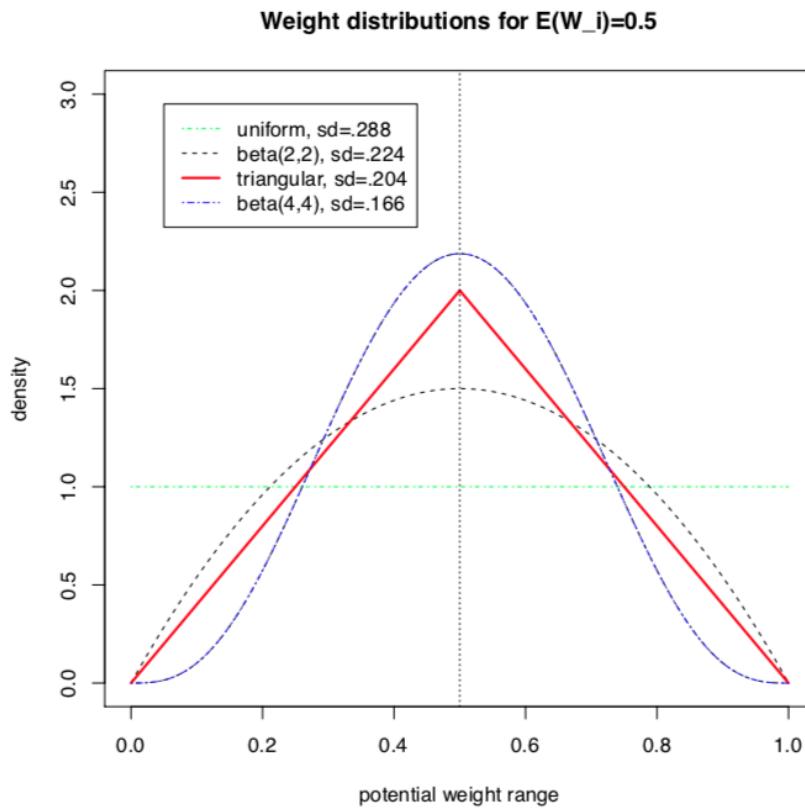
### 18.1 Modelling uncertain criteria significance weights

Let us consider the significance weights of a family  $F$  of  $m$  criteria to be independent random variables  $W_j$ , distributing the potential significance weights of each criterion  $j = 1, \dots, m$  around a mean value  $E(W_j)$  with variance  $V(W_j)$ .

Choosing a specific stochastic model of uncertainty is usually application specific. In the limited scope of this chapter, we will illustrate the consequence of this design decision on the resulting outranking modelling with four slightly different models for taking into account the uncertainty with which we know the numerical significance weights: *uniform*, *triangular*, and two models of *Beta laws*, one more widespread and, the other, more concentrated.

When considering, for instance, that the potential range of a significance weight is distributed between 0 and two times its mean value, we obtain the following random variates (Bisdorff, 2020):

- A continuous *uniform* distribution on the range 0 to  $2E(W_j)$ . Thus  $W_j \sim U(0, 2E(W_j))$  and  $V(W_j) = 1/3(E(W_j))^2$ ;
- A *symmetric beta* distribution with, for instance, parameters  $\alpha = 2$  and  $\beta = 2$ . Thus,  $W_j \sim Beta(2, 2) \times 2E(W_j)$  and  $V(W_j) = 1/5(E(W_j))^2$ ;
- A *symmetric triangular* distribution on the same range with mode  $E(W_j)$ . Thus  $W_j \sim Tr(0, 2E(W_j), E(W_j))$  with  $V(W_j) = 1/6(E(W_j))^2$ ;
- A *narrower beta* distribution with, for instance, parameters  $\alpha = 4$  and  $\beta = 4$ . Thus  $W_j \sim Beta(4, 4) \times 2E(W_j)$ ,  $V(W_j) = 1/9(E(W_j))^2$ .



**Fig. 18.1** Four models of uncertain criteria significance weights

It is worthwhile noticing in Fig. 18.1 that these four uncertainty models all admit the same expected value,  $E(W_j)$ , however, with a respective variance which goes decreasing from  $1/3$ , to  $1/9$  of the square of  $E(W_j)$  (Bisdorff, 2014).

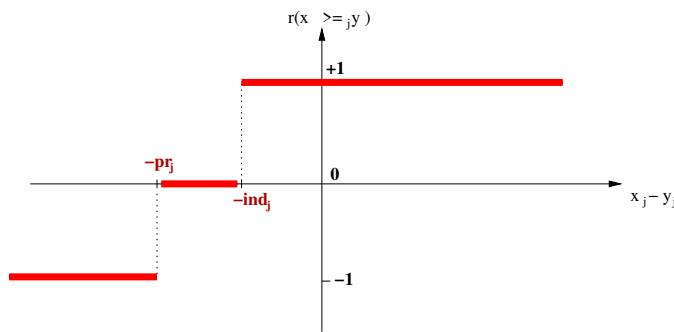
## 18.2 Bipolar-valued likelihood of outranking situations

Let  $A = \{x, y, z, \dots\}$  be a finite set of  $n$  potential decision actions, evaluated on  $F = \{1, \dots, m\}$  – a finite and coherent family of  $m$  performance criteria. On each criterion  $j \in F$ , the decision actions are evaluated on a real performance scale  $[0; M_j]$ , supporting an upper-closed indifference threshold  $ind_j$  and a lower-closed preference threshold  $pr_j$  such that  $0 \leq ind_j < pr_j \leq M_j$ . The marginal performance of object  $x$  on criterion  $j$  is denoted  $x_j$ . Each criterion  $j$  is thus characterising a marginal double threshold order  $\geq_j$  on  $A$ :

$$r(x \geq_j y) = \begin{cases} +1 & \text{if } x_j - y_j \leq -ind_j, \\ -1 & \text{if } x_j - y_j \leq -pr_j, \\ 0 & \text{otherwise.} \end{cases} \quad (18.1)$$

Semantics of the marginal bipolar-valued characteristic function:

- $+1$  signifies  $x$  is performing at least as well as  $y$  on criterion  $j$ ,
- $-1$  signifies that  $x$  is not performing at least as well as  $y$  on criterion  $j$ ,
- $0$  signifies that it is unclear whether, on criterion  $j$ ,  $x$  is performing at least as well as  $y$ .



**Fig. 18.2** Bipolar-valued outranking characteristic function

Each criterion  $j$  in  $F$  contributes the random significance  $W_j$  of his “*performing at least as well as*” characteristic  $r(x \geq_j y)$  to the global characteristic  $\tilde{r}(x \geq y)$  in the

following way:

$$\tilde{r}(x \geq y) = \sum_{j \in F} W_j \times r(x \geq_j y) \quad (18.2)$$

Thus,  $\tilde{r}(x \geq y)$  becomes a simple sum of positive or negative independent random variables with known means and variances where  $\tilde{r}(x \geq y) > 0$  signifies  $x$  is globally performing at least as well as  $y$ ,  $\tilde{r}(x \geq y) < 0$  signifies that  $x$  is not globally performing at least as well as  $y$ , and  $\tilde{r}(x \geq y) = 0$  signifies that it is unclear whether  $x$  is globally performing at least as well as  $y$ .

From the *Central Limit Theorem* (CLT), we know that such a sum of random variables leads, with  $m$  getting large, to a Gaussian distribution  $Y$  with

$$E(Y) = \sum_{j \in F} (E(W_j) \times r(x \geq_j y)), \text{ and} \quad (18.3)$$

$$V(Y) = \sum_{j \in F} (V(W_j) \times |r(x \geq_j y)|). \quad (18.4)$$

And the *likelihood of validation*, respectively *invalidation* of an “*performing at least as well as*” situation, denoted  $lh(x \geq y)$ , may hence be assessed by the probability  $P(Y > 0) = 1.0 - P(Y \leq 0)$  that  $Y$  takes a positive, resp.  $P(Y < 0)$  takes a negative value. In the bipolar-valued case here, we can judiciously make usage of the standard Gaussian *error function*, i.e. the bipolar  $2P(Z) - 1.0$  version of the standard Gaussian  $P(Z)$  probability distribution function (Press et al, 2007):

$$lh(x \geq y) = -\text{erf}\left(\frac{1}{\sqrt{2}} \frac{-E(Y)}{\sqrt{V(Y)}}\right) \quad (18.5)$$

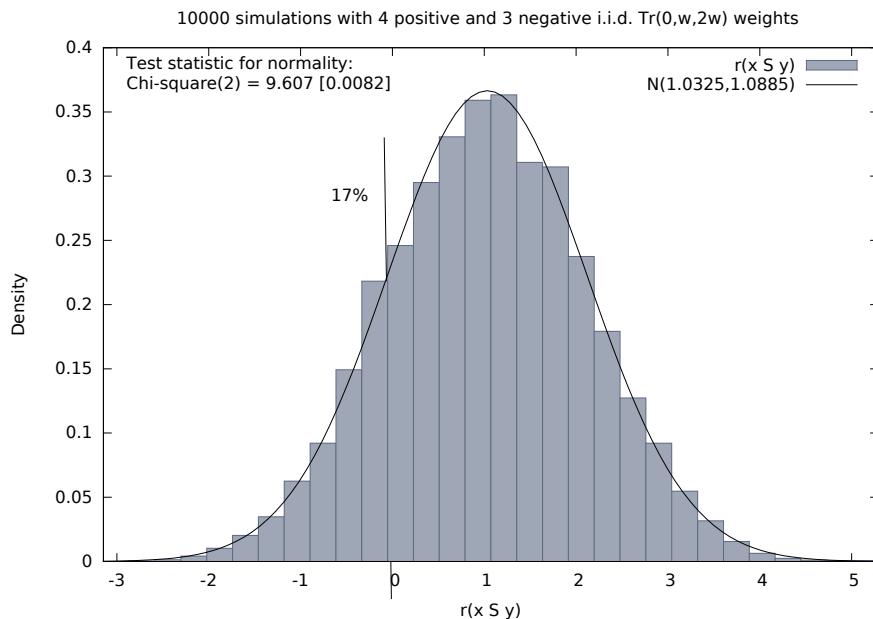
The range of the bipolar-valued  $lh(x \geq y)$  hence becomes  $[-1.0; +1.0]$ , and  $-lh(x \geq y) = lh(x \not\geq y)$ , i.e. a *negative likelihood* represents the likelihood of the correspondent *negated “performing at least as well as”* situation. A likelihood of  $+1.0$  (resp.  $-1.0$ ) means the corresponding preferential situation appears *certainly validated* (resp. invalidated).

*Example 18.1.* Let  $x$  and  $y$  be evaluated wrt 7 equisignificant criteria; Four criteria positively support that  $x$  is *as least as good performing* than  $y$  and three criteria support that  $x$  is *not at least as good* performing than  $y$ . Suppose  $E(W_j) = w$  for  $j = 1, \dots, 7$  and  $W_j \sim Tr(0, 2w, w)$  for  $j = 1, \dots, 7$ . The expected value of the global “*performing at least as well as*” characteristic value becomes:  $E(\tilde{r}(x \geq y)) = 4w - 3w = w$  with a variance  $V(\tilde{r}(x \geq y)) = 7\frac{1}{6}w^2$ .

If  $w = 1$ ,  $E(\tilde{r}(x \geq y)) = 1$  and  $sd(\tilde{r}(x \geq y)) = 1.08$ . By the CLT, the bipolar likelihood of the *at least as good* performing situation becomes:  $lh(x \geq y) = 0.66$ , which corresponds to a global support of  $(0.66 + 1.0)/2 = 83\%$  of the criteria significance weights.

A *Monte Carlo* simulation with 10000 runs empirically confirms the effective convergence to a Gaussian (see Fig. 18.3 realised with *gretl*<sup>1</sup>). Indeed,  $\tilde{r}(x \geq y) \sim Y = \mathcal{N}(1.03, 1.089)$ , with an empirical probability of observing a negative majority margin of about 17%.

<sup>1</sup> The Gnu Regression, Econometrics and Time-series Library <http://gretl.sourceforge.net/>.



**Fig. 18.3** Distribution of 10 000 random outranking characteristic values.

### 18.3 Confidence level of outranking digraphs

**Definition 18.1 (Confident outranking situation).** Following the classical outranking approach (see (Bisdorff, 2013)), we say, from an epistemic perspective, that decision action  $x$  outranks decision action  $y$  at *confidence* level  $\alpha\%$ , when:

- An expected majority of criteria validates, at confidence level  $\alpha\%$  or higher, a global “*performing at least as well as*” situation between  $x$  and  $y$ , and
  - no considerably less performing is observed on a discordant criterion.

Dually, decision action  $x$  *does not outrank* decision action  $y$  at confidence level  $\alpha\%$ , when:

- An expected minority of criteria at confidence level  $-\alpha\%$  or lesser, validates a global “*performing at least as well as*” situation between  $x$  and  $y$ , and
  - no considerably better performing situation is observed on a discordant criterion.

Otherwise, the situation is indeterminate.

**Time for a coded example** Let us consider the following random performance tableau.

```
1 >>> from randomPerfTabs import RandomPerformanceTableau
2 >>> t = RandomPerformanceTableau(
3 ...         numberOfRows=7, \
4 ...         numberOfCriteria=7, seed=100)
5 >>> t.showPerformanceTableau(Transposed=True)
```

***** performance tableau *****								
criteria	weights	'a1'	'a2'	'a3'	'a4'	'a5'	'a6'	'a7'
'g1'	1	15.17	44.51	57.87	58.00	24.22	29.10	96.58
'g2'	1	82.29	43.90	NA	35.84	29.12	34.79	62.22
'g3'	1	44.23	19.10	27.73	41.46	22.41	21.52	56.90
'g4'	1	46.37	16.22	21.53	51.16	77.01	39.35	32.06
'g5'	1	47.67	14.81	79.70	67.48	NA	90.72	80.16
'g6'	1	69.62	45.49	22.03	33.83	31.83	NA	48.80
'g7'	1	82.88	41.66	12.82	21.92	75.74	15.45	6.05

For the corresponding confident outranking digraph, we assume uncertain criteria significance weights of *triangular* distribution and require a confidence level of  $\alpha = 90\%$ .

The `ConfidentBipolarOutrankingDigraph` class provides such a construction.

```

1 >>> from outrankingDigraphs import\
2 ...           ConfidentBipolarOutrankingDigraph
3 >>> g90 = ConfidentBipolarOutrankingDigraph(t,\ 
4 ...           distribution ='triangular',confidence=90)
5 >>> g90
6 *----- Object instance description -----*
7 Instance class      : ConfidentBipolarOutrankingDigraph
8 Instance name       : rel_randomperftab_CLT
9 Actions             : 7
10 Criteria            : 7
11 Size                : 15
12 Uncertainty model  : triangular(a=0,b=2w)
13 Likelihood domain   : [-1.0;+1.0]
14 Confidence level    : 0.80 (90.0%)
15 Confident majority  : 0.14 (57.1%)
16 Determinateness (%) : 62.07
17 Valuation domain    : [-1.00;1.00]
18 Attributes          : ['name', 'bipolarConfidenceLevel',
19 ...                      'distribution', 'betaParameter', 'actions',
20 ...                      'order', 'valuationdomain', 'criteria',
21 ...                      'evaluation', 'concordanceRelation',
22 ...                      'vetos', 'negativeVetos',
23 ...                      'largePerformanceDifferencesCount',
24 ...                      'likelihoods', 'confidenceCutLevel',
25 ...                      'relation', 'gamma', 'notGamma']
```

The resulting 90%-confident expected outranking relation is shown below.

**Listing 18.1** 90%-confident outranking relation with triangular distributed significance weights

```

1 >>> g90.showRelationTable(LikelihoodDenotation=True)
2 * ---- Outranking Relation Table ----
3 r/(lh) | 'a1'  'a2'  'a3'  'a4'  'a5'  'a6'  'a7'
4 -----
5 'a1' | +0.00  +0.71  +0.29  +0.29  +0.29  +0.29  +0.00
6 | ( - )  (+1.00) (+0.95) (+0.95) (+0.95) (+0.95) (+0.65)
7 'a2' | -0.71  +0.00  -0.29  +0.00  +0.00  +0.29  -0.57
8 | (-1.00) ( - )  (-0.95) (-0.65) (+0.73) (+0.95) (-1.00)
9 'a3' | -0.29  +0.29  +0.00  -0.29  +0.00  +0.00  -0.29
10 | (-0.95) (+0.95) ( - )  (-0.95) (-0.73) (-0.00) (-0.95)
11 'a4' | +0.00  +0.00  +0.57  +0.00  +0.29  +0.57  -0.43
12 | (-0.00) (+0.65) (+1.00) ( - )  (+0.95) (+1.00) (-0.99)
```

```

13   'a5' | -0.29  +0.00  +0.00  +0.00  +0.00  +0.29  -0.29
14   | (-0.95) (-0.00) (+0.73) (-0.00) ( - ) (+0.99) (-0.95)
15   'a6' | -0.29  +0.00  +0.00  -0.29  +0.00  +0.00  +0.00
16   | (-0.95) (-0.00) (+0.73) (-0.95) (+0.73) ( - ) (-0.00)
17   'a7' | +0.00  +0.71  +0.57  +0.43  +0.29  +0.00  +0.00
18   | (-0.65) (+1.00) (+1.00) (+0.99) (+0.95) (-0.00) ( - )
19   Valuation domain : [-1.000; +1.000]
20   Uncertainty model : triangular(a=2.0,b=2.0)
21   Likelihood domain : [-1.0;+1.0]
22   Confidence level : 0.80 (90.0%)
23   Confident majority : 0.14 (57.1%)
24   Determinateness : 0.24 (62.1%)

```

The ( $lh$ ) figures, indicated in the table above, correspond to bipolar likelihoods and the required bipolar confidence level equals  $(0.90 + 1.0)/2 = 0.80$  (see Line 22 above). Action  $a_1$  thus confidently outranks all other actions, except  $a_7$  where the actual likelihood ( $+0.65$ ) is lower than the required one ( $0.80$ ) and we furthermore observe a considerable counter-performance on criterion  $g_1$ .

Notice also the lack of confidence in the outranking situations we observe between action  $a_2$  and actions  $a_4$  and  $a_5$ . In the deterministic case we would have  $r(a_2 \geq a_4) = -0.143$  and  $r(a_2 \geq a_5) = +0.143$ . All outranking situations with a characteristic value lower or equal to  $abs(0.143)$ , i.e. a majority support of  $1.143/2 = 57.1\%$  and less, appear indeed to be *not confident* at  $\alpha$  level 90% (see Line 23 above).

Fig. 18.4 shows the corresponding strict 90%-confident outranking digraph, oriented by its initial and terminal strict prekernels.

```

1 >>> gcd90 = ~ (-g90)
2 >>> gcd90.showPreKernels()
3 *--- Computing preKernels ---*
4 Dominant preKernels :
5   ['a1', 'a7']
6     independence : 0.0
7     dominance : 0.2857
8     absorbency : -0.7143
9     covering : 0.800
10 Absorbent preKernels :
11   ['a2', 'a5', 'a6']
12     independence : 0.0
13     dominance : -0.2857
14     absorbency : 0.2857
15     covered : 0.583
16 >>> gcd90.exportGraphViz(fileName='confidentOutranking',
17 ...     bestChoice=['a1', 'a7'], \
18 ...     worstChoice=['a2', 'a5', 'a6'])
19 *---- exporting a dot file for GraphViz tools ----*
20   Exporting to confidentOutranking.dot
21   dot -Grankdir=BT -Tpng confidentOutranking.dot \
22     -o confidentOutranking.png

```

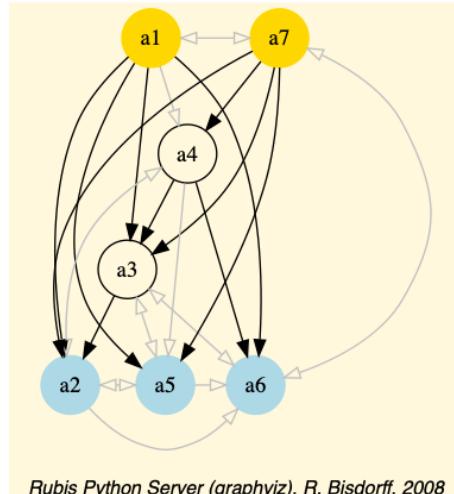
Now, what becomes this 90%-confident outranking digraph when we require a stronger confidence level of, say 99%?

**Listing 18.2** 99%-confident outranking relation

```

1 >>> g99 = ConfidentBipolarOutrankingDigraph(t,\
```

**Fig. 18.4** 90%-confident strict outranking digraph oriented by its initial and terminal prekernels



Rubis Python Server (graphviz), R. Bisdorff, 2008

```

2 ...      distribution='trinangular',confidence=99)
3 >>> g99.showRelationTable()
4 * ----- Outranking Relation Table -----
5 r/(lh) | 'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'
6 -----
7 'a1' | +0.00  +0.71  +0.00  +0.00  +0.00  +0.00  +0.00
8 | ( - )  (+1.00) (+0.95) (+0.95) (+0.95) (+0.95) (+0.65)
9 'a2' | -0.71  +0.00  +0.00  +0.00  +0.00  +0.00  -0.57
10 | (-1.00) ( - )  (-0.95) (-0.65) (+0.73) (+0.95) (-1.00)
11 'a3' | +0.00  +0.00  +0.00  +0.00  +0.00  +0.00  +0.00
12 | (-0.95) (+0.95) ( - )  (-0.95) (-0.73) (-0.00) (-0.95)
13 'a4' | +0.00  +0.00  +0.57  +0.00  +0.00  +0.57  -0.43
14 | (-0.00) (+0.65) (+1.00) ( - )  (+0.95) (+1.00) (-0.99)
15 'a5' | +0.00  +0.00  +0.00  +0.00  +0.00  +0.29  +0.00
16 | (-0.95) (-0.00) (+0.73) (-0.00) ( - )  (+0.99) (-0.95)
17 'a6' | +0.00  +0.00  +0.00  +0.00  +0.00  +0.00  +0.00
18 | (-0.95) (-0.00) (+0.73) (-0.95) (+0.73) ( - )  (-0.00)
19 'a7' | +0.00  +0.71  +0.57  +0.43  +0.00  +0.00  +0.00
20 | (-0.65) (+1.00) (+1.00) (+0.99) (+0.95) (-0.00) ( - )
21 Valuation domain : [-1.000; +1.000]
22 Uncertainty model : triangular(a=2.0,b=2.0)
23 Likelihood domain : [-1.0;+1.0]
24 Confidence level : 0.98 (99.0%)
25 Confident majority : 0.29 (64.3%)
26 Determinateness : 0.13 (56.6%)

```

At 99% confidence, the minimal required significance majority support amounts to 64.3% (see Listing 18.2 Line 25 above). As a result, most outranking situations don't get anymore validated, like the outranking situations between action a1 and actions a3, a4, a5 and a6 (see Line 5 above). The overall epistemic determination of the digraph consequently drops from .1% to 56.6% (see Line 25).

Finally, what becomes the previous 90%-confident outranking digraph when the uncertainty concerning the criteria significance weights is modelled with a larger variance, like *uniform* variates (see Line 2 below).

**Listing 18.3** 90%-confident outranking relation with uniform variates

```

1 >>> gu90 = ConfidentBipolarOutrankingDigraph(t,\n2 ...           confidence=90,distribution='uniform')\n3 >>> gu90.showRelationTable()\n4   * ---- Outranking Relation Table -----\n5   r/(lh) |  'a1'  'a2'  'a3'  'a4'  'a5'  'a6'  'a7'\n6   -----|-----\n7   'a1' | +0.00  +0.71  +0.29  +0.29  +0.29  +0.29  +0.00\n8   | ( - )  (+1.00) (+0.84) (+0.84) (+0.84) (+0.84) (+0.49)\n9   'a2' | -0.71  +0.00  -0.29  +0.00  +0.00  +0.29  -0.57\n10  | (-1.00) ( - )  (-0.84) (-0.49) (+0.56) (+0.84) (-1.00)\n11  'a3' | -0.29  +0.29  +0.00  -0.29  +0.00  +0.00  -0.29\n12  | (-0.84) (+0.84) ( - )  (-0.84) (-0.56) (-0.00) (-0.84)\n13  'a4' | +0.00  +0.00  +0.57  +0.00  +0.29  +0.57  -0.43\n14  | (-0.00) (+0.49) (+1.00) ( - )  (+0.84) (+1.00) (-0.95)\n15  'a5' | -0.29  +0.00  +0.00  +0.00  +0.00  +0.29  -0.29\n16  | (-0.84) (-0.00) (+0.56) (-0.00) ( - )  (+0.92) (-0.84)\n17  'a6' | -0.29  +0.00  +0.00  -0.29  +0.00  +0.00  +0.00\n18  | (-0.84) (-0.00) (+0.56) (-0.84) (+0.56) ( - )  (-0.00)\n19  'a7' | +0.00  +0.71  +0.57  +0.43  +0.29  +0.00  +0.00\n20  | (-0.49) (+1.00) (+1.00) (+0.95) (+0.84) (-0.00) ( - )\n21  Valuation domain  : [-1.000; +1.000]\n22  Uncertainty model : uniform(a=2.0,b=2.0)\n23  Likelihood domain : [-1.0;+1.0]\n24  Confidence level   : 0.80 (90.0%)\n25  Confident majority : 0.14 (57.1%)\n26  Determinateness    : 0.24 (62.1%)

```

Despite lower likelihood values (see the *g90* relation in Listing table above), we keep the same confident majority level of 57.1% (see Listing 18.3 Line 25 above) and, hence, also the same 90%-confident outranking digraph.

For concluding, it is worthwhile noticing again that it is in fact the *neutral* value of our bipolar-valued epistemic logic that allows us to easily handle  $\alpha\%$  confidence or not of outranking situations when confronted with uncertain criteria significances. Remarkable furthermore is the usage, the standard Gaussian error function (erf) provides by delivering *signed* likelihood values immediately concerning either a positive relational statement, or when negative, its negated version.

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# Chapter 19

## Robustness analysis of outranking digraphs

**Abstract** The required cardinal significance weights of the performance criteria represent the '*Achilles*' heel of our outranking approach. Rarely will indeed a decision maker be cognitively competent for suggesting precise decimal-valued criteria significance weights. More often, the decision problem will involve more or less equally important decision objectives with more or less equi-significant criteria per objective. In this chapter, we study the robustness of the outranking digraph when the criteria significance weights faithfully indicate solely an order of importance.

### 19.1 Cardinal or ordinal criteria significance weights?

A random example of such a decision problem with equally important decision objectives and equi-significant criteria may be generated with the Random3ObjectivesPerformanceTableau class .

**Listing 19.1** Generate a Random 3 Objectives Performance Tableau

```
1 >>> from randomPerfTabs import \
2 ...           Random3ObjectivesPerformanceTableau
3 >>> pt = Random3ObjectivesPerformanceTableau(\
4 ...           numberOfActions=7, \
5 ...           numberOfCriteria=9, seed=102)
6 >>> pt
7      ----- PerformanceTableau instance description -----
8      Instance class    : Random3ObjectivesPerformanceTableau
9      Seed              : 102
10     Instance name    : random3ObjectivesPerfTab
11     Actions           : 7
12     Objectives        : 3
13     Criteria          : 9
14     NA proportion (%): 0.0
15     Attributes        : ['name', 'valueDigits', 'BigData',
16                           'OrdinalScales', 'missingDataProbability',
17                           'negativeWeightProbability', 'randomSeed',
```

```

18         'sumWeights', 'valuationPrecision',
19         'commonScale', 'objectiveSupportingTypes',
20         'actions', 'objectives', 'criteriaWeightMode',
21         'criteria', 'evaluation', 'weightPreorder']
22 >>> pt.showObjectives()
23 *----- show objectives -----*
24 Eco: Economical aspect
25     ec1 criterion of objective Eco 8
26     ec4 criterion of objective Eco 8
27     ec8 criterion of objective Eco 8
28     Total weight: 24.00 (3 criteria)
29 Soc: Societal aspect
30     so2 criterion of objective Soc 12
31     so7 criterion of objective Soc 12
32     Total weight: 24.00 (2 criteria)
33 Env: Environmental aspect
34     en3 criterion of objective Env 6
35     en5 criterion of objective Env 6
36     en6 criterion of objective Env 6
37     en9 criterion of objective Env 6
38     Total weight: 24.00 (4 criteria)

```

In Listing 19.1 one may notice a performance tableau `pt` describing seven decision alternatives that are assessed with respect to three equally important decision objectives concerning:

1. An *economical* aspect (Line 24) with a coalition of three performance criteria of significance weight 8,
2. A *societal* aspect (Line 29) with a coalition of two performance criteria of significance weight 12, and
3. An *environmental* aspect (Line 33) with a coalition of four performance criteria of significance weight 6.

The question we tackle is the following: How *dependent* on the actual values of the significance weights appears to be the corresponding bipolar-valued outranking digraph? In the previous Chapter 18 we assumed that the criteria significance weights were random variables. Here, we shall assume that we know for sure only the preordering of the significance weights. In the random outranking digraph, generated in List. 19.1 on the preceding page, we observe three increasing weight equivalence classes.

**Listing 19.2** The significance weights preorder

```

1 >>> pt.showWeightPreorder()
2     ['en3', 'en5', 'en6', 'en9'] (6) <
3     ['ec1', 'ec4', 'ec8'] (8) <
4     ['so2', 'so7'] (12)

```

How robust appear now the validated outranking and outranked situations when allowing all possible significance weights that are compatible with the weights preorder shown in Listing 19.2 above?

## 19.2 Qualifying the stability of outranking situations

Let us construct the bipolar-valued outranking digraph corresponding to the random 3-Objectives performance tableau  $pt$ .

**Listing 19.3** Example Bipolar Outranking Digraph

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> g = BipolarOutrankingDigraph(pt)
3 >>> g.showRelationTable()
4 * ----- Relation Table -----
5   r(>=) | 'p1'  'p2'  'p3'  'p4'  'p5'  'p6'  'p7'
6 -----
7   'p1' | +1.00 -0.42 +0.00 -0.69 +0.39 +0.11 -0.06
8   'p2' | +0.58 +1.00 +0.83 +0.00 +0.58 +0.58 +0.58
9   'p3' | +0.25 -0.33 +1.00 +0.00 +0.50 +1.00 +0.25
10  'p4' | +0.78 +0.00 +0.61 +1.00 +1.00 +1.00 +0.67
11  'p5' | -0.11 -0.50 -0.25 -0.89 +1.00 +0.11 -0.14
12  'p6' | +0.22 -0.42 +0.00 -1.00 +0.17 +1.00 -0.11
13  'p7' | +0.22 -0.50 +0.17 -0.06 +0.78 +0.42 +1.00

```

In Listing 19.3 Lines 7-13, we notice on the principal diagonal of the relation table the certainly validated reflexive terms  $+1.00$ ; they are trivially independent of any significance weights. Now, we know for sure that *unanimous* outranking situations are also completely independent of the significance weights. And, all outranking situations that are supported by a majority significance in each coalition of equi-significant criteria are as well independent of the actual importance we attach to each individual criteria coalition. We are furthermore able to effectively test if an outranking situation is in fact independent of all the potential significance weights that are compatible with the given preordering of the weights shown in Listing 19.2 (Bisdorff, 2014). Mind that usually one also obtains outranking situations that are *dependent* on the precise cardinal weights we allocate to the criteria significances.

Such a stability denotation of outranking situations may be inspected by using the `StabilityDenotation=True` flag with the common `showRelationTable()` method.

**Listing 19.4** Bipolar-valued outranking relation table with stability denotation

```

1 >>> g.showRelationTable(StabilityDenotation=True)
2 * ----- Relation Table -----
3   r/(stab) | 'p1'  'p2'  'p3'  'p4'  'p5'  'p6'  'p7'
4 -----
5   'p1'    | +1.00 -0.42 +0.00 -0.69 +0.39 +0.11 -0.06
6   | (+4)  (-2)  (+0)  (-3)  (+2)  (+2)  (-1)
7   'p2'    | +0.58 +1.00 +0.83  0.00 +0.58 +0.58 +0.58
8   | (+2)  (+4)  (+3)  (+2)  (+2)  (+2)  (+2)
9   'p3'    | +0.25 -0.33 +1.00  0.00 +0.50 +1.00 +0.25
10  | (+2)  (-2)  (+4)  (0)  (+2)  (+2)  (+1)
11  'p4'    | +0.78  0.00 +0.61 +1.00 +1.00 +1.00 +0.67
12  | (+3)  (-1)  (+3)  (+4)  (+4)  (+4)  (+2)
13  'p5'    | -0.11 -0.50 -0.25 -0.89 +1.00 +0.11 -0.14
14  | (-2)  (-2)  (-2)  (-3)  (+4)  (+2)  (-2)
15  'p6'    | +0.22 -0.42  0.00 -1.00 +0.17 +1.00 -0.11

```

16		(+2) (-2) (+1) (-2) (+2) (+4) (-2)
17	'p7'	+0.22 -0.50 +0.17 -0.06 +0.78 +0.42 +1.00
18		(+2) (-2) (+1) (-1) (+3) (+2) (+4)

In Listing 19.4 we may hence distinguish the following bipolar-valued stability levels:

- ±4: *unanimous* outranking, resp. outranked situation. The pairwise trivial reflexive outrankings, for instance, all show this stability level;
- ±3: validated outranking, resp. outranked situation in *each* coalition of equisignificant criteria. This is, for instance, the case for the outranking situation observed between alternatives p1 and p4 (see Listing 19.4 Lines 6 and 12);
- ±2: validated outranking, resp. outranked situation with *all* potential significance weights that are *compatible* With the given significance preorder (see Listing 19.2). This is the case when comparing alternatives p1 and p2 (see Lines 6 and 8);
- ±1: validated outranking, resp. outranked situation with the *precisely given decimal* significance weights, a situation we may observe between alternatives p3 and p7 (see Lines 10 and 16);
- 0: indeterminate outranking situation, like the one between alternatives p1 and p3 (see Lines 6 and 10).

It is worthwhile noticing that, in the one limit case where all performance criteria appear equi-significant –there is given a single equivalence class containing all the criteria significance weights– one may only distinguish stability levels ±4 and ±3. In the other limit case, when all performance criteria admit different significance weights, the significance weights may be linearly ordered without ties and no stability level +3 or -3 can be observed.

As mentioned above, all reflexive comparisons trivially confirm an unanimous outranking situation: all decision alternatives are indeed always “*performing as well as*” themselves. But there appear also two non reflexive unanimous outranking situations: when comparing, for instance, alternative p4 with alternatives p5 and p6 (see Listing 19.4 Lines 14 and 16). Let us inspect the details of how alternatives p4 and p5 compare.

```

1 >>> g.showPairwiseComparison('p4','p5')
2 *----- pairwise comparison -----*
3 Comparing actions : (p4, p5)
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ecl 8.00 85.19 46.75 +38.44 | 5.00 10.00 +8.00
6 ec4 8.00 72.26 8.96 +63.30 | 5.00 10.00 +8.00
7 ec8 8.00 44.62 35.91 +8.71 | 5.00 10.00 +8.00
8 en3 6.00 80.81 31.05 +49.76 | 5.00 10.00 +6.00
9 en5 6.00 49.69 29.52 +20.17 | 5.00 10.00 +6.00
10 en6 6.00 66.21 31.22 +34.99 | 5.00 10.00 +6.00
11 en9 6.00 50.92 9.83 +41.09 | 5.00 10.00 +6.00
12 so2 12.00 49.05 12.36 +36.69 | 5.00 10.00 +12.00
13 so7 12.00 55.57 44.92 +10.65 | 5.00 10.00 +12.00
14 Valuation in range: -72.00 to +72.00; -----
15 global concordance: +72.00

```

Alternative  $p_4$  is indeed performing unanimously “*at least as well as*” alternative  $p_5$  and  $r(p_4 \succsim p_5) = 72/72 = +1.00$  (see Listing 19.4 Line 11).

The converse comparison does not, however, deliver such an unanimous outranked situation.

```

1 >>> g.showPairwiseComparison('p5','p4')
2 *----- pairwise comparison ----*
3 Comparing actions : (p5, p4)
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ec1 8.00 46.75 85.19 -38.44 | 5.00 10.00 -8.00
6 ec4 8.00 8.96 72.26 -63.30 | 5.00 10.00 -8.00
7 ec8 8.00 35.91 44.62 -8.71 | 5.00 10.00 +0.00
8 en3 6.00 31.05 80.81 -49.76 | 5.00 10.00 -6.00
9 en5 6.00 29.52 49.69 -20.17 | 5.00 10.00 -6.00
10 en6 6.00 31.22 66.21 -34.99 | 5.00 10.00 -6.00
11 en9 6.00 9.83 50.92 -41.09 | 5.00 10.00 -6.00
12 so2 12.00 12.36 49.05 -36.69 | 5.00 10.00 -12.00
13 so7 12.00 44.92 55.57 -10.65 | 5.00 10.00 -12.00
14 Valuation in range: -72.00 to +72.00; -----
15 global concordance: -64.00

```

The converse comparison only qualifies at stability level  $-3$  (see Listing 19.4 Line 13);  $r(p_5 \succsim p_4) = -64/72 = -0.89$ . On criterion  $ec8$  we observe indeed a small negative performance difference of  $-8.71$  (see Line 7 above) which is effectively below the supposed preference discrimination threshold of  $10.00$ . Yet, the outranked situation is supported by a majority of criteria in each decision objective. Hence, the reported preferential situation is completely independent of any chosen significance weights.

Let us now consider a comparison, like the one between alternatives  $p_2$  and  $p_1$ , that is qualified at stability level  $+2$ , resp.  $-2$ .

**Listing 19.5** Comparison of alternatives  $p_2$  and  $p_1$

```

1 >>> g.showPairwiseOutrankings('p2','p1')
2 *----- pairwise comparison ----*
3 Comparing actions : (p2, p1)
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ec1 8.00 89.77 38.11 +51.66 | 5.00 10.00 +8.00
6 ec4 8.00 86.00 22.65 +63.35 | 5.00 10.00 +8.00
7 ec8 8.00 89.43 77.02 +12.41 | 5.00 10.00 +8.00
8 en3 6.00 20.79 58.16 -37.37 | 5.00 10.00 -6.00
9 en5 6.00 23.83 31.40 -7.57 | 5.00 10.00 +0.00
10 en6 6.00 18.66 11.41 +7.25 | 5.00 10.00 +6.00
11 en9 6.00 26.65 44.37 -17.72 | 5.00 10.00 -6.00
12 so2 12.00 89.12 22.43 +66.69 | 5.00 10.00 +12.00
13 so7 12.00 84.73 28.41 +56.32 | 5.00 10.00 +12.00
14 Valuation in range: -72.00 to +72.00; -----
15 global concordance: +42.00
16 *----- pairwise comparison ----*
17 Comparing actions : ('p1', 'p2')
18 crit. wght. g(x) g(y) diff | ind pref r()
19 ec1 8.00 38.11 89.77 -51.66 | 5.00 10.00 -8.00
20 ec4 8.00 22.65 86.00 -63.35 | 5.00 10.00 -8.00

```

```

21      ec8    8.00  77.02  89.43  -12.41 | 5.00  10.00  -8.00
22      en3    6.00  58.16  20.79  +37.37 | 5.00  10.00  +6.00
23      en5    6.00  31.40  23.83  +7.57 | 5.00  10.00  +6.00
24      en6    6.00  11.41  18.66  -7.25 | 5.00  10.00  +0.00
25      en9    6.00  44.37  26.65  +17.72 | 5.00  10.00  +6.00
26      so2   12.00  22.43  89.12  -66.69 | 5.00  10.00  -12.00
27      so7   12.00  28.41  84.73  -56.32 | 5.00  10.00  -12.00
28      Valuation in range: -72.00 to +72.00;  -----
29                                         global concordance: -30.00

```

In both comparisons, the performances observed with respect to the environmental decision objective are not validating with a significant majority the outranking, resp. outranked situation. Hence, the stability of the reported preferential situations is in fact dependent on choosing significance weights that are compatible with the given significance weights preorder (see Listing 19.2).

Let us finally inspect a comparison that is only qualified at stability level +1, like the one between alternatives p7 and p3.

**Listing 19.6** Comparison of alternatives p7 and p3

```

1 >>> g.showPairwiseOutrankings('p7','p3')
2 *----- pairwise comparison -----*
3 Comparing actions : ('p7', 'p3')
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ec1 8.00 15.33 80.19 -64.86 | 5.00 10.00 -8.00
6 ec4 8.00 36.31 68.70 -32.39 | 5.00 10.00 -8.00
7 ec8 8.00 38.31 91.94 -53.63 | 5.00 10.00 -8.00
8 en3 6.00 30.70 46.78 -16.08 | 5.00 10.00 -6.00
9 en5 6.00 35.52 27.25 +8.27 | 5.00 10.00 +6.00
10 en6 6.00 69.71 1.65 +68.06 | 5.00 10.00 +6.00
11 en9 6.00 13.10 14.85 -1.75 | 5.00 10.00 +6.00
12 so2 12.00 68.06 58.85 +9.21 | 5.00 10.00 +12.00
13 so7 12.00 58.45 15.49 +42.96 | 5.00 10.00 +12.00
14 Valuation in range: -72.00 to +72.00;  -----
15                                         global concordance: +12.00
16 *----- pairwise comparison -----*
17 Comparing actions : ('p3', 'p7')
18 crit. wght. g(x) g(y) diff | ind pref r()
19 ec1 8.00 80.19 15.33 +64.86 | 5.00 10.00 +8.00
20 ec4 8.00 68.70 36.31 +32.39 | 5.00 10.00 +8.00
21 ec8 8.00 91.94 38.31 +53.63 | 5.00 10.00 +8.00
22 en3 6.00 46.78 30.70 +16.08 | 5.00 10.00 +6.00
23 en5 6.00 27.25 35.52 -8.27 | 5.00 10.00 +0.00
24 en6 6.00 1.65 69.71 -68.06 | 5.00 10.00 -6.00
25 en9 6.00 14.85 13.10 +1.75 | 5.00 10.00 +6.00
26 so2 12.00 58.85 68.06 -9.21 | 5.00 10.00 +0.00
27 so7 12.00 15.49 58.45 -42.96 | 5.00 10.00 -12.00
28 Valuation in range: -72.00 to +72.00;  -----
29                                         global concordance: +18.00

```

In both cases, choosing only significances that are compatible with the given weights preorder will not always result in positively validated outranking situations.

### 19.3 Computing the stability denotation of outranking situations

Stability levels  $\pm 4$  and  $\pm 3$  are, the case given, easy to detect. Detecting a stability level  $\pm 2$  is far less obvious. Now, it is precisely again the bipolar-valued epistemic characteristic domain that will give us a way to implement an effective test for stability level  $+2$  and  $-2$  (Bisdorff, 2004a,b).

Let us consider the significance equivalence classes we observe in the given weights preorder. Here we observe three weight classes: 6, 8, and 12, in increasing order (see Listing 19.2). In the pairwise comparisons, shown above, these equivalence classes may appear positively or negatively, besides the indeterminate significance of value 0.00. We thus get the following ordered bipolar list of significance weights:  $W = [-12, -8, -6, 0, 6, 8, 12]$ .

In all the pairwise marginal comparisons shown in the previous Section, we may observe that each one of the nine criteria assigns one precise item out of this list  $W$ . Let us denote  $q[i]$  the number of criteria assigning item  $W[i]$ , and  $Q[i]$  the cumulative sums of these  $q[i]$  counts, where  $i$  is an index in the range of the length of list  $W$ .

In the comparison of alternatives  $p_2$  and  $p_1$ , for instance (see Listing 19.5), we observe the following counts:

$W[i]$	-12	-8	-6	0	6	8	12
$q[i]$	0	0	2	1	1	3	2
$Q[i]$	0	0	2	3	4	7	9

Let us denote  $-q$  and  $-Q$  the reversed versions of the  $q$  and the  $Q$  lists. We thus obtain the following result.

$W[i]$	-12	-8	-6	0	6	8	12
$-q[i]$	2	3	1	1	2	0	0
$-Q[i]$	2	5	6	7	9	9	9

Now, a pairwise outranking situation will be qualified at stability level  $+2$ , i.e. positively validated with any significance weights that are compatible with the given weights preorder, when for all  $i$ , we observe  $Q[i] \leq -Q[i]$  and there exists one  $i$  such that  $Q[i] < -Q[i]$ . Similarly, a pairwise outranked situation will be qualified at stability level  $-2$ , when for all  $i$ , we observe  $Q[i] \geq -Q[i]$  and there exists one  $i$  such that  $Q[i] > -Q[i]$  (Bisdorff, 2004b).

We may verify, for instance, that the outranking situation observed between alternatives  $p_2$  and  $p_1$  does indeed verify this *first order distributional dominance* condition.

$W[i]$	-12	-8	-6	0	6	8	12
$Q[i]$	0	0	2	3	4	7	9
$-Q[i]$	2	5	6	7	9	9	9

Notice that outranking situations qualified at stability levels  $\pm 4$  and  $\pm 3$ , evidently also verify the stability level  $\pm 2$  test above. The outranking situation between alternatives  $p_7$  and  $p_3$  does not, however, verify this test (see Listing 19.6).

$W[i]$	-12	-8	-6	0	6	8	12
$q[i]$	0	3	1	0	3	0	2
$Q[i]$	0	3	4	4	7	7	9
$-Q[i]$	2	2	5	5	6	9	9

This time, not all the  $Q[i]$  are *lower or equal* than the corresponding  $-Q[i]$  terms. Hence the outranking situation between  $p_7$  and  $p_3$  is not positively validated with all potential significance weights that are compatible with the given weights pre-order.

Using this stability denotation, we may, hence, define the following *robust* version of a bipolar-valued outranking digraph.

## 19.4 Robust bipolar-valued outranking digraphs

### Definition 19.1 (Robust outranking situation).

- We say that decision alternative  $x$  *robustly outranks* decision alternative  $y$  when:
  - $x$  positively outranks  $y$  at stability level  $+2$  or higher and
  - we may not observe any considerable counter-performance of  $x$  on a discordant criterion.
- Dually, we say that decision alternative  $x$  *does not robustly outrank* decision alternative  $y$  when:
  - $x$  is positively outranked by  $y$  at stability level  $-2$  or lower and
  - we may not observe any considerable better performance of  $x$  on a discordant criterion.
- Otherwise the outranking situation is indeterminate.

The corresponding *robust* outranking digraph may be computed as follows with the RobustOutrankingDigraph class:

### Listing 19.7 Computing a robust outranking digraph

```

1 >>> from outrankingDigraphs import \
2 ...                    RobustOutrankingDigraph
3 >>> rg = RobustOutrankingDigraph(pt) # same pt
4 >>> rg
5      ----- Object instance description -----
6      Instance class      : RobustOutrankingDigraph
7      Instance name       : robust_random3ObjectivesPerfTab
8      Actions             : 7
9      Criteria            : 9
10     Size                : 22

```

```

11 Determinateness (%) : 68.45
12 Valuation domain : [-1.00;1.00]
13 Attributes : ['name', 'methodData', 'actions',
14     'order', 'criteria', 'evaluation',
15     'vetos', 'valuationdomain',
16     'cardinalRelation', 'ordinalRelation',
17     'equisignificantRelation', 'unanimousRelation',
18     'relation', 'gamma', 'notGamma']
19 >>> rg.showRelationTable()
20 * ---- Relation Table ----
21 r/(stab) | 'p1' 'p2' 'p3' 'p4' 'p5' 'p6' 'p7'
22 -----|-----
23 'p1' | +1.00 -0.42 +0.00 -0.69 +0.39 +0.11 +0.00
24 | (+4) (-2) (+0) (-3) (+2) (+2) (-1)
25 'p2' | +0.58 +1.00 +0.83 +0.00 +0.58 +0.58 +0.58
26 | (+2) (+4) (+3) (+2) (+2) (+2) (+2)
27 'p3' | +0.25 -0.33 +1.00 +0.00 +0.50 +1.00 +0.00
28 | (+2) (-2) (+4) (+0) (+2) (+2) (+1)
29 'p4' | +0.78 +0.00 +0.61 +1.00 +1.00 +1.00 +0.67
30 | (+3) (-1) (+3) (+4) (+4) (+4) (+2)
31 'p5' | -0.11 -0.50 -0.25 -0.89 +1.00 +0.11 -0.14
32 | (-2) (-2) (-2) (-3) (+4) (+2) (-2)
33 'p6' | +0.22 -0.42 +0.00 -1.00 +0.17 +1.00 -0.11
34 | (+2) (-2) (+1) (-2) (+2) (+4) (-2)
35 'p7' | +0.22 -0.50 +0.00 +0.00 +0.78 +0.42 +1.00
36 | (+2) (-2) (+1) (-1) (+3) (+2) (+4)

```

We may notice that all outranking situations, qualified at stability level  $\pm 1$ , are now put to an *indeterminate* status. In the example here, three positive outrankings get dropped: between  $p_3$  and  $p_7$ , between  $p_7$  and  $p_3$ , and between  $p_6$  and  $p_3$ , where the last situation is actually already doubtful because of a veto situation (see Listing 19.7 Lines 22-35). Three negative outrankings get dropped as well: between  $p_1$  and  $p_7$ , between  $p_4$  and  $p_2$ , and between  $p_7$  and  $p_4$  (see Lines 22-35).

Notice by the way that outranking or outranked situations, although qualified at level  $\pm 2$  or  $\pm 3$  may nevertheless become doubtful because of considerable performance differences. We may observe such a doubtful situation when comparing, for instance, alternatives  $p_2$  and  $p_4$  (see Listing 19.7 Lines 24-25).

**Listing 19.8** Comparing alternatives  $p_2$  and  $p_4$

```

1 >>> rg.showPairwiseComparison('p2','p4')
2 *----- pairwise comparison -----*
3 Comparing actions : (p2, p4)
4 crit. wght. g(x) g(y) diff | ind pref r() | v veto
5 -----
6 ec1 8.00 89.77 85.19 +4.58 | 5.00 10.00 +8.00 |
7 ec4 8.00 86.00 72.26 +13.74 | 5.00 10.00 +8.00 |
8 ec8 8.00 89.43 44.62 +44.81 | 5.00 10.00 +8.00 |
9 en3 6.00 20.79 80.81 -60.02 | 5.00 10.00 -6.00 | 60.00 -1.00
10 en5 6.00 23.83 49.69 -25.86 | 5.00 10.00 -6.00 |
11 en6 6.00 18.66 66.21 -47.55 | 5.00 10.00 -6.00 |
12 en9 6.00 26.65 50.92 -24.27 | 5.00 10.00 -6.00 |
13 so2 12.00 89.12 49.05 +40.07 | 5.00 10.00 +12.00 |
14 so7 12.00 84.73 55.57 +29.16 | 5.00 10.00 +12.00 |
15 Valuation in range: -72.00 to +72.00; -----
16 global concordance: +24.00

```

Despite being robust, the apparent positive outranking situation between alternatives  $p_2$  and  $p_4$  becomes doubtful because of a considerable counter-performance ( $-60.02$ ) of  $p_2$  observed on criterion  $en3$ , a negative difference which exceeds slightly the assumed veto discrimination threshold  $v = 60.00$  (see Listing 19.8 Line 9).

We may finally compare in Fig. 19.1 the *standard* and the *robust* version of the corresponding strict outranking digraphs, both oriented by their respective identical initial and terminal prekernels.

```

1 >>> rg.showPreKernels()
2     *--- Computing preKernels ---*
3     Dominant preKernels :
4     ['p2', 'p4']
5     independence : 0.00
6     dominance   : 0.667
7     absorbency   : -0.50
8     covering     : 1.000
9     Absorbent preKernels :
10    ['p5']
11    independence : 1.0
12    dominance   : -0.889
13    absorbency   : 0.167
14    covered      : 1.000
15    ['p6']
16    independence : 1.0
17    dominance   : -1.0
18    absorbency   : 0.111
19    covered      : 1.000

```

Standard strict outranking digraph

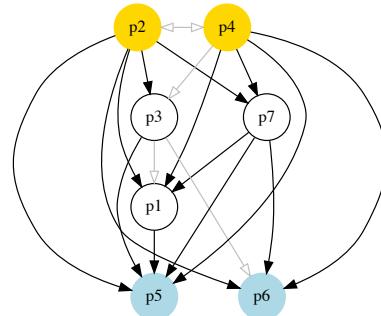


Diagram3 (graphviz), R. Bisdorff, 2020

Robust strict outranking digraph

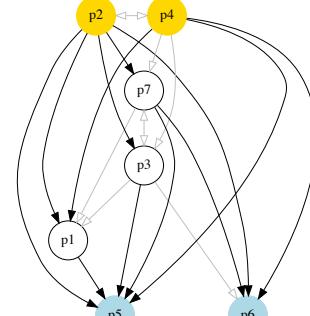


Diagram3 (graphviz), R. Bisdorff, 2020

**Fig. 19.1** Standard versus robust strict outranking digraphs oriented by their initial and terminal prekernels

The robust version (right in Fig. 19.1) drops two strict outranking situations: between  $p_4$  and  $p_7$  and between  $p_7$  and  $p_1$ . The remaining 14 strict outranking (resp. outranked) situations are now all verified at a stability level of  $+2$  and more (resp.  $-2$  and less). They remain stably validated, hence, with all potential significance weights that are compatible with the given significance weights preordering (see Listing 19.2).

To appreciate the apparent partial ordering of both the standard and the robust strict outranking digraphs shown in Fig. 19.1, let us have a final heat map view in Fig. 19.2 on the underlying performance tableau ordered by the NETFLOWS ranking rule actually applied to the robust version of the outranking digraph (see the `outrankingModel='this'` flag in Line 4 below).

**Listing 19.9** Computing a robust performance heatmap view

```

1 >>> rg.showHTMLPerformanceHeatmap(\n2 ...           Correlations=True,\n3 ...           colorLevels=5,\n4 ...           outrankingModel='this',\n5 ...           rankingRule='NetFlows')
```

### Heatmap of Performance Tableau 'robust\_random3ObjectivesPerfTab'

criteria	ec4	so2	ec1	ec8	en9	en3	so7	en6	en5
<b>weights</b>	+8.00	+12.00	+8.00	+8.00	+6.00	+6.00	+12.00	+6.00	+6.00
<b>tau<sup>(*)</sup></b>	+0.55	+0.52	+0.45	+0.38	+0.31	+0.29	+0.24	+0.05	+0.05
<b>p4</b>	72.26	49.05	85.19	44.62	50.92	80.81	55.57	66.21	49.69
<b>p2</b>	86.00	89.12	89.77	89.43	26.65	20.79	84.73	18.66	23.83
<b>p3</b>	68.70	58.85	80.19	91.94	14.85	46.78	15.49	1.65	27.25
<b>p7</b>	36.31	68.06	15.33	38.31	13.10	30.70	58.45	69.71	35.52
<b>p1</b>	22.65	22.43	38.11	77.02	44.37	58.16	28.41	11.41	31.40
<b>p6</b>	75.76	48.59	21.06	29.63	32.96	12.54	26.40	36.04	43.09
<b>p5</b>	8.96	12.36	46.75	35.91	9.83	31.05	44.92	31.22	29.52

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: **this**, Ranking rule: **NetFlows**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.960**

Mean marginal correlation (a) : **+0.338**

Standard marginal correlation deviation (b) : **+0.168**

Ranking fairness (a) - (b) : **+0.170**

**Fig. 19.2** Robust heatmap of the random 3 objectives performance tableau ordered by the NETFLOWS ranking rule.

As the initial prekernel  $\{p_2, p_4\}$  is in the robust outranking digraph validated at least at stability level  $\pm 2$ , recommending alternatives  $p_4$ , as well as  $p_2$ , as potential best choices, appears robustly justified. Alternative  $p_4$  represents indeed an overall *best compromise choice* between all decision objectives, whereas alternative  $p_2$  gives an

unanimous best choice with respect to two out of three decision objectives. Up to the decision maker to make his final choice.

## 19.5 Characterising unopposed multiobjective outranking situations

When facing a performance tableau involving multiple decision objectives, the robustness level  $\pm 3$  may lead to distinguishing what we call *unopposed* outranking situations, like the one shown in the previous section between alternative  $p_4$  and  $p_1$  ( $r(p_4 \succ p_1) = +0.78$ , see Listing 19.4 Line 11), namely preferential situations that are more or less validated or invalidated by all the decision objectives.

**Definition 19.2 (Unopposed outranking situation).**

- We say that decision alternative  $x$  is *unopposedly outranked* decision alternative  $y$  when  $x$  positively outranks  $y$  on one or more decision objectives without  $x$  being positively outranked by  $y$  on any decision objective.
- Dually, we say that decision alternative  $x$  is *unopposedly outranked by* decision alternative  $y$  when  $x$  is positively outranked by  $y$  on one or more decision objectives without  $x$  positively outranking  $y$  on any decision objective.

Let us reconsider, for instance, the performance tableau  $pt$  with three decision objectives already seen in Listing 19.1:

```

1  >>> pt.showObjectives()
2  ----- show objectives -----
3  Eco: Economical aspect
4      ec1 criterion of objective Eco 8
5      ec4 criterion of objective Eco 8
6      ec8 criterion of objective Eco 8
7      Total weight: 24.00 (3 criteria)
8  Soc: Societal aspect
9      so2 criterion of objective Soc 12
10     so7 criterion of objective Soc 12
11     Total weight: 24.00 (2 criteria)
12  Env: Environmental aspect
13     en3 criterion of objective Env 6
14     en5 criterion of objective Env 6
15     en6 criterion of objective Env 6
16     en9 criterion of objective Env 6
17     Total weight: 24.00 (4 criteria)

```

We notice in this example three decision objectives of equal importance (see Lines 3,13,17). What will be the outranking situations that are positively (resp. negatively) validated for each one of the decision objectives taken individually ?

We may obtain such unopposed multiobjective outranking situations by operating an epistemic *average fusion* (see the `symmetricAverage()` method ) of the marginal outranking digraphs restricted to the coalition of criteria supporting each one of the decision objectives (see Listing 19.10 below).

**Listing 19.10** Computing unopposed outranking situations

```

1 >>> from outrankingDigraphs import\                                BipolarOutrankingDigraph
2 ...                                                              
3 >>> geco = BipolarOutrankingDigraph(pt,\                         objectivesSubset=['Eco'])
4 ...                                                              
5 >>> gsoc = BipolarOutrankingDigraph(pt,\                         objectivesSubset=['Soc'])
6 ...                                                              
7 >>> genv = BipolarOutrankingDigraph(pt,\                         objectivesSubset=['Env'])
8 ...                                                              
9 >>> from digraphs import FusionLDigraph
10 >>> objectiveWeights = \
11 ...                           [pt.objectives[obj]['weight']\ \
12 ...                                for obj in t.objectives]
13 >>> uopg = FusionLDigraph([geco,gsoc,genv],\ \
14 ...                           operator='o-average',\ \
15 ...                           weights=objectiveWeights)
16 >>> uopg.showRelationTable(ReflexiveTerms=False)
17 * ---- Relation Table ----
18   r | 'p1'   'p2'   'p3'   'p4'   'p5'   'p6'   'p7'
19 -----
20 'p1' |   -   +0.00  +0.00  -0.69  +0.39  +0.11  +0.00
21 'p2' |  +0.00   -   +0.83  +0.00  +0.00  +0.00  +0.00
22 'p3' |  +0.00  -0.33   -   +0.00  +0.50  +0.00  +0.00
23 'p4' |  +0.78  +0.00  +0.61   -   +1.00  +1.00  +0.67
24 'p5' |  -0.11  +0.00  +0.00  -0.89   -   +0.11  +0.00
25 'p6' |  +0.00  +0.00  +0.00  -0.44  +0.17   -   +0.00
26 'p7' |  +0.00  +0.00  +0.00  +0.00  +0.78  +0.42   -
27 Valuation domain: [-1.0; 1.0]

```

Positive (resp. negative)  $r(x \succsim y)$  characteristic values, like  $r(p1 \succsim p5) = +0.39$  (see Listing 19.10 Line 20), show hence only outranking situations being validated (resp. invalidated) by one or more decision objectives without being invalidated (resp. validated) by any other decision objective.

For easily computing this kind of *unopposed multiobjective* outranking digraphs, the `outrankingDigraphs` module conveniently provides a corresponding `UnOpposedBipolarOutrankingDigraph` constructor .

**Listing 19.11** Computing unopposed outranking digraphs

```

16         'criteria', 'methodData',
17         'evaluation', 'order', 'runTimes',
18         'relation', ...
19         'gamma', 'notGamma']
20 >>> uopg.computeOppositeness(InPercents=True)
21 {'standardSize': 23, 'unopposedSize': 13,
22  'oppositeness': 43.47826086956522}

```

The resulting *unopposed* outranking digraph keeps in fact 13 (see Listing 19.11 Lines 20-22) out of the 23 positively validated *standard* outranking situations, leading to a degree of *oppositeness* –preferential disagreement between decision objectives– of  $(1.0 - 13/23) = 0.4348$ .

We may now, for instance, verify the unopposed status of the outranking situation observed between alternatives  $p_1$  and  $p_5$ .

**Listing 19.12** Example of unopposed multiobjective outranking situation

```

1 >>> uopg.showPairwiseComparison('p1','p5')
2 *----- pairwise comparison -----*
3 Comparing actions : ('p1', 'p5')
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ec1 8.00 38.11 46.75 -8.64 | 5.00 10.00 +0.00
6 ec4 8.00 22.65 8.96 +13.69 | 5.00 10.00 +8.00
7 ec8 8.00 77.02 35.91 +41.11 | 5.00 10.00 +8.00
8 en3 6.00 58.16 31.05 +27.11 | 5.00 10.00 +6.00
9 en5 6.00 31.40 29.52 +1.88 | 5.00 10.00 +6.00
10 en6 6.00 11.41 31.22 -19.81 | 5.00 10.00 -6.00
11 en9 6.00 44.37 9.83 +34.54 | 5.00 10.00 +6.00
12 so2 12.00 22.43 12.36 +10.07 | 5.00 10.00 +12.00
13 so7 12.00 28.41 44.92 -16.51 | 5.00 10.00 -12.00
14 Valuation in range: -72.00 to +72.00; -----
15                                     global concordance: +28.00

```

In Listing 19.12 we see that alternative  $p_1$  does indeed positively outrank alternative  $p_5$  from the economic perspective ( $r(p_1 \succsim_{Eco} p_5) = +16/24$ ) as well as from the environmental perspective ( $r(p_1 \succsim_{Env} p_5) = +12/24$ ). Whereas, from the societal perspective, both alternatives appear incomparable ( $r(p_1 \succsim_{Soc} p_5) = 0/24$ ).

When proportionally equal criteria significance weights per objective are given, these outranking situations appear hence *stable* with respect to all possible total importance weights we could allocate to the decision objectives.

This gives way for computing multiobjective *Pareto efficient* choice recommendations.

## 19.6 Computing Pareto efficient multiobjective choices

Indeed, best choice recommendations, computed from an *unopposed multiobjective* outranking digraph, deliver the case given *Pareto efficient* choices.

**Listing 19.13** Pareto efficient multiobjective choice

```

1 >>> uopg.showBestChoiceRecommendation()

```

```

2 Best choice recommendation(s) (BCR)
3 (in decreasing order of determinateness)
4 Credibility domain: [-1.00,1.00]
5 === >> potential best choice(s)
6 choice : ['p2', 'p4', 'p7']
7 independence : 0.00
8 dominance : 0.33
9 absorbency : 0.00
10 covering (%) : 33.33
11 determinateness (%) : 50.00
12 === >> potential worst choice(s)
13 choice : ['p3', 'p5', 'p6', 'p7']
14 independence : 0.00
15 dominance : -0.61
16 absorbency : 0.11
17 covered (%) : 33.33
18 determinateness (%) : 50.00

```

Our previous *robust* best choice recommendation (p2 and p4, see Fig. 19.1) remains, in this example here, *stable*. We recover indeed the best choice recommendation [p2, p4, p7] (see Listing 19.13 Line 6). Yet, notice that decision alternative p7 appears to be at the same time a potential *best* as well as a potential *worst* choice recommendation (see Line 13), a consequence of p7 being completely *incomparable* to the other decision alternatives when restricting the comparability to only unopposed strict outranking situations.

We may visualize this kind of Pareto efficient result in Fig. 19.3.

```

1 >>> (~(-uopg)).exportGraphViz(fileName = 'unopDigraph', \
2 ...                                bestChoice = ['p2','p4'], \
3 ...                                worstChoice = ['p3','p5','p6'])
4 *---- exporting a dot file for GraphViz tools ----*
5 Exporting to unopDigraph.dot
6 dot -Grankdir=BT -Tpng unopDigraph.dot -o unopDigraph.png

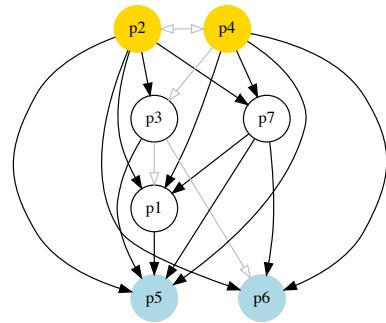
```

In order to make now an eventual best unique choice, a decision maker will necessarily have to weigh, in a second stage of the decision aiding process, the relative importance of the individual decision objectives.

In a *social choice* context, where decision objectives would match different political parties, Pareto efficient best choice recommendations represent in fact *multi-partisan* social choices that may judiciously temper plurality tyranny effects. This idea is developed in the following Chapter 20.

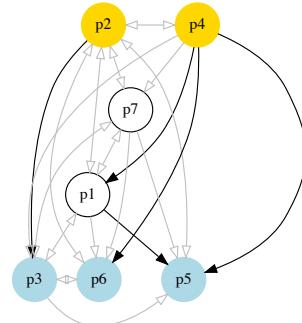
For concluding, let us mention that it is precisely again our bipolar-valued logical characteristic framework that provides us here with a first order distributional dominance test for effectively qualifying the stability level  $\pm 2$  robustness of an outranking digraph when facing performance tableaux with criteria of only ordinal-valued significance. A real-world application of our stability analysis with such a kind of performance tableau may be consulted in (Bisdorff, 2015).

Standard strict outranking digraph



Digraph3 (graphviz), R. Bisdorff, 2020

Unopposed strict outranking digraph



Digraph3 (graphviz), R. Bisdorff, 2020

**Fig. 19.3** Standard versus *unopposed* strict outranking digraphs oriented by best and worst choice recommendations

## References

- Bisdorff R (2004a) Concordant outranking with multiple criteria of ordinal significance. *4OR* 2(4):293–308, URL <http://hdl.handle.net/10993/23721> 281
- Bisdorff R (2004b) Preference aggregation with multiple criteria of ordinal significance. *Annales du LAMSADE* 3:25–44, URL <http://hdl.handle.net/10993/42420> 281
- Bisdorff R (2014) On confident outrankings with multiple criteria of uncertain significance. In: Mousseau V, Pirlot M (eds) DAP’2014 From Multiple Criteria Decision Aid to Preference Learning, Ecole Centrale de Paris, pp 1–6, URL <http://hdl.handle.net/10993/23910> 277
- Bisdorff R (2015) The EURO 2004 Best Poster Award: Choosing the Best Poster in a Scientific Conference. In: Bisdorff R, Dias L, Mousseau V, Pirlot M (eds) Evaluation and Decision Models with Multiple Criteria: Case Studies, Springer, pp 117–166 289



## Chapter 20

# Tempering plurality tyranny effects in social choice

**Abstract** In a *social choice* context, where decision objectives would match different political parties, Pareto efficient choice recommendations represent in fact *multipartisan* social choices that may judiciously deliver the primary selection in a two stage election system. Our outranking model is based on bipolar approvals-disapprovals of “*as well as performing as*” statements. A similar approach is put into practice with bipolar approval-disapproval voting systems. When converting such approval-disapproval voting ballots into corresponding performance records, we obtain a  $(-1, 0, 1)$ -valued evaluative voting system. We eventually show that in a bipolar-approval voting system, the winner tends to be among the more or less multipartisan candidates.

## Introduction

*The choice of a voting procedure shapes the democracy in which we live.*  
– [Baujard, Gavrel, Igersheim, Laslier, and Lebon \(2013\)](#).

Tempering plurality tyranny effects is, from the seminal work by B. Roy on [\(Bennayoun et al, 1966\)](#), achieved in the outranking approach by taking into account considerable negative performance differences that may render doubtful otherwise positive “*at least as well as performing*” situations. In the social choice context with general elections, such a polarisation is not feasable as the genuine uninominal voting ballots do not contain a rich enough preferential information. Yet, when matching decision objectives with political parties, the Pareto efficient outranking situations seen in Section 19.6, correspond to multipartisan “*at least as well approved as*” situations when pairwisely comparing the eligible candidates. A best choice recommendation based of the corresponding multipartisan strict “*better approved as*” situations may, the case given, deliver a convincing primary selection in a two-stage election.

## 20.1 Two-stage elections with multipartisan primary selection

To compute multipartisan social choices we need to, first, convert a given linear voting profile with pre-election polls into a corresponding performance tableau.

We shall illustrate this point with a voting profile we discussed already in Chapter 7.

**Listing 20.1** Example of a 3 parties voting profile

```

1 >>> from votingProfiles import RandomLinearVotingProfile
2 >>> lvp = RandomLinearVotingProfile(numberOfCandidates=15,
3 ...                               numberOfVoters=1000,
4 ...                               WithPolls=True,
5 ...                               partyRepartition=0.5,
6 ...                               other=0.1,
7 ...                               seed=0.9189670954954139)
8 >>> lvp
9      *----- VotingProfile instance description -----*
10     Instance class   : RandomLinearVotingProfile
11     Instance name    : randLinearProfile
12     Candidates       : 15
13     Voters          : 1000
14     Attributes       : ['name', 'seed', 'candidates',
15                           'voters', 'WithPolls', 'RandomWeights',
16                           'sumWeights', 'poll1', 'poll2',
17                           'other', 'partyRepartition',
18                           'linearBallot', 'ballot']
19 >>> lvp.showRandomPolls()
20 Random repartition of voters
21 Party_1 supporters : 460 (46.0%)
22 Party_2 supporters : 436 (43.6%)
23 Other voters       : 104 (10.4%)
24 *----- random polls -----
25 Party-1(46.0%) | Party-(43.6%) | expected
26
27   a06 : 19.91% | a11 : 22.94% | a06 : 15.00%
28   a07 : 14.27% | a08 : 15.65% | a11 : 13.08%
29   a03 : 10.02% | a04 : 15.07% | a08 : 09.01%
30   a13 : 08.39% | a06 : 13.40% | a07 : 08.79%
31   a15 : 08.39% | a03 : 06.49% | a03 : 07.44%
32   a11 : 06.70% | a09 : 05.63% | a04 : 07.11%
33   a01 : 06.17% | a07 : 05.10% | a01 : 05.06%
34   a12 : 04.81% | a01 : 05.09% | a13 : 05.04%
35   a08 : 04.75% | a12 : 03.43% | a15 : 04.23%
36   a10 : 04.66% | a13 : 02.71% | a12 : 03.71%
37   a14 : 04.42% | a14 : 02.70% | a14 : 03.21%
38   a05 : 04.01% | a15 : 00.86% | a09 : 03.10%
39   a09 : 01.40% | a10 : 00.44% | a10 : 02.34%
40   a04 : 01.18% | a05 : 00.29% | a05 : 01.97%
41   a02 : 00.90% | a02 : 00.21% | a02 : 00.51%

```

In this example (see Listing 20.1 Lines 18-), we obtained 460 Party-1 supporters (46%), 436 Party-2 supporters (43.6%) and 104 other voters (10.4%). Favorite candidates of Party-1 supporters, with more than 10%, appeared to be a06 (19.91%),

a07 (14.27%) and a03 (10.02%). Whereas for Party-2 supporters, favorite candidates appeared to be a11 (22.94%), followed by a08 (15.65%), a04 (15.07%) and a06 (13.4%).

We may convert this linear voting profile into a `PerformanceTableau` object where each political party matches to a decision objective.

**Listing 20.2** Converting a voting profile into a performance tableau

```

1 >>> lvp.save2PerfTab('votingPerfTab')
2 >>> from perfTabs import PerformanceTableau
3 >>> vpt = PerformanceTableau('votingPerfTab')
4 >>> vpt
5     *----- PerformanceTableau instance description -----
6     Instance class      : PerformanceTableau
7     Instance name       : votingPerfTab
8     Actions             : 15
9     Objectives          : 3
10    Criteria            : 1000
11    Attributes          : ['name', 'actions', 'objectives',
12                           'criteria', 'weightPreorder', 'evaluation']
13 >>> vpt.objectives
14     OrderedDict([
15         ('party0', {'name': 'other', 'weight': Decimal('104'),
16                     'criteria': ['v0003', 'v0008', 'v0011', ...]}),
17         ('party1', {'name': 'party 1', 'weight': Decimal('460'),
18                     'criteria': ['v0002', 'v0006', 'v0007', ...]}),
19         ('party2', {'name': 'party 2', 'weight': Decimal('436'),
20                     'criteria': ['v0001', 'v0004', 'v0005', ...]})])
21

```

In Listing 20.2 we first store the linear voting in a `PerformanceTableau` format (see Line 1). In Line 3, we reload this performance tableau data. The three parties of the linear voting profile represent three decision objectives and the 1000 voters are distributed as 10000 performance criteria according to the party they support.

In order to operate now a *primary multipartisan selection* of potential election winners, we compute the corresponding unopposed multiobjective outranking digraph (see Section 19.5).

**Listing 20.3** Computing unopposed multiobjective outranking situations

```

1 >>> from outrankingDigraphs import \
2 ...      UnOpposedBipolarOutrankingDigraph
3 >>> uog = UnOpposedBipolarOutrankingDigraph(vpt)
4 >>> uog
5     *----- Object instance description -----
6     Instance class      : UnOpposedBipolarOutrankingDigraph
7     Instance name       : unopposed_outrankings
8     Actions             : 15
9     Criteria            : 1000
10    Size                : 34
11    Oppositeness (%)   : 67.31
12    Determinateness (%) : 57.61
13    Valuation domain    : [-1.00;1.00]
14    Attributes          : ['name', 'actions', 'valuationdomain',

```

```

15             'objectives', 'criteria',
16             'methodData',
17             'evaluation', 'order', 'runTimes',
18             'relation',
             'marginalRelationsRelations',
             'gamma', 'notGamma']

```

From the potential 105 pairwise outranking situations, we keep 34 positively validated outranking situations, leading to a degree of *oppositeness* between political parties of 67.31%.

We may visualize the corresponding bipolar-valued relation table by orienting the list of candidates with the help of the initial and terminal prekernels.

**Listing 20.4** Computing unopposed multiobjective outranking situations

```

1 >>> uog.showPreKernels()
2     *--- Computing preKernels ---*
3     Dominant preKernels :
4     ['a11', 'a06', 'a13', 'a15']
5         independence : 0.0
6         dominance : 0.18
7         absorbency : -0.66
8         covering : 0.43
9     Absorbent preKernels :
10    ['a02', 'a04', 'a14', 'a03']
11        independence : 0.0
12        dominance : 0.0
13        absorbency : 0.37
14        covered : 0.46
15 >>> orientedCandidatesList = ['a06','a11','a13','a15',\
16 ...           'a01','a05','a07','a08','a09','a10','a12',\
17 ...           'a02','a03','a04','a14']
18 >>> uog.showHTMLRelationTable(
19 ...     actionsList=orientedCandidatesList,\n
20 ...     tableTitle='Unopposed three-partisan outrankings')

```

In Fig. 20.1, we may notice that the dominating outranking prekernel  $\{a06, a11, a13, a15\}$  gathers in fact a multipartisan selection of potential election winners. It is worthwhile noticing that in the majority margins obtained from a linear voting profile do verify the zero-sum rule:  $(r(x \succsim y) + r(y \succsim x) = 0.0)$ . To each positive outranking situation corresponds indeed an equivalent negative converse situation and the resulting outranking and strict outranking digraphs are the same.

When restricting now, in a secondary election stage, the set of eligible candidates to this dominating prekernel, we may compute the actual best social choice.

**Listing 20.5** Recommending the secondary election winner

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> g2 = BipolarOutrankingDigraph(vpt,\n3 ...     actionsSubset=['a06','a11','a13','a15'])
4 >>> g2.showRelationTable(ReflexiveTerms=False)
5     * --- Relation Table ---
6     r   | 'a06'  'a11'  'a13'  'a15'
7     -----|-----

```

## Multipartisan outranking situations

<b>r(x S y)</b>	<b>a06</b>	<b>a11</b>	<b>a13</b>	<b>a15</b>	<b>a01</b>	<b>a05</b>	<b>a07</b>	<b>a08</b>	<b>a09</b>	<b>a10</b>	<b>a12</b>	<b>a02</b>	<b>a03</b>	<b>a04</b>	<b>a14</b>
<b>a06</b>	–	0.00	0.00	0.00	0.44	0.00	0.25	0.00	0.00	0.00	0.56	0.86	0.29	0.00	0.57
<b>a11</b>	0.00	–	0.00	0.00	0.00	0.55	0.00	0.18	0.59	0.51	0.39	0.80	0.00	0.42	0.47
<b>a13</b>	0.00	0.00	–	0.00	0.00	0.52	-0.27	0.00	0.00	0.00	0.00	0.77	0.00	0.00	0.16
<b>a15</b>	0.00	0.00	0.00	–	0.00	0.39	0.00	0.00	0.00	0.00	0.00	0.66	0.00	0.00	0.00
<b>a01</b>	-0.44	0.00	0.00	0.00	–	0.00	0.00	0.00	0.00	0.00	0.00	0.77	0.00	0.00	0.20
<b>a05</b>	0.00	-0.55	-0.52	-0.39	0.00	–	0.00	-0.47	0.00	-0.12	0.00	0.37	0.00	0.00	0.00
<b>a07</b>	-0.25	0.00	0.27	0.00	0.00	0.00	–	0.00	0.00	0.00	0.30	0.83	0.00	0.00	0.38
<b>a08</b>	0.00	-0.18	0.00	0.00	0.00	0.47	0.00	–	0.00	0.00	0.00	0.77	0.00	0.29	0.00
<b>a09</b>	0.00	-0.59	0.00	0.00	0.00	0.00	0.00	0.00	–	0.00	0.00	0.55	0.00	0.00	0.00
<b>a10</b>	0.00	-0.51	0.00	0.00	0.00	0.12	0.00	0.00	0.00	–	0.00	0.50	0.00	0.00	0.00
<b>a12</b>	-0.56	-0.39	0.00	0.00	0.00	0.00	-0.30	0.00	0.00	0.00	–	0.72	0.00	0.00	0.10
<b>a02</b>	-0.86	-0.80	-0.77	-0.66	-0.77	-0.37	-0.83	-0.77	-0.55	-0.50	-0.72	–	0.00	0.00	0.00
<b>a03</b>	-0.29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	–	0.00	0.00
<b>a04</b>	0.00	-0.42	0.00	0.00	0.00	0.00	0.00	-0.29	0.00	0.00	0.00	0.00	0.00	–	0.00
<b>a14</b>	-0.57	-0.47	-0.16	0.00	-0.20	0.00	-0.38	0.00	0.00	0.00	-0.10	0.00	0.00	0.00	–

Valuation domain: [-1.00; +1.00]

**Fig. 20.1** Relation table of multipartisan outranking digraph

```

8   'a06' |   -    +0.10  +0.48  +0.52
9   'a11' | -0.10   -    +0.27  +0.29
10  'a13' | -0.48 -0.27   -    +0.19
11  'a15' | -0.52 -0.29 -0.19   -
12  Valuation domain: [-1.0; 1.0]
13 >>> g2.computeCondorcetWinners()
14  ['a06']
15 >>> g2.computeCopelandRanking()
16  ['a06', 'a11', 'a13', 'a15']

```

Candidate a06 appears clearly to be the winner of this election. Notice by the way that the restricted pairwise outranking relation shown in Listing 20.5 models a linear ordering of the preselected candidates.

We may eventually check the quality of this best choice by noticing that candidate a06 represents indeed the *simple majority*, the *instant-run-off*, the BORDA, as well as the CONDORCET winner of the initially given linear voting profile lvp.

```

1 >>> lvp.computeSimpleMajorityWinner()
2  ['a06']
3 >>> lvp.computeInstantRunoffWinner()
4  ['a06']
5 >>> lvp.computeBordaWinners()
6  ['a06']
7 >>> from votingProfiles import MajorityMarginsDigraph
8 >>> cd = MajorityMarginsDigraph(lvp)
9 >>> cd.computeCondorcetWinners()
10 ['a06']

```

In our example voting profile here, the multipartisan primary selection stage appears quite effective in reducing the number of eligible candidates to four out of a set of 15 candidates without btw rejecting the actual winning candidate.

However, in a very *divisive* two major parties system, like in the US, where preferences of the supporters of one major party appear to be very opposite to the preferences of the supporters of the other major party, the multipartisan outranking digraph will become nearly indeterminate.

In Listing 20.6 below we generate such a divisive kind of linear voting profile with the help of the `DivisivePolitics` flag <sup>1</sup> (see Lines 4 and 13-19 in Listing 20.6). When now converting the voting profile into a performance tableau (Lines 20-21), we may compute the corresponding unopposed outranking digraph.

**Listing 20.6** A divisive two-party example of a random linear voting profile

```

1  >>> from votingProfiles import RandomLinearVotingProfile
2
3  >>> lvp = RandomLinearVotingProfile(\ 
4  ...     numberOfCandidates=7,numberOfVoters=500,\ 
5  ...     WithPolls=True, partyRepartition=0.4,other=0.2,\ 
6  ...     DivisivePolitics=True, seed=1)
7
8  Random repartition of voters
9  Party-1 supporters : 240 (48.00%)
10 Party-2 supporters : 160 (32.00%)
11 Other voters       : 100 (20.00%)
12 *----- random polls -----
13
14  a2 : 30.84%    |  a1 : 30.84%    |  a2 : 15.56%
15  a3 : 23.67%    |  a4 : 23.67%    |  a3 : 12.91%
16  a7 : 17.29%    |  a6 : 17.29%    |  a7 : 11.43%
17  a5 : 11.22%    |  a5 : 11.22%    |  a1 : 11.00%
18  a6 : 09.79%    |  a7 : 09.79%    |  a6 : 10.23%
19  a4 : 04.83%    |  a3 : 04.83%    |  a4 : 09.89%
20  a1 : 02.37%    |  a2 : 02.37%    |  a5 : 08.98%
21 >>> lvp.save2PerfTab('divisiveExample')
22 >>> dvp = PerformanceTableau('divisiveExample')
23 >>> from outrankingDigraphs import \
24  ...     UnOpposedBipolarOutrankingDigraph
25 >>> uodg = UnOpposedBipolarOutrankingDigraph(dvp)
26 >>> uodg
27 *----- Object instance description -----*
28  Instance class : UnOpposedBipolarOutrankingDigraph
29  Instance name  : divisiveExample
30  Actions        : 7
31  Criteria       : 500
32  Size           : 0
33  Oppositeness (%) : 100.00
34  Determinateness (%) : 50.00
35  Valuation domain : [-1.00;1.00]
```

<sup>1</sup> The `RandomLinearVotingProfile` constructor provides a `DivisivePolitics` flag (*False* by default) for generating random linear voting profiles based on a divisive polls structure.

With an oppositeness degree of 100.0% (see Listing 20.6 Lines 31-32), the preferential disagreement between the political parties is complete, and the unopposed outranking digraph `uodg` becomes completely indeterminate as shown in the relation table below.

```

1 >>> uodg.showRelationTable(ReflexiveTerms=False)
2   * ----- Relation Table -----
3   r  | 'a1'  'a2'  'a3'  'a4'  'a5'  'a6'  'a7'
4   -----|-----
5   'a1' |   -  +0.00 +0.00 +0.00 +0.00 +0.00 +0.00
6   'a2' |  +0.00   -  +0.00 +0.00 +0.00 +0.00 +0.00
7   'a3' |  +0.00 +0.00   -  +0.00 +0.00 +0.00 +0.00
8   'a4' |  +0.00 +0.00 +0.00   -  +0.00 +0.00 +0.00
9   'a5' |  +0.00 +0.00 +0.00 +0.00   -  +0.00 +0.00
10  'a6' |  +0.00 +0.00 +0.00 +0.00 +0.00   -  +0.00
11  'a7' |  +0.00 +0.00 +0.00 +0.00 +0.00 +0.00   -
12
12 Valuation domain: [-1.0; 1.0]

```

As a consequence, a multipartisan primary selection, computed with a `showBestChoiceRecommendation()` method, will keep the complete initial set of eligible candidates and, hence, becomes *ineffective* (see Listing 20.7 Line 6).

**Listing 20.7** Example of ineffective primary multipartisan selection

```

1 >>> uodg.showBestChoiceRecommendation()
2 Rubis best choice recommendation(s) (BCR)
3 (in decreasing order of determinateness)
4 Credibility domain: [-1.00,1.00]
5 === >> ambiguous choice(s)
6   choice          : ['a1','a2','a3','a4','a5','a6','a7']
7   independence    : 0.00
8   dominance       : 1.00
9   absorbency      : 1.00
10  covered (%)     : 100.00
11  determinateness (%) : 50.00
12    - most credible action(s) = { }

```

With such kind of divisive voting profile, there may indeed not always exist an obvious winner. In Listing 20.8 below, we see, for instance, that the simple majority winner is `a2` (Line 2), whereas the instant-run-off winner is `a6` (Line 4).

**Listing 20.8** Example of non obvious secondary selection

```

1 >>> lvp.computeSimpleMajorityWinner()
2   ['a2']
3 >>> lvp.computeInstantRunoffWinner()
4   ['a6']
5 >>> from votingProfiles import MajorityMarginsDigraph
6 >>> cg = MajorityMarginsDigraph(lvp)
7 >>> cg.showRelationTable(ReflexiveTerms=False)
8   * ----- Relation Table -----
9   r() | 'a1' 'a2' 'a3' 'a4' 'a5' 'a6' 'a7'
10  -----|-----
11  'a1' |   -  -68  -90  -46  -68  -88  -84
12  'a2' |  +68   -  -32  +80  +46   -6  -24

```

```

13   'a3' | +90 +32 - +58 +46 +4 +8
14   'a4' | +4 -80 -58 - -16 -68 -72
15   'a5' | +68 -46 -46 +16 - -26 -64
16   'a6' | +88 +6 -4 +68 +26 - -2
17   'a7' | +84 +24 -8 +72 +64 +2 -
18 Valuation domain: [-500;+500]
19 >>> cg.computeCondorcetWinners()
20   ['a3']
21 >>> lvp.computeBordaWinners()
22   ['a3','a7']
23 >>> cg.computeCopelandRanking()
24   ['a3', 'a7', 'a6', 'a2', 'a5', 'a4', 'a1']

```

But in our example here, we are lucky. When constructing with the pairwise majority margins digraph (Line 6), a CONDORCET winner, namely  $a_3$  becomes apparent (Lines 13,20), which is also one of the two BORDA winners (Line 22). More interesting even is to notice that the apparent majority margins digraph models in fact a linear ranking  $[a_3, a_7, a_6, a_2, a_5, a_4, a_1]$  of all the eligible candidates, as shown with a COPELAND ranking rule (Line 24).

We may eventually visualize in Fig. 20.2 this linear ranking with a graphviz drawing where we drop all transitive arcs (Line 1) and orient the drawing with CONDORCET winner  $a_3$  and loser  $a_1$  (Line 2 below).

```

1 >>> cg.closeTransitive(Reverse=True)
2 >>> cg.exportGraphViz('divGraph', \
3 ...           bestChoice=['a3'],worstChoice=['a1'])
4   *---- exporting a dot file for GraphViz tools -----
5   Exporting to divGraph.dot
6   dot -Grankdir=BT -Tpng divGraph.dot -o divGraph.png

```

## 20.2 Bipolar approval voting systems

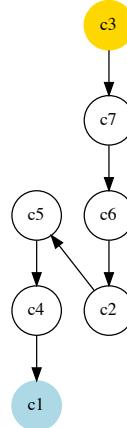
Traditionally, most of the voting systems in use in the World, do only collect approval votes and abstentions, dismissing potentially strong disapproval opinions, not to be assimilated to voting abstentions. Collecting both explicit approvals (+1) and explicit disapprovals (-1) essentially enriches the expression of voters' preferences.

In the `votingProfiles` module we provide a `BipolarApprovalVotingProfile` class for handling voting results where, for each eligible candidate  $c$ , the voters are invited to *approve* (+1), *disapprove* (-1), or *ignore* (0) the statement that candidate  $c$  should win the election (Baujard et al, 2013).

File `bpApVotingProfile.py` contains an example of such a bipolar approval voting profile concerning 100 voters and 15 eligible candidates<sup>2</sup>. We may inspect its content as follows.

---

<sup>2</sup> The file `bpApVotingProfile.py` may be found in the `examples` directory of the DI-GRAPH3 resources.



**Fig. 20.2** The linear ranking modelled by the majority margins digraph.

*Digraph3 (graphviz), R. Bisдорff, 2020*

**Listing 20.9** Bipolar approval voting profiles

```

1 >>> from votingProfiles import \
2 ...                               BipolarApprovalVotingProfile
3 >>> bavp = BipolarApprovalVotingProfile('bpApVotingProfile')
4 >>> bavp
5 *----- VotingProfile instance description -----*
6     Instance class    : BipolarApprovalVotingProfile
7     Instance name     : bpApVotingProfile
8     Candidates        : 15
9     Voters            : 100
10    Attributes        : ['name', 'candidates', 'voters',
11                          'approvalBallot', 'netApprovalScores',
12                          'ballot']
```

Beside the `candidates` and `voters` attributes, we discover in Listing 20.10 Line 11 above the `approvalBallot` attribute which gathers bipolar approval votes. Its content is the following.

**Listing 20.10** Inspecting a bipolar approval ballot

```

1 >>> bavp.approvalBallot
2     {'v001':
3         {'a01': Decimal('0'),
4          ...
5         'a04': Decimal('1'),
6          ...
7         'a15': Decimal('0')
8     },
9     'v002':
10    {'a01': Decimal('-1'),
```

```

11     'a02': Decimal('0'),
12     ...
13     'a15': Decimal('1')
14     },
15     ...
16 'v100':
17     {'a01': Decimal('0'),
18     'a02': Decimal('1'),
19     ...
20     'a15': Decimal('1')
21     }
22 }
```

Let us denote  $A_v$  the set of candidates approved by voter  $v$ . In the `approvalBallot` attribute we hence record in fact the bipolar-valued truth characteristic values  $r(C \in A_v)$  of the statements that candidate  $C$  is *approved* by voter  $v$ . In Listing 20.10 Line 5, we observe for instance that voter `v001` positively approves candidate `a04`. And, in Line 10, we see that voter `v002` negatively approves, i.e. positively disapproves candidate `a01`.

We may now consult with the `showApprovalResults()` method and the `showDisapprovalResults()` method how many approvals or disapprovals each candidate receives.

```

1 >>> bavp.showApprovalResults()
2     Approval results
3     Candidate: 'a12' obtains 34 votes
4     Candidate: 'a05' obtains 30 votes
5     Candidate: 'a03' obtains 28 votes
6     Candidate: 'a14' obtains 27 votes
7     Candidate: 'a11' obtains 27 votes
8     Candidate: 'a04' obtains 27 votes
9     Candidate: 'a01' obtains 27 votes
10    Candidate: 'a13' obtains 24 votes
11    Candidate: 'a07' obtains 24 votes
12    Candidate: 'a15' obtains 23 votes
13    Candidate: 'a02' obtains 23 votes
14    Candidate: 'a09' obtains 22 votes
15    Candidate: 'a08' obtains 22 votes
16    Candidate: 'a10' obtains 21 votes
17    Candidate: 'a06' obtains 21 votes
18    Total approval votes: 380
19    Approval proportion: 380/1500 = 0.25
20 >>> bavp.showDisapprovalResults()
21     Disapproval results
22     Candidate: 'a12' obtains 16 votes
23     Candidate: 'a03' obtains 22 votes
24     Candidate: 'a09' obtains 23 votes
25     Candidate: 'a04' obtains 24 votes
26     Candidate: 'a06' obtains 24 votes
27     Candidate: 'a13' obtains 24 votes
28     Candidate: 'a11' obtains 25 votes
29     Candidate: 'a02' obtains 26 votes
30     Candidate: 'a07' obtains 26 votes
```

```

31 Candidate: 'a08' obtains 26 votes
32 Candidate: 'a05' obtains 27 votes
33 Candidate: 'a10' obtains 27 votes
34 Candidate: 'a14' obtains 27 votes
35 Candidate: 'a15' obtains 27 votes
36 Candidate: 'a01' obtains 32 votes
37 Total disapproval votes: 376
38 Disapproval proportion: 376/1500 = 0.25

```

In Lines 3 and 22 above, we may see that, of all potential candidates, it is Candidate a12 who receives the highest number of approval votes (34) and the lowest number of disapproval votes (16). Total number of approval, respectively disapproval, votes approaches more or less a proportion of 25% of the  $100 \times 15 = 1500$  potential approval votes. About 50% of the latter remain hence ignored.

When operating now, for each candidate  $C$ , the difference between the number of approval and the number of disapproval votes he receives, we obtain per candidate a corresponding *net approval* score; in fact, the bipolar truth characteristic value of the statement “*candidate  $C$  should win the election*”.

$$r(\text{Candidate } C \text{ should win the election}) = \sum_v (r(C \in A_v)). \quad (20.1)$$

These bipolar characteristic values are stored in the `netApprovalScores` attribute and may be printed out with the `showNetApprovalScores()` method:

```

1 >>> bavp.showNetApprovalScores()
2 Net Approval Scores
3 Candidate: 'a12' obtains 18 net approvals
4 Candidate: 'a03' obtains 6 net approvals
5 Candidate: 'a05' obtains 3 net approvals
6 Candidate: 'a04' obtains 3 net approvals
7 Candidate: 'a11' obtains 2 net approvals
8 Candidate: 'a14' obtains 0 net approvals
9 Candidate: 'a13' obtains 0 net approvals
10 Candidate: 'a09' obtains -1 net approvals
11 Candidate: 'a07' obtains -2 net approvals
12 Candidate: 'a06' obtains -3 net approvals
13 Candidate: 'a02' obtains -3 net approvals
14 Candidate: 'a15' obtains -4 net approvals
15 Candidate: 'a08' obtains -4 net approvals
16 Candidate: 'a01' obtains -5 net approvals
17 Candidate: 'a10' obtains -6 net approvals

```

We observe in Line 3 above that Candidate a12, with a net approval score of  $34 - 16 = 18$ , represents indeed the *best approved* candidate for winning the election. With a net approval score of  $28 - 22 = 6$ , Candidate a03 appears 2nd-best approved. The net approval scores define hence a potentially weak ranking on the set of eligible election candidates, and the winner(s) of the election is(are) determined by the first-ranked candidate(s).

### 20.3 Pairwise comparison of bipolar approval votes

The approval votes of each voter define now on the set of eligible candidates three ordered categories: his approved (+1), his ignored (0) and his disapproved (-1) ones. Within each of these three categories we consider the voter's actual preferences as *not communicated*, i.e. as missing data. This gives for each voter a partially determined strict order which we find in the `ballot` attribute.

```

1  >>> bavp.ballot['v001']['a12']
2  {'a02': Decimal('1'), 'a11': Decimal('1'),
3  'a14': Decimal('1'), 'a04': Decimal('0'),
4  'a06': Decimal('1'), 'a05': Decimal('1'),
5  'a12': Decimal('0'), 'a13': Decimal('0'),
6  'a15': Decimal('1'), 'a01': Decimal('1'),
7  'a08': Decimal('1'), 'a07': Decimal('1'),
8  'a09': Decimal('0'), 'a03': Decimal('1'),
9  'a10': Decimal('0')}

```

For voter  $v001$ , for instance, the best approved candidate  $a12$  is strictly preferred to candidates:  $a01, a02, a03, a05, a06, a07, a08, a11, a14$  and  $a15$ . No candidate is preferred to  $a12$  and the comparison with  $a04, a09, a10$  and  $a13$  is not communicated, hence indeterminate. Mind by the way that the reflexive comparison of  $a12$  with itself is, as usual, ignored, i.e. indeterminate. Each voter  $v$  defines thus a partially determined transitive strict preference relation denoted  $\succ_v$  on the eligible candidates.

For each pair of eligible candidates, we aggregate the previous individual voter's preferences into a truth characteristic of the statement: "candidate  $x$  is *better approved than* candidate  $y$ ", denoted  $r(x \succ y)$ :

$$r(x \succ y) = \sum_v (r(x \succ_v y)). \quad (20.2)$$

We say that candidate  $x$  is *better approved than* Candidate  $y$  when  $r(x \succ y) > 0$ , i.e. there is a majority of voters who approve *more* and disapprove *less*  $x$  than  $y$ . Vice-versa, we say that candidate  $x$  is *not better approved than* candidate  $y$  when  $r(x \succ y) < 0$ , i.e. there is a majority of voters who disapprove more and approve less  $x$  than  $y$ . This computation is achieved with the `MajorityMarginsDigraph` constructor.

```

1 >>> from votingProfiles import MajorityMarginsDigraph
2 >>> m = MajorityMarginsDigraph(bavp)
3 >>> m
4 *----- Digraph instance description -----*
5  Instance class      : MajorityMarginsDigraph
6  Instance name       : rel_bpApVotingProfile
7  Digraph Order       : 15
8  Digraph Size        : 97
9  Valuation domain    : [-100.00;100.00]
10 Determinateness (%) : 52.55
11 Attributes          : ['name', 'actions',
12                           'criteria', 'ballot',

```

```

13     'valuationdomain', 'relation',
14     'order', 'gamma', 'notGamma']

```

The resulting digraph  $m$  contains 97 positively validated relations (see Line 8 above) and (see Line 9) for all pairs  $(x, y)$  of eligible candidates,  $r(x \succ y)$  takes value in an valuation range from  $-100.00$  (all voters opposed) to  $+100.00$  (unanimously supported).

We may inspect these pairwise  $r(x \succ y)$  values in a browser view:

```
>>> m.showHTMLRelationTable(relationName='r(x > y)')
```

$r(x \geq y)$	a01	a02	a03	a04	a05	a06	a07	a08	a09	a10	a11	a12	a13	a14	a15
a01	-	1	-10	-10	-8	-2	-2	-3	-3	2	-5	-19	-4	-2	2
a02	-1	-	-5	-4	-9	-2	5	1	-7	3	-2	-20	-1	-4	-2
a03	10	5	-	1	4	6	5	10	7	11	0	-5	2	6	9
a04	10	4	-1	-	2	5	7	8	-1	4	0	-7	8	2	8
a05	8	9	-4	-2	-	4	6	7	4	2	1	-17	6	0	3
a06	2	2	-6	-5	-4	-	-1	2	-3	5	-4	-13	-1	-1	2
a07	2	-5	-5	-7	-6	1	-	7	-4	2	-5	-17	-2	-2	4
a08	3	-1	-10	-8	-7	-2	-7	-	-2	2	-2	-16	0	0	-1
a09	3	7	-7	1	-4	3	4	2	-	3	0	-18	-4	0	2
a10	-2	-3	-11	-4	-2	-5	-2	-2	-3	-	-6	-15	-4	-4	2
a11	5	2	0	0	-1	4	5	2	0	6	-	-15	4	0	5
a12	19	20	5	7	17	13	17	16	18	15	15	-	12	13	18
a13	4	1	-2	-8	-6	1	2	0	4	4	-4	-12	-	1	4
a14	2	4	-6	-2	0	1	2	0	0	4	0	-13	-1	-	-1
a15	-2	2	-9	-8	-3	-2	-4	1	-2	-2	-5	-18	-4	1	-

Valuation domain: [-100; +100]

**Fig. 20.3** The bipolar-valued pairwise majority margins

In Fig. 20.3 it gets apparent that candidate  $a_{12}$  is a CONDORCET winner, i.e. the candidate who beats all the other candidates and, with the given voting profile  $g_{avp}$ , should without doubt win the election. This strongly confirms the first-ranked result obtained with the previous net approval scoring.

Let us eventually compute, with the help of the NETFLOWS ranking rule<sup>3</sup>, a linear ranking of the 15 eligible candidates and compare the result with the net approval scores ranking.

**Listing 20.11** Comparing the net approval and the NETFLOWS rankings

```

1 >>> from linearOrders import NetFlowsOrder
2 >>> nf = NetFlowsOrder(m,Comments=True)
3 >>> print('NetFlows versus Net Approval Ranking')

```

<sup>3</sup> See Section 8.3.

```

4 >>> print('Candidate\tNetFlows score\tNet Approval score')
5 >>> for item in nf.netFlows:
6 ...     print( '%9s\t %+.3f\t %+.1f' %
7 ...             (item[1], item[0], \
8 ...              bavp.netApprovalScores[item[1]]))
9 NetFlows versus Net Approval Ranking
10 Candidate    NetFlows score  Net Approval score
11 'a12'        +410.000      +18.0
12 'a03'        +142.000      +6.0
13 'a04'        +98.000       +3.0
14 'a05'        +54.000       +3.0
15 'a11'        +34.000       +2.0
16 'a09'        -16.000       -1.0
17 'a14'        -20.000       +0.0
18 'a13'        -22.000       +0.0
19 'a06'        -50.000       -3.0
20 'a07'        -74.000       -2.0
21 'a02'        -96.000       -3.0
22 'a08'        -102.000      -4.0
23 'a15'        -110.000      -4.0
24 'a10'        -122.000      -6.0
25 'a01'        -126.000      -5.0

```

In Listing 20.11 we may notice that the NETFLOWS rule delivers a ranking that is very similar to the one previously obtained with the corresponding *Net Approval* scores. Only minor inversions do appear, like in the midfield, where candidate  $a09$  advances before candidates  $a13$  and  $a14$  (see Line 16), and  $a6$  and  $a07$  swap their positions 9 and 10 (see Line 19). And, the two last-ranked candidates  $a10$  and  $a01$  also swap their positions.

This confirms again the pertinence of the net approval scoring approach for finding the winner in a bipolar approving voting system. Yet, voting by approving (+1), disapproving (-1) or ignoring (0) eligible candidates, may also be seen as a performance evaluation of the eligible candidates on a  $\{-1, 0, 1\}$ -graded ordinal scale.

## 20.4 Three-valued evaluative voting systems

Following such an epistemic perspective, we may effectively convert the given `BipolarApprovalVotingProfile` instance into a `PerformanceTableau` instance, so as to get access to a corresponding outranking decision aiding approach.

Mind that, contrary to the majority margins of the “*better approved than*” relation, all voters consider now the approved candidates to be all equivalent (+1). Same is true for the disapproved (-1), respectively the ignored (0) candidates. The voter’s marginal preferences model this time a complete preorder with three equivalence classes.

From the saved file `AVPerfTab.py` (see Line 1 below), we may construct an outranking relation on the eligible candidates with our standard `BipolarOutrankingDigraph` constructor. The semantics of this outranking relation are the following:

- We say that Candidate  $x$  *outranks* Candidate  $y$  when there is a majority of voters who consider  $x$  *at least as well evaluated as*  $y$  (see Listing 20.12 Line 3 below);
- We say that Candidate  $x$  is *outranked by* Candidate  $y$  when there is a majority of voters who consider  $x$  *not at least as well evaluated as*  $y$ ;
- Otherwise, the outranking situation is indeterminate.

**Listing 20.12** Computing the outranking digraph

```

1 >>> bavp.save2PerfTab(fileName='AVPerfTab',valueDigits=0)
2     ---- Saving as performance tableau in AVPerfTab.py ----*
3 >>> from outrankingDigraphs import\
4         ...                                BipolarOutrankingDigraph
5 >>> odg = BipolarOutrankingDigraph('AVPerfTab')
6 >>> odg
7     *----- Object instance description -----*
8     Instance class      : BipolarOutrankingDigraph
9     Instance name       : rel_AVPerfTab
10    Actions            : 15
11    Criteria           : 100
12    Size               : 210
13    Determinateness (%) : 69.29
14    Valuation domain   : [-1.00;1.00]
15    Attributes         : ['name', 'actions', 'order',
16                           'criteria', 'evaluation', 'NA',
17                           'valuationdomain', 'relation',
18                           'gamma', 'notGamma', ...]

```

The size ( $210 = 15 \times 14$ ) of the resulting outranking digraph `odg`, shown in Listing 20.12 Line 11 above, reveals that the corresponding “*at least as good evaluated as*” relation models actually a trivial *complete* digraph. All candidates appear to be *equally* at least as well evaluated and the strict “*better evaluated than*” (codual) outranking digraph becomes in fact empty. The converted performance tableau does apparently not contain sufficiently discriminatory performance evaluations for supporting any strict preference situations.

Yet, we may nevertheless try to apply again the NETFLOWS ranking rule to this complete outranking digraph `odg` and print side by side the corresponding NETFLOWS scores and the previous Net Approval scores.

**Listing 20.13** Comparing the NETFLOWS and the Net Approval rankings

```

1 >>> from linearOrders import NetFlowsOrder
2 >>> nf = NetFlowsOrder(odg)
3 >>> print('NetFlows versus Net Approval Ranking')
4 >>> print('Candidate\tNetFlows Score\tNet Approval Score')
5 >>> for item in nf.netFlows:
6     ...     print('%9s\t%+.3f\t%+.0f' %
7     ...           ('\\'',item[1], '\\'', item[0], \
8     ...             bavp.netApprovalScores[item[1]]))
9     NetFlows versus Net Approval Ranking
10    Candidate    NetFlows score Net Approval score
11    'a12        +4.100          +18.0
12    'a03        +1.420          +6.0

```

13	'a04	+0.980	+3.0
14	'a05	+0.540	+3.0
15	'a11	+0.340	+2.0
16	'a09	-0.160	-1.0
17	'a14	-0.200	+0.0
18	'a13	-0.220	+0.0
19	'a06	-0.500	-3.0
20	'a07	-0.740	-2.0
21	'a02	-0.960	-3.0
22	'a08	-1.020	-4.0
23	'a15	-1.100	-4.0
24	'a10	-1.220	-6.0
25	'a01	-1.260	-5.0

Despite its apparent poor strict preference discriminating power, we obtain in Listing 20.13 here NETFLOWS scores that are directly proportional (divided by 100) to the scores obtained with the *better approved than* majority margins digraph  $m$  (see Listing 20.11).

Encouraged by this positive result, we may furthermore compute as well a best choice recommendation.

**Listing 20.14** Computing a best social choice recommendation

```

1 >>> odg.showBestChoiceRecommendation(CoDual=True)
2 Rubis best choice recommendation(s) (BCR)
3 (in decreasing order of determinateness)
4 Credibility domain: [-1.00,1.00]
5 === >> ambiguous best choice(s)
6 * choice      : ['a01','a02','a03','a04','a05',
7                   'a06','a07','a08','a09','a10',
8                   'a11','a12','a13','a14','a15']
9 independence   : 0.06
10 dominance     : 1.00
11 absorbency    : 1.00
12 covering (%)  : 100.00
13 determinateness (%) : 61.13
14 - most credible action(s) = {
15   'a12': 0.44, 'a03': 0.34, 'a04': 0.30,
16   'a14': 0.28, 'a13': 0.24, 'a06': 0.24,
17   'a11': 0.20, 'a10': 0.20, 'a07': 0.20,
18   'a01': 0.20, 'a08': 0.18, 'a05': 0.18,
19   'a15': 0.14, 'a09': 0.14, 'a02': 0.06, }
20 === >> ambiguous worst choice(s)
21 * choice      : ['a01','a02','a03','a04','a05',
22                   'a06','a07','a08','a09','a10',
23                   'a11','a12','a13','a14','a15']
24 independence   : 0.06
25 dominance     : 1.00
26 absorbency    : 1.00
27 covered (%)   : 100.00
28 determinateness (%) : 63.73
29 - most credible action(s) = {
30   'a13': 0.36, 'a06': 0.36, 'a15': 0.34,
31   'a01': 0.34, 'a08': 0.32, 'a07': 0.30,

```

```

32     'a02': 0.30, 'a14': 0.28, 'a11': 0.28,
33     'a09': 0.28, 'a04': 0.26, 'a10': 0.24,
34     'a05': 0.20, 'a03': 0.20, 'a12': 0.06, }
```

The strict outranking digraph ( $\sim(-odg)$ ) being actually *empty*, we obtain a unique *ambiguous* –best as well as last– choice recommendation which trivially retains all fifteen candidates (see Listing 20.14 Lines 6-8 above). Yet, the bipolar-valued best choice membership characteristic vector reveals that, among all the fifteen potential winners, it is indeed Candidate a12 the most credible one with a 72% majority of voters’ support (see Line 15,  $(0.44 + 1.0)/2 = 0.72$ ); followed by Candidate a03 (67%) and Candidate a04 (65%). Similarly, Candidates a13 and a06 represent the most credible losers with a 68% majority voters’ support (Line 30).

We observe here empirically that *evaluative* voting systems, using three-valued ordinal performance scales, match closely a corresponding bipolar approval voting systems. The latter voting system models, however, more faithfully the very preferential information that is expressed with *approved*, *disapproved* and *ignored* statements. The corresponding evaluation on a three-graded scale, being value (numbers) based, cannot express the fact that in bipolar approval voting systems there is no preferential information given concerning the pairwise comparison of all approved, respectively disapproved or ignored candidates.

Let us finally illustrate how bipolar approval voting systems may favour multipartisan supported candidates. We shall therefore compare *bipolar approval* versus *uninominal plurality* election results when considering a highly divisive and partisan political context.

## 20.5 Favouring multipartisan candidates

In modern democracy, politics are largely structured by political parties and activists. Let us so consider a bipolar approval voting profile dvp where the random voter behaviour is simulated from two pre-electoral polls concerning a political scene with essentially two major highly competing parties, like the one existing in the US.

**Listing 20.15** A random bipolar approval voting profile in a divisive political context

```

1 >>> dvp = RandomBipolarApprovalVotingProfile(\n2     ...             numberOfCandidates=15,\n3     ...             numberOfVoters=100,\n4     ...             approvalProbability=0.25,\n5     ...             disapprovalProbability=0.25,\n6     ...             WithPolls=True,\n7     ...             partyRepartition=0.5,\n8     ...             other=0.05,\n9     ...             DivisivePolitics=True,\n10    ...             seed=200)\n11 >>> dvp.showRandomPolls()\n12     Random repartition of voters\n13     Party-1 supporters : 45 (45.00%)
```

```

14 Party-2 supporters : 49 (49.00%)
15 Other voters       : 6 (06.00%)
16 ----- random polls -----
17 Party-1(45.0%) | Party-2(49.0%) | expected
18 -----
19 'a05' : 24.10% | 'a07' : 24.10% | 'a07' : 11.87%
20 'a14' : 23.48% | 'a10' : 23.48% | 'a10' : 11.60%
21 'a03' : 15.13% | 'a01' : 15.13% | 'a05' : 10.91%
22 'a12' : 07.55% | 'a04' : 07.55% | 'a14' : 10.67%
23 'a08' : 07.11% | 'a09' : 07.11% | 'a01' : 07.67%
24 'a15' : 04.37% | 'a13' : 04.37% | 'a03' : 07.09%
25 'a11' : 03.99% | 'a02' : 03.99% | 'a04' : 04.55%
26 'a06' : 03.80% | 'a06' : 03.80% | 'a09' : 04.49%
27 'a02' : 02.79% | 'a11' : 02.79% | 'a12' : 04.32%
28 'a13' : 02.63% | 'a15' : 02.63% | 'a08' : 04.30%
29 'a09' : 02.24% | 'a08' : 02.24% | 'a06' : 03.57%
30 'a04' : 01.89% | 'a12' : 01.89% | 'a13' : 03.32%
31 'a01' : 00.57% | 'a03' : 00.57% | 'a15' : 03.25%
32 'a10' : 00.20% | 'a14' : 00.20% | 'a02' : 03.21%
33 'a07' : 00.14% | 'a05' : 00.14% | 'a11' : 03.16%

```

In Listing 20.15, the divisive political situation is reflected by the fact that Party-1 and Party-2 supporters show strict reversed preferences. The leading candidates of Party-1 (a05 and a14) are last choices for Party-2 supporters and, Candidates a07 and a10, leading candidates for Party-2 supporters, are similarly the least choices for Party-1 supporters.

No clear winner may be guessed from these pre-election polls. As Party-2 shows however slightly more supporters than Party-1, the expected winner in an uninominal plurality or instant-runoff voting system will be Candidate 'a07', i.e, the leading candidate of majority Party-2 (see below).

```

1 >>> dvp.computeSimpleMajorityWinner()
2 ['a07']
3 >>> dvp.computeInstantRunoffWinner()
4 ['a07']

```

Now, in a corresponding bipolar approval voting system, Party-1 supporters will usually approve their leading candidates and disapprove the leading candidates of Party-2. Vice versa, Party-2 supporters will usually approve their leading candidates and disapprove the leading candidates of Party-1. Let us consult the resulting approval votes per candidate.

```

1 >>> dvp.showApprovalResults()
2     Candidate: 'a07' obtains 30 votes
3     Candidate: 'a10' obtains 28 votes
4     Candidate: 'a05' obtains 28 votes
5     Candidate: 'a01' obtains 28 votes
6     Candidate: 'a03' obtains 26 votes
7     Candidate: 'a02' obtains 26 votes
8     Candidate: 'a12' obtains 25 votes
9     Candidate: 'a14' obtains 24 votes
10    Candidate: 'a13' obtains 24 votes
11    Candidate: 'a09' obtains 21 votes

```

```

12 Candidate: 'a04' obtains 21 votes
13 Candidate: 'a08' obtains 19 votes
14 Candidate: 'a06' obtains 17 votes
15 Candidate: 'a15' obtains 15 votes
16 Candidate: 'a11' obtains 12 votes
17 Total approval votes: 344
18 Approval proportion: 344/1500 = 0.23

```

When considering only the approval votes, we find confirmed above that the leading candidate of Party-2 obtains in this simulation a plurality of approval votes. In uninominal plurality or instant-runoff voting systems, this candidate wins hence the election, quite to the despair of Party-1 supporters. As a foreseeable consequence, this election result will be more or less aggressively contested which leads to a loss of popular trust in democratic elections and institutions.

If we look however on the corresponding disapprovals, we discover that, not surprisingly, the leading candidates of both parties collect by far the highest number of disapproval votes.

```

1 >>> dvp.showDisapprovalResults()
2 Candidate: 'a02' obtains 14 votes
3 Candidate: 'a04' obtains 14 votes
4 Candidate: 'a13' obtains 14 votes
5 Candidate: 'a06' obtains 15 votes
6 Candidate: 'a09' obtains 15 votes
7 Candidate: 'a08' obtains 16 votes
8 Candidate: 'a11' obtains 16 votes
9 Candidate: 'a15' obtains 18 votes
10 Candidate: 'a12' obtains 20 votes
11 Candidate: 'a01' obtains 29 votes
12 Candidate: 'a03' obtains 30 votes
13 Candidate: 'a10' obtains 37 votes
14 Candidate: 'a07' obtains 44 votes
15 Candidate: 'a14' obtains 45 votes
16 Candidate: 'a05' obtains 49 votes
17 Total disapproval votes: 376
18 Disapproval proportion: 376/1500 = 0.25

```

Balancing now approval against disapproval votes will favour the moderate, bipartisan supported, candidates.

```

1 >>> dvp.showNetApprovalScores()
2 Net Approval Scores
3 Candidate: 'a02' obtains 12 net approvals
4 Candidate: 'a13' obtains 10 net approvals
5 Candidate: 'a04' obtains 7 net approvals
6 Candidate: 'a09' obtains 6 net approvals
7 Candidate: 'a12' obtains 5 net approvals
8 Candidate: 'a08' obtains 3 net approvals
9 Candidate: 'a06' obtains 2 net approvals
10 Candidate: 'a01' obtains -1 net approvals
11 Candidate: 'a15' obtains -3 net approvals
12 Candidate: 'a11' obtains -4 net approvals
13 Candidate: 'a03' obtains -4 net approvals
14 Candidate: 'a10' obtains -9 net approvals

```

```
15 Candidate: 'a07' obtains -14 net approvals
16 Candidate: 'a14' obtains -21 net approvals
17 Candidate: 'a05' obtains -21 net approvals
```

Candidate 'a02', appearing in the pre-electoral polls in the midfield (in position 7 for Party-2 and in position 9 for Party-1 supporters, see Listing 20.15), shows indeed the highest net approval score. Second highest net approval score obtains Candidate 'a13', in position 6 for Party-2 and in position 10 for Party-1 supporters.

Fig. 20.4, showing the NETFLOWS ranked relation table of the “*better approved than*” majority margins digraph, confirms below this net approval scoring result.

```
1 >>> m = MajorityMarginsDigraph(dvp)
2 >>> m.showHTMLRelationTable(\n3 ...     actionsList=m.computeNetFlowsRanking(),\n4 ...     relationName='r(x > y)')
```

r(x > y)	a02	a13	a04	a09	a12	a08	a06	a01	a11	a15	a03	a10	a07	a14	a05
<b>a02</b>	-	6	5	6	9	5	10	12	12	14	11	15	22	22	23
<b>a13</b>	-6	-	2	5	2	5	8	10	14	10	9	13	18	23	20
<b>a04</b>	-5	-2	-	0	2	2	3	7	7	8	11	13	18	21	18
<b>a09</b>	-6	-5	0	-	0	2	5	5	11	9	5	13	16	21	16
<b>a12</b>	-9	-2	-2	0	-	4	6	2	9	6	10	5	10	23	25
<b>a08</b>	-5	-5	-2	-2	-4	-	4	0	8	9	7	5	11	21	20
<b>a06</b>	-10	-8	-3	-5	-6	-4	-	-2	5	5	5	6	13	17	18
<b>a01</b>	-12	-10	-7	-5	-2	0	2	-	1	-1	3	8	11	9	13
<b>a11</b>	-12	-14	-7	-11	-9	-8	-5	-1	-	1	-2	7	14	13	16
<b>a15</b>	-14	-10	-8	-9	-6	-9	-5	1	-1	-	0	3	10	14	14
<b>a03</b>	-11	-9	-11	-5	-10	-7	-5	-3	2	0	-	-3	7	16	16
<b>a10</b>	-15	-13	-13	-13	-5	-5	-6	-8	-7	-3	3	-	3	6	7
<b>a07</b>	-22	-18	-18	-16	-10	-11	-13	-11	-14	-10	-7	-3	-	3	5
<b>a14</b>	-22	-23	-21	-21	-23	-21	-17	-9	-13	-14	-16	-6	-3	-	0
<b>a05</b>	-23	-20	-18	-16	-25	-20	-18	-13	-16	-14	-16	-7	-5	0	-

Valuation domain: [-100; +100]

**Fig. 20.4** The pairwise *better approved than* majority margins

Candidate 'a02' appears indeed *better approved than* any other candidate (CONDORCET winner); and, the leading candidates of Party-1, 'a05' and 'a14', are *less approved than* any other candidates (weak CONDORCET losers).

```
1 >>> m.computeCondorcetWinners()
2      ['a02']
3 >>> m.computeWeakCondorcetLosers()
4      ['a05','a14']
```

We see this result furthermore confirmed when computing the corresponding best, respectively worst choice recommendation.

```

1 >>> m.showBestChoiceRecommendation()
2 Rubis best choice recommendation(s) (BCR)
3 (in decreasing order of determinateness)
4 Credibility domain: [-100.00,100.00]
5 === >> potential best choice(s)
6   * choice          : ['a02']
7     independence    : 100.00
8     dominance       : 5.00
9     absorbency      : -23.00
10    covering (%)    : 100.00
11    determinateness (%) : 52.50
12    - most credible action(s) = { 'a02': 5.00, }
13 === >> potential worst choice(s)
14   * choice          : ['a05', 'a14']
15     independence    : 0.00
16     dominance       : -23.00
17     absorbency      : 5.00
18     covered (%)     : 100.00
19     determinateness (%) : 50.00
20     - most credible action(s) = { }

```

Candidate a02, being actually a CONDORCET winner, gives an initial prekernel of digraph  $m$ , whereas Party-1 leading Candidates a05 and a14, both being weak CONDORCET losers, give together a terminal prekernel. They hence represent our *best choice*, respectively, *worst choice* recommendations for winning this simulated election.

Let us conclude by predicting that, for leading political candidates in an aggressively divisive political context, the perspective to easily fail election with bipolar approval voting systems, might or will induce a change in the usual way of running electoral campaigns. Political parties and politicians, who avoid aggressive competitive propaganda and instead propose multipartisan collaborative social choices, will be rewarded with better election results than any kind of extremism. It could mean the end of sterile political obstructions and war like electoral battles. *Let's do it !*.

It is worthwhile noticing again the essential structural and computational role, the ignored value is again playing in bipolar approval voting systems. This epistemic and logical *neutral* term is needed indeed for handling in a consistent and efficient manner *not communicated votes* and/or *indeterminate* preferential statements.

## References

- Baujard A, Gavrel F, Iggersheim H, Laslier JF, Lebon I (2013) Approval Voting, Evaluation Voting: An Experiment during the 2012 French Presidential Election. *Revue Économique* (Presses de Sciences Po) 64(2):345–356 [293](#), [300](#)
- Benayoun R, Roy B, Sussmann B (1966) ELECTRE: une méthode pour guider le choix en présence de points de vue multiples. Tech. Rep. 49, Société d’Economie et de Mathématique Appliquée, Direction Scientifique [293](#)

**Part V**  
**Working with undirected graphs**

The last part introduces Python resources for working with undirected graphs. Its aim is to eventually show the operational benefits one may get when implementing vertex adjacency with bipolar-valued characteristics. Several special graph models and algorithms are illustrated: –Q-coloring, –MIS and clique enumeration, –line graphs and computing maximal matchings, –grid graphs and computing the Ising model, – $n$ -cycle graphs and computing their non-isomorphic MISs, –Spanning tree graphs and graph forests, –Generating and recognizing split, interval and permutation graphs.

# Chapter 21

## Bipolar-valued undirected graphs

**Abstract** The chapter introduces bipolar-valued undirected graphs and illustrates several special graph models and algorithms like Q-coloring, MIS and clique enumeration, line graphs and maximal matchings, grid graphs, and  $n$ -cycle graphs with their non-isomorphic maximal independent sets of vertices.

### 21.1 Implementing simple graphs

In the DIGRAPH3 `graphs` module, the root `Graph` class provides a generic simple graph model, without loops and multiple links. A given object of this `Graph` type contains at least the following attributes:

1. `name`: usually the name of the stored instance,
2. `vertices`: a dictionary with `name` and `shortName` attributes,
3. `order`: the number of vertices,
4. `valuationDomain`: a dictionary with three entries: the minimum ( $-1$ , means certainly no link), the median ( $0$ , means missing information) and the maximum characteristic value ( $+1$ , means certainly a link),
5. `edges` : a dictionary with *frozensets*<sup>1</sup> of pairs of vertices as keys carrying a characteristic value in the range of the previous valuation domain,
6. `size`: the number of positive edges,
7. `gamma`: a dictionary containing the direct neighbors of each vertex, automatically added by the object constructor.

#### Example Python terminal session

A random crisp graph instance can be generated with the `RandomGraph` class (see Lines 1-2 in List. 21.1 on the next page below).

---

<sup>1</sup> [Python documentation](#)

**Listing 21.1** Generating a random graph instance

```

1 >>> from graphs import RandomGraph
2 >>> g = RandomGraph(order=7, edgeProbability=0.5)
3 >>> g.save(fileName='tutorialGraph')

```

The saved Graph instance, named `tutorialGraph.py`, is encoded as shown in Listing 21.2.

**Listing 21.2** Stored instance of a random graph

```

1  # Graph instance saved in Python format
2  from decimal import Decimal
3  vertices = {
4      'v1': {'shortName': 'v1', 'name': 'random vertex'},
5      'v2': {'shortName': 'v2', 'name': 'random vertex'},
6      'v3': {'shortName': 'v3', 'name': 'random vertex'},
7      'v4': {'shortName': 'v4', 'name': 'random vertex'},
8      'v5': {'shortName': 'v5', 'name': 'random vertex'},
9      'v6': {'shortName': 'v6', 'name': 'random vertex'},
10     'v7': {'shortName': 'v7', 'name': 'random vertex'},
11 }
12 valuationDomain = {
13     'min':Decimal('-1'), 'med':Decimal('0'),
14     'max':Decimal('1')}
15 edges = {
16     frozenset(['v1','v2']) : Decimal('-1'),
17     frozenset(['v1','v3']) : Decimal('-1'),
18     frozenset(['v1','v4']) : Decimal('-1'),
19     frozenset(['v1','v5']) : Decimal('1'),
20     frozenset(['v1','v6']) : Decimal('-1'),
21     frozenset(['v1','v7']) : Decimal('-1'),
22     frozenset(['v2','v3']) : Decimal('1'),
23     frozenset(['v2','v4']) : Decimal('1'),
24     frozenset(['v2','v5']) : Decimal('-1'),
25     frozenset(['v2','v6']) : Decimal('1'),
26     frozenset(['v2','v7']) : Decimal('-1'),
27     frozenset(['v3','v4']) : Decimal('-1'),
28     frozenset(['v3','v5']) : Decimal('-1'),
29     frozenset(['v3','v6']) : Decimal('-1'),
30     frozenset(['v3','v7']) : Decimal('-1'),
31     frozenset(['v4','v5']) : Decimal('1'),
32     frozenset(['v4','v6']) : Decimal('-1'),
33     frozenset(['v4','v7']) : Decimal('1'),
34     frozenset(['v5','v6']) : Decimal('1'),
35     frozenset(['v5','v7']) : Decimal('-1'),
36     frozenset(['v6','v7']) : Decimal('-1'),
37 }

```

The stored graph instance may be reloaded and plotted with the `exportGraphViz()` method (see Fig. 21.1 on the next page):

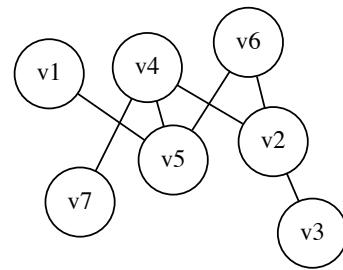
```

1 >>> g = Graph('tutorialGraph')
2 >>> g.exportGraphViz()
3     ----- exporting a dot file for GraphViz tools -----
4     Exporting to tutorialGraph.dot

```

```
5   fdp -Tpng tutorialGraph.dot -o tutorialGraph.png
```

**Fig. 21.1** Example simple graph instance



*Digraph3 (graphviz), R. Bisdorff, 2019*

Properties, like the gamma function –the cover relation–, vertex degrees and neighbourhood depths are shown with a `showShort()` method.

**Listing 21.3** Inspecting a graph instance

```

1  >>> g.showShort()
2  *---- short description of the graph ----*
3  Name : 'tutorialGraph'
4  Vertices : ['v1', 'v2', 'v3', 'v4', 'v5', 'v6', 'v7']
5  Valuation domain : {'min': -1, 'med': 0, 'max': 1}
6  Gamma function :
7  v1 -> ['v5']
8  v2 -> ['v6', 'v4', 'v3']
9  v3 -> ['v2']
10 v4 -> ['v5', 'v2', 'v7']
11 v5 -> ['v1', 'v6', 'v4']
12 v6 -> ['v2', 'v5']
13 v7 -> ['v4']
14 degrees      : [0, 1, 2, 3, 4, 5, 6]
15 distribution  : [0, 3, 1, 3, 0, 0, 0]
16 nbh depths   : [0, 1, 2, 3, 4, 5, 6, 'inf.']
17 distribution  : [0, 0, 1, 4, 2, 0, 0, 0]
  
```

A Graph instance corresponds bijectively to a symmetric Digraph instance and we can convert from one to the other with the `graph2Digraph()`, and vice versa, with the `digraph2Graph()` method (see List. 21.4). Thus, all computing resources of the Digraph class suitable for symmetric digraphs become readily available, and vice versa.

**Listing 21.4** Conversion between graphs and digraphs

```
1  >>> dg = g.graph2Digraph()
```

```

2 >>> dg.showRelationTable(ndigits=0,ReflexiveTerms=False)
3     * ---- Relation Table ----
4     S | 'v1'  'v2'  'v3'  'v4'  'v5'  'v6'  'v7'
5     ---|-----
6     'v1' |   -   -1   -1   -1    1   -1   -1
7     'v2' |   -1   -   1    1   -1    1   -1
8     'v3' |   -1   1   -   -1   -1   -1   -1
9     'v4' |   -1   1   -1   -   1   -1    1
10    'v5' |    1   -1   -1    1   -   1   -1
11    'v6' |   -1   1   -1   -1    1   -   -1
12    'v7' |   -1   -1   -1    1   -1   -1   -
13 >>> g1 = dg.digraph2Graph()
14 >>> g1
15     *----- Digraph instance description -----*
16     Instance class      : Digraph
17     Instance name       : tutorialGraph
18     Digraph Order       : 7
19     Digraph Size        : 14
20     Valuation domain   : [-1.00;1.00]
21     Determinateness (%) : 100.00
22     Attributes          : ['name', 'order', 'actions',
23                               'valuationdomain', 'relation',
24                               'gamma', 'notGamma']

```

## 21.2 Q-coloring of a graph

With the `Q_Coloring` class, a 3-coloring of the tutorial graph `g` may be computed with a Gibbs sampler (Geman and Geman, 1984) and plotted as shown in Listing 21.5.

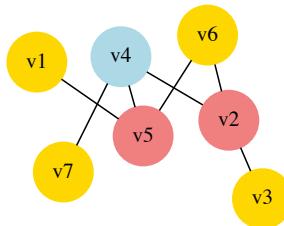
**Listing 21.5** Computing a 3-coloring of the random graph  $g$

Actually, with the given tutorial graph instance, a 2-coloring is already feasible (see Fig. 21.2).

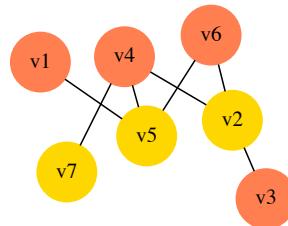
```

1 >>> qc = Q_Coloring(g,colors=['gold','coral'])
2   Running a Gibbs Sampler for 42 step !
3   The q-coloring with 2 colors is feasible !!
4 >>> qc.showConfiguration()
5     v5 gold
6     v3 coral
7     v7 gold
8     v2 gold
9     v4 coral
10    v1 coral
11    v6 coral
12 >>> qc.exportGraphViz('tutorial-2-coloring')
13   Exporting to tutorial-2-coloring.dot
14   fdp -Tpng tutorial-2-coloring.dot\
15                               -o tutorial-2-coloring.png

```



Digraph3 (graphviz), R. Bisdorff, 2019



Digraph3 (graphviz), R. Bisdorff, 2019

**Fig. 21.2** 3-Coloring and 2-Coloring of the tutorial graph

## 21.3 MIS and clique enumeration

2-colorings define sets of independent vertices that are maximal in cardinality –the MISs. The `showMIS()` method computes and prints out such sets. The result is stored in a `missets` attribute (see Listing 21.6)

**Listing 21.6** Computing and printing the maximal independent sets of graph `g`

```

1 >>> g = Graph('tutorialGraph')
2 >>> g.showMIS()
3   *---  Maximal Independent Sets  ---*

```

```

4      ['v2', 'v5', 'v7']
5      ['v3', 'v5', 'v7']
6      ['v1', 'v2', 'v7']
7      ['v1', 'v3', 'v6', 'v7']
8      ['v1', 'v3', 'v4', 'v6']
9      number of solutions: 5
10     cardinality distribution
11     card.: [0, 1, 2, 3, 4, 5, 6, 7]
12     freq.: [0, 0, 0, 3, 2, 0, 0, 0]
13     execution time: 0.00032 sec.
14     Results in self.misset
15 >>> g.misset
16     [frozenset({'v7', 'v2', 'v5'}),
17      frozenset({'v3', 'v7', 'v5'}),
18      frozenset({'v1', 'v2', 'v7'}),
19      frozenset({'v1', 'v6', 'v7', 'v3'}),
20      frozenset({'v1', 'v6', 'v4', 'v3'})]

```

A MIS in the dual `-g` of a graph instance `g`, corresponds to a maximal *clique*, i.e. a maximal complete subgraph in `g`. Maximal cliques may be computed and printed with the `showCliques()` method. The result is stored in a `clique` attribute as shown in Listing 21.7.

**Listing 21.7** Computing and printing the maximal independent sets of graph `g`

```

1 >>> g.showCliques()
2     *--- Maximal Cliques ---*
3     ['v2', 'v3']
4     ['v4', 'v7']
5     ['v2', 'v4']
6     ['v4', 'v5']
7     ['v1', 'v5']
8     ['v2', 'v6']
9     ['v5', 'v6']
10    number of solutions: 7
11    cardinality distribution
12    card.: [0, 1, 2, 3, 4, 5, 6, 7]
13    freq.: [0, 0, 7, 0, 0, 0, 0, 0]
14    execution time: 0.00049 sec.
15    Results in self.cliques
16 >>> g.cliques
17     [frozenset({'v2', 'v3'}), frozenset({'v4', 'v7'}),
18      frozenset({'v2', 'v4'}), frozenset({'v4', 'v5'}),
19      frozenset({'v1', 'v5'}), frozenset({'v6', 'v2'}),
20      frozenset({'v6', 'v5'})]

```

## 21.4 Line graphs and maximal matchings

The `graphs` module also provides a `LineGraph` constructor. A *line graph* represents the adjacencies between edges of the given graph instance. In Listing 21.8 on the next page we compute for instance the line graph of the 5-cycle graph.

**Listing 21.8** Computing the line graph of the 5-cycle graph

```

1 >>> from graphs import CycleGraph, LineGraph
2 >>> g = CycleGraph(order=5)
3 >>> g
4     *----- Graph instance description -----*
5     Instance class    : CycleGraph
6     Instance name     : cycleGraph
7     Graph Order       : 5
8     Graph Size        : 5
9     Valuation domain : [-1.00; 1.00]
10    Attributes       : ['name', 'order',
11                      'vertices', 'valuationDomain',
12                      'edges', 'size', 'gamma']
13 >>> lg = LineGraph(g)
14 >>> lg
15     *----- Graph instance description -----*
16     Instance class    : LineGraph
17     Instance name     : line-cycleGraph
18     Graph Order       : 5
19     Graph Size        : 5
20     Valuation domain : [-1.00; 1.00]
21     Attributes       : ['name', 'graph',
22                      'valuationDomain', 'vertices',
23                      'order', 'edges', 'size', 'gamma']
24 >>> lg.showShort()
25     *---- short description of the graph ----*
26     Name              : 'line-cycleGraph'
27     Vertices          : [frozenset({'v1', 'v2'}),
28                           frozenset({'v1', 'v5'}), frozenset({'v2', 'v3'}),
29                           frozenset({'v3', 'v4'}), frozenset({'v4', 'v5'})]
30     Valuation domain : {'min': Decimal('-1'),
31                           'med': Decimal('0'), 'max': Decimal('1')}
32     Gamma function   :
33     frozenset({'v1', 'v2'}) ->
34     [frozenset({'v2', 'v3'}), frozenset({'v1', 'v5'})]
35     frozenset({'v1', 'v5'}) ->
36     [frozenset({'v1', 'v2'}), frozenset({'v4', 'v5'})]
37     frozenset({'v2', 'v3'}) ->
38     [frozenset({'v1', 'v2'}), frozenset({'v3', 'v4'})]
39     frozenset({'v3', 'v4'}) ->
40     [frozenset({'v2', 'v3'}), frozenset({'v4', 'v5'})]
41     frozenset({'v4', 'v5'}) ->
42     [frozenset({'v4', 'v3'}), frozenset({'v1', 'v5'})]
43     degrees          : [0, 1, 2, 3, 4]
44     distribution     : [0, 0, 5, 0, 0]
45     nbh depths      : [0, 1, 2, 3, 4, 'inf.']
46     distribution     : [0, 0, 5, 0, 0, 0]

```

Iterated line graph constructions are usually expanding, except for chordless cycles, where the same cycle is repeated, and for non-closed paths, where iterated line graphs progressively reduce one by one the number of vertices and edges and eventually become an empty graph.

Notice in Listing 21.9 that the MISs in a line graph provide *maximal matchings* –maximal sets of independent edges– of the original graph.

**Listing 21.9** Computing the MISs of the line graph of the 8-cycle graph

```

1 >>> c8 = CycleGraph(order=8)
2 >>> lc8 = LineGraph(c8)
3 >>> lc8.showMIS()
4     Maximal Independent Sets ---
5     [{frozenset({'v3', 'v4'}), frozenset({'v5', 'v6'}), frozenset({'v1', 'v8'})}]
6     [{frozenset({'v2', 'v3'}), frozenset({'v5', 'v6'}), frozenset({'v1', 'v8'})}]
7     [{frozenset({'v8', 'v7'}), frozenset({'v2', 'v3'}), frozenset({'v5', 'v6'})}]
8     [{frozenset({'v8', 'v7'}), frozenset({'v2', 'v3'}), frozenset({'v4', 'v5'})}]
9     [{frozenset({'v7', 'v6'}), frozenset({'v3', 'v4'}), frozenset({'v1', 'v8'})}]
10    [{frozenset({'v2', 'v1'}), frozenset({'v8', 'v7'}), frozenset({'v4', 'v5'})}]
11    [{frozenset({'v2', 'v1'}), frozenset({'v7', 'v6'}), frozenset({'v4', 'v5'})}]
12    [{frozenset({'v2', 'v1'}), frozenset({'v7', 'v6'}), frozenset({'v3', 'v4'})}]
13    [{frozenset({'v7', 'v6'}), frozenset({'v2', 'v3'}), frozenset({'v1', 'v8'})},
14      frozenset({'v4', 'v5'})]
15    [{frozenset({'v2', 'v1'}), frozenset({'v8', 'v7'}), frozenset({'v3', 'v4'})},
16      frozenset({'v5', 'v6'})]
17    number of solutions: 10
18    cardinality distribution
19    card.: [0, 1, 2, 3, 4, 5, 6, 7, 8]
20    freq.: [0, 0, 0, 8, 2, 0, 0, 0, 0]
21    execution time: 0.00029 sec.

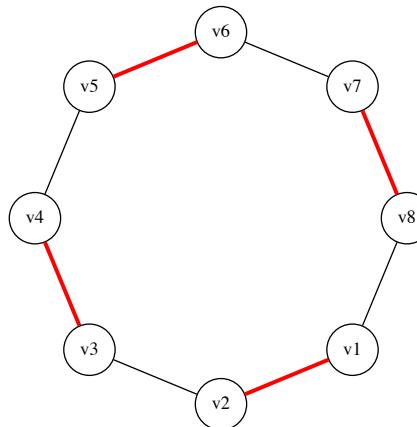
```

**Fig. 21.3** A perfect maximum matching of the 8-cycle graph.

Every vertex of the 8-cycle is

adjacent to a matching edge.

Odd cycle graphs do not admit any perfect matching



*Digraph3 (graphviz), R. Bisdorff, 2019*

The two last MISs of cardinality 4 (see Lines 13-16 in List. 21.9) give perfect maximum matchings of the 8-cycle graph as shown in Figure 21.3.

**Listing 21.10** Computing maximum matchings in the 8-cycle graph

```

1 >>> maxMatching = c8.computeMaximumMatching()
2 >>> c8.exportGraphViz(fileName='maxMatchingcycleGraph', \

```

```

3 ...                               matching=maxMatching)
4 *---- exporting a dot file for GraphViz tools ----*
5 Exporting to maxMatchingcycleGraph.dot
6 Matching: {frozenset({'v1', 'v2'}),
7             frozenset({'v5', 'v6'}),
8             frozenset({'v3', 'v4'}),
9             frozenset({'v7', 'v8'}) }
10 circo -Tpng maxMatchingcycleGraph.dot\
11             -o maxMatchingcycleGraph.png

```

## 21.5 Grids and the ISING model

Special classes of graphs, like the `GridGraph` class provide  $n \times m$  rectangular or triangular grids. With a *Gibbs* sampler<sup>2</sup> the `IsingModel` class can simulate in Figure 21.4 on the next page an ISING model on, for instance, a  $15 \times 15$  grid (Ising, 1925).

**Listing 21.11** Simulating an Ising model on a the  $15 \times 15$  rectangular grid

```

1 >>> from graphs import GridGraph, IsingModel
2 >>> g = GridGraph(n=15, m=15)
3 >>> g.showShort()
4 *---- show short -----*
5 Grid graph : grid-6-6
6 n : 6
7 m : 6
8 order : 36
9 >>> im = IsingModel(g, beta=0.3, nSim=100000)
10 Running a Gibbs Sampler for 100000 step !
11 >>> im.exportGraphViz(colors=['lightblue', 'lightcoral'])
12 *---- exporting a dot file for GraphViz tools -----*
13 Exporting to grid-15-15-ising.dot
14 fdp -Tpng grid-15-15-ising.dot -o grid-15-15-ising.png

```

## 21.6 Simulating METROPOLIS random walks

Finally, we provide the `MetropolisChain` class, a specialization of the `Graph` class, implementing a generic Monte Carlo Markov Chain (MCMC) sampler for simulating random walks on a graph following given transition probabilities `probs = {'v1': x, 'v2': y, ...}` for visiting each vertex (see List. 21.12) (Metropolis et al, 1953).

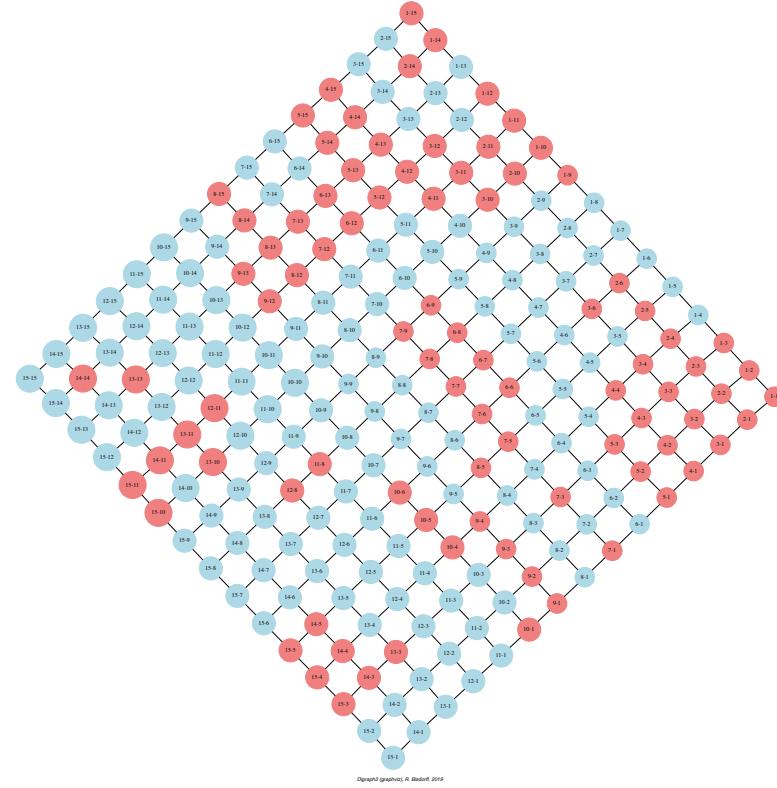
**Listing 21.12** Simulating random walks on a graph

```

1 >>> from graphs import Graph

```

<sup>2</sup> Geman and Geman (1984)



**Fig. 21.4** Ising model of the 15x15 grid graph.

```

2 >>> g = Graph(numberOfVertices=5,edgeProbability=0.5)
3 >>> g.showShort()
4     *----- short description of the graph -----*
5     Name          : 'randomGraph'
6     Vertices      : ['v1', 'v2', 'v3', 'v4', 'v5']
7     Valuation domain : {'max': 1, 'med': 0, 'min': -1}
8     Gamma function :
9     v1 -> ['v2', 'v3', 'v4']
10    v2 -> ['v1', 'v4']
11    v3 -> ['v5', 'v1']
12    v4 -> ['v2', 'v5', 'v1']
13    v5 -> ['v3', 'v4']
14 >>> # initialize a potential stationary probability vector
15 >>> n = g.order
16 >>> i = 0
17 >>> for v in g.vertices:

```

```

18 ...     probs[v] = (n - i) / (n * (n+1) / 2)
19 ...     i += 1

```

The `checkSampling()` method of the `MetropolisChain` class (see below) generates a random walk of  $nSim = 30000$  steps on the given graph and records by the way the observed relative frequency with which each vertex is passed by. In Listing 21.13 we notice how well the random walk is matching the given stationary probability vector.

**Listing 21.13** Checking the quality of the MCMC sampler

```

1 >>> from graphs import MetropolisChain
2 >>> met = MetropolisChain(g,probs)
3 >>> frequency = met.checkSampling(verticesList[0],nSim=30000)
4 >>> for v in verticesList:
5 ...     print(v,probs[v],frequency[v])
6     v1 0.3333 0.3343
7     v2 0.2666 0.2680
8     v3 0.2 0.2030
9     v4 0.1333 0.1311
10    v5 0.0666 0.0635

```

In this example, the stationary transition probability distribution (see List. 21.13 above), shown by the `showTransitionMatrix()` method in Listing 21.14 is quite adequately simulated.

**Listing 21.14** Printing the transition probability distribution

```

1 >>> met.showTransitionMatrix()
2 * ----- Transition Matrix -----
3   Pij | 'v1'   'v2'   'v3'   'v4'   'v5'
4   -----|-----
5   'v1' | 0.23   0.33   0.30   0.13   0.00
6   'v2' | 0.42   0.42   0.00   0.17   0.00
7   'v3' | 0.50   0.00   0.33   0.00   0.17
8   'v4' | 0.33   0.33   0.00   0.08   0.25
9   'v5' | 0.00   0.00   0.50   0.50   0.00

```

For the reader interested in algorithmic applications of Markov Chains we recommend consulting [Häggström \(2002\)](#).

## 21.7 Computing the non isomorphic MISs of the n-cycle graph

Due to the public success of our common work with Jean-Luc Marichal ([Bisdorff and Marichal, 2008](#)), we present in this chapter an example Python session for computing the non isomorphic maximal independent sets (MISs) of the 12-cycle graph, i.e. a `CirculantDigraph` class instance of order 12 and symmetric circulants 1 and  $-1$ .

```

1 >>> from digraphs import CirculantDigraph
2 >>> c12 = CirculantDigraph(order=12,circulants=[1,-1])

```

```

3 >>> c12 # 12-cycle digraph instance
4 *----- Digraph instance description -----*
5 Instance class : CirculantDigraph
6 Instance name : c12
7 Digraph Order : 12
8 Digraph Size : 24
9 Valuation domain : [-1.0, 1.0]
10 Determinateness : 100.000
11 Attributes : ['name', 'order', 'circulants',
12 'actions', 'valuationdomain',
13 'relation', 'gamma', 'notGamma']
```

Such  $n$ -cycle graphs – see the 12-cycle graph in Figure 21.5 on page 331 – are also provided as undirected graph instances by the `CycleGraph` class.

```

1 >>> from graphs import CycleGraph
2 >>> cg12 = CycleGraph(order=12)
3 >>> cg12
4 *----- Graph instance description -----*
5 Instance class : CycleGraph
6 Instance name : cycleGraph
7 Graph Order : 12
8 Graph Size : 12
9 Valuation domain : [-1.0, 1.0]
10 Attributes : ['name', 'order', 'vertices',
11 'valuationDomain', 'edges',
12 'size', 'gamma']
```

A non-isomorphic MIS of the 12-cycle graph corresponds in fact to a set of isomorphic MISs, i.e. an orbit of MISs under the automorphism group of the 12-cycle graph. In Listing 21.15 we are now first computing all maximal independent sets that are detectable in the 12-cycle digraph with the `showMIS()` method.

**Listing 21.15** Computing the MISs of the 12-cycle graph

```

1 >>> c12.showMIS(withListing=False)
2 *--- Maximal independent choices ---
3 number of solutions: 29
4 cardinality distribution
5 card.: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
6 freq.: [0, 0, 0, 0, 3, 24, 2, 0, 0, 0, 0, 0]
7 Results in c12.misset
```

In the 12-cycle graph, we observe 29 labelled MISs: 3 of cardinality 4, 24 of cardinality 5, and 2 of cardinality 6. In case of  $n$ -cycle graphs with  $n > 20$ , as the cardinality of the MISs becomes big, it is preferable to use, as shown in Listing 21.16 on the facing page, the `perrinMIS` shell command<sup>3</sup> compiled from C and installed along with all the DIGRAPH3 python modules for computing the set of MISs observed in the graph.

<sup>3</sup> The `perrinMIS` shell command may be installed system wide with the command `textttmake installPerrin` from the main DIGRAPH3 directory. It is stored by default into `/usr/local/bin/`. This may be changed with the `INSTALLDIR` flag. The command `make installPerrinUser` installs it instead without sudo into the user's private `.bin` directory.

**Listing 21.16** Computing the MISs with the perrinMIS shell command

```

1 ...$ echo 12 | /usr/local/bin/perrinMIS
2 # -----
3 # Generating MIS set of Cn with the      #
4 # Perrin sequence algorithm.           #
5 # Temporary files used.               #
6 # even versus odd order optimised.   #
7 # RB December 2006                   #
8 # Current revision Dec 2018          #
9 # -----
10 Input cycle order ? <-- 12
11 mis 1 : 100100100100
12 mis 2 : 010010010010
13 mis 3 : 001001001001
14 ...
15 ...
16 ...
17 mis 27 : 001001010101
18 mis 28 : 101010101010
19 mis 29 : 010101010101
20 Cardinalities:
21 0 : 0
22 1 : 0
23 2 : 0
24 3 : 0
25 4 : 3
26 5 : 24
27 6 : 2
28 7 : 0
29 8 : 0
30 9 : 0
31 10 : 0
32 11 : 0
33 12 : 0
34 Total: 29
35 execution time: 0 sec. and 2 millisec.

```

Reading in the result of the perrinMIS shell command, stored in a file called by default curd.dat, may be operated with the readPerrinMisset () method .

```

1 >>> c12.readPerrinMisset(file='curd.dat')
2 >>> c12.misset
3     {frozenset({'5', '7', '10', '1', '3'}),
4      frozenset({'9', '11', '5', '2', '7'}),
5      frozenset({'7', '2', '4', '10', '12'}),
6      ...
7      ...
8      ...
9      frozenset({'8', '4', '10', '1', '6'}),
10     frozenset({'11', '4', '1', '9', '6'}),
11     frozenset({'8', '2', '4', '10', '12', '6'})
12 }

```

For computing the corresponding non-isomorphic MISs, we actually need the automorphism group of the 12-cycle graph. The Digraph class therefore provides the

`automorphismGenerators()` method which adds automorphism group generators to a `DiGraph` class instance with the help of the external shell `dreadnaut` shell command from the *nauty* software package<sup>4</sup>.

**Listing 21.17** Computing the automorphism group generators

```

1 >>> c12.automorphismGenerators()
2     ...
3     Permutations
4     {'1':'1', '2':'12', '3':'11', '4':'10',
5      '5':'9', '6':'8', '7':'7', '8':'6', '9':'5',
6      '10':'4', '11':'3', '12':'2'}
7     {'1':'2', '2':'1', '3':'12', '4':'11', '5':'10',
8      '6':'9', '7':'8', '8':'7', '9':'6', '10':'5',
9      '11':'4', '12':'3'}
10 >>> print('grpsize = ', c12.automorphismGroupSize)
11     grpsize = 24

```

The 12-cycle graph's automorphism group, of size 24, is generated with both the permutations shown in Listing 21.17.

The `showOrbits()` method renders now the labelled representatives of each of the four orbits of isomorphic MISs observed in the 12-cycle graph (see Lines 7-10 below).

**Listing 21.18** Computing the MISs orbits of the 12-cycle graph

```

1 >>> c12.showOrbits(c12.misset, withListing=False)
2     ...
3     *---- Global result ----
4     Number of MIS: 29
5     Number of orbits : 4
6     Labelled representatives and cardinality:
7     1: ['2','4','6','8','10','12'], 2
8     2: ['2','5','8','11'], 3
9     3: ['2','4','6','9','11'], 12
10    4: ['1','4','7','9','11'], 12
11    Symmetry vector
12    stabilizer size: [1, 2, 3, ..., 8, 9, ..., 12, 13, ...]
13    frequency      : [0, 2, 0, ..., 1, 0, ..., 1, 0, ...]

```

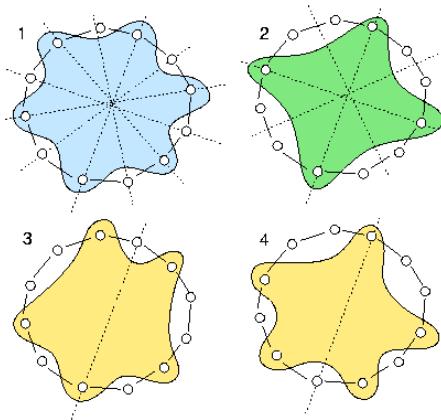
The corresponding group stabilizers' sizes and frequencies: orbit 1 with 6 symmetry axes, orbit 2 with 4 symmetry axes, and orbits 3 and 4 both with one symmetry axis (see Lines 12-13 in List. 21.18), are illustrated in the corresponding unlabelled graphs of Figure 21.5 on the next page.

The non isomorphic MISs in the 12-cycle graph represent in fact all the ways one may write the number 12 as the circular sum of '2's and '3's without distinguishing opposite directions of writing. The first orbit corresponds to writing six times a '2'; the second orbit corresponds to writing four times a '3'. The third and fourth orbit

---

<sup>4</sup> The `automorphismGenerators` method uses the `dreadnaut` shell command from the *nauty* software package. See <https://www3.cs.stonybrook.edu/~algorith/implement/nauty/implementation.shtml>. On Mac OS there exist `dmg` installers and on Ubuntu Linux or Debian, one may easily install it with `...$ sudo apt-get install nauty`.

**Fig. 21.5** The symmetry axes of the four non isomorphic MISs of the 12-cycle graph



correspond to writing two times a '3' and three times a '2'. There are two non isomorphic ways to do this latter circular sum. Either separating the '3's by one and two '2's, or by zero and three '2's (Bisdorff and Marichal, 2008).

The next chapter is devoted more specifically to tree graphs and graph forests.

## References

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# Chapter 22

## On tree graphs and graph forests

**Abstract** The chapter specifically addresses working with tree graphs and graph forests. We illustrate how to generate and recognize random tree graphs and how to compute the centres of a tree and draw a rooted and oriented tree. Finally, algorithms for computing spanning trees and forests are presented.

### 22.1 Generating random tree graphs

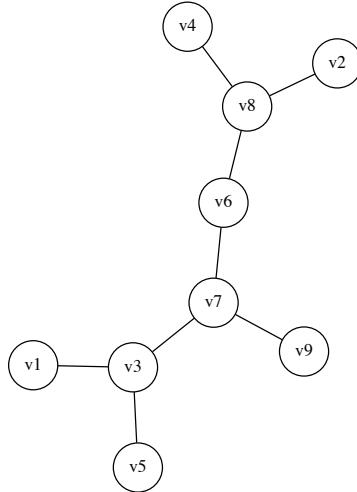
Using the `RandomTree` class from the `graphs` module, we may, for instance, generate a random tree graph with 9 vertices.

**Listing 22.1** Generating a random tree graph

```
1 >>> from graphs import RandomTree
2 >>> rt = RandomTree(order=9, seed=100)
3 >>> rt
4     *----- Graph instance description -----*
5     Instance class    : RandomTree
6     Instance name     : randomTree
7     Graph Order       : 9
8     Graph Size        : 8
9     Valuation domain : [-1.00; 1.00]
10    Attributes       : ['name', 'order',
11                        'vertices', 'valuationDomain',
12                        'edges', 'prueferCode',
13                        'size', 'gamma']
14    *----- RandomTree specific data -----*
15    Pruefer code    : ['v3','v8','v8','v3','v7','v6','v7']
16 >>> rt.exportGraphViz('tutRandomTree')
17    *----- exporting a dot file for GraphViz tools --*
18    Exporting to tutRandomTree.dot
19    neato -Tpng tutRandomTree.dot -o tutRandomTree.png
```

In Fig. 22.1 on the following page one may notice that a tree graph of order  $n > 2$  always contains  $n - 1$  edges (see Lines 7 and 8 in Listing 22.1) and its structure is

**Fig. 22.1** Random tree graph instance of order 9. One may distinguish vertices like  $v_1$ ,  $v_2$ ,  $v_4$ ,  $v_5$  or  $v_9$  of degree 1, called the *leaves* of the tree, and vertices like  $v_3$ ,  $v_6$ ,  $v_7$  or  $v_8$  of degree 2 or more, called the *nodes* of the tree



*Digraph3 (graphviz), R. Bisdorff, 2019*

entirely characterised by a corresponding PRÜFER *code*, i.e. a list of vertices keys of length  $n - 2$ . See, for instance in Line 15 the code `['v3', 'v8', 'v8', 'v3', 'v7', 'v6', 'v7']` corresponding to our sample tree graph `rt`.

Each position of the code indicates the parent of the remaining leaf with the smallest vertex label. Vertex  $v_3$  is thus the parent of  $v_1$  and we drop leaf  $v_1$ ,  $v_8$  is now the parent of leaf  $v_2$  and we drop  $v_2$ , vertex  $v_8$  is again the parent of leaf  $v_4$  and we drop  $v_4$ , vertex  $v_3$  is the parent of leaf  $v_5$  and we drop  $v_5$ ,  $v_7$  is now the parent of leaf  $v_3$  and we may drop  $v_3$ ,  $v_6$  becomes the parent of leaf  $v_8$  and we drop  $v_8$ ,  $v_7$  becomes the parent of leaf  $v_6$  and we may drop  $v_6$ . The two eventually remaining vertices,  $v_7$  and  $v_9$ , give the last link in the reconstructed tree (Barthélemy and Guenoche, 1991).

It is, as well possible to, first, generate a random PRÜFER code of length  $n - 2$  from a set of  $n$  vertices (see Listing 22.2 Lines 1-9) and then, construct the corresponding tree of order  $n$  by reversing the procedure illustrated above. The resulting tree graph is shown in Fig. 22.2 on the next page.

**Listing 22.2** Generating a tree graph with a random PRÜFER code.

```

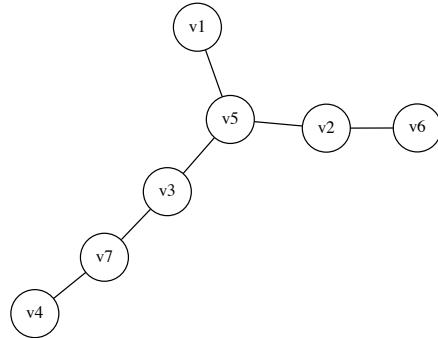
1 >>> verticesList = ['v1', 'v2', 'v3', 'v4', 'v5', 'v6', 'v7']
2 >>> n = len(verticesList)
3 >>> from random import seed, choice
4 >>> seed(101)
5 >>> code = []
6 >>> for k in range(n-2):
7 ...     code.append( choice(verticesList) )
8 >>> print(code)
  
```

```

9      ['v5', 'v7', 'v2', 'v5', 'v3']
10 >>> rt = RandomTree(prueferCode=code)
11 >>> rt
12 *----- Graph instance description -----*
13 Instance class      : RandomTree
14 Instance name       : randomTree
15 Graph Order         : 7
16 Graph Size          : 6
17 Valuation domain   : [-1.00; 1.00]
18 Attributes : ['name', 'order', 'vertices',
19                 'valuationDomain', 'edges',
20                 'prueferCode', 'size', 'gamma']
21 *---- RandomTree specific data ----*
22 Pruefer code : ['v5', 'v7', 'v2', 'v5', 'v3']
23 >>> rt.exportGraphViz('tutPruefTree')
24 *---- exporting a dot file for GraphViz tools -----*
25 Exporting to tutPruefTree.dot
26 neato -Tpng tutPruefTree.dot -o tutPruefTree.png

```

**Fig. 22.2** Tree instance generated with a random PRÜFER code ['v5', 'v7', 'v2', 'v5', 'v3']



*Digraph3 (graphviz), R. Bisdorff, 2019*

Following from the bijection between a labelled tree and its PRÜFER code, we actually know that there exist  $n^{n-2}$  different tree graphs with the same  $n$  vertices.

Given a genuine graph, how can we recognize that it is in fact a tree instance ?

## 22.2 Recognizing tree graphs

Given a graph  $g$  of order  $n$  and size  $s$ , the following 5 assertions A1, A2, A3, A4 and A5 are all equivalent (see (Barthélemy and Guenoche, 1991)):

A1  $g$  is a tree;

- A2  $g$  is without (chordless) cycles and  $n = s + 1$ ;
- A3  $g$  is connected and  $n = s + 1$ ;
- A4 Any two vertices of  $g$  are always connected by a unique path;
- A5  $g$  is connected and dropping any single edge will always disconnect  $g$ .

Assertion A3, for instance, gives a simple test for recognizing a tree graph. In case of a *lazy evaluation* of the test in Listing 22.3 Line 3 below, it is opportune, from a computational complexity perspective, to first, check the order and size of the graph, before checking its potential connectedness. We provide the `isTree()` method for computing both these tests (see Line 8 in Listing 22.3 below).

**Listing 22.3** Recognizing a tree graph.

```

1 >>> from graphs import RandomGraph
2 >>> g = RandomGraph(order=8, edgeProbability=0.3, seed=62)
3 >>> if g.order == (g.size + 1) and g.isConnected():
4 ...     print('The graph is a tree ?', True)
5 ... else:
6 ...     print('The graph is a tree ?', False)
7     The graph is a tree ? True
8 >>> g.isTree()
9     True

```

The random graph of order 8 and edge probability 30%, generated with seed 62, is actually a tree graph instance, as confirmed by its graphviz drawing shown in Fig. 22.3 on the facing page.

```

1 >>> g.exportGraphViz('test62')
2     *---- exporting a dot file for GraphViz tools ---*
3     Exporting to test62.dot
4     fdp -Tpng test62.dot -o test62.png

```

Yet, we still have to recover its corresponding PRÜFER code. Therefore, we may use the `tree2Pruefer()` method from the `TreeGraph` class. But, first, the instance class of graph  $g$  is changed to the `TreeGraph` type (see Line 2 in Listing 22.4).

**Listing 22.4** Computing the PRÜFER code of a tree graph instance.

```

1 >>> from graphs import TreeGraph
2 >>> g.__class__ = TreeGraph
3 >>> g.tree2Pruefer()
4     ['v6', 'v1', 'v2', 'v1', 'v2', 'v5']

```

In Fig. 22.3 on the facing page, we noticed that vertex  $v2$  is actually situated in the *centre* of the tree with a neighborhood depth of 2. Centres of a graph are the vertices with minimal neighbourhood depth. For finding such centre(s), the `Graph` class provides the `computeGraphCentres()` method. (see Lines 1-2 in Listing 22.4). Knowing now the centre of graph  $g$ , we may draw a correspondingly rooted and oriented tree with the `exportOrientedTreeGraphViz()` method from the `TreeGraph` class (see Fig. 22.4 on the facing page).

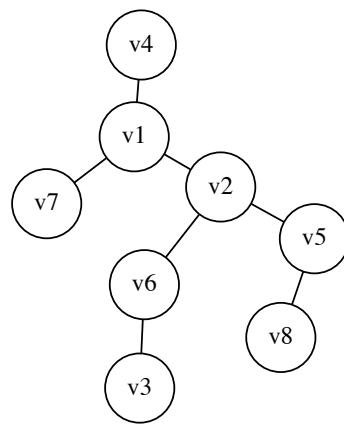
**Listing 22.5** Computing the centres of a tree and drawing a rooted and oriented tree.

```

1 >>> g.computeGraphCentres()

```

**Fig. 22.3** Recognizing a tree instance. We may notice that vertex  $v_2$  is actually situated in the *centre* of the tree with a neighborhood depth of 2



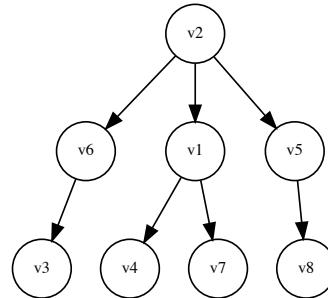
*Digraph3 (graphviz), R. Bisdorff, 2019*

```

2     {'v2': 2}
3 >>> g.exportOrientedTreeGraphViz(\n4     ...     fileName='rootedTree', root='v2')\n5     *---- exporting a dot file for GraphViz tools\n6     Exporting to rootedTree.dot\n7     dot -Grankdir=TB -Tpng rootedTree.dot -o rootedTree.png

```

**Fig. 22.4** Drawing an oriented tree rooted at its centre



*Digraph3 (graphviz)*  
R. Bisdorff, 2020

Let us now turn our attention toward a major application of tree graphs, namely *spanning trees* and *forests* related to graph traversals.

## 22.3 Spanning trees and forests

With the `RandomSpanningTree` class we may generate, from a given connected graph `g` instance, *uniform* random instances of a spanning tree by using WILSON's algorithm (see Listing 22.6 and Fig. 22.5 on the facing page).

**Listing 22.6** Generating uniform random spanning trees.

```

1 >>> from graphs import RandomGraph, \
2 ...                               RandomSpanningTree
3 >>> g = RandomGraph(order=9, \
4 ...                           edgeProbability=0.4, seed=100)
5 >>> spt = RandomSpanningTree(g)
6 >>> spt
7     *----- Graph instance description -----*
8     Instance class      : RandomSpanningTree
9     Instance name       : randomGraph_randomSpanningTree
10    Graph Order         : 9
11    Graph Size          : 8
12    Valuation domain   : [-1.00; 1.00]
13    Attributes          : ['name','vertices','order',
14                           'valuationDomain',
15                           'edges','size','gamma',
16                           'dfs','date', 'dfsx',
17                           'prueferCode']
18    *---- RandomTree specific data ----*
19    Pruefer code        : ['v7','v9','v5','v1','v8','v4','v9']
20 >>> spt.exportGraphViz(fileName='randomSpanningTree', \
21 ...                           WithSpanningTree=True)
22    *---- exporting a dot file for GraphViz tools ----*
23    Exporting to randomSpanningTree.dot
24    [[v1', 'v5', 'v6', 'v5', 'v1', 'v8', 'v9', 'v3', 'v9', 'v4',
25     'v7', 'v2', 'v7', 'v4', 'v9', 'v8', 'v1']]
26    neato -Tpng randomSpanningTree.dot \
27     -o randomSpanningTree.png

```

WILSON's algorithm only works for connected graphs (Wilson, 1996). More general, and in case of a not connected graph, the `RandomSpanningForest` class generates a not necessarily uniform random instance of a *spanning forest* – one or more random tree graphs – generated from a random *depth first search* of the graph components' traversals (see Listing 22.7 and Fig. 22.6 on page 340).

**Listing 22.7** Computing spanning forests over disconnected graphs.

```

1 >>> g = RandomGraph(order=15, \
2 ...                           edgeProbability=0.1, seed=140)
3 >>> g.computeComponents()
4  [{v12', 'v01', 'v13'}, {'v02', 'v06'}, ...

```

**Fig. 22.5** Random spanning tree

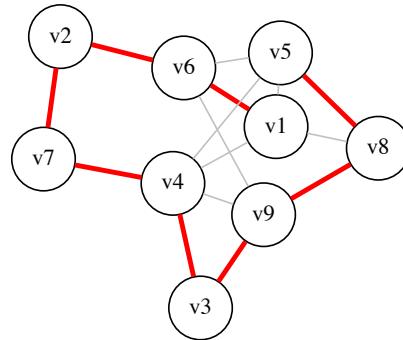


Diagram3 (graphviz), R. Bisdorff, 2019

```

5      {'v08','v03','v07'}, {'v15','v11','v10','v04','v05'},
6      {'v09','v14'}]
7 >>> spf = RandomSpanningForest(g, seed=100)
8 >>> spf.exportGraphViz(fileName='spanningForest', \
9 ...                           WithSpanningTree=True)
10     *----- exporting a dot file for GraphViz tools -----*
11     Exporting to spanningForest.dot
12     [['v03','v07','v08','v07','v03'],
13     ['v13','v12','v13','v01','v13'],
14     ['v02','v06','v02'],
15     ['v15','v11','v04','v11','v15',
16     'v10','v05','v10','v15'],
17     ['v09','v14','v09']]
18     neato -Tpng spanningForest.dot -o spanningForest.png

```

## 22.4 Maximum determined spanning forests

In case of valued graphs –supporting weighted edges– we may finally construct a *most determined* spanning tree (or forest if not connected) using KRUSKAL’s greedy minimum-spanning-tree algorithm on the dual valuation of the graph (Kruskal, 1956)<sup>1</sup>.

In Listing 22.8 we generate, for instance, a randomly valued graph with five vertices and seven edges bipolar-valued in [-1.0; 1.0].

<sup>1</sup> KRUSKAL’s algorithm is a minimum-spanning-tree algorithm which finds an edge of the least possible weight that connects any two trees in the forest.

**Fig. 22.6** Random spanning forest

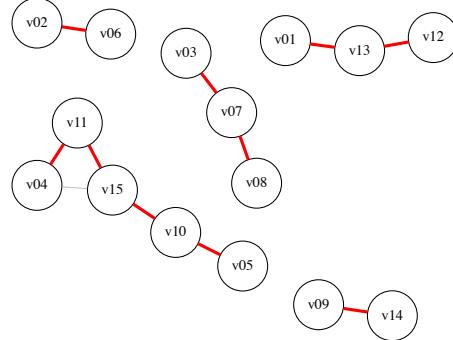


Diagram3 (graphviz), R. Bisdorff, 2019

**Listing 22.8** Generating randomly bipolar-valued graphs.

```

1 >>> from graphs import RandomValuationGraph
2 >>> g = RandomValuationGraph(order=5, seed=2)
3 >>> g
4 *----- Graph instance description -----*
5 Instance class      : RandomValuationGraph
6 Instance name       : randomGraph
7 Graph Order         : 5
8 Graph Size          : 7
9 Valuation domain   : [-1.00; 1.00]
10 Attributes         : ['name', 'order',
11                  'vertices', 'valuationDomain',
12                  'edges', 'size', 'gamma']

```

To inspect the edges' actual weights, we first transform the graph into a corresponding digraph (see Line 1 in Listing 22.9) and use the `showRelationTable()` method (see Line 2) for printing its symmetric adjacency matrix.

**Listing 22.9** Symmetric relation table

```

1 >>> dg = g.graph2Digraph()
2 >>> dg.showRelationTable()
3 *----- Relation Table -----*
4   S  |  'v1'  'v2'  'v3'  'v4'  'v5'
5   ---|-----
6   'v1' |  0.00  0.91  0.90 -0.89 -0.83
7   'v2' |  0.91  0.00  0.67  0.47  0.34
8   'v3' |  0.90  0.67  0.00 -0.38  0.21
9   'v4' | -0.89  0.47 -0.38  0.00  0.21
10  'v5' | -0.83  0.34  0.21  0.21  0.00
11 Valuation domain: [-1.00;1.00]

```

To compute the most determined spanning tree or forest, we can use the `BestDeterminedSpanningForest` constructor.

**Listing 22.10** Computing best determined spanning forests.

```

1 >>> from graphs import\
2             BestDeterminedSpanningForest
3 >>> mt = BestDeterminedSpanningForest(g)
4 >>> print(mt)
5     *----- Graph instance description -----*
6     Instance class      : BestDeterminedSpanningForest
7     Instance name       : bdSpanningForest
8     Graph Order         : 5
9     Graph Size          : 4
10    Valuation domain   : [-1.00; 1.00]
11    Attributes          : ['name','vertices','order',
12                           'valuationDomain',
13                           'edges','size','gamma',
14                           'dfs','date',
15                           'averageTreeDetermination']
16    *---- best determined spanning tree ----*
17    Depth first search path(s) :
18    [['v1','v2','v4','v2','v5','v2','v1','v3','v1']]
19    Average determination(s) : [Decimal('0.655')]

```

The random graph  $g$  is connected and, hence, admits a single spanning tree of maximum mean determination  $= (0.47 + 0.91 + 0.90 + 0.34)/4 = 0.655$  (see Lines 9, 6 and 10 in Listing 22.8 on page 339 and Fig. 22.7 on the following page).

```

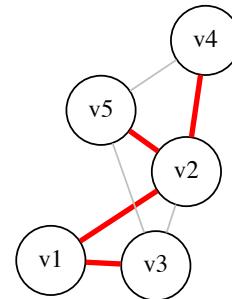
1 >>> mt.exportGraphViz(\
2 ...      fileName='bestDeterminedspanningTree', \
3 ...      WithSpanningTree=True)
4     *---- exporting a dot file for GraphViz tools ---*
5     Exporting to spanningTree.dot
6     [['v4','v2','v1','v3','v1','v2','v5','v2','v4']]
7     neato -Tpng bestDeterminedSpanningTree.dot \
8           -o bestDeterminedSpanningTree.png

```

One may easily verify that all other potential spanning trees, including instead the edges  $\{v3, v5\}$  and/or  $\{v4, v5\}$ , will show a lower average determination.

Chapter 23 on page 345, the last on undirected graphs, is devoted to different models of perfect graphs, namely split, interval, comparability and permutation graphs.

**Fig. 22.7** Best determined spanning tree



*Digraph3 (graphviz), R. Bisdorff, 2019*

## References

- Barthélemy J, Guenoche A (1991) Trees and Proximities Representations. Wiley  
[334](#), [335](#)
- Kruskal J (1956) On the shortest spanning subtree of a graph and the traveling salesman problem. Proceedings of the American Mathematical Society 7:48–50 [339](#)
- Wilson DB (1996) Generating random spanning trees more quickly than the cover time. In: Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia PA), ACM New York, pp 296–303 [338](#)



## Chapter 23

# About split, interval and permutation graphs

**Abstract** The last chapter of this book eventually presents some famous classes of perfect graphs, namely comparability, interval, permutation and split graphs. We first present an example of a graph which is at the same time a triangulated, a comparability, a split and a permutation graph. The importance to be an interval is illustrated with BERGE's mystery story. We discuss furthermore the generation of permutation graphs and close with how to recognize that a given graph is in fact a permutation graph.

### 23.1 A 'multiply' perfect graph

A graph  $g$  is called:

- BERGE or *perfect* when  $g$  and its dual  $-g$  both don't contain any chordless odd cycles of length greater than 3 (Berge, 1963; Chudnovsky, Robertson, Seymour, and Robin, 2006);
- *Triangulated* graph when  $g$  does not contain any *chordless cycle* of length 4 and more;

**Definition 23.1 (Some perfect graph classes).** Following Martin Golumbic (see Golumbic (2004, p. 149)), we call a given graph  $g$ :

- *Comparability* graph when  $g$  is *transitively orientable*;
- *Interval* graph when  $g$  is *triangulated* and its dual  $-g$  is a *comparability* graph;
- *Permutation* graph when  $g$  and its dual  $-g$  both are *comparability* graphs;
- *Split* graph when  $g$  and its dual  $-g$  both are *triangulated* graphs.

To illustrate these *perfect* graph classes, we will generate from 8 intervals, randomly chosen in the default integer range  $[0, 10]$ , a `RandomIntervalIntersectionsGraph` instance  $g$  (see Line 2 below).

```
1 >>> from graphs import \
2 ...           RandomIntervalIntersectionsGraph
```

```

3 >>> g = RandomIntervalIntersectionsGraph(\
4 ...                               order=8, seed=100)
5 >>> g
6 *----- Graph instance description -----*
7 Instance class      : RandomIntervalIntersectionsGraph
8 Instance name       : randIntervalIntersections
9 Seed                : 100
10 Graph Order        : 8
11 Graph Size         : 23
12 Valuation domain  : [-1.0; 1.0]
13 Attributes          : ['seed', 'name', 'order',
14   'intervals', 'vertices', 'valuationDomain',
15   'edges', 'size', 'gamma']
16 >>> print(g.intervals)
17 [(2,7), (2,7), (5,6), (6,8), (1,8), (1,1), (4,7), (0,10)]

```

With  $\text{seed} = 100$ , we obtain here an *interval graph*, in fact a *perfect graph*, which is *conjointly* a *triangulated*, a *comparability*, a *split* and a *permutation graph* (see Listing 23.1 Lines 6, 10, 14 and Fig. 23.1 on the next page).

**Listing 23.1** Testing perfect graph categories.

```

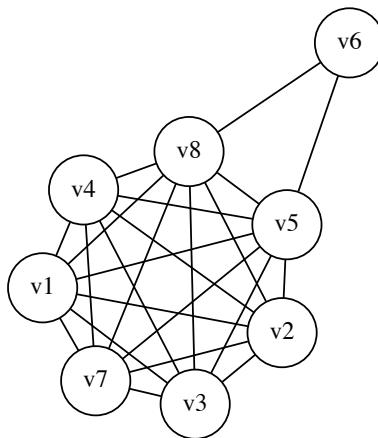
1 >>> g.isPerfectGraph(Comments=True)
2 Graph randIntervalIntersections is perfect !
3 >>> g.isIntervalGraph(Comments=True)
4 Graph 'randIntervalIntersections' is triangulated.
5 Graph 'dual_randIntervalIntersections' is transitively orientable.
6 => Graph 'randIntervalIntersections' is an interval graph.
7 >>> g.isSplitGraph(Comments=True)
8 Graph 'randIntervalIntersections' is triangulated.
9 Graph 'dual_randIntervalIntersections' is triangulated.
10 => Graph 'randIntervalIntersections' is a split graph.
11 >>> g.isPermutationGraph(Comments=True)
12 Graph 'randIntervalIntersections' is transitively orientable.
13 Graph 'dual_randIntervalIntersections' is transitively orientable.
14 => Graph 'randIntervalIntersections' is a permutation graph.
15 >>> print(g.computePermutation())
16 ['v5', 'v6', 'v4', 'v2', 'v1', 'v3', 'v7', 'v8']
17 ['v8', 'v6', 'v1', 'v2', 'v3', 'v4', 'v7', 'v5']
18 [8, 2, 6, 5, 7, 4, 3, 1]
19 >>> g.exportGraphViz('randomSplitGraph')
20 *---- exporting a dot file for GraphViz tools -----
21 Exporting to randomSplitGraph.dot
22 fdp -Tpng randomSplitGraph.dot -o randomSplitGraph.png

```

Notice however that the four properties:

1.  $g$  is a *comparability* graph;
  2.  $g$  is a *co-comparability* graph, i.e.  $-g$  is a comparability graph;
  3.  $g$  is a *triangulated* graph;
  4.  $g$  is a *co-triangulated* graph, i.e.  $-g$  is a comparability graph;
- are independent of one another [Golumbic \(2004, p. 275\)](#).

**Fig. 23.1** A conjointly triangulated, comparability, interval, permutation and split graph. In the Figure here one may readily recognize the essential characteristic of split graphs, namely being always splittable into two disjoint subgraphs: an *independent choice*  $\{v_6\}$  and a *clique*  $\{v_1, v_2, v_3, v_4, v_5, v_7, v_8\}$  – which explains their name



*Digraph3 (graphviz), R. Bisdorff, 2019*

## 23.2 Who is the liar?

Claude Berge's famous mystery story related by [Golumbic \(2004, p. 20\)](#) may well illustrate the importance of being an *interval graph*.

Suppose the file `berge.py`<sup>1</sup> contains the following Graph instance data:

```

1  vertices = {
2      'A': {'name': 'Abe', 'shortName': 'A'},
3      'B': {'name': 'Burt', 'shortName': 'B'},
4      'C': {'name': 'Charlotte', 'shortName': 'C'},
5      'D': {'name': 'Desmond', 'shortName': 'D'},
6      'E': {'name': 'Eddie', 'shortName': 'E'},
7      'I': {'name': 'Ida', 'shortName': 'I'},
8  }
9  valuationDomain = {'min':-1,'med':0,'max':1}
10 edges = {
11     frozenset(['A','B']) : 1,
12     frozenset(['A','C']) : -1,
13     frozenset(['A','D']) : 1,
14     frozenset(['A','E']) : 1,
15     frozenset(['A','I']) : -1,
16     frozenset(['B','C']) : -1,
17     frozenset(['B','D']) : -1,
18     frozenset(['B','E']) : 1,
19     frozenset(['B','I']) : 1,

```

<sup>1</sup> The file `berge.py` is provided in the `examples` directory of the DIGRAPH3 resources.

```

20     frozenset(['C','D']) : 1,
21     frozenset(['C','E']) : 1,
22     frozenset(['C','I']) : 1,
23     frozenset(['D','E']) : -1,
24     frozenset(['D','I']) : 1,
25     frozenset(['E','I']) : 1,
26 }
```

Six professors (labeled *A*, *B*, *C*, *D*, *E* and *I*) had been to the library on the day that a rare tractate was stolen. Each entered once, stayed for some time, and then left. If two professors were in the library at the same time, then at least one of them saw the other. Detectives questioned the professors and gathered the testimonies that *Abe* saw *Burt* and *Eddie*; *Burt* saw *Abe* and *Ida*; *Charlotte* saw *Desmond* and *Ida*; *Desmond* saw *Abe* and *Ida*; *Eddie* saw *Burt* and *Ida*; and *Ida* saw *Charlotte* and *Eddie*. This data is gathered in the previous file, where each positive edge  $\{x,y\}$  models the testimony that, either  $x$  saw  $y$ , or  $y$  saw  $x$ .

```

1 >>> from graphs import Graph
2 >>> g = Graph('berge')
3 >>> g.showShort()
4     ----- short description of the graph -----
5     Name          : 'berge'
6     Vertices      : ['A', 'B', 'C', 'D', 'E', 'I']
7     Valuation domain : {'min': -1, 'med': 0, 'max': 1}
8     Gamma function :
9     A -> ['D', 'B', 'E']
10    B -> ['E', 'I', 'A']
11    C -> ['E', 'D', 'I']
12    D -> ['C', 'I', 'A']
13    E -> ['C', 'B', 'I', 'A']
14    I -> ['C', 'E', 'B', 'D']
15 >>> g.exportGraphViz('bergel')
16     ----- exporting a dot file for GraphViz tools -----
17     Exporting to berge1.dot
18     fdp -Tpng berge1.dot -o berge1.png
```

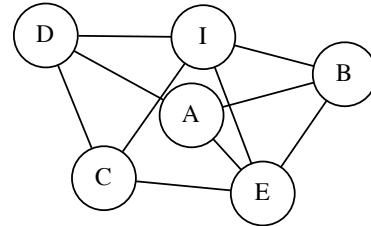
From graph theory we know that time interval intersections graphs must in fact be interval graph instances, i.e. *triangulated* and *co-comparative* graphs. The testimonies graph shown in Fig. 23.2 on the facing page should therefore not contain any chordless cycle of four and more vertices. Now, the presence or not of such chordless cycles in the testimonies graph may be checked with the `computeChordlessCycles()` method (Bisdorff, 2010).

```

1 >>> g.computeChordlessCycles()
2 Chordless cycle certificate: ['D','C','E','A','D']
3 Chordless cycle certificate: ['D','I','E','A','D']
4 Chordless cycle certificate: ['D','I','B','A','D']
5  [(['D','C','E','A','D'],frozenset({'C','D','E','A'})),
6   (['D','I','E','A','D'],frozenset({'D','E','I','A'})),
7   (['D','I','B','A','D'],frozenset({'D','B','I','A'}))]
```

We see in the Listing above three intersection cycles of length 4, which is impossible to occur on the linear time line. Obviously one professor lied!

**Fig. 23.2** Graph representation of the testimonies of the professors



*Digraph3 (graphviz), R. Bisdorff, 2019*

And it is *Desmond*. If we doubt his testimony that he saw *Abe* (see Line 1 below), we obtain indeed in Fig. 23.3 on the next page a triangulated graph instance whose dual is a comparability graph.

```

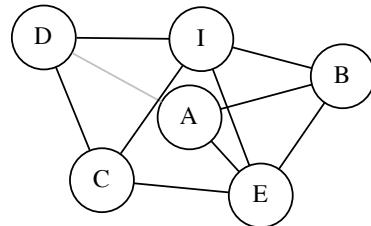
1 >>> g.setEdgeValue( ('D','A'), 0)
2 >>> g.showShort()
3     *---- short description of the graph ----*
4     Name          : 'berge'
5     Vertices      : ['A','B','C','D','E','I']
6     Valuation domain : {'med': 0,'min': -1,'max': 1}
7     Gamma function :
8     A -> ['B', 'E']
9     B -> ['A', 'I', 'E']
10    C -> ['I', 'E', 'D']
11    D -> ['I', 'C']
12    E -> ['A', 'I', 'B', 'C']
13    I -> ['B', 'E', 'D', 'C']
14 >>> g.isIntervalGraph(Comments=True)
15     Graph 'berge' is triangulated.
16     Graph 'dual_berge' is transitively orientable.
17     => Graph 'berge' is an interval graph.
18 >>> g.exportGraphViz('berge2')
19     *---- exporting a dot file for GraphViz tools ----*
20     Exporting to berge2.dot
21     fdp -Tpng berge2.dot -o berge2.png

```

### 23.3 Generating permutation graphs

A graph is called a *permutation* or *inversion* graph if there exists a permutation of its list of vertices such that the graph is isomorphic to the inversions operated by the

**Fig. 23.3** The triangulated testimonies graph



*Digraph3 (graphviz), R. Bisdorff, 2019*

permutation in this list (see [Golumbic \(2004, Chapter 7, pp 157-170\)](#)). This kind of graphs (see Fig. 23.4 on the facing page) is also part of the class of perfect graphs.

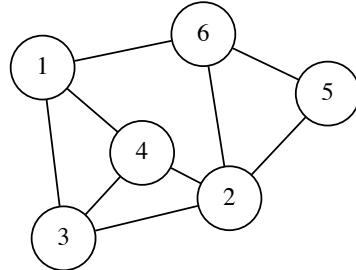
```

1 >>> from graphs import PermutationGraph
2 >>> g = PermutationGraph(permuation = [4,3,6,1,5,2])
3 >>> g
4 *----- Graph instance description -----*
5 Instance class : PermutationGraph
6 Instance name : permutationGraph
7 Graph Order : 6
8 Permutation : [4, 3, 6, 1, 5, 2]
9 Graph Size : 9
10 Valuation domain : [-1.00; 1.00]
11 Attributes : ['name', 'vertices', 'order',
12                 'permuation', 'valuationDomain',
13                 'edges', 'size', 'gamma']
14 >>> g.isPerfectGraph()
15 True
16 >>> g.exportGraphViz(fileName='permutationGraph')
17 *---- exporting a dot file for GraphViz tools ----*
18 Exporting to permutationGraph.dot
19 fdp -Tpng permutationGraph.dot \
20             -o permutationGraph.png

```

By using color sorting queues, the minimal vertex coloring shown in Fig. 23.5 on the next page is computable in  $O(n \log(n))$  (Golumbic, 2004).

**Fig. 23.4** The default Di-GRAPH3 permutation graph



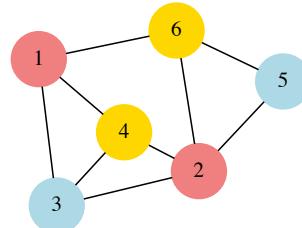
*Digraph3 (graphviz), R. Bisdorff, 2019*

```

10 *---- exporting a dot file for GraphViz tools ----*
11 Exporting to coloredPermutationGraph.dot
12 fdp -Tpng coloredPermutationGraph.dot \
13   -o coloredPermutationGraph.png

```

**Fig. 23.5** Minimal vertex coloring of the permutation graph.



*Digraph3 (graphviz), R. Bisdorff, 2019*

The correspondingly colored *matching diagram* of the nine inversions –the actual edges of the permutation graph–, which are induced by the given [4,3,6,1,5,2] permutation, may as well be drawn with the graphviz neato layout and explicitly positioned horizontal lists of vertices as shown in Fig. 23.6 on the following page.

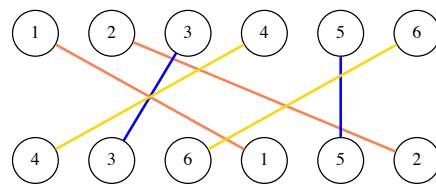
```

1 >>> g.exportPermutationGraphViz('matchingDiagram', \
2 ...                               WithEdgeColoring=True)
3 *---- exporting a dot file for GraphViz tools ----*
4   Exporting to matchingDiagram.dot
5   neato -n -Tpng matchingDiagram.dot \

```

```
6      -o matchingDiagram.png
```

**Fig. 23.6** Colored matching diagram of the permutation  $[4,3,6,1,5,2]$



Digraph3 (graphviz), R. Bisdorff, 2019

As mentioned before, a permutation graph and its dual are *transitively orientable*. The `transitiveOrientation()` method constructs from a given permutation graph a digraph where each edge of the permutation graph is converted into an arc oriented in increasing alphabetic order of the adjacent vertices' keys (see [Golumbic \(2004\)](#)). This orientation of the edges of a permutation graph is always transitive and delivers a *transitive ordering* of the vertices as shown in Fig. 23.7 on the next page.

```

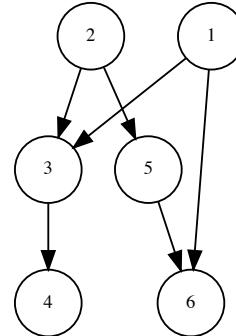
1 >>> dg = g.transitiveOrientation()
2 >>> dg
3     *----- Digraph instance description -----*
4     Instance class      : TransitiveDigraph
5     Instance name       : oriented_permutationGraph
6     Digraph Order       : 6
7     Digraph Size        : 9
8     Valuation domain   : [-1.00; 1.00]
9     Determinateness    : 100.000
10    Attributes          : ['name', 'order', 'actions',
11                           'valuationdomain', 'relation',
12                           'gamma', 'notGamma', 'size']
13 >>> print('Transitivity degree: %.3f' %\
14 ...           dg.computeTransitivityDegree() )
15     Transitivity degree: 1.000
16 >>> dg.exportGraphViz(fileName='orientedPermGraph')
17     *---- exporting a dot file for GraphViz tools -----*
18     Exporting to orientedPermGraph.dot
19     dot -Grankdir=TB -Tpng orientedPermGraph.dot\
20                           -o orientedPermGraph.png
```

The dual of a permutation graph is again a permutation graph and as such also transitively orientable.

```

1 >>> dgd = (-g).transitiveOrientation()
2 >>> print('Dual transitivity degree: %.3f' %\
3 ...           dgd.computeTransitivityDegree() )
4     Dual transitivity degree: 1.00
```

**Fig. 23.7** The transitive orientation of the permutation graph



*DiGraph3 (graphviz)  
R. Bisdorff, 2020*

## 23.4 Recognizing permutation graphs

Now, a given graph  $g$  is a permutation graph if and only if both  $g$  and  $-g$  are transitively orientable. This property gives a polynomial test procedure (in  $O(n^3)$ ) due to the transitivity check for recognizing permutation graphs.

Let us consider, for instance, the following random graph of order 8 generated with an edge probability of 40% and a random seed equal to 4335.

```

1 >>> from graphs import RandomGraph
2 >>> g = RandomGraph(order=8,
3 ...                           edgeProbability=0.4, seed=4335)
4 >>> g
5 *----- Graph instance description -----*
6     Instance class      : RandomGraph
7     Instance name       : randomGraph
8     Seed                 : 4335
9     Edge probability    : 0.4
10    Graph Order         : 8
11    Graph Size          : 10
12    Valuation domain   : [-1.00; 1.00]
13    Attributes          : ['name', 'order', 'vertices',
14                           'valuationDomain', 'seed',
15                           'edges', 'size', 'gamma',
16                           'edgeProbability']
17 >>> g.isPerfectGraph()
18     True
19 >>> g.exportGraphViz(fileName='randomGraph4335')
20 *---- exporting a dot file for GraphViz tools ----*
21     Exporting to randomGraph4335.dot
  
```

```
22 fdp -Tpdf randomGraph4335.dot -o randomGraph4335.pdf
```

**Fig. 23.8** Random graph of order 8 generated with edge probability 0.4.

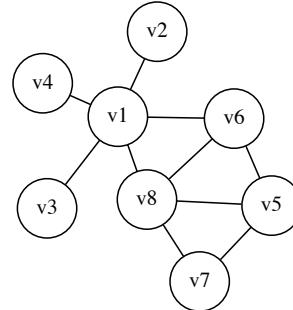


Diagram (graphviz), R. Bisdorff, 2019

If the random perfect graph instance  $g$ , shown in Fig. 23.8, is indeed a permutation graph,  $g$  and its dual  $-g$  are both *transitively orientable*, i.e. comparability graphs (Golumbic, 2004). With the `isComparabilityGraph()` test, we may easily check this fact. This method proceeds indeed by trying to construct a transitive neighbourhood decomposition of a given graph instance and, if successful, stores the resulting edge orientations into a `edgeOrientations` attribute (Golumbic, 2004, p.129-132).

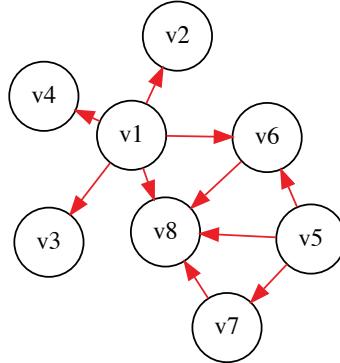
```

1 >>> if g.isComparabilityGraph():
2 ...     print(g.edgeOrientations)
3 {('v1', 'v1'): 0, ('v1', 'v2'): 1, ('v2', 'v1'): -1, ('v1', 'v3'): 1,
4 ('v3', 'v1'): -1, ('v1', 'v4'): 1, ('v4', 'v1'): -1, ('v1', 'v5'): 0,
5 ('v5', 'v1'): 0, ('v1', 'v6'): 1, ('v6', 'v1'): -1, ('v1', 'v7'): 0,
6 ('v7', 'v1'): 0, ('v1', 'v8'): 1, ('v8', 'v1'): -1, ('v2', 'v2'): 0,
7 ('v2', 'v3'): 0, ('v3', 'v2'): 0, ('v2', 'v4'): 0, ('v4', 'v2'): 0,
8 ('v2', 'v5'): 0, ('v5', 'v2'): 0, ('v2', 'v6'): 0, ('v6', 'v2'): 0,
9 ('v2', 'v7'): 0, ('v7', 'v2'): 0, ('v2', 'v8'): 0, ('v8', 'v2'): 0,
10 ('v3', 'v3'): 0, ('v3', 'v4'): 0, ('v4', 'v3'): 0, ('v3', 'v5'): 0,
11 ('v5', 'v3'): 0, ('v3', 'v6'): 0, ('v6', 'v3'): 0, ('v3', 'v7'): 0,
12 ('v7', 'v3'): 0, ('v3', 'v8'): 0, ('v8', 'v3'): 0, ('v4', 'v4'): 0,
13 ('v4', 'v5'): 0, ('v5', 'v4'): 0, ('v4', 'v6'): 0, ('v6', 'v4'): 0,
14 ('v4', 'v7'): 0, ('v7', 'v4'): 0, ('v4', 'v8'): 0, ('v8', 'v4'): 0,
15 ('v5', 'v5'): 0, ('v5', 'v6'): 1, ('v6', 'v5'): -1, ('v5', 'v7'): 1,
16 ('v7', 'v5'): -1, ('v5', 'v8'): 1, ('v8', 'v5'): -1, ('v6', 'v6'): 0,
17 ('v6', 'v7'): 0, ('v7', 'v6'): 0, ('v6', 'v8'): 1, ('v8', 'v6'): -1,
18 ('v7', 'v7'): 0, ('v7', 'v8'): 1, ('v8', 'v7'): -1, ('v8', 'v8'): 0}
19 >>> g.exportEdgeOrientationsGraphViz('transOrientGraph')
20 ----- exporting a dot file for GraphViz tools -----
21   Exporting to transOrientGraph.dot
22   fdp -Tpng transOrientGraph.dot \
      -o transOrientGraph.png

```

The resulting orientation of the edges of  $g$ , shown in Fig. 23.9 on the next page, is indeed transitive. The same procedure applied to the dual graph  $gd = -g$  gives as

**Fig. 23.9** Transitive neighbourhoods of the graph  $g$



*Digraph3 (graphviz), R. Bisdorff, 2019*

well a transitive orientation to the edges of  $-g$  shown in Fig 23.10 on the following page.

```

1 >>> gd = -g
2 >>> if gd.isComparabilityGraph():
3 ...     print(gd.edgeOrientations)
4 {('v1','v1'): 0, ('v1','v2'): 0, ('v2','v1'): 0, ('v1','v3'): 0,
5 ('v3','v1'): 0, ('v1','v4'): 0, ('v4','v1'): 0, ('v1','v5'): 1,
6 ('v5','v1'): -1, ('v1','v6'): 0, ('v6','v1'): 0, ('v1','v7'): 1,
7 ('v7','v1'): -1, ('v1','v8'): 0, ('v8','v1'): 0, ('v2','v2'): 0,
8 ('v2','v3'): -2, ('v3','v2'): 2, ('v2','v4'): -3, ('v4','v2'): 3,
9 ('v2','v5'): 1, ('v5','v2'): -1, ('v2','v6'): 1, ('v6','v2'): -1,
10 ('v2','v7'): 1, ('v7','v2'): -1, ('v2','v8'): 1, ('v8','v2'): -1,
11 ('v3','v3'): 0, ('v3','v4'): -3, ('v4','v3'): 3, ('v3','v5'): 1,
12 ('v5','v3'): -1, ('v3','v6'): 1, ('v6','v3'): -1, ('v3','v7'): 1,
13 ('v7','v3'): -1, ('v3','v8'): 1, ('v8','v3'): -1, ('v4','v4'): 0,
14 ('v4','v5'): 1, ('v5','v4'): -1, ('v4','v6'): 1, ('v6','v4'): -1,
15 ('v4','v7'): 1, ('v7','v4'): -1, ('v4','v8'): 1, ('v8','v4'): -1,
16 ('v5','v5'): 0, ('v5','v6'): 0, ('v6','v5'): 0, ('v5','v7'): 0,
17 ('v7','v5'): 0, ('v5','v8'): 0, ('v8','v5'): 0, ('v6','v6'): 0,
18 ('v6','v7'): 1, ('v7','v6'): -1, ('v6','v8'): 0, ('v8','v6'): 0,
19 ('v7','v7'): 0, ('v7','v8'): 0, ('v8','v7'): 0, ('v8','v8'): 0}
20 >>> gd.exportEdgeOrientationsGraphViz('transOrientDualGraph')
21 *---- exporting a dot file for Graphviz tools ----*
22 Exporting to transOrientDualGraph.dot
23 fdp -Tpng transOrientDualGraph.dot \
      -o transOrientDualGraph.png

```

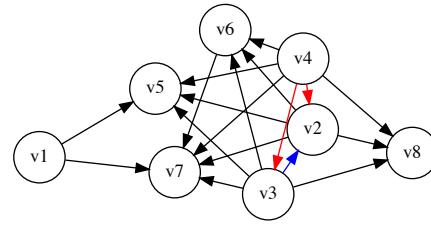
Let us recheck these facts by explicitly constructing transitively oriented digraph instances with the `computeTransitivelyOrientedDigraph()` method.

```

1 >>> og = g.computeTransitivelyOrientedDigraph(\n              PartiallyDetermined = True)
2 ...
3 >>> print('Transitivity degree: %.3f' %\n4 ...          (og.transitivityDegree))
5   Transitivity degree: 1.000

```

**Fig. 23.10** Transitive neighbourhoods of the dual graph  $-\mathbf{g}$ . It is worthwhile noticing that the orientation of  $\mathbf{g}$  is achieved with a single neighbourhood decomposition, covering all the vertices. Whereas, the orientation of the dual graph  $-\mathbf{g}$  here needs a decomposition into three subsequent neighbourhoods marked in black, red and blue



Digraph3 (graphviz), R. Bisdorff, 2019

```

6  >>> ogd = (-g).computeTransitivelyOrientedDigraph(\n7      ...                           PartiallyDetermined = True)\n8  >>> print('Transitivity degree: %.3f' %\n9      ...                           (ogd.transitivityDegree))\n10 >>> Transitivity degree: 1.000

```

The `PartiallyDetermined = True` flag (see Lines 2 and 7 above) is required here in order to orient only the actual edges of the graphs. Relations between vertices not linked by an edge are put to the indeterminate characteristic value 0. This allows us to compute, later on, convenient disjunctive digraph fusions.

As both graphs are indeed transitively orientable (see Lines 5 and 10 above), we may conclude that the given random graph  $\mathbf{g}$  is actually a permutation graph instance. Yet, we still need to find now its corresponding permutation. We therefore implement a recipe given by Martin Golumbic (Golumbic, 2004, p. 159).

We will first *fuse* both  $\mathbf{og}$  and  $\mathbf{ogd}$  orientations above with an *epistemic disjunction* operated with the symmetric  $\text{o-max}$  operator (see Section 2.5 on page 24).

```

1  >>> from digraphs import FusionDigraph\n2  >>> f1 = FusionDigraph(og,ogd,operator='o-max')\n3  >>> s1 = f1.computeCopelandRanking()\n4  >>> print(s1)\n5  ['v5','v7','v1','v6','v8','v4','v3','v2']

```

With the `computeCopelandRanking()` method, we obtain the linear ordering  $[v5, v7, v1, v6, v8, v4, v3, v2]$  of the vertices (see Line 5 above).

We reverse now the orientation of the edges in  $\mathbf{og}$  (see  $-\mathbf{og}$  in Line 1 below) in order to generate, again by disjunctive fusion, the *inversions* that are produced by the permutation we are looking for.

Computing again a ranking with the COPELAND rule, reveals the permuted list of vertices we are looking for (see Line 4 below).

```

1  >>> f2 = FusionDigraph((-og),ogd,operator='o-max')\n2  >>> s2 = f2.computeCopelandRanking()\n3  >>> print(s2)\n4  ['v8','v7','v6','v5','v4','v3','v2','v1']

```

Vertex  $v_8$  is put from position 5 to position 1, vertex  $v_7$  is put from position 2 to position 2, vertex  $v_6$  from position 4 to position 3, vertex  $v_5$  from position 1 to position 4, etc ... . We generate these position swaps for all vertices and obtain thus the required permutation (see Line 5 below).

```

1 >>> permutation = [0 for j in range(g.order)]
2 >>> for j in range(g.order):
3 ...     permutation[s2.index(s1[j])] = j+1
4 >>> print(permutation)
5 [5, 2, 4, 1, 6, 7, 8, 3]

```

It is worthwhile noticing by the way that transitive orientations of a given graph and its dual are usually *not unique* and, so may also be the resulting permutations. However, they all correspond to isomorphic graphs (Golumbic, 2004). In our case here, we observe two different permutations and their reverses::

```

1 s1: ['v1', 'v4', 'v3', 'v2', 'v5', 'v6', 'v7', 'v8']
2 s2: ['v4', 'v3', 'v2', 'v8', 'v6', 'v1', 'v7', 'v5']
3 (s1 -> s2): [2, 3, 4, 8, 6, 1, 7, 5]
4 (s2 -> s1): [6, 1, 2, 3, 8, 5, 7, 4]

```

And

```

1 s3: ['v5', 'v7', 'v1', 'v6', 'v8', 'v4', 'v3', 'v2']
2 s4: ['v8', 'v7', 'v6', 'v5', 'v4', 'v3', 'v2', 'v1']
3 (s3 -> s4): [5, 2, 4, 1, 6, 7, 8, 3]
4 (s4 -> s3) = [4, 2, 8, 3, 1, 5, 6, 7]

```

The `computePermutation()` method does directly operate all these steps: - computing transitive orientations, - ranking their epistemic fusion and, - delivering a corresponding permutation.

```

1 >>> g.computePermutation(Comments=True)
2 ['v1', 'v2', 'v3', 'v4', 'v5', 'v6', 'v7', 'v8']
3 ['v2', 'v3', 'v4', 'v8', 'v6', 'v1', 'v7', 'v5']
4 [2, 3, 4, 8, 6, 1, 7, 5]

```

We may finally check that, for instance, the two permutations [2, 3, 4, 8, 6, 1, 7, 5] and [4, 2, 8, 3, 1, 5, 6, 7] observed above, will generate corresponding *isomorphic* permutation graphs.

```

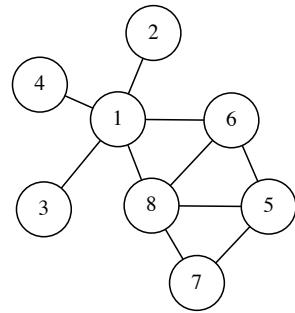
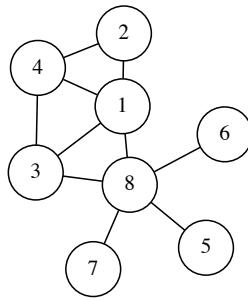
1 >>> gtesta = PermutationGraph(\n2 ...     permutation=[2, 3, 4, 8, 6, 1, 7, 5])
3 >>> gtestb = PermutationGraph(\n4 ...     permutation=[4, 2, 8, 3, 1, 5, 6, 7])
5 >>> gtesta.exportGraphViz('gtesta')
6 >>> gtestb.exportGraphViz('gtestb')

```

And, we indeed recover in Fig. 23.11 on the following page indeed two isomorphic copies of the original random graph (compare with Fig. 23.8 on page 354).

[2, 3, 4, 8, 6, 1, 7, 5]

[4, 2, 8, 3, 1, 5, 6, 7]

*Digraph3 (graphviz), R. Bisdorff, 2019**Digraph3 (graphviz), R. Bisdorff, 2019***Fig. 23.11** Isomorphic permutation graphs.

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