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# Algorithmic Decision Making with Python Programming Resources

A computational journey from multiple-criteria performance records to decision algorithms via bipolar-valued outranking digraphs

September 22, 2021

Springer



*This book is dedicated to my colleague and  
dear friend, the late Prof. Marc ROUBENS.*



## Preface

The reader will find in this monograph a series of tutorials and advanced topics originally written over the last decade as documentation parts for the DIGRAPH3 collection of Python modules. These programming resources – like the `outrankingDigraphs` module – were essentially used both for the computational verification of decision algorithms and for the preparation and illustration of the Algorithmic Decision Theory course taught at the University of Luxembourg from 2010 to 2020. Some resources, like the `randomNumbers` module, served for preparing and illustrating the Lectures of a Computational Statistics Course. Curious readers will also discover some resources –the `arithmetics` module– used for preparing and illustrating a first Semester Course on Discrete Mathematics.

The DIGRAPH3 Python programming resources are useful in the field of Algorithmic Decision Theory and more specifically for the outranking approach of Multiple-Criteria Decision Aidng (MCDA). In this latter scientific field, we address essentially three kinds of usage.

First, we present algorithms and illustrate computing tools for solving either, a multiple-criteria best choice selection or, a dual worst choice rejection problem. We also tackle the problem of how to list a set of items with multiple incommensurable performance criteria either, from the best to the worst (ranking problem), or from the worst to the best (ordering problem). Finally, we present order-statistical algorithms for relative or absolute quantiles-rating of multiple-criteria performance records.

It is necessary to mention that the DIGRAPH3 ressources do not provide a professional Python software library. The collection of Python modules, I describe in this book, was not built following any professional software development methodology. The design of classes and methods was kept as simple and elementary as was opportune for the author. Sophisticated and cryptic overloading of classes, methods and variables is more or less avoided all over. A simple copy, paste and ad hoc customisation development strategy was generally preferred. As a consequence, the DIGRAPH3 modules keep a large part of independence. Furthermore, the development of the DIGRAPH3 modules being spread over two decades, the programming style did evolve with growing experience and the changes and enhancement coming up with the ongoing new releases of the standard Python3 libraries. The required

backward compatibility necessarily introduced so with time some notation and programming technique changes.

The purpose of this book is to present in a single monograph the scientific enhancements the author has contributed over the past two decades to the outranking approach-based multiple-criteria decision aiding field - and that are either left unpublished or published in very specialised media only and difficult to access.

This monograph should provide the reader with a self-contained series of tutorials which explain how to solve multiple-criteria selection, as well as ranking or rating decision problems. If successful in this aim, the curious reader will effectively install the DIGRAPH3 programming resources on their laptop and try out and redo for themselves the proposed computations.

The material in this book is valuable for master students and doctoral candidates in Computer Science, Mathematics, Engineering Sciences or Computational Management Sciences taking a course on Algorithmic Decision Theory, Multiple-Criteria Decision Aiding or Decision Analysis. Some experience in computer programming, in particular with Python, will assist the reader, but it is not a prerequisite. The many coding examples shown throughout the text are purposely kept elementary from a programming point of view.

Chapters presenting algorithms for ranking multiple-criteria performance records from best to worst –especially when facing big performance tableaux– will be of computational interest for designers of web recommender systems.

Similarly, the relative and absolute quantiles-rating algorithms, discussed and illustrated in several chapters, will be of practical interest for public or private performance auditors.

Finally, the monograph does not provide any mathematical developments or proofs. Those readers interested in the mathematical background of our decision algorithms are invited to consult the references provided at chapter level. Full texts of most of these references may be downloaded from the open access <https://orbi.lu.uni.lu/> repository of the University of Luxembourg.

**Acknowledgments**

This monograph contains many ideas, methods and tools that are not only the author's. They have been shared and enhanced with friends, colleagues and students: *Pascal Bouvry, Denis Bouyssou, Luis Dias, Claude Lamboray, Patrick Meyer, Vincent Mousseau, Alex Olteanu, Marc Pirlot, the late Bernard Roy, Ulrich Sorger, Alexis Tsoukiàs, Thomas Veneziano* and especially, the late *Marc Roubens*.

The UL HPC team, and more specifically *Valentin Plugaru* and *Sébastien Varette*, helped furthermore with mastering the multiprocessing experiments on the HPC platforms.

Their help is gratefully acknowledged.

Luxembourg, Autumn 2021

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# Introduction

## The editing strategy

The reader will find in the five parts of this monograph several series of tutorials and advanced topics that present and illustrate computational methods and tools mainly useful in the field of Multiple-Criteria Decision Aiding and Decision Analysis. These methods and tools were designed and implemented first in Python2 and then in Python3 by the author over the last two decades in order to support both the computational verification and validation of decision algorithms as well as the preparation and illustration of a Master Course on Algorithmic Decision Theory taught at the University of Luxembourg from 2010 to 2020.

Each chapter illustrates a specific preference modelling aspect, like building a best choice recommendation, ranking a set of potential decision alternatives, or computing the winner of an election. In order to keep parts and chapters more or less self-contained, definitions and explanations of major concepts, like bipolar-valued digraphs, multiple criteria performance tableaux and outranking situations, may appear several times in the monograph.

Explicit Python programming examples, purposely kept elementary, are shown in numerous terminal session style listings. A complete list of the numbered listings, shown over all the chapters, is printed in the Appendix. These programming examples were all checked against errors with the `doctest` module of the standard Python3 library and should work effectively as such either, in a Python3 interactive terminalconsole, or for sure in an `ipython` console<sup>1</sup>. Note that the layout of console `print(...)` outcomes has been edited in some listings for easing their reading. Some chapters will rely on a given data file that is made available in the `examples` directory of the DIGRAPH3 resources.

For similarly easing their reading, most chapters do not provide mathematical developments and proofs. Readers interested in such details are invited to consult the references listed separately at the end of each chapter. The author's references pro-

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<sup>1</sup> IP[i]: IPython interactive computing, <https://ipython.org>

vide full text access to preprints on the open access <https://orbilu.uni.lu/> repository of the University of Luxembourg.

Readers interested in the technical aspects of the organisation and implementation of the collection of DIGRAPH3 Python modules are invited to consult the extensive reference manual: <https://digraph3.readthedocs.io/en/latest/techDoc.html>, assisted by a search page covering the whole DIGRAPH3 documentation.

## Organisation of the book

The book is organised into five parts.

**Part I** presents three chapters introducing the DIGRAPH3 programming resources and the main formal objects discussed in this book, namely *bipolar-valued digraphs* and, in particular, *outranking digraphs*.

In Chapter 1, the reader will gain contact with the DIGRAPH3 Python resources. First are given the installation instructions and the list of the main DIGRAPH3 Python modules with their purpose. A Python terminal session using the root digraphs module eventually illustrates how to generate, save and inspect a random crisp digraph.

Chapter 2 introduces the bipolar-valued digraph model –the root type of all our digraph models–. A randomly bipolar-valued digraph instance is generated. Drawing the digraph, separating its asymmetric and symmetric parts, or its border and inner parts, is illustrated. The initial digraph instance may be reconstructed by epistemic disjunctive fusion from these respective parts. Dual, converse and codual transforms, as well as symmetric and transitive closures are presented. Complete, empty and indeterminate digraphs are eventually presented.

Chapter 3 presents the bipolar-valued outranking digraph –the main formal object used and discussed in this monograph–. After illustrating its hybrid type –it is conjointly a multiple-criteria performance tableau and a bipolar-valued digraph modelling the outranking situations between the given performance records– pairwise comparisons and the recoding of the digraph characteristic valuation are illustrated. The codual transform of the outranking digraph renders the corresponding strict outranking digraph, i.e. its asymmetric part.

**Part II** illustrates in eight methodological chapters multiple-criteria performance evaluation models and decision algorithms. These chapters are mostly problem oriented.

Chapter 4 presents the RUBIS best choice recommender system. The approach is illustrated with building a best office-location recommendation. We show how to explore a given performance tableau and compute the corresponding outranking digraph. After presenting the pragmatic principles that govern our best choice recommendation algorithm we solve the best office-location choice problem.

Chapter 5 illustrates a way of creating a new multiple-criteria performance tableau by editing a given template file containing 5 decision alternatives, 3 decision

objectives and 6 performance criteria. We discuss in detail how to edit the decision alternatives, the decision objectives, the family of performance criteria, and finally, the evaluations of the decision alternatives on the performance criteria.

Chapter 6 describes the DIGRAPH3 resources for generating random multiple-criteria performance tableaux. These random performance tableaux instances, mainly meant for illustration and training purposes, were serving the preparation and illustration of the Algorithmic Decision Theory Course lectures. The random generators propose several useful models like a Cost-Benefit tableau, a three Objectives —economic, societal and environmental— tableau, and an academic performance tableau.

Chapter 7 is more specifically devoted to handling linear voting profiles and computing the winner of such election results like the simple majority or the instant-run-off winner. By following CONDORCET’s recipe, we consider pairwise comparisons of election candidates and balance the number of times the first beats the second against the number of times the second beats the first in order to obtain a majority margins digraph, in fact a bipolar-valued digraph. When the voters express contradictory linear voting profiles one may naturally observe cyclic social preferences without seeing any paradox in this situation. Finally, the chapter presents a more politically realistical generator for random linear voting profiles by taking into account pre-election polls.

Chapter 8 introduces several algorithms for solving the ranking problem with a bipolar-valued outranking digraph. The COPELAND, NETFLOWS, KEMENY, SLATER, KOHLER and the RANKEDPAIRS ranking rules are illustrated with the help of a random outranking digraph. The fitness of their respective ranking result is measured with a bipolar-valued version of KENDALL’s ordinal correlation index.

Chapter 9 applies order statistics for sorting a set  $X$  of  $n$  potential decision alternatives, evaluated on  $m$  incommensurable performance criteria, into  $q$  quantile equivalence classes. The sorting algorithm is based on pairwise outranking characteristics involving the quantile class limits observed on each criterion. Thus we may implement a weak ordering algorithm of complexity  $O(nmq)$ .

Chapter 10 addresses the problem of rating multiple-criteria performance records of a set of potential decision alternatives with respect to performance quantiles learned from similar decision alternatives observed in the past. We show how to learn performance quantiles from historical performance tableaux. New performance records may now be rated with respect to these learned quantile norms.

Chapter 11 tackles the ranking of big multiple-criteria performance tableaux with thousands or millions of records. To effectively compute rankings from performance tableaux of these sizes, the chapter proposes a collection of cythonized –C-compiled and optimised– modules that may be run on Linux Debian HPC equipement as available, for instance, at the University of Luxembourg.

**Part III** delivers three realistic algorithmic decision making case studies.

Chapter 12 presents a case study concerning the building of a best choice recommendation for Alice, a German student who wants some advice concerning the choice of her future University studies. We present Alice’s performance tableau — potential foreign language study programs, her decision objectives, performance

criteria and performance evaluations—and build a best choice recommendation for her. A thorough robustness analysis confirms a very best choice.

In Chapter 13 we are resolving with our DIGRAPH3 resources a ranking decision problem based on published data from the Times Higher Education (THE) World University Rankings 2016 by Computer Science (CS) subject. We first have a look into the THE multiple-criteria ranking data with short Python scripts. In a second section, we relax the commensurability hypothesis of the ranking criteria and show how to similarly rank with multiple incommensurable performance criteria of solely ordinal significance. A third section is finally devoted to introduce quality measures for qualifying ranking results.

Chapter 14 presents and discusses how to rate with the help of our DIGRAPH3 resources the apparent student enrolment quality of higher education institutions. The multiple-criteria performance tableau, we use, is inspired by a 2004 student survey published by DER SPIEGEL magazine and concerning nearly 50,000 students, enrolled in one of fifteen popular academic subjects, like German Studies, Life Sciences, Psychology, Law or Computer Science.

In Chapter 15, we propose a series of decision problems of various difficulties which may serve as exercises and exam questions for an Algorithmic Decision Theory or Multiple-Criteria Decision Analysis course. They cover selection, ranking and rating problems.

**Part IV** presents in five chapters more advanced topics showing some pearls of bipolar-valued epistemic logic.

Starting from a motivating decision problem about how to list, from the best to the worst, a set of movies that are star-rated by journalists and movie critics, Chapter 16 shows that KENDALL’s ordinal correlation index tau can be extended to a relational bipolar-valued equivalence measure of bipolar-valued digraphs. This finding gives way, on the one hand, to measure the fitness and fairness of multiple-criteria ranking rules. On the other hand, it provides a tool for illustrating preference divergences between decision objectives and/or performance criteria.

We illustrate in Chapter 17, first, the concept of graph kernel, i.e. maximal independent set of vertices. In non-symmetric digraphs the kernel concept becomes richer and separates into initial and terminal kernels. In, furthermore, lateralised outranking digraphs, initial and terminal kernels become separate and may deliver suitable first resp. last choice recommendations. After commenting the tractability of kernel computations, we close the chapter with the solving of bipolar-valued kernel equation systems.

In Chapter 18 we propose to link a qualifying significance majority for outranking situations with a required  $\alpha\%$ -confidence level. We model therefore the significance weights as random variables following more or less widespread distributions around a mean value that corresponds to the given deterministic significance weights. As the bipolar-valued random credibility of an outranking situation hence results from the simple sum of positive or negative independent random variables, we can apply the Central Limit Theorem (CLT) for computing the bipolar-valued likelihood that the expected significance majority margin will indeed be positive, respectively negative.

In Chapter 19 we study the robustness of the outranking digraph when the criteria significance weights faithfully indicate solely an order of importance. The required cardinal significance weights of the performance criteria represent actually the 'Achilles' heel of the outranking approach. Rarely will indeed a decision maker be cognitively competent for suggesting precise decimal-valued criteria significance weights. This approach leads furthermore to the concept of unopposed or Pareto efficient multiobjective choices.

In a social choice context, where decision objectives would match different political parties, such Pareto efficient choices represent in fact multipartisan social choices. Chapter 20 shows that they may judiciously deliver the primary selection in a two stage election system. The outranking model is based on bipolar approvals-disapprovals of “*at least as well evaluated as*” statements. A similar approach is put into practice with bipolar approval-disapproval voting systems. When converting such approval-disapproval voting ballots into corresponding performance records, one obtains a  $(-1, 0, 1)$ -valued evaluative voting system. We eventually show in this chapter that in such bipolar voting systems, the election winner tends to be among the more or less multipartisan candidates.

**Part V** illustrates in three chapters computational resources for working with simple undirected graphs.

Chapter 21 introduces bipolar-valued undirected graphs and illustrates several special graph models and algorithms like Q-coloring, maximal independent set (MIS) and clique enumeration, line graphs and maximal matchings, grid graphs, and n-cycle graphs with their non-isomorphic MISs.

Chapter 22 specifically addresses working with tree graphs and graph forests. We illustrate how to generate and recognise random tree graphs and how to compute the centres of a tree and draw a rooted and oriented tree. Finally, algorithms for computing spanning trees and forests are presented.

Chapter 23 eventually presents some famous classes of BERGE graphs, namely comparability, interval, permutation and split graphs. We first present an example of an interval graph which is at the same time a triangulated, a comparability, a split and a permutation graph. The importance of being an interval graph is illustrated with *Cl. Berge's* mystery story. We discuss furthermore the generation of permutation graphs and close with how to recognise that a given graph is in fact a permutation graph.

## Highlights

Contrary to what is generally thought, it is the preparation of the multiple-criteria performance tableau that takes most of the decision analysis time, not running any decision algorithms. Designing adequate performance evaluating criteria functions for each decision objective and collecting meaningful and precise evaluations is crucial for the success of the decision making. This is a very critical and essential step. Chapters 4, 5 and 12 illustrate and discuss in detail coherent multiple-criteria

performance tableaux. In order to discover more examples of potential performance tableaux, we provide in Chapter 6 random generators for several common kinds of performance tableaux.

Once the multiple-criteria performance tableau is ready, starts the thrilling step of discovering the resulting outranking relation. Are there many chordless outranking circuits? What is its degree of symmetry? What is its degree of transitivity? If the number of potential decision alternatives is small –less than 30– one can try, in the case of a selection problem, to compute prekernels in order to find potential first or last choice decision alternatives? Chapters 4, 12 and 17 are illustrating and discussing this challenging computational problem.

Comparing various ranking rules working on bipolar-valued outranking relations constructed from performance tableaux of various kinds: Cost-Benefit, 3-Objectives, academic a.-o., has made us confident about the fact that convincing criteria for judging the quality of a ranking result may not to be found alone by mathematical properties, like KEMENY optimality or CONDORCET consistency. More useful seams to be the fair balancing of decision objectives and performance criteria. In this respect it is the NETFLOWS ranking rule which appears to be most effective and often gives fairly balanced multiple-criteria rankings. Chapter 8 on ranking rules, the ranking and rating case studies of Chapters 13 and 14, and Chapter 16 on bipolar-valued relational equivalence of digraphs illustrate and discuss this important topic.

The bipolar-valued epistemic logic, in which our decision algorithms are computing and expressing their decision solutions, provides effective assistance for coping with missing data and imprecise performance evaluations. Chapters 14 and 16 illustrate this advantage. An efficient robustness analysis becomes furthermore available for handling, on the one side, uncertain criteria significance weights leading in Chapter 18 to  $\alpha\%$ -confident outranking digraphs. On the other side, Chapter 19 illustrates how to compute robust outranking digraphs and decision solutions when solely ordinal criteria significance weights are given. In Chapter 20, the same kind of robustness analysis proposes strategies for tempering plurality tyranny effects in social choice problems by favouring multipartisan candidates, like two-stage elections with multipartisan primary selection of candidates or bipolar approval-disapproval voting systems.

Noticing the efficiency of the bipolar-valued epistemic logical framework for handling outranking digraphs, we could not resist making in Chapters 21 and 22 an excursion into the domain of simple undirected graphs and tree graphs. The beautiful book on Algorithmic Graph Theory and Perfect Graphs by *M. Ch. Golumbic* gave eventually the opportunity to tackle in the last Chapter 23 some famous classes of BERGE graphs.

It is my hope that the reader, by going on, will find the same astonishment and enchantment as I experienced when discovering the simplicity, efficiency and elegance of handling bipolar-valued outranking digraphs and graphs with Python programming resources. Extending the bipolar-valued epistemic logical framework to

other computational science domains will prove valuable, I am sure, for many future scientific works.



**Part I**

**Introduction to the DIGRAPH3 Python  
resources**

The first part contains three chapters for introducing the DIGRAPH3 software collection of Python programming resources. The first chapter is devoted to the installation of the DIGRAPH3 Python modules and running a first Python terminal session using the DIGRAPH3 3 resources. The second chapter introduces bipolar-valued digraphs, the root type of all available specialised digraphs. The third chapter finally introduces the main formal objects of this book, namely bipolar-valued outranking digraphs.

# Chapter 1

## Working with the DIGRAPH3 Python resources

**Abstract** The chapter is devoted to a first contact with the DIGRAPH3 Python resources. Following the installation instructions, we list the main Python modules with their purpose and eventually illustrate in a first Python terminal session how to generate, save and inspect a random crisp digraph.

### 1.1 Installing the DIGRAPH3 resources

Using the DIGRAPH3 Python modules is easy<sup>1</sup>. You only need to have installed on your system the Python programming language of version 3 (readily available under Linux and Mac OS). Notice that, from Version 3.3 on, the Python standard decimal module implements very efficiently in C its Decimal class. Now, Decimal numbers are mainly used in the DIGRAPH3 characteristic valuation functions, which makes the recent Python-3.7+ versions much faster (more than twice as fast) when extensive digraph operations are performed. Several download options (easiest under Linux or Mac OS-X) are given; either, by using a git client and clone a working copy from the `github.com` directory:

```
...$ git clone https://github.com/rbisdorff/Digraph3
```

or from the `sourceforge.net` directory:

```
...$ git clone https://git.code.sf.net/p/digraph3/code Digraph3
```

It is also possible with a browser access, to download either, from the `github.com` link or, from the `sourceforge.net` link above the latest distribution zip archive and extract it.

On Linux or Mac OS, `...$ cd` to the cloned or extracted `Digraph3` directory. The following shell command installs (with `sudo !`) the DIGRAPH3 modules in the current running python environment:

```
.../Digraph3$ make install
```

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<sup>1</sup> See the technical description of the DIGRAPH3 programming resources, [Bisdorff \(2021\)](#).

Python-3.8 (or later) environment is recommended (see the `makefile` for adapting the `make install` command to your running python environment).

Whereas the following shell command installs the DIGRAPH3 modules in an activated virtual python environment:

```
.../Digraph3$ make installVenv
```

If the *cython*<sup>2</sup> C-compiled modules for Big Data applications are required, it is necessary to previously install the *Cython* package in the running Python environment:

```
...$ python3 -m pip install cython
```

It is recommended to run a test suite:

```
.../Digraph3$ make tests
```

Test results are stored in the `Digraph3/test` directory. Notice that the `python3 pytest` package is required:

```
...$ python3 -m pip install pytest
```

A verbose (with `stdout` not captured) `pytest` suite may be run as follows:

```
.../Digraph3$ make verboseTests
```

When the GNU parallel shell tool<sup>3</sup> is installed and multiple cores are detected, the tests may be executed in multiple processing mode:

```
.../Digraph3$ make pTests
```

Individual module `pytest` suites are also provided (see the `makefile`), like the one for the `outrankingDigraphs` module (see Chap. 2):

```
.../Digraph3$ make outrankingDigraphsTests
```

### Dependencies:

To be fully functional, the DIGRAPH3 resources mainly need:

- The graphviz tools ([Gansner and North, 2000](#))<sup>4</sup>, and
- The R statistics resources<sup>5</sup> to be installed.
- When exploring digraph isomorphisms, the nauty isomorphism testing program is required ([McKay and Piperno, 2013](#)). On linux you may try: `sudo apt install nauty`. For Mac OS X, corresponding `dmg` installers are available for downloading.
- Two specific methods of the `OutrankingDigraph` class for clustering performance criteria or decision alternatives require furthermore the `calmat` matrix computing resource to be installed (see the `calmat` directory in the DIGRAPH3 resources).

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<sup>2</sup> <https://cython.org>

<sup>3</sup> <https://www.gnu.org/software/parallel>

<sup>4</sup> <https://graphviz.org>

<sup>5</sup> <https://www.r-project.org>

## 1.2 Organisation of the DIGRAPH3 Python modules

The main data handling modules of the DIGRAPH3 resources are the following:

1. `digraphs`: main part of the DIGRAPH3 source code with the root `Digraph` class.
2. `graphs`: resources for handling undirected graphs with the root `Graph` class and a bridge to the `digraphs` module resources.
3. `perfTabs`: tools for handling multiple-criteria performance tableaux with root `PerformanceTableau` class.
4. `outrankingDigraphs`: root module for handling outranking digraphs with the abstract root `OutrankingDigraph` class and the main `BipolarOutrankingDigraph` class.<sup>6</sup>
5. `votingProfiles`: classes and methods for handling voting ballots and computing election results with main `LinearVotingProfile` class.

Various random generators are provided by the following modules:

1. `randomDigraphs`: various random digraph models like random crisp digraphs (`RandomDigraph` class) or random bipolar-valued digraphs (`RandomValuationDigraph` class).
2. `randomPerfTabs`: various implemented random performance tableau models, like Cost-Benefit tableaux (`RandomCBPerformanceTableau` class) or 3-Objectives tableaux (`Random3ObjectivesPerformanceTableau` class).
3. `randomNumbers`: additional random number generators, not available in the standard Python `random.py` library, like a discrete random variable (`DiscreteRandomVariable` class) or a Cauchy random variable (`CauchyRandomVariable` class)

Following modules provide tools for *sorting*, *ranking* and *rating* problems:

1. `sortingDigraphs`: tools for solving sorting problems with the root `SortingDigraph` class and the main `QuantilesSortingDigraph` class;
2. `linearOrders`: tools for solving linearly ranking or ordering problems with the root `LinearOrder` class;
3. `transitiveDigraphs`: additional tools for handing transitive digraphs with root `TransitiveDigraph` class.

Tools for specifically handling Big Data are eventually provided by the following modules:

1. `performanceQuantiles`: incremental representation of large performance tableaux via binned cumulated density functions per criteria; Depends on the `randomPerfTabs` module.

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<sup>6</sup> Notice that the `outrankingDigraph` class defines a hybrid object type, inheriting conjointly from the `Digraph` class and the `PerformanceTableau` class.

2. `sparseOutrankingDigraphs`: sparse implementation design for bipolar-valued outranking digraphs of order  $> 500$ .
3. *Cythonized* modules: C-compiled and optimised Python modules for handling big performance tableaux and bipolar outranking digraphs of order  $> 1000$ .

Readers interested in technical implementation details are invited to consult the reference manual of the DIGRAPH3 resources, where they will find the documentation and complete source code of all DIGRAPH3 modules, classes and methods ([Bisdorff, 2021](#)).

### 1.3 Starting a DIGRAPH3 terminal session

After downloading the DIGRAPH3 resources, one may start an interactive Python3 terminal session in the `Digraph3` directory.

```
$HOME/.../Digraph3$ python3
Python 3.9.6 (v3.9.6:db3ff76dal, Jun 28 2021, 11:49:53)
[Clang 6.0 (clang-600.0.57)] on darwin
Type "help", "copyright", "credits" or
"license" for more information.
>>>
```

For exploring the classes and methods provided by the DIGRAPH3 modules enter the Python3 commands following the session prompts marked with `>>>` or `...`; the lines without a prompt are output from the Python3 interpreter. Python class names and boolean parameters start by convention with a capital case; names of other Python objects, like modules, methods and variables start with a lower case. All Python names and code are shown in a typewriting font.

**Listing 1.1** Generating a digraph instance

```
1 >>> from randomDigraphs import RandomDigraph
2 >>> dg = RandomDigraph(order=5, arcProbability=0.5, \
3 ...                               seed=101)
4 >>> dg
5 *----- Digraph instance description -----*
6 Instance class      : RandomDigraph
7 Instance name       : randomDigraph
8 Digraph Order       : 5
9 Digraph Size        : 12
10 Valuation domain   : [-1.00; 1.00]
11 Determinateness (%) : 100.00
12 Attributes         : ['actions','valuationdomain',
13                           'relation','order','name',
14                           'gamma','notGamma']
```

In Listing 1.1 we import, for instance, from the `randomDigraphs` module the `RandomDigraph` class in order to generate a random digraph object `dg` of order 5 and arc probability of 50%. The resulting digraph of *order* 5 –number of nodes

called (decision) *actions*– and *size* 12 –number of arcs– is completely determined (see Line 11).

The content of dg may be saved in a file named `tutorialDigraph.py`.

```
1 >>> dg.save('tutorialDigraph')
2 *---- Saving digraph in file: <tutorialDigraph.py> ----*
```

with the following content:

**Listing 1.2** A stored digraph instance

```
1 from decimal import Decimal
2 from collections import OrderedDict
3 actions = OrderedDict([
4     ('a1', {'shortName': 'a1',
5         'name': 'random decision action'}),
6     ('a2', {'shortName': 'a2',
7         'name': 'random decision action'}),
8     ('a3', {'shortName': 'a3',
9         'name': 'random decision action'}),
10    ('a4', {'shortName': 'a4',
11        'name': 'random decision action'}),
12    ('a5', {'shortName': 'a5',
13        'name': 'random decision action'}),
14])
15 valuationdomain = {
16    'min': Decimal('-1.0'),
17    'med': Decimal('0.0'),
18    'max': Decimal('1.0'),
19    'hasIntegerValuation': True, # representation format
20}
21 relation = {
22    'a1': {'a1':Decimal('-1.0'), 'a2':Decimal('-1.0'),
23            'a3':Decimal('1.0'), 'a4':Decimal('-1.0'),
24            'a5':Decimal('-1.0'),},
25    'a2': {'a1':Decimal('1.0'), 'a2':Decimal('-1.0'),
26            'a3':Decimal('-1.0'), 'a4':Decimal('1.0'),
27            'a5':Decimal('1.0'),},
28    'a3': {'a1':Decimal('1.0'), 'a2':Decimal('-1.0'),
29            'a3':Decimal('-1.0'), 'a4':Decimal('1.0'),
30            'a5':Decimal('-1.0'),},
31    'a4': {'a1':Decimal('1.0'), 'a2':Decimal('1.0'),
32            'a3':Decimal('1.0'), 'a4':Decimal('-1.0'),
33            'a5':Decimal('-1.0'),},
34    'a5': {'a1':Decimal('1.0'), 'a2':Decimal('1.0'),
35            'a3':Decimal('1.0'), 'a4':Decimal('-1.0'),
36            'a5':Decimal('-1.0'),},
37}
```

In the DIGRAPH3 resources, all digraph object instances are of root Digraph type ([see the technical description](#)) and contain at least the following attributes (see Listing 1.1 on the preceding page Lines 12-14):

1. A name attribute, holding usually the actual name of the stored instance that was used to create the instance;

2. An ordered dictionary of digraph nodes called `actions` (decision alternatives) with at least a `name` attribute;
3. An `order` attribute containing the number of graph nodes (length of the actions dictionary) automatically added by the object constructor;
4. A logical characteristic `valuationDomain` dictionary with three decimal entries: the minimum ( $-1.0$ , means certainly false), the median ( $0.0$ , means missing information) and the maximum characteristic value ( $+1.0$ , means certainly true);
5. A double dictionary called `relation` and indexed by an oriented pair of actions (nodes) and carrying a decimal characteristic value in the range of the previous valuation domain;
6. Its associated `gamma` attribute, a dictionary containing the direct successors, respectively predecessors of each action, automatically added by the object constructor;
7. Its associated `notGamma` attribute, a dictionary containing the actions that are not direct successors respectively predecessors of each action, automatically added by the object constructor.

## 1.4 Inspecting a digraph object

Different `show...` methods, like the `showShort()` method , reveal us now that `dg` is a crisp, irreflexive and connected digraph of order five (see List. 1.3 Lines 1, 16, 26, 29).

**Listing 1.3** Random crisp digraph object

```

1 >>> dg.showShort()
2 *----- show short -----*
3 Digraph           : tutorialDigraph
4 Actions          : OrderedDict([
5   ('a1', {'shortName': 'a1', 'name': 'random decision action'}),
6   ('a2', {'shortName': 'a2', 'name': 'random decision action'}),
7   ('a3', {'shortName': 'a3', 'name': 'random decision action'}),
8   ('a4', {'shortName': 'a4', 'name': 'random decision action'}),
9   ('a5', {'shortName': 'a5', 'name': 'random decision action'})
10  ])
11 Valuation domain :
12  'min': Decimal('-1.0'),
13  'max': Decimal('1.0'),
14  'med': Decimal('0.0'), 'hasIntegerValuation': True
15  }
16 >>> dg.showRelationTable()
17 * ----- Relation Table -----
18 S    |  'a1'  'a2'  'a3'  'a4'  'a5'
19 -----|-----
20 'a1' |    -1    -1     1    -1    -1
21 'a2' |     1    -1    -1     1     1
22 'a3' |     1    -1    -1     1    -1

```

```

23   'a4' |   1     1     1    -1    -1
24   'a5' |   1     1     1    -1    -1
25 Valuation domain: [-1;+1]
26 >>> dg.showComponents()
27 *--- Connected Components ---*
28 1: ['a1', 'a2', 'a3', 'a4', 'a5']
29 >>> dg.showNeighborhoods()
30 Neighborhoods:
31   Gamma :
32 'a1': in => {'a2', 'a4', 'a3', 'a5'}, out => {'a3'}
33 'a2': in => {'a5', 'a4'}, out => {'a1', 'a4', 'a5'}
34 'a3': in => {'a1', 'a4', 'a5'}, out => {'a1', 'a4'}
35 'a4': in => {'a2', 'a3'}, out => {'a1', 'a3', 'a2'}
36 'a5': in => {'a2'}, out => {'a1', 'a3', 'a2'}
37 Not Gamma :
38 'a1': in => set(), out => {'a2', 'a4', 'a5'}
39 'a2': in => {'a1', 'a3'}, out => {'a3'}
40 'a3': in => {'a2'}, out => {'a2', 'a5'}
41 'a4': in => {'a1', 'a5'}, out => {'a5'}
42 'a5': in => {'a1', 'a4', 'a3'}, out => {'a4'}

```

The `exportGraphViz()` method generates in the current working directory a `tutorialDigraph.dot` file and a `tutorialdigraph.png` picture of the tutorial digraph `dg` (see Fig. 1.1), if the `graphviz` tools are installed on your system ([Gansner and North, 2000](#)).

```

1 >>> dg.exportGraphViz('tutorialDigraph')
2 *---- exporting a dot file do GraphViz tools -----
3 Exporting to tutorialDigraph.dot
4 dot -Grankdir=BT -Tpng tutorialDigraph.dot -o
      tutorialDigraph.png

```

**Fig. 1.1** The tutorial crisp digraph. The `exportGraphViz()` method is depending on drawing tools from <https://graphviz.org>. On Linux Ubuntu or Debian you may try `'sudo apt-get install graphviz'` to install them. For Mac OSX There are ready `dmg` installers available



*Digraph3 (graphviz), R. Bisdorff, 2020*

Further methods are provided for inspecting this random Digraph object dg, like the following `showStatistics()` method.

**Listing 1.4** Inspecting a Digraph object

```

1 >>> dg.showStatistics()
2     *----- general statistics -----*
3     for digraph           : <tutorialDigraph.py>
4     order                 : 5 nodes
5     size                  : 12 arcs
6     undetermined          : 0 arcs
7     determinateness (%)   : 100.0
8     arc density            : 0.60
9     double arc density     : 0.40
10    single arc density    : 0.40
11    absence density       : 0.20
12    strict single arc density: 0.40
13    strict absence density : 0.20
14    nbr. of components    : 1
15    nbr. of strong components : 1
16    transitivity degree (%) : 60.0
17    : [0, 1, 2, 3, 4, 5]
18    outdegrees distribution : [0, 1, 1, 3, 0, 0]
19    indegrees distribution  : [0, 1, 2, 1, 1, 0]
20    mean outdegree         : 2.40
21    mean indegree          : 2.40
22    : [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]
23    symmetric degrees dist. : [0, 0, 0, 0, 1, 4, 0, 0, 0, 0, 0]
24    mean symmetric degree  : 4.80
25    outdegrees concentration index : 0.1667
26    indegrees concentration index : 0.2333
27    symdegrees concentration index : 0.0333
28    : [0, 1, 2, 3, 4, 'inf']
29    neighbourhood depths distribution: [0, 1, 4, 0, 0, 0]
30    mean neighbourhood depth        : 1.80
31    digraph diameter               : 2
32    agglomeration distribution     :
33    a1 : 58.33
34    a2 : 33.33
35    a3 : 33.33
36    a4 : 50.00
37    a5 : 50.00
38    agglomeration coefficient      : 45.00

```

The preceding `show...()` methods usually rely upon corresponding `compute` methods, like: `computeSize()`, `computeDeterminateness()`, or `computeTransitivityDegree()`.

**Listing 1.5** Various `compute...()` methods.

```

1 >>> dg.computeSize()
2 12
3 >>> dg.computeDeterminateness(InPercents=True)
4 Decimal('100.00')
5 >>> dg.computeTransitivityDegree(InPercents=True)
6 Decimal('60.00')

```

Mind that `show...()` methods output their results in the *Python* console. We provide also some `showHTML...()` methods which output their results in a system browser's tab or window.

```
1 >>> dg.showHTMLRelationMap(relationName='r(x,y)', \
2 ...                                rankingRule=None)
```

**Fig. 1.2** Browsing the relation map of the tutorial digraph. + indicates a certainly valid and – indicates a certainly invalid relation. Here we find confirmed again that our random digraph instance `dg`, is indeed a crisp, i.e. 100% determined irreflexive digraph instance

## Relation Map

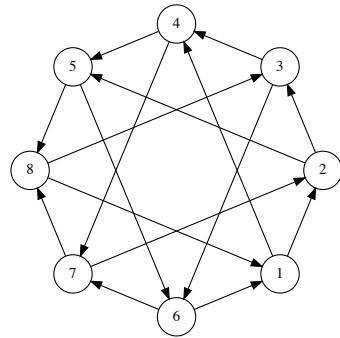
### Ranking rule: Alphabetic

r(x,y)	a1	a2	a3	a4	a5
a1	–	–	+	–	–
a2	+	–	–	+	+
a3	+	–	–	+	–
a4	+	+	+	–	–
a5	+	+	+	–	–

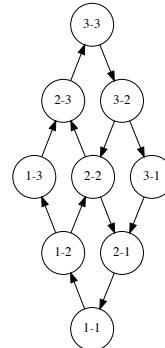
  

Semantics	
+	certainly valid
.	valid
□	indeterminate
–	invalid
–	certainly invalid

Some special types of digraph instances, like the `CirculantDigraph` or the `GridDigraph` classes, are readily available (see Fig. 1.3).



Digraph3 (graphviz), R. Bisendorff, 2020



Digraph3 (graphviz), R. Bisendorff, 2020

**Fig. 1.3** The circulant [1,3] digraph and the 3x3 grid digraph

**Listing 1.6** Circulant digraphs and  $n \times m$  grid digraphs

```

1 >>> from digraphs import CirculantDigraph, GridDigraph
2 >>> c8 = CirculantDigraph(order=8, circulants=[1, 3])
3 >>> c8.exportGraphViz('c8')
4     ----- exporting a dot file for GraphViz tools -----
5     Exporting to c8.dot
6     circo -Tpng c8.dot -o c8.png
7 >>> grid3 = GridDigraph(n=3, m=3, \
8                         hasMedianSplitOrientation=True)
9 >>> grid3.exportGraphViz('grid3')
10    ----- exporting a dot file for GraphViz tools -----
11    Exporting to grid3.dot
12    dot -Grankdir=BT -Tpng grid3.dot -o grid3.png

```

The next Chapter 2 will introduce the fondamental *bipolar-valued digraph* model which is *root object type* to all the digraph models implemented in the DIGRAPH3 modules (Bisdorff, 2021).

## References

- Bisdorff R (2021) Technical documentation of the Digraph3 collection of Python modules. URL <https://digraph3.readthedocs.io/en/latest/techDoc.html> 11, 14, 20
- Gansner E, North S (2000) An open graph visualization system and its applications to software engineering. SOFTWARE - PRACTICE AND EXPERIENCE 30(11):1203–1233, URL <https://graphviz.org/documentation/> 12, 17
- McKay B, Piperno A (2013) Practical Graph Isomorphism, II. J Symbolic Computation 60:94–112, URL <https://www.cs.sunysb.edu/~algorith/implement/nauty/implement.shtml> 12

## Chapter 2

# Working with bipolar-valued digraphs

**Abstract** The chapter introduces bipolar-valued digraphs, the fundamental root type of all the specialised digraphs implemented in the DIGRAPH3 modules. With the help of a randomly valued digraph, we illustrate some basic digraph manipulation methods, like drawing the digraph, dividing the digraph into its asymmetric and symmetric parts, separating the border from the inner part, computing associated dual, converse and codual digraphs, and operating symmetric and transitive closures.

### 2.1 Random bipolar-valued digraphs

In Listing 2.1, we generate a uniformly random  $[-1.0; +1.0]$ -valued digraph of order 7, denoted rdg and modelling, for instance, a binary relation  $S(x,y)$  defined on the set of nodes of rdg. For this purpose, the DIGRAPH3 resources provide in the randomDigraphs module a specific RandomValuationDigraph class (Bisdorff, 2021b).

**Listing 2.1** Random bipolar-valued digraph instance

```
1 >>> from randomDigraphs import RandomValuationDigraph
2 >>> rdg = RandomValuationDigraph(order=7)
3 >>> rdg.save('tutRandValDigraph')
4 >>> from digraphs import Digraph
5 >>> rdg = Digraph('tutRandValDigraph')
6 >>> rdg
7     *----- Digraph instance description -----*
8     Instance class      : Digraph
9     Instance name       : tutRandValDigraph
10    Digraph Order       : 7
11    Digraph Size        : 22
12    Valuation domain   : [-1.00;1.00]
13    Determinateness (%) : 75.24
14    Attributes          : ['name','actions','order',
15                           'valuationdomain','relation',
```

```
16     'gamma', 'notGamma' ]
```

With the `save()` method (see List. 2.1 on the preceding page Line 3) we keep for future use a backup version of `rdg` which is saved into a file named `tutRandValDigraph.py` in the current working directory. The genuine `Digraph` class constructor can reload the `rdg` object from the stored file (Line 5). One can easily inspect the content of a `Digraph` object with its name `rdg` (Line 6). The digraph size 22 indicates the number of positively valued arcs (Line 11). The valuation domain is uniformly distributed in the interval  $[-1.0; 1.0]$ , the mean absolute arc valuation being  $(0.7524 \times 2) - 1.0 = 0.5048$  (Line 13).

As mentioned in the previous Chapter 1, all objects of `Digraph` type contain at least the list of attributes shown here in Lines 14-16:

- A name (string) and a dictionary of `actions` (digraph nodes),
- An `order` (integer) attribute containing the number of nodes,
- A `valuationdomain` dictionary and a double dictionary `relation` representing the adjacency table of the digraph relation,
- A `gamma` and a `notGamma` dictionary containing the direct neighbourhood and not neighbourhood of each node.

The `Digraph` class provides some generic `show...()` methods for exploring the content of a given `Digraph` object, like the `showRelationTable()`, the `showComponents()` and the `showNeighborhoods()` methods.

**Listing 2.2** Example of random valuation digraph

```
1 >>> rdg.showRelationTable()
2     * ---- Relation Table ----*
3     r(xSy) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
4     -----|-----
5     '1'   | 0.00 -0.48  0.70  0.86  0.30  0.38  0.44
6     '2'   | -0.22  0.00 -0.38  0.50  0.80 -0.54  0.02
7     '3'   | -0.42  0.08  0.00  0.70 -0.56  0.84 -1.00
8     '4'   | 0.44 -0.40 -0.62  0.00  0.04  0.66  0.76
9     '5'   | 0.32 -0.48 -0.46  0.64  0.00 -0.22 -0.52
10    '6'   | -0.84  0.00 -0.40 -0.96 -0.18  0.00 -0.22
11    '7'   | 0.88  0.72  0.82  0.52 -0.84  0.04  0.00
12 >>> rdg.showComponents()
13     *---- Connected Components ---*
14     1: ['1', '2', '3', '4', '5', '6', '7']
15 >>> rdg.showNeighborhoods()
16     *---- Neighborhoods -----*
17     Gamma:
18     '1': in => {'5', '7', '4'}, out => {'5', '7', '6', '3', '4'}
19     '2': in => {'7', '3'}, out => {'5', '7', '4'}
20     '3': in => {'7', '1'}, out => {'6', '2', '4'}
21     '4': in => {'5', '7', '1', '2', '3'}, out => {'5', '7', '1', '6'}
22     '5': in => {'1', '2', '4'}, out => {'1', '4'}
23     '6': in => {'7', '1', '3', '4'}, out => set()
24     '7': in => {'1', '2', '4'}, out => {'1', '2', '3', '4', '6'}
25     Not Gamma:
26     '1': in => {'6', '2', '3'}, out => {'2'}
27     '2': in => {'5', '1', '4'}, out => {'1', '6', '3'}
28     '3': in => {'5', '6', '2', '4'}, out => {'5', '7', '1'}
```

```

29   '4': in => {'6'}, out => {'2','3'}
30   '5': in => {'7','6','3'}, out => {'7','6','2','3'}
31   '6': in => {'5','2'}, out => {'5','7','1','3','4'}
32   '7': in => {'5','6','3'}, out => {'5'}

```

Positive characteristic values indicate the presence of an arc whereas negative ones indicate their absence. 0.0 values indicate an indeterminate situation: there may be or not be an arc (see List. 2.2 on the preceding page Lines 5-11). Mind that some `Digraph` class methods will ignore the *reflexive* links by considering that they are *indeterminate*, i.e. the characteristic value  $r(xSx)$  for all action  $x$  is set to the *median*, i.e. *indeterminate* value 0.0 in this case (Bisdorff, 2004).

## 2.2 Graphviz drawings

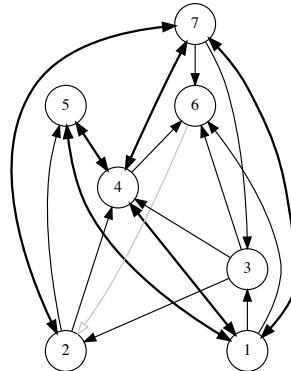
An even better insight into the `Digraph` object `rdg` is given by looking in Figure 2.1 at its `graphviz` drawing (Gansner and North, 2000)<sup>1</sup>.

```

1 >>> rdg.exportGraphViz('tutRandValDigraph')
2 *---- exporting a dot file for GraphViz tools ----*
3 Exporting to tutRandValDigraph.dot
4 dot -Grankdir=BT -Tpng tutRandValDigraph.dot\
      -o tutRandValDigraph.png
5

```

**Fig. 2.1** The tutorial random valuation digraph. Double links are drawn in bold black with an arrowhead at each end, whereas single asymmetric links are drawn in black with an arrowhead showing the direction of the link. Notice the undetermined relational situation ( $r(6S2) = 0.00$ ) observed between nodes 6 and 2. The corresponding link is marked in gray with an open arrowhead in the drawing



Digraph3 (graphviz), R. Bisdorff, 2020

---

<sup>1</sup> The `exportGraphViz()` method is depending on drawing tools from the `graphviz` software (<https://graphviz.org/>). On Linux Ubuntu or Debian you may try `sudo apt-get install graphviz` to install them. There are ready `dmg` installers for Mac OSX.

### 2.3 Asymmetric and symmetric parts

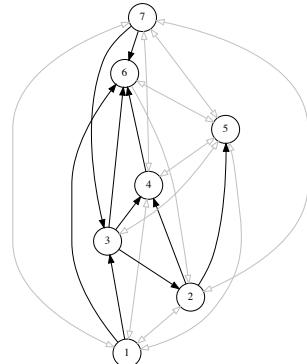
Extracting both the *symmetric* part as well as the *asymmetric* part of digraph `rdg` may be done with the help of two corresponding classes (see List. 2.3).

**Listing 2.3** Computing asymmetric and symmetric Parts

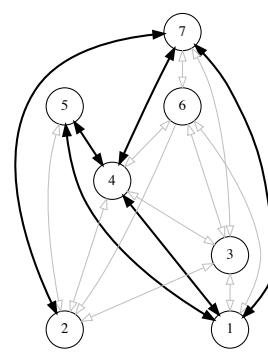
```

1 >>> from digraphs import AsymmetricPartialDigraph,\ 
2 ...                                SymmetricPartialDigraph
3 >>> asymDg = AsymmetricPartialDigraph(rdg)
4 >>> asymDg.exportGraphViz()
5 >>> symDg = SymmetricPartialDigraph(rdg)
6 >>> symDg.exportGraphViz()
```

Asymmetric Part



Symmetric Part



**Fig. 2.2** Asymmetric and symmetric part of the tutorial random valuation digraph

The class constructors of the partial objects `asymDg` and `symDg` set to the indeterminate characteristic value all non-asymmetric, respectively non-symmetric links between nodes (see Fig. 2.2).

Here below, for illustration the source code of the `_constructRelation` constructor of the `AsymmetricPartialDigraph` class:

**Listing 2.4** Computing the asymmetric part of a bipolar-valued relation

```

1 def _constructRelation(self):
2     actions = self.actions
3     Min = self.valuationdomain['min']
4     Max = self.valuationdomain['max']
5     Med = self.valuationdomain['med']
6     relationIn = self.relation
7     relationOut = {}
8     for a in actions:
9         relationOut[a] = {}
10        for b in actions:
11            if a != b:
```

```

12         if relationIn[a][b] >= Med and relationIn[b][a] <= Med:
13             relationOut[a][b] = relationIn[a][b]
14         elif relationIn[a][b] <= Med and relationIn[b][a] >= Med:
15             relationOut[a][b] = relationIn[a][b]
16         else:
17             relationOut[a][b] = Med
18     else: # reflexive links are ignored
19         relationOut[a][b] = Med
20
return relationOut

```

## 2.4 Border and inner parts

We can also extract the *border* –the part of a digraph induced by the union of its initial and terminal prekernels (see Chap. 17)– as well as, the *inner part* –the complement of the border– with the help of two corresponding classes: `GraphBorder` and `GraphInner` (see List. 2.5).



**Fig. 2.3** Border and inner part of a linear order oriented by terminal and initial kernels

Figure 2.3 shows the border and inner parts of the linear ordering obtained from the tutorial random valuation digraph `rdg` with the NETFLOWS ranking rule (see Sec. 8.3).

**Listing 2.5** Border and inner part of a linear order

```

1 >>> from digraphs import GraphBorder, GraphInner
2 >>> from linearOrders import NetFlowsOrder
3 >>> nf = NetFlowsOrder(rdg)
4 >>> nf.netFlowsOrder
5   ['6', '4', '5', '3', '2', '1', '7']
6 >>> bnf = GraphBorder(nf)

```

```

7 >>> bnf.exportGraphViz(lastChoice=['6'],firstChoice=['7'])
8 >>> inf = GraphInner(nf)
9 >>> inf.exportGraphViz(lastChoice=['6'],firstChoice=['7'])

```

The drawings in Figure 2.3 on the previous page are oriented with the terminal node 6 (lastChoice parameter) and initial node 7 (firstChoice parameter) (see List. 2.5 on the preceding page Lines 7 and 9).

The class constructors of the partial digraphs `bnf` and `inf` (see Lines 3 and 6) set to the *indeterminate* characteristic value all links not in the *border*, respectively *not* in the *inner* part (see Fig. 2.3 on the previous page). Being much *denser* than a linear order, the actual inner part of our tutorial random valuation digraph `rdg` is reduced to a single arc between nodes 3 and 4 (see Fig. 2.4). Indeed, a complete digraph on the limit has no inner part (privacy!) at all, whereas empty and indeterminate digraphs admit both, an empty border and an empty inner part.



**Fig. 2.4** Border and inner part of the tutorial random valuation digraph `rdg`

## 2.5 Fusion by epistemic disjunction

The `Digraph` object `rdg` can be restored from both partial objects `asymDg` and `symDg`, or as well from the border `bg` and the inner part `ig`, with a *bipolar disjunctive fusion* operator `o-max`, also called *epistemic disjunction*, available via the `FusionDigraph` class.

### Listing 2.6 Epistemic fusion of partial digraphs

```

1 >>> from digraphs import FusionDigraph
2 >>> fusDg = FusionDigraph(asymDg,symDg,operator='o-max')

```

```

3 >>> # fusDg = FusionDigraph(bg, ig, operator='o-max')
4 >>> fusDg.showRelationTable()
5   * ----- Relation Table -----
6   r(xSy) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
7   -----|-----
8   '1'   |  0.00 -0.48  0.70  0.86  0.30  0.38  0.44
9   '2'   | -0.22  0.00 -0.38  0.50  0.80 -0.54  0.02
10  '3'   | -0.42  0.08  0.00  0.70 -0.56  0.84 -1.00
11  '4'   |  0.44 -0.40 -0.62  0.00  0.04  0.66  0.76
12  '5'   |  0.32 -0.48 -0.46  0.64  0.00 -0.22 -0.52
13  '6'   | -0.84  0.00 -0.40 -0.96 -0.18  0.00 -0.22
14  '7'   |  0.88  0.72  0.82  0.52 -0.84  0.04  0.00

```

The epistemic fusion operator  $\text{o-max}$  (see List. 2.6 on the preceding page Line 2) is defined as follows:

### Definition 2.1 (Disjunctive epistemic fusion operator $\text{o-max}$ ).

Let  $r$  and  $r'$  characterise two bipolar-valued epistemic situations:

- $\text{o-max}(r, r') = \max(r, r')$  when both  $r \geq 0.0$  and  $r' \geq 0.0$ , i.e.  $r$  and  $r'$  are positively valid or indeterminate;;
- $\text{o-max}(r, r') = \min(r, r')$  when both  $r \leq 0.0$  and  $r' \leq 0.0$ , i.e.  $r$  and  $r'$  are positively invalid or indeterminate;
- $\text{o-max}(r, r') = 0.0$ , i.e. indeterminate otherwise.

Mind that the  $\text{o-max}$  operator, like a mean operator, is *not associative* when more than 2 operands are given. In order to make the  $\text{o-max}$  fusion univocal, the following rule is applied: –first, all positive and negative terms are separately aggregated, –then the  $\text{o-max}$  fusion is applied to both aggregates.

## 2.6 Dual, converse and codual digraphs

The `digraphs` module provides class constructors for computing the *dual* (negated relation <sup>2</sup>), the *converse* (transposed relation) and the *codual* (transposed and negated relation) of the `Digraph` instance `rdg` (see List. 2.7 Lines 4, 16, 29).

### Listing 2.7 Computing associated dual, converse and codual digraphs

```

1 >>> from digraphs import \
2 ...           DualDigraph, ConverseDigraph, CoDualDigraph
3 >>> # dual of rdg
4 >>> ddg = DualDigraph(rdg)
5 >>> ddg.showRelationTable()
6   -r(xSy) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
7   -----|-----
8   '1'   |  0.00  0.48 -0.70 -0.86 -0.30 -0.38 -0.44
9   '2'   |  0.22  0.00  0.38 -0.50  0.80  0.54 -0.02

```

---

<sup>2</sup> Not to be confused with the dual graph of a plane graph  $g$  that has a vertex for each face of  $g$ . Here we mean the *less than* (strict converse) relation corresponding to a *greater or equal* relation, or the *less than or equal* relation corresponding to a (strict) *better than* relation.

```

10   '3' |  0.42  0.08  0.00 -0.70  0.56 -0.84  1.00
11   '4' | -0.44  0.40  0.62  0.00 -0.04 -0.66 -0.76
12   '5' | -0.32  0.48  0.46 -0.64  0.00  0.22  0.52
13   '6' |  0.84  0.00  0.40  0.96  0.18  0.00  0.22
14   '7' |  0.88 -0.72 -0.82 -0.52  0.84 -0.04  0.00
15 >>> # converse of rdg
16 >>> cdg = ConverseDigraph(rdg)
17 >>> cdg.showRelationTable()
18   * ---- Relation Table ----
19   r(ySx) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
20   -----|-----
21   '1' |  0.00 -0.22 -0.42  0.44  0.32 -0.84  0.88
22   '2' | -0.48  0.00  0.08 -0.40 -0.48  0.00  0.72
23   '3' |  0.70 -0.38  0.00 -0.62 -0.46 -0.40  0.82
24   '4' |  0.86  0.50  0.70  0.00  0.64 -0.96  0.52
25   '5' |  0.30  0.80 -0.56  0.04  0.00 -0.18 -0.84
26   '6' |  0.38 -0.54  0.84  0.66 -0.22  0.00  0.04
27   '7' |  0.44  0.02 -1.00  0.76 -0.52 -0.22  0.00
28 >>> # codual of rdg
29 >>> cddg = CoDualDigraph(rdg)
30 >>> cddg.showRelationTable()
31   * ---- Relation Table ----
32   -r(ySx) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
33   -----|-----
34   '1' |  0.00  0.22  0.42 -0.44 -0.32  0.84 -0.88
35   '2' |  0.48  0.00 -0.08  0.40  0.48  0.00 -0.72
36   '3' | -0.70  0.38  0.00  0.62  0.46  0.40 -0.82
37   '4' | -0.86 -0.50 -0.70  0.00 -0.64  0.96 -0.52
38   '5' | -0.30 -0.80  0.56 -0.04  0.00  0.18  0.84
39   '6' | -0.38  0.54 -0.84 -0.66  0.22  0.00 -0.04
40   '7' | -0.44 -0.02  1.00 -0.76  0.52  0.22  0.00

```

Computing the *dual*, respectively the *converse* of a digraph, may also be done with prefixing the `_neg_` (`-`) or the `_invert_` (`~`) operator. The *codual* of a Digraph object may, hence, as well be computed with a *composition* (in either order) of both operations (see Line 3 below).

**Listing 2.8** Computing the dual, the converse and the codual of a digraph

```

1 >>> ddg = -rdg    # dual of rdg
2 >>> cdg = ~rdg   # converse of rdg
3 >>> cddg = ~(~rdg) # = -(~(rdg)) codual of rdg
4 >>> (-(~rdg)).showRelationTable()
5   * ---- Relation Table ----
6   -r(ySx) | '1'   '2'   '3'   '4'   '5'   '6'   '7'
7   -----|-----
8   '1' |  0.00  0.22  0.42 -0.44 -0.32  0.84 -0.88
9   '2' |  0.48  0.00 -0.08  0.40  0.48  0.00 -0.72
10  '3' | -0.70  0.38  0.00  0.62  0.46  0.40 -0.82
11  '4' | -0.86 -0.50 -0.70  0.00 -0.64  0.96 -0.52
12  '5' | -0.30 -0.80  0.56 -0.04  0.00  0.18  0.84
13  '6' | -0.38  0.54 -0.84 -0.66  0.22  0.00 -0.04
14  '7' | -0.44 -0.02  1.00 -0.76  0.52  0.22  0.00

```

## 2.7 Symmetric and transitive closures

Symmetric and transitive closures, by default in-site methods, are also available (see Fig. 2.5). Note that it is a good idea, before going ahead with these in-site operations, that irreversibly modify the original `rdg` object, to previously make a backup version of `rdg`. The simplest storage method, always provided by the generic `Digraph.save()` method, writes out in a named file the Python content of the `Digraph` object in string representation (see Sec. 1.3 on page 14).

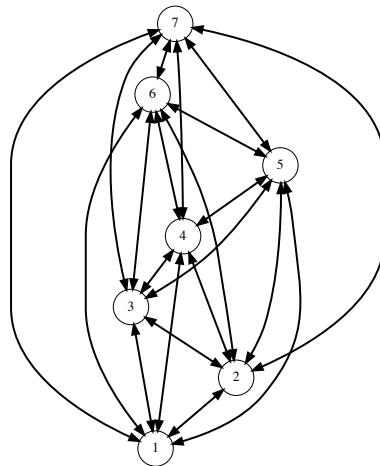
**Listing 2.9** Symmetric and transitive closures

```

1 >>> rdg.save('tutRandValDigraph')
2 >>> rdg.closeSymmetric(InSite=True)
3 >>> rdg.closeTransitive(InSite=True)
4 >>> rdg.exportGraphViz('strongComponents')

```

**Fig. 2.5** Symmetric and transitive closure of the tutorial random valuation digraph `rdg`



*Digraph3 (graphviz), R. Bisдорff, 2020*

The `closeSymmetric()` method (see List. 2.9 Line 2), of complexity  $O(n^2)$  where  $n$  denotes the digraph's order, changes, on the one hand, all single pairwise links it may detect into double links by operating a disjunction of the pairwise relations. On the other hand, the `closeTransitive()` method (see Line 3), implements the Roy-Warshall transitive closure algorithm of complexity  $O(n^3)$  (Roy, 1959 and Warshall, 1962).

The same `closeTransitive()` with a `Reverse = True` flag may be readily used for eliminating all transitive arcs from a transitive digraph instance. We make usage of this feature when drawing Hasse diagrams of `TransitiveDigraph` objects.

## 2.8 Strong components

As the original digraph `rdg` was connected (see above the result of the `showShort()` command), both the symmetric and the transitive closures operated together, will necessarily produce a single strong component, i.e. a **complete** digraph. It is sometimes useful to collapse all strong components in a given digraph and construct the so *collapsed* digraph. Using the `StrongComponentsCollapsedDigraph` constructor here will render a single hyper-node gathering all the original nodes (see Line 7 below).

**Listing 2.10** Computing the strong components in a digraph

```

1 >>> from digraphs import StrongComponentsCollapsedDigraph
2 >>> sc = StrongComponentsCollapsedDigraph(rdg)
3 >>> sc.showAll()
4     *---- show detail ----*
5     Digraph           : tutRandValDigraph_Scc
6     *---- Actions ----*
7     ['_7_1_2_6_5_3_4_']
8     *---- Relation Table ----
9     S      | 'Scc_1'
10    -----|-----
11    'Scc_1' | 0.00
12     *---- strong Components ----*
13     short      content
14     'Scc_1'     '_7_1_2_6_5_3_4_'
15     *---- Neighborhoods ----*
16     Gamma       :
17     'frozenset({'7','1','2','6','5','3','4'})':
18             in => set(), out => set()
19     Not Gamma   :
20     'frozenset({'7','1','2','6','5','3','4'})':
21             in => set(), out => set()

```

## 2.9 CSV storage

Sometimes it is required to exchange the graph valuation data in CSV format with a statistical package like **R**<sup>3</sup>. For this purpose it is possible to export the digraph data into a CSV file. The valuation domain is hereby normalised by default to the range  $[-1.0, 1.0]$  and the diagonal is put by default to the minimal value  $-1.0$ .

```

1 >>> rdg = Digraph('tutRandValDigraph')
2 >>> rdg.saveCSV('tutRandValDigraph')
3     # content of file tutRandValDigraph.csv
4     "d", "1", "2", "3", "4", "5", "6", "7"
5     "1", -1.0, 0.48, -0.7, -0.86, -0.3, -0.38, -0.44
6     "2", 0.22, -1.0, 0.38, -0.5, -0.8, 0.54, -0.02

```

---

<sup>3</sup> <https://www.r-project.org/>

```

7 "3", 0.42, -0.08, -1.0, -0.7, 0.56, -0.84, 1.0
8 "4", -0.44, 0.4, 0.62, -1.0, -0.04, -0.66, -0.76
9 "5", -0.32, 0.48, 0.46, -0.64, -1.0, 0.22, 0.52
10 "6", 0.84, 0.0, 0.4, 0.96, 0.18, -1.0, 0.22
11 "7", -0.88, -0.72, -0.82, -0.52, 0.84, -0.04, -1.0

```

It is possible to reload a Digraph instance from its previously saved CSV file content.

```

1 >>> from digraphs import CSVDigraph
2 >>> rdgcsv = CSVDigraph('tutRandValDigraph')
3 >>> rdgcsv.showRelationTable(ReflexiveTerms=False)
4 * ---- Relation Table -----
5 r(xSy) | '1' '2' '3' '4' '5' '6' '7'
6 -----|-----
7 '1' | - -0.48 0.70 0.86 0.30 0.38 0.44
8 '2' | -0.22 - -0.38 0.50 0.80 -0.54 0.02
9 '3' | -0.42 0.08 - 0.70 -0.56 0.84 -1.00
10 '4' | 0.44 -0.40 -0.62 - 0.04 0.66 0.76
11 '5' | 0.32 -0.48 -0.46 0.64 - -0.22 -0.52
12 '6' | -0.84 0.00 -0.40 -0.96 -0.18 - -0.22
13 '7' | 0.88 0.72 0.82 0.52 -0.84 0.04 -

```

It is as well possible with the `showHTMLRelationTable()` method to show in Figure 2.6 a coloured version of the valued relation table in a system browser window.

```

1 >>> rdgcsv.showHTMLRelationTable(tableTitle="Tutorial random
digraph")

```

**Fig. 2.6** The valued relation table shown in a browser window. Positive arcs are shown in green and negative arcs in red. Indeterminate –zero-valued– arcs, like the reflexive diagonal ones or the arc between node 6 and node 2, are shown in gray

## Tutorial random digraph

r(x S y)	1	2	3	4	5	6	7
<b>1</b>	0.00	-0.48	0.70	0.86	0.30	0.38	0.44
<b>2</b>	-0.22	0.00	-0.38	0.50	0.80	-0.54	0.02
<b>3</b>	-0.42	0.08	0.00	0.70	-0.56	0.84	-1.00
<b>4</b>	0.44	-0.40	-0.62	0.00	0.04	0.66	0.76
<b>5</b>	0.32	-0.48	-0.46	0.64	0.00	-0.22	-0.52
<b>6</b>	-0.84	0.00	-0.40	-0.96	-0.18	0.00	-0.22
<b>7</b>	0.88	0.72	0.82	0.52	-0.84	0.04	0.00

## 2.10 Complete, empty and indeterminate digraphs

Let us finally mention some special universal classes of digraphs that are readily available in the `digraphs` module<sup>4</sup>, like:

- the `CompleteDigraph` class,
- the `EmptyDigraph` class and
- the `IndeterminateDigraph` class,

which set all pairwise characteristic values respectively to the *maximum*, the *minimum*, respectively to the *median* indeterminate value.

**Listing 2.11** Complete, empty and indeterminate digraphs

```

1 >>> from digraphs import CompleteDigraph, EmptyDigraph, \
2 ...           IndeterminateDigraph
3 >>> # the empty digraph
4 >>> e = EmptyDigraph(order=5)
5 >>> e.showRelationTable()
6     * ---- Relation Table ----
7     S   |   '1'   '2'   '3'   '4'   '5'
8     --- -|-----
9     '1' |   -1.00  -1.00  -1.00  -1.00  -1.00
10    '2' |   -1.00  -1.00  -1.00  -1.00  -1.00
11    '3' |   -1.00  -1.00  -1.00  -1.00  -1.00
12    '4' |   -1.00  -1.00  -1.00  -1.00  -1.00
13    '5' |   -1.00  -1.00  -1.00  -1.00  -1.00
14 >>> e.showNeighborhoods()
15 Neighborhoods:
16     Gamma      :
17     '1': in => set(), out => set()
18     '2': in => set(), out => set()
19     '5': in => set(), out => set()
20     '3': in => set(), out => set()
21     '4': in => set(), out => set()
22     Not Gamma :
23     '1': in => {'2','4','5','3'}, out => {'2','4','5','3'}
24     '2': in => {'1','4','5','3'}, out => {'1','4','5','3'}
25     '5': in => {'1','2','4','3'}, out => {'1','2','4','3'}
26     '3': in => {'1','2','4','5'}, out => {'1','2','4','5'}
27     '4': in => {'1','2','5','3'}, out => {'1','2','5','3'}
28 >>> # the indeterminate digraph
29 >>> i = IndeterminateDigraph()
30     * ---- Relation Table ----
31     S   |   '1'   '2'   '3'   '4'   '5'
32     --- -|-----
33     '1' |   0.00  0.00  0.00  0.00  0.00
34     '2' |   0.00  0.00  0.00  0.00  0.00
35     '3' |   0.00  0.00  0.00  0.00  0.00
36     '4' |   0.00  0.00  0.00  0.00  0.00
37     '5' |   0.00  0.00  0.00  0.00  0.00
38 >>> i.showNeighborhoods()
39 Neighborhoods:

```

---

<sup>4</sup> See [Bisdorff, 2021a](#)

```

40      Gamma      :
41      '1': in => set(), out => set()
42      '2': in => set(), out => set()
43      '5': in => set(), out => set()
44      '3': in => set(), out => set()
45      '4': in => set(), out => set()
46      Not Gamma :
47      '1': in => set(), out => set()
48      '2': in => set(), out => set()
49      '5': in => set(), out => set()
50      '3': in => set(), out => set()
51      '4': in => set(), out => set()

```

Mind the subtle difference between the neighbourhoods of an *empty* and the neighbourhoods of an *indeterminate* Digraph instance. In the first kind, the neighbourhoods are known to be completely *empty* (see List. 2.11 on the preceding page Lines 23-27) whereas, in the latter, *nothing is known* about the actual neighbourhoods of the nodes (see Lines 47-51). These two cases illustrate why in the case of *bipolar-valued* digraphs, we may sometimes need both a `gamma` and a `notGamma` attribute.

In the following Chapter 3 we introduce the main formal object of this book, namely *bipolar-valued outranking* digraphs.

## Notes

It was *Denis Bouyssou* who first suggested us end of the nineties, when we started to work in Prolog on the computation of fuzzy digraph kernels with finite domain constraint solvers, that the 50% criteria significance majority was a very special value to be carefully taken into account. The converging solution vectors of the fixpoint kernel equations furthermore confirmed this special status of the 50% majority (see Chap. 17). These early insights led to the seminal articles on bipolar-valued epistemic logic where we introduced split truth/falseness semantics for a multi-valued logical processing of fuzzy preference modelling (Bisdorff, 2000, 2002). The characteristic valuation domain remained however the classical fuzzy [0.0; 1.0] valuation domain.

It is only in 2004, when we succeeded in assessing the stability of the outranking digraph when solely ordinal criteria significance weights are given, that it became clear and evident for us that the characteristic valuation domain had to be shifted to a [-1.0; +1.0]-valued domain (see Chap. 19 and Bisdorff (2004)). In this bipolar valuation, the 50% majority threshold corresponds now to the median 0.0 value, characterising with the correct zero value an epistemic indeterminateness – no knowledge– situation. Furthermore, identifying truth and falseness directly by the sign of the characteristic value revealed itself to be very efficient not only from a computational point of view, but also from scientific and semiotic perspectives.

A positive (resp. negative) characteristic value now attest a logically valid (resp. invalid) statement and a negative affirmation now means a positive refutation and vice versa. Furthermore, the median zero value gives way to efficiently handling partial objects –like the border or the assymmetric part of a digraph– and, even more important from a practical decision making point of view, any missing data.

The bipolar  $[-1.0; +1.0]$ -valued characteristic domain opened so the way to important new operations and concepts, like the disjunctive epistemic fusion operation seen in Section 2.5 on page 26 that confers the outranking digraph a logically sound and epistemically correct definition (Bisdorff, 2013). KENDALL’s ordinal correlation index could be extended to a bipolar-valued relational equivalence index between digraphs (see Chap. 16 and Bisdorff, 2012). Making usage of the bipolar-valued Gaussian error function naturally led to defining a bipolar-valued likelihood function, where a positive, resp. negative, value gives the likelihood of an affirmation, resp. a refutation (see Chap. 18 and Bisdorff, 2014).

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## Chapter 3

# Working with outranking digraphs

*“The rule for the combination of independent concurrent arguments takes a very simple form when expressed in terms of the intensity of belief ... It is this: Take the sum of all the feelings of belief which would be produced separately by all the arguments pro, subtract from that the similar sum for arguments con, and the remainder is the feeling of belief which ought to have the whole. This is a proceeding which men often resort to, under the name of balancing reasons.” – C.S. Peirce, The probability of induction (1878)*

**Abstract** In this chapter, we introduce the main formal object of this book, namely the bipolar-valued outranking digraph. With a randomly generated multiple-criteria performance tableau, we construct the corresponding bipolar-valued outranking relation from pairwise comparisons. The resulting bipolar-valued outranking characteristics may be recoded. Finally, the codual outranking digraph gives us the associated strict outranking relation.

### 3.1 The hybrid outranking digraph model

In the `outrankingDigraphs` module, the `BipolarOutrankingDigraph` class provides our standard *outranking digraph* constructor. Such an instance represents a *hybrid* object of both the `PerformanceTableau` type *and* the `OutrankingDigraph` type (Bisdorff, 2021). A given `BipolarOutrankingDigraph` object contains hence always at least the following attributes:

1. `actions`: an ordered dictionary describing the potential decision actions or alternatives with `name` and `comment` attributes,
2. `objectives`: a possibly empty ordered dictionary of decision objectives with `name` and `comment` attributes, describing the multiple preference dimensions involved in the decision problem,

3. `criteria`: an ordered dictionary of performance criteria, i.e. *preferentially independent* and *non-redundant* decimal-valued evaluation functions used for assessing the performance of each potential decision action with respect to a decision objective,
4. `evaluation`: a double dictionary gathering performance evaluations for each decision alternative on each criterion function.
5. `valuationdomain`: a dictionary with three entries: the minimum ( $-1.0$ , *certainly outranked*), the median ( $0.0$ , *indeterminate*) and the maximum characteristic value ( $+1.0$ , *certainly outranking*),
6. `relation`: a double dictionary defined on the Cartesian product of the set of decision alternatives capturing the credibility of the pairwise *outranking situation* computed on the basis of the performance differences observed between couples of decision alternatives on the given family of criteria functions.

Let us consider, for instance, a random bipolar-valued outranking digraph with seven decision actions denoted  $a_1, a_2, \dots, a_7$ . We need therefore, first, to generate in Listing 3.1 a corresponding random performance tableau.

**Listing 3.1** Generating a random performance tableau.

```

1 >>> from perfTabs import RandomPerformanceTableau
2 >>> pt = RandomPerformanceTableau(numberOfActions=7, \
3 ...                                     seed=100)
4 >>> pt
5      *----- PerformanceTableau instance description -----*
6      Instance class      : RandomPerformanceTableau
7      Seed                 : 100
8      Instance name       : randomperftab
9      Actions              : 7
10     Criteria             : 7
11     NaN proportion (%) : 6.1
12 >>> pt.showActions()
13      *----- show digraphs actions -----*
14      key: a1
15      name:      action 1
16      comment:   RandomPerformanceTableau() generated.
17      key: a2
18      name:      action 2
19      comment:   RandomPerformanceTableau() generated.
20      ...
21      ...
22      key: a7
23      name:      action 7
24      comment:   RandomPerformanceTableau() generated.

```

In this example we consider a family of seven *equisignificant* cardinal *criteria functions*  $g_1, g_2, \dots, g_7$ , measuring the performance of each alternative on a rational scale from 0.0 (worst) to 100.00 (best). In order to capture the evaluation procedures' potential *uncertainty* and *imprecision*, each criterion function  $g_1$  to  $g_7$  admits three performance *discrimination thresholds* of 2.5, 5.0 and 80.0 pts for warranting respectively any *indifference*, *preference* or *considerable performance difference* situation.

**Listing 3.2** Inspecting the performance criteria.

```

1 >>> pt.showCriteria()
2     *---- criteria ----*
3     g1 'RandomPerformanceTableau() instance'
4         Scale = [0.0, 100.0]
5         Weight = 1.0
6         Threshold ind : 2.50 + 0.00x ; percentile: 4.76
7         Threshold pref : 5.00 + 0.00x ; percentile: 9.52
8         Threshold veto : 80.00 + 0.00x ; percentile: 95.24
9     g2 'RandomPerformanceTableau() instance'
10        Scale = [0.0, 100.0]
11        Weight = 1.0
12        Threshold ind : 2.50 + 0.00x ; percentile: 6.67
13        Threshold pref : 5.00 + 0.00x ; percentile: 6.67
14        Threshold veto : 80.00 + 0.00x ; percentile: 100.00
15        ...
16        ...
17     g7 'RandomPerformanceTableau() instance'
18        Scale = [0.0, 100.0]
19        Weight = 1.0
20        Threshold ind : 2.50 + 0.00x ; percentile: 0.00
21        Threshold pref : 5.00 + 0.00x ; percentile: 4.76
22        Threshold veto : 80.00 + 0.00x ; percentile: 100.00

```

On criteria function `g1` we observe, for instance, about 5% of *indifference* situations, about 90% of *preference* situations and about 5% of *considerable* performance difference situations (see Lines 6-8 in List. 3.2).

The individual *performance evaluation* of all decision alternatives on each criterion are gathered in a *performance table*.

**Listing 3.3** Inspecting the performance evaluations

```

1 >>> pt.showPerformanceTableau()
2     *---- performance tableau ----*
3     criteria | 'a1'  'a2'  'a3'  'a4'  'a5'  'a6'  'a7'
4     -----|-----
5     'g1'   |  15.2  44.5  57.9  58.0  24.2  29.1  96.6
6     'g2'   |  82.3  43.9   NA   35.8  29.1  34.8  62.2
7     'g3'   |  44.2  19.1  27.7  41.5  22.4  21.5  56.9
8     'g4'   |  46.4  16.2  21.5  51.2  77.0  39.4  32.1
9     'g5'   |  47.7  14.8  79.7  67.5   NA  90.7  80.2
10    'g6'   |  69.6  45.5  22.0  33.8  31.8   NA  48.8
11    'g7'   |  82.9  41.7  12.8  21.9  75.7  15.4   6.0

```

It is noteworthy to mention the three *missing data* (NA) cases: action `a3` is missing, for instance, an evaluation on criterion `g2` (see Line 6 in List. 3.3).

## 3.2 The bipolar-valued outranking digraph

Given the previous random performance tableau `pt`, the `BipolarOutrankingDigraph` class constructor computes the corresponding *bipolar-valued outranking digraph*.

**Listing 3.4** Example of random bipolar-valued outranking digraph.

```

1 >>> from outrankingDigraphs import \
2     ...                                BipolarOutrankingDigraph
3 >>> odg = BipolarOutrankingDigraph(pt)
4 >>> odg
5     *----- Object instance description -----*
6     Instance class      : BipolarOutrankingDigraph
7     Instance name       : rel_randomperftab
8     Actions             : 7
9     Criteria            : 7
10    Size                : 20
11    Determinateness (%) : 63.27
12    Valuation domain    : [-1.00;1.00]
13    Attributes          : [
14        'name', 'actions',
15        'criteria', 'evaluation', 'NA',
16        'valuationdomain', 'relation',
17        'order', 'gamma', 'notGamma', ...
18    ]

```

The resulting digraph contains 20 positive (valid) outranking relations. And, the mean majority criteria significance support of all the pairwise outranking situations is 63.3% (see Lines 10-11 in List 3.4).

We can inspect the complete  $[-1.0, +1.0]$ -valued adjacency table with the `showRelationTable()` method().

**Listing 3.5** Inspecting the valued adjacency table.

```

1 >>> odg.showRelationTable()
2     *----- Relation Table -----
3     r(x,y) |  'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'
4     -----|-----
5     'a1' | +1.00  +0.71  +0.29  +0.29  +0.29  +0.29  +0.00
6     'a2' | -0.71  +1.00  -0.29  -0.14  +0.14  +0.29  -0.57
7     'a3' | -0.29  +0.29  +1.00  -0.29  -0.14  +0.00  -0.29
8     'a4' | +0.00  +0.14  +0.57  +1.00  +0.29  +0.57  -0.43
9     'a5' | -0.29  +0.00  +0.14  +0.00  +1.00  +0.29  -0.29
10    'a6' | -0.29  +0.00  +0.14  -0.29  +0.14  +1.00  +0.00
11    'a7' | +0.00  +0.71  +0.57  +0.43  +0.29  +0.00  +1.00
12
12     Valuation domain: [-1.0; 1.0]

```

The `BipolarOutrankingDigraph` class constructor computes from the given performance tableau `pt` the characteristic value  $r(x \succsim y)$  of a *pairwise outranking relation*  $x \succsim y$  in a default *valuation domain*  $[-1.0, +1.0]$  with the *median* value 0.0 acting as *indeterminate* characteristic value.

The semantics of  $r(x \succsim y)$  are the following:

**Definition 3.1 (Semantics of the outranking characteristic function).**

- When  $r(x \succsim y) > 0.0$ , it is *True* that  $x$  outranks  $y$ , i.e. alternative  $x$  is “*at least as well evaluated as*” alternative  $y$  on a weighted majority of criteria **and** there is no considerable negative performance difference observed in disfavour of  $x$ ,
- When  $r(x \succsim y) < 0.0$ , it is *False* that  $x$  outranks  $y$ , i.e. alternative  $x$  is **only “evaluated at least as well as** alternative  $y$  on a weighted minority of criteria **and** there is no considerable positive performance difference observed in favour of  $x$ ,
- When  $r(x \succsim y) = 0.0$ , it is **indeterminate** whether  $x$  outranks  $y$  or not.

### 3.3 Pairwise comparisons

From above given semantics, we notice in line 5 in Listing 3.5 on the preceding page that  $a_1$  outranks  $a_2$ :  $r(a_1 \succsim a_2) + 0.71$ , but not  $a_7$ :  $r(a_1 \succsim a_7) = 0.0$ . In order to comprehend the characteristic values shown in the outranking relation table, we can inspect with the `showPairwiseComparison()` method the details of the pairwise multiple-criteria comparison between, for instance, alternatives  $a_1$  and  $a_2$ .

**Listing 3.6** Inspecting a pairwise multiple-criteria comparison

```

1 >>> odg.showPairwiseComparison('a1','a2')
2 *----- pairwise comparison -----*
3 Comparing actions : (a1, a2)
4 crit. wght. g(a1)   g(a2)   diff   | ind     pref    r()
5 -----      -----|-----|-----|-----|-----|-----|-----|-----|
6   g1    1.00  15.17  44.51 -29.34 | 2.50  5.00  -1.00
7   g2    1.00  82.29  43.90 +38.39 | 2.50  5.00   +1.00
8   g3    1.00  44.23  19.10 +25.13 | 2.50  5.00   +1.00
9   g4    1.00  46.37  16.22 +30.15 | 2.50  5.00   +1.00
10  g5    1.00  47.67  14.81 +32.86 | 2.50  5.00   +1.00
11  g6    1.00  69.62  45.49 +24.13 | 2.50  5.00   +1.00
12  g7    1.00  82.88  41.66 +41.22 | 2.50  5.00   +1.00
13 Valuation in range: -7.00 to +7.00;
14 r(a1,a2): +5.00/7.00 = +0.71           +5.00

```

The outranking characteristic value  $r(a_1 \succsim a_2)$  represents the relative *majority margin* resulting from the difference between the significance weights of the criteria in favour and the significance weights of the criteria in disfavour of the statement that alternative  $a_1$  is “*at least as well evaluated as*” alternative  $a_2$ . No considerable performance difference being observed, no polarising situation is triggered in this pairwise comparison.

Such a polarised situation is however observed when we compare the evaluations of alternatives  $a_1$  and  $a_7$  with the `showPairwiseOutrankings()` method.

**Listing 3.7** Pairwise comparison with considerable performance difference

```

1 >>> odg.showPairwiseOutrankings('a1','a7')
2 *----- pairwise comparison -----*
3 Comparing actions : (a1, a7)

```

	crit.	wght.	g(a1)	g(a7)	diff		ind	pref	r()		v	veto
<hr/>												
6	g1	1.00	15.17	96.58	-81.41		2.50	5.00	-1.00		80.00	-1.00
7	g2	1.00	82.29	62.22	+20.07		2.50	5.00	+1.00			
8	g3	1.00	44.23	56.90	-12.67		2.50	5.00	-1.00			
9	g4	1.00	46.37	32.06	+14.31		2.50	5.00	+1.00			
10	g5	1.00	47.67	80.16	-32.49		2.50	5.00	-1.00			
11	g6	1.00	69.62	48.80	+20.82		2.50	5.00	+1.00			
12	g7	1.00	82.88	6.05	+76.83		2.50	5.00	+1.00			
<hr/>												
14	Valuation in range: -7.00 to +7.00; r(x,y) = +1/7 => 0.0											
15	*----- pairwise comparison -----*											
16	Comparing actions : (a1, a7)											
17	crit.	wght.	g(a7)	g(a1)	diff		ind	pref	r()		v	veto
18	<hr/>											
19	g1	1.00	96.58	15.17	+81.41		2.50	5.00	+1.00		80.00	+1.00
20	g2	1.00	62.22	82.29	-20.07		2.50	5.00	-1.00			
21	g3	1.00	56.90	44.23	+12.67		2.50	5.00	+1.00			
22	g4	1.00	32.06	46.37	-14.31		2.50	5.00	-1.00			
23	g5	1.00	80.16	47.67	+32.49		2.50	5.00	+1.00			
24	g6	1.00	48.80	69.62	-20.82		2.50	5.00	-1.00			
25	g7	1.00	6.05	82.88	-76.83		2.50	5.00	-1.00			
<hr/>												
27	Valuation in range: -7.00 to +7.00; r(x,y) = -1/7 => 0.0											

This time, we observe a  $(1/7 + 1)/2 = 57.1\%$  majority of criteria significance warranting a “*at least as well evaluated as*” situation between alternative a1 and alternative a7. Yet, we also observe a considerable *negative* performance difference on criterion g1 (see Line 6 in List. 3.7 on the preceding page). Both contradictory facts trigger eventually in Line 14 an *indeterminate* outranking situation. The inverse polarisation effect appears when considering in Lines 19-25 the converse performance differences between alternative a7 and alternative a1. The considerable better performing situation on criterion g1 makes doubtful the otherwise “*not at least as well evaluated as*” situation (see Lines 19 and 27).

Notice that the occurrence in a pairwise comparison of conjointly considerable positive and negative performance differences will also trigger an indeterminate outranking situation. When observing at the same time a positive (resp. negative) “*at least as well evaluated as*” situation and one or more considerable positive (resp. negative) performance difference, the outranking situation gets validated (resp. invalidated) for certain (Bisdorff, 2013).

### 3.4 Recoding the characteristic valuation domain

All outranking digraphs, being of root `Digraph` type, inherit the methods available under this latter class. The characteristic valuation domain of a digraph can, for instance, be recoded with the `recodeValuation()` method to the *integer* range  $[-7, +7]$ , i.e. plus or minus the total significance weights of the family of criteria considered in this example instance `odg`.

**Listing 3.8** Recoding the digraph valuation

```
1 >>> odg.recodeValuation(-7,+7)
2 >>> odg.valuationdomain['hasIntegerValuation'] = True
```

```

3 >>> Digraph.showRelationTable(odg,ReflexiveTerms=False)
4   * ----- Relation Table -----
5   r(x,y) | 'a1' 'a2' 'a3' 'a4' 'a5' 'a6' 'a7'
6   -----|-----
7   'a1' | - +5 +2 +2 +2 +2 0
8   'a2' | -5 - -1 -1 +1 +2 -4
9   'a3' | -1 +2 - -1 -1 0 -1
10  'a4' | 0 +1 +4 - +2 +4 -3
11  'a5' | -1 0 +1 0 - +2 -1
12  'a6' | -1 0 +1 -1 +1 - 0
13  'a7' | 0 +5 +4 +3 +2 0 -
14 Valuation domain: [-7;+7]

```

Notice in Listing 3.8 on the preceding page that the self comparison characteristics  $r(x \succsim x)$  may be ignored by setting the `ReflexiveTerms` Boolean parameter to `False`. Mind that the trivial reflexive terms of outranking relations are ignored in some of the DIGRAPH3 methods.

### 3.5 The strict outranking digraph

From theory we know that a bipolar-valued outranking digraph is *weakly complete*, i.e. if  $r(x \succsim y) < 0.0$  then  $r(y \succsim x) \geq 0.0$ . From this property follows that a bipolar-valued outranking digraph verifies the *coduality principle*: the *dual*<sup>1</sup> –strict negation– of the *converse* –inverse– of the outranking relation ( $x \succsim y$ ) corresponds to its asymmetric *strict outranking* part ( $x \succ y$ ) (Bisdorff (2013, 2020)).

Visualising the *codual (strict)* outranking digraph may be done with a graphviz drawing<sup>2</sup>.

```

1 >>> cdodg = -(~odg)
2 >>> cdodg.exportGraphViz('codualOdg')
3   *----- exporting a dot file for GraphViz tools -----
4   Exporting to codualOdg.dot
5   dot -Grankdir=BT -Tpng codualOdg.dot -o codualOdg.png

```

Many more tools for exploiting bipolar-valued outranking digraphs are available in the DIGRAPH3 resources (Bisdorff, 2021).

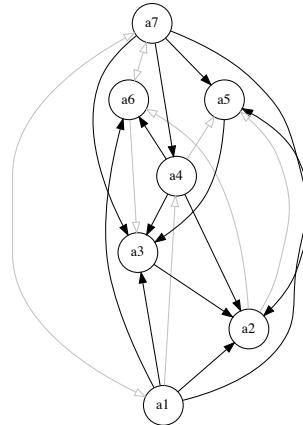
In the next methodological Part II, we present and discuss multiple-criteria evaluation models and decision algorithms, like building a best choice recommendation, determining the winner of an election, computing linear rankings or quantile ratings with multiple incommensurable criteria.

---

<sup>1</sup> Not to be confused with the dual graph of a plane graph  $g$  that has a vertex for each face of  $g$ . Here we mean the *less than* (strict converse) relation corresponding to a *greater or equal* relation, or the *less than or equal* relation corresponding to a (strict) *better than* relation.

<sup>2</sup> The `exportGraphViz()` method is depending on drawing tools from the graphviz software (<https://graphviz.org/>). On Linux Ubuntu or Debian you may try `sudo apt-get install graphviz` to install them. There are ready `dmg` installers for Mac OSX.

**Fig. 3.1** The strict (codual) outranking digraph. It becomes readily clear now from the picture that both alternatives  $a_1$  and  $a_7$  are *not outranked* by any other alternatives. Hence,  $a_1$  and  $a_7$  appear as *weak CONDORCET winners* and may be recommended as potential *best* decision alternatives in this illustrative preference modelling example



Digraph3 (graphviz), R. Bisdorff, 2020

## Notes

The seminal work on outranking digraphs goes back to the seventies and eighties when *Bernard Roy* joined the just starting University Paris-Dauphine and founded there the '*Laboratoire d'Analyse et de Modélisation de Systèmes pour l'Aide à la Décision*' (LAMSADE). The LAMSADE became the major site in the development of the outranking approach to multiple-criteria decision aiding. ([Roy and Bouyssou, 1993](#)).

The ongoing success of the original *outranking* concept stems from the fact that it is rooted in a sound pragmatism. The multiple-criteria performance tableau, necessarily associated with a given outranking digraph, is indeed convincingly objective and meaningful ([Roy, 1991](#)). And, ideas from social choice theory gave initially the insight that a pairwise voting mechanism à la CONDORCET could provide an order-statistical tool for aggregating a set of preference points of view into what *Marc Barbut* called the *central CONDORCET point* of view ([de Caritat, Marquis de Condorcet, 1784](#) and [Barbut, 1980](#)); in fact the median of the multiple preference points of view, at minimal absolute KENDALL's ordinal correlation distance from all individual points of view (see Chap. 16).

Considering thus each performance criterion as a subset of unanimous voters and balancing the votes in favour against considerable counter-performances in disfavour gave eventually rise to the concept of *outranking situation*, a distinctive feature of the Multiple-Criteria Decision Aiding approach ([Bisdorff et al., 2015](#)). A modern definition would be : an alternative  $x$  is said to *outrank* alternative  $y$  when – a *significant majority* of criteria confirm that alternative  $x$  has to be considered as *at least as well evaluated as* an alternative  $y$  (the *concordance argument*); and – no discordant criterion opens to significant doubt the validity of the previous confirmation.

tion by revealing a considerable counter-performance of alternative  $x$  compared to  $y$  (the *discordance* argument).

If the concordance argument was always well received, the discordance argument however, very confused in the beginning (Benayoun et al., 1966), could only be handled in an epistemically correct and logically sound way by using a bipolar-valued epistemic logic (see Def. 3.1 on page 41 and Bisdorff, 2013). The outranking situation had consequently to receive an explicit negative definition: An alternative  $x$  is said to *do not outrank* an alternative  $y$  when – a *significant majority* of criteria confirm that alternative  $x$  has to be considered as *not at least as well evaluated as* alternative  $y$ ; and – no discordant criterion opens to significant doubt the validity of the previous confirmation by revealing a considerable *better* performance of alternative  $x$  compared to  $y$ .

Furthermore, the initial conjunctive aggregation of the concordance and discordance arguments had to be replaced by a disjunctive epistemic fusion operation, polarising in a logically sound and epistemically correct way the concordance with the discordance argument. This way, bipolar-valued outranking digraphs gained two very useful properties from a measure theoretical perspective. They are *weakly complete*; incomparability situations are no longer attested by the absence of positive outranking relations, but instead by epistemic indeterminateness. And they verify the *coduality principle*: the negation of the epistemic “*at least as well evaluated as*” situation corresponds formally to the strict converse epistemic “*less well evaluated than*” situation.

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**Part II**

**Evaluation models and decision algorithms**

The second and methodological part of the book presents in eight chapters multiple-criteria evaluation models and decision algorithms for selecting a best decision alternative, for ranking the potential decision alternative from best to worst, and for relative or absolute quantile rating with the help of bipolar-valued outranking digraphs. We also show how to edit a new multiple-criteria performance tableau and present several models of random performance tableau generators. A last chapter is devoted to HPC ranking of big performance tableaux.

## Chapter 4

# Building a best choice recommendation

“... *The goal of our research was to design a resolution method ... that is easy to put into practice, that requires as few and reliable hypotheses as possible, and that meets the needs [of the decision maker]...*”

—(Benayoun, Roy, and Sussmann, 1966).

**Abstract** This chapter presents the RUBIS best choice recommender system. Our approach is illustrated with a best office location selection problem. We show how to explore the given performance tableau and compute the corresponding outranking digraph. After presenting the pragmatic principles that govern our best choice recommendation algorithm we solve the best office location choice problem.

### 4.1 What office-location to choose?

A SME, specialised in printing and copy services, has to move into new offices, and its CEO has gathered seven *potential new office locations* (see Tab. 4.1).

**Table 4.1** The potential new office locations

ID	Name	Address	Comment
A	Ave	Avenue de la liberté	High standing city center
B	Bon	Bonnevoie	Industrial environment
C	Ces	Cessange	Residential suburb location
D	Dom	Dommeldange	Industrial suburb environment
E	Bel	Esch-Belval	New and ambitious urbanisation far from the city
F	Fen	Fentange	Out in the countryside
G	Gar	Avenue de la Gare	Main city shopping street

Three *decision objectives*, in order of decreasing importance, are guiding the CEO's choice:

1. *maximise* the future turnover of the SME,
2. *minimise* the future yearly costs induced by the moving,
3. *maximise* the new working conditions.

The decision consequences to take into account for evaluating the potential new office locations with respect to each one of the three objectives are modelled by the *family of performance criteria*<sup>1</sup> shown in Table 4.2 below.

**Table 4.2** The family of performance criteria

Objective	ID	Name	Weight	Comment
Yearly costs	Ct	Costs	45	Annual rent, charges, and cleaning
Future turnover	Pr	Proximity	32	Distance from town center
Future turnover	V	Visibility	26	Circulation of potential customers
Future turnover	St	Standing	23	Image and presentation
Working conditions	W	Space	10	Working space
Working conditions	Cf	Comfort	6	Quality of office equipment
Working conditions	P	Parking	3	Available parking facilities

In Table 4.2 we notice that the *Costs* criterion admits the highest significance weight (45), followed by the *Future turnover* criteria (32, 26, 23). The *Working conditions* criteria are the less significant (10, 6, 3)<sup>2</sup>. It follows that the CEO considers *maximising the future turnover* the most important objective ( $(32 + 26 + 23) = 81$ ), followed by minimising the future yearly costs (45), and less important, *maximising working conditions* ( $(10 + 6 + 3) = 19$ ).

The evaluations of the seven potential new locations on each performance criterion are gathered in a *performance table* shown in Table 4.3 on the facing page. All criteria, except the *Costs* Criterion, admit for evaluation a qualitative satisfaction scale from 0% (weakest) to 100% (highest). One may thus notice that location A (Ave) is the most expensive, but also 100% satisfying the *Proximity* as well as the *Standing* criterion. Whereas location C (Ces) is the cheapest one; providing however no satisfaction at all on both the *Standing* and the *Working Space* criteria.

Concerning yearly costs, we suppose that the CEO is indifferent up to a performance difference of 1000.00€, and he actually prefers a location when there is at least a positive difference of 2500.00€. The evaluations observed on the six qualitative criteria (measured in percentages of satisfaction) are very subjective and rather imprecise. The CEO is hence *indifferent* up to a satisfaction difference of 10%,

<sup>1</sup> See Roy, 2000

<sup>2</sup> These criteria weights were supposedly established with a swing weighing MCDA method (Keeney and Raiffa, 1976).

**Table 4.3** Performance evaluations of the potential office locations

Criterion	A	B	C	D	E	F	G
Costs	35.0K€	17.8K€	6.7K€	14.1K€	34.8K€	18.6K€	12.0K€
Proximity	100	20	80	70	40	0	60
Visibility	60	80	70	50	60	0	100
Standing	100	10	0	30	90	70	20
Working space	75	30	0	55	100	0	50
Working comfort	0	100	10	30	60	80	50
Parking	90	30	100	90	70	0	80

and he claims a significant *preference* when the satisfaction difference is at least of 20%. Furthermore, a satisfaction difference of 80% represents for him a *considerably large* performance difference, triggering the case given a *polarisation* of the preferential situation (Bisdorff, 2013).

In view of Table 4.3, what is now the office location we may recommend to the CEO as **best choice**?

## 4.2 The given performance tableau

The file `officeChoice.py`, stored in the `examples` directory of the DI-GRAF3 resources, provides a corresponding `PerformanceTableau` object. We can inspect its actual content with the computing resources provided by the `perfTabs` module .

**Listing 4.1** Inspecting the `officeChoice` performance tableau.

```

1 >>> from perfTabs import PerformanceTableau
2 >>> pt = PerformanceTableau('officeChoice')
3 >>> pt
4 *----- PerformanceTableau instance description -----*
5   Instance class      : PerformanceTableau
6   Instance name       : officeChoice
7   Actions             : 7
8   Objectives          : 3
9   Criteria            : 7
10  NaN proportion (%) : 0.0
11  Attributes          : ['name', 'actions', 'objectives',
12                           'criteria', 'weightPreorder',
13                           'NA', 'evaluation']
14 >>> pt.showPerformanceTableau()
15 *---- performance tableau ----*
16 Criteria| 'Ct'      'Cf'      'P'       'Pr'      'St'      'V'       'W'
17 Weights | 45.00     6.00     3.00    32.00    23.00    26.00    10.00
18 -----| -----

```

19	'Ave'		-35000.00	0.00	90.00	100.00	100.00	60.00	75.00
20	'Bon'		-17800.00	100.00	30.00	20.00	10.00	80.00	30.00
21	'Ces'		-6700.00	10.00	100.00	80.00	0.00	70.00	0.00
22	'Dom'		-14100.00	30.00	90.00	70.00	30.00	50.00	55.00
23	'Bel'		-34800.00	60.00	70.00	40.00	90.00	60.00	100.00
24	'Fen'		-18600.00	80.00	0.00	0.00	70.00	0.00	0.00
25	'Gar'		-12000.00	50.00	80.00	60.00	20.00	100.00	50.00

We thus recover all the input data shown in Section 4.1. Notice that the *negative* evaluations of the *Costs* criterion indicate a negative preference direction: the *lower* the costs, the *better* it is.

The `showCriteria()` method evaluates the actual preference discrimination we observe on each performance criterion.

**Listing 4.2** Inspecting the performance criteria.

```

1 >>> pt.showCriteria(IntegerWeights=True)
2 *---- criteria ----*
3 Ct 'Costs'
4   Preference direction: min
5   Scale = (0.00, 50000.00)
6   Weight = 45
7   Threshold ind : 1000.00 + 0.00x ; percentile: 9.52
8   Threshold pref : 2500.00 + 0.00x ; percentile: 14.29
9 Cf 'Comfort'
10  Preference direction: max
11  Scale = (0.00, 100.00)
12  Weight = 6
13  Threshold ind : 10.00 + 0.00x ; percentile: 9.52
14  Threshold pref : 20.00 + 0.00x ; percentile: 28.57
15  Threshold veto : 80.00 + 0.00x ; percentile: 90.48
16 ...
17 ...

```

On the *Costs* criterion, 9.5% of the performance differences are considered insignificant and 14.3% below the preference discrimination threshold (see Lines 7-8 in List. 4.2). On the qualitative *Comfort* criterion, we observe again 9.5% of insignificant performance differences (line 13). Due to the imprecision in the subjective evaluations, we notice here 28.6% of performance differences below the preference discrimination threshold (Line 14). Furthermore,  $100.0 - 90.5 = 9.5\%$  of the performance differences are judged *considerably large* (Line 15) and will trigger hence a polarisation of the concerned outranking situations (Bisdorff, 2013). Same information is available for all the other criteria.

A colourful comparison of all the evaluations is shown in Figure 4.1 on the next page by the *heatmap* statistics, illustrating the respective quantile class of each evaluation. As the set of potential alternatives is tiny, we choose here a classification into performance quintiles.

```

1 >>> pt.showHTMLPerformanceHeatmap(colorLevels=5, \
2 ...                                         rankingRule=None)

```

Location Ave shows extreme and contradictory evaluations: highest *Costs* and no *Working Comfort* on the one hand, and total satisfaction with respect to *Standing*,

### Heatmap of Performance Tableau 'officeChoice'

criteria	Ct	Pr	V	St	W	Cf	P
<b>weights</b>	+45.00	+32.00	+26.00	+23.00	+10.00	+6.00	+3.00
<b>Ave</b>	-35000	100	60	100	75	0	90
<b>Bon</b>	-17800	20	80	10	30	100	30
<b>Ces</b>	-6700	80	70	0	0	10	100
<b>Dom</b>	-14100	70	50	30	55	30	90
<b>Bel</b>	-34800	40	60	90	100	60	70
<b>Fen</b>	-18600	0	0	70	0	80	0
<b>Gar</b>	-12000	60	100	20	50	50	80

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

Fig. 4.1 Unranked heatmap of the office choice performance tableau

*Proximity* and *Parking facilities* on the other hand. Similar, but opposite, situation is given for location *Ces*: unsatisfactory *Working Space*, no *Standing* and no *Working Comfort* on the one hand, and lowest *Costs*, best *Proximity* and *Parking facilities* on the other hand. Contrary to these contradictory alternatives, we observe two appealing compromise alternatives: locations *Dom* and *Gar*. Finally, location *Fen* is clearly the less satisfactory alternative of all.

To help now the CEO choosing the best office location, we are going to compute pairwise outranking situations on the set of potential decision alternatives (see Bisдорff, 2013).

### 4.3 Computing the outranking digraph

#### Definition 4.1 (Outranking situation).

For two potential decision alternatives  $x$  and  $y$ :

- “Alternative  $x$  outranks alternative  $y$ ”, denoted  $(x \succ y)$ , is given when:
  1. A *majority* of criteria significance warrants that alternative  $x$  is *at least as well evaluated as* alternative  $y$ , and
  2. *No considerable negative performance difference* is observed on any criterion.
- “Alternative  $x$  does not outrank  $y$ ”, denoted  $(x \not\succ y)$ , is given when:
  1. Only a *minority* of criteria significance warrants that alternative  $x$  is *at least as well evaluated as* alternative  $y$ , and
  2. *No considerable positive performance difference* is observed on any criterion.
- Otherwise, the outranking situation between alternatives  $x$  and  $y$  is considered to be *indeterminate*.

The credibility of each pairwise outranking situation, denoted  $r(x \succsim y)$ , is measured in a bipolar significance valuation  $[-1.0, 1.0]$ , where *positive* terms  $r(x \succsim y) > 0.0$  indicate a *validated outranking*, and *negative* terms  $r(x \succsim y) < 0.0$  indicate an *invalidated outranking*, i.e. a *validated outranked* situation. The *median* value  $r(x \succsim y) = 0.0$  represents an *indeterminate* situation (see Bisдорff, 2004 and Bisдорff, 2013).

For computing such a bipolar-valued binary outranking relation from the given performance tableau `pt`, we use the `BipolarOutrankingDigraph` class from the `outrankingDigraphs` module. The corresponding `showHTMLRelationTable()` method shows here the resulting bipolar-valued adjacency matrix in a system browser window (see Fig. 4.2).

**Listing 4.3** Computing a bipolar-valued outranking digraph

```
1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> bod = BipolarOutrankingDigraph(pt)
3 >>> bod.showHTMLRelationTable()
```

**Fig. 4.2** Bipolar-valued adjacency matrix. In the resulting outranking relation we may notice, on the one hand, that Alternative D is *positively outranking* all other potential office locations. On the other hand, alternatives A (the most expensive) and C (the cheapest) are *not outranked* by any other location

## Valued Adjacency Matrix

r(x S y)	A	B	C	D	E	F	G
A	-	0.00	1.00	0.30	0.78	0.00	0.00
B	0.00	-	0.00	-0.56	0.00	1.00	-0.60
C	0.00	0.00	-	0.46	0.00	1.00	0.10
D	0.10	0.56	0.02	-	0.46	1.00	0.25
E	0.52	0.00	0.00	-0.10	-	1.00	-0.42
F	0.00	-1.00	-1.00	-1.00	-1.00	-	-1.00
G	0.00	0.92	-0.10	1.00	0.54	1.00	-

Valuation domain: [-1.00; +1.00]

Alternative D gives a CONDORCET winner<sup>3</sup>, whereas alternatives A (the most expensive) and C (the cheapest) give in fact *weak* CONDORCET winners.

```
1 >>> bod.computeCondorcetWinners()
2 ['D']
3 >>> bod.computeWeakCondorcetWinners()
4 ['A', 'C', 'D']
```

From theory, we know that outranking digraphs are *weakly complete*, i.e. for all  $x$  and  $y$  in  $X$ ,  $r(x \succsim y) < 0.0$  implies that  $r(y \succsim x) \geq 0.0$ . And, they verify the *coduality principle*:  $r(x \not\succsim y) = r(y \not\succsim x)$  (Bisдорff, 2013).<sup>4</sup>

<sup>3</sup> See Chapter 7 on computing the winner of an election.

<sup>4</sup> Not to be confused with the dual graph of a plane graph  $g$  that has a vertex for each face of  $g$ . Here we mean the *less than* (strict converse) relation corresponding to a *greater or equal* relation, or the *less than or equal* relation corresponding to a (strict) *better than* relation.

**Definition 4.2 (Strict outranking situation).**

For two potential decision alternatives  $x$  and  $y$ , “ $x$  strictly outranks alternative  $y$ ”, denoted  $(x \succ y)$ , when “ $x$  outranks alternative  $y$ ” ( $(x \succsim y) > 0.0$ ) and “ $y$  does not outrank alternative  $x$ ” ( $(y \succsim x) < 0.0$ ).

Following from the coduality principle, the strict outranking digraph `bodcd` is build from the outranking digraph `bod` with a codual transform (see Sec. 2.6).

```

1 >>> bodcd = ~(-bod) # codual tranform
2 >>> bodcd
3 *----- Object instance description -----*
4 Instance class      : BipolarOutrankingDigraph
5 Instance name       : converse-dual-rel_officeChoice
6 Actions             : 7
7 Criteria            : 7
8 Size                : 10
9 Determinateness (%) : 72.38
10 Valuation domain   : [-1.00;1.00]
```

## 4.4 Designing a best choice recommender system

Solving a best-choice problem consists traditionally in finding *the* unique best decision alternative. We adopt here instead a modern recommender system’s approach which shows a non empty subset of decision alternatives which contains by construction the potential best alternative(s).

The five *pragmatic principles* for computing such a *best-choice recommendation* (BCR) are the following:

- P1: *Elimination for well motivated reasons*; each eliminated alternative has to be strictly outranked by at least one alternative in the BCR.
- P2: *Minimal size*; the BCR must be as limited in cardinality as possible.
- P3: *Efficient and informative*; The BCR must not contain a self-contained sub-recommendation.
- P4: *Effectively better*; the BCR must not be ambiguous in the sense that it may not be both a first choice as well as a last choice recommendation.
- P5: *Maximally determined*; the BCR is, of all potential best-choice recommendations, the most determined one in the sense of the epistemic characteristics of the bipolar-valued outranking relation.

Let  $X$  be the set of potential decision alternatives. Let  $Y$  be a non empty subset of  $X$ , called a *choice* in the strict outranking digraph  $G(X, r(\succ))$ . We can now qualify a BCR  $Y$  in following terms:

- $Y$  is called strictly *outranking* (resp. *outranked*) when for all not selected alternative  $x$  there exists an alternative  $y \in Y$  retained such that  $r(y \succ x) > 0.0$  (resp.  $r(y \prec x) > 0.0$ ). Such a choice verifies principle P1.

- $Y$  is called *weakly independent* when for all  $x \neq y$  in  $Y$  we observe  $r(x \succsim y) \leq 0.0$ . Such a choice verifies principles P3 (*internal stability*).
- $Y$  is conjointly a strictly *outranking* (resp. *outranked*) **and** *weakly independent* choice. Such a choice is called an *initial* (resp. *terminal*) *prekernel*<sup>5</sup>. The initial prekernel now verifies principles P1, P2, P3 and P4.
- To finally verify principle P5, we recommend among all potential initial prekernels, a *most determined* one, i.e. a strictly *outranking* and *weakly independent* choice supported by the highest criteria significance. And in this most determined initial prekernel we eventually retain the alternative(s) that are included with highest criteria significance<sup>6</sup>.

Mind that a given strict outranking digraph may not always admit prekernels. This is the case when the digraph contains chordless circuits of odd length (see Chap. 17). Luckily, our strict outranking digraph `bodcd` here does not show any chordless outranking circuits; a fact we can check with the `computeChordlessCircuits()` method followed by the `showChordlessCircuits()` method<sup>7</sup>.

```

1 >>> bodcd.computeChordlessCircuits()
2 []
3 >>> bodcd.showChordlessCircuits()
4 No circuits observed in this digraph.

```

When observing chordless odd outranking circuits, we need to break them open with the `BrokenCocsDigraph` class at their weakest link, before enumerating the prekernels. (Bisdorff, 2021).

We are ready now for building a best best choice recommendation.

## 4.5 Computing the RUBIS best choice recommendation

The `showBestChoiceRecommendation()` method computes the RUBIS best choice recommendation directly from the outranking digraph `bod`. By default this method is operating on the *codual* (strict) outranking digraph where chordless odd circuits have been bopen up (see the `CoDual` and `BrokenCocs` parameters in List. 4.4 Line 2):

**Listing 4.4** Computing the best choice recommendation

```

1 >>> bod.showBestChoiceRecommendation(\_
2 ...     CoDual=True, BrokenCocs=True # default
3 ...     settings\_
4 ...         ChoiceVector = True)
4 * --- First and last choice recommendation(s) ---

```

<sup>5</sup> See Chap. 17 on computing kernels in digraphs

<sup>6</sup> See Sec. 17.6

<sup>7</sup> The `computeChordlessCircuits()` and `showChordlessCircuits()` methods are separate because there are various methods available for enumerating the chordless circuits in a digraph (Bisdorff, 2010).

```

5   (in decreasing order of determinateness)
6   Credibility domain: [-1.00,1.00]
7   === >> potential first choice(s)
8   * choice           : ['A', 'C', 'D']
9   independence      : 0.00
10  dominance         : 0.10
11  absorbency        : 0.00
12  covering (%)     : 41.67
13  determinateness (%) : 50.59
14  - characteristic vector = {
15    'D': +0.02, 'G': 0.00, 'C': 0.00, 'A': 0.00,
16    'F': -0.02, 'E': -0.02, 'B': -0.02, }
17  === >> potential last choice(s)
18  * choice           : ['A', 'F']
19  independence      : 0.00
20  dominance         : -0.52
21  absorbency        : 1.00
22  covered (%)       : 50.00
23  determinateness (%) : 50.00
24  - characteristic vector = {
25    'G': 0.00, 'F': 0.00, 'E': 0.00, 'D': 0.00,
26    'C': 0.00, 'B': 0.00, 'A': 0.00, }
```

It is interesting to notice in Line 8 above that the RUBIS *first choice recommendation* consists actually in the previously mentioned set of weak CONDORCET winners: A, C and D (see Fig. 4.2 on page 54). In the corresponding prekernel characteristic vector (see Lines 3, 15 and Sec. 17.6), representing the bipolar credibility degree with which each alternative may indeed be included in, or excluded from this recommendation, we find that alternative D is the only positively validated one, whereas both extreme alternatives –A (the most expensive) and C (the cheapest)—stay in an indeterminate situation (see Bisdorff et al., 2006; Bisdorff, 2006). They may *be or not be* potential best choices. Notice furthermore that compromise alternative G, while not actually being included in this strictly outranking prekernel, shows as well an indeterminate situation with respect to *being or not being* recommended as potential best choice. Alternatives B, E and F are all negatively included, i.e. positively excluded, from this best choice recommendation (see Line 16).

To inspect why alternative D is the only positive best choice recommendation, we shall compare now the evaluations of alternatives D and G in a pairwise perspective.

**Listing 4.5** Inspecting pairwise comparison between alternatives G and D

```

1 >>> bod.showPairwiseComparison('G','D')
2 *----- pairwise comparison -----
3 Comparing actions : ('G', 'D')
4 crit. wght.    g(x)    g(y)    diff. |    ind.    pref.    concord.
5 =====
6 Costs 45.00 -12000.00 -14100.00 +2100.00 | 1000.00 2500.00 +45.00
7 Comf.  6.00    50.00   30.00    +20.00 |   10.00  20.00 +6.00
8 Park.  3.00    80.00   90.00   -10.00 |   10.00  20.00 +3.00
9 Prox. 32.00   60.00   70.00   -10.00 |   10.00  20.00 +32.00
10 Stdg. 23.00   20.00   30.00   -10.00 |   10.00  20.00 +23.00
11 Visi. 26.00   100.00  50.00    +50.00 |   10.00  20.00 +26.00
12 Spac. 10.00   50.00   55.00   -5.00 |   10.00  20.00 +10.00
13 =====
14 Valuation in range: -145.00 to +145.00; global concordance: +145.00
```

In Listing 4.5 on the previous page, we notice that, with the given preference discrimination thresholds, alternative G is actually *certainly at least as well evaluated as* alternative D:  $r(G \succsim D) = +145/145 = +1.0$ .

Yet, we must as well acknowledge in Listing 4.6, that the cheapest alternative C is in fact *strictly outranking* alternative G:  $r(C \succsim G) = +15/145 > 0.0$ , and  $r(G \succsim C) = -15/145 < 0.0$ .

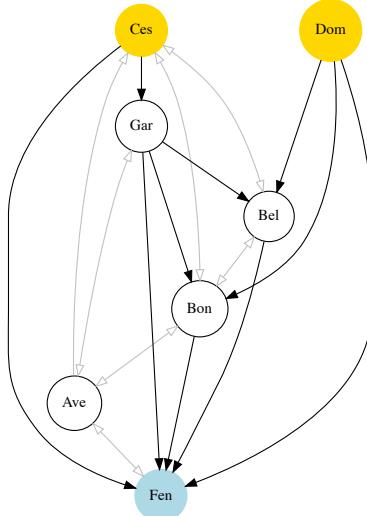
**Listing 4.6** Inspecting pairwise comparison between alternatives C and G

```

1 >>> bod.showPairwiseComparison('C','G')
2 *----- pairwise comparison -----
3 Comparing actions : (C,G) / (G,C)
4 crit. wght. g(x) g(y) diff. | ind. pref. (C, G) / (G, C)
5 =====
6 'C' 45.00 -6700.00 -12000.00 +5300.00 | 1000.00 2500.00 +45.00/-45.00
7 'Cf' 6.00 10.00 50.00 -40.00 | 10.00 20.00 -6.00/+6.00
8 'P' 3.00 100.00 80.00 +20.00 | 10.00 20.00 +3.00/-3.00
9 'Pr' 32.00 80.00 60.00 +20.00 | 10.00 20.00 +32.00/-32.00
10 'St' 23.00 0.00 20.00 -20.00 | 10.00 20.00 -23.00/+23.00
11 'V' 26.00 70.00 100.00 -30.00 | 10.00 20.00 -26.00/+26.00
12 'W' 10.00 0.00 50.00 -50.00 | 10.00 20.00 -10.00/+10.00
13
14 Valuation in range: -145 to +145; r(C >= G)/r(G >= c): +15.00/-15.00

```

**Fig. 4.3** Best office choice recommendation from strict outranking digraph. Notice that location A (Ave) (the most expensive) is appearing *incomparable* to all the other alternatives



Digraph3 (graphviz), R. Bischoff, 2020

Let us finally notice in Listing 4.4 on page 56 Line 18 that both alternatives A and F are reported as potential last choice recommendation. Yet, this last choice recommendation appears to be globally indeterminate (Lines 25-26). This confirms the *incomparability* status of alternative A (see Fig. 4.3).

```

1 >>> bodcd.exportGraphViz(fileName='bestOfficeChoice', \
2 ...                           firstChoice=['C','D'], \
3 ...                           lastChoice=['F']) \
4     *---- exporting a dot file for GraphViz tools -----*
5     Exporting to bestOfficeChoice.dot
6     dot -Grankdir=BT -Tpng bestOfficeChoice.dot \
7           -o bestOfficeChoice.png

```

## 4.6 Weakly ordering the outranking digraph

To get a global insight in the overall strict outranking situations, we may use the `RankingByChoosingDigraph` class imported from the `transitive-Digraphs` module for computing a *ranking-by-choosing* result from the codual, i.e. the strict outranking digraph instance `bodcd` (see above). If the computing node supports multiple processor cores, *first* and *last* choosing iterations may be run in parallel (see Line 4 in List. 4.7).

**Listing 4.7** Ranking-by-choosing the outranking digraph

```

1 >>> from transitiveDigraphs import \
2 ...                           RankingByChoosingDigraph
3 >>> rbc = RankingByChoosingDigraph(bodcd)
4 Threading ... # multiprocessing if 2 cores are available
5 Exiting computing threads
6 >>> rbc.showRankingByChoosing()
7 Ranking by Choosing and Rejecting
8     1st ranked ['D']
9     2nd ranked ['C', 'G']
10    2nd last ranked ['B', 'C', 'E']
11    1st last ranked ['A', 'F']
12 >>> rbc.exportGraphViz('officeChoiceRanking')
13     *---- exporting a dot file for GraphViz tools -----*
14     Exporting to officeChoiceRanking.dot
15     dot -Grankdir=TB -Tpng officeChoiceRanking.dot \
16           -o officeChoiceRanking.png

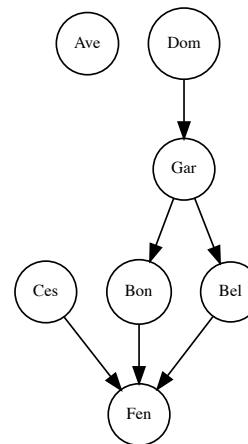
```

The best choice recommendation hence depends on the very importance the CEO is attaching to each of his decision objectives. In the given setting here, where he considers that *maximising the future turnover* is the most important objective followed by *minimising the Costs* and, less important, *maximising the working conditions*, location D represents actually the *best compromise*. However, if *Costs* do not play much a role, it would be perhaps better to decide to move to the most advantageous location A; or if, on the contrary, *Costs* do matter a lot, moving to the cheapest alternative C could definitely represent a more convincing recommendation.

It might be worth editing the criteria significance weights in the `officeChoice.py` data file in such a way that:

- All three decision objectives are considered *equally important*, and
- All criteria under each objective are considered *equi-significant*.

**Fig. 4.4** *Ranking-by-choosing the potential office locations.* In this *ranking-by-choosing* method, where we operate the *epistemic fusion* of iterated (strict) first and last choices, compromise alternative Dom is now ranked before compromise alternative Gar. The overall partial ordering result shows again the important fact that the most expensive location Ave, and the cheapest location Ces, due to their contradictory performances, appear both *incomparable* with most of the other alternatives.



Digraph3 (graphviz)  
R. Bisdorff, 2020

What will become the best choice recommendation under this working hypothesis?<sup>8</sup>

In the next Chapter 5 we precisely show how to edit a new performance tableau from a given template file.

## Notes

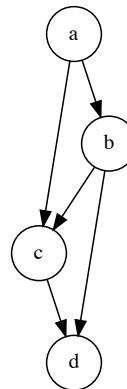
Following a seminar presentation in 2005 at the LAMSADE<sup>9</sup>, where the author promoted the use of kernels of the outranking digraph as suitable candidates for delivering best choice recommendations (Bisdorff, 2005), a critical discussion started about the methodological requirement for a convincing best choice recommendation to be internally stable (pragmatic principle P3). Denis Bouyssou illustrated his doubts with the potential outranking digraph shown in Figure 4.5 on the facing page.

His commentary was the following: Adding alternative  $d$  to the set of potential best choice candidates is not convincing as there exists in the given digraph the node

<sup>8</sup> See also the notes of Lecture 7 from the MICS Algorithmic Decision Theory course (Bisdorff, 2020).

<sup>9</sup> Laboratoires d'Analyse et de Modélisation de Systèmes d'Aide à la Décision, Université Paris-Dauphine, UMR 7243 CNRS

**Fig. 4.5** *The internal stability of a best choice recommendation in question.* The only kernel of this digraph is the pair {a, d}; yet, it is an ambiguous recommendation, as {a, d} is conjointly an outranking and outranked choice. If the instability of the best choice recommendation is, however, not considered a problem then the choice {a, b} shows the most convincing strict outranking quality and could be considered in priority for recommendation as potential best choice candidates



Digraph3 (graphviz), R. Bisdorff, 2020

b, which is better evaluated than d. The argument that the incomparability between a and d should favour d as potential best choice is interesting but another hypothesis could be that b perhaps outranks a. In this latter case, it seems clear that the actual best choice recommendation should be reduced to node b, unless one disposes of other information, like a performance tableau and/or the actual computation method of the outranking situations. In any case, one has to be very clear about the available information when judging a best choice procedure.

It became thereafter obvious for us all that both the lack of a specific performance tableau as well as the lack of a precisely defined algorithm for computing valid outranking situations do not allow to judge if a given digraph does indeed model a potential outranking relation. In our present bipolar-valued epistemic approach, a valid outranking digraph instance, following from a given performance tableau and the disjunctive epistemic fusion construction of the outranking relation (see Chap. 3), will necessarily verify the weak completeness condition and the coduality principle. As a consequence, incomparability situations are now modelled by epistemic indeterminateness not by the actual absence of a reciprocal outranking relation.

The digraph put forward by D. Bouyssou in the October 2005 discussion is not weakly complete –node a is not outranking node d and vice versa– and does hence not represent, in our present sense, a valid outranking digraph instance. Yet, it may be a partial tournament and as such it could be a strict outranking digraph, i.e. the asymmetric part –the codual– of a valid outranking digraph. In this case, nodes a and d –the kernel of the strict outranking digraph– would actually for sure outrank each other and, hence, represent both indifferently the natural best choice recommendation. However, in this not strict codual digraph, node a becomes also the

unique CONDORCET winner –outranking for sure all other nodes– and gives hence the evident unique best choice recommendation.

Only after 2013, when the weak completeness and the coduality properties of the outranking digraph were discovered, became it obvious that the initial prekernels of the strict outranking digraph, coupled with the solution of the corresponding kernel equation system, were in fact delivering the most convincing best choice recommendations (see Chap. 17, Sec. 20.4 and Bisdorff, 2013). It stays an interesting open mathematical problem to show (or not) that both necessary conditions: –weak completeness and coduality– are also sufficient for qualifying any bipolar-valued digraph as potential instance of an outranking digraph.

## References

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# Chapter 5

## How to create a new multiple-criteria performance tableau

**Abstract** The chapter illustrates a way of creating a new `PerformanceTableau` instance by editing a given template with 5 decision alternatives, 3 decision objectives and 6 performance criteria. We discuss in detail editing the decision alternatives, the decision objectives, the family of performance criteria, and finally, the evaluations of the decision alternatives on the performance criteria.

### 5.1 Editing a template file

For easing the editing of a new multiple-criteria performance tableau, we provide the following `perfTab_Template.py` file in the `examples` directory of the DIGRAPH3 resources.

**Listing 5.1** `PerformanceTableau` object template

```
1 ######
2 # Digraph3 documentation
3 # Template for creating a new PerformanceTableau instance
4 # (C) R. Bisдорff Mar 2021
5 # Digraph3/examples/perfTab_Template.py
6 #####
7 from decimal import Decimal
8 from collections import OrderedDict
9 #####
10 # edit the decision actions
11 # avoid special characters, like '_', '/' or ':',
12 # in action identifiers and short names
13 actions = OrderedDict([
14     ('a1', {
15         'shortName': 'action1',
16         'name': 'decision alternative a1',
17         'comment': 'some specific features of this alternative',
18     }),
19     ...
20     ...
21 ])
22 #####
23 # edit the decision objectives
24 # adjust the list of performance criteria
25 # and the total weight (sum of the criteria weights)
```

```

26     # per objective
27     objectives = OrderedDict([
28         ('obj1', {
29             'name': 'decision objective obj1',
30             'comment': "some specific features of this objective",
31             'criteria': ['g1', 'g2'],
32             'weight': Decimal('6'),
33         }),
34         ...
35         ...
36     })
37 #####
38     # edit the performance criteria
39     # adjust the objective reference
40     # Left Decimal of a threshold = constant part and
41     # right Decimal = proportional part of the threshold
42     criteria = OrderedDict([
43         ('g1', {
44             'shortName': 'crit1',
45             'name': "performance criteria 1",
46             'objective': 'obj1',
47             'preferenceDirection': 'max',
48             'comment': 'measurement scale type and unit',
49             'scale': (Decimal('0.0'), Decimal('100.0')),
50             'thresholds': {'ind': (Decimal('2.50'), Decimal('0.0')),
51                             'pref': (Decimal('5.00'), Decimal('0.0')),
52                             'veto': (Decimal('60.00'), Decimal('0.0'))},
53             },
54             'weight': Decimal('3'),
55         )),
56         ...
57         ...
58     })
59 #####
60     # default missing data symbol = -999
61     NA = Decimal('-999')
62 #####
63     # edit the performance evaluations
64     # criteria to be minimized take negative evaluations
65     evaluation = {
66         'g1': {
67             'a1': Decimal("41.0"),
68             'a2': Decimal("100.0"),
69             'a3': Decimal("63.0"),
70             'a4': Decimal("23.0"),
71             'a5': NA,
72         },
73         # g2 is of ordinal type and scale 0-10
74         'g2': {
75             'a1': Decimal("4"),
76             'a2': Decimal("10"),
77             'a3': Decimal("6"),
78             'a4': Decimal('2'),
79             'a5': Decimal('9'),
80         },
81         # g3 has preferenceDirection = 'min'
82         'g3': {
83             'a1': Decimal("-52.2"),
84             'a2': NA,
85             'a3': Decimal("-47.3"),
86             'a4': Decimal('-35.7'),
87             'a5': Decimal('-68.00'),
88         },
89         ...
90         ...
91     }
92 #####

```

The template file, shown in Listing 5.1 on page 65, contains first the instructions to import the required standard Python `Decimal` and `OrderedDict` classes (see Lines 7-8). Four main sections are following: the potential decision *actions*, the decision *objectives*, the performance *criteria*, and finally the performance *evaluations*.

## 5.2 Editing the decision alternatives

Decision alternatives are stored in attribute `actions` under the `OrderedDict` format. The `OrderedDict` object keeps this initial order when iterating over the decision alternatives<sup>1</sup>.

Required attributes of each decision alternative, besides the object identifier<sup>2</sup>, are: `shortName`, `name` and `comment` (see Lines 15-17 in List. 5.1 on page 65). The `shortName` attribute is essentially used when showing the performance tableau or the performance heatmap in a browser view.

The random performance tableau models, introduced in Chapter 6, use the `actions` attribute for storing special features of the decision alternatives. The *Cost-Benefit* model, for instance, uses a `type` attribute for distinguishing between *advantageous*, *neutral* and *cheap* alternatives (see Sec. 6.3). The *3-Objectives* model keeps a detailed record of the performance profile per decision objective and the corresponding random generators per performance criteria as shown in Listing 5.2.

**Listing 5.2** Example of decision alternative description

```

1 >>> from randomPerfTabs import \
2 ...                 Random3ObjectivesPerformanceTableau
3 >>> t = Random3ObjectivesPerformanceTableau()
4 >>> t.actions
5 OrderedDict([
6     ('p01', {'shortName': 'p01',
7             'name': 'action p01 Eco~ Soc- Env+',
8             'comment': 'random public policy',
9             'Eco': 'fair',
10            'Soc': 'weak',
11            'Env': 'good',
12            'profile': {'Eco':'fair',
13                      'Soc':'weak',
14                      'Env':'good'}
15            'generators': {'ec01': ('triangular', 50.0, 0.5),
16                           'so02': ('triangular', 30.0, 0.5),
17                           'en03': ('triangular', 70.0, 0.5),
18                           ...
19            },
20        })

```

---

<sup>1</sup> See the `OrderedDict` description in the Python documentation (Python Software Foundation, 2021).

<sup>2</sup> Mind that *graphviz* drawings require node identifier strings without any special characters like `'_'` or `'/'`.

```

21     ) ,
22     ...
23 )

```

The second section of the template file concerns the decision *objectives*.

### 5.3 Editing the decision objectives

The minimal required attributes of the ordered decision *objectives* dictionary, besides the individual objective identifiers, are `name`, `comment`, `criteria`: the list of significant performance criteria, and `weight`: the importance of the decision objective. The latter attribute contains the sum of the *significance* weights of the objective's criteria list (see Lines 27-?? in List [5.1 on page 65](#)).

The `objectives` attribute is methodologically useful for specifying the performance criteria significance in building decision recommendations. Mostly, we assume indeed that decision objectives are all equally important and the performance criteria are equi-significant per objective. This is, for instance, the default setting in the random 3-*Objectives* performance tableau model.

**Listing 5.3** Example of decision objectives' description

```

1 >>> # t = Random3ObjectivesPerformanceTableau()
2 >>> t.objectives
3 OrderedDict([
4     ('Eco',
5         {'name': 'Economical aspect',
6          'comment': 'Random3ObjectivesPerformanceTableau generated',
7          'criteria': ['ec01', 'ec06', 'ec09'],
8          'weight': Decimal('48')}),
9     ('Soc',
10        {'name': 'Societal aspect',
11         'comment': 'Random3ObjectivesPerformanceTableau generated',
12         'criteria': ['so02', 'so12'],
13         'weight': Decimal('48')}),
14     ('Env',
15        {'name': 'Environmental aspect',
16         'comment': 'Random3ObjectivesPerformanceTableau generated',
17         'criteria': ['en03', 'en04', 'en05', 'en07',
18                     'en08', 'en10', 'en11', 'en13'],
19         'weight': Decimal('48')})
20 ])

```

The importance weight sums up to 48 for each one of the three example decision objectives shown in Listing 5.3 so that the significance weight of each one of the 3 economic criteria is set to 16, of both societal criteria is set to 24, and of each one of the 6 environmental criteria is set to 8 (see Lines 8, 13 and 19).

Mind that the `objectives` attribute is always present in a `PerformanceTableau` object instance, even when empty. In this case, we consider that each performance criterion canonically represents its own decision objective. The criterion

significance weight equals in this case the corresponding decision objective's importance weight.

The third section of the template file concerns now the *performance criteria*.

## 5.4 Editing the family of performance criteria

In order to assess how well each potential decision alternative is satisfying a given decision objective, we need *performance criteria*, i.e. decimal-valued evaluation functions gathered in an ordered `criteria` dictionary. The required attributes (see List. 5.4), besides the criteria identifiers, are the usual `shortName`, `name` and `comment`. Specific for a criterion are furthermore the `objective` reference, the significance weight, the evaluation scale (minimum and maximum performance values), the `preferenceDirection` ('`max`' or '`min`') and the performance discrimination thresholds attributes.

**Listing 5.4** Example of performance criteria description

```

1  criteria = OrderedDict([
2      ('g1', {
3          'shortName': 'crit1',
4          'name': "performance criteria 1",
5          'comment': 'measurement scale type and unit',
6          'objective': 'obj1',
7          'weight': Decimal('3'),
8          'scale': (Decimal('0.0'), Decimal('100.0')),
9          'preferenceDirection': 'max',
10         'thresholds': {'ind': (Decimal('2.50'), Decimal('0.0')),
11                         'pref': (Decimal('5.00'), Decimal('0.0')),
12                         'veto': (Decimal('60.00'), Decimal('0.0'))
13                     },
14     }),
15     ...
16 ])

```

In our bipolar-valued outranking approach, all performance criteria implement *decimal-valued* evaluation functions, where preferences are either *increasing* or *decreasing* with measured performances. In order to model a *coherent* performance tableau, the family of decision criteria must satisfy two methodological requirements:

1. *Independence*: Each decision criterion implements an evaluation that is *functionally independent* of the evaluation of the other decision criteria, i.e. the performance evaluated on one of the criteria does not *constrain* in any sense the performance evaluated on any other criterion.
2. *Non redundancy*: Each performance criterion is only *significant* for a *single* decision objective.

For taking into account any, usually *unavoidable, imprecision* of the performance evaluation procedures, we may specify three performance *discrimination*

*thresholds*: an *indifference* (*ind*), a *preference* (*pref*) and a *considerable performance difference* (*veto*) threshold (see List. 5.4 on the previous page Lines 10-12). The left decimal number of a threshold description tuple indicates a *constant part*, whereas the right decimal number indicates a *proportional part*.

On the template performance criterion *g1*, shown in Listing 5.4 on the preceding page, we observe for instance an evaluation scale from 0.0 to 100.0 with a constant *indifference* threshold of 2.5, a constant *preference* threshold of 5.0, and a constant *considerable performance difference* threshold of 60.0. The latter threshold will trigger, the case given, a *polarisation* of the outranking statement (Bisdorff, 2013).

In a random *Cost-Benefit* performance tableau model we may obtain by default the following description of a cardinal *Costs* criterion:

**Listing 5.5** Example of cardinal *Costs* criterion

```

1 >>> from randomPerfTabs import \
2 ...           RandomCBPerformanceTableau
3 >>> tcb = RandomCBPerformanceTableau()
4 >>> tcb.showObjectives()
5 *----- decision objectives -----*
6 C: Costs
7   c1 random cardinal cost criterion 6
8   Total weight: 6.00 (1 criteria)
9 ...
10 ...
11 >>> tcb.criteria
12 OrderedDict([
13   ('c1', {'preferenceDirection': 'min',
14           'scaleType': 'cardinal',
15           'objective': 'C',
16           'shortName': 'c1',
17           'name': 'random cardinal cost criterion',
18           'scale': (0.0, 100.0),
19           'weight': Decimal('6'),
20           'randomMode': ['triangular', 50.0, 0.5],
21           'comment': 'Evaluation generator: triangular law ...',
22           'thresholds': OrderedDict([
23             ('ind', (Decimal('1.49'), Decimal('0'))),
24             ('pref', (Decimal('3.7'), Decimal('0'))),
25             ('veto', (Decimal('67.71'), Decimal('0'))),
26             [])),
27           }),
28 ...
29 ...
30 ])

```

Criterion *c1* appears here to be a cardinal criterion to be minimised and significant for the *Costs* (*C*) decision objective (see Line 13 in List. 5.5). The *showCriteria()* prints out the corresponding performance discrimination thresholds.

```

1 >>> tcb.showCriteria(IntegerWeights=True)
2 *----- criteria -----*
3   c1 'Costs/random cardinal cost criterion'

```

```

4     Preference direction: min
5     Scale = (0.0, 100.0)
6     Weight = 6
7     Threshold ind : 1.49 + 0.00x ; percentile: 5.13
8     Threshold pref : 3.70 + 0.00x ; percentile: 10.26
9     Threshold veto : 67.71 + 0.00x ; percentile: 96.15
10    ...
11    ...

```

The *indifference* threshold on this criterion amounts to a constant value of 1.49 (Line 6 above). More or less 5% of the observed performance differences on this criterion appear hence to be *insignificant*. Similarly, with a preference threshold of 3.70, about 90% of the observed performance differences are preferentially *significant* (Line 7). Furthermore,  $100.0 - 96.15 = 3.85\%$  of the observed performance differences appear to be *considerable* (Line 8) and will trigger, the case given, a *polarisation* of the corresponding outranking situations.

After the performance criteria descriptions, we are ready for recording the actual *performance evaluations*.

## 5.5 Editing the performance evaluations

The individual evaluations of each decision alternative on each decision criterion are recorded in a double *criterion*  $\times$  *action* dictionary called `evaluation` (see List 5.5). As we may encounter cases of missing data, we previously define a *missing data* symbol `NA` which is set to a value disjoint from all the measurement scales, by default `Decimal(' -999 ')` (see Line 2 in List 5.6).

**Listing 5.6** Editing performance evaluations

```

1 #-----
2 NA = Decimal(' -999 ')
3 #-----
4 evaluation = {
5     'g1': {
6         'a1': Decimal("41.0"),
7         'a2': Decimal("100.0"),
8         'a3': Decimal("63.0"),
9         'a4': Decimal('23.0'),
10        'a5': NA, # missing data
11    },
12    ...
13    ...
14    # g3 has preferenceDirection = 'min'
15    'g3': {
16        'a1': Decimal(" -52.2"), # negative grades
17        'a2': NA,
18        'a3': Decimal(" -47.3"),
19        'a4': Decimal(' -35.7 '),
20        'a5': Decimal(' -68.00 '),
21    },

```

```

22     ...
23     ...
24 }
```

Notice in Listing 5.6 on the previous page that on a criterion with preferenceDirection = 'min' all performance evaluations are recorded as *negative* values (Lines 16 and following).

We can now inspect below the eventually edited complete template performance tableau `perfTab_Template.py` with the `showPerformanceTableau()` method.

```

1 >>> from perfTabs import PerformanceTableau
2 >>> pt = PerformanceTableau('perfTab_Template')
3 >>> pt.showPerformanceTableau(ndigits=1)
4 *---- performance tableau ----*
5 Criteria | 'g1'   'g2'   'g3'   'g4'   'g5'   'g6'
6 Actions  |   3     3     6     2     2     2
7 -----|-----
8 'action1' | 41.0   4.0   -52.2  71.0   63.0   22.5
9 'action2' | 100.0  10.0   NA    89.0   30.7   75.0
10 'action3' | 63.0   6.0   -47.3  55.4   63.5   NA
11 'action4' | 23.0   2.0   -35.7  83.5   37.5   54.9
12 'action5' |  NA    9.0   -68.0  10.0   88.0   75.0
```

Computing below the associated outranking digraph `bod` allows checking the potential presence of any polarised outranking situations.

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> bod = BipolarOutrankingDigraph(pt)
3 >>> bod.showPolarisations()
4 *---- Negative polarisations ----*
5 number of negative polarisations : 1
6 1: r(a4 >= a2) = -0.44
7     criterion: g1
8     Considerable performance difference : -77.00
9     Veto discrimination threshold       : -60.00
10    Polarisation: r(a4 >= a2) = -0.44 ==> -1.00
11 *---- Positive polarisations ----*
12 number of positive polarisations : 1
13 1: r(a2 >= a4) = 0.56
14     criterion: g1
15     Considerable performance difference : +77.00
16     Counter-veto threshold           : 60.00
17     Polarisation: r(a2 >= a4) = 0.56 ==> +1.00
```

Indeed, due to the considerable positive performance difference (+77.00) observed on performance criterion `g1`, alternative `a2` “*for sure outranks*” alternative `a4`, respectively `a4` “*does for sure not outrank*” `a2`.

## 5.6 Inspecting the template outranking relation

In Listing 5.7, the `showRelationTable()` method prints out the outranking relation table.

**Listing 5.7** The template outranking relation

```

1 >>> bod.showRelationTable()
2 * ---- Relation Table ----
3   r | 'a1'   'a2'   'a3'   'a4'   'a5'
4   ---|-----
5 'a1' | +1.00 -0.44 -0.22 -0.11 +0.06
6 'a2' | +0.44 +1.00 +0.33 +1.00 +0.28
7 'a3' | +0.67 -0.33 +1.00 +0.00 +0.17
8 'a4' | +0.11 -1.00 +0.00 +1.00 +0.06
9 'a5' | -0.06 -0.06 -0.17 -0.06 +1.00

```

One may notice in this outranking relation table above that decision alternative  $a_2$  positively *outranks* all the other four alternatives (Line 6). Similarly, alternative  $a_5$  is positively *outranked* by all the other alternatives (see Line 9). We can orient this way the *graphviz* drawing of the template outranking digraph.

```

1 >>> bod.exportGraphViz(fileName= 'template', \
2 ...                           firstChoice =['a2'], \
3 ...                           lastChoice=['a5'])
4 *---- exporting a dot file for GraphViz tools ----*
5   Exporting to template.dot
6   dot -Grankdir=BT -Tpng template.dot -o template.png

```

In Figure 5.1 on the next page, alternatives *action3* ( $a_3$ ) and *action4* ( $a_4$ ) appear actually *incomparable*. In Listing 5.7, their pairwise outranking characteristics show indeed the *indeterminate* value 0.00 (Lines 7-8).

Checking their pairwise comparison may be done with the `showPairwiseComparison()` method:

```

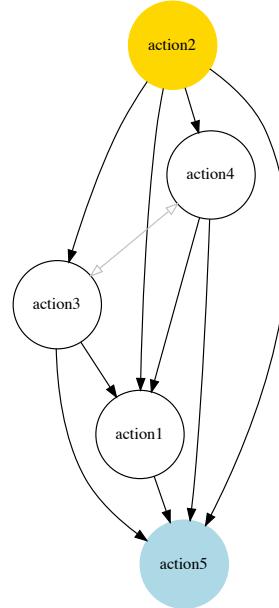
1 >>> bod.showPairwiseComparison('a3', 'a4')
2 *----- pairwise comparison ----*
3 Comparing actions : ('a3', 'a4')
4 crit. wght. g(a3) g(a4) diff | ind pref r() |
5 -----
6 'g1' 3.00 63.00 23.00 +40.00 | 2.50 5.00 +3.00 |
7 'g2' 3.00 6.00 2.00 +4.00 | 0.00 1.00 +3.00 |
8 'g3' 6.00 -47.30 -35.70 -11.60 | 0.00 10.00 -6.00 |
9 'g4' 2.00 55.40 83.50 -28.10 | 2.09 4.18 -2.00 |
10 'g5' 2.00 63.50 37.50 +26.00 | 0.00 10.00 +2.00 |
11 'g6' NA 54.90
12 Outranking characteristic value: r(a3 >= a4) = +0.00
13 Valuation in range: -18.00 to +18.00

```

The incomparability situation between  $a_3$  and  $a_4$  results in fact from a perfect balancing of positive (+8) and negative (-8) criteria significance weights.

The five decision alternatives can finally be ranked with a heatmap browser view following the COPELAND ranking rule (see Sec. 8.2), a rule which consistently reproduces the partial outranking order shown in Figure 5.1 on the next page.

**Fig. 5.1** The template outranking digraph. The digraph `bod` models in fact a *partial order* on the five potential decision alternatives



Digraph3 (graphviz), R. Bisdorff, 2020

```

1 >>> bod.showHTMLPerformanceHeatmap(ndigits=1,\n2 ...     colorLevels=5, Correlations=True,\n3 ...     rankingRule='Copeland',\n4 ...     pageTitle=\n5 ...         'Heatmap of the template performance tableau')

```

Due to an 8 against 7 *plurality tyranny* effect, the COPELAND ranking rule, essentially based on crisp majority outranking counts, puts here alternative *action5* (a5) last, despite its excellent evaluations observed on criteria g<sub>2</sub>, g<sub>5</sub> and g<sub>6</sub>.

A slightly *fairer* ranking result may be obtained in Figure 5.3 on the facing page with the NETFLOWS ranking rule (see Sec. 8.3).

```

1 >>> bod.showHTMLPerformanceHeatmap(ndigits=1,\n2 ...     colorLevels=5, Correlations=True,\n3 ...     rankingRule='NetFlows',\n4 ...     pageTitle=\n5 ...         'Heatmap of the template performance tableau')

```

It might be opportune to furthermore study the robustness of the apparent outranking situations when assuming *uncertain* or solely *ordinal* criteria significance weights (see Chap. 18 and Chap. 19).

### Heatmap of the template performance tableau

criteria	crit4	crit1	crit3	crit2	crit6	crit5
<b>weights</b>	+2.00	+3.00	+6.00	+3.00	+2.00	+2.00
<b>tau(*)</b>	+0.60	+0.40	+0.35	+0.20	+0.10	-0.60
<b>action2</b>	89.0	100.0	NA	10.0	75.0	30.7
<b>action3</b>	55.4	63.0	-47.3	6.0	NA	63.5
<b>action4</b>	83.5	23.0	-35.7	2.0	54.9	37.5
<b>action1</b>	71.0	41.0	-52.2	4.0	22.5	63.0
<b>action5</b>	10.0	NA	-68.0	9.0	75.0	88.0

Color legend:

<b>quantile</b>	20.00%	40.00%	60.00%	80.00%	100.00%
-----------------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: **standard**, Ranking rule: **Copeland**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+1.000**

Mean marginal correlation (a) : **+0.228**

Standard marginal correlation deviation (b) : **+0.322**

Ranking fairness (a) - (b) : **-0.094**

Fig. 5.2 COPELAND ranked heatmap of the template performance tableau

### Heatmap of the template performance tableau

criteria	crit2	crit6	crit1	crit4	crit3	crit5
<b>weights</b>	+3.00	+2.00	+3.00	+2.00	+6.00	+2.00
<b>tau(*)</b>	+0.60	+0.50	+0.40	+0.20	-0.05	-0.20
<b>action2</b>	10.0	75.0	100.0	89.0	NA	30.7
<b>action3</b>	6.0	NA	63.0	55.4	-47.3	63.5
<b>action5</b>	9.0	75.0	NA	10.0	-68.0	88.0
<b>action4</b>	2.0	54.9	23.0	83.5	-35.7	37.5
<b>action1</b>	4.0	22.5	41.0	71.0	-52.2	63.0

Color legend:

<b>quantile</b>	20.00%	40.00%	60.00%	80.00%	100.00%
-----------------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: **standard**, Ranking rule: **NetFlows**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.920**

Mean marginal correlation (a) : **+0.206**

Standard marginal correlation deviation (b) : **+0.286**

Ranking fairness (a) - (b) : **-0.081**

Fig. 5.3 NETFLOWS ranked heatmap of the template performance tableau

### References

- Bisdorff R (2013) On polarizing outranking relations with large performance differences. Journal of Multi-Criteria Decision Analysis, Wiley 20:3–12, URL <http://hdl.handle.net/10993/24570>

Python Software Foundation (2021) Python documentation. URL <https://docs.python.org/3/> 67

# Chapter 6

## Generating random performance tableaux

**Abstract** The chapter describes the DIGRAPH3 `randomPerfTabs` module for generating random multiple-criteria performance tableaux. The module proposes several useful random models, like a *Cost-Benefit* tableau, a three Objectives – *economic*, *societal* and *environmental*– tableau, and an *academic* performance tableau.

### 6.1 Introduction

The `randomPerfTabs` module provides several classes for generating random performance tableaux models of different kind, mainly for the purpose of testing implemented methods and tools presented and discussed in the Algorithmic Decision Theory course lectures at the University of Luxembourg. This chapter introduces the most useful models.

The simplest generic class, called `RandomPerformanceTableau`, generates a set of  $n$  decision actions, a family of  $m$  real-valued performance criteria, ranging by default from 0.0 to 100.0, associated with default discrimination thresholds: 2.5 (`ind`), 5.0 (`pref`) and 60.0 (`veto`). The generated random evaluations are by default Beta (2, 2) distributed on each measurement scale.

One of the most useful models, called `RandomCBPerformanceTableau`, proposes a performance tableau involving two decision objectives, named *Costs* (to be minimised) respectively *Benefits* (to be maximised); its purpose being to generate more or less contradictory performances on these two, usually conflicting, objectives. *Low costs* will randomly be correlated with *low benefits*, whereas *high costs* will randomly be correlated with *high benefits*.

Many public policy decision problems involve three often conflicting decision objectives taking into account *economical*, *societal* as well as *environmental* aspects. For this type of performance tableau model, we provide a specific class, called `Random3ObjectivesPerformanceTableau`.

Deciding which students, based on the grades obtained in a number of examinations, validate or not their academic studies, is the genuine decision practice of universities and academies. To thoroughly study these kind of decision problems, we provide a corresponding performance tableau model, called `RandomAcademicPerformanceTableau`, which gathers grades obtained by a given number of students in a given number of weighted courses.

## 6.2 Random standard performance tableaux

The `RandomPerformanceTableau` class, the simplest of the kind, specialises the generic `PerformanceTableau` class, and takes the following parameters:

- `numberOfActions` := number of decision actions.
- `numberOfCriteria` := number of performance criteria.
- `weightDistribution` :=
  - 'random' (default) | 'fixed' | 'equisignificant':
    - If 'random', weights are uniformly selected randomly from the given weight scale;
    - If 'fixed', the `weightScale` must provided a corresponding weights distribution;
    - If 'equi-significant', all criterion weights are put to unity.
- `weightScale` := `[Min, Max]` (default =`(1, numberOfCriteria)`).
- `IntegerWeights` := `True` (default) | `False` (normalised to proportions of 1.0).
- `commonScale` := `[a, b]`; common performance measuring scales (default = `[0.0, 100.0]`)
- `commonThresholds` := `[(q0, q1), (p0, p1), (v0, v1)]`; indifference (`q`), preference (`p`) and considerable performance difference (`v`) discrimination thresholds. For each threshold type  $x \in \{q, p, v\}$ , the float  $x_0$  value represents a *constant percentage* of the common scale and the float  $x_1$  value a *proportional value* of the actual performance measure. Default values are `[(2.5, 0.0), (5.0, 0.0), (60.0, 0, 0)]`.
- `commonMode` := common distribution of random performance measurements<sup>1</sup>:
  - ('beta', `None` (default setting),  $(\alpha, \beta)$ ), a beta generator with default  $\alpha = 2$  and  $\beta = 2$  parameters.
  - ('uniform', `None`, `None`), uniformly distributed float values on the given common scales' range `[Min, Max]`;
  - ('normal',  $\mu$ ,  $\sigma$ ), truncated Gaussian distribution, by default  $\mu = (b - a)/2$  and  $\sigma = (b - a)/4$ ;

---

<sup>1</sup> See Lecture 3 of the Computational Statistic Course ([Bisdorff, 2020](#)).

- (`'triangular'`, *mode*, *repartition*), generalised triangular distribution with a probability repartition parameter specifying the probability mass accumulated until the mode value. By default, *mode* =  $(b - a)/2$  and *repartition* = 0.5.<sup>2</sup>
- *valueDigits* := integer, precision of performance measurements (2 decimal digits by default).
- *missingDataProbability* :=  $0.0 \leq \text{float} \leq 1.0$ ; probability of missing performance evaluation on a criterion for an alternative (default 0.025).
- *NA* := `Decimal` (default = `-999`); missing data symbol.

**Code example:**

**Listing 6.1** Generating a random performance tableau

```

1 >>> from randomPerfTabs import RandomPerformanceTableau
2 >>> t = RandomPerformanceTableau(numberOfActions=21, \
3 ...                               numberOfCriteria=13, seed=100)
4 >>> t.actions
5   {'a01': {
6     'comment': 'RandomPerformanceTableau() generated.',
7     'name': 'random decision action'
8   },
9   'a02': { ... },
10  ...
11 }
12 >>> t.criteria
13   {'g01': {
14     'thresholds': {
15       'ind' : (Decimal('10.0'), Decimal('0.0')),
16       'veto': (Decimal('80.0'), Decimal('0.0')),
17       'pref': (Decimal('20.0'), Decimal('0.0'))},
18     'scale': [0.0, 100.0],
19     'weight': Decimal('1'),
20     'name': 'RandomPerformanceTableau() instance',
21     'comment': "Arguments: weightDistribution=random;
22                           weightScale=(1, 1);
23                           commonMode=None"
24   },
25   'g02': { ... },
26   ...
27 }
28 >>> t.NA
29   Decimal('-999')
30 >>> t.evaluation
31   {'g01': {'a01': Decimal('15.17'),
32             'a02': Decimal('44.51'),
33             'a03': Decimal('-999'), # missing evaluation
34             ... },
35   ...
36 }
```

---

<sup>2</sup> The `randomNumbers` module provides for this purpose the `ExtendedTriangularRandomVariable` class.

```
37 >>> t.showHTMLPerformanceTableau()
```

## Performance table randomperftab

criteria	g01	g02	g03	g04	g05	g06	g07	g08	g09	g10	g11	g12	g13
weight	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
a01	15.17	46.37	82.88	41.14	59.94	41.19	58.68	44.73	22.19	64.64	34.93	42.36	17.55
a02	44.51	16.22	41.66	53.58	31.39	65.22	71.96	57.84	78.08	77.37	8.30	63.41	61.55
a03	NA	21.53	12.82	56.93	26.80	48.03	54.35	62.42	94.27	73.57	71.11	21.81	56.90
a04	58.00	51.16	21.92	65.57	59.02	44.77	37.49	58.39	80.79	55.39	46.44	19.57	39.22
a05	24.22	77.01	75.74	83.87	40.85	8.55	85.44	67.34	57.40	39.08	64.83	29.37	96.39
a06	29.10	39.35	15.45	34.99	49.12	11.49	28.44	52.89	64.24	62.92	58.28	32.02	10.25
a07	96.58	32.06	6.05	49.56	NA	66.06	41.64	13.08	38.31	24.82	48.39	57.03	42.91
a08	82.29	47.67	9.96	79.43	29.45	84.17	31.99	90.88	39.58	50.78	61.88	44.40	48.26
a09	43.90	14.81	60.55	42.37	6.72	56.14	34.20	51.54	21.79	79.13	50.95	93.16	81.89
a10	38.75	79.70	27.88	42.39	71.88	66.09	58.33	58.88	17.10	44.25	48.73	30.63	52.73
a11	35.84	67.48	38.81	33.75	26.87	64.10	71.95	62.72	NA	85.80	58.37	49.33	NA
a12	29.12	13.97	67.45	38.60	48.30	11.87	NA	57.76	74.86	26.57	48.80	43.57	7.68
a13	34.79	90.72	38.93	57.38	64.14	97.86	91.16	43.80	33.68	38.98	28.87	63.36	60.03
a14	62.22	80.16	19.26	62.34	60.96	24.72	73.63	71.21	56.43	46.12	26.09	51.43	12.86
a15	44.23	69.62	94.95	34.95	63.46	52.97	98.84	78.74	36.64	65.12	22.46	55.52	68.79
a16	19.10	45.49	65.63	64.96	50.57	55.91	10.02	34.70	29.31	50.15	70.68	62.57	71.09
a17	27.73	22.03	48.00	79.38	23.35	74.03	58.74	59.42	50.95	82.27	49.20	43.27	38.61
a18	41.46	33.83	7.97	75.11	49.00	55.70	64.99	38.47	49.86	17.45	28.08	35.21	67.81
a19	22.41	NA	34.86	49.30	65.18	39.84	81.16	NA	55.99	66.55	55.38	43.08	29.72
a20	21.52	69.98	71.81	43.74	24.53	55.39	52.67	13.67	66.80	57.46	70.81	5.41	76.05
a21	56.90	48.80	31.66	15.31	40.57	58.14	70.19	67.23	61.10	31.04	60.72	22.39	70.38

Fig. 6.1 Browser view on random performance tableau instance

Best and worst evaluation on each criterion are marked in *light green*, respectively in *light red*. Notice that missing (NA) evaluations are recorded in a performance tableau by default as Decimal ('-999') value (see List. 6.1 on the previous page Lines 28-29).

### 6.3 Random Cost-Benefit performance tableaux

The `randomPerfTabs` module provides a `RandomCBPerformanceTableau` class for generating random *Costs* versus *Benefits* organised performance tableaux. The random generator is following the directives below:

- Three types of decision actions are distinguished: *cheap*, *neutral* and *expensive* ones with an equal proportion of 1/3. Two types of weighted criteria are

also distinguished: *Costs* criteria to be *minimised*, and *Benefits* criteria to be *maximised*, in the proportions 1/3 respectively 2/3;

- Random performances on each type of criteria are drawn, either from an ordinal scale [0; 10], or from a cardinal scale [0.0; 100.0], following a parametric triangular law of mode: 30% performance for cheap, 50% for neutral, and 70% performance for expensive decision actions, with constant probability repartition 0.5 on each side of the respective mode;
- Costs criteria use mostly cardinal scales (3/4), whereas Benefits criteria use mostly ordinal scales (2/3);
- The sum of weights of the Costs criteria equals by default the sum of weights of the Benefits criteria: `weighDistribution = 'equiobjectives'`;
- On cardinal criteria, both of cost or of benefit type, following constant performance discrimination quantiles are instantiated: 5% indifferent situations, 90% preference situations, and 5% considerable performance difference situations.

#### Parameters:

- If `numberOfActions == None`, a uniform random number between 10 and 31 of *cheap*, *neutral* or *advantageous* actions (equal 1/3 probability each type) actions is instantiated. Minimal number of decision actions required is 3;
- If `numberOfCriteria == None`, a uniform random number between 5 and 21 of cost or benefit criteria (1/3 respectively 2/3 probability) is instantiated;
- `weightDistribution := 'equisignificant'` (default) | 'equi-objectives' | 'fixed' | 'random';
- `default weightScale` for 'random' weight distribution is  $1 - \text{numberOfCriteria}$ ;
- All *cardinal* criteria are evaluated with decimals between 0.0 and 100.0 whereas *ordinal* criteria are evaluated with integers between 0 and 10.
- `commonThresholds` is obsolete. Preference discrimination is specified as *percentiles* of concerned performance differences (see below).
- `commonPercentiles := {'ind':5, 'pref':10, 'veto':95}` are expressed in percents (reversed for vetoes), and only concern cardinal criteria.
- `missingDataProbability := 0.0 ≤ float ≤ 1.0`; probability of missing performance evaluation on a criterion for an alternative (default 0.025).
- `NA := Decimal` (default = -999); missing data symbol.

#### Example Python session:

**Listing 6.2** Generating a random Cost-Benefit performance tableau

```

1 >>> from randomPerfTabs import \
2 ...                 RandomCBPerformanceTableau
3 >>> t = RandomCBPerformanceTableau(
4 ...     numberOfActions=7, \
5 ...     numberOfCriteria=5, \
6 ...     weightDistribution='equiobjectives', \

```

```

7     commonPercentiles={'ind':0.05,
8     ...
9             'pref':0.10,\ 
10            'veto':0.95},\
11    seed=100)
12  >>> t.showActions()
13  ----- show decision action -----
14  key: a1
15  short name: alc
16  name: random cheap decision action
17  key: a2
18  short name: a2n
19  name: random neutral decision action
20  ...
21  key: a7
22  short name: a7a
23  name: random advantageous decision action
24  >>> t.showCriteria()
25  ----- criteria -----
26  b1 'random ordinal benefit criterion'
27  Preference direction: max
28  Scale = (0, 10)
29  Weight = 3
30  ...
31  c1 'random cardinal cost criterion'
32  Preference direction: min
33  Scale = (0.0, 100.0)
34  Weight = 2
35  Threshold ind : 1.76 + 0.00x ; percentile: 9.5
36  Threshold pref : 2.16 + 0.00x ; percentile: 14.3
37  Threshold veto : 73.19 + 0.00x ; percentile: 95.2
38  ...
39 }
```

In Listing 6.2 on the preceding page one may notice the three types of decision actions (see Lines 12-22), as well as the two types (Lines 24-36) of criteria with either an *ordinal* or a *cardinal* performance measuring scale. In the latter case, by default about 5% of the random performance differences will be below the *indifference* and 10% below the *preference* discriminating threshold. About 5% will be *considerably large*. More statistics about the generated performance evaluations can be inspected with the `showStatistics()` method.

```

1  >>> t.showStatistics()
2  ----- Performance tableau summary statistics -----
3  Instance name      : randomCBperftab
4  Actions           : 7
5  Criteria          : 5
6  Criterion name   : b1
7  Criterion weight  : 3
8  criterion scale   : 0.00 - 10.00
9  mean evaluation   : 5.14
10 standard deviation: 2.64
11 maximal evaluation: 8.00
12 quantile Q3 (x_75): 8.00
13 median evaluation : 6.50
14 quantile Q1 (x_25): 3.50
```

```

15      minimal evaluation : 1.00
16      mean absolute difference : 2.94
17      standard difference deviation : 3.74
18      ...
19      Criterion name : c1
20      Criterion weight : 2
21      criterion scale : -100.00 - 0.00
22      mean evaluation : -49.32
23      standard deviation : 27.59
24      maximal evaluation : 0.00
25      quantile Q3 (x_75) : -27.51
26      median evaluation : -35.98
27      quantile Q1 (x_25) : -54.02
28      minimal evaluation : -91.87
29      mean absolute difference : 28.72
30      standard difference deviation : 39.02
31      ...

```

A heatmap view with 5 color levels gives the result shown in Figure 6.2.

```

1 >>> t.showHTMLPerformanceHeatmap(colorLevels=5,\n2 ...     rankingRule=None,\n3 ...     pageTitle='Random Cost-Benefit Performance Tableau')

```

## RandomCost-Benefit Performance Tableau

criteria	b1	b2	c1	c2	c3
weights	+3.00	+3.00	+2.00	+2.00	+2.00
a1c	2.00	20.66	-28.65	-79.26	-3.19
a2n	3.00	47.34	-15.47	-52.85	-44.47
a3n	3.00	57.44	-81.69	-77.77	-66.35
a4c	2.00	69.06	-77.23	-57.77	-34.80
a5a	7.00	35.23	-57.33	-39.65	-43.45
a6n	9.00	52.77	-39.33	-58.08	-71.82
a7a	5.00	92.47	-79.57	-34.87	-87.54

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

**Fig. 6.2** Unordered heatmap of a random Cost-Benefit performance tableau

Such a performance tableau may be stored and re-accessed as follows.

```

1 >>> t.save('temp')
2      ----- saving performance tableau in XMCDA 2.0 format
3      -----
3      File: temp.py saved !
4 >>> from perfTabs import PerformanceTableau
5 >>> t = PerformanceTableau('temp')

```

## 6.4 Random three objectives performance tableaux

The randomPerfTabs module provides a Random3ObjectivesPerformanceTableau class for generating random performance tableaux concerning public policies evaluated with respect to three decision objectives taking respectively into account *economical*, *societal* as well as *environmental* aspects. Each potential public policy is qualified randomly as performing *weak* (-), *fair* (~) or *good* (+) with respect to each one of the three objectives.

**Generator directives are the following:**

- `numberOfActions = 20` (default), minimal number required is 3;
- `numberOfCriteria = 13` (default),
- `weightDistribution = 'equiobjectives'` (default) | 'random' | 'equisignificant',
- `weightScale = (1,numberOfCriteria)`: only used when random criterion weights are requested,
- `integerWeights = True` (default): `False` gives normalised rational weights,
- `commonScale = (0.0,100.0)`,
- `commonThresholds = [(5.0,0.0), (10.0,0.0), (60.0,0.0)]`: Performance discrimination thresholds may be set for 'ind', 'pref' and 'veto' thresholds,
- `commonMode = ['triangular','variable',0.5]`: random number generators of various other types ('uniform','beta') are available. If the mode of the 'triangular' distribution is set to 'variable', three modes at 0.3(-), 0.5(~), respectively 0.7(+) of the common scale span are set at random for each coalition and action.
- `valueDigits = 2` (default): evaluations are encoded as decimals,
- `missingDataProbability = 0.05` (default): random insertion of missing values with given probability,
- `NA := Decimal (default = Decimal(' -999'))`; missing data symbol,
- `seed = None` (default).

**Example Python session:**

**Listing 6.3** Generating a random 3 Objectives performance tableau

```

1 >>> from randomPerfTabs import \
2 ...             Random3ObjectivesPerformanceTableau
3 >>> t = Random3ObjectivesPerformanceTableau(\n4 ...             numberOfActions=7,\n5 ...             numberOfCriteria=13,\n6 ...             weightDistribution='equiobjectives',\n7 ...             seed=120)\n8 >>> t.showObjectives()\n9     ----- show objectives -----"\n10     Eco: Economical aspect\n11     ec01 criterion of objective Eco 18\n12     ec05 criterion of objective Eco 18

```

```

13     ec06 criterion of objective Eco 18
14     ec12 criterion of objective Eco 18
15     Total weight: 72.00 (4 criteria)
16     Soc: Societal aspect
17         so02 criterion of objective Soc 24
18         so11 criterion of objective Soc 24
19         so13 criterion of objective Soc 24
20         Total weight: 72.00 (3 criteria)
21     Env: Environmental aspect
22         en03 criterion of objective Env 12
23         en04 criterion of objective Env 12
24         en07 criterion of objective Env 12
25         en08 criterion of objective Env 12
26         en09 criterion of objective Env 12
27         en10 criterion of objective Env 12
28     Total weight: 72.00 (6 criteria)

```

In Listing 6.3 on the facing page, we notice that four *equisignificant* criteria (ec01, ec05, ec06, and ec12) assess, for instance, the performance of the public policies from an *economic* point of view (Lines 10-15). Three *equisignificant* criteria do the same from a *societal* (Lines 16-20), and six from an *environmental* point of view (Lines 21-28). The 'equiobjectives' directive results hence in a balanced total weight (72.00) for each decision objective (see Line 6).

Variable *triangular* modes: 0.3, 0.5 or 0.7 of the span of the measure scale, give a different performance status for each public policy with respect to the three decision objectives.

```

1 >>> t.showActions()
2     key: p1
3         short name: p1
4         name:           action p1 Eco- Soc+ Env+
5         profile:      {'Eco':'weak', 'Soc':'good', 'Env':'good'}
6     key: p2
7         short name: p2
8         name:           action p2 Eco- Soc- Env-
9         profile:      {'Eco':'weak', 'Soc':'weak', 'Env':'weak'}
10    ...
11    key: p7
12        short name: p7
13        name:           action p7 Eco+ Soc- Env-
14        profile:      {'Eco':'good', 'Soc':'weak', 'Env':'weak'}

```

Policy p1, for instance, will probably show *good* performances with respect to the *societal* and *environmental* aspects, and *weak* performances with respect to the *economic* aspect, whereas policy p2, weak with respect to all the three objectives, will probably appear among the weakest policies.

We may inspect in Figure 6.3 on the next page the given random three-objectives performance tableau with the `showHTMLPerformanceTableau()` method.

```

1 >>> t.showHTMLPerformanceTableau()

```

Light green cells show the highest and light red the lowest evaluations. Policy p1 shows thus the weakest performance (17.09/100.00) on the environmental cri-

## Performance table random3ObjectivesPerfTab

criteria	ec01	ec05	ec06	ec12	en03	en04	en07	en08	en09	en10	so02	so11	so13
weight	18.00	18.00	18.00	18.00	12.00	12.00	12.00	12.00	12.00	12.00	24.00	24.00	24.00
p1	31.05	26.95	35.91	32.85	68.88	70.52	22.94	50.04	17.06	32.52	20.90	NA	62.37
p2	12.54	11.31	29.63	9.77	43.09	36.04	32.96	54.35	82.25	22.29	75.76	51.42	27.96
p3	30.70	58.45	38.31	24.89	72.36	87.02	30.57	80.17	90.38	86.60	54.51	43.21	6.58
p4	16.18	61.70	60.27	15.10	26.63	66.28	12.84	58.78	86.68	57.92	NA	71.91	70.18
p5	86.00	89.43	44.41	18.32	31.10	74.55	29.89	40.26	26.86	41.91	23.83	57.41	22.41
p6	77.65	94.24	24.75	65.64	3.85	36.15	47.36	81.22	72.54	80.54	27.25	37.04	34.11
p7	72.26	44.62	70.55	29.82	52.69	37.80	50.06	28.27	28.53	29.50	29.81	41.38	26.31

Fig. 6.3 Browser view on the given random three-objectives performance tableau

terion en09, whereas policy p3 shows the best performances on four out of the six *environmental* criteria.

No trivial best choice becomes apparent when looking at the performance tableau shown in Figure 6.3. Let us therefore compute a RUBIS best choice recommendation (see Chapter 4 on page 49).

**Listing 6.4** What is the public policy to recommend as best choice ?

```

1 >>> from outrankingDigraphs import \
2 ...           BipolarOutrankingDigraph
3 >>> g = BipolarOutrankingDigraph(t)
4 >>> g.showBestChoiceRecommendation()
5 ****
6 Rubis best choice recommendation(s) (BCR)
7 (in decreasing order of determinateness)
8 Credibility domain: [-1.00,1.00]
9 === >> potential first choice(s)
10 * choice          : ['p3', 'p4', 'p5', 'p6']
11 independence      : 0.00
12 dominance        : 0.17
13 absorbency       : -1.00
14 covering (%)     : 41.67
15 determinateness (%) : 52.98
16 - most credible action(s) = { 'p3': 0.17, }
17 === >> potential last choice(s)
18 * choice          : ['p1', 'p2', 'p5', 'p7']
19 independence      : 0.00
20 dominance        : -0.44
21 absorbency       : 0.19
22 covered (%)      : 41.67
23 determinateness (%) : 50.79
24 - most credible action(s) = { 'p7': 0.06, }
```

Policy p3 gives the most credible best choice recommendation with the support of a 57.5% majority of significance, whereas policy p7 gives a credible last choice recommendation . Policy p5 represents an ambiguous first and last choice candidate.

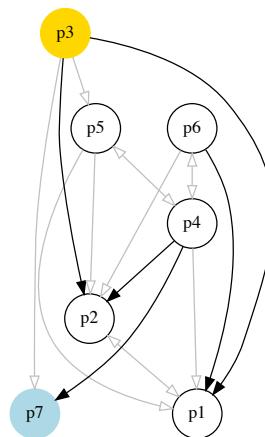
A drawing of the strict outranking digraph oriented by first and last choices shows in Figure 6.4 the complete preferential picture.

```

1 >>> (~(-g)).exportGraphViz(\n2 ...           fileName='3ObjPerfTabBestChoice',\n3 ...           firstChoice=['p3'],lastChoice=['p7'])\n4     *---- exporting a dot file for GraphViz tools -----*\n5     Exporting to 3ObjPerfTabBestChoice.dot\n6     dot -Grankdir=BT -Tpng 3ObjPerfTabBestChoice.dot\\n7             -o 3ObjPerfTabBestChoice.png

```

**Fig. 6.4** The strict outranking digraph oriented by first and last choices. Policy p5 is indeed incomparable –in a strict outranking sense– to all the other six policies. Policies p1, p2 and p7 appear strictly outranked



Digraph3 (graphviz), R. Bisдорff, 2020

A heatmap view on the COPELAND ranked performance tableau confirms the best choice recommendation (see Section 8.2 on page 110).

```

1 >>> t.showHTMLPerformanceHeatmap(Correlations=True, \
2 ...                           colorLevels=5,ndigits=1, \
3 ...                           rankingRule='Copeland')

```

The heatmap view with its COPELAND ranking based on the bipolar outranking digraph confirms policy's p3 as first-ranked. Notice also that the three strict outranked policies do effectively appear in the last positions.

## 6.5 Random academic performance tableaux

The RandomAcademicPerformanceTableau class generates performance tableaux with random grades for a given number of students in different courses.

### Heatmap of Performance Tableau 'random3ObjectivesPerfTab'

criteria	en10	ec05	en09	en04	en08	ec01	ec12	ec06	so11	so02	en07	so13	en03
<b>weights</b>	+12.00	+18.00	+12.00	+12.00	+12.00	+18.00	+18.00	+18.00	+24.00	+24.00	+12.00	+24.00	+12.00
<b>tau(*)</b>	+0.79	+0.52	+0.45	+0.40	+0.38	+0.14	+0.12	+0.10	+0.02	-0.02	-0.10	-0.10	-0.17
<b>p3</b>	86.6	58.5	90.4	87.0	80.2	30.7	24.9	38.3	43.2	54.5	30.6	6.6	72.4
<b>p4</b>	57.9	61.7	86.7	66.3	58.8	16.2	15.1	60.3	71.9	NA	12.8	70.2	26.6
<b>p6</b>	80.5	94.2	72.5	36.1	81.2	77.7	65.6	24.8	37.0	27.2	47.4	34.1	3.9
<b>p5</b>	41.9	89.4	26.9	74.5	40.3	86.0	18.3	44.4	57.4	23.8	29.9	22.4	31.1
<b>p7</b>	29.5	44.6	28.5	37.8	28.3	72.3	29.8	70.5	41.4	29.8	50.1	26.3	52.7
<b>p1</b>	32.5	26.9	17.1	70.5	50.0	31.1	32.9	35.9	NA	20.9	22.9	62.4	68.9
<b>p2</b>	22.3	11.3	82.2	36.0	54.4	12.5	9.8	29.6	51.4	75.8	33.0	28.0	43.1

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: *Ordinal (Kendall) correlation between marginal criterion and global ranking relation*

Outranking model: **standard**, Ranking rule: **Copeland**

*Ordinal (Kendall) correlation between global ranking and global outranking relation: +0.896*

Mean marginal correlation (a) : **+0.161**

Standard marginal correlation deviation (b) : **+0.258**

Ranking fairness (a) - (b) : **-0.098**

**Fig. 6.5** Browser view on the COPELAND ranked performance tableau

#### Generator directives:

- `numberOfStudents := Integer` (default 10)
- `numberOfCourses := Integer` (default 5)
- `weightDistribution := 'equisignificant' — 'random'` (default),
- `weightScale := 1, 1 - numberOfCourses` (default when random)),
- `IntegerWeights := Boolean` (`True = default`),
- `commonScale := (Integer,integer) (0,20)` (default),
- `ndigits := Integer` (default 0),
- `WithTypes := Boolean` (default False),
- `commonMode := ('triangular',xm=14,r=0.25)` (default),
- `commonThresholds := ' ind':(0,0), ' pref':(1,0)` (default),
- `missingDataProbability := 0.0` (default),
- `NA := Decimal` (default = -999); missing data symbol.

When Parameter `WithTypes` is set to `True`, the students are randomly allocated to one of the following four categories –weak (1/6), fair (1/3), good (1/3), and excellent (1/3)– in the bracketed proportions. In a default 0 – 20 grading range, the random range of a weak student is 0 – 10, of a fair student 4 – 16, of a good student 8 – 20, and of an excellent student 12 – 20. The random grading generator follows in this case a double triangular probability law with *mode* (*xm*) equal to the middle of the random range and median repartition (*r* = 0.5) of probability each side of the mode.

#### Listing 6.5 Generating a random academic performance tableau

```
1 >>> from randomPerfTabs import RandomAcademicPerformanceTableau
2 >>> t = RandomAcademicPerformanceTableau(\
```

```

3 ...           numberOfStudents=11,\ 
4 ...           numberOfCourses=7, missingDataProbability=0.03,\ 
5 ...           WithTypes=True, seed=100)
6 >>> t
7 *----- PerformanceTableau instance description -----*
8 Instance class   : RandomAcademicPerformanceTableau
9 Seed             : 100
10 Instance name   : randstudPerf
11 Actions          : 11
12 Criteria         : 7
13 NA proportion(%) : 5.2
14 Attributes       : ['randomSeed', 'name', 'actions',
15                   'criteria', 'evaluation', 'weightPreorder']
16 >>> t.showPerformanceTableau()
17 *---- performance tableau ----*
18 Courses | 'm1' 'm2' 'm3' 'm4' 'm5' 'm6' 'm7'
19 ECTS    | 2   1   3   4   1   1   5
20 -----
21 's01f' | 12  13  15  08  16  06  15
22 's02g' | 10  15  20  11  14  15  18
23 's03g' | 14  12  19  11  15  13  11
24 's04f' | 13  15  12  13  13  10  06
25 's05e' | 12  14  13  16  15  12  16
26 's06g' | 17  13  10  14  NA  15  13
27 's07e' | 12  12  12  18  NA  13  17
28 's08f' | 14  12  09  13  13  15  12
29 's09g' | 19  14  15  13  09  13  16
30 's10g' | 10  12  14  17  12  16  09
31 's11w' | 10  10  NA  10  10  NA  08
32 >>> t.weightPreorder
33 [ ['m2', 'm5', 'm6'], ['m1'], ['m3'], ['m4'], ['m7'] ]

```

The example random tableau, generated for instance above with `missingDataProbability = 0.03`, `WithTypes = True` and `seed = 100` (see [List 6.5 on the facing page](#) Lines 2-5), results in a set of two excellent (`s05e` and `s07e`), five good (`s02g`, `s03g`, `s06g`, `s09g` and `s10g`), three fair (`s01f`, `s04f` and `s08f`) and one weak (`s11w`) students. Notice that six students get a grade below the course validating threshold of 10 and we observe four missing grades (NA), two in course `m5` and, one in courses `m3` and `m6` (see Lines 21-31).

A statistical summary of the students' grades obtained in the highest weighted course, namely `m7`, is shown with the `showCourseStatistics()` method.

#### **Listing 6.6** Student performance summary statistics per course

```

1 >>> t.showCourseStatistics('m7')
2 *---- Summary performance statistics ----*
3 Course name   : m7
4 Course weight : 5
5 Students      : 11
6 Grading scale : 0.00 - 20.00
7 Missing evaluations : 0
8 Mean evaluation   : 12.82
9 Standard deviation : 3.79
10 Maximal evaluation : 18.00

```

```

11   Quantile Q3 (x_75)      : 16.25
12   Median evaluation       : 14.00
13   Quantile Q1 (x_25)      : 10.50
14   Minimal evaluation      : 6.00
15   Mean absolute difference : 4.30
16   Standard difference deviation : 5.35

```

In Listing 6.6 on the previous page, the Course m7 evaluation statistics show a mean grade of 12.82 and a median grade of 14. Maximal (resp. minimal) grade is 18 (resp. 6).

## Ranking the students

criteria	m7	m4	m2	m3	m1	m6	m5
<b>weights</b>	+5.00	+4.00	+1.00	+3.00	+2.00	+1.00	+1.00
<b>tau<sup>(*)</sup></b>	+0.73	+0.31	+0.29	+0.20	+0.11	+0.09	+0.07
<b>s07e</b>	17.00	18.00	12.00	12.00	12.00	13.00	NA
<b>s02g</b>	18.00	11.00	15.00	20.00	10.00	15.00	14.00
<b>s09g</b>	16.00	13.00	14.00	15.00	19.00	13.00	9.00
<b>s05e</b>	16.00	16.00	14.00	13.00	12.00	12.00	15.00
<b>s06g</b>	13.00	14.00	13.00	10.00	17.00	15.00	NA
<b>s03g</b>	11.00	11.00	12.00	19.00	14.00	13.00	15.00
<b>s10g</b>	9.00	17.00	12.00	14.00	10.00	16.00	12.00
<b>s01f</b>	15.00	8.00	13.00	15.00	12.00	6.00	16.00
<b>s08f</b>	12.00	13.00	12.00	9.00	14.00	15.00	13.00
<b>s04f</b>	6.00	13.00	15.00	12.00	13.00	10.00	13.00
<b>s11w</b>	8.00	10.00	10.00	NA	10.00	NA	10.00

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\* **tau**: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: **standard**, Ranking rule: **NetFlows**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.887**

**Fig. 6.6** Ranking the students in a performance heatmap view

With a ranked heatmap view on all the grades we now get in Figure 6.6 a global pictures of the performance of all the eleven students. The ranking shown here, produced with the NETFLOWS ranking rule (see Sec. 8.3), reports a high correlation of +0.887 with the corresponding bipolar-valued outranking digraph (see Chap. 16).

The NETFLOWS ranking represents also a rather *fair weighted consensus* between the individual courses' marginal rankings as is made apparent with the showRankingConsensusQuality() method in Listing 6.7.

**Listing 6.7** Consensus quality of the students's ranking

```

1 >>> from outrankingDigraphs import\
2     ...          BipolarOutrankingDigraph
3 >>> g = BipolarOutrankingDigraph(t)
4 >>> t.showRankingConsensusQuality(\n
5     ...          g.computeNetFlowsRanking())
6     Consensus quality of ranking:
7     ['s07', 's02', 's09', 's05', 's06', 's03', 's10',
8     's01', 's08', 's04', 's11']
9     Criterion (weight): correlation
10    -----
11     'm7' (0.294): +0.727
12     'm4' (0.235): +0.309
13     'm2' (0.059): +0.291
14     'm3' (0.176): +0.200
15     'm1' (0.118): +0.109
16     'm6' (0.059): +0.091
17     'm5' (0.059): +0.073
18     Summary:
19     Weighted mean marginal correlation (a): +0.361
20     Standard deviation (b)                 : +0.248
21     Ranking fairness (a)-(b)              : +0.113

```

The correlation with the marginal course rankings follows precisely the order of the given ECTS weights. Course m7, with the highest relative weight (0.294) is most present (+0.727) in the global NETFLOWS ranking and no marginal course ranking appears negatively correlated with this global ranking. The mean weighted correlation is +0.361.

Useful multiple-criteria ranking rules are presented and discussed in detail in Chapter 8. The next Chapter 7 is concerned with yet another kind of performance evaluation models which is discussed in social choice theory, namely *linear voting profiles*.

## References

Bisdorff R (2020) Lecture 3: Continuous Random Variables. In: Lectures of the Computational Statistics Course, University of Luxembourg, URL <http://hdl.handle.net/10993/37870> 78



# Chapter 7

## Who wins the election ?

**Abstract** This chapter is more specifically devoted to handling linear voting profiles and computing the winner of such election results. By following CONDORCET’s recipe, we consider pairwise comparisons of election candidates and balance the number of times the first beats the second against the number of times the second beats the first. Thus we obtain the majority margins digraph, in fact a bipolar-valued digraph. When the voters express contradictory linear voting profiles one naturally observes cyclic social preferences without seeing any paradox in this situation. Finally we present a more politically realistic generator for random linear voting profiles which takes into account pre-election polls.

### 7.1 Linear voting profiles

The `votingProfiles` module provides resources like the `LinearVotingProfile` class for handling election results (Bisdorff, 2020a). To illustrates these resources let us consider elections involving a set of eligible candidates and a set of weighted voters, who express their voting preferences in a complete linear ranking (without ties) of the eligible candidates. The module provides a `RandomLinearVotingProfile` class for generating random instances of such `LinearVotingProfile` type. In the interactive Python session shown in Listing 7.1, a random linear voting profile is generated, for instance, for the election of 3 candidates by 5 voters:

**Listing 7.1** Example of random linear voting profile

```
1 >>> from votingProfiles import \
2 ...                               RandomLinearVotingProfile
3 >>> lvp = RandomLinearVotingProfile(numberOfVoters=5, \
4 ...                                         numberOfCandidates=3, \
5 ...                                         RandomWeights=True)
6 >>> lvp.candidates
7   OrderedDict([ ('c1',{'name':'Candidate 1'}), \
8                 ('c2',{'name':'Candidate 2'})]),
```

```

9          ('c3',{'name':'Candidate 3'}) ])
10 >>> lvp.voters
11  OrderedDict([('v1',{'weight': 2}),
12                 ('v2',{'weight': 3}),
13                 ('v3',{'weight': 1}),
14                 ('v4',{'weight': 5}),
15                 ('v5',{'weight': 4}))])
16 >>> lvp.linearBallot
17  {'v1': ['c2', 'c1', 'c3'],
18   'v2': ['c3', 'c1', 'c2'],
19   'v3': ['c1', 'c3', 'c2'],
20   'v4': ['c1', 'c2', 'c3'],
21   'v5': ['c3', 'c1', 'c2']}

```

The `LinearVotingProfile` data concerning the eligible candidates and the voters is internally stored in two ordered dictionaries: attribute `candidates` (Line 6) and attribute `voters` (Line ??). Notice that in this random example, the five voters are weighted. The linear voting ballots are stored in a standard dictionary: attribute `linearBallot` (Line 16). These ballots can be inspected with the `showLinearBallots()` method.

**Listing 7.2** Showing linear voting profiles

```

1 >>> lvp.showLinearBallots()
2     voters(weight)      candidates rankings
3     v1(2):      ['c2', 'c1', 'c3']
4     v2(3):      ['c3', 'c1', 'c2']
5     v3(1):      ['c1', 'c3', 'c2']
6     v4(5):      ['c1', 'c2', 'c3']
7     v5(4):      ['c3', 'c1', 'c2']
8     nbr. of voters: 15

```

Editing of this linear voting profile may be done by saving first the data into a file, then edit this file, and reload it again.

```

1 >>> lvp.save(fileName='tutorialLinearVotingProfile1')
2     *** Saving linear profile in file:
3                     <tutorialLinearVotingProfile1.py> ***
4 >>> from votingProfiles import LinearVotingProfile
5 >>> lvp = LinearVotingProfile('tutorialLinearVotingProfile1')

```

## 7.2 Computing the winner

The `computeUninominalVotes()` and the `computeSimpleMajorityWinner()` methods compute *uninominal votes*, i.e. how many times a candidate was ranked first, and who is consequently the *simple majority* (plurality) winner(s) in this election.

```

1 >>> lvp.computeUninominalVotes()
2  {'c1': 6, 'c2': 2, 'c3': 7}
3 >>> lvp.computeSimpleMajorityWinner()

```

```
4   ['c3']
```

As we observe no absolute majority (8/15) of votes for any one of the three candidates, one may look, with the `computeInstantRunoffWinner()` method, for the *instant runoff* winner instead ([Bisdorff, 2020a](#)).

**Listing 7.3** Example Instant Run Off Winner

```
1 >>> lvp.computeInstantRunoffWinner(Comments=True)
2   Half of the Votes = 7.50
3   ==> stage = 1
4     remaining candidates ['c1', 'c2', 'c3']
5     uninominal votes {'c1': 6, 'c2': 2, 'c3': 7}
6     minimal number of votes = 2
7     maximal number of votes = 7
8     candidate to remove = c2
9     remaining candidates = ['c1', 'c3']
10  ==> stage = 2
11  remaining candidates ['c1', 'c3']
12  uninominal votes {'c1': 8, 'c3': 7}
13  minimal number of votes = 7
14  maximal number of votes = 8
15  candidate c1 obtains an absolute majority
16  Instant run off winner: ['c1']
```

In Listing 7.3 no candidate obtains at stage 1 an absolute majority of votes. Candidate `c2` obtains the minimal number of votes (2/15) and is, hence, eliminated. At stage 2, candidate `c1` eventually obtains an absolute majority of the votes (8/15) and is hence elected.

One may also follow the *Chevalier de Borda*'s advice and, after a *rank analysis* of the linear ballots, compute the **BORDA score** -the average rank- of each candidate and hence determine the **BORDA winner(s)** ([de Borda, 1781](#)).

**Listing 7.4** Example of BORDA rank scores

```
1 >>> lvp.computeRankAnalysis()
2 { 'c1': [6, 9, 0], 'c2': [2, 5, 8], 'c3': [7, 1, 7] }
3 >>> v.computeBordaScores()
4 OrderedDict([
5   ('c1', {'BordaScore': 24, 'averageBordaScore': 1.6}),
6   ('c3', {'BordaScore': 30, 'averageBordaScore': 2.0}),
7   ('c2', {'BordaScore': 36, 'averageBordaScore': 2.4}) ])
8 >>> lvp.computeBordaWinners()
9 ['c1']
```

In Listing 7.4, Candidate `c1` obtains the minimal BORDA score, followed by candidate `c3` and finally candidate `c2`. The corresponding BORDA *rank analysis table* may be printed out with the `showRankAnalysisTable()` method.

**Listing 7.5** Rank analysis example with BORDA scores

```
1 >>> lvp.showRankAnalysisTable()
2 *----- Borda rank analysis tableau -----*
3   candi- | alternative-to-rank |      Borda
4   dates   |    1      2      3      | score  average
```

5	'c1'	6	9	0	24/15	1.60
6	'c3'	7	1	7	30/15	2.00
7	'c2'	2	5	8	36/15	2.40

In our randomly generated election results, we are lucky: Candidate  $c_1$  is both the instant runoff winner and the BORDA winner (see List. 7.3 on the previous page and List. 7.4). However, one could also follow the *Marquis de Condorcet*'s advice, and compute the *majority margins* obtained by voting for each individual pair of candidates (de Caritat, Marquis de Condorcet, 1784).

### 7.3 The majority margins digraph

Candidate  $c_1$ , for instance, is ranked four times before and once behind candidate  $c_2$  (see List. 7.2 on page 94). Hence, due to the voters' weights, the corresponding *majority margin*  $M(c_1, c_2)$  amounts to  $(3 + 1 + 5 + 4) - (2) = +11$ . These pairwise *majority margins* define on the set of candidates what we call a *majority margins digraph*. The MajorityMarginsDigraph class is handling such kind of digraphs.

**Listing 7.6** Example of *Majority Margins* digraph

```

1 >>> from votingProfiles import MajorityMarginsDigraph
2 >>> mmdg = MajorityMarginsDigraph(lvp,\n3 ...                                IntegerValuation=True)
4 >>> mmdg
5 *----- Digraph instance description -----
6 Instance class      : MajorityMarginsDigraph
7 Instance name       : rel_randomLinearVotingProfile1
8 Digraph Order       : 3
9 Digraph Size        : 3
10 Valuation domain   : [-15.00;15.00]
11 Determinateness (%) : 55.56
12 Attributes          : ['name', 'actions', 'voters',
13                           'ballot', 'valuationdomain',
14                           'relation', 'order',
15                           'gamma', 'notGamma']
16 >>> mmdg.showAll()
17 *---- show detail -----*
18 Digraph           : rel_randomLinearVotingProfile1
19 *---- Candidates ----*
20 ['c1', 'c2', 'c3']
21 *---- Characteristic valuation domain ----*
22 {'max': Decimal('15.0'),
23  'med': Decimal('0'),
24  'min': Decimal('-15.0'),
25  'hasIntegerValuation': True}
26 *---- majority margins ----*
27 M(x,y) | 'c1'  'c2'  'c3'
-----|-----

```

```

29      'c1' | 0   11   1
30      'c2' | -11  0   -1
31      'c3' | -1   1   0
32  Valuation domain: [-15;+15]

```

Notice in Listing 7.6 Lines 29-31 that, in the case of linear voting profiles, majority margins always verify a *zero sum property*:  $M(x,y) + M(y,x) = 0$  for all candidates  $x$  and  $y$ . This is not true in general for arbitrary voting profiles. The *majority margins* digraph of linear voting profiles defines in fact a *weak tournament* and belongs, hence, to the class of *self-codual*<sup>1</sup> bipolar-valued digraphs (see Sec. 2.6).

A candidate  $x$ , showing a positive majority margin  $M(x,y)$ , is beating candidate  $y$  with an absolute majority in a pairwise voting. Hence, a candidate showing only positive terms in their row in the *majority margins* digraph relation table, beats all other candidates with absolute majority of votes. CONDORCET recommends to declare this candidate (is always unique, why?) the winner of the election. Here we are lucky, it is again candidate `c1` who is the CONDORCET winner (see Line 29 in List. 7.6 on the preceding page). This result is confirmed below by the `computeCondorcetWinners()` method.

```

1 >>> mmdg.computeCondorcetWinners()
2 ['c1']

```

By seeing the majority margins like a *bipolar-valued characteristic function* of a global voters' preference relation defined on the set of eligible candidates, we may reuse all operational resources of the generic `Digraph` class<sup>2</sup>, and especially its `exportGraphViz()` method for visualising in Figure 7.1 on the following page the election result.

```

1 >>> mmdg.exportGraphViz(\n2 ...           fileName='tutorialLinearBallots', \
3 ...           graphType='pdf')
4 *----- exporting a dot file for GraphViz tools -----*
5 Exporting to tutorialLinearBallots.dot
6 dot -Grankdir=BT -Tpng tutorialLinearBallots.dot \
7           -o tutorialLinearBallots.pdf

```

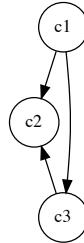
That a majority margins digraph models a linear ranking of the eligible candidates, as shown in Figure 7.1 on the next page, represents a very unlikely event. Usually, when aggregating linear ballots, there appear cyclic social preferences.

---

<sup>1</sup> The class of self-codual bipolar-valued digraphs consists of all weakly asymmetric digraphs, i.e. digraphs containing only asymmetric and/or indeterminate links. Limit cases consists of, on the one side, full tournaments with indeterminate reflexive links, and, on the other side, fully indeterminate digraphs. In this class, the converse (inverse  $\sim$ ) operator is indeed identical to the dual (negation  $-$ ) one.

<sup>2</sup> See Chap. 2

**Fig. 7.1** Visualising an election result. In the Figure we notice that the *majority margins* digraph from our example linear voting profile models in fact a linear ranking of the candidates:  $c_1 > c_3 > c_2$ , the same actually as modelled by the BORDA scores (see List. 7.4 on page 95)



Digraph3 (graphviz), R. Bisendorff, 2020

## 7.4 Cyclic social preferences

Let us consider for instance in Listing 7.7 the following linear voting profile  $v$  and construct the corresponding majority margins digraph.

**Listing 7.7** Example of cyclic social preferences

```

1  >>> v.showLinearBallots()
2      voters(weight): candidates rankings
3          v1(1): ['c1', 'c3', 'c5', 'c2', 'c4']
4          v2(1): ['c1', 'c2', 'c4', 'c3', 'c5']
5          v3(1): ['c5', 'c2', 'c4', 'c3', 'c1']
6          v4(1): ['c3', 'c4', 'c1', 'c5', 'c2']
7          v5(1): ['c4', 'c2', 'c3', 'c5', 'c1']
8          v6(1): ['c2', 'c4', 'c5', 'c1', 'c3']
9          v7(1): ['c5', 'c4', 'c3', 'c1', 'c2']
10         v8(1): ['c2', 'c4', 'c5', 'c1', 'c3']
11         v9(1): ['c5', 'c3', 'c4', 'c1', 'c2']
12
13 >>> mmdg = MajorityMarginsDigraph(v)
14 >>> mmdg.showRelationTable(ReflexiveTerms=False)
15     * ----- Relation Table -----*
16     M(x,y) | 'c1' 'c2' 'c3' 'c4' 'c5'
17     -----|-----
18     'c1' | - 1 -1 -5 -3
19     'c2' | -1 - 1 1 -1
20     'c3' | 1 -1 - -3 -1
21     'c4' | 5 -1 3 - 1
22     'c5' | 3 1 1 -1 -
22     Valuation domain: [-9;+9]
  
```

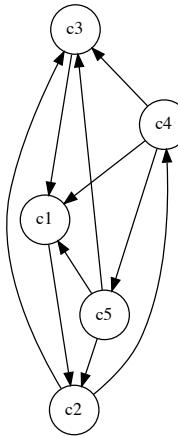
Now there doesn't exist anymore a completely positive row in the relation table (see Lines 17-21 in List. 7.7). No one of the five candidates is beating all the others with an absolute majority of votes. There is no CONDORCET winner anymore. In fact, when looking in Figure 7.2 on the next page at a graphviz drawing of this *majority margins* digraph, we may observe *cyclic* voters's preferences, like  $c_1 > c_2 > c_3 > c_1$ .

```

1 >>> mmdg.exportGraphViz('cycles',graphType='pdf')
2 *---- exporting a dot file for GraphViz tools ----*
3 Exporting to cycles.dot
4 dot -Grankdir=BT -Tpng cycles.dot -o cycles.pdf

```

**Fig. 7.2** Cyclic social preferences.  $c_1 > c_2 > c_3 > c_1$  for instance



Digraph3 (graphviz), R. Bisendorff, 2020

There may be many circuits appearing in a *majority margins* digraph. The `computeChordlessCircuits` method detects and enumerates all minimal chordless circuits in a `Digraph` instance.

```

1 >>> mmdg.computeChordlessCircuits()
2 [(['c2', 'c3', 'c1'], frozenset({'c2', 'c3', 'c1'})),
3  (['c2', 'c4', 'c5'], frozenset({'c2', 'c5', 'c4'})),
4  (['c2', 'c4', 'c1'], frozenset({'c2', 'c1', 'c4'}))]

```

In our example voting profile  $v$ , we actually obtain three chordless social preference circuits and determining the winner of this election result becomes non trivial. There is, for instance, no more any instant runoff winner.

```

1 >>> v.computeInstantRunoffWinner(Comments=True)
2 Total number of votes = 9.000
3 Half of the Votes = 4.50
4 ==> stage = 1
5     remaining candidates ['c2', 'c5', 'c1', 'c4', 'c3']
6     uninominal votes {'c2': 2.0, 'c5': 3.0, 'c1': 2.0, 'c4': 1.0, 'c3': 1.0}
7     minimal number of votes = 1.0
8     maximal number of votes = 3.0
9     candidate to remove = c3
10    candidate to remove = c4

```

```

11     remaining candidates =  ['c2', 'c5', 'c1']
12 ==> stage = 2
13     remaining candidates ['c2', 'c5', 'c1']
14     uninominal votes {'c2': 3.0, 'c5': 3.0, 'c1': 3.0}
15     minimal number of votes = 3.0
16     maximal number of votes = 3.0
17     candidate to remove = c1
18     candidate to remove = c5
19     candidate to remove = c2
20     remaining candidates = []
21 []

```

CONDORCET's approach for determining the winner of an election is *not decisive* in this example voting profile. One needs therefore, the case given, exploiting more sophisticated approaches for finding the winner of the election on the basis of the given linear ballots (see Chap. 8 and Bisendorff et al., 2008).

The NETFLOWS ranked heatmap view on voting profiles with the `showHTMLVotingHeatmap()` method is one such tool for showing convincing voting results (see Sec. 8.3).

**Listing 7.8** NETFLOWS ranked heatmap view on a voting profile

```

1 >>> v.showHTMLVotingHeatmap(rankingRule='Netflows', \
2 ...                               colorLevels=3,Correlations=True)

```

**Fig. 7.3** Visualising a linear voting profile in a NETFLOWS ranked heatmap. As the number of voters is usually much larger than the number of eligible candidates, the voting heatmap is by default transposed into a voters  $\times$  candidates view. Notice that the importance weights of the voters are *negative*, which means that the preference direction of the criteria (in this case the individual voters) is *decreasing* (min), i.e. goes from lowest (best) rank to highest (worst) rank

## Voting Heatmap

criteria	weight	tau*	c4	c5	c2	c3	c1
v5	-1.00	+0.60	1	4	2	3	5
v3	-1.00	+0.60	3	1	2	4	5
v8	-1.00	+0.40	2	3	1	5	4
v7	-1.00	+0.40	2	1	5	3	4
v6	-1.00	+0.40	2	3	1	5	4
v9	-1.00	+0.20	3	1	5	2	4
v4	-1.00	+0.00	2	4	5	1	3
v2	-1.00	-0.40	3	5	2	4	1
v1	-1.00	-0.80	5	3	4	2	1

Color legend:

quantile	33.33%	66.66%	100.00%
----------	--------	--------	---------

(\* tau: Ordinal (Kendall) correlation between

marginal criterion and global ranking relation

Outranking model: robust, Ranking rule: NetFlows

Ordinal (Kendall) correlation between

global ranking and global outranking relation: +0.778

It is worthwhile noticing in Figure 7.3 that the compromise NETFLOWS ranking  $c4 > c5 > c2 > c3 > c1$ , shown in this heatmap, results in the highest possible *ordinal correlation* index of +0.778 with the majority margins digraph (see

Chap. 16). This NETFLOWS ranking result corresponds also to the BORDA scores ranking<sup>3</sup>.

**Listing 7.9** Rank analysis table with BORDA scores

```

1 >>> v.showRankAnalysisTable()
2     *----- Borda rank analysis tableau -----*
3     candi- | alternative-to-rank           |      Borda
4     dates  |   1    2    3    4    5   | score  average
5     -----|-----
6     'c4'  |   1    4    3    0    1   | 23    2.56
7     'c5'  |   3    0    3    2    1   | 25    2.78
8     'c2'  |   2    3    0    1    3   | 27    3.00
9     'c3'  |   1    2    2    2    2   | 29    3.22
10    'c1'  |   2    0    1    4    2   | 31    3.44

```

Let us eventually notice in the rank analysis table, shown in Listing 7.9 above, that the uninominal plurality winner would be, with three votes, candidate  $c_5$ , whereas the best NETFLOWS and BORDA ranked candidate  $c_1$  obtains with candidate  $c_3$  only one vote.

But do represent our example linear voting profiles here realistic election outcomes?

## 7.5 On generating realistic random linear voting profiles

By default, the `RandomLinearVotingProfile` class generates random linear voting profiles where every candidate has the same uniform probability to be ranked at a certain position by all the voters. Each voter's random linear ballot is indeed generated via a uniform shuffling of the list of candidates.

In reality, political election data are quite different. There usually will be different favorite and marginal candidates for each political party. To simulate these aspects with our random generator, we are using two random exponentially distributed polls of the candidates and consider a bipartisan political landscape with a certain random balance (default theoretical party repartition = 0.50) between the two sets of potential party supporters. A certain theoretical proportion (default = 0.1) will not support any party.

Let us generate in Listing 7.10 such a linear voting profile for an election with 1000 voters and 15 candidates.

**Listing 7.10** Generating a linear voting profile with random polls

```

1 >>> from votingProfiles import \
2 ...             RandomLinearVotingProfile
3 >>> lvp = RandomLinearVotingProfile(\ 
4 ...             numberOfCandidates=15, \
5 ...             numberOfVoters=1000, \

```

---

<sup>3</sup> Mind that BORDA scores require the unrealistic working hypothesis that one knows how to precisely code in numbers the marginal linear ranks per voter.

```

6 ...           WithPolls=True, \
7 ...           partyRepartition=0.5, \
8 ...           other=0.1, \
9 ...           seed=0.9189670954954139)
10 >>> lvp
11 *----- VotingProfile instance description -----*
12 Instance class : RandomLinearVotingProfile
13 Instance name  : randLinearProfile
14 Candidates    : 15
15 Voters        : 1000
16 Attributes     : ['name', 'seed', 'candidates',
17                      'voters', 'RandomWeights',
18                      'sumWeights', 'poll1', 'poll2',
19                      'bipartisan', 'linearBallot',
20                      'ballot']
21 >>> lvp.showRandomPolls()
22 Random repartition of voters
23 Party-1 supporters : 460 (46.0%)
24 Party-2 supporters : 436 (43.6%)
25 Other voters       : 104 (10.4%)
26 *----- random polls -----*
27 Party-1(46.0%) | Party-2(43.6%) | expected
28 -----
29 c06 : 19.91% | c11 : 22.94% | c06 : 15.00%
30 c07 : 14.27% | c08 : 15.65% | c11 : 13.08%
31 c03 : 10.02% | c04 : 15.07% | c08 : 09.01%
32 c13 : 08.39% | c06 : 13.40% | c07 : 08.79%
33 c15 : 08.39% | c03 : 06.49% | c03 : 07.44%
34 c11 : 06.70% | c09 : 05.63% | c04 : 07.11%
35 c01 : 06.17% | c07 : 05.10% | c01 : 05.06%
36 c12 : 04.81% | c01 : 05.09% | c13 : 05.04%
37 c08 : 04.75% | c12 : 03.43% | c15 : 04.23%
38 c10 : 04.66% | c13 : 02.71% | c12 : 03.71%
39 c14 : 04.42% | c14 : 02.70% | c14 : 03.21%
40 c05 : 04.01% | c15 : 00.86% | c09 : 03.10%
41 c09 : 01.40% | c10 : 00.44% | c10 : 02.34%
42 c04 : 01.18% | c05 : 00.29% | c05 : 01.97%
43 c02 : 00.90% | c02 : 00.21% | c02 : 00.51%

```

In this example (see above Lines 21 and following), we obtain 460 Party-1 supporters (46%), 436 Party-2 supporters (43.6%) and 104 other voters (10.4%). Favorite candidates of Party-1 supporters, with more than 10%, appear to be `c06` (19.91%), `c07` (14.27%) and `c03` (10.02%). Whereas for Party-2 supporters, favorite candidates appear to be `c11` (22.94%), followed by `c08` (15.65%), `c04` (15.07%) and `c06` (13.4%). Being *first* choice for Party-1 supporters and *fourth* choice for Party-2 supporters, this candidate `c06` is a natural candidate for clearly winning this election game (see List. 7.11).

**Listing 7.11** The uninominal and BORDA election winner

```

1 >>> lvp.computeSimpleMajorityWinner()
2 ['c06']
3 >>> lvp.computeInstantRunoffWinner()
4 ['c06']

```

```

5 >>> lvp.computeBordaWinners()
6 ['c06']

```

Is candidate *c06* also a CONDORCET winner? To verify, we start by creating the corresponding *majority margins* digraph *mmdg* with the help of the *MajorityMarginsDigraph* class. The created digraph instance contains 15 *actions* –the candidates– and 104 *oriented arcs* –the *positive majority margins*– (see List. 7.12 Lines 7–8).

**Listing 7.12** A majority margins digraph constructed from a linear voting profile

```

1 >>> from votingProfiles import MajorityMarginsDigraph
2 >>> mmdg = MajorityMarginsDigraph(lvp)
3 >>> mmdg
4 *----- Digraph instance description -----*
5 Instance class      : MajorityMarginsDigraph
6 Instance name       : rel_randLinearProfile
7 Digraph Order       : 15
8 Digraph Size        : 104
9 Valuation domain    : [-1000.00;1000.00]
10 Determinateness (%) : 67.08
11 Attributes          : ['name', 'actions', 'voters',
12                           'ballot', 'valuationdomain',
13                           'relation', 'order',
14                           'gamma', 'notGamma']

```

The *showHTMLRelationTable()* method visualise the resulting pairwise majority margins by showing the HTML formated version of the *mmdg* relation table in a browser view.

```

1 >>> mmdg.showHTMLRelationTable(\n2 ...           tableTitle='Pairwise majority margins',\n3 ...           relationName='M(x,y)')

```

**Fig. 7.4** Browsing the majority margins. Light green cells contain the positive majority margins, whereas light red cells contain the negative majority margins

### Pairwise majority margins

M(x,y)	c01	c02	c03	c04	c05	c06	c07	c08	c09	c10	c11	c12	c13	c14	c15
<b>c01</b>	-	768	-138	108	478	-436	-198	-140	238	440	-268	148	50	202	218
<b>c02</b>	-768	-	796	-484	-368	-858	-828	-772	-546	-496	-800	-722	-768	-696	-658
<b>c03</b>	138	796	-	160	590	-286	-80	-8	372	522	-158	280	210	360	338
<b>c04</b>	-108	484	-160	-	184	-370	-180	-288	160	136	-420	16	-62	56	30
<b>c05</b>	-478	368	-590	-184	-	-730	-640	-472	-234	-116	-550	-442	-522	-376	-386
<b>c06</b>	436	858	286	370	730	-	248	234	574	692	102	556	482	566	520
<b>c07</b>	198	828	80	180	640	-248	-	0	358	602	-94	304	266	384	420
<b>c08</b>	140	772	8	288	472	-234	0	-	436	396	-176	276	134	298	244
<b>c09</b>	-238	546	-372	-160	234	-574	-358	-436	-	116	-594	-126	-194	-90	-14
<b>c10</b>	-440	496	-522	-136	116	-692	-602	-396	-116	-	-510	-310	-442	-304	-266
<b>c11</b>	268	800	158	420	550	-102	94	176	594	510	-	388	268	474	292
<b>c12</b>	-148	722	-280	-16	442	-556	-304	-276	126	310	-388	-	-92	100	148
<b>c13</b>	-50	768	-210	62	522	-482	-266	-134	194	442	-268	92	-	158	186
<b>c14</b>	-202	696	-360	-56	376	-566	-384	-298	90	304	-474	-100	-158	-	68
<b>c15</b>	-218	658	-338	-30	386	-520	-420	-244	14	266	-292	-148	-186	-68	-

Valuation domain: [-1000; +1000]

A complete light green *row* in Figure 7.4 on the preceding page reveals a CONDORCET winner, whereas a complete light green *column* reveals a CONDORCET loser. We recover again candidate c06 as CONDORCET winner<sup>4</sup>, whereas the obvious CONDORCET loser is here candidate c02, the candidate with the lowest support in both parties (see List. 7.10 on page 101 Line 43).

With a same bipolar *first-ranked* and *last-ranked* selection procedure, we may weakly rank the candidates (with possible ties) by iterating these *first-ranked* and *last-ranked* choices among the remaining candidates (Bisdorff, 1999).

Before showing the *ranking-by-choosing* result, we have to, first, compute the iterated bipolar selection procedure (see List. 7.13).

**Listing 7.13** Ranking by iterating choosing the *first* and *last* remaining candidates

```

1 >>> cdg.showRankingByChoosing()
2 Ranking by Choosing and Rejecting
3   1st best ranked ['c06']
4     2nd best ranked ['c11']
5       3rd best ranked ['c07', 'c08']
6         4th best ranked ['c03']
7           5th best ranked ['c01']
8             6th best ranked ['c13']
9               7th first ranked ['c04']
10              7th last ranked ['c12']
11                6th last ranked ['c14']
12                  5th last ranked ['c15']
13                    4th last ranked ['c09']
14                      3rd last ranked ['c10']
15                        2nd last ranked ['c05']
16                          1st last ranked ['c02']

```

The first selection concerns c06 (first) and c02 (last), followed by c11 (first) opposed to c05 (last), and so on, until there remains at iteration step 7 a last pair of candidates, namely [c04, c12] (see Lines 9-10).

Notice furthermore the 3rd-best ranked candidates at iteration step 3 (see Line 5), namely the pair (c07, c08). Both candidates represent indeed conjointly the *3rd best ranked* choice. We obtain hence a *weak ranking*, i.e. a ranking with a tie.

Let us mention that the *instant runoff* procedure, we used before when operated with a `Comments=True` parameter setting, will deliver a more or less similar *reversed linear ordering-by-rejecting* result, namely [c02, c10, c14, c05, c09, c13, c12, c15, c04, c01, c08, c03, c07, c11, c06], ordered from the *last* to the *first* choice.

Remarkable about both these *ranking-by-choosing* or *ordering-by-rejecting* results is the fact that the random voting behaviour, simulated here with the help of two discrete random variables<sup>5</sup>, defined respectively by the two random party polls, is rendering a ranking that is more or less in accordance with the simulated balance

<sup>4</sup> The concept of CONDORCET winner –a generalisation of absolute majority winners– proposed by CONDORCET in 1784, is an early historical example of initial digraph kernel (see Chap. 17).

<sup>5</sup> Discrete random variables with a given empirical probability law (here the polls) are provided in the `randomNumbers` module by the `DiscreteRandomVariable` class (Bisdorff, 2020b).

of the polls: Party-1 supporters : 460; Party-2 supporters: 436 (see List. 7.10 on page 101 Lines 29-43 third column). Despite a random voting behaviour per voter, the given polls apparently show a *very strong incidence* on the eventual election result. In order to avoid any manipulation of the election outcome, public media are therefore in some countries not allowed to publish polls during the last weeks before a general election.

Mind that the specific *ranking-by-choosing* procedure, we use here on the majority margins digraph, operates the selection procedure by extracting at each step *initial* and *terminal* prekernels, i.e. difficult operational problems (see Chap. 17 and Bisдорff, 1999); A technique that does not allow in general to tackle voting profiles with much more than 30 candidates.

Next Chapter 8 on multiple incommensurable criteria ranking methods presents more methods and tools for ranking from pairwise majority margins when a larger number of potential candidates or decision alternatives is given.

## References

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# Chapter 8

## Ranking with multiple incommensurable criteria

”... whether we are deciding between buying different commodity baskets, or making choices about what to do on a holiday, or deciding for whom to vote for in an election, we are inescapably involved in evaluating alternatives with non-commensurable aspects.”

—Amartya Sen, Idea of Justice, ([Sen, 2009](#))

**Abstract** The DIGRAPH3 python resources provide several algorithms for solving the multiple incommensurable criteria ranking problem via bipolar-valued outranking digraphs. The COPELAND, NETFLOWS, KEMENY, SLATER, KOHLER, and the RANKEDPAIRS ranking rules are introduced and illustrated with a random outranking digraph.

### 8.1 The ranking problem

We need to rank without ties a set  $X$  of items (usually decision alternatives) that are evaluated on multiple incommensurable performance criteria; yet, for which we know their pairwise bipolar-valued *strict outranking* characteristics, i.e.  $r(x \succsim y)$  for all  $x, y \in X$  (see Sec. 3.5 and ([Bisdorff, 2013](#))).

Let us consider in Listing 8.1 a didactic outranking digraph  $g$  generated from a random *Cost-Benefit* performance tableau (see Sec. 6.3) concerning 9 decision alternatives evaluated on 13 performance criteria. The BipolarOutrankingDigraph computes the corresponding bipolar-valued *outranking digraph*  $g$  and a codual transform gives us the requested strict outranking digraph  $gcd$  with the pairwise  $r(x \succsim y)$  characteristic values (see Lines 11-19 below).

**Listing 8.1** Random bipolar-valued strict outranking relation characteristics

```
1 >>> from randomPerfTabs import RandomCBPerformanceTableau
2 >>> t = RandomCBPerformanceTableau(numberOfActions=9, \
3 ...           numberOfCriteria=13, seed=200)
```

```

4 >>> from outrankingDigraphs import BipolarOutrankingDigraph
5 >>> g = BipolarOutrankingDigraph(t)
6 >>> qcd = ~(-g) # strict (codual) outranking digraph
7 >>> gcd.showRelationTable(ReflexiveTerms=False)
8 * ---- Relation Table ----
9 r(>) | 'a1' 'a2' 'a3' 'a4' 'a5' 'a6' 'a7' 'a8' 'a9'
10 -----|-----
11 'a1' | - 0.00 +0.10 -1.00 -0.13 -0.57 -0.23 +0.10 +0.00
12 'a2' | -1.00 - 0.00 +0.00 -0.37 -0.42 -0.28 -0.32 -0.12
13 'a3' | -0.10 0.00 - -0.17 -0.35 -0.30 -0.17 -0.17 +0.00
14 'a4' | 0.00 0.00 -0.42 - -0.40 -0.20 -0.60 -0.27 -0.30
15 'a5' | +0.13 +0.22 +0.10 +0.40 - +0.03 +0.40 -0.03 -0.07
16 'a6' | -0.07 -0.22 +0.20 +0.20 -0.37 - +0.10 -0.03 -0.07
17 'a7' | -0.20 +0.28 -0.03 -0.07 -0.40 -0.10 - +0.27 +1.00
18 'a8' | -0.10 -0.02 -0.23 -0.13 -0.37 +0.03 -0.27 - +0.03
19 'a9' | 0.00 +0.12 -1.00 -0.13 -0.03 -0.03 -1.00 -0.03 -

```

Some ranking rules will work on the associated CONDORCET digraph, i.e. the corresponding *median cut* polarised strict outranking digraph.

**Listing 8.2** Median cut polarised strict outranking relation characteristics

```

1 >>> ccd = PolarisedOutrankingDigraph(gcd,\n2 ...           level=g.valuationdomain['med'],\n3 ...           KeepValues=False, StrictCut=True)\n4 >>> ccd.showRelationTable(ReflexiveTerms=False,\n5 ...           IntegerValues=True)\n6 *---- Relation Table ----\n7 r(>)_med | 'a1' 'a2' 'a3' 'a4' 'a5' 'a6' 'a7' 'a8' 'a9'\n8 -----|-----\n9 'a1' | - 0 +1 -1 -1 -1 -1 +1 0\n10 'a2' | -1 - +0 0 -1 -1 -1 -1 -1\n11 'a3' | -1 0 - -1 -1 -1 -1 -1 0\n12 'a4' | 0 0 -1 - -1 -1 -1 -1 -1\n13 'a5' | +1 +1 +1 +1 - +1 +1 -1 -1\n14 'a6' | -1 -1 +1 +1 -1 - +1 -1 -1\n15 'a7' | -1 +1 -1 -1 -1 -1 - +1 +1\n16 'a8' | -1 -1 -1 -1 -1 +1 -1 - +1\n17 'a9' | 0 +1 -1 -1 -1 -1 -1 -1 -

```

Unfortunately, such crisp median-cut CONDORCET digraphs, associated with a given strict outranking digraph, only exceptionally model a linear ordering. Usually, pairwise majority comparisons do not render a *complete* or, at least, a *transitive* partial order. There may even frequently appear *cyclic* outranking situations (see Sec. 7.4).

To discover how *difficult* this ranking problem can get, let us have a look in Figure. 8.1 on the next page at the corresponding strict outranking digraph *graphviz* drawing<sup>1</sup>.

```

1 >>> gcd.exportGraphViz(fileName='rankingTutorial')
2 *---- exporting a dot file for GraphViz tools -----*

```

---

<sup>1</sup> The `exportGraphViz()` method is depending on drawing tools from graphviz software (<https://graphviz.org/>).

```

3 Exporting to rankingTutorial.dot
4 dot -Grankdir=BT -Tpng rankingTutorial.dot\
5 -o rankingTutorial.png

```

**Fig. 8.1** The strict outranking relation  $\succ$ . The relation shown here is, for instance, *not transitive*: alternative  $a_8$  outranks alternative  $a_6$  and alternative  $a_6$  outranks  $a_4$ , however,  $a_8$  does not outrank  $a_4$ . Furthermore, alternatives  $a_6$ ,  $a_7$  and  $a_8$  show a cyclic outranking relation



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The `computeTransitivityDegree()` method computes the *transitivity degree* of the outranking digraph `gcd` shown in Figure 8.1, i.e. the ratio of the number of closed transitive triples  $-x \succ y, y \succ z$  and  $x \succ z-$  over the number of triples  $x \succ y, y \succ z$  (see below Lines 3-4).

```

1 >>> gcd.computeTransitivityDegree(Comments=True)
2 Transitivity degree of graph <codual_rel_randomCBperftab>
3 triples x>y>z: 78, closed: 38, open: 40
4 closed/triples = 0.487

```

With only 49% of the required transitive arcs, the strict outranking digraph `gcd` is hence very far from being transitive; a serious problem when a linear ordering of the decision alternatives is looked for.

The `computeChordlessCircuits()` method, followed by the `showChordlessCircuits()` method, can check furthermore whether there appear any cyclic outranking situations.<sup>2</sup>

```

1 >>> gcd.computeChordlessCircuits()
2 >>> gcd.showChordlessCircuits()
3   1 circuit(s).
4   *---- Chordless circuits ----*
5   1: ['a6', 'a7', 'a8'], credibility : 0.033

```

There is one chordless circuit detected in the given strict outranking digraph `gcd`, namely alternative  $a_6$  outranks alternative  $a_7$ , the latter outranks  $a_8$ , and  $a_8$  outranks again alternative  $a_6$  (see Fig. 8.1 on the previous page). Any potential linear ordering of these three alternatives will, in fact, always contradict somehow the given outranking relation.

Now, several heuristic ranking rules have been proposed for constructing a linear ordering which is closest in some specific sense to a given outranking relation. The DIGRAPH3 resources provide some of the most common of these ranking rules, like the COPELAND, KEMENY, SLATER, KOHLER, and the RANKEDPAIRS ranking rules.

## 8.2 The COPELAND ranking

---

### Algorithm 8.1 COPELAND ranking rule (Copeland, 1951)

---

*in* : median-cut strict outranking digraph  $G(X, \succ_{>0})$ ,  
*out* : linear ranking of the set  $X$  of decision actions.

1. The COPELAND ranking rule computes for each alternative a score resulting from the sum for all non-reflexive pairs of alternatives of the differences between the crisp *strict outranking* characteristics  $r(x \succ_{>0} y)$  and the crisp *strict outranked* characteristics  $r(y \succ_{>0} x)$ .
  2. The alternatives are ranked in decreasing order of these COPELAND scores; ties, the case given, being resolved with a lexicographical rule applied to the identifiers of the decision actions.
- 

In Listing 8.3 on the facing page we show the ranking of the given strict outranking digraph `gcd` obtained with the `CopelandRanking` class from the `linearOrders` module.

---

<sup>2</sup> The `computeChordlessCircuits()` and `showChordlessCircuits()` methods are separate because there are various methods available for enumerating the chordless circuits in a digraph (Bisdorff, 2010).

**Listing 8.3** Computing a COPELAND Ranking

```

1 >>> from linearOrders import CopelandRanking
2 >>> cop = CopelandRanking(gcd, Comments=True)
3 Copeland decreasing scores
4     a5 : +12
5     a1 : +2
6     a6 : +2
7     a7 : +2
8     a8 :  0
9     a4 : -3
10    a9 : -3
11    a3 : -5
12    a2 : -7
13 Copeland Ranking:
14 ['a5','a1','a6','a7','a8','a4','a9','a3','a2']

```

Alternative  $a_5$  obtains here the best COPELAND score (+12), followed by alternatives  $a_1$ ,  $a_6$  and  $a_7$  with same score (+2); following the lexicographic rule,  $a_1$  is hence ranked before  $a_6$  and  $a_6$  before  $a_7$ . Same situation is observed for  $a_4$  and  $a_9$  with a score of  $-3$  (see List. 8.3 Lines 4-12). The COPELAND ranking rule is in fact *invariant* under the *codual* transform (see Sec. 2.6) and renders a same linear order indifferently from digraphs  $g$  or  $\text{gcd}$ .

The COPELAND rule works well, furthermore on any strict outranking digraph that models a linear (partial) order on the *median cut* strict outranking digraph  $\text{ccd}$  (Dias and Lamboray, 2010). In Listing 8.4, the `computeRankingCorrelation()` method, coupled with the `showCorrelation()` method, indicate the ordinal correlation of the COPELAND ranking result, shown in Listing 8.3 Line 14, with the given outranking digraph  $g$  (see Chap. 16 and Bisdorff, 2012).

**Listing 8.4** Checking the ordinal quality of the COPELAND ranking

```

1 >>> corr = g.computeRankingCorrelation(cop.copelandRanking)
2 >>> g.showCorrelation(corr)
3 Correlation indexes:
4     Crisp ordinal correlation : +0.463
5     Valued equivalence       : +0.107
6     Epistemic determination   :  0.230

```

With an epistemic determination level of 0.230 (Line 6), the crisp ordinal correlation –KENDALL  $\tau$ – index of +0.463 is in fact supported by 61.5%( $100.0 \times (1.0 + 0.23)/2$ ) of the strict outranking situations. Furthermore, the bipolar-valued *relational equivalence* between the strict outranking relation and the COPELAND ranking equals +0.107, i.e. a *majority* of 55.35% of the criteria significance supports the relational equivalence between the given strict outranking relation and the COPELAND ranking (see Chap. 16).

The COPELAND scores, shown in Listing 8.3 deliver actually only a *weak ranking*, i.e. a ranking with ties. This weak ranking may be constructed with the `WeakCopelandOrder` class from the `transitiveDigraphs` module.

**Listing 8.5** Computing a weak COPELAND ranking

```

1 >>> from transitiveDigraphs import WeakCopelandOrder

```

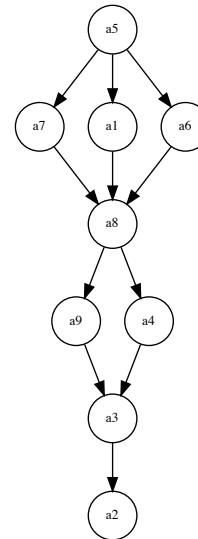
```

2 >>> wcop = WeakCopelandOrder(g)
3 >>> wcop.showRankingByChoosing()
4 Ranking by Choosing and Rejecting
5   1st ranked ['a5']
6     2nd ranked ['a1', 'a6', 'a7']
7       3rd ranked ['a8']
8         3rd last ranked ['a4', 'a9']
9           2nd last ranked ['a3']
10          1st last ranked ['a2']

```

In Listing 8.5 on the preceding page, the `WeakCopelandOrder` class models the weak ranking actually delivered by the COPELAND scores (see List. 8.3 on the previous page). We may draw its corresponding skeleton<sup>3</sup>.

**Fig. 8.2** Drawing of the weak COPELAND ranking. The drawing shows the skeleton of the weak ranking produced by the corresponding ties of the COPELAND scores



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R. Bisendorff, 2020

```

1 >>> wcop.exportGraphViz(fileName='weakCopelandRanking')
2 *---- exporting a dot file for GraphViz tools ----*
3 Exporting to weakCopelandRanking.dot
4 dot -Grankdir=TB -Tpng weakCopelandRanking.dot \
      -o weakCopelandRanking.png

```

Let us now consider a similar ranking rule, but working directly on the criteria *significance majority margins*, i.e. the *bipolar-valued* outranking characteristic valuation.

---

<sup>3</sup> The skeleton of a transitive relation drops the transitivity induced arcs.

### 8.3 The NETFLOWS ranking

The bipolar-valued version of the COPELAND ranking rule, we call NETFLOWS<sup>4</sup>, computes for each alternative  $x \in X$  a *net flow* score.

---

#### Algorithm 8.2 NETFLOWS ranking rule

---

*in* : bipolar-valued strict outranking digraph  $G(X, \succsim)$ ,  
*out* : linear ranking of the set  $X$  of decision actions.

1. The NETFLOWS rule computes for each alternative  $x \in X$  the sum over all non-reflexive  $y \in X$  of the differences between the *strict outranking* characteristics  $r(x \succsim y)$  and the *strict outranked* characteristics  $r(y \succsim x)$ .
  2. The alternatives are ranked in decreasing order of these NETFLOWS scores; ties, the case given, being resolved with a lexicographical rule applied to the identifiers of the decision actions.
- 

The NetFlowsRanking class from the linearOrders module computes this linear ranking from the given strict outranking digraph gcd.

#### Listing 8.6 Computing a NETFLOWS ranking

```

1 >>> from linearOrders import NetFlowsRanking
2 >>> nf = NetFlowsRanking(gcd, Comments=True)
3   Net flow scores :
4     a5 : +3.600
5     a7 : +2.800
6     a6 : +1.300
7     a3 : +0.033
8     a1 : -0.400
9     a8 : -0.567
10    a4 : -1.283
11    a9 : -2.600
12    a2 : -2.883
13  NetFlows Ranking:
14  ['a5','a7','a6','a3','a1','a8','a4','a9','a2']
15 >>> cop.copelandRanking # comparing both
16  ['a5','a1','a6','a7','a8','a4','a9','a3','a2']

```

In Listing 8.6 here, the NETFLOWS scores actually deliver directly a linear ranking *without ties* which is rather different from the one delivered by the COPELAND rule (compare Lines 14 and 16). It may happen, however, that we obtain, as with the COPELAND scores above, only a ranking with ties, which may then be resolved similarly by following a lexicographic rule applied to the identifiers of the decision alternatives. In such cases, it is possible to construct again a *weak ranking* with the corresponding WeakNetFlowsOrder class from the transitiveDigraphs module.

---

<sup>4</sup> This ranking rule is also known under the name PROMETHEE ranking rule (Brans and Vincke, 1985).

It is worthwhile noticing again, that similar to the COPELAND ranking rule seen before, the NETFLOWS ranking rule is also *invariant* under the codual transform (see Sec. 2.6) and delivers the same ranking result indifferently from digraph  $g$  or  $gcd$ .

The NETFLOWS ranking result appears to be better correlated (+0.638 vs. +0.463) with the given strict outranking relation than its crisp cousin, the COPELAND ranking (see Line 4 in List. 8.7).

**Listing 8.7** Checking the quality of the NETFLOWS Ranking

```

1 >>> corr = gcd.computeOrdinalCorrelation(nf)
2 >>> gcd.showCorrelation(corr)
3 Correlation indexes:
4   Extended Kendall tau      : +0.638
5   Epistemic determination   :  0.230
6   Bipolar-valued equivalence : +0.147

```

Indeed, the ordinal correlation index of +0.638 leads to a bipolar-valued *relational equivalence* characteristics of +0.147, i.e. a majority of 57.35% of the criteria significance supports the relational equivalence between the given outranking digraphs  $g$  or  $gcd$  and the corresponding NETFLOWS ranking (see Chap. 16). The weaker ordinal ranking quality of the COPELAND rule (+0.463) stems in this example here essentially from the *weakness* of the actual COPELAND ranking result and our subsequent *arbitrary* lexicographic resolution of the many ties given by the COPELAND scores (see Fig. 8.2 on page 112).

To appreciate now the ordinal correlations of both the COPELAND and the NETFLOWS rankings with the underlying strict outranking relation, it is useful to consider the '*optimal*' KEMENY and SLATER ranking rules.

## 8.4 KEMENY rankings

A KEMENY ranking  $k$  is a linear ranking without ties of the set of  $n$  decision alternatives  $X$  which is *closest*, in the sense of the ordinal KENDALL distance<sup>5</sup>, to the given valued outranking digraph  $g$  (Kemeny, 1959).

Formally:

$$k = \operatorname{argmax}_{p \in \mathcal{P}(X)} \sum_{i \neq j} (r(p[i] \succsim p[j]) - r(p[j] \succsim p[i])), \quad (8.1)$$

where  $\mathcal{P}(X)$  denotes the set of all permutations of  $X$  and  $i, j = 0, \dots, n$ .

The KemenyRanking class from the linearOrders module computes such a ranking which is highest possible correlated with the underlying strict outranking relation. The KEMENY rule is also *invariant* under the codual transform.

Mind that the KemenyRanking class constructor, in order to find a KEMENY ranking, has to compute NETFLOWS scores for every permutation of the list of

---

<sup>5</sup> See Chap. 16 and Bis dorff, 2012

decision alternatives (see Eq. 8.1 on the preceding page). Therefore the class is limited, by default, to digraphs of order up to 7 (Bisdorff, 2021). In Listing 8.8 Line 2, the `orderLimit` parameter allows to raise this limit.

**Listing 8.8** Computing a KEMENY ranking

```

1 >>> from linearOrders import KemenyRanking
2 >>> ke = KemenyRanking(gcd,orderLimit=9)
3 >>> # default orderLimit is 7
4 >>> ke.showRanking()
5  ['a5','a6','a7','a3','a9','a4','a1','a8','a2']
6 >>> corr = gcd.computeOrdinalCorrelation(ke)
7 >>> gcd.showCorrelation(corr)
8   Correlation indexes:
9     Extended Kendall tau      : +0.779
10    Epistemic determination   :  0.230
11    Bipolar-valued equivalence : +0.179

```

So, +0.779 represents the *highest possible* ordinal correlation index –*fitness*– any potential linear ranking can achieve with the given pairwise outranking digraph (see List. 8.8 Line 9).

A KEMENY ranking may not be unique. In our example here, we obtain in fact two such KEMENY rankings with a same *maximal* KEMENY index of 12.92.

**Listing 8.9** Optimal KEMENY rankings

```

1 >>> ke.maximalRankings
2  [[['a5','a6','a7','a3','a8','a9','a4','a1','a2'],
3   ['a5','a6','a7','a3','a9','a4','a1','a8','a2']]
4 >>> ke.maxKemenyIndex
5  Decimal('12.9166667')

```

Figure 8.3 on the following page shows the partial order defined by the epistemic disjunction of both optimal KEMENY rankings obtained with the `RankingsFusion` class (see Sec. 2.5).

**Listing 8.10** Computing the epistemic disjunction of all optimal KEMENY rankings

```

1 >>> from transitiveDigraphs import RankingsFusion
2 >>> wke = RankingsFusion(ke,ke.maximalRankings)
3 >>> wke.exportGraphViz(fileName='tutorialKemeny')
4 *---- exporting a dot file for GraphViz tools ----*
5 Exporting to tutorialKemeny.dot
6 dot -Grankdir=TB -Tpng tutorialKemeny.dot -o tutorialKemeny.png

```

To retain now a specific representative among all the potential rankings with maximal KEMENY index, we will choose, with the help of the `showRankingConsensusQuality()` method, the one proposing the best criteria consensus.

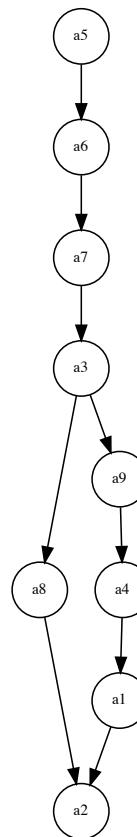
**Listing 8.11** Computing the consensus quality of the first KEMENY ranking

```

1 >>> q.showRankingConsensusQuality(ke.maximalRankings[0])
2 Consensus quality of ranking:
3  ['a5','a6','a7','a3','a8','a9','a4','a1','a2']
4  crit. (weight): tau | crit. (weight): tau
5  -----
6  b09 (0.050) : +0.361 | b04 (0.050) : +0.333

```

**Fig. 8.3** Epistemic disjunction of optimal KEMENY rankings. It is interesting to notice that both KEMENY rankings only differ in their respective positioning of alternative  $a_8$ ; either before or after alternatives  $a_9$ ,  $a_4$  and  $a_1$



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```

7   b08 (0.050) : +0.292 | b01 (0.050) : +0.264
8   c01 (0.167) : +0.250 | b03 (0.050) : +0.222
9   b07 (0.050) : +0.194 | b05 (0.050) : +0.167
10  c02 (0.167) : +0.000 | b10 (0.050) : +0.000
11  b02 (0.050) : -0.042 | b06 (0.050) : -0.097
12  c03 (0.167) : -0.167
13  Summary:
14      Weighted mean marginal correlation (a) : +0.099
15      Standard deviation (b) : +0.177
16      Ranking fairness (a)-(b) : -0.079
  
```

In Listing 8.11 on the previous page are shown the ordinal correlation of the first Kemeny ranking with all the marginal strict outranking relations per performance criterion (see Lines 6-12). The Benefit criterion  $b_09$  is highest correlated (+0.361,

Line 6), whereas the Costs criterion c03 is lowest correlated ( $-167$ , Line 12) with this KEMENY ranking.

The ranking consensus quality of the second KEMENY ranking is shown in Listing 8.12 below.

**Listing 8.12** Computing the consensus quality of the second KEMENY ranking

```

1 >>> g.showRankingConsensusQuality(ke.maximalRankings[1])
2 Consensus quality of ranking:
3  ['a5','a6','a7','a3','a9','a4','a1','a8','a2']
4    crit. (weight): tau | crit. (weight): tau
5  -----
6    b09 (0.050) : +0.306 | b08 (0.050) : +0.236
7    c01 (0.167) : +0.194 | b07 (0.050) : +0.194
8    c02 (0.167) : +0.167 | b04 (0.050) : +0.167
9    b03 (0.050) : +0.167 | b01 (0.050) : +0.153
10   b05 (0.050) : +0.056 | b02 (0.050) : +0.014
11   b06 (0.050) : -0.042 | c03 (0.167) : -0.111
12   b10 (0.050) : -0.111
13 Summary:
14   Weighted mean marginal correlation (a) : +0.099
15   Standard deviation (b) : +0.132
16   Ranking fairness (a)-(b) : -0.033

```

The Benefit criterion b09 is again highest correlated (+0.306, Line 6) with this KEMENY ranking, but the weakest correlated criterion is now Benefit criterion b10 ( $-0.111$ , Line 12).

Both KEMENY rankings show the same *weighted mean marginal correlation* (+0.099) with all thirteen performance criteria. However, the second ranking shows a slightly lower *standard deviation*: +0.132 versus +0.177, resulting in a slightly *fairer* ranking result:  $-0.033$  versus  $-0.079$  (see List. 8.11 on page 115 Lines 14-16 and List. 8.12 Lines 14-16).

When several rankings with maximal KEMENY index are given, the KemenyRanking class constructor instantiates the ranking with *highest* mean marginal correlation and, in case of ties, with *lowest* weighted standard deviation. Here we obtain ranking: [a5, a6, a7, a3, a9, a4, a1, a8, a2] (see Line 5 in List. 8.8 on page 115).

## 8.5 SLATER rankings

The SLATER ranking rule is identical to the KEMENY rule, but it is working, instead, on the CONDORCET–median cut polarised–digraph *ccd* (Slater, 1961). The SLATER rule is again *invariant* under the codual transform and the SlaterRanking class from the linearOrders module delivers hence indifferently on *g* or *gcd* the following results:

**Listing 8.13** Computing a SLATER ranking

```

1 >>> from linearOrders import SlaterRanking
2 >>> sl = SlaterRanking(gcd,orderLimit=9)
3 >>> sl.slaterRanking
4  ['a5','a6','a4','a1','a3','a7','a8','a9','a2']

```

```

5 >>> corr = gcd.computeRankingCorrelation(sl.slaterRanking)
6 >>> sl.showCorrelation(corr)
7 Correlation indexes:
8   Extended Kendall tau      : +0.676
9   Epistemic determination   :  0.230
10  Bipolar-valued equivalence : +0.156
11 >>> len(sl.maximalRankings)
12    7

```

We notice in Listing 8.13 above that the SLATER ranking shown in Line 4 represents a rather good fit (+0.676), slightly better apparently than the NET-FLOWS ranking result (+0.638). However, there are in fact 7 such optimal SLATER rankings (see Line 12). The corresponding epistemic disjunction, again with the RankingsFusion class from the `transitiveDigraphs` module, gives the partial ordering shown in Figure 8.4 on the facing page:

**Listing 8.14** Computing the epistemic disjunction of optimal SLATER rankings

```

1 >>> from transitiveDigraphs import RankingsFusion
2 >>> slw = RankingsFusion(sl, sl.maximalRankings)
3 >>> slw.exportGraphViz(fileName='tutorialSlater')
4 *----- exporting a dot file for GraphViz tools -----*
5 Exporting to tutorialSlater.dot
6 dot -Grankdir=TB -Tpng tutorialSlater.dot\
7           -o tutorialSlater.png

```

The KEMENY and SLATER ranking rules are furthermore computationally *difficult* problems and effective ranking results are only computable for tiny outranking digraphs (< 20 objects) (see Eq. 8.1 on page 114).

More computationally efficient ranking heuristics, like the COPELAND and NET-FLOWS rules, are therefore needed in practice. Let us finally, after these *ranking-by-scoring* strategies, also present two popular *ranking-by-choosing* rules.

## 8.6 The KOHLER ranking-by-choosing rule

---

### Algorithm 8.3 The KOHLER ranking-by-choosing rule (Kohler, 1978)

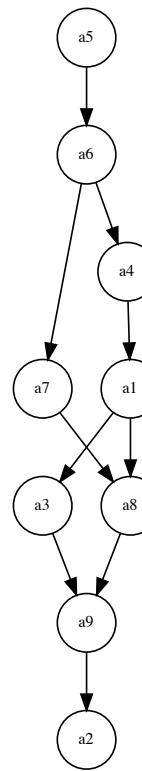
---

*in* : bipolar-valued digraph  $G(X, \succsim)$ ,  
*out* : linear ranking of the set  $X$  of decision actions.

At step  $i$  ( $i$  goes from 1 to  $n$ ) do the following:

1. Compute for each row of the bipolar-valued *strict* outranking relation table (see List. 8.1 on page 107) the smallest value;
  2. Select the row where this minimum is maximal. Ties are resolved in lexicographic order;
  3. Put the selected decision alternative at rank  $i$ ;
  4. Delete the corresponding row and column from the relation table and restart until the table is empty.
-

**Fig. 8.4** Epistemic disjunction of optimal SLATER rankings. What precise SLATER ranking result should we hence adopt?



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**Listing 8.15** Computing a KOHLER ranking

```

1 >>> from linearOrders import KohlerRanking
2 >>> kocd = KohlerRanking(gcd)
3 >>> kocd.showRanking()
4     ['a5','a7','a6','a3','a9','a8','a4','a1','a2']
5 >>> corr = gcd.computeOrdinalCorrelation(kocd)
6 >>> gcd.showCorrelation(corr)
7     Correlation indexes:
8         Extended Kendall tau      : +0.747
9         Epistemic determination   : 0.230
10        Bipolar-valued equivalence : +0.172
  
```

With this *min-max* lexicographic ranking-by-choosing strategy, we obtain a correlation result (+0.747) that is until now clearly the nearest to an optimal KEMENY ranking (see List. 8.8 on page 115). Only two adjacent pairs: (a6, a7) and (a8,

a9) are actually inverted here. Notice that the KOHLER ranking rule, contrary to the previously mentioned rules, is **not** invariant under the codual transform and requires to work on the strict outranking digraph  $g_{\text{CD}}$  for a better correlation result.

```
1 >>> ko = KohlerRanking(g)
2 >>> corr = g.computeOrdinalCorrelation(ko)
3 >>> g.showCorrelation(corr)
4     Correlation indexes:
5         Crisp ordinal correlation : +0.483
6         Epistemic determination :  0.230
7         Bipolar-valued equivalence : +0.111
```

But the KOHLER rule has a *dual* version, the prudent *Arrow-Raynaud* ordering-by-choosing rule, where a corresponding *max-min* strategy, when used on the *non-strict* dual outranking digraph  $-g$ , for ordering from *last* to *first* produces a similar ranking result (Arrow and Raynaud, 1986).

Noticing that the NETFLOWS score of an alternative  $x$  represents in fact a bipolar-valued characteristic of the assertion “*alternative  $x$  is ranked first*”, we may enhance the KOHLER rule by replacing the simple *min-max* strategy with an *iterated maximal* NETFLOWS score selection.

**Algorithm 8.4** The iterated NETFLOWS ranking-by-choosing rule

*in* : bipolar-valued digraph  $G(X, \succsim)$ ,  
*out* : linear ranking of the set  $X$  of  $n$  decision actions.

At step  $i$  ( $i$  goes from 1 to  $n$ ) do the following:

1. Compute for each row of the bipolar-valued outranking relation table (see List. 8.1 on page 107) the corresponding NETFLOWS score;
  2. Select the row where this score is *maximal*, ties being resolved by lexicographic order;
  3. Put the corresponding decision alternative at rank  $i$ ;
  4. Delete the corresponding row and column from the relation table and restart until the table is empty.

The `IteratedNetFlowsRanking` class from the `linearOrders` module computes this ranking result.

**Listing 8.16** Ranking-by-choosing with iterated maximal NETFLOWS scores

```

14             'name', 'actions', 'order',
15             'valuationdomain', 'relation',
16             'gamma', 'notGamma']
17 >>> inf.iteratedNetFlowsRanking
18     ['a5','a7','a6','a3','a4','a1','a8','a9','a2']
19 >>> corr = g.computeRankingCorrelation(\n
20 ...             inf.iteratedNetFlowsRanking)
21 >>> g.showCorrelation(corr)
22 Correlation indexes:
23     Crisp ordinal correlation : +0.743
24     Epistemic determination   : 0.230
25     Bipolar-valued equivalence : +0.171

```

Like the KOHLER rule, the iterated NETFLOWS rule has also a dual *ordering-by-choosing* version, where instead of selecting at each step  $i$  the row with maximal NETFLOWS score, we choose the row with the *minimal* NETFLOWS score. Both the ranking and ordering result are computed by the IteratedNetFlowsRanking class (see [Lines 12 on the preceding page-13 on the facing page](#) in [List. 8.16 on the preceding page](#)).

```

1 >>> inf.iteratedNetFlowsOrdering
2     ['a2','a9','a1','a4','a3','a8','a7','a6','a5']
3 >>> corr = g.computeOrderCorrelation(\n
4 ...             inf.iteratedNetFlowsOrdering)
5 >>> g.showCorrelation(corr)
6 Correlation indexes:
7     Crisp ordinal correlation : +0.751
8     Epistemic determination   : 0.230
9     Bipolar-valued equivalence : +0.173

```

The iterated NETFLOWS ranking and its dual, the iterated NETFLOWS ordering, do not usually deliver both the same result. In our example outranking digraph  $g$  for instance, it is the *ordering-by-choosing* result that obtains a slightly better correlation with the given outranking digraph (+0.751), a result that is also slightly better than the original KOHLER ranking result (+0.747, see [List. 8.15 on page 119 Line 8 on page 119](#)).

With different *ranking-by-choosing* and *ordering-by-choosing* results, it may be useful to *fuse* now, similar to what we have done before with KEMENY's and SLATER's optimal rankings, both, the iterated NETFLOWS ranking and ordering into a partial ranking. But we are hence back to the practical problem of what linear ranking should we eventually retain?

Let us finally mention another interesting *ranking-by-choosing* approach.

## 8.7 The RANKEDPAIRS ranking rule

*N. Tideman*'s ranking-by-choosing heuristic, the RANKEDPAIRS rule, working best this time on the non strict outranking digraph  $g$ , is based on a *prudent incremental* construction of linear orders that avoids on the fly any cycling outranking situations ([Tideman, 1987](#)).

**Algorithm 8.5** The RANKEDPAIRS ranking rule

---

*in* : bipolar-valued digraph  $G(X, \succsim)$ ,  
*out* : linear ranking of the set  $X$  of decision actions.

1. Rank the ordered pairs  $(x, y)$  of alternatives in decreasing order of  $r(x \succsim y) + r(y \not\succsim x)$ ;
  2. Consider the pairs in that order (ties are resolved by a lexicographic rule):
    - if the next pair does not create a *circuit* with the pairs already blocked, block this pair;
    - if the next pair creates a *circuit* with the already blocked pairs, skip it.
- 

With our didactic outranking digraph  $g$ , we get the following result.

**Listing 8.17** Computing a RANKEDPAIRS ranking

```

1 >>> from linearOrders import RankedPairsRanking
2 >>> rp = RankedPairsRanking(g)
3 >>> rp.showRanking()
4 ['a5','a6','a7','a3','a8','a9','a4','a1','a2']

```

The RANKEDPAIRS rule renders in our example here luckily one of the two optimal KEMENY ranking, as we may verify below.

```

1 >>> ke.maximalRankings
2 [[['a5','a6','a7','a3','a8','a9','a4','a1','a2'],
3  ['a5','a6','a7','a3','a9','a4','a1','a8','a2']]]
4 >>> corr = g.computeOrdinalCorrelation(rp)
5 >>> g.showCorrelation(corr)
6 Correlation indexes:
7   Extended Kendall tau      : +0.779
8   Epistemic determination   : 0.230
9   Bipolar-valued equivalence : +0.179

```

Similar to KOHLER's rule, the RANKEDPAIRS rule has also a prudent dual version, the *Dias-Lamboray ordering-by-choosing* rule, which produces, when working this time on the dual outranking digraph  $-g$  a similar ranking result (see Lamboray, 2009; Dias and Lamboray, 2010).

Besides of not providing a unique linear ranking, the *ranking-by-choosing* rules, as well as their duals, the *ordering-by-choosing* rules, are unfortunately not scalable to larger outranking digraphs (of orders  $> 100$ ). For such bigger outranking digraphs, with several hundred or thousands of alternatives, only the COPELAND and the NETFLOWS *ranking-by-scoring* rules, with a polynomial complexity of  $O(n^2)$ , where  $n$  is the order of the outranking digraph, remain in fact tractable. Furthermore, as computing the COPELAND and NETFLOWS scores may be done separately per alternative, the latter ranking rules can right away be processed in parallel when multiprocessing resources are available.

The physical necessity to write down a list of items in a linear sequence renders the ranking decision problem very important in practice. However, a relative rating of such items into performance quantiles classes would be, from the very preference modelling perspective, more expressive and faithful. This is the subject of the following Chapter 9.

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# Chapter 9

## Rating by sorting into relative performance quantiles

**Abstract** In this chapter, we apply order statistics for sorting a set  $X$  of  $n$  potential decision alternatives, evaluated on  $m$  incommensurable performance criteria, into  $q$  quantile equivalence classes. The sorting algorithm is based on pairwise outranking characteristics involving the quantile class limits observed on each criterion. Thus we may implement a weak ordering algorithm of complexity  $O(nmq)$ .

### 9.1 Quantile sorting on a single performance criterion

A single criterion sorting category  $K$  is a (usually) lower-closed interval  $[m_k; M_k[$  on a real-valued performance measurement scale, with  $m_k \leq M_k$ . If  $x$  is a measured performance on this scale, we may distinguish three sorting situations:

1.  $x < m_k$  and  $x < M_k$ : The performance  $x$  is lower than category  $K$ .
2.  $x \geq m_k$  and  $x < M_k$ : The performance  $x$  belongs to category  $K$ .
3.  $x > m_k$  and  $x \geq M_k$ : The performance  $x$  is higher than category  $K$ .

As the relation  $<$  is the dual of  $\geq$  ( $\geq$ ), it will be sufficient to check that  $x \geq m_k$  as well as  $x \not\geq M_k$  are true for  $x$  to be considered a member of category  $K$ .

Upper-closed categories (in a more mathematical integration style) may as well be considered. In this case it is sufficient to check that  $m_k \not\geq x$  as well as  $M_k \geq x$  are true for  $x$  to be considered a member of category  $K$ . It is worthwhile noticing that a category  $K$  such that  $m_k = M_k$  is hence always empty by definition. In order to be able to properly sort over the complete range of values to be sorted, we will need to use a special, two-sided closed last, respectively first, category.

Let  $K = K_1, \dots, K_q$  be a non trivial partition of the criterion's performance measurement scale into  $q \geq 2$  ordered categories  $K_k$  – i.e. lower-closed intervals  $[m_k; M_k[$  – such that  $m_k < M_k$ ,  $M_k = m_{k+1}$  for  $k = 0, \dots, q-1$  and  $M_q = \infty$ . And, let  $A = \{a_1, a_2, a_3, \dots\}$  be a finite set of not all equal performance measures observed on the scale in question.

**Property:** For all performance measure  $x \in A$  there exists now a unique  $k$  such that  $x \in K_k$ . If we assimilate, like in descriptive statistics, all the measures gathered

in a category  $K_k$  to the central value of the category – i.e.  $(m_k + M_k)/2$  – the sorting result will hence define a weak order (complete preorder) on  $A$ .

Let  $Q = \{Q_0, Q_1, \dots, Q_q\}$  denote the set of  $q+1$  increasing order-statistical quantiles –like quartiles or deciles– we may compute from the ordered set  $A$  of performance measures observed on a performance scale. If  $Q_0 = \min(X)$ , the following intervals:  $[Q_0; Q_1], [Q_1; Q_2], \dots, [Q_{q-1}; \infty]$  define a set of  $q$  lower-closed sorting categories. And, in the case of upper-closed categories, if  $Q_q = \max(X)$ , we obtain the intervals  $]-\infty; Q_1], ]Q_1; Q_2], \dots, ]Q_{q-1}; Q_q]$ . The corresponding sorting of  $A$  will result, in both cases, in a repartition of all measures  $x$  into the  $q$  quantile categories  $K_k$  for  $k = 1, \dots, q$ .

**Example:** Let  $A = \{a_7 = 7.03, a_{15} = 9.45, a_{11} = 20.35, a_{16} = 25.94, a_{10} = 31.44, a_9 = 34.48, a_{12} = 34.50, a_{13} = 35.61, a_{14} = 36.54, a_{19} = 42.83, a_5 = 50.04, a_2 = 59.85, a_{17} = 61.35, a_{18} = 61.61, a_3 = 76.91, a_6 = 91.39, a_1 = 91.79, a_4 = 96.52, a_8 = 96.56, a_{20} = 98.42\}$  be a set of 20 increasing performance measures observed on a given criterion. The lower-closed category limits we obtain with quartiles ( $q = 4$ ) are:  $Q_0 = 7.03 = a_7, Q_1 = 34.485, Q_2 = 54.945$  (median performance), and  $Q_3 = 91.69$ . And the sorting into these four categories defines on  $A$  a complete preorder with the following four equivalence classes:  $K_1 = \{a_7, a_{10}, a_{11}, a_{10}, a_{15}, a_{16}\}$ ,  $K_2 = \{a_5, a_9, a_{13}, a_{14}, a_{19}\}$ ,  $K_3 = \{a_2, a_3, a_6, a_{17}, a_{18}\}$ , and  $K_4 = \{a_1, a_4, a_8, a_{20}\}$ .

## 9.2 Sorting into quantiles with multiple performance criteria

Let us now suppose that we are given a performance tableau with a set  $X$  of  $n$  decision alternatives evaluated on a coherent family of  $m$  performance criteria associated with the corresponding outranking relation  $\succsim$  defined on  $X$ . We denote  $x_j$  the performance of alternative  $x$  observed on criterion  $j$ .

Suppose furthermore that we want to sort the decision alternatives into  $q$  upper-closed quantile equivalence classes. We therefore consider a series :  $k = k/q$  for  $k = 0, \dots, q$  of  $q+1$  equally spaced quantiles, like quartiles: 0, 0.25, 0.5, 0.75, 1; quintiles: 0, 0.2, 0.4, 0.6, 0.8, 1; or deciles: 0, 0.1, 0.2, ..., 0.9, 1, for instance.

The upper-closed  $\mathbf{q}^k$  class corresponds to the  $m$  quantile intervals  $]q_j(p_{k-1}); q_j(p_k)]$  observed on each criterion  $j$ , where  $k = 2, \dots, q$ ,  $q_j(p_q) = \max_X(x_j)$ , and the first class gathers all performances below or equal to  $Q_j(p_1)$ .

The lower-closed  $\mathbf{q}_k$  class corresponds to the  $m$  quantile intervals  $[q_j(p_{k-1}); q_j(p_k)[$  observed on each criterion  $j$ , where  $k = 1, \dots, q-1$ ,  $q_j(p_0) = \min_X(x_j)$ , and the last class gathers all performances above or equal to  $Q_j(p_{q-1})$ .

We call **q-tiles** a complete series of  $k = 1, \dots, q$  upper-closed  $\mathbf{q}^k$ , respectively lower-closed  $\mathbf{q}_k$ , multiple-criteria quantile classes.

**Property:** With the help of the bipolar-valued characteristic of the outranking relation  $r(x \succsim y)$  we may compute as follows the bipolar-valued characteristic of the assertion:  $x$  belongs to upper-closed  $q$ -tiles class  $\mathbf{q}^k$  class, resp. lower-closed class  $\mathbf{q}_k$  (Bisdorff, 2020):

$$r(x \in \mathbf{q}^k) = \min [-r(\mathbf{q}(p_{q-1}) \succsim x), r(\mathbf{q}(p_q) \succsim x)] \quad (9.1)$$

$$r(x \in \mathbf{q}_k) = \min [r(x \succsim \mathbf{q}(p_{q-1})), -r(x \succsim \mathbf{q}(p_q))] \quad (9.2)$$

The `min` operator implements the logical conjunction and, the outranking relation  $\succsim$  verifying the coduality principle,  $-r(\mathbf{q}(p_{q-1}) \succsim x) = r(\mathbf{q}(p_{q-1}) \prec x)$ , resp.  $-r(x \succsim \mathbf{q}(p_q)) = r(x \prec \mathbf{q}(p_q))$ .

The `QuantilesSortingDigraph` class from the `sortingDigraphs` module can compute, for instance, such a quintiling of a given random performance tableau (Bisdorff, 2021).

**Listing 9.1** Computing a quintiles sorting result

```

1 >>> from randomPerfTabs import RandomPerformanceTableau
2 >>> t = RandomPerformanceTableau(numberOfActions=50,seed=5)
3 >>> from sortingDigraphs import QuantilesSortingDigraph
4 >>> qs = QuantilesSortingDigraph(t,limitingQuantiles=5)
5 >>> qs
6     *----- Object instance description -----*
7     Instance class   : QuantilesSortingDigraph
8     Instance name    : sorting_with_5-tile_limits
9     Actions          : 50
10    Criteria         : 7
11    Categories       : 5
12    Lowerclosed      : False
13    Size              : 841
14    Valuation domain : [-1.00;1.00]
15    Determinateness (%) : 81.39
16    Attributes        : ['actions','actionsOrig',
17                          'criteria','evaluation','runTimes','name',
18                          'limitingQuantiles','LowerClosed',
19                          'categories','criteriaCategoryLimits',
20                          'profiles','profilelimits','hasNoVeto',
21                          'valuationdomain','nbrThreads','relation',
22                          'categoryContent','order','gamma','notGamma']
23    *----- Constructor run times (in sec.) -----*
24    # Threads        : 1
25    Total time       : 0.03120
26    Data input       : 0.00300
27    Compute profiles : 0.00075
28    Compute relation : 0.02581
29    Weak Ordering    : 0.00052

```

5-tiling the 50 decision alternatives results in 5 ordered upper-closed categories of decision alternatives –the quintiles– supported by a 81.39% majority of criteria significance.

The `showCriteriaQuantileLimits()` method shows in Listing 9.2 the quintile limits separating on each criterion the respective evaluations of the 50 decision alternatives into the 5 quintiles. Highest evaluations are observed on criterion `g1` (see Line 6 below).

**Listing 9.2** Inspecting the quantile limits

```

1 >>> qs.showCriteriaQuantileLimits()
2     Quantile Class Limits (q = 5)
3     Upper-closed classes
4     crit.          0.20      0.40      0.60      0.80      1.00
5     *-----
6     g1            31.35    41.09    58.53    71.91    98.08
7     g2            27.81    39.19    49.87    61.66    96.18

```

8	g3	25.10	34.78	49.45	63.97	92.59
9	g4	24.61	37.91	53.91	71.02	89.84
10	g5	26.94	36.43	52.16	72.52	96.25
11	g6	23.94	44.06	54.92	67.34	95.97
12	g7	30.94	47.40	55.46	69.04	97.10

And with the `showSorting()` method we can inspect the actual quintiles sorting result.

#### Listing 9.3 Computing a quintiles sorting result

```

1 >>> qs.showSorting()
2     *--- Sorting results in descending order ---*
3     ]0.80 - 1.00]: ['a22']
4     ]0.60 - 0.80]: ['a03','a07','a08','a11','a14','a17',
5                         'a19','a20','a29','a32','a33','a37',
6                         'a39','a41','a42','a49']
7     ]0.40 - 0.60]: ['a01','a02','a04','a05','a06','a08',
8                         'a09','a16','a17','a18','a19','a21',
9                         'a24','a27','a28','a30','a31','a35',
10                        'a36','a40','a43','a46','a47','a48',
11                        'a49','a50']
12    ]0.20 - 0.40]: ['a04','a10','a12','a13','a15','a23',
13                         'a25','a26','a34','a38','a43','a44',
14                         'a45','a49']
15    ] < - 0.20]: ['a44']

```

Most of the decision alternatives (26) are gathered in the median quintile  $]0.40 - 0.60]$  class, whereas the highest quintile  $]0.80 - 1.00]$  and the lowest quintile  $] < -0.20]$  classes gather each one a unique decision alternative (a22, resp. a44).

Inspecting the details of the corresponding sorting characteristics may be done with the `showSortingCharacteristics()` method.

#### Listing 9.4 Bipolar-valued sorting characteristics (extract)

```

1 >>> qs.valuationdomain
2     {'min': Decimal('-1.0'), 'med': Decimal('0'),
3      'max': Decimal('1.0')}
4 >>> qs.showSortingCharacteristics()
5     x in q^k           r(q^{k-1} < x)   r(q^k >= x)   r(x in q^k)
6     a22 in ]< - 0.20]   1.00          -0.86          -0.86
7     a22 in ]0.20 - 0.40]   0.86          -0.71          -0.71
8     a22 in ]0.40 - 0.60]   0.71          -0.71          -0.71
9     a22 in ]0.60 - 0.80]   0.71          -0.14          -0.14
10    a22 in ]0.80 - 1.00]   0.14          1.00          0.14
11    ...
12    ...
13    a44 in ]< - 0.20]   1.00          0.00          0.00
14    a44 in ]0.20 - 0.40]   0.00          0.57          0.00
15    a44 in ]0.40 - 0.60]  -0.57          0.86          -0.57
16    a44 in ]0.60 - 0.80]  -0.86          0.86          -0.86
17    a44 in ]0.80 - 1.00]  -0.86          0.86          -0.86
18    ...
19    ...
20    a49 in ]< - 0.20]   1.00          -0.43          -0.43
21    a49 in ]0.20 - 0.40]   0.43          0.00          0.00
22    a49 in ]0.40 - 0.60]   0.00          0.00          0.00
23    a49 in ]0.60 - 0.80]   0.00          0.57          0.00
24    a49 in ]0.80 - 1.00]  -0.57          0.86          -0.57

```

In Listing 9.4 on the preceding page Line 10, alternative a22 verifies indeed positively both sorting conditions only for the highest quintile [0.80 – 1.00] class. Whereas alternatives a44 and a49, for instance, weakly verify both sorting conditions for two, resp. three, adjacent quintile classes (Lines 13-14 and 21-23).

Quantiles sorting results, indeed, verify always the following properties (Bisdorff, 2020):

*Property 9.1 (Formal properties of a quantiles sorting result).*

1. *Coherence*: Each object is sorted into a non-empty subset of *adjacent q-tiles* classes. An alternative that would *miss* evaluations on all the criteria will be sorted conjointly in all *q-tiled* classes.
2. *Uniqueness*: When  $r(x \in \mathbf{q}^k) \neq 0.0$  for  $k = 1, \dots, q$ , the performance  $x$  is sorted into *exactly one single q-tiled* class.
3. *Separability*: Computing the sorting result for performance  $x$  is *independent* from the computing of the other performances' sorting results. This property gives access to efficient parallel processing of class membership characteristics.

The *q-tiles* sorting result leaves us hence with more or less *overlapping* ordered quantile equivalence classes. For constructing now a linearly ranked *q-tiles* partition of  $X$ , we may apply three strategies:

1. *Average* (default): In decreasing lexicographic order of the average of the lower and upper quantile limits and the upper quantile class limit;
2. *Optimistic*: In decreasing lexicographic order of the upper and lower quantile class limits;
3. *Pessimistic*: In decreasing lexicographic order of the lower and upper quantile class limits;

**Listing 9.5** Weakly ranking the quintiles sorting result

```

1 >>> qs.showQuantileOrdering(strategy='average')
2     ]0.80-1.00] : ['a22']
3     ]0.60-0.80] : ['a03','a07','a11','a14','a20','a29',
4                     'a32','a33','a37','a39','a41','a42']
5     ]0.40-0.60] : ['a08','a17','a19']
6     ]0.20-0.80] : ['a49']
7     ]0.40-0.60] : ['a01','a02','a05','a06','a09','a16',
8                     'a18','a21','a24','a27','a28','a30',
9                     'a31','a35','a36','a40','a46','a47',
10                    'a48','a50']
11    ]0.20-0.60] : ['a04','a43']
12    ]0.20-0.40] : ['a10','a12','a13','a15','a23','a25',
13                     'a26','a34','a38','a45']
14    ] < -0.40] : ['a44']
```

Following, for instance, the *average* ranking strategy, we find confirmed in the weak ranking shown in Listing 9.5 that alternative a49 is indeed sorted into three adjacent quintiles classes, namely ]0.20 – 0.80] (see Line 6) and precedes the ]0.40 – 0.60] class, of same average of lower and upper limits.

The QuantilesSortingDigraph class constructor models hence a linearly ordered decomposition of the corresponding bipolar-valued outranking digraph. This decomposition leads us to a new *sparse pre-ranked* outranking digraph model.

### 9.3 The sparse pre-ranked outranking digraph model

Following the methodological requirements of the outranking approach, a given outranking digraph is, necessarily associated with a corresponding performance tableau (see Chap. 3). And, we can use this associated performance tableau for linearly decomposing the set of potential decision alternatives into ordered quantiles equivalence classes by using the quantiles sorting algorithm seen in the previous Section.

In the coding example shown in Listing 9.6 below, we generate, first (Lines 2-3), a simple performance tableau of 75 decision alternatives and, secondly (Lines 4-6), construct with the PreRankedOutrankingDigraph class from sparseOutrankingDigraphs module the corresponding sparse outranking digraph called prg. Notice by the way the BigData flag (Line 3) used for generating a parsimoniously commented performance tableau (Bisdorff, 2021).

**Listing 9.6** Computing a *pre-ranked* sparse outranking digraph

```

1 >>> from randomPerfTabs import RandomPerformanceTableau
2 >>> tp = RandomPerformanceTableau(numberOfActions=75,\n3 ...                                BigData=True, seed=100)
4 >>> from sparseOutrankingDigraphs import \
5 ...                                PreRankedOutrankingDigraph
6 >>> prg = PreRankedOutrankingDigraph(Tp, quantiles=5)
7 >>> prg
8      ----- Object instance description -----
9      Instance class      : PreRankedOutrankingDigraph
10     Instance name       : randomperftab_pr
11     Actions             : 75
12     Criteria            : 7
13     Sorting by          : 5-Tiling
14     Ordering strategy   : average
15     Components          : 9
16     Minimal order       : 1
17     Maximal order       : 25
18     Average order       : 8.3
19     fill rate           : 20.432%
20     Attributes          :
21     ['actions','criteria','evaluation','NA','name',
22     'order','runTimes','dimension','sortingParameters',
23     'valuationdomain','profiles','categories','sorting',
24     'decomposition','nbrComponents','components',
25     'fillRate','minimalComponentSize','maximalComponentSize',
... ]

```

The ordering of the 5-tiling result is following the *average* lower and upper quintile limits strategy (see List. 9.6 Line 14). We obtain here 9 ordered components of

minimal order 1 and maximal order 25. The corresponding *pre-ranked decomposition* may be inspected as follows.

**Listing 9.7** The quantiles decomposition of a pre-ranked outranking digraph

```

1 >>> prg.showDecomposition()
2     *---- quantiles decomposition in decreasing order---*
3     c1. ]0.80-1.00] : [5, 42, 43, 47]
4     c2. ]0.60-1.00] : [73]
5     c3. ]0.60-0.80] : [1, 4, 13, 14, 22, 32, 34, 35, 40,
6                     41, 45, 61, 62, 65, 68, 70, 75]
7     c4. ]0.40-0.80] : [2, 54]
8     c5. ]0.40-0.60] : [3, 6, 7, 10, 15, 18, 19, 21, 23, 24,
9                     27, 30, 36, 37, 48, 51, 52, 56, 58,
10                    63, 67, 69, 71, 72, 74]
11    c6. ]0.20-0.60] : [8, 11, 25, 28, 64, 66]
12    c7. ]0.20-0.40] : [12, 16, 17, 20, 26, 31, 33, 38, 39,
13                     44, 46, 49, 50, 53, 55]
14    c8. ] -<-0.40] : [9, 29, 60]
15    c9. ] -<-0.20] : [57, 59]
```

The highest quintile class ( $]80\% - 100\%]$ ) contains decision alternatives 5, 42, 43 and 47. Lowest quintile class ( $]-20\%]$ ) gathers alternatives 57 and 59 (see List 9.7 Lines 3 and 15). The resulting sparse outranking relation is shown with the `showHTMLRelationMap()` method in a heatmap browser view.

```

1 >>> prg.showHTMLRelationMap()
```

In Figure 9.1 on the following page we easily recognise the 9 linearly ordered quantile equivalence classes. *Green* and *light-green* pixels show positive *outranking* situations, whereas positive *outranked* –negative *outranking*– situations are shown in *red* and *light-red*. Indeterminate situations appear in white. In each one of the 9 quantile equivalence classes we recover the corresponding bipolar-valued outranking *sub-relation*, which leads to an actual *fill-rate* of 20.4% (see List 9.4 on page 128 Line 19).

Computing the ordinal correlation between the sparse and the standard outranking digraph allows to check how faithful the sparse model represents the complete outranking relation.

```

1 >>> g = BipolarOutrankingDigraph(tp)
2 >>> corr = prg.computeOrdinalCorrelation(g)
3 >>> g.showCorrelation(corr)
4   Correlation indexes:
5     Crisp ordinal correlation : +0.863
6     Epistemic determination   : 0.315
7     Bipolar-valued equivalence : +0.272
```

The ordinal correlation index between the standard and the sparse outranking digraphs is quite high (+0.863) and their bipolar-valued equivalence is supported by a mean criteria significance majority of  $(1.0 + 0.272)/2 = 64\%$ .

It is worthwhile noticing in Listing 9.6 on the facing page (Lines 20-) that sparse pre-ranked outranking digraphs do not contain a `relation` attribute. The access



**Fig. 9.1** The relation map of a sparse outranking digraph

to pairwise outranking characteristic values is here provided via a corresponding `relation()` function.

**Listing 9.8** Functional binary relation characteristics

```

1 def relation(self,x,y):
2     """
3         Dynamic construction of the global
4         outranking characteristic function r(x,y).
5     """
6     Min = self.valuationdomain['min']
7     Med = self.valuationdomain['med']
8     Max = self.valuationdomain['max']
9     if x == y:
10         return Med

```

```

11     cx = self.actions[x]['component']
12     cy = self.actions[y]['component']
13     if cx == cy:
14         return self.components[cx]['subGraph'].relation[x][y]
15     elif self.components[cx]['rank'] >
16         self.components[cy]['rank']:
17         return Min
18     else:
19         return Max

```

In Listing 9.8 on the preceding page Lines 9-10, all reflexive situations are set to the *indeterminate* value. When two decision alternatives belong to a same component –quantile equivalence class– we access the relation attribute of the corresponding outranking sub-digraph (Lines 13-14). Otherwise we just check the respective ranks of the components (Lines 15-18).

## 9.4 Ranking pre-ranked sparse outranking digraphs

Each one of these nine ordered components is locally ranked by using a suitable ranking rule. Best operational results, both in run times and quality, are more or less equally obtained with the COPELAND and the NETFLOWS rules (see Sec. 8.2 and Sec. 8.3). The eventually obtained linear ordering (from the worst to best) is stored in a `prg.boostedOrder` attribute. A reversed linear order (from the best to the worst) is stored in a `prg.boostedRanking` attribute.

```

1 >>> prg.boostedRanking
2 [43, 47, 42, 5, 73, 65, 68, 32, 62, 70, 35, 22, 75, 45, 1,
3   61, 41, 34, 4, 13, 40, 14, 2, 54, 63, 37, 56, 71, 69, 36,
4   19, 72, 15, 48, 6, 30, 74, 3, 21, 58, 52, 18, 7, 24, 27,
5   23, 67, 51, 10, 25, 11, 8, 64, 28, 66, 53, 12, 31, 39, 55,
6   20, 46, 49, 16, 44, 26, 38, 33, 17, 50, 29, 60, 9, 59, 57]

```

Alternative 43 appears *first-ranked*, whereas alternative 57 is *last-ranked* (see Line 2 and 6 above).

The quality of this ranking result may be assessed by computing with the `computeRankingCorrelation()` method its ordinal correlation index with the standard outranking digraph `g`.

```

1 >>> corr = g.computeRankingCorrelation(prg.boostedRanking)
2 >>> g.showCorrelation(corr)
3 Correlation indexes:
4   Crisp ordinal correlation : +0.807
5   Epistemic determination   : 0.315
6   Bipolar-valued equivalence : +0.254

```

One may also verify below that the COPELAND ranking obtained, for instance, from the standard outranking digraph is as well highly correlated (+0.822) with the one obtained from the sparse outranking digraph.

```

1 >>> from linearOrders import CopelandOrder

```

```

2     >>> cop = CopelandOrder(g)
3     >>> print(cop.computeRankingCorrelation(prg.boostedRanking))
4     {'correlation': 0.822, 'determination': 1.0}

```

Noticing the computational efficiency of the quantiles sorting construction, coupled with the separability property of the quantile class membership characteristics computation, we will make usage of the `PreRankedOutrankingDigraph` constructor in Chapter 11 for HPC ranking big and even huge performance tableaux (Bisdorff, 2016). The next Chapter 10 addresses now the problem of absolute rating into learned quantile performance norms.

## References

- Bisdorff R (2016) Computing linear rankings from trillions of pairwise outranking situations. In: Busa-Fekete R, Hüllermeier E, Mousseau V, Pfannschmidt K (eds) From Multiple Criteria Decision Aid to Preference Learning , DAP'2016, University of Paderborn (Germany), pp 1–16, URL <http://hdl.handle.net/10993/28613> 134
- Bisdorff R (2020) Lecture 10: On quantiles rating with multiple incommensurable criteria. In: Lectures of the Algorithmic Decision Theory Course, University of Luxembourg, URL <http://hdl.handle.net/10993/37933> 126, 129
- Bisdorff R (2021) Technical documentation of the Digraph3 collection of Python modules. URL <https://digraph3.readthedocs.io/en/latest/techDoc.html> 127, 130

# Chapter 10

## Rating by ranking with learned performance quantile norms

**Abstract** We address in Chapter 10 the problem of rating multiple-criteria performances of a set of potential decision alternatives with respect to performance quantiles learned from historical performance data gathered from similar decision alternatives observed in the past. We show how to learn performance quantiles from historical performance tableaux. New performance records may now be rated with respect to these quantile norms.

### 10.1 The absolute rating problem

To illustrate the *absolute rating* problem we face, consider for a moment that, in a given decision making problem we observe, for instance, in Table 10.1 below, the multi-criteria performance evaluations of two potential decision alternatives, named a1001 and a1010, evaluated on 7 *incommensurable* performance criteria: 2 *Costs* criteria c1 and c2 (to *minimise*) and 5 *Benefits* criteria b1 to b5 (to *maximise*).

**Table 10.1** Multi-criteria performances of two potential decision alternatives

Criterion (weight)	b1 (2)	b2 (2)	b3 (2)	b4 (2)	b5 (2)	c1 (5)	c2 (5)
a1001	37.0	2	2	61.0	31.0	-4	-40.0
a1010	32.0	9	6	55.0	51.0	-4	-35.0

The performance on *Benefits* criteria b1, b4 and b5 is measured on a cardinal scale from 0.0 (worst) to 100.0 (best) whereas, the performance on the *Benefits* criteria b2 and b3 and on the *Costs* criterion c1 is measured on an ordinal scale from 0 (worst) to 10 (best), respectively -10 (worst) to 0 (best). The performance on the *Costs* criterion c2 is eventually measured on a cardinal *negative* scale from -100.00 (worst) to 0.0 (best). The two *Costs* criteria are equi-significant (weight

5). Similarly, the five Benefits criteria are also equi-significant (weight 2). The *importance* (sum of significance weights:  $2 \times 5 = 10$ ) of the *Costs* criteria is hence equivalent to the *importance* (sum of significance weights:  $5 \times 2 = 10$ ) of the *Benefits* criteria.

The non trivial decision problem we now face, is to decide, how the previous two multiple-criteria performance records of alternatives a1001, respectively a1010, may be rated: *excellent* ?, *good* ?, or *fair* ?; perhaps even, *weak* ? or *very weak* ? when compared with similar multi-criteria performance records one has already rated into quantiles in the past.

To solve this *absolute* rating problem, first, we need to estimate multi-criteria *performance quantiles* from historical records ([Bisdorff, 2020](#)).

## 10.2 Incremental learning of historical performance quantiles

Suppose that we see flying in random multiple-criteria performances from a given model of random performance tableau (see Chap. 5). The question we address here is to estimate empirical performance quantiles on the basis of so far observed performance vectors. For this task, we are inspired by [Chambers, James, Lambert, and Vander Wiel \(2006\)](#), who present an efficient algorithm for incrementally updating a quantile-binned cumulative distribution function (CDF) with newly observed CDFs<sup>1</sup>.

The `PerformanceQuantiles` class, using the `IncrementalQuantilesEstimator` class from the `randomNumbers` module, implements such a performance quantiles estimation based on a given performance tableau ([Bisdorff, 2021](#)).

Its main attributes are:

- Ordered `objectives` and a `criteria` dictionaries from a valid performance tableau instance;
- A list `quantileFrequencies` of quantile frequencies like:
  - `quartiles` [0.0, 0.25, 0.5, 0.75, 1.0],
  - `quintiles` [0.0, 0.2, 0.4, 0.6, 0.8, 1.0] or
  - `deciles` [0.0, 0.1, 0.2, ... 1.0] for instance;
- An ordered dictionary `limitingQuantiles` of so far estimated *lower* (default) or *upper* quantile class limits for each frequency per criterion;
- An ordered dictionary `historySizes` for keeping track of the number of evaluations seen so far per criterion. Missing data, the case given, make these sizes vary from criterion to criterion.

Below, we show an example Python session concerning 900 decision alternatives randomly generated with a *Cost-Benefit* performance tableau model (see Sec. 5.3) from which are also drawn the performances of alternatives a1001 and a1010 shown in Table 10.1 on the previous page above.

---

<sup>1</sup> We have adapted in Python a C++ implementation published by ([Press, Teukolsky, Vetterling, and Flannery, 2007](#), Chapter 5).

**Listing 10.1** Computing performance quantiles from a given performance tableau

```

1 >>> from performanceQuantiles import PerformanceQuantiles
2 >>> from randomPerfTabs import RandomCBPerformanceTableau
3 >>> nbrActions=900
4 >>> nbrCrit = 7
5 >>> seed = 100
6 >>> pt = RandomCBPerformanceTableau(numberOfActions=nbrActions,\n7 ...                               numberOfCriteria=nbrCrit,seed=seed)
8 >>> pq = PerformanceQuantiles(pt,\n9 ...                               numberOfBins = 'quartiles',\n10 ...                             LowerClosed=True)
11 >>> pq
12 *----- PerformanceQuantiles instance description -----*
13     Instance class      : PerformanceQuantiles
14     Instance name       : 4-tiled_performances
15     Objectives          : 2
16     Criteria            : 7
17     Quantiles           : 4
18     History sizes       : {'c1': 887,'b1': 888,'b2': 891,'b3': 895,
19                           'b4': 892,'c2': 893,'b5': 887}
20     Attributes          : ['perfTabType','valueDigits',
21                           'actionsTypeStatistics',
22                           'objectives', 'BigData',
23                           'missingDataProbability',
24                           'criteria', 'LowerClosed', 'name',
25                           'quantilesFrequencies', 'historySizes',
26                           'limitingQuantiles', 'cdf']

```

The `PerformanceQuantiles` class parameter `numberOfBins` (see List. 10.1 Line 9 above), choosing the wished number of quantile frequencies, may be either *quartiles* (4 bins), *quintiles* (5 bins), *deciles* (10 bins), *dodeciles* (20 bins) or any other integer number of quantile bins. The quantile bins may be either *lower closed* (default) or *upper-closed*.

Inspecting the estimated quartile limits may be operated with the `showLimitingQuantiles()` method.

**Listing 10.2** Printing out the estimated quartile limits

```

1 >>> pq.showLimitingQuantiles(ByObjectives=True)
2     ---- Historical performance quantiles -----
3     Costs
4     criteria | weights | '0.00' '0.25' '0.50' '0.75' '1.00'
5     -----|-----
6     'c1'   |    5    | -10     -7     -5     -3     0
7     'c2'   |    5    | -96.37 -70.65 -50.10 -30.00 -1.43
8     Benefits
9     criteria | weights | '0.00' '0.25' '0.50' '0.75' '1.00'
10    -----|-----
11    'b1'   |    2    | 1.99    29.82  49.44  70.73  99.83
12    'b2'   |    2    | 0        3       5       7       10
13    'b3'   |    2    | 0        3       5       7       10
14    'b4'   |    2    | 3.27    30.10  50.82  70.89  98.05
15    'b5'   |    2    | 0.85    29.08  48.55  69.98  97.56

```

Both objectives are *equally important*; the sum of weights (10) of the *Costs* criteria balance the sum of weights (10) of the *Benefits* criteria (see List. 10.2 on the previous page column 2). The preference direction of the *Costs* criteria  $c_1$  and  $c_2$  is *negative*; the lesser the costs, the better it is, whereas all the *Benefits* criteria  $b_1$  to  $b_5$  show *positive* preference directions, i.e. the higher the benefits, the better it is. The columns entitled '0.00', resp. '1.00' show the *quartile*  $Q_0$ , resp.  $Q_4$ , i.e. the *worst*, resp. *best* performance observed so far on each criterion. Column '0.50' shows the *median* ( $Q_2$ ) performance observed on the criteria.

New decision alternatives with random multiple-criteria performance vectors from the same random performance tableau model as  $pt$  (see List. 10.1 on the preceding page) may now be generated with a generic `RandomPerformanceGenerator` class from the `randomPerfTabs` module (Bisdorff, 2021).<sup>2</sup>

**Listing 10.3** Generating 100 new random decision alternatives of the same model

```
1 >>> from randomPerfTabs import RandomPerformanceGenerator
2 >>> rpg = RandomPerformanceGenerator(tp, seed=seed)
3 >>> newTab = rpg.randomPerformanceTableau(100)
```

The so far estimated historical quantile limits must, first, be updated with this newly arriving 100 data records:

```
1 >>> # Updating the quartile norms shown above
2 >>> pq.updateQuantiles(newTab, historySize=None)
```

Parameter `historySize` of the `updateQuantiles()` method (Line 2 above) allows to *balance* the *new* evaluations against the *historical* ones.

With `historySize = None` (the default setting), the balance in the example above is 900/1000 (90%, the weight of historical data) against 100/1000 (10%, the weight of the new incoming observations). Setting `historySize = 0`, for instance, will ignore all historical data (0/100 against 100/100) and restart building the quantile estimation with solely the new incoming data. The `showHTMLLimitingQuantiles()` method shows the updated quantile limits in a browser view (see Fig. 10.1 on the next page).

```
1 >>> # showing the updated quantile limits in a browser view
2 >>> pq.showHTMLLimitingQuantiles(Transposed=True)
```

## 10.3 Rating-by-ranking new performances with quantile norms

For *rating* a newly given set of decision alternatives with the help of empirical performance quantiles estimated from historical data, the `sortingDigraphs` module provides the `LearnedQuantilesRatingDigraph` class, a specialisation

---

<sup>2</sup> The `RandomPerformanceGenerator` class works for the *standard* performance tableau model (see Sec. 5.2), the *Cost-Benefit* model (see Sec. 5.3), and the 3-objectives model (see Sec. 5.4).

**Fig. 10.1** Showing updated quartiles limits per criterion.

The 0.25 column shows the first quartile ( $Q_1$ ) limits, the 0.50 column shows the second quartile ( $Q_2$ ) limits and the 0.75 column shows the third quartile ( $Q_3$ ) limits. Column 0.00 (resp. 1.00) shows the minimum (resp. maximum) performance on each criterion

## Performance quantiles

Sampling sizes between 986 and 995.

criterion	0.00	0.25	0.50	0.75	1.00
b1	1.99	28.77	49.63	75.27	99.83
b2	0.00	2.94	4.92	6.72	10.00
b3	0.00	2.90	4.86	8.01	10.00
b4	3.27	35.91	58.58	72.00	98.05
b5	0.85	32.84	48.09	69.75	99.00
c1	-10.00	-7.35	-5.39	-3.38	0.00
c2	-96.37	-72.22	-52.27	-33.99	-1.43

of the `QuantilesSortingDigraph` class (see Chap. 9). The absolute rating result is computed by *ranking* the new performance records together with the learned historical quantile limits.

The class constructor requires a valid `PerformanceQuantiles` instance and, by default, uses the COPELAND or the NETFLOWS ranking rule, whichever fits best in an ordinal correlation sense with the underlying outranking digraph.

It is important to notice that the `LearnedQuantilesRatingDigraph` class, contrary to the generic `OutrankingDigraph` class, does not inherit from the generic `PerformanceTableau` class, but instead from the `PerformanceQuantiles` class (Bisdorff, 2021).

The actions attribute in such a `LearnedQuantilesRatingDigraph` class instance contains not only the newly given decision alternatives, but also the historical quantile limits (attribute profiles) obtained from a given `PerformanceQuantiles` class instance, i.e. estimated quantile bins' performance limits from historical performance data.

We reconsider now the `PerformanceQuantiles` object instance `pq` as computed in the previous section. Let `newActions` be a list of 10 new decision alternatives generated with the same random performance tableau generator `rpg` and including, for our didactical purpose, the two decision alternatives `a1001` and `a1010` mentioned at the beginning.

**Listing 10.4** Computing the absolute rating of 10 new decision alternatives

```

1 >>> from sortingDigraphs import LearnedQuantilesRatingDigraph
2 >>> newActions = rpg.randomActions(10)
3 >>> lqr = LearnedQuantilesRatingDigraph(pq,newActions,\n4 ...                                         rankingRule='best')
5 >>> lqr
6     ----- Object instance description
7     Instance class      : LearnedQuantilesRatingDigraph
8     Instance name       : normedRatingDigraph
9     Criteria            : 7
10    Quantile profiles   : 4

```

```

11 Lower-closed bins      : True
12 New actions           : 10
13 Size                  : 93
14 Determinateness (%)   : 76.1
15 Ranking rule          : Copeland
16 Ordinal correlation   : +0.95
17 Attributes: ['runTimes','objectives','criteria',
18     'LowerClosed','quantilesFrequencies',
19     'limitingQuantiles','historySizes','cdf','name',
20     'newActions','evaluation','categories',
21     'criteriaCategoryLimits','profiles','profileLimits',
22     'hasNoVeto','actions','completeRelation','relation',
23     'concordanceRelation','valuationdomain','order',
24     'gamma','notGamma','rankingRule','rankingCorrelation',
25     'rankingScores','actionsRanking','ratingCategories',
26     'ratingRelation','relationOrig']
27 *---- Constructor run times (in sec.)
28 Threads               : 1
29 Total time            : 0.02218
30 Data input             : 0.00134
31 Quantile classes      : 0.00008
32 Compute profiles       : 0.00021
33 Compute relation       : 0.01869
34 Compute rating          : 0.00186
35 Compute sorting         : 0.00000

```

Data input to the `LearnedQuantilesRatingDigraph` class constructor are a valid `PerformanceQuantiles` object `pq` and a `newActions` list of newly generated decision alternatives with the same random generator `rpg` (see List. 10.4 on the preceding page Line 2-3).

The `actionsSubset` parameter of the `showPerformanceTableau()` method allows a look at the digraph's nodes, here called `newActions`.

#### **Listing 10.5** Performance tableau of the new incoming decision alternatives

```

1 >>> lqr.showPerformanceTableau(actionsSubset=lqr.newActions)
2 *---- performance tableau ----*
3 criteria | a1001 a1002 a1003 a1004 a1005 a1006 a1007 a1008 a1009 a1010
4 -----
5   'b1' | 37.0 27.0 24.0 16.0 42.0 33.0 39.0 64.0 42.0 32.0
6   'b2' | 2.0 5.0 8.0 3.0 3.0 3.0 6.0 5.0 4.0 9.0
7   'b3' | 2.0 4.0 2.0 1.0 6.0 3.0 2.0 6.0 6.0 6.0
8   'b4' | 61.0 54.0 74.0 25.0 28.0 20.0 20.0 49.0 44.0 55.0
9   'b5' | 31.0 63.0 61.0 48.0 30.0 39.0 16.0 96.0 57.0 51.0
10  'c1' | -4.0 -6.0 -8.0 -5.0 -1.0 -5.0 -1.0 -6.0 -6.0 -4.0
11  'c2' | -40.0 -23.0 -37.0 -37.0 -24.0 -27.0 -73.0 -43.0 -94.0 -35.0

```

Among the 10 new incoming decision alternatives, we recognise alternatives `a1001` (see column 2) and `a1010` (see last column) we have shown in Table 10.1 on page 135.

The `actions` dictionary of a `LearnedQuantilesRatingDigraph` class instance includes, besides the 10 performance evaluations of the ten new alternatives, also the closed lower limits of the four quartile classes:  $m1 = [0.0 - [, m2 = [0.25 - [, m3 = [0.5 - [, m4 = [0.75 - [$ . We find these limits in the `profiles` attribute (see List. 10.6 on the facing page below).

**Listing 10.6** Showing the limiting profiles of the rating quantiles

```

1 >>> lqr.showPerformanceTableau(actionsSubset=lqr.profiles)
2     *----- Quartiles limit profiles -----*
3     criteria | 'm1'   'm2'   'm3'   'm4'
4     -----|-----
5     'b1'   | 2.0    28.8   49.6   75.3
6     'b2'   | 0.0    2.9    4.9    6.7
7     'b3'   | 0.0    2.9    4.9    8.0
8     'b4'   | 3.3    35.9   58.6   72.0
9     'b5'   | 0.8    32.8   48.1   69.7
10    'c1'   | -10.0  -7.4   -5.4   -3.4
11    'c2'   | -96.4  -72.2  -52.3  -34.0

```

The main run time is spent by the `LearnedQuantilesRatingDigraph` class constructor in computing a bipolar-valued outranking relation on the extended actions set including both the new alternatives as well as the quartile class limits (see List. 10.4 on page 139 Lines 23-29).<sup>3</sup>

The actual rating procedure will rely on a complete ranking of the new decision alternatives as well as the quartile class limits obtained from the corresponding bipolar-valued outranking digraph. Two efficient and scalable ranking rules, the COPELAND and its valued version, the NETFLOWS rule may be used for this purpose. The `rankingRule` parameter allows to choose one of both. With `rankingRule='best'` (see List. 10.4 on page 139 Line 3) the `LearnedQuantilesRatingDigraph` constructor will choose the ranking rule that results in the highest ordinal correlation with the given outranking relation (see Chap. 22 and (Bisdorff, 2012)).

In this rating example, the COPELAND rule appears to be the more appropriate ranking rule.

**Listing 10.7** COPELAND ranking of new alternatives and historical quartile limits

```

1 >>> lqr.rankingRule
2   'Copeland'
3 >>> lqr.actionsRanking
4   ['m4', 'a1005', 'a1010', 'a1002', 'a1008', 'a1006', 'a1001',
5   'a1003', 'm3', 'a1007', 'a1004', 'a1009', 'm2', 'm1']
6 >>> lqr.showCorrelation(lqr.rankingCorrelation)
7   Correlation indexes:
8     Crisp ordinal correlation : +0.945
9     Epistemic determination   :  0.522
10    Bipolar-valued equivalence : +0.493

```

We achieve here in Listing. 10.7 a linear ranking without ties (from best to worst) of the digraph's actions set, i.e. including the new decision alternatives as well as the quartile limits  $m_1$  to  $m_4$ , which is very close in an ordinal sense ( $\tau = 0.945$ ) to the underlying strict outranking relation.

---

<sup>3</sup> In case of large volumes, i.e. many new decision alternatives and centile classes for instance, a multi-threading version may be used when multiple processing cores are available (Bisdorff, 2021).

The eventual rating procedure is based in this example on the *lower* quartile limits, such that the quartile classes' contents are filtered out in increasing order of the *quartiles*.

```

1 >>> lqr.ratingCategories
2 OrderedDict([
3     ('m2', ['a1007','a1004','a1009']),
4     ('m3', ['a1005','a1010','a1002','a1008',
5             'a1006','a1001','a1003'])
6 ])

```

We notice above that no new decision alternative is actually rated into the lowest  $[0.0 - 0.25[$ , respectively highest  $[0.75 - [$  quartile classes. Indeed, the absolute rating result is shown with the `showQuantilesRating()` method:

**Listing 10.8** Absolute quartiles rating result

```

1 >>> lqr.showQuantilesRating()
2      ----- Quartiles rating result -----
3      [0.50 - 0.75[ ['a1005', 'a1010', 'a1002', 'a1008',
4                  'a1006', 'a1001', 'a1003']
5      [0.25 - 0.50[ ['a1007', 'a1004', 'a1009']

```

The same result may also be shown in a browser view with the `showHTMLRatingHeatmap()` method using a specialised rating heatmap format (see Fig. 10.2 on the facing page):

```

1 >>> lqr.showHTMLRatingHeatmap(\'
2 ...                 pageTitle='Heatmap of Quartiles Rating', \
3 ...                 Correlations=True, colorLevels=5)

```

Using furthermore a specialised version of the `exportGraphViz()` method allows drawing in Figure 10.3 on page 144 the same rating result in a Hasse diagram format.<sup>4</sup>

```

1 >>> lqr.exportRatingGraphViz('quartileRatingDigraph')
2      ----- exporting a dot file for GraphViz tools -----
3      Exporting to quartileRatingDigraph.dot
4      dot -Grankdir=TB -Tpng quartileRatingDigraph.dot -o
            quartileRatingDigraph.png

```

We have actually solved now the *absolute rating* problem stated at the beginning. Decision alternatives  $a_{1001}$  and  $a_{1010}$  (see Tab. 10.1 on page 135) are both rated into the same third quartile Q3 class (see Fig. 10.3 on page 144), even if the COPELAND ranking, obtained from the underlying strict outranking digraph (see Fig. 10.2 on the facing page), suggests that alternative  $a_{1010}$  is effectively *better performing than* alternative  $a_{1001}$ .

A *preciser* rating result may indeed be achieved when using *deciles* instead of *quartiles* for estimating the historical marginal cumulative distribution functions.

---

<sup>4</sup> Note that the graphviz dot file was post-edited in order to mark in blue alternatives  $a_{1001}$  and  $a_{1010}$ .

**Fig. 10.2 Heatmap of absolute quartiles ranking.** The quartile equivalence classes appear lower-closed. No alternative is rated into the Q1 class ( $[0.00 - 0.25]$ ) and no alternative is rated into the Q4 class ( $[0.75 - 1.00]$ )

### Heatmap of Quartiles Rating

Ranking rule: Copeland; Ranking correlation: 0.938

criteria	c2	b3	c1	b4	b1	b2	b5
weights	5	2	5	2	2	2	2
tau(*)	+0.64	+0.54	+0.43	+0.37	+0.37	+0.35	+0.34
[0.75 -	-30.00	7.00	-3.00	70.89	70.73	7.00	69.98
<b>a1005c</b>	-24.00	6.00	-1.00	28.00	42.00	3.00	30.00
<b>a1010n</b>	-35.00	6.00	-4.00	55.00	32.00	9.00	51.00
<b>a1002c</b>	-23.00	4.00	-6.00	54.00	27.00	5.00	63.00
<b>a1008n</b>	-43.00	6.00	-6.00	49.00	64.00	5.00	96.00
<b>a1006c</b>	-27.00	3.00	-5.00	20.00	33.00	3.00	39.00
<b>a1001c</b>	-40.00	2.00	-4.00	61.00	37.00	2.00	31.00
<b>a1003a</b>	-37.00	2.00	-8.00	74.00	24.00	8.00	61.00
[0.50 -	-50.10	5.00	-5.00	50.82	49.44	5.00	48.55
<b>a1007c</b>	-73.00	2.00	-1.00	20.00	39.00	6.00	16.00
<b>a1004c</b>	-37.00	1.00	-5.00	25.00	16.00	3.00	48.00
<b>a1009n</b>	-94.00	6.00	-6.00	44.00	42.00	4.00	57.00
[0.25 -	-70.65	3.00	-7.00	30.10	29.82	3.00	29.08
[0.00 -	-96.37	0.00	-10.00	3.27	1.99	0.00	0.85

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation.

**Listing 10.9** Absolute deciles rating result

```

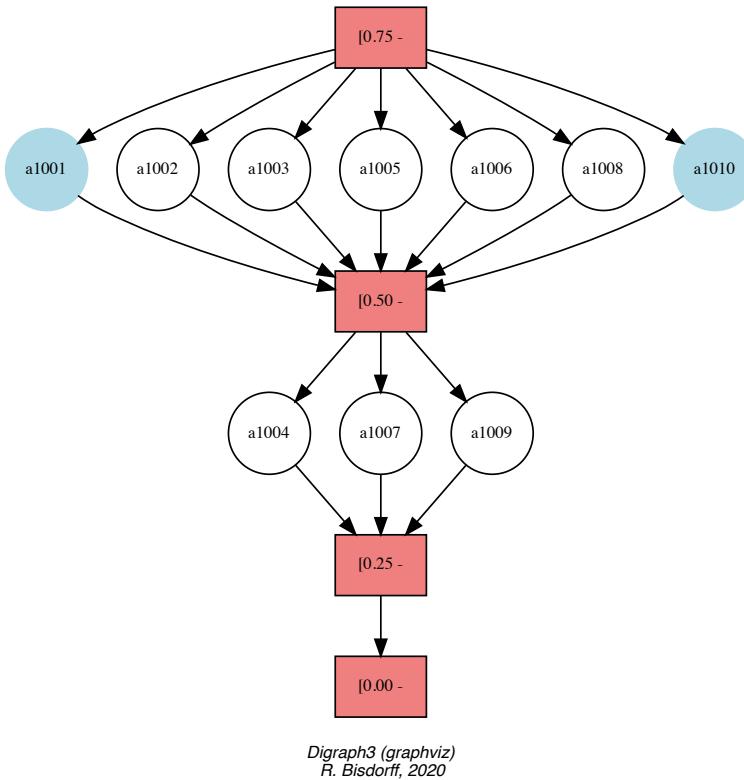
1 >>> pq1 = PerformanceQuantiles(pt, numberOfBins = 'deciles', \
2 ...                                LowerClosed=True)
3 >>> pq1.updateQuantiles(newTab, historySize=None)
4 >>> lqr1 =
5     LearnedQuantilesRatingDigraph(pq1, newActions, rankingRule='best')
6 >>> lqr1.showQuantilesRating()
7 *----- Deciles rating result -----
8 [0.60 - 0.70[ ['a1005', 'a1010', 'a1008', 'a1002']
9 [0.50 - 0.60[ ['a1006', 'a1001', 'a1003']
10 [0.40 - 0.50[ ['a1007', 'a1004']
11 [0.30 - 0.40[ ['a1009']]
```

Compared with the quartiles rating result, we notice in Listing 10.9 that the seven alternatives (a1001, a1002, a1003, a1005, a1006, a1008 and a1010), rated before into the third quartile class  $[0.50 - 0.75[$ , are now divided up: alternatives a1002, a1005, a1008 and a1010 attain now the 7th decile class  $[0.60 - 0.70[$ , whereas alternatives a1001, a1003 and a1006 attain only the 6th decile class  $[0.50 - 0.60[$ . Of the three Q2  $[0.25 - 0.50[$  rated alternatives (a1004, a1007 and a1009), alternatives a1004 and a1007 are now rated into the 5th decile class  $[0.40 - 0.50[$  and a1009 is lowest rated into the 4th decile class  $[0.30 - 0.40[$ .

A browser heatmap view in Figure 10.4 on page 145 more conveniently illustrates this refined rating result.

```

1 >>> lqr1.showHTMLRatingHeatmap (\
```



**Fig. 10.3** Absolute quartiles rating digraph

```

2 ...     pageTitle='Heatmap of the deciles rating', \
3 ...     colorLevels=5, Correlations=True)

```

To avoid having to recompute performance deciles from historical data when wishing to refine a rating result, it is useful, depending on the actual size of the historical data, to initially compute performance quantiles with a relatively high number of bins, for instance *dodeciles* or even *centiles*. It is then possible to interpolate on the fly *quartiles* or *deciles* for instance, when constructing the rating digraph.

**Listing 10.10** From deciles interpolated quartiles rating result

```

1 >>> lqr2 = LearnedQuantilesRatingDigraph(pq1,newActions,
2 ...             quantiles='quartiles')
3 >>> lqr2.showQuantilesRating()
4     ----- Deciles rating result -----
5     [0.50 - 0.75[ ['a1005', 'a1010', 'a1002', 'a1008',

```

**Fig. 10.4 Heatmap of absolute deciles rating.** Decision alternatives a1001 and a1010 are now, as expected, rated in the 6<sup>th</sup> decile (D6), respectively in the 7<sup>th</sup> decile (D7)

### Heatmap of Deciles rating

Ranking rule: NetFlows; Ranking correlation: **0.960**

criteria	c2	b3	c1	b1	b5	b2	b4
<b>weights</b>	5	2	5	2	2	2	2
<b>tau(*)</b>	0.67	0.65	0.58	0.57	0.53	0.53	0.48
[0.90 -]	-20.32	7.73	-2.53	86.83	82.16	7.66	82.04
[0.80 -]	-29.70	7.26	-3.35	79.30	75.15	6.64	74.66
[0.70 -]	-37.97	6.67	-4.14	70.95	60.20	5.88	69.76
<b>a1005c</b>	-24.00	6.00	-1.00	42.00	30.00	3.00	28.00
<b>a1010n</b>	-35.00	6.00	-4.00	32.00	51.00	9.00	55.00
<b>a1008n</b>	-43.00	6.00	-6.00	64.00	96.00	5.00	49.00
<b>a1002c</b>	-23.00	4.00	-6.00	27.00	63.00	5.00	54.00
[0.60 -]	-44.23	5.92	-5.04	60.56	56.01	5.37	62.23
<b>a1006c</b>	-27.00	3.00	-5.00	33.00	39.00	3.00	20.00
<b>a1001c</b>	-40.00	2.00	-4.00	37.00	31.00	2.00	61.00
<b>a1003a</b>	-37.00	2.00	-8.00	24.00	61.00	8.00	74.00
[0.50 -]	-52.22	4.64	-6.02	49.56	48.07	4.83	58.45
<b>a1007c</b>	-73.00	2.00	-1.00	39.00	16.00	6.00	20.00
<b>a1004c</b>	-37.00	1.00	-5.00	16.00	48.00	3.00	25.00
[0.40 -]	-60.50	3.84	-6.69	39.61	40.16	4.25	49.82
<b>a1009n</b>	-94.00	6.00	-6.00	42.00	57.00	4.00	44.00
[0.30 -]	-67.14	3.12	-7.32	30.85	34.33	3.30	40.89
[0.20 -]	-77.07	2.55	-7.94	23.84	29.57	2.27	30.45
[0.10 -]	-83.04	1.99	-8.48	16.64	16.91	1.58	24.78
[0.00 -]	-96.37	0.00	-10.00	1.99	0.85	0.00	3.27

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation.

```

6      'a1006', 'a1001', 'a1003']
7  [0.25 - 0.50[ ['a1004', 'a1007', 'a1009']]

```

With the `quantiles` parameter (see List. 10.10 on the preceding page Line 2), we recover by interpolation the same quartiles rating as obtained directly with historical performance quartiles (see List. 10.8 on page 142). Mind that a correct interpolation of quantiles from a given cumulative distribution function requires more or less uniform distributions of observations in each bin. More generally, in the case of industrial production monitoring problems, for instance, where large volumes of historical performance data may be available, it can be of interest to estimate even more precisely the marginal cumulative distribution functions, especially when *tail* rating results, i.e. distinguishing *very best*, or *very weak* multiple-criteria performances, become a critical issue. Similarly, the `historySize` parameter may be used for monitoring on the fly *unstable* random multiple-criteria performance data (Chambers et al., 2006).

## References

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# Chapter 11

## HPC ranking of big performance tableaux

**Abstract** The sparse outranking digraph, introduced in Chapter 9 is suitable for tackling the ranking of big multiple-criteria performance tableaux with thousands or millions of records. To effectively compute rankings from performance tableaux of these sizes, we propose in this chapter a collection of C-compiled and optimised DIGRAPH3 modules that may be run on HPC equipment as available, for instance at the University of Luxembourg.

### 11.1 C-compiled Python modules

The DIGRAPH3 collection provides cythonized<sup>1</sup>, i.e. C-compiled and optimised versions of the main python modules for tackling multiple-criteria decision problems facing very large sets of decision alternatives ( $> 10000$ ). Such problems appear usually with a combinatorial organisation of the potential decision alternatives, as is frequently the case in bioinformatics for instance. If HPC facilities with nodes supporting numerous cores ( $> 20$ ) and big RAM ( $> 50\text{GB}$ ) are available, ranking up to several millions of alternatives becomes effectively tractable (see (Bisdorff, 2016)).

Four cythonized DIGRAPH3 modules, prefixed with the letter `c` and taking a `.pyx` extension, are provided with their corresponding setup tools in the `cython` directory of the DIGRAPH3 resources, namely:

```
cRandPerfTabs.pyx,  
cIntegerOutrankingDigraphs.pyx,  
cIntegerSortingDigraphs.pyx, and  
cSparseIntegerOutrankingDigraphs.pyx.
```

Their automatic compilation and installation, alongside the standard DIGRAPH3 `python3` modules, requires the `cython` compiler:

```
...$ python3 -m pip install cython
```

---

<sup>1</sup> CYTHON C-extension for Python, <https://cython.org/>

and a C compiler:

```
...$ sudo apt install gcc on Ubuntu, for instance.
```

These cythonized modules, specifically designed for being run on HPC clusters, require the Unix *forking* start method of subprocesses and therefore, due to forking problems on Mac OS platforms, may only operate safely on Linux platforms (see start methods of the [Python multiprocessing library](#)).

## 11.2 Big Data performance tableaux

In order to efficiently type the C variables, the `cRandPerfTabs` module provides the root `cPerformanceTableau` class, but, with *integer* action keys, *float* performance evaluations, *integer* criteria weights and *float* discrimination thresholds. And, to limit as much as possible memory occupation of class instances, all the usual verbose comments are dropped from the description of the actions and `criteria` dictionaries. A random instance may be generated with the `cRandomPerformanceTableau` class:

**Listing 11.1** Big data performance tableau format

```

1 >>> from cRandPerfTabs import cRandomPerformanceTableau
2 >>> t = cRandomPerformanceTableau(numberOfActions=4,\n3 ...                                numberOfCriteria=2)
4 >>> t
5     ----- PerformanceTableau instance description -----
6     Instance class      : cRandomPerformanceTableau
7     Seed                : None
8     Instance name       : cRandomperftab
9     Actions             : 4
10    Criteria            : 2
11    Attributes          : ['randomSeed', 'name', 'actions',
12                           'criteria', 'evaluation', 'weightPreorder']
13 >>> t.actions
14     OrderedDict([(1, {'name': '#1'}), (2, {'name': '#2'}),
15                  (3, {'name': '#3'}), (4, {'name': '#4'})])
16 >>> t.criteria
17     OrderedDict([
18         ('g1', {'name': 'RandomPerformanceTableau() instance',
19                  'thresholds': {'ind': (10.0, 0.0),
20                               'pref': (20.0, 0.0),
21                               'veto': (80.0, 0.0)},
22                  'scale': (0.0, 100.0),
23                  'weight': 1,
24                  'preferenceDirection': 'max'}),
25         ('g2', {'name': 'RandomPerformanceTableau() instance',
26                  'thresholds': {'ind': (10.0, 0.0),
27                               'pref': (20.0, 0.0),
28                               'veto': (80.0, 0.0)},
29                  'scale': (0.0, 100.0),
30                  'weight': 1,
31                  'preferenceDirection': 'max'}))
32 >>> t.evaluation
33     {'g1': {1: 35.17, 2: 56.4, 3: 1.94, 4: 5.51},
34      'g2': {1: 95.12, 2: 90.54, 3: 51.84, 4: 15.42}}
35 >>> t.showPerformanceTableau()
36     Criteria |   'g1'     'g2'
37     Actions  |     1       1
38     -----|-----

```

```

39      '#1' | 91.18  90.42
40      '#2' | 66.82  41.31
41      '#3' | 35.76  28.86
42      '#4' | 7.78   37.64

```

Several methods for converting from the Big Data model to the standard model and vice versa are provided by the `cPerformanceTableau` class.

```

1 >>> t1 = t.convert2Standard()
2 >>> t1.convertWeight2Decimal()
3 >>> t1.convertEvaluation2Decimal()
4 >>> t1
5 *----- PerformanceTableau instance description -----
6   Instance class    : PerformanceTableau
7   Seed              : None
8   Instance name     : std_cRandomperftab
9   Actions           : 4
10  Criteria          : 2
11  Attributes        : ['name', 'actions', 'criteria',
12                      'weightPreorder', 'evaluation',
13                      'randomSeed']

```

## 11.3 C-implemented integer-valued outranking digraphs

The C compiled version of the `BipolarOutrankingDigraph` class takes integer  $r$ -characteristic values.

**Listing 11.2** Constructing big bipolar-valued outranking digraphs

```

1 >>> from cRandPerfTabs import cRandomPerformanceTableau
2 >>> t = cRandomPerformanceTableau(numberOfActions=1000,
3 ...                                         numberOfCriteria=2, seed=100)
4 >>> from cIntegerOutrankingDigraphs import \
5 ...                                         IntegerBipolarOutrankingDigraph
6 >>> g = IntegerBipolarOutrankingDigraph(t,\n7 ...                                         Threading=True, nbrCores=4)
8 >>> g
9 *----- Object instance description -----
10  Instance class   : IntegerBipolarOutrankingDigraph
11  Instance name    : rel_cRandomperftab
12  Actions          : 1000
13  Criteria         : 2
14  Size             : 460795
15  Determinateness  : 55.975
16  Valuation domain : {'min': -2, 'med': 0, 'max': 2,
17                  'hasIntegerValuation': True}
18 *--- Constructor run times (in sec.) ---
19  Total time       : 4.10093
20  Data input       : 0.00955
21  Compute relation : 3.33437
22  Gamma sets       : 0.75699
23  Size             : 465024

```

```

24 Determinateness   : 56.877
25 Attributes       : ['name', 'actions', 'criteria',
26           'totalWeight', 'valuationdomain',
27           'methodData', 'evaluation',
28           'order', 'runTimes', 'nbrThreads',
29           'relation', 'gamma', 'notGamma']

```

On a classical intel-i7 equipped PC with four single threaded cores, the `IntegerBipolarOutrankingDigraph` class constructor takes about four seconds for computing a *million* pairwise outranking characteristic values (see List. 11.2 on the previous page). In a similar setting, the standard `BipolarOutrankingDigraph` constructor operates more than two times slower.

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> t1 = t.convert2Standard()
3 >>> g1 = BipolarOutrankingDigraph(t1,\n4 ...                           Threading=True, nbrCores=4)
5 >>> g1
6 *----- Object instance description -----*
7 Instance class      : BipolarOutrankingDigraph
8 Instance name       : rel_std_cRandomperftab
9 Actions             : 1000
10 Criteria            : 2
11 Size                : 460795
12 Determinateness (%) : 77.99
13 Valuation domain    : [-1.00;1.00]
14 Attributes          : ['name','actions','valuationdomain',
15           'criteria','methodData','evaluation',
16           'NA','order','runTimes','nbrThreads',
17           'relation','gamma','notGamma']
18 ---- Constructor run times (in sec.) ----
19 Total time          : 14.92166
20 Data input          : 0.01506
21 Compute relation    : 13.51562
22 Gamma sets          : 1.39096
23 Threads              : 4

```

By far, most of the run time is in each case needed for computing the individual pairwise outranking characteristic values: 4.10 sec. versus 14.92 sec. Notice also below the memory occupations –about 108MB for `g` and 216MB for `g1`– of both outranking digraph instances.

```

1 >>> from digraphsTools import total_size
2 >>> total_size(g)
3 107503580
4 >>> total_size(g1)
5 211519995
6 >>> total_size(g1.relation)/total_size(g1)
7 0.67
8 >>> total_size(g1.gamma)/total_size(g1)
9 0.23

```

The standard Decimal valued `BipolarOutrankingDigraph` instance `g1` thus nearly doubles the memory occupation of the corresponding `IntegerBipo-`

larOutrankingDigraph g instance (see Line 3 and 5 above). 90% of this memory occupation is due to the g1.relation (67%) and the g1.gamma (23%) dictionaries. And these ratios quadratically grow with the digraph order. To limit the object sizes for really big outranking digraphs, we need to abandon the complete implementation of adjacency tables and gamma functions.

## 11.4 The sparse implementation of big outranking digraphs

The idea is to first decompose the complete outranking relation into an ordered collection of equivalent quantile performance classes. Let us consider for this illustration a random performance tableau with 100 decision alternatives evaluated on 7 criteria.

```
1 >>> from cRandPerfTabs import cRandomPerformanceTableau
2 >>> t = cRandomPerformanceTableau(numberOfActions=100, \
3 ...                                     numberOfCriteria=7, seed=100)
```

The cSparseIntegerOutrankingDigraphs module provides the SparseIntegerOutrankingDigraph class for sorting the 100 decision alternatives into overlapping quartile classes and rank all 100 with respect to the average quantile limits.

**Listing 11.3** Constructing the sparse integer outranking digraph

```
1 >>> from cSparseIntegerOutrankingDigraphs import \
2 ...                               SparseIntegerOutrankingDigraph
3 >>> sg = SparseIntegerOutrankingDigraph(t, quantiles=4)
4 >>> sg
5      ----- Object instance description -----
6      Instance class      : SparseIntegerOutrankingDigraph
7      Instance name       : cRandomperftab_mp
8      Actions             : 100
9      Criteria            : 7
10     Sorting by          : 4-Tiling
11     Ordering strategy   : average
12     Ranking rule         : Copeland
13     Components           : 6
14     Minimal order        : 1
15     Maximal order        : 35
16     Average order        : 16.7
17     fill rate            : 24.970%
18     ----- Constructor run times (in sec.) -----
19     Nbr of threads       : 1
20     Total time           : 0.08212
21     QuantilesSorting    : 0.01481
22     Preordering           : 0.00022
23     Decomposing          : 0.06707
24     Ordering              : 0.00000
25     Attributes            : ['runTimes', 'name', 'actions',
26                               'criteria', 'evaluation', 'order', 'dimension',
```

```

27     'sortingParameters', 'nbrOfCPUs',
28     'valuationdomain', 'profiles', 'categories',
29     'sorting', 'minimalComponentSize',
30     'decomposition', 'nbrComponents', 'nd',
31     'components', 'fillRate',
32     'maximalComponentSize', 'componentRankingRule',
33     'boostedRanking']

```

We obtain in this example here a decomposition into 6 linearly ordered components with a maximal component size of 35 for component c3.

```

1 >>> sg.showDecomposition()
2 *--- quantiles decomposition in decreasing order---*
3   c1. ]0.75-1.00] : [3,22,24,34,41,44,50,53,56,62,93]
4   c2. ]0.50-1.00] : [7,29,43,58,63,81,96]
5   c3. ]0.50-0.75] : [1,2,5,8,10,11,20,21,25,28,30,33,
6           35,36,45,48,57,59,61,65,66,68,70,
7           71,73,76,82,85,89,90,91,92,94,95,97]
8   c4. ]0.25-0.75] : [17,19,26,27,40,46,55,64,69,87,98,100]
9   c5. ]0.25-0.50] : [4,6,9,12,13,14,15,16,18,23,31,32,
10          37,38,39,42,47,49,51,52,54,60,67,72,
11          74,75,77,78,80,86,88,99]
12   c6. ]<-0.25]    : [79,83,84]

```

A restricted outranking relation is stored for each component with more than one alternative. The showRelationMap() prints out below the relation map of the sparse digraph sg for the 75 first-ranked alternatives. (see Fig. 11.1 on the facing page):

```
1 >>> sg.showRelationMap(toIndex=75)
```

With a fill rate of 25%, the memory occupation of this sparse outranking digraph sg instance takes now only 769kB, compared to the 1.7MB required by a corresponding standard BipolarOutrankingDigraph instance (see List.11.3 on the previous page).

```
1 >>> print('%.0fkB' % (total_size(sg) /1024) )
2 769kB
```

For sparse outranking digraphs, the adjacency table is implemented as a dynamic relation() function instead of a double dictionary.

**Listing 11.4** The relation() function of a sparse outranking digraph

```

1 def relation(self, int x, int y):
2     """
3     *Parameters:
4         * x (int action key),
5         * y (int action key).
6     Dynamic construction of the global outranking
7     characteristic function *r(x S y)*.
8     """
9     cdef int Min, Med, Max, rx, ry
10    Min = self.valuationdomain['min']
11    Med = self.valuationdomain['med']
12    Max = self.valuationdomain['max']
13    if x == y:
14        return Med
15    cx = self.actions[x]['component']
16    cy = self.actions[y]['component']

```

**Fig. 11.1** Sparse quartiles-sorting decomposed outranking relation (extract). Legend:  
*outranking for certain* ( $\top$ );  
*outranked for certain* ( $\perp$ );  
*more or less outranking* (+);  
*more or less outranked* (-);  
*indeterminate*



```

17     #print(self.components)
18     rx = self.components[cx]['rank']
19     ry = self.components[cy]['rank']
20     if rx == ry:
21         try:
22             rxi = self.components[cx]['subGraph'].relation
23             return rxi[x][y]
24         except AttributeError:
25             componentRanking = self.components[cx]['componentRanking']
26             if componentRanking.index(x) < componentRanking.index(y):
27                 return Max
28             else:
29                 return Min
30     elif rx > ry:
31         return Min
32     else:
33         return Max

```

## 11.5 Quantiles ranking of big performance tableaux

The `boostedRanking` attribute of the sparse outranking digraph `sg` contains the global linear ranking of the 100 decision alternatives. This ranking is computed by locally ranking the individual 6 components with the COPELAND rule (see List. 11.3 on page 151 Lines 12 and 33).

```

1 >>> sg.boostedRanking
2 [22, 53, 3, 34, 56, 62, 24, 44, 50, 93, 41, 63, 29, 58,
3   96, 7, 43, 81, 91, 35, 25, 76, 66, 65, 8, 10, 1, 11, 61,
4   30, 48, 45, 68, 5, 89, 57, 59, 85, 82, 73, 33, 94, 70,
5   97, 20, 92, 71, 90, 95, 21, 28, 2, 36, 87, 40, 98, 46, 55,
6   100, 64, 17, 26, 27, 19, 69, 6, 38, 4, 37, 60, 31, 77, 78,
7   47, 99, 18, 12, 80, 54, 88, 39, 9, 72, 86, 42, 13, 23, 67,
8   52, 15, 32, 49, 51, 74, 16, 14, 75, 79, 83, 84]
```

When actually computing linear rankings of the complete set of alternatives, the local outranking relations per component are of no practical usage, and we may furthermore reduce the memory occupation of the resulting digraph by:

1. refining the ordering of the quantile classes by taking into account how well an alternative is outranking the lower limit of its quantile class, respectively the upper limit of its quantile class is *not* outranking the alternative;
2. dropping the local outranking digraphs and keeping for each component only a locally ranked list of alternatives.

We provide therefore the `cQuantilesRankingDigraph` class from the `cSparseIntegerOutrankingDigraphs` module.

**Listing 11.5** Ranking the sparse integer outranking digraph

```

1 >>> from cSparseIntegerOutrankingDigraphs import \
2 ...           cQuantilesRankingDigraph
3 >>> qr = cQuantilesRankingDigraph(t, 4)
4 >>> qr
5 ----- Object instance description -----
6 Instance class      : cQuantilesRankingDigraph
7 Instance name       : cRandomperftab_mp
8 Actions             : 100
9 Criteria            : 7
10 Sorting by         : 4-Tiling
11 Ordering strategy : optimal
12 Ranking rule       : Copeland
13 Components          : 47
14 Minimal order      : 1
15 Maximal order      : 10
16 Average order      : 2.1
17 fill rate          : 2.566%
18 ----- Constructor run times (in sec.) -----
19 Nbr of threads      : 1
20 Total time          : 0.03702
21 QuantilesSorting    : 0.01785
22 Preordering          : 0.00022
23 Decomposing         : 0.01892
```

```

24     Attributes      : ['runTimes', 'name',
25     'actions', 'order', 'dimension', 'sortingParameters',
26     'nbrOfCPUs', 'valuationdomain', 'profiles', 'categories',
27     'sorting', 'minimalComponentSize', 'decomposition',
28     'nbrComponents', 'nd', 'components', 'fillRate',
29     'maximalComponentSize', 'componentRankingRule',
      'boostedRanking']

```

With this *optimised* quantile ordering strategy, we obtain now 47 performance equivalence classes (see [List. 11.5 on the preceding page](#) Line 13).

**Listing 11.6** The ordered components of the sparse outranking digraph

```

1 >>> qr.components
2     OrderedDict([
3         ('c01', {'rank': 1,
4             'lowQtileLimit': ']0.75',
5             'highQtileLimit': '1.00]',
6             'componentRanking': [53]}),
7         ('c02', {'rank': 2,
8             'lowQtileLimit': ']0.75',
9             'highQtileLimit': '1.00]',
10            'componentRanking': [3, 23, 63, 50]}),
11        ('c03', {'rank': 3,
12            'lowQtileLimit': ']0.75',
13            'highQtileLimit': '1.00]',
14            'componentRanking': [34, 44, 56, 24, 93, 41]}),
15        ...
16        ...
17        ...
18        ('c45', {'rank': 45,
19            'lowQtileLimit': ']0.25',
20            'highQtileLimit': '0.50]',
21            'componentRanking': [49]}),
22        ('c46', {'rank': 46,
23            'lowQtileLimit': ']0.25',
24            'highQtileLimit': '0.50]',
25            'componentRanking': [52, 16, 86]}),
26        ('c47', {'rank': 47,
27            'lowQtileLimit': ']<',
28            'highQtileLimit': '0.25]',
29            'componentRanking': [79, 83, 84]})))
30 >>> print('%.0fkB' % (total_size(qr)/1024) )
31     208kB

```

We observe in Listing 11.6 an even more considerably less voluminous memory occupation 208kB, compared to the 769kB of the `SparseIntegerOutrankingDigraph` instance.

It is opportune, however, to measure the loss of quality of the resulting COPELAND ranking when working with sparse outranking digraphs.

**Listing 11.7** Measuring the loss of quality with respect to the standard outranking digraph

```

1 >>> from cIntegerOutrankingDigraphs import\
2     ...
          IntegerBipolarOutrankingDigraph

```

```

3 >>> ig = IntegerBipolarOutrankingDigraph(t)
4 >>> print('Complete outranking : %+.3f' \
5 ...     % (ig.computeOrderCorrelation(\ \
6 ...         ig.computeCopelandOrder()['correlation'])))
7     Complete outranking : +0.747
8
9 >>> print('Sparse 4-tiling : %+.3f' \
10 ...    % (ig.computeOrderCorrelation(\ \
11 ...        list(reversed(sg.boostedRanking))['correlation'])))
12     Sparse 4-tiling      : +0.717
13
14 >>> print('Optimzed sparse 4-tiling: %+.3f' \
15 ...    % (ig.computeOrderCorrelation(\ \
16 ...        list(reversed(qr.boostedRanking)))\ \
17 ...        ['correlation'])))
18     Optimzed sparse 4-tiling: +0.705

```

The best ranking correlation with the pairwise outranking situations (+0.747) is naturally given when we apply the COPELAND rule to the standard outranking digraph (see List. 10.7 on page 141 Line 7). When applying the same rule to the sparse 4-tiled outranking digraph, one gets a correlation of +0.717 (Line 12), and when applying the COPELAND rule to the optimised sparse 4-tiled digraph, we still obtain a correlation of +0.705 (Line 18). These optimistic results actually depend on the number of quantiles we use as well as on the given model of random performance tableau.

In case of `Random3ObjectivesPerformanceTableau` instances, for instance, we would get in a similar setting an even better standrad outranking correlation of +0.86, a sparse 4-tiling correlation of +0.82, and an optimised sparse 4-tiling correlation of +0.81.

## 11.6 HPC quantiles ranking records

Following from the separability property of the  $q$ -tiles sorting of each action into a  $q$ -tiles class, the  $q$ -sorting algorithm may be safely split into as much threads as are multiple processing cores available for working in parallel (see Prop. 9.1). Furthermore, the ranking procedure being local to each diagonal component, these procedures may as well be safely processed in parallel threads on each component restricted outranking digraph.

Using the HPC platform of the University of Luxembourg – <https://hpc-docs.uni.lu/> (Varrette et al., 2014)– the following run times for very big ranking problems could be achieved both on:

- *Iris* -skylake nodes with 28 cores:  
(see <https://hpc-docs.uni.lu/systems/iris>), and
- the 3TB -bigmem *Gaia-183* node with 64 cores  
(see <https://hpc.uni.lu/systems/gaia>),

by running the cythonized Python modules in an Intel compiled virtual Python 3.6.5 environment [GCC Intel(R) 17.0.1 –enable-optimizations c++ gcc 6.3 mode] on Debian 8 Linux.

**Fig. 11.2** *HPC-UL Ranking Performance Records (Spring 2018)*. On the big memory equipped Gaia-183 node with 64 cores we were able to rank one million, resp. 6 million decision alternatives in about 2 minutes, resp. 41 minutes

$\gtrless^q$ order	outranking size	$q$	fill rate	nbr. cores	run time
5 000	$25 \times 10^6$	4	0.005%	28	0.5"
10 000	$1 \times 10^8$	4	0.001%	28	1"
100 000	$1 \times 10^{10}$	5	0.002%	28	10"
<b>1 000 000</b>	<b><math>1 \times 10^{12}</math></b>	<b>6</b>	<b>0.001%</b>	<b>64</b>	<b>2'</b>
3 000 000	$9 \times 10^{12}$	15	0.004%	64	13'
<b>6 000 000</b>	<b><math>36 \times 10^{12}</math></b>	<b>15</b>	<b>0.002%</b>	<b>64</b>	<b>41'</b>

## References

- Bisdorff R (2016) Computing linear rankings from trillions of pairwise outranking situations. In: Busa-Fekete R, Hüllermeier E, Mousseau V, Pfannschmidt K (eds) From Multiple Criteria Decision Aid to Preference Learning , DAP'2016, University of Paderborn (Germany), pp 1–16, URL <http://hdl.handle.net/10993/28613> 147
- Varrette S, Bouvry P, Cartiaux H, Georgatos F (2014) Management of an Academic HPC Cluster: The UL Experience. In: Intl. Conf. on High Performance Computing & Simulation (HPCS 2014), Bologna (Italy), IEEE, pp 959–967 156



**Part III**

**Evaluation and decision case studies**

The third part with four chapters proposes decision making case studies illustrating different performance evaluation models and decision problems. Chapter 12 concerns building a best choice recommendation for helping a young student to select a foreign language study program. Chapter 13, inspired by the THE World University Rankings, presents and discusses the multiple incomensurable criteria ranking of academic Computer Science Departments. Inspired by the 2004 data published by DER SPIEGEL magazine, Chapter 14 showcases a multiple-criteria rating problem about the student enrolment quality of German universities. Chapter 15 lists finally some exercises for students taking a course on Algorithmic Decision Theory.

## Chapter 12

### Alice's best choice: A selection case study

**Abstract** The chapter presents a case study concerning the building of a best choice recommendation for Alice, a German student who wants some advice concerning the choice of her future University studies. We present Alice's performance tableau –potential foreign language study programs, her decision objectives, performance criteria and performance evaluations– and build a best choice recommendation for her. A thorough robustness analysis confirms a very best choice.



**Fig. 12.1** Alice D., 19 years old German student finishing her secondary studies in Köln (Germany), desires to undertake foreign languages studies.

This case study is inspired by a multiple criteria decision analysis case study ([Düscher, 2001](#)).

## 12.1 The decision problem

Alice will probably receive her "Abitur" with satisfactory and/or good marks and wants to start her further studies thereafter. She would not mind staying in Köln, yet is ready to move elsewhere if necessary. The length of the higher studies do concern her, as she wants to earn her life as soon as possible. Her parents however agree to financially support her study fees as well as her living costs during her studies.

Alice has already identified 10 potential study programs.

**Table 12.1** The potential study programs

ID	Diploma	Institution	City
T-UD	Qualified translator (T)	University (UD)	Düsseldorf
T-FHK	Qualified translator (T)	Higher Technical School (FHK)	Köln
T-FHM	Qualified translator (T)	Higher Technical School (FHM)	München
I-FHK	Graduate interpreter (I)	Higher Technical School (FHK)	Köln
T-USB	Qualified translator (T)	University (USB)	Saarbrücken
I-USB	Graduate interpreter (I)	University (USB)	Saarbrücken
T-UHB	Qualified translator (T)	University (UHB)	Heidelberg
I-UHB	Graduate interpreter (I)	University (UHB)	Heidelberg
S-HKK	Specialised secretary (S)	Chamber of Commerce (HKK)	Köln
C-HKK	Foreign correspondent (C)	Chamber of Commerce (HKK)	Köln

In Table 12.1 we notice that Alice considers three *Graduate Interpreter* studies (8 or 9 Semesters), respectively in Köln, in Saarbrücken or in Heidelberg; and five *Qualified translator* studies (8 or 9 Semesters), respectively in Köln, in Düsseldorf, in Saarbrücken, in Heidelberg or in Munich. She also considers two short (4 Semesters) study programs at the Chamber of Commerce in Köln.

Four **decision objectives** of more or less equal importance are guiding Alice's choice:

1. *maximize* the attractiveness of the study place (GEO),
2. *maximize* the attractiveness of her further studies (LEA),
3. *minimize* her financial dependency on her parents (FIN),
4. *maximize* her professional perspectives (PRA).

The decision consequences Alice wishes to take into account for evaluating the potential study programs with respect to each of the four objectives are modelled by the following *coherent family of criteria* (Roy, 1991; Roy and Bouyssou, 1993). Such a family of performance criteria verifies:

1. *Exhaustiveness*: No argument acceptable to Alice can be put forward to justify a preference in favour of study program  $x$  versus program  $y$  when  $x$  and  $y$  have the same performance level on each of the performance criteria;
2. *Cohesiveness*: Alice recognises that program  $x$  must be preferred to program  $y$  whenever the performance level of  $x$  is significantly better than that of  $y$  on one

of the criteria of positive weight, performance levels of  $x$  and  $y$  being the same on each of the other criteria;

3. *Non-redundancy*: One of the above requirements is violated if one of the performance criteria is left out from the family.

**Table 12.2** Alice's family of performance criteria

ID	Name	Comment	Objective	Weight
DH	Proximity	Distance in km to her home (min)	GEO	3
BC	Big City	Number of inhabitants (max)	GEO	3
AS	Studies	Attractiveness of the studies (max)	LEA	6
SF	Fees	Annual study fees (min)	FIN	2
LC	Living	Monthly living costs (min)	FIN	2
SL	Length	Length of the studies (min)	FIN	2
AP	Profession	Attractiveness of the profession (max)	PRA	2
AI	Income	Annual income after studying (max)	PRA	2
PR	Prestige	Occupational prestige (max)	PRA	2

Within each decision objective, the performance criteria are considered to be *equisignificant*. Hence, the four decision objectives get a same *importance weight* of 6 (see Tab. 12.2 Column 5).

## 12.2 The performance tableau

The actual evaluations of Alice's potential study programs are stored in a file named `AliceChoice.py` of PerformanceTableau format<sup>1</sup>.

**Listing 12.1** Alice's performance tableau

```

1 >>> from perfTabs import PerformanceTableau
2 >>> t = PerformanceTableau('AliceChoice')
3 >>> t.showObjectives()
4     *----- decision objectives -----"
5     GEO: Geographical aspect
6     DH Distance to parent's home 3
7     BC Number of inhabitants      3
8     Total weight: 6 (2 criteria)
9     LEA: Learning aspect
10    AS Attractiveness of the study program 6
11    Total weight: 6.00 (1 criteria)

```

---

<sup>1</sup> The performance tableau `AliceChoice.py` may be found in the `examples` directory of the DIGRAPH3 resources ([Bisdorff, 2021](#)).

```

12     FIN: Financial aspect
13         SF Annual registration fees    2
14         LC Monthly living costs      2
15         SL Study time              2
16         Total weight: 6.00 (3 criteria)
17     PRA: Professional aspect
18         AP Attractiveness of the profession      2
19         AI Annual professional income after studying 2
20         OP Occupational Prestige            2
21         Total weight: 6.00 (3 criteria)

```

Details of the performance criteria may be consulted in the browser view below.

```
1 >>> t.showHTMLCriteria()
```

### AliceChoice: Family of Criteria

#	Identifier	Name	Comment	Weight	Scale			Thresholds (ax + b)		
					direction	min	max	indifference	preference	veto
1	AI	Annual professional income after studying	Professional aspect measured in x / 1000 Euros	2.00	max	0.00	50.00	0.00x + 0.00	0.00x + 1.00	
2	AP	Attractiveness of the profession	Professional aspect subjectively measured on a three-level scale: 0 (weak), 1 (fair), 2 (good)	2.00	max	0.00	2.00	0.00x + 0.00	0.00x + 1.00	
3	AS	Attractiveness of the study program	Learning aspect subjectively measured from 0 (weak) to 10 (excellent)	6.00	max	0.00	10.00	0.00x + 0.00	0.00x + 1.00	0.00x + 7.00
4	BC	Number of inhabitants	Geographical aspect: measured in x / 1000	3.00	max	0.00	2000.00	0.01x + 0.00	0.05x + 0.00	
5	DH	Distance to parent's home	Geographical aspect measured in km	3.00	min	0.00	1000.00	0.00x + 0.00	0.00x + 10.00	
6	LC	Monthly living costs	Financial aspect measured in Euros	2.00	min	0.00	1000.00	0.00x + 0.00	0.00x + 100.00	
7	OP	Occupational Prestige	Professional aspect measured in SIOPS points	2.00	max	0.00	100.00	0.00x + 0.00	0.00x + 10.00	
8	SF	Annual registration fees	Financial aspect measured in Euros	2.00	min	400.00	4000.00	0.00x + 0.00	0.00x + 100.00	
9	SL	study time	Financial aspect measured in number of semesters	2.00	min	0.00	10.00	0.00x + 0.00	0.00x + 0.50	

Fig. 12.2 Alice's performance criteria

It is worthwhile noticing in Figure 12.2 that, on her subjective attractiveness scale of the study programs (criterion AS), Alice considers a performance differences of 7 points to be *considerable* and triggering, the case given, an outranking polarisation (Bisdorff, 2013). Notice also the proportional *indifference* (1%) and *preference* (5%) discrimination thresholds shown on criterion BC (number of inhabitants).

Alice is subjectively evaluating the *Attractiveness* of the studies (criterion AS) on an ordinal scale from *weak* (0) to *excellent* (10). Similarly, she is subjectively evaluating the *Attractiveness* of the respective professions (criterion AP) on a three level ordinal scale from *weak* (0), *fair* (1) to *good* (2). Considering the *Occupational Prestige* (criterion OP), she looked up the SIOPS<sup>2</sup>. All the other evaluation data she found on the internet.

In the *heatmap* view shown in Figure 12.3 on the next page, we may now consult Alice's performance evaluations.

<sup>2</sup> Standard International Occupational Prestige Scale (Ganzeboom and Treiman, 1996).

```

1 >>> t.showHTMLPerformanceHeatmap(\n
2 ...     colorLevels=5,Correlations=True,ndigits=0)

```

### Heatmap of Performance Tableau 'AliceChoice'

criteria	AS	AP	SF	OP	AI	DH	LC	BC	SL
weights	+6.00	+2.00	+2.00	+2.00	+2.00	+3.00	+2.00	+3.00	+2.00
tau(*)	+0.71	+0.64	+0.36	+0.36	+0.24	+0.03	-0.04	-0.07	-0.24
I-FHK	8	2	-400	62	35	0	0	1015	-8
I-USB	8	2	-400	62	45	-269	-1000	196	-9
T-FHK	5	1	-400	62	35	0	0	1015	-8
I-UHB	8	2	-400	62	45	-275	-1000	140	-9
T-UD	5	1	-400	62	45	-41	-1000	567	-9
T-USB	5	1	-400	62	45	-260	-1000	196	-9
T-FHM	4	1	-400	62	35	-631	-1000	1241	-8
T-UHB	5	1	-400	62	45	-275	-1000	140	-9
C-HKK	2	0	-4000	44	30	0	0	1015	-4
S-HKK	1	0	-4000	44	30	0	0	1015	-4

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Ranking rule: NetFlows

Ordinal (Kendall) correlation between global ranking and global outranking relation: +0.692

**Fig. 12.3** Heatmap of Alice's performance tableau

Her ten potential study programs are ordered with the NETFLOWS ranking rule applied to the corresponding bipolar-valued outranking digraph (see Sec. 8.3). *Graduate interpreter* studies in Köln (I-FHK) or Saarbrücken (I-USB), followed by *Qualified Translator* studies in Köln (T-FHK) appear to be Alice's most preferred alternatives. The least attractive for her appear to be studies at the Chamber of Commerce of Köln (C-HKK, S-HKK). Notice by the way that evaluations on performance criteria to be *minimised*, like *Distance to Home* (criterion DH) or *Study time* (criterion SL), are registered as *negative* values, so that smaller measures are, in this case, preferred to larger ones.

It is finally interesting to observe in Figure 12.3 (third row) that the *most significant* performance criteria, appear to be for Alice, on the one side, the *Attractiveness* of the study program (criterion AS, tau = +0.72) followed by the *Attractiveness* of the future profession (criterion AP, tau = +0.62). On the other side, *Study times* (criterion SL, tau = -0.24), *Big city* (criterion BC, tau = -0.07) as well as *Monthly living costs* (criterion LC, tau = -0.04) appear to be *not so significant* (see Chap. 16).

### 12.3 Building a best choice recommendation

Let us now have a look at the resulting pairwise outranking situations.

**Listing 12.2** Computing Alice's outranking digraph

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> dg = BipolarOutrankingDigraph(t)
3 >>> dg
4     *----- Object instance description -----*
5     Instance class      : BipolarOutrankingDigraph
6     Instance name       : rel_AliceChoice
7     Actions             : 10
8     Criteria            : 9
9     Size                : 67
10    Determinateness (%) : 73.91
11    Valuation domain   : [-1.00;1.00]
12 >>> dg.computeSymmetryDegree(Comments=True)
13     Symmetry degree of graph <rel_AliceChoice> : 0.49

```

From Alice's performance tableau we obtain in the digraph `dg` 67 positively validated pairwise outranking situations, supported by a 74% majority of criteria significance (see Lines 9-10 in List. 12.2).

Due to the poorly discriminating performance evaluations, nearly half of these outranking situations (see Line 12) are *symmetric* and reveal actually *more or less indifference* situations between the potential study programs. This is well illustrated in the *relation map* of the outranking digraph shown in Figure 12.4.

## Outranking relation map

### Ranking rule: Copeland



**Fig. 12.4** COPELAND ranked outranking relation map

```

1 >>> dg.showHTMLRelationMap(\n2 ...             tableTitle='Outranking relation map', \n3 ...             rankingRule='Copeland')
```

We have mentioned that Alice considers a performance difference of 7 points on the *Attractiveness of studies* criterion AS to be considerable which triggers, the case given, a potential polarisation of the outranking characteristics. In Figure 12.4 on the preceding page, these polarisations appear in the last column and last row. The `showPolarisations()` method is useful for inspecting the occurrence of outranking polarisations.

**Listing 12.3** Inspecting polarised outranking situations

```

1 >>> dg.showPolarisations()\n2 ----- Negative polarisations -----*\n3     number of negative polarisations : 3\n4     1: r(S-HKK >= I-FHK) = -0.17\n5         criterion: AS\n6         Considerable performance difference : -7.00\n7         Veto discrimination threshold       : -7.00\n8         Polarisation: r(S-HKK >= I-FHK) = -0.17 ==> -1.00\n9     2: r(S-HKK >= I-USB) = -0.17\n10        criterion: AS\n11        Considerable performance difference : -7.00\n12        Veto discrimination threshold       : -7.00\n13        Polarisation: r(S-HKK >= I-USB) = -0.17 ==> -1.00\n14     3: r(S-HKK >= I-UHB) = -0.17\n15        criterion: AS\n16        Considerable performance difference : -7.00\n17        Veto discrimination threshold       : -7.00\n18        Polarisation: r(S-HKK >= I-UHB) = -0.17 ==> -1.00\n19 ----- Positive polarisations -----*\n20     number of positive polarisations: 3\n21     1: r(I-FHK >= S-HKK) = 0.83\n22         criterion: AS\n23         Considerable performance difference : 7.00\n24         Counter-veto threshold           : 7.00\n25         Polarisation: r(I-FHK >= S-HKK) = 0.83 ==> +1.00\n26     2: r(I-USB >= S-HKK) = 0.17\n27         criterion: AS\n28         Considerable performance difference : 7.00\n29         Counter-veto threshold           : 7.00\n30         Polarisation: r(I-USB >= S-HKK) = 0.17 ==> +1.00\n31     3: r(I-UHB >= S-HKK) = 0.17\n32         criterion: AS\n33         Considerable performance difference : 7.00\n34         Counter-veto threshold           : 7.00\n35         Polarisation: r(I-UHB >= S-HKK) = 0.17 ==> +1.00
```

In Listing 12.3, we see that *considerable* performance differences concerning the *Attractiveness of the studies* (AS criterion) are indeed observed between the *Specialised Secretary* study programm offered in Köln and the *Graduate Interpreter* study programs offered in Köln, Saarbrücken and Heidelberg. They polarise, hence, three *more or less invalid* outranking situations to *certainly invalid* (Lines 8, 13,

18) and corresponding three *more or less valid* converse outranking situations to *certainly valid* ones (Lines 25, 30, 35).

One may finally notice in the relation map, shown in Figure 12.4 on page 166, that the four best-ranked study programs, I-FHK, I-USB, I-UHB and T-FHK, are in fact CONDORCET winners (see List. 12.4 Line 2), i.e. they are all four *indifferent* one of the other **and** positively *outranking* all other alternatives, a result confirmed in Listing 12.4 below by our RUBIS best choice recommendation (Chap. 4 and Bis-dorff, Meyer, and Roubens, 2008).

**Listing 12.4** Alice's best choice recommendation

```

1 >>> dg.computeCondorcetWinners()
2   ['I-FHK', 'I-UHB', 'I-USB', 'T-FHK']
3 >>> dg.showBestChoiceRecommendation()
4   Best choice recommendation(s) (BCR)
5     (in decreasing order of determinateness)
6     Credibility domain: [-1.00,1.00]
7   === >> potential first choice(s)
8     choice           : ['I-FHK', 'I-UHB', 'I-USB', 'T-FHK']
9     independence    : 0.17
10    dominance       : 0.08
11    absorbency      : -0.83
12    covering (%)   : 62.50
13    determinateness (%) : 68.75
14    most credible action(s) = {'I-FHK': 0.75,'T-FHK': 0.17,
15                                'I-USB': 0.17,'I-UHB': 0.17}
16   === >> potential last choice(s)
17     choice           : ['C-HKK', 'S-HKK']
18     independence    : 0.50
19     dominance       : -0.83
20     absorbency      : 0.17
21     covered (%)    : 100.00
22     determinateness (%) : 58.33
23     most credible action(s) = {'S-HKK': 0.17,'C-HKK': 0.17}
```

Most credible best choice among the four best-ranked study programs eventually becomes the *Graduate Interpreter* study program at the Technical High School in Köln (see Line 14) supported by a  $(0.75 + 1)/2.0 = 87.5\%(18/24)$  majority of global criteria significance<sup>3</sup>

In the relation map, shown in Figure 12.4 on page 166, we see in the left lower corner that the *asymmetric part* of the outranking relation, i.e. the corresponding *strict* outranking relation, is actually *transitive* (see Line 3 in List. 12.5 below). Hence, a *graphviz* drawing of its *skeleton*, oriented by the previous *first*, respectively *last* choice, may well illustrate our *best choice recommendation*. The *Reverse=True* flag in Line 4 eliminates the transitivity induced arcs from di-graph *dgc*.

**Listing 12.5** Alice's strict best choice recommendation

```

1 >>> dgc = ~ (-dg)
```

---

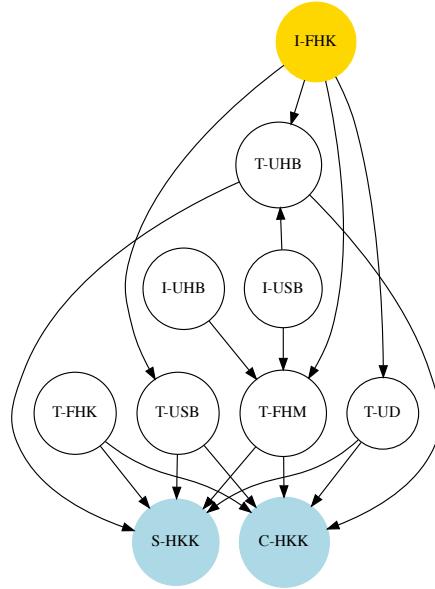
<sup>3</sup> See Sec. 4.5 on computing best choice recommendations and Sec. 17.6 on solving kernel equation systems.

```

2 >>> dgcd.isTransitive()
3     True
4 >>> dgcd.closeTransitive(Reverse=True, InSite=True)
5 >>> dgcd.exportGraphViz('aliceBestChoice', \
6 ...                         firstChoice=['I-FHK'], \
7 ...                         lastChoice=['S-HKK', 'C-HKK'])
8 *----- exporting a dot file for GraphViz tools -----*
9 Exporting to aliceBestChoice.dot
10 dot -Grankdir=BT -Tpng aliceBestChoice.dot -o
      aliceBestChoice.png

```

**Fig. 12.5** Alice's best choice recommendation. We notice that the *Graduate Interpreter* studies come first, followed by the *Qualified Translator* studies. Last come the *Chamber of Commerce*'s specialised studies. This confirms again the high significance that Alice attaches to the *attractiveness* of her further studies and of her future profession (see criteria AS and AP in Fig. 12.3 on page 165)



Digraph3 (graphviz), R. Bisdorff, 2020

Let us now, for instance, check the pairwise outranking situations observed between the first and second-ranked alternative, i.e. *Graduate Interpreter* studies in Köln versus *Graduate Interpreter* studies in Saarbrücken (see I-FHK and I-USB in Fig. 12.3 on page 165).

```

1 >>> dg.showHTMLPairwiseOutrankings('I-FHK', 'I-USB')

```

In a similar way, one may finally compute a *weak ranking* of all the potential study programs with the help of the RankingByChoosingDigraph class from the transitiveDigraphs module, that computes a bipolar ranking by conjointly *best-choosing* and *last-rejecting* (Bisdorff, 1999).

**Fig. 12.6** Comparing the first and second best-ranked study programs. The Köln alternative is at least as well evaluated as the Saarbrücken alternative on all the performance criteria, except the Annual income (of significance 2/24). Conversely, the Saarbrücken alternative is clearly outperformed from the geographical (0/6) as well as from the financial perspective (2/6)

## Pairwise Comparison

### Comparing actions : (I-FHK,I-USB)

crit	wght.	g(x)	g(y)	diff	ind	pref	concord	v	polarisation
AI	2.00	35.00	+45.00	-10	0.00	1.00	-2.00		
AP	2.00	2.00	+2.00	0	0.00	1.00	+2.00		
AS	6.00	8.00	+8.00	0	0.00	1.00	+6.00		
BC	3.00	1015.00	+196.00	819	10.15	50.75	+3.00		
DH	3.00	0.00	-269.00	269	0.00	10.00	+3.00		
LC	2.00	0.00	-1000.00	1000	0.00	100.00	+2.00		
OP	2.00	62.00	+62.00	0	0.00	10.00	+2.00		
SF	2.00	-400.00	-400.00	0	0.00	100.00	+2.00		
SL	2.00	-8.00	-9.00	1	0.00	0.50	+2.00		

Valuation in range: -24.00 to +24.00; global concordance: +20.00

## Pairwise Comparison

### Comparing actions : (I-USB,I-FHK)

crit	wght.	g(x)	g(y)	diff	ind	pref	concord	v	polarisation
AI	2.00	45.00	+35.00	10	0.00	1.00	+2.00		
AP	2.00	2.00	+2.00	0	0.00	1.00	+2.00		
AS	6.00	8.00	+8.00	0	0.00	1.00	+6.00		
BC	3.00	196.00	+1015.00	-819	10.15	50.75	-3.00		
DH	3.00	-269.00	+0.00	-269	0.00	10.00	-3.00		
LC	2.00	-1000.00	+0.00	-1000	0.00	100.00	-2.00		
OP	2.00	62.00	+62.00	0	0.00	10.00	+2.00		
SF	2.00	-400.00	-400.00	0	0.00	100.00	+2.00		
SL	2.00	-9.00	-8.00	-1	0.00	0.50	-2.00		

Valuation in range: -24.00 to +24.00; global concordance: +4.00

**Listing 12.6** Weakly ranking by bipolar best-choosing and last-rejecting

```

1 >>> from transitiveDigraphs import\
2 ...                               RankingByChoosingDigraph
3 >>> rbc = RankingByChoosingDigraph(dg)
4 >>> rbc.showRankingByChoosing()
5   Ranking by Choosing and Rejecting
6     1st ranked ['I-FHK']
7     2nd ranked ['I-USB']
8     3rd ranked ['I-UHB']
9     4th ranked ['T-FHK']
10    5th ranked ['T-UD']
11    5th last ranked ['T-UD']
12    4th last ranked ['T-UHB', 'T-USB']
13    3rd last ranked ['T-FHM']
14    2nd last ranked ['C-HKK']
15    1st last ranked ['S-HKK']

```

In Listing 12.6, we find confirmed that the *Interpreter* studies appear all preferred to the *Translator* studies. Furthermore, the *Interpreter* studies in Saarbrücken appear preferred to the same studies in Heidelberg. The Köln alternative is apparently the preferred one of all the *Translater* studies. And, the *Foreign Correspondent* and the *Specialised Secretary* studies appear second-last and last ranked.

Yet, how robust are our findings with respect to potential settings of the decision objectives' importance and the performance criteria significance weights?

## 12.4 Robustness analysis

Alice considers her four decision objectives as being *more or less* equally important. Here we have, however, allocated *strictly equal* importance weights with *strictly equi-significant* criteria per objective. How robust is the previous best choice recommendation when, now, we would consider the importance of the objectives and, hence, the significance of the respective performance criteria to be *more or less uncertain*?

To answer this question, we consider the respective criteria significance weights  $w_j$  for  $j = 1, \dots, 9$  to be *triangular random variables* in the range 0 to  $2w_j$  with mode =  $w_j$ . Computing a corresponding 90%-*confident* outranking digraph may be done with the help of the `ConfidentBipolarOutrankingDigraph` class (see Chap. 18).

**Listing 12.7** Computing the 90% confident outranking digraph

```

1 >>> from outrankingDigraphs import \
2 ...           ConfidentBipolarOutrankingDigraph
3 >>> cdg = ConfidentBipolarOutrankingDigraph(t,\ 
4 ...           distribution='triangular',confidence=90.0)
5 >>> cdg
6 *----- Object instance description -----*
7 Instance class      : ConfidentBipolarOutrankingDigraph
8 Instance name       : rel_AliceChoice_CLT
9 Actions             : 10
10 Criteria            : 9
11 Size                : 44
12 Valuation domain    : [-1.00;1.00]
13 Uncertainty model   : triangular(a=0,b=2w)
14 Likelihood domain   : [-1.0;+1.0]
15 Confidence level     : 90.0%
16 Confident majority   : 14/24 (58.3%)
17 Determinateness (%) : 68.19

```

Of the original 67 valid outranking situations, 44 outranking situations are retained as being 90%-*confident* (see List. 12.7 Line 11). The corresponding 90%-*confident qualified majority* of criteria significance amounts to  $14/24 = 58.3\%$  (Line 16).

Concerning now a 90%-*confident* best choice recommendation, we are lucky.

**Listing 12.8** Computing the 90%-*confident* best choice recommendation.

```

1 >>> cdg.computeCondorcetWinners()
2 ['I-FHK']
3 >>> cdg.showBestChoiceRecommendation()
4 ****
5 Best choice recommendation(s) (BCR)
6 (in decreasing order of determinateness)
7 Credibility domain: [-1.00,1.00]
8 ==> potential best choice(s)
9 choice          : ['I-FHK','I-UHB','I-USB',
10                      'T-FHK','T-FHM']

```

```

11      independence      : 0.00
12      dominance        : 0.42
13      absorbency       : 0.00
14      covering (%)     : 20.00
15      determinateness (%) : 61.25
16      - most credible action(s) = { 'I-FHK': 0.75, }

```

The *Graduate Interpreter* studies in Köln remain indeed a 90%-confident CONDORCET winner (see Fig. 12.7 on the next page Line 2). Hence, the same study program also remains our 90%-confident most credible best choice supported by a comfortable 18/24(87.5%) majority of the global criteria significance (see Lines 9-10 and 16 in List. 12.8 on the preceding page).

When previously comparing the two best-ranked study programs (see Fig. 12.6 on page 170), we have observed that I-FHK actually positively outranks I-USB on all four decision objectives. When admitting equi-significant criteria significances per objective, this outranking situation is hence valid independently of the importance weights Alice may allocate to each of her decision objectives.

The `UnOpposedBipolarOutrankingDigraph` class constructor from the `outrankingDigraphs` module computes these *unopposed* outranking situations (see Sec. 19.5).

#### **Listing 12.9** Computing the unopposed outranking situations

```

1 >>> from outrankingDigraphs import \
2 ...                               UnOpposedBipolarOutrankingDigraph
3 >>> uop = UnOpposedBipolarOutrankingDigraph(t)
4 >>> uop
5      *----- Object instance description -----*
6      Instance class      : UnOpposedBipolarOutrankingDigraph
7      Instance name       : AliceChoice_unopposed_outrankings
8      Actions             : 10
9      Criteria            : 9
10     Size                : 28
11     Oppositeness (%)   : 58.21
12     Determinateness (%) : 62.94
13     Valuation domain    : [-1.00;1.00]
14 >>> uop.isTransitive()
15     True

```

28 out the 67 standard outranking situations remain valid, which leads to an *oppositeness degree* of  $(1.0 - 28/67) = 58.21\%$  (see Lines 10-11 in List. 12.9). Remarkable furthermore is that this unopposed outranking digraph `uop` is actually *transitive*, i.e. modelling a *partial ranking* of the study programs (Lines 14-15).

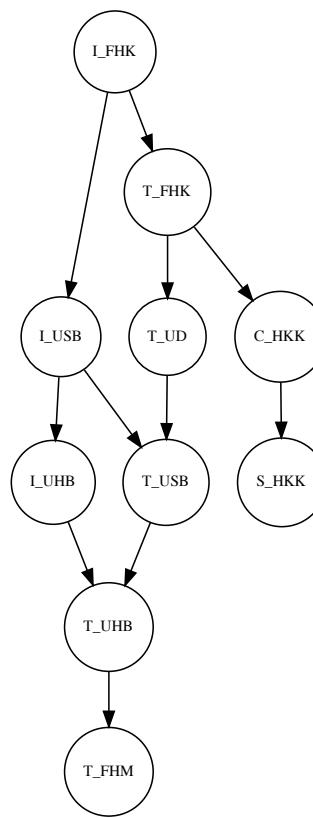
The `exportGraphViz()` method of the `TransitiveDigraph` class draws the corresponding partial ranking.

```

1 >>> from transitiveDigraphs import TransitiveDigraph
2 >>> TransitiveDigraph.exportGraphViz(uop,\ 
3 ...           fileName='choiceUnopposed')
4      *---- exporting a dot file for GraphViz tools -----*
5      Exporting to choiceUnopposed.dot
6      dot -Grankdir=TB -Tpng choiceUnopposed.dot -o
          choiceUnopposed.png

```

**Fig. 12.7** Unopposed partial ranking of the potential study programs. Again, when equally significant performance criteria are assumed per decision objective, we observe in the Figure here that *I-FHK* remains the stable best choice, *independently* of the actual importance weights that Alice may wish to allocate to her four decision objectives



Digraph3 (graphviz)  
R. Bisendorff, 2020

In view of her performance tableau shown in Figure 12.3 on page 165, *Graduate Interpreter* studies at the Technical High School Köln, thus, represent definitely **Alice's very best choice**.

For further reading about the RUBIS *Best Choice* methodology, one may consult in Bisendorff (2015) the study of a *real decision-aiding case* about choosing a best poster in a scientific conference.

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## Chapter 13

# The best academic Computer Science Depts: A ranking case study

**Abstract** In this case study, we are solving with our DIGRAPH3 resources a ranking decision problem based on published data from the *Times Higher Education* (THE) *World University Rankings* 2016 by *Computer Science* (CS) subject. Several hundred academic CS Departments, from all over the world, were ranked that year following an overall numerical score based on the weighted average of five performance criteria: *Teaching* (the learning environment, 30%), *Research* (volume, income and reputation, 30%), *Citations* (research influence, 27.5%), *International outlook* (staff, students, and research, 7.5%), and *Industry income* (innovation, 5%). To illustrate our DIGRAPH3 programming resources, we shall first have a look into the THE multiple-criteria ranking data with short Python scripts. In a second section, we shall relax the commensurability hypothesis of the ranking criteria and show how to similarly rank with multiple incommensurable performance criteria of solely ordinal significance. A third section is finally devoted to introduce quality measures for qualifying ranking results.

### 13.1 The THE performance tableau

For this decision making case study, we use an extract of the published TIMES HIGHER EDUCATION (THE) World University rankings 2016 by Computer Science (CS) subject, concerning the 75 first-ranked academic Institutions<sup>1</sup>. The multiple-criteria performance tableau data, collected from the THE web pages, is stored in a file named `the-cs-2016.py` of PerformanceTableau format<sup>2</sup>.

**Listing 13.1** Performance tableau of the

```
1 >>> from perfTabs import PerformanceTableau  
2 >>> cspt = PerformanceTableau('the_cs_2016')
```

---

<sup>1</sup> [THE World University Rankings 2016 by Computer Science subject](#)

<sup>2</sup> The performance tableau `the-cs-2016.py` is available in the `examples` directory of the DIGRAPH3 resources ([Bisdorff, 2021](#)).

```

3 >>> cspt
4 *----- PerformanceTableau instance description -----*
5 Instance class      : PerformanceTableau
6 Instance name       : the-cs-2016
7 Actions             : 75
8 Objectives          : 5
9 Criteria            : 5
10 NaN proportion (%) : 0.0
11 Attributes          : ['name','description','actions',
12                      'objectives','criteria',
13                      'weightPreorder','NA','evaluation']

```

Potential decision alternatives, in our case here, are the 75 THE best-ranked CS Departments in 2016, all of them located at world renowned Institutions, like California Institute of Technology, Swiss Federal Institute of Technology Zurich, Technical University München, University of Oxford or the National University of Singapore (see List. 13.2 below).

Instead of using prefigured DIGRAPH3 `show...()` methods, readily available for inspecting `PerformanceTableau` instances, Listing 13.2 illustrates how to write small Python scripts for printing out its content.

**Listing 13.2** Printing the CS Departments

```

1 >>> for x in cspt.actions:
2 ...     print('%s:%s (%s)' % \
3 ...           (x,cspt.actions[x]['name'],cspt.actions[x]['comment']) )
4 albt: University of Alberta (CA)
5 anu: Australian National University (AU)
6 ariz: Arizona State University (US)
7 bjup: Beijing University (CN)
8 bro: Brown University (US)
9 calt: California Institute of Technology (US)
10 cbu: Columbia University (US)
11 chku: Chinese University of Hong Kong (HK)
12 cihk: City University of Hong Kong (HK)
13 cir: University of California at Irvine (US)
14 cmel: Carnegie Mellon University (US)
15 cou: Cornell University (US)
16 csb: University of California at Santa Barbara (US)
17 csd: University Of California at San Diego (US)
18 dut: Delft University of Technology (NL)
19 eind: Eindhoven University of Technology (NL)
20 ens: Superior Normal School at Paris (FR)
21 epfl: Swiss Federal Institute of Technology Lausanne (CH)
22 epfr: Polytechnic school of Paris (FR)
23 ethz: Swiss Federal Institute of Technology Zurich (CH)
24 frei: University of Freiburg (DE)
25 git: Georgia Institute of Technology (US)
26 glas: University of Glasgow (UK)
27 hels: University of Helsinki (FI)
28 hkpu: Hong Kong Polytechnic University (CN)
29 hkst: Hong Kong University of Science and Technology (HK)
30 hku: Hong Kong University (HK)
31 humb: Berlin Humboldt University (DE)
32 icl: Imperial College London (UK)
33 indis: Indian Institute of Science (IN)
34 itmo: ITMO University (RU)
35 kcl: King's College London (UK)
36 kist: Korea Adv. Institute of Science and Technology (KR)
37 kit: Karlsruhe Institute of Technology (DE)
38 kth: KTH Royal Institute of Technology (SE)
39 kuj: Kyoto University (JP)

```

```

40 kul: Catholic University Leuven (BE)
41 lms: Lomonosov Moscow State University (RU)
42 man: University of Manchester (UK)
43 mcp: University of Maryland College Park (US)
44 mel: University of Melbourne (AU)
45 mil: Polytechnic University of Milan (IT)
46 mit: Massachusetts Institute of Technology (US)
47 naji: Nanjing University (CN)
48 ntu: Nanyang Technological University of Singapore (SG)
49 ntw: National Taiwan University (TW)
50 nyu: New York University (US)
51 oxf: University of Oxford (UK)
52 pud: Purdue University (US)
53 qut: Queensland University of Technology (AU)
54 rcu: Rice University (US)
55 rwth: RWTH Aachen University (DE)
56 shJi: Shanghai Jiao Tong University (CN)
57 sing: National University of Singapore (SG)
58 sou: University of Southampton (UK)
59 stut: University of Stuttgart (DE)
60 tech: Technion - Israel Institute of Technology (IL)
61 tlavi: Tel Aviv University (IR)
62 tsu: Tsinghua University (CN)
63 tub: Technical University of Berlin (DE)
64 tud: Technical University of Darmstadt (DE)
65 tum: Technical University of Muenchen (DE)
66 ucl: University College London (UK)
67 ued: University of Edinburgh (UK)
68 uiu: University of Illinois at Urbana-Champaign (US)
69 unlu: University of Luxembourg (LU)
70 unsw: University of New South Wales (AU)
71 unt: University of Toronto (CA)
72 uta: University of Texas at Austin (US)
73 utj: University of Tokyo (JP)
74 utw: University of Twente (NL)
75 uwa: University of Waterloo (CA)
76 wash: University of Washington (US)
77 wtu: Vienna University of Technology (AUS)
78 zhej: Zhejiang University (CN)

```

The THE authors base their 2016 ranking on five decision objectives ([Times Higher Education, 2016](#)).

### **Listing 13.3** The THE ranking objectives

```

1 >>> for obj in cspt.objectives:
2 ...     print('%s: %s (%.1f%%),\n\t%s' % \
3 ...           (obj,cspt.objectives[obj]['name'], \
4 ...            cspt.objectives[obj]['weight'], \
5 ...            cspt.objectives[obj]['comment'])) \
6 ...
7 Teaching: Best learning environment (30.0%)
8 Reputation survey; Staff-to-student ration;
9 Doctorate-to-student ratio;
10 Doctorate-to-academic-staff ratio;
11 Institutional income.
12 Research: Highest volume and reputation (30.0%)
13 Reputation survey;
14 Research income;
15 Research productivity.
16 Citations: Highest research influence (27.5%)
17 Impact.
18 International outlook: Most international staff,
19             students and research (7.5%)
20 Proportions of international students;
21 Proportions of international staff;
22 International collaborations.

```

```

23 Industry income: Best knowledge transfer (5.0%)
24 Volume.

```

With a cumulated importance of 87% (see above), *Teaching*, *Research* and *Citations* represent clearly the major ranking objectives. *International outlook* and *Industry income* are considered of minor importance (12.5%).

THE authors do, unfortunately, not publish the detail of their performance assessments for evaluating CS Depts with respect to each one of performance criteria per ranking objective.<sup>3</sup>. The THE 2016 ranking publication reveals solely a compound assessment on a single performance criteria per ranking objective. The five retained performance criteria may be printed out as follows.

```

1 >>> for g in cspt.criteria:
2 ...     print('%.1f%%' % \
3 ...         (g,cspt.criteria[g]['name'],cspt.criteria[g]['comment'],\
4 ...          cspt.criteria[g]['weight']))
5 ...     gtch: Teaching, The learning environment (30.0%)
6 ...     gres: Research, Volume, income and reputation (30.0%)
7 ...     gcit: Citations, Research influence (27.5%)
8 ...     gint: International outlook, In staff, students and research
9 ...         (7.5%)
9 ...     gind: Industry income, knowledge transfer (5.0%)

```

The largest part (87.5%) of criteria significance is, hence canonically, allocated to the major ranking criteria: *Teaching* (30%), *Research* (30%) and *Citations* (27.5%). The small remaining part (12.5%) goes to *International outlook* (7.5%) and *Industry income* (5%).

In order to render commensurable these performance criteria, the THE authors replace, per criterion, the actual performance evaluation obtained by each University with the corresponding *quantile* observed in the cumulative distribution of the performance evaluations obtained by all the surveyed institutions ([Times Higher Education, 2016](#)). The THE ranking is eventually determined by an *overall score* per University which corresponds to the weighted average of these five criteria quantiles, as illustrated in Listing 13.4.

**Listing 13.4** Computing the THE overall scores

```

1 >>> theScores = []
2 >>> for x in cspt.actions:
3 ...     xscore = Decimal('0')
4 ...     for g in cspt.criteria:
5 ...         xscore += cspt.evaluation[g][x] * \
6 ...             (cspt.criteria[g]['weight']/Decimal('100'))
7 ...     theScores.append((xscore,x))

```

In Listing 13.5 on the next page (Lines 15-16), we may thus notice that, in the 2016 edition of the THE World University rankings by CS subject, the Swiss Federal

---

<sup>3</sup> THE gives some insight on the subject and significance of the actual ranking criteria used for evaluating along each ranking objective on her website ([Times Higher Education, 2016](#))

Institute of Technology Zürich is first-ranked with an overall score of 92.9; followed by the California Institute of Technology (overall score: 92.4)<sup>4</sup>.

**Listing 13.5** Printing the ranked performance table

```

1 >>> theScores.sort(reverse = True)
2 >>> print('## Univ \tgtch gres gcit gint gind overall')
3 >>> print('-----')
4 >>> i = 1
5 >>> for it in theScores:
6 ...     x = it[1]
7 ...     xScore = it[0]
8 ...     print('%2d: %s' % (i,x), end=' \t')
9 ...     for g in cspt.criteria:
10 ...         print('.1f' % (cspt.evaluation[g][x]),end=' ')
11 ...         print(' .1f' % xScore)
12 ...     i += 1
13     ## Univ \tgtch gres gcit gint gind overall
14 -----
15     1: ethz    89.2   97.3   97.1   93.6   64.1    92.9
16     2: calt   91.5   96.0   99.8   59.1   85.9    92.4
17     3: oxf    94.0   92.0   98.8   93.6   44.3    92.2
18     4: mit    87.3   95.4   99.4   73.9   87.5    92.1
19     5: git    87.2   99.7   91.3   63.0   79.5    89.9
20     6: cmel   88.1   92.3   99.4   58.9   71.1    89.4
21     7: icl    90.1   87.5   95.1   94.3   49.9    89.0
22     8: epfl   86.3   91.6   94.8   97.2   42.7    88.9
23     9: tum    87.6   95.1   87.9   52.9   95.1    87.7
24    10: sing   89.9   91.3   83.0   95.3   50.6    86.9
25    11: cou    81.6   94.1   99.7   55.7   45.7    86.6
26    12: ucl    85.5   90.3   87.6   94.7   42.4    86.1
27    13: wash   84.4   88.7   99.3   57.4   41.2    85.6
28    14: hkst   74.3   92.0   96.2   84.4   55.8    85.5
29    15: ntu    76.6   87.7   90.4   92.9   86.9    85.5
30    16: ued    85.7   85.3   89.7   95.0   38.8    85.0
31    17: unt    79.9   84.4   99.6   77.6   38.4    84.4
32    18: uiu    85.0   83.1   99.2   51.4   42.2    83.7
33    19: mcp    79.7   89.3   94.6   29.8   51.7    81.5
34    20: cbu    81.2   78.5   94.7   66.9   45.7    81.3
35    21: tsu    88.1   90.2   76.7   27.1   85.9    80.9
36    22: csd    75.2   81.6   99.8   39.7   59.8    80.5
37    23: uwa    75.3   82.6   91.3   72.9   41.5    80.0
38    24: nyu    71.1   77.4   99.4   78.0   39.8    79.7
39    25: uta    72.6   85.3   99.6   31.6   49.7    79.6
40    26: kit    73.8   85.5   84.4   41.3   76.8    77.9
41    27: bju    83.0   85.3   70.1   30.7   99.4    77.0
42    28: csb    65.6   70.9   94.8   72.9   74.9    76.2
43    29: rwth   77.8   85.0   70.8   43.7   89.4    76.1
44    30: hku    77.0   73.0   77.0   96.8   39.5    75.4
45    31: pud    76.9   84.8   70.8   58.1   56.7    75.2
46    32: kist   79.4   88.2   64.2   31.6   92.8    74.9
47    33: kcl    45.5   94.6   86.3   95.1   38.3    74.8
48    34: chku   64.1   69.3   94.7   75.6   49.9    74.2
49    35: epfr   81.7   60.6   78.1   85.3   62.9    73.7
50    36: dut    64.1   78.3   76.3   69.8   90.1    73.4
51    37: tub    66.2   82.4   71.0   55.4   99.9    73.3
52    38: utj    92.0   91.7   48.7   25.8   49.6    72.9
53    39: cir    68.8   64.6   93.0   65.1   40.4    72.5
54    40: ntw    81.5   79.8   66.6   25.5   67.6    72.0
55    41: anu    47.2   73.0   92.2   90.0   48.1    70.6
56    42: rcu    64.1   53.8   99.4   63.7   46.1    69.8
57    43: mel    56.1   70.2   83.7   83.3   50.4    69.7
58    44: lms    81.5   68.1   61.0   31.1   87.8    68.4

```

<sup>4</sup> The author's own Computer Science Dept at the University of Luxembourg was ranked on position 63 with an overall score of 58.0.

59	45: ens	71.8	40.9	98.7	69.6	43.5	68.3
60	46: wtu	61.8	73.5	73.7	51.9	62.2	67.9
61	47: tech	54.9	71.0	85.1	51.7	40.1	67.1
62	48: bro	58.5	54.9	96.8	52.3	38.6	66.5
63	49: man	63.5	71.9	62.9	84.1	42.1	66.3
64	50: zhej	73.5	70.4	60.7	22.6	75.7	65.3
65	51: frei	54.2	51.6	89.5	49.7	99.9	65.1
66	52: unsw	60.2	58.2	70.5	87.0	44.3	63.6
67	53: kuj	75.4	72.8	49.5	28.3	51.4	62.8
68	54: sou	48.2	60.7	75.5	87.4	43.2	62.1
69	55: shJi	66.9	68.3	62.4	22.8	38.5	61.4
70	56: itmo	58.0	32.0	98.7	39.2	68.7	60.5
71	57: kul	35.2	55.8	92.0	46.0	88.3	60.5
72	58: glas	35.2	52.5	91.2	85.8	39.2	59.8
73	59: utw	38.2	52.8	87.0	69.0	60.0	59.4
74	60: stut	54.2	60.6	61.1	36.3	97.8	58.9
75	61: naji	51.4	76.9	48.8	39.7	74.4	58.6
76	62: tud	46.6	53.6	75.9	53.7	66.5	58.3
77	63: unlu	35.2	44.2	87.4	99.7	54.1	58.0
78	64: qut	45.5	42.6	82.8	75.2	63.0	58.0
79	65: hkpu	46.8	36.5	91.4	73.2	41.5	57.7
80	66: albt	39.2	53.3	69.9	91.9	75.4	57.6
81	67: mil	46.4	64.3	69.2	44.1	38.5	57.5
82	68: heils	48.8	49.6	80.4	50.6	39.5	57.4
83	69: cihk	42.4	44.9	80.1	76.2	67.9	57.3
84	70: tlavu	34.1	57.2	89.0	45.3	38.6	57.2
85	71: indis	56.9	76.1	49.3	20.1	41.5	57.0
86	72: ariz	28.4	61.8	84.3	59.3	42.0	56.8
87	73: kth	44.8	42.0	83.6	71.6	39.2	56.4
88	74: humb	48.4	31.3	94.7	41.5	45.5	55.3
89	75: eind	32.4	48.4	81.5	72.2	45.8	54.4

It is important to notice that a ranking by weighted average scores requires *commensurable ranking criteria* of precise decimal significance and on which precise decimal performance evaluations are given. It is very unlikely that the THE 2016 performance assessments verify indeed these conditions. Here we show how to relax these methodological requirements –precise commensurable criteria and decimal evaluations– by following instead a bipolar-valued epistemic logic based ranking methodology (see Chap. 8).

## 13.2 Ranking with multiple criteria of ordinal significance

Let us, first, have a critical look in Figure 13.1 on the facing page at the THE performance criteria.

```
1 >>> cspt.showHTMLCriteria(Sorted=False)
```

Considering a very likely imprecision of the performance evaluation procedure, followed by some potential violation of uniform distributed quantile classes, we assume here that a performance quantile difference of up to 2.5% is *insignificant*, whereas a difference of 5% and more warrants a *clearly better*, resp. *clearly less good* performance. With quantiles 94%, resp. 87.3%, Oxford's CS teaching environment, for instance, is thus clearly better evaluated than that of the MIT (see List. 13.5 on the previous page Lines 27-28). We shall furthermore assume that a *considerable* performance quantile difference of 60%, observed on the three major

## the\_cs\_2016: Family of Criteria

#	Identifier	Name	Comment	Weight	Scale			Thresholds (ax + b)		
					direction	min	max	indifference	preference	veto
1	gtch	Teaching	The learning environment	30.00	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	0.00x + 60.00
2	gres	Research	Volume, income and reputation	30.00	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	0.00x + 60.00
3	gcit	Citations	Research influence	27.50	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	0.00x + 60.00
4	gint	International outlook	In staff, students and research	7.50	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	
5	gind	Industry income	Innovation	5.00	max	0.00	100.00	0.00x + 2.50	0.00x + 5.00	

**Fig. 13.1** The THE ranking criteria

ranking criteria: *Teaching*, *Research* and *Citations*, will trigger the polarisation of pairwise outranking, respectively outranked situations (Bisdorff, 2013).

The effect these performance discrimination thresholds induce on the outranking modelling can be inspected with the `showCriteria()` method.

### Listing 13.6 Inspecting the performance discrimination thresholds

```

1 >>> cspt.showCriteria()
2     ----- criteria -----
3     gtch 'Teaching'
4         Scale = (Decimal('0.00'), Decimal('100.00'))
5         Weight = 0.300
6         Threshold ind : 2.50 + 0.00x ; percentile: 8.07
7         Threshold pref : 5.00 + 0.00x ; percentile: 15.75
8         Threshold veto : 60.00 + 0.00x ; percentile: 99.75
9     gres 'Research'
10        Scale = (Decimal('0.00'), Decimal('100.00'))
11        Weight = 0.300
12        Threshold ind : 2.50 + 0.00x ; percentile: 7.86
13        Threshold pref : 5.00 + 0.00x ; percentile: 16.14
14        Threshold veto : 60.00 + 0.00x ; percentile: 99.21
15     gcit 'Citations'
16        Scale = (Decimal('0.00'), Decimal('100.00'))
17        Weight = 0.275
18        Threshold ind : 2.50 + 0.00x ; percentile: 11.82
19        Threshold pref : 5.00 + 0.00x ; percentile: 22.99
20        Threshold veto : 60.00 + 0.00x ; percentile: 100.00
21     gint 'International outlook'
22        Scale = (Decimal('0.00'), Decimal('100.00'))
23        Weight = 0.075
24        Threshold ind : 2.50 + 0.00x ; percentile: 6.45
25        Threshold pref : 5.00 + 0.00x ; percentile: 11.75
26     gind 'Industry income'
27        Scale = (Decimal('0.00'), Decimal('100.00'))
28        Weight = 0.050
29        Threshold ind : 2.50 + 0.00x ; percentile: 11.82
30        Threshold pref : 5.00 + 0.00x ; percentile: 21.51

```

Between 6% and 12% of the observed quantile differences are, hence, considered to be *insignificant*. Similarly, between 77% and 88% are considered to be *significant*. Less than 1% correspond to *considerable* quantile differences on both the *Teaching* and *Research* criteria; actually triggering an epistemic polarisation effect (Bisdorff, 2013).

Beside the likely imprecise performance discrimination, the *precise decimal significance weights*, as allocated by the THE authors to the five ranking criteria are, as well, quite *questionable*. Criteria significance weights may carry sometimes hidden strategies for rendering the performance evaluations commensurable in view of a numerical computation of the overall ranking scores. The eventual ranking result is, the case given, as much depending on the precise values of the given criteria significance weights as, vice versa, the given precise significance weights are depending on the subjectively expected and accepted ranking results<sup>5</sup>. We will therefore drop these precise decimal weights and, instead, only require a corresponding criteria significance preorder:  $gtch = gres > gcit > gint > gind$ . *Teaching environment* and *Research volume and reputation* are equally considered most significant, followed by *Research influence*. Then comes *International outlook in staff, students and research* and, least significant finally, *Industry income and innovation*.

Both these working hypotheses: –performance discrimination thresholds and – solely ordinal criteria significance, give us way to a ranking methodology based on *robust pairwise outranking* situations (see Chap. 19 and (Bisdorff, 2004)).

### **Definition 13.1 (Robust outranking situations with ordinal criteria significance weights).**

- We say that CS Dept  $x$  *robustly outranks* CS Dept  $y$  when  $x$  positively outranks  $y$  with **all** significance weight vectors that are compatible with the significance preorder:  $gtch = gres > gcit > gint > gind$ ;
- We say that CS Dept  $x$  is *robustly outranked* by CS Dept  $y$  when  $x$  is positively outranked by  $y$  with **all** significance weight vectors that are compatible with the significance preorder:  $gtch = gres > gcit > gint > gind$ ;
- Otherwise, CS Depts  $x$  and  $y$  are considered to be *incomparable*.

A corresponding digraph constructor is provided by the RobustOutranking-Digraph class.

#### **Listing 13.7 Computing the robust outranking digraph**

```

1 >>> from outrankingDigraphs import RobustOutrankingDigraph
2 >>> rdg = RobustOutrankingDigraph(cspt)
3 >>> rdg
4     ----- Object instance description -----
5     Instance class      : RobustOutrankingDigraph
6     Instance name       : robust_the_cs_2016
7     Actions             : 75
8     Criteria            : 5
9     Size                : 2993

```

---

<sup>5</sup> In a social choice context, this potential double bind between voting profiles and election result, corresponds to voting manipulation strategies.

```

10   Determinateness (%) : 78.16
11   Valuation domain    : [-1.00;1.00]
12 >>> rdg.computeIncomparabilityDegree(Comments=True)
13   Incomparability degree (%) of digraph <robust_the_cs_2016>:
14     links x<->y: 2775, incomparable: 102, comparable: 2673
15     (incomparable/links) = 0.037
16 >>> rdg.computeTransitivityDegree(Comments=True)
17   Transitivity degree of digraph <robust_the_cs_2016>:
18     triples x>y>z: 405150, closed: 218489, open: 186661
19     (closed/triples) = 0.539
20 >>> rdg.computeSymmetryDegree(Comments=True)
21   Symmetry degree (%) of digraph <robust_the_cs_2016>:
22     arcs x>y: 2673, symmetric: 320, asymmetric: 2353
23     (symmetric/arcs) = 0.12

```

In the resulting digraph instance `rdg` (see Line 9 in Listing 13.7 on the facing page), we observe 2993 such robust pairwise outranking situations validated with a mean significance of 78% (Line 10). Unfortunately, in the case here, they do not deliver any complete linear ranking. The robust outranking digraph `rdg` contains 102 incomparability situations (3.7%, Line 15); nearly half of its transitive closure is missing (46.1%, Line 19) and 12% of the positive outranking situations correspond in fact to symmetric indifference situations (Line 23).

Worse even, the digraph `rdg` admits a really high number of outranking circuits<sup>6</sup>.

#### **Listing 13.8** Inspecting outranking circuits

```

1 >>> rdg.computeChordlessCircuits()
2 >>> rdg.showChordlessCircuits()
3 *---- Chordless circuits ----*
4 145 circuits.
5 1: ['albt','unlu','ariz','hels'], cred. : 0.300
6 2: ['albt','tlavu','hels'], cred. : 0.150
7 3: ['anu', 'man', 'itmo'], cred. : 0.250
8 4: ['anu', 'zhej', 'rcu'], cred. : 0.250
9 ...
10 ...
11 82: ['csb','epfr','rwth'], cred. : 0.250
12 83: ['csb','epfr','pud','nyu'], cred. : 0.250
13 84: ['csd','kcl','kist'], cred. : 0.250
14 ...
15 ...
16 142: ['kul','qut','mil'], cred. : 0.250
17 143: ['lms','rcu','tech'], cred. : 0.300
18 144: ['mil','stut','qut'], cred. : 0.300
19 145: ['mil','stut','tud'], cred. : 0.300

```

Among the 145 detected robust outranking circuits reported in Listing 13.8, we notice, for instance, two outranking circuits of length 4 (see circuits 1 and 83).

---

<sup>6</sup> The `computeChordlessCircuits()` and `showChordlessCircuits()` methods are separate because there are various methods available for enumerating the chordless circuits in a digraph (Bisdorff, 2010).

Let us inspect below the bipolar-valued robust outranking characteristics of the first circuit.

**Listing 13.9** Showing the relation table with stability denotation

```

1 >>> rdg.showRelationTable(actionsSubset=\
2 ...      ['albt','unlu','ariz','hels'],\
3 ...      Sorted=False)
4 * ----- Relation Table -----
5 r/(stab) | 'albt' 'unlu' 'ariz' 'hels'
6 -----|-----
7 'albt' | +1.00 +0.30 +0.00 +0.00
8 | (+4) (+2) (-1) (-1)
9 'unlu' | +0.00 +1.00 +0.40 +0.00
10 | (+0) (+4) (+2) (-1)
11 'ariz' | +0.00 -0.12 +1.00 +0.40
12 | (+1) (-2) (+4) (+2)
13 'hels' | +0.45 +0.00 -0.03 +1.00
14 | (+2) (+1) (-2) (+4)
15 Valuation domain: [-1.0; 1.0]
16 Stability denotation semantics:
17 +4|-4 : unanimous outranking | outranked situation;
18 +2|-2 : outranking | outranked situation validated
19     with all potential significance weights that are
20     compatible with the given significance preorder;
21 +1|-1 : validated outranking | outranked situation
22     with the given significance weights;
23 0   : indeterminate relational situation.

```

In Listing 13.9, we notice that the robust outranking circuit [albt, unlu, ariz, hels] will reappear with all potential criteria significance weight vectors that are compatible with given preorder:  $\text{gtch} = \text{gres} > \text{gcit} > \text{gint} > \text{gind}$ . Notice also the  $(\pm 1)$  marked outranking situations, like the one between albt and ariz. The statement that “*Arizona State University strictly outranks University of Alberta*” is in fact valid with the precise THE criteria significance weights, but not with all potential significance weights vectors that are compatible with the given significance preorder. All these  $(\pm 1)$  marked outranking situations become hence *doubtful* ( $r(x \sim y) = 0.00$ ) and the corresponding CS Depts, like University of Alberta and Arizona State University, become *incomparable* in a robust outranking sense.

Showing many incomparabilities and indifferences; not being transitive and containing many robust outranking circuits; all these relational characteristics, make that no ranking algorithm, applied to digraph  $\text{rdg}$ , does exist that would produce a *unique* optimal linear ranking result. Methodologically, we are only left with ranking heuristics. In Chapter 8 on ranking with multiple criteria we have now seen several potential heuristic ranking rules that can be used for ranking with an outranking digraph; yet, they may deliver potentially more or less diverging results. Considering the order of digraph  $\text{rdg}$  (75) and the largely unequal THE criteria significance weights, we rather opt, in this tutorial, for the NETFLOWS ranking rule<sup>7</sup>. Its com-

<sup>7</sup> The reader might try other ranking rules, like the COPELAND or KOHLER rules. Mind that the latter ranking-by-choosing rule is more complex (see Chap. 8).

plexity in  $O(n^2)$  is indeed quite tractable and, by avoiding potential *tyranny of short majority* effects, the NETFLOWS rule may specifically take the ranking criteria significance weights into a more fairly balanced account.

The NETFLOWS ranking result of the CS Depts can be directly computed with the `computeNetFlowsRanking()` method.

**Listing 13.10** Computing a robust NETFLOWS ranking

```

1 >>> nfRanking = rdg.computeNetFlowsRanking()
2 >>> nfRanking
3 ['ethz','calt','mit', 'oxf', 'cmel','git', 'epfl',
4 'icl', 'cou', 'tum', 'wash', 'sing','hkst','ucl',
5 'uiu', 'unt', 'ued', 'ntu', 'mcp', 'csd', 'cbu',
6 'uta', 'tsu', 'nyu', 'uwa', 'csb', 'kit', 'utj',
7 'bju', 'kcl', 'chku','kist', 'rwth','pud', 'epfr',
8 'hku', 'rcu', 'cir', 'dut', 'ens', 'ntw', 'anu',
9 'tub', 'mel', 'lms', 'bro', 'frei','wtu', 'tech',
10 'itmo','zhej','man', 'kuj', 'kul', 'unsw','glas',
11 'utw', 'unlu','naji','sou', 'hkpu','qut', 'humb',
12 'shJi','stut','tud', 'tlavu','cihk','albt','indis',
13 'ariz','kth', 'hels','eind', 'mil']
```

We actually obtain in Listing 13.10 a very similar ranking result as the one obtained with the THE average scores. The same group of seven Depts: ethz, calt, mit, oxf, cmel, git and epfl, is top-ranked. And a same group of Depts: tlavu, cihk, indis, ariz, kth, hels, eind, and mil appears at the end of the list.

We can print out the difference between the overall scores based THE ranking and the NETFLOWS ranking above with the following short Python script, where we make use of an ordered Python dictionary with net-flow scores, stored in the `rdg.netFlowsRankingDict` attribute by the previous computation.

**Listing 13.11** Comparing the robust NETFLOWS ranking with the THE ranking

```

1 >>> # rdg.netFlowsRankingDict: ordered dictionary with net flow
2 >>> # scores stored in rdg by the computeNetFlowsRanking() method
3 >>> # theScores = [(xScore_1,x_1), (xScore_2,x_2),... ]
4 >>> # is sorted in decreasing order of xscores_i
5 >>> print(
6 ... ' NetFlows ranking  gtch gres gcit gint gind   THE ranking')
7 >>> for i in range(75):
8 ...     x = nfRanking[i]
9 ...     xScore = rdg.netFlowsRankingDict[x]['netFlow']
10 ...    theScore, theX = theScores[i]
11 ...    print('%2d: %s (%.2f)  % (i+1,x,xScore), end=' \t')
12 ...    for g in rdg.criteria:
13 ...        print('%.1f ' % (t.evaluation[g][x]),end=' ')
14 ...    print(' %.2f' % (theX,theScore) )
15 NetFlows ranking  gtch gres gcit gint gind   THE ranking
16 1: ethz (116.95) 89.2 97.3 97.1 93.6 64.1   ethz (92.88)
17 2: calt (116.15) 91.5 96.0 99.8 59.1 85.9   calt (92.42)
18 3: mit (112.72) 87.3 95.4 99.4 73.9 87.5   oxf (92.20)
19 4: oxf (112.00) 94.0 92.0 98.8 93.6 44.3   mit (92.06)
20 5: cmel (101.60) 88.1 92.3 99.4 58.9 71.1   git (89.88)
21 6: git (93.40) 87.2 99.7 91.3 63.0 79.5   cmel (89.43)
22 7: epfl (90.88) 86.3 91.6 94.8 97.2 42.7   icl (89.00)
23 8: icl (90.62) 90.1 87.5 95.1 94.3 49.9   epfl (88.86)
24 9: cou (84.60) 81.6 94.1 99.7 55.7 45.7   tum (87.70)
25 10: tum (80.42) 87.6 95.1 87.9 52.9 95.1   sing (86.86)
```

26	11: wash (76.28)	84.4	88.7	99.3	57.4	41.2	cou (86.59)
27	12: sing (73.05)	89.9	91.3	83.0	95.3	50.6	ucl (86.05)
28	13: hkst (71.05)	74.3	92.0	96.2	84.4	55.8	wash (85.60)
29	14: ucl (66.78)	85.5	90.3	87.6	94.7	42.4	hkst (85.47)
30	15: uiu (64.80)	85.0	83.1	99.2	51.4	42.2	ntu (85.46)
31	16: unt (62.65)	79.9	84.4	99.6	77.6	38.4	ued (85.03)
32	17: ued (58.67)	85.7	85.3	89.7	95.0	38.8	unt (84.42)
33	18: ntu (57.88)	76.6	87.7	90.4	92.9	86.9	uiu (83.67)
34	19: mcp (54.08)	79.7	89.3	94.6	29.8	51.7	mcp (81.53)
35	20: csd (46.62)	75.2	81.6	99.8	39.7	59.8	cbu (81.25)
36	21: cbu (44.27)	81.2	78.5	94.7	66.9	45.7	tsu (80.91)
37	22: uta (43.27)	72.6	85.3	99.6	31.6	49.7	csd (80.45)
38	23: tsu (42.42)	88.1	90.2	76.7	27.1	85.9	uwa (80.02)
39	24: nyu (35.30)	71.1	77.4	99.4	78.0	39.8	nyu (79.72)
40	25: uwa (28.88)	75.3	82.6	91.3	72.9	41.5	uta (79.61)
41	26: csb (18.18)	65.6	70.9	94.8	72.9	74.9	kit (77.94)
42	27: kit (16.32)	73.8	85.5	84.4	41.3	76.8	bju (77.04)
43	28: utj (15.95)	92.0	91.7	48.7	25.8	49.6	csb (76.23)
44	29: bju (15.45)	83.0	85.3	70.1	30.7	99.4	rwth (76.06)
45	30: kcl (11.95)	45.5	94.6	86.3	95.1	38.3	hku (75.41)
46	31: chku (9.43)	64.1	69.3	94.7	75.6	49.9	pud (75.17)
47	32: kist (7.30)	79.4	88.2	64.2	31.6	92.8	kist (74.94)
48	33: rwth (5.00)	77.8	85.0	70.8	43.7	89.4	kcl (74.81)
49	34: pud (2.40)	76.9	84.8	70.8	58.1	56.7	chku (74.23)
50	35: epfr (-1.70)	81.7	60.6	78.1	85.3	62.9	epfr (73.71)
51	36: hku (-3.83)	77.0	73.0	77.0	96.8	39.5	dut (73.44)
52	37: rcu (-6.38)	64.1	53.8	99.4	63.7	46.1	tub (73.25)
53	38: cir (-8.20)	68.8	64.6	93.0	65.1	40.4	utj (72.92)
54	39: dut (-8.85)	64.1	78.3	76.3	69.8	90.1	cir (72.50)
55	40: ens (-8.97)	71.8	40.9	98.7	69.6	43.5	ntw (72.00)
56	41: ntw (-11.15)	81.5	79.8	66.6	25.5	67.6	anu (70.57)
57	42: anu (-11.50)	47.2	73.0	92.2	90.0	48.1	rcu (69.79)
58	43: tub (-12.20)	66.2	82.4	71.0	55.4	99.9	mel (69.67)
59	44: mel (-23.98)	56.1	70.2	83.7	83.3	50.4	lms (68.38)
60	45: lms (-25.43)	81.5	68.1	61.0	31.1	87.8	ens (68.35)
61	46: bro (-27.18)	58.5	54.9	96.8	52.3	38.6	wtu (67.86)
62	47: frei (-34.42)	54.2	51.6	89.5	49.7	99.9	tech (67.06)
63	48: wtu (-35.05)	61.8	73.5	73.7	51.9	62.2	bro (66.49)
64	49: tech (-37.95)	54.9	71.0	85.1	51.7	40.1	man (66.33)
65	50: itmo (-38.50)	58.0	32.0	98.7	39.2	68.7	zhej (65.34)
66	51: zhej (-43.70)	73.5	70.4	60.7	22.6	75.7	frei (65.08)
67	52: man (-44.83)	63.5	71.9	62.9	84.1	42.1	unsw (63.65)
68	53: kuj (-47.40)	75.4	72.8	49.5	28.3	51.4	kuj (62.77)
69	54: kul (-49.98)	35.2	55.8	92.0	46.0	88.3	sou (62.15)
70	55: unsw (-54.88)	60.2	58.2	70.5	87.0	44.3	shJi (61.35)
71	56: glas (-56.98)	35.2	52.5	91.2	85.8	39.2	itmo (60.52)
72	57: utw (-59.27)	38.2	52.8	87.0	69.0	60.0	kul (60.47)
73	58: unlu (-60.08)	35.2	44.2	87.4	99.7	54.1	glas (59.78)
74	59: naji (-60.52)	51.4	76.9	48.8	39.7	74.4	utw (59.40)
75	60: sou (-60.83)	48.2	60.7	75.5	87.4	43.2	stut (58.85)
76	61: hkpu (-62.05)	46.8	36.5	91.4	73.2	41.5	naji (58.61)
77	62: qut (-66.17)	45.5	42.6	82.8	75.2	63.0	tud (58.28)
78	63: humb (-68.10)	48.4	31.3	94.7	41.5	45.5	unlu (58.04)
79	64: shJi (-69.72)	66.9	68.3	62.4	22.8	38.5	qut (57.99)
80	65: stut (-69.90)	54.2	60.6	61.1	36.3	97.8	hkpu (57.69)
81	66: tud (-70.83)	46.6	53.6	75.9	53.7	66.5	albt (57.63)
82	67: tlavu (-71.50)	34.1	57.2	89.0	45.3	38.6	mil (57.47)
83	68: cihk (-72.20)	42.4	44.9	80.1	76.2	67.9	hels (57.40)
84	69: albt (-72.33)	39.2	53.3	69.9	91.9	75.4	cihk (57.33)
85	70: indis (-72.53)	56.9	76.1	49.3	20.1	41.5	tlavu (57.19)
86	71: ariz (-75.10)	28.4	61.8	84.3	59.3	42.0	indis (57.04)
87	72: kth (-77.10)	44.8	42.0	83.6	71.6	39.2	ariz (56.79)
88	73: hels (-79.55)	48.8	49.6	80.4	50.6	39.5	kth (56.36)
89	74: eind (-82.85)	32.4	48.4	81.5	72.2	45.8	humb (55.34)
90	75: mil (-83.67)	46.4	64.3	69.2	44.1	38.5	eind (54.36)

The first inversion we observe in Listing 13.11 on page 185 (Lines 18-19) concerns Oxford University and the MIT, switching positions 3 and 4. Most inversions are similarly short and concern only switching very close positions in either way. There are some slightly more important inversions concerning, for instance, the Hong Kong University CS Dept, ranked into position 30 in the THE ranking and here in the position 36 (Line 51). The opposite situation may also happen; the Berlin Humboldt University CS Dept, occupying the 74th position in the THE ranking, advances in the robust NETFLOWS ranking to position 63 (Line 78).

In our bipolar-valued epistemic framework, the NETFLOWS score of any CS Dept  $x$  corresponds to the criteria significance support for the logical statement “ $x$  is first-ranked”. Formally

$$r(x \text{ is first-ranked}) = \sum_{y \neq x} r((x \succ y) + (y \prec x)) = \sum_{y \neq x} (r(x \succ y) - r(y \succ x)). \quad (13.1)$$

Using the robust outranking characteristics of digraph `rdg`, we can thus explicitly compute, for instance, ETH Zürich’s NETFLOWS score, denoted `nfx` below.

```

1 >>> x = 'ethz'
2 >>> nfx = Decimal('0')
3 >>> for y in rdg.actions:
4     ...     if x != y:
5     ...         nfx += (rdg.relation[x][y] \
6     ...             - rdg.relation[y][x])
7 >>> print(x, nfx)
8     ethz 116.950

```

In Listing 13.11 on page 185 (Line 16), one may now verify that ETH Zürich obtains indeed the highest NETFLOWS score 116.95, and gives, hence the *most credible* first-ranked CS Dept of the 75 potential candidates.

Yet, how may we now convince the reader, that the outranking based ranking result here appears more objective and trustworthy, than the classic value theory based THE ranking by average quantile scores?

### 13.3 How to judge the quality of a ranking result?

In a multiple-criteria based ranking problem, inspecting pairwise marginal performance differences may give objectivity to global preferential statements. That a CS Dept  $x$  convincingly outranks Dept  $y$  can conveniently be checked with the `showPairwiseOutrankings()` method.. The ETH Zürich CS Dept is, for instance, first ranked before Caltech’s Dept in both previous rankings. Lest us check the preferential reasons.

**Listing 13.12** Comparing pairwise criteria performances

```

1 >>> rdg.showPairwiseOutrankings('ethz','calt')
2     ----- pairwise comparisons -----
3     Valuation in range: -100.00 to +100.00
4     Comparing actions : ('ethz', 'calt')

```

	crit.	wght.	g(ethz)	g(calt)	diff		ind	pref	r()	
<hr/>										
7	'gcit'	27.50	97.10	99.80	-2.70		2.50	5.00	+0.00	
8	'gind'	5.00	64.10	85.90	-21.80		2.50	5.00	-5.00	
9	'gint'	7.50	93.60	59.10	+34.50		2.50	5.00	+7.50	
10	'gres'	30.00	97.30	96.00	+1.30		2.50	5.00	+30.00	
11	'gtch'	30.00	89.20	91.50	-2.30		2.50	5.00	+30.00	
<hr/>										
13							r(x >= y): +62.50			
14	crit.	wght.	g(calt)	g(ethz)	diff		ind	pref	r()	
15							<hr/>			
16	'gcit'	27.50	99.80	97.10	+2.70		2.50	5.00	+27.50	
17	'gind'	5.00	85.90	64.10	+21.80		2.50	5.00	+5.00	
18	'gint'	7.50	59.10	93.60	-34.50		2.50	5.00	-7.50	
19	'gres'	30.00	96.00	97.30	-1.30		2.50	5.00	+30.00	
20	'gtch'	30.00	91.50	89.20	+2.30		2.50	5.00	+30.00	
<hr/>										
21							r(y >= x): +85.00			
22										

A significant positive performance difference (+34.50), concerning the *International outlook* criterion (of 7.5% significance), is observed in Listing 13.12 on the preceding page in favour of the ETH Zürich Dept (Line 9 above). Similarly, a significant positive performance difference (+21.80), concerning the *Industry income* criterion (of 5% significance), is observed, this time, in favour of the Caltech Dept (Line 17). The former, larger positive, performance difference, observed on a more significant criterion, gives so far a first convincing argument of 12.5% significance for putting ETH Zürich first, before Caltech. Yet, the slightly positive performance difference (+2.70, Line 16) between Caltech and ETH Zürich on the *Citations* criterion (of 27.5% significance) confirms an “*at least as well evaluated as*” situation in favour of the Caltech Dept.

The inverse negative performance difference (−2.70, Line 7), however, is neither *significant* ( $< -5.00$ ), nor *insignificant* ( $> -2.50$ ), and does hence neither confirm nor infirm a “*not at least as well evaluated as*” situation in disfavour of ETH Zürich. We observe here a convincing argument of 27.5% significance for ranking Caltech first, before ETH Zürich.

Notice finally, that, on the *Teaching* and *Research* criteria of total significance 60%, both Depts do, with performance differences ( $< |2.50|$ ), perform one as well as the other. As these two major performance criteria together necessarily admit always the highest significance with the imposed significance weight preorder: *gtch* = *gres* > *gcit* > *gint* > *gind*, both outranking situations get in fact globally confirmed at stability level +2 (see Chap. 19).

A browser view of the corresponding robust relation map, ordering the CS Depts again with the same NETFLOWS ranking rule, well illustrates all such *stable outranking* situations.

```
1 >>> rdg.showHTMLRelationMap(\n2 ...                 tableTitle='Robust Outranking Map',\n3 ...                 rankingRule='NetFlows')
```

In Figure 13.2 on the next page, *dark green*, resp. *light green* marked positions show certainly, resp. positively valid outranking situations, whereas *dark red*, resp. *light red* marked positions show certainly, respectively positively valid outranked situations. In the left upper corner one may verify that the five top-ranked Depts



**Fig. 13.2** Relation map of the robust outranking relation

([ethz, calt, oxf, mit, cmel]) are in fact mutually outranking each other and thus are all indifferent one to another. They give even robust CONDORCET winners by robustly outranking all other Depts.

Notice by the way that no certainly valid robust outranking (dark green) and no certainly valid robust outranked situations (dark red) appear in Figure 13.3 on page 191 below, resp. above the principal diagonal; none of these robust preferential situations are hence violated by the robust NETFLOWS ranking. The non reflexive *white* positions in the relation map, like the one between the Georgia Institute of Technology and the MIT, mark outranking or outranked situations that are not robust with respect to the given significance weight preorder. They become, hence, doubtful and are set to the indeterminate characteristic value 0.0.

By measuring the ordinal correlation with the underlying pairwise global and marginal robust outranking situations, the quality of the robust NETFLOWS ranking result can be formally evaluated with the `computeRankingCorrelation()` method and the `showCorrelation()` method (see Chap. 16).

#### Listing 13.13 Measuring the quality of the NETFLOWS ranking result

```
>>> corrf = rdg.computeRankingCorrelation(nfRanking)
```

```

2 >>> rdg.showCorrelation(corrnf)
3   Correlation indexes:
4     Crisp ordinal correlation : +0.901
5     Epistemic determination   :  0.563
6     Bipolar-valued equivalence : +0.507

```

**Listing 13.13 on the previous page** Line 4 indicates that the NETFLOWS ranking result is indeed highly correlated (+0.901, in KENDALL's tau index sense) with the pairwise global robust outranking relation. Their bipolar-valued *relational equivalence* index (+0.51, Line 6) indicates a more than 75% criteria significance support.

With the `showRankingConsensusQuality()` method, we can also check how the NETFLOWS ranking rule is actually balancing the five ranking objectives.

**Listing 13.14** Measuring the consensus quality of the NETFLOWS ranking result

```

1 >>> rdg.showRankingConsensusQuality(nfRanking)
2   Criterion (weight): correlation
3   -----
4     gtch (0.300): +0.660
5     gres (0.300): +0.638
6     gcit (0.275): +0.370
7     gint (0.075): +0.155
8     gind (0.050): +0.101
9   Summary:
10    Weighted mean marginal correlation (a): +0.508
11    Standard deviation (b)                 : +0.187
12    Ranking fairness (a)-(b)              : +0.321

```

The ordinal correlation indexes with the marginal performance criterion rankings are nearly respecting the given significance weights preorder:  $gtch \approx gres > gcit > gint > gind$  (see Lines 4-8 above). The mean marginal ordinal correlation index is quite high (+0.51). Coupled with a low standard deviation (0.187), we obtain a quite fairly balanced ranking result (Lines 10-12).

We can furthermore inspect with the `showCriteriaCorrelationTable()` method the mutual correlation indexes observed between the individual criterion based marginal robust outranking relations.

**Listing 13.15** Showing the ordinal correlation between the marginal criterion relations

```

1 >>> rdg.showCriteriaCorrelationTable()
2   Criteria ordinal correlation index
3     | gcit      gind      gint      gres      gtch
4     |-----|-----|-----|-----|
5   gcit | +1.00    -0.11    +0.24    +0.13    +0.17
6   gind |           +1.00    -0.18    +0.15    +0.15
7   gint |           |       +1.00    +0.04    -0.00
8   gres |           |           +1.00    +0.67
9   gtch |           |           |       +1.00

```

Slightly contradictory ( $-0.11$ ) appear the *Citations* and *Industrial income* criteria (Line 5 Column 3 in List. 13.15). Due perhaps to potential confidentiality clauses, it seems perhaps not always possible to publish industrially relevant research results in highly ranked journals. However, criteria *Citations* and *International outlook* show a slightly positive correlation (+0.24, Column 4), whereas the

*International outlook* criterion shows no apparent correlation with both the major *Teaching* and *Research* criteria. The latter are however highly correlated (+0.67. Line 9 Column 6).

A Principal Component Analysis may well illustrate with the `export3DplotOfCriteriaCorrelation()` method the previous findings (Bisdorff, 2020).

```
>>> rdg.export3DplotOfCriteriaCorrelation(graphType='pdf')
```



Fig. 13.3 3D PCA plot of the pairwise criteria correlation table

In Figure 13.3, one may notice first that more than 80% of the total variance of the previous correlation table is explained by the apparent opposition between the marginal outrankings of criteria: *Teaching*, *Research* and *Industry income* on the left side, and the marginal outrankings of criteria: *Citations* and *International outlook* on the right side. Notice also in the left lower corner the nearly identical positions of the marginal outrankings of the major *Teaching* and *Research* criteria. In the factors

2 and 3 plot, about 30% of the total variance is captured by the opposition between the marginal outrankings of the *Teaching* and *Research* criteria and the marginal outrankings of the *Industrial income* criterion. Finally, in the factors 1 and 3 plot, nearly 15% of the total variance is explained by the opposition between the marginal outrankings of the *International outlook* criterion and the marginal outrankings of the *Citations* criterion.

It is, finally, interesting to similarly assess the ordinal correlation between the THE average scores-based ranking and the robust outranking digraph.

**Listing 13.16** Computing the ordinal quality of the THE ranking

```

1 >>> # theScores = [(xScore_1,x_1), (xScore_2,x_2),... ]
2 >>> # is sorted in decreasing order of xscores
3 >>> theRanking = [item[1] for item in theScores]
4 >>> corrthe = rdg.computeRankingCorrelation(theRanking)
5 >>> rdg.showCorrelation(corrthe)
6     Correlation indexes:
7         Crisp ordinal correlation : +0.907
8         Epistemic determination : 0.563
9         Bipolar-valued equivalence : +0.511
10 >>> rdg.showRankingConsensusQuality(theRanking)
11     Criterion (weight): correlation
12     -----
13     gtch (0.300): +0.683
14     gres (0.300): +0.670
15     gcit (0.275): +0.319
16     gint (0.075): +0.161
17     gind (0.050): +0.106
18     Summary:
19     Weighted mean marginal correlation (a): +0.511
20     Standard deviation (b) : +0.210
21     Ranking fairness (a)-(b) : +0.302

```

The THE ranking result is similarly correlated (+0.907, Line 7 in List. 13.16) with the pairwise global robust outranking situations. By its overall weighted scoring rule, the THE ranking naturally induces marginal criterion correlations that are compatible with the given significance weight preorder (Lines 13-17). Notice that the mean marginal correlation is of a similar value (+0.51, Line 19) as the robust NETFLOWS ranking. Yet, its standard deviation is slightly higher, which leads to a less fairer balancing of the three major ranking criteria.

To conclude, let us emphasise, that, without any commensurability hypothesis and by taking, furthermore, into account, first, the always present more or less imprecision of any performance evaluation and, secondly, solely ordinal criteria significance weights, we may obtain here with our robust outranking approach a very similar ranking result with a slightly better preference modelling quality. A convincing heatmap view of the 25 first-ranked Institutions may eventually be generated in the default system browser with following command.

```

1 >>> rdg.showHTMLPerformanceHeatmap(
2 ...             WithActionNames=True,\ 
3 ...             outrankingModel='this',\ 
4 ...             rankingRule='NetFlows',\

```

```

5 ...           ndigits=1, \
6 ...           Correlations=True, \
7 ...           fromIndex=0, toIndex=25)

```

### Heatmap of Performance Tableau 'robust\_the\_cs\_2016'

criteria	gtch	gres	gcit	gint	gind
<b>weights</b>	+30.00	+30.00	+27.50	+7.50	+5.00
<b>tau<sup>(*)</sup></b>	+0.66	+0.64	+0.37	+0.15	+0.10
<b>Swiss Federal Institute of Technology Zürich (ethz)</b>	89.2	97.3	97.1	93.6	64.1
<b>California Institute of Technology (caltech)</b>	91.5	96.0	99.8	59.1	85.9
<b>Massachusetts Institute of Technology (mit)</b>	87.3	95.4	99.4	73.9	87.5
<b>University of Oxford (oxf)</b>	94.0	92.0	98.8	93.6	44.3
<b>Carnegie Mellon University (cmel)</b>	88.1	92.3	99.4	58.9	71.1
<b>Georgia Institute of Technology (git)</b>	87.2	99.7	91.3	63.0	79.5
<b>Swiss Federal Institute of Technology Lausanne (epfl)</b>	86.3	91.6	94.8	97.2	42.7
<b>Imperial College London (icl)</b>	90.1	87.5	95.1	94.3	49.9
<b>Cornell University (cou)</b>	81.6	94.1	99.7	55.7	45.7
<b>Technical University of München (tum)</b>	87.6	95.1	87.9	52.9	95.1
<b>University of Washington (wash)</b>	84.4	88.7	99.3	57.4	41.2
<b>National University of Singapore (sing)</b>	89.9	91.3	83.0	95.3	50.6
<b>Hong Kong University of Science and Technology (hkst)</b>	74.3	92.0	96.2	84.4	55.8
<b>University College London (ucl)</b>	85.5	90.3	87.6	94.7	42.4
<b>University of Illinois at Urbana-Champaign (uiu)</b>	85.0	83.1	99.2	51.4	42.2
<b>University of Toronto (unt)</b>	79.9	84.4	99.6	77.6	38.4
<b>University of Edinburgh (ued)</b>	85.7	85.3	89.7	95.0	38.8
<b>Nanyang Technological University of Singapore (ntu)</b>	76.6	87.7	90.4	92.9	86.9
<b>University of Maryland College Park (mcp)</b>	79.7	89.3	94.6	29.8	51.7
<b>University Of California at San Diego (csd)</b>	75.2	81.6	99.8	39.7	59.8
<b>Columbia University (cbu)</b>	81.2	78.5	94.7	66.9	45.7
<b>University of Texas at Austin (uta)</b>	72.6	85.3	99.6	31.6	49.7
<b>Tsinghua University (tsu)</b>	88.1	90.2	76.7	27.1	85.9
<b>New York University (nyu)</b>	71.1	77.4	99.4	78.0	39.8
<b>University of Waterloo (uwa)</b>	75.3	82.6	91.3	72.9	41.5

Color legend:

quantile	14.29%	28.57%	42.86%	57.14%	71.43%	85.71%	100.00%
----------	--------	--------	--------	--------	--------	--------	---------

(\* **tau**: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: **this**, Ranking rule: **NetFlows**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.901**

Mean marginal correlation (a) : **+0.508**

Standard marginal correlation deviation (b) : **+0.187**

Ranking fairness (a) - (b) : **+0.321**

**Fig. 13.4** Extract of a heatmap browser view on the NETFLows ranking result

As an exercise, the reader is invited to try out other robust outranking based ranking heuristics. Notice also that we have not challenged in this case study the THE provided criteria significance preorder. It would be very interesting to consider the five ranking objectives as equally important and, consequently, consider the ranking criteria to be equisignificant. Curious to see the ranking results under such settings.

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## **Chapter 14**

# **The best students, where do they study? A rating case study**

**Abstract** In 2004, the German magazine *Der Spiegel*, with the help of *McKinsey & Company* and *AOL*, conducted an extensive online survey, assessing the apparent quality of German University students. The eventually published results by the *Spiegel* magazine concerned nearly 50,000 students, enrolled in one of fifteen popular academic subjects, like *German Studies*, *Life Sciences*, *Psychology*, *Law* or *CS*. Based on this published data, we present and discuss in this chapter, how to *rate* with the help of our DIGRAPH3 software ressources the apparent global *enrolment quality* of new performance records.

### **14.1 The rating problem**

In the 2004 DER SPIEGEL survey, more than 80,000 students, by participating, were questioned on their 'Abitur' and university exams' marks, time of studies and age, grants, awards and publications, IT proficiency, linguistic skills, practical work experience, foreign mobility and civil engagement. Each student received in return a quality score through a specific weighing of the collected data which depended on the subject the student is mainly studying. Publishing only those subject-University combinations, where at least 18 students had correctly filled in the questionnaire, left 41 German Universities where, for at least eight out of the fifteen subjects, an average enrolment quality score could be determined ([Friedmann et al., 2004; DER SPIEGEL, 2004](#)).

We suppose in this rating case study that five German universities: U1, U2, U3, U4 and U5 conducted in 2005 a similar survey among their enrolled students which gave the following enrolment quality scores per academic subject:

In Table [14.1 on the following page](#), the fifteen popular academic subjects are grouped into topical 'Faculties': - *Humanities*; - *Law, Economics & Management*; - *Life Sciences & Medicine*; - *Natural Sciences & Mathematics*; and - *Technology*. None of the five universities has students enrolled in all the fifteen subjects. Univer-

**Table 14.1** Enrolment quality scores per academic scores

Subject	U1	U2	U3	U4	U5
bio	NA	53.10	49.70	52.20	55.20
med	NA	NA	NA	49.50	55.50
math	56.80	54.70	56.30	58.60	61.30
phys	58.90	59.80	53.90	59.10	60.90
chem	52.00	50.10	54.20	53.60	56.70
germ	51.40	53.50	51.40	53.30	61.40
pol	NA	54.00	NA	50.80	59.60
soc	59.10	51.50	55.60	51.00	52.20
psy	57.70	NA	54.40	62.70	59.80
law	NA	NA	41.90	NA	51.10
eco	49.60	NA	NA	NA	54.40
mgt	54.00	53.40	50.70	49.60	NA
info	55.40	52.60	55.80	54.60	NA
elec	56.10	54.50	NA	57.20	NA
mec	54.30	55.20	NA	54.40	NA

sity U1 has, for instance, no students in Life Sciences & Medicine, and in Law and Politology. Whereas the University U5 does not offer any Technology subjects.

The average enrolment quality scores of the five universities, shown in Table 14.1 are stored in a file named `ratingCaseStudy.py` of PerformanceTableau format. The `showHTMLPerformanceTableau()` method produces in Figure 14.1 on the facing page a colourful browser view of these performance records.

```

1 >>> from perfTabs import PerformanceTableau
2 >>> pt = PerformanceTableau('ratingCaseStudy')
3 >>> pt.showHTMLPerformanceTableau(Transposed=True, \
4 ...                                     title='Average enrolment scores')
```

With the best score in nine out of fifteen subjects, university U5 presents the highest global enrolment quality of the five, whereas university U3, with five lowest scores, shows the lowest student enrolment quality of the five.

The university administrations would like to know now how their respective enrolment quality scores are to be appreciated in view the results of the 2004 DER SPIEGEL survey. Are they among the top universities, the midfield or the bottom group?

**Fig. 14.1** Student enrolment quality scores per subject. The light green, resp. light red, figures indicate highest, resp. lowest, score among the five universities.

## Average enrolment scores

criterion	U1	U2	U3	U4	U5
bio	NA	53.10	49.70	52.20	55.20
chem	52.00	50.10	54.20	53.60	56.70
eco	49.60	NA	NA	NA	54.40
elec	56.10	54.50	NA	57.20	NA
germ	51.40	53.50	51.40	53.30	61.40
info	55.40	52.60	55.80	54.60	NA
law	NA	NA	41.90	NA	51.10
math	56.80	54.70	56.30	58.60	61.30
mec	54.30	55.20	NA	54.40	NA
med	NA	NA	NA	49.50	55.50
mgt	54.00	53.40	50.70	49.60	NA
phys	58.90	59.80	53.90	59.10	60.90
pol	NA	54.00	NA	50.80	59.50
psy	57.70	NA	54.40	62.70	59.80
soc	59.10	51.50	55.60	51.00	52.20

## 14.2 The 2004 performance quintiles

The estimated lower-closed performance quintiles of the 2004 average enrolment quality per academic subject are stored in a file named `historicalQuintiles.py`<sup>1</sup> which can be reloaded with the `PerformanceQuantiles` class..

**Listing 14.1** Inspecting stored historical performance quantiles

```

1 >>> from performanceQuantiles import PerformanceQuantiles
2 >>> pq = PerformanceQuantiles('historicalQuintiles')
3 >>> pq
4     *---- PerformanceQuantiles instance description ----*
5     Instance class      : PerformanceQuantiles
6     Instance name       : historicalQuintiles
7     Objectives          : 4
8     Criteria            : 15
9     Quantiles           : 5
10    History sizes       : {'germ': 39, 'pol': 34, 'psy': 34,
11                           'soc': 32, 'law': 32, 'eco': 21,
12                           'mgt': 34, 'bio': 34, 'med': 28,
13                           'phys': 37, 'chem': 35, 'math': 27,
14                           'info': 33, 'elec': 14, 'mec': 13}
15    Attributes          : ['name', 'objectives', 'NA', 'criteria',
16                           'quantilesFrequencies', 'historySizes',
17                           'LowerClosed', 'limitingQuantiles',
18                           'cdf', 'perfTabType']
```

<sup>1</sup> The file may be found in the examples directory of the DIGRAPH3 resources.

The history sizes, reported in Listing 14.1 on the previous page, indicate the number of Universities evaluated in the 2004 survey in each one of the popular fifteen subjects. *German Studies*, for instance, were evaluated for 39 out of 41 Universities, whereas *Electrical* and *Mechanical Engineering* were only evaluated for 14, respectively 13 Institutions. None of the fifteen subjects were evaluated in all the 41 Universities<sup>2</sup>.

Details of the fifteen academic subjects –the performance criteria– may be consulted in a browser view (see Fig. 14.2).

```
1 >>> pt.showHTMLCriteria()
```

#	Identifier	Name	Comment	Weight	Scale			Thresholds (ax + b)		
					direction	min	max	indifference	preference	veto
1	bio	Life Sciences	Life Sciences & Medicine	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
2	chem	Chemistry	Natural Sciences & Mathematics	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
3	eco	Economics	Law, Economics & Management	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
4	elec	Electrical Engineering	Technology	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
5	germ	German Studies	Humanities	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
6	info	Computer Science	Technology	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
7	law	Law Studies	Law, Economics & Management	1.00	max	35.00	65.00	0.00x + 0.10	0.00x + 0.50	
8	math	Mathematics	Natural Sciences & Mathematics	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
9	mec	Mechanical Engineering	Technology	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
10	med	Medicine	Life Sciences & Medicine	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
11	mgt	Management	Law, Economics & Management	1.00	max	40.00	80.00	0.00x + 0.10	0.00x + 0.50	
12	phys	Physics	Natural Sciences & Mathematics	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	
13	pol	Politology	Humanities	1.00	max	50.00	70.00	0.00x + 0.10	0.00x + 0.50	
14	psy	Psychology	Humanities	1.00	max	50.00	70.00	0.00x + 0.10	0.00x + 0.50	
15	soc	Sociology	Humanities	1.00	max	45.00	65.00	0.00x + 0.10	0.00x + 0.50	

**Fig. 14.2** The fifteen academic subjects taken into account for assessing the student enrolment quality

All fifteen subjects are considered equally significant (see Column Weight). The average scores in most subjects vary from 45 to 65. In some subjects, however, like *Law Studies* (35.0 – 65 – 0) and *Politology* (50.0 – 70.0) a different variability is observed. The average enrolment scores per subject are hence incommensurable and global average enrolment scores over all subjects become meaningless.

To take furthermore into account the potential and very likely imprecision of the quality scores' computation, we assume that, for all subjects, an average enrolment quality score difference of 0.1 is *insignificant*, whereas a difference of 0.5 is sufficient to positively attest a *better* enrolment quality. No considerable performance difference are assumed.

The `showLimitingQuantiles()` method prints in Listing 14.2 the estimated quintile limits of the 2004 survey.

**Listing 14.2** Estimated quintile limits of the 2004 survey

```
1 >>> pq.showLimitingQuantiles()
2     ----- performance quantiles -----
3     criteria | weights | '0.00' '0.20' '0.40' '0.60' '0.80' '1.00'
4     ----- |-----|
```

<sup>2</sup> It would have been interesting to estimate such quantile limits from the individual quality scores of all the nearly 50,000 surveyed students. But this data was not public (Friedmann et al., 2004).

5	'bio'		1.0		45.00	50.40	51.80	53.14	55.04	57.10
6	'chem'		1.0		45.00	53.30	54.20	55.80	57.40	58.80
7	'eco'		1.0		49.60	52.14	53.38	54.28	56.94	60.80
8	'elec'		1.0		50.10	54.08	55.34	56.54	57.64	60.20
9	'germ'		1.0		45.00	52.14	54.02	55.84	57.48	61.40
10	'info'		1.0		45.00	54.40	55.44	56.68	58.10	59.80
11	'law'		1.0		39.10	42.30	45.08	46.30	47.26	51.10
12	'math'		1.0		51.60	56.54	57.76	59.44	61.00	63.10
13	'mec'		1.0		51.90	54.02	54.48	55.18	56.54	57.80
14	'med'		1.0		45.00	49.20	49.84	51.10	52.42	60.10
15	'mgt'		1.0		47.50	52.16	52.98	54.68	55.96	68.00
16	'phys'		1.0		53.90	58.78	59.90	60.96	61.96	62.80
17	'pol'		1.0		50.80	54.62	56.18	57.78	59.78	65.90
18	'psy'		1.0		52.50	57.58	58.46	59.80	60.94	64.10
19	'soc'		1.0		45.00	51.70	53.92	55.42	56.26	59.80

We see confirmed again the incommensurability between the subjects, we noticed already in the apparent enrolment quality scoring, especially between *Law Studies* (39.1 – 51.1) and *Politology* (50.5 – 65.9).<sup>3</sup>

### 14.3 Rating-by-ranking with lower-closed quintile limits

We add, now, the estimated quintile limits to the enrolment quality records of the 5 Universities and rank, by using the COPELAND rule, all these records conjointly together with the help of the LearnedQuantilesRatingDigraph class from the sortingDigraphs module.

```

1 >>> from sortingDigraphs import \
2 ...                               LearnedQuantilesRatingDigraph
3 >>> lqr = LearnedQuantilesRatingDigraph(pq,pt,\ 
4 ...                                         rankingRule='Copeland')
5 >>> lqr
6 *----- Object instance description -----*
7 Instance class      : LearnedQuantilesRatingDigraph
8 Instance name       : learnedRatingDigraph
9 Criteria            : 15
10 Quantiles           : 5
11 Lower-closed bins   : True
12 New actions          : 5
13 Size                : 44
14 Determinateness (%) : 77.6
15 Ranking rule         : Copeland
16 Ordinal correlation : +0.97

```

The resulting ranking of the 5 Universities including the lower-closed quintile score limits may be well illustrated with the showHTMLRatingHeatmap() method.

```

1 >>> lqr.showHTMLRatingHeatmap(colorLevels=5,\ 
2 ...                           Correlations=True,\ 

```

<sup>3</sup> The *Spiegel* authors opted therefore for a simple 3-tiling of the Universities per evaluated academic subject, followed by an average BORDA scores based global ranking (Friedmann et al., 2004).

```
3 ... ndigits=1, rankingRule='Copeland')
```

### Heatmap of Performance Tableau 'learnedRatingDigraph'

Ranking rule: Copeland; Ranking correlation: 0.973

criteria	phys	math	germ	chem	bio	mgt	psy	info	pol	law	elec	med	eco	mec	soc
weights	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
tau(*)	+0.79	+0.77	+0.77	+0.66	+0.62	+0.59	+0.54	+0.52	+0.46	+0.41	+0.41	+0.40	+0.39	+0.38	+0.36
[0.80 -]	62.0	61.0	57.5	57.4	55.0	56.0	60.9	58.1	59.8	47.3	57.6	52.4	56.9	56.5	56.3
U5	60.9	61.3	61.4	56.7	55.2	NA	59.8	NA	59.5	51.1	NA	55.5	54.4	NA	52.2
[0.60 -]	61.0	59.4	55.8	55.8	53.1	54.7	59.8	56.7	57.8	46.3	56.5	51.1	54.3	55.2	55.4
[0.40 -]	59.9	57.8	54.0	54.2	51.8	53.0	58.5	55.4	56.2	45.1	55.3	49.8	53.4	54.5	53.9
U1	58.9	56.8	51.4	52.0	NA	54.0	57.7	55.4	NA	NA	56.1	NA	49.6	54.3	59.1
U4	59.1	58.6	53.3	53.6	52.2	49.6	62.7	54.6	50.8	NA	57.2	49.5	NA	54.4	51.0
U2	59.8	54.7	53.5	50.1	53.1	53.4	NA	52.6	54.0	NA	54.5	NA	NA	55.2	51.5
[0.20 -]	58.8	56.5	52.1	53.3	50.4	52.2	57.6	54.4	54.6	42.3	54.1	49.2	52.1	54.0	51.7
U3	53.9	56.3	51.4	54.2	49.7	50.7	54.4	55.8	NA	41.9	NA	NA	NA	NA	55.6
[0.00 -]	53.9	51.6	1.0	2.0	1.0	47.5	52.5	3.0	50.8	39.1	50.1	3.0	49.6	51.9	1.0

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation.

**Fig. 14.3** Heatmap view of the quintiles rating-by-ranking result

The ordinal correlation (+0.97) of the COPELAND ranking with the underlying bipolar-valued outranking digraph is very high (see Fig. 14.3 Row 1). Most correlated subjects with this *rating-by-ranking* result appear to be *Physics* (+0.79), *Mathematics* (+0.77) and *German Studies* (+0.77). *Sociology* (+0.36) is the less correlated subjects (see Row 4).

From the actual ranking position of the lower 5-tiling limits, we may now immediately deduce the quintile enrolment quality equivalence classes. No university reaches the highest quintile ([0.80 –]), whereas U3 is rated into the lowest quintile ([0.00 – 0.20]). The other three universities U1, U2 and U4 are rated in the second quintile ([0.20 – 0.40]). The rating result may be easily printed out with the `showQuantilesRating()` method.

**Listing 14.3** Showing the quintiling of the enrolment quality of the 5 Universities

```
1 >>> lqr.showQuantilesRating()
2 *---- Quantiles rating result ----*
3 [0.60 - 0.80[ ['U5']
4 [0.20 - 0.40[ ['U1', 'U4', 'U2']
5 [0.00 - 0.20[ ['U3']
```

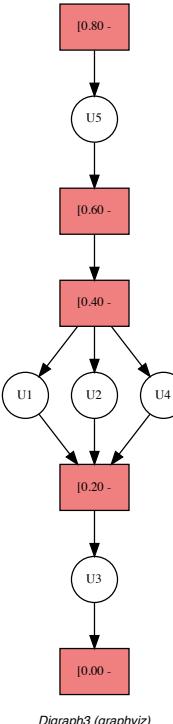
A corresponding *graphviz* drawing with the special `exportRatingByRankingGraphViz()` method may well illustrate in Figure 14.4 on the next page these enrolment quality equivalence classes.

```

1 >>> lqr.exportRatingByRankingGraphViz('ratingResult')
2     *---- exporting a dot file for GraphViz tools -----*
3     Exporting to ratingResult.dot
4     dot -Grankdir=TB -Tpdf dot -o ratingResult.png

```

**Fig. 14.4** Drawing of the quintiles rating-by-ranking result



We have noticed in Chapter 8, that there is not a unique optimal rule for ranking from a given outranking digraph like digraph `lqr` here. The COPELAND rule, for instance, has the advantage of being CONDORCET consistent, i.e. when the outranking digraph models a linear ranking, this ranking will necessarily be the result of the COPELAND rule. When this is not the case, and especially when the outranking digraph shows chordless circuits, all potential ranking rules may give very divergent ranking results, and sometimes substantially divergent rating-by-ranking results.

The `computeChordlessCircuits()` and `showChordlessCircuits()` methods allow to check this issue<sup>4</sup>.

```

1 >>> lqr.computeChordlessCircuits()
2 >>> lqr.showChordlessCircuits()
3 *---- Chordless circuits ----*
4 1 circuits.
5 1: ['U1', 'U2', 'U4'] , credibility : 0.200

```

Indeed, there appears an outranking circuit among the three universities U1, U2 and U4 rated into the 2nd quintile. It is, hence, interesting, to verify if the epistemic fusion of the rating-by-ranking results, one may obtain when applying three different ranking rules, like the KEMENY, COPELAND and NETFLOWS rules, does actually confirm our rating-by-ranking result shown in Figure 14.4 on the preceding page. For this purpose we make use in Listing 14.4 of the `RankingsFusionDigraph` class.

**Listing 14.4** Computing the epistemic fusion of two rating-by-ranking results

```

1 >>> # lqr.actionsRanking is Copeland ranked
2 >>> from linearOrders import KemenyOrder,\ 
3 ...                                NetFlowsOrder
4 >>> ke = KemenyOrder(lqr,orderLimit=10)
5 >>> nf = NetFlowsOrder(lqr)
6 >>> from transitiveDigraphs import\
7 ...                                RankingsFusionDigraph
8 >>> rankings = [lqr.actionsRanking,\ 
9 ...                  nf.netFlowsRanking,\ 
10 ...                 ke.kemenyRanking]
11 >>> rankings
12 [[m5,'u5','m4','m3','u1','u2','u4','m2','u3','m1'],
13 ['m5','u5','m4','m3','u1','u4','u2','u3','m2','m1'],
14 ['m5','u5','m4','m3','u1','u4','u2','m2','u3','m1']]
15 >>> rf = RankingsFusionDigraph(lqr,rankings)
16 >>> rf.exportGraphViz(fileName='fusionResult',\
17 ...                   WithRatingDecoration=True)
18 exporting a dot file for GraphViz tools
19 Exporting to fusionResult.dot
20 dot -Grankdir=TB -Tpng fusionResult.dot\
21           -o fusionResult.png

```

The fusion of the three rating-by-ranking results is shown in Figure 14.5 on the facing page where we see that they diverge solely in their rating-by-ranking of Universities U2, U3 and U4. In Listing 14.5 on the next page Lines 12-14, one may more precisely notice that the KEMENY and COPELAND rules rate U3 in the first quintile ( $[0.00 - 0.20]$ ), whereas the NETFLOWS rule puts U3 together with U1, U2 and U4 into the second quintile ( $[0.20 - 0.40]$ ). And, the KEMENY rule inverts the position of U2 and U4.

---

<sup>4</sup> The `computeChordlessCircuits()` and `showChordlessCircuits()` methods are separate because there are various methods available for enumerating the chordless circuits in a digraph (Bisdorff, 2010).

**Fig. 14.5** Disjunctive fusion of the KEMENY, COPELAND and NETFLOWS rankings. They diverge in their rating-by-ranking of Universities U2, U3 and U4



Digraph3 (graphviz)  
R. Bisdorff, 2020

In Listing 14.5, the `showRankingConsensusQuality()` method reveals finally how fair KEMENY and COPELAND rules actually balance the fifteen academic subjects.

**Listing 14.5** Checking the consensus quality of KEMENY and COPELAND rankings

```

1 >>> lqr.showRankingConsensusQuality(ke.kemenyRanking)
2 Consensus quality of ranking:
3 ['m5', 'u5', 'm4', 'm3', 'u1', 'u2', 'u4', 'm2', 'u3', 'm1']
4 criterion (weight): correlation
5 -----
6 phys (0.067): +0.833
7 germ (0.067): +0.789
8 math (0.067): +0.722
9 bio (0.067): +0.667
10 mgt (0.067): +0.633
  
```

```

11 chem (0.067): +0.611
12 psy (0.067): +0.544
13 pol (0.067): +0.500
14 info (0.067): +0.478
15 mec (0.067): +0.422
16 law (0.067): +0.411
17 soc (0.067): +0.400
18 med (0.067): +0.400
19 eco (0.067): +0.389
20 elec (0.067): +0.367
21 Summary:
22 Weighted mean marginal correlation (a): +0.544
23 Standard deviation (b) : +0.150
24 Ranking fairness (a)-(b) : +0.394
25 >>> lqr.showRankingConsensusQuality(lqr.actionsRanking)
26 Consensus quality of ranking:
27 ['m5', 'u5', 'm4', 'm3', 'u1', 'u4', 'u2', 'm2', 'u3', 'm1']
28 criterion (weight): correlation
29 -----
30 phys (0.067): +0.789
31 math (0.067): +0.767
32 germ (0.067): +0.767
33 chem (0.067): +0.656
34 bio (0.067): +0.622
35 mgt (0.067): +0.589
36 psy (0.067): +0.544
37 info (0.067): +0.522
38 pol (0.067): +0.456
39 law (0.067): +0.411
40 elec (0.067): +0.411
41 med (0.067): +0.400
42 eco (0.067): +0.389
43 mec (0.067): +0.378
44 soc (0.067): +0.356
45 Summary:
46 Weighted mean marginal correlation (a): +0.537
47 Standard deviation (b) : +0.148
48 Ranking fairness (a)-(b) : +0.389

```

The consensus quality of the two rating-by-ranking results do not sensibly differ one from the other. They both show a similar high mean marginal correlation (+0.544, +0.537), similar standard deviations (+0.150, +0.148) and, hence a similar ranking fairness (+0.394, +0.389).

To furthermore check the quality of our COPELAND rating-by-ranking result, we shall below compute a direct rating-by-sorting into the historic 2004 quintiles of the enrolment quality scores, without making use of any outranking digraph based ranking rule.

## 14.4 Rating by quintiles sorting

In the case study here, the five Universities U<sub>1</sub>, U<sub>2</sub>, U<sub>3</sub>, U<sub>4</sub> and U<sub>5</sub> represent the decision actions: *where to study*. We say now that University  $x$  is sorted into the lower-closed quintile  $k$  when the performance record of  $x$  positively outranks the lower limit record  $\mathbf{q}(p_{k-1})$  of quintile  $q$  and  $x$  does not positively outrank the upper limit record  $\mathbf{q}(p_k)$  of quintile  $q$ , for  $q = 1, \dots, 5$  (see List. 14.1 on page 197).

With the help of the bipolar-valued characteristic of the outranking relation  $r(x \succsim y)$  we may indeed compute the bipolar-valued characteristic of the assertion: “ $x$  belongs to the lower-closed quintile class  $\mathbf{q}_k$ ”:

$$r(x \in \mathbf{q}_k) = \min [r(x \succsim \mathbf{q}(p_{k-1})), r(x \not\sim \mathbf{q}(p_k))] \quad (14.1)$$

where  $k = 1, \dots, 5$  and  $(p_k)$  denote the respective quintile proportions: 0.20, 0.40, 0.60, 0.80 and 1.00. As bipolar-valued outranking digraphs verify the coduality principle,  $r(x \not\sim \mathbf{q}(p_k)) = r(x \text{precnsim} \mathbf{q}(p_k))$  (see Sec. 9.2).

The `showSortingCharacteristics()` method gives a precise look in Listing 14.6 on these quintiles sorting characteristics.

**Listing 14.6** Showing quantiles sorting characteristics

	<code>lqr.showSortingCharacteristics()</code>			
1	>>> lqr.showSortingCharacteristics()			
2	<code>x in K_k</code>	<code>r(x &gt;= m_k)</code>	<code>r(x &lt; M_k)</code>	<code>r(x in K_k)</code>
3	U5 in [0.00 - 0.20[	0.73	-0.73	-0.73
4	U5 in [0.20 - 0.40[	0.73	-0.60	-0.60
5	U5 in [0.40 - 0.60[	0.60	-0.60	-0.60
6	U5 in [0.60 - 0.80[	0.60	0.00	0.00
7	U5 in [0.80 - <[	0.00	1.00	0.00
8				
9	U2 in [0.00 - 0.20[	0.73	-0.13	-0.13
10	U2 in [0.20 - 0.40[	0.13	0.20	0.13
11	U2 in [0.40 - 0.60[	-0.20	0.47	-0.20
12	U2 in [0.60 - 0.80[	-0.47	0.73	-0.47
13	U2 in [0.80 - <[	-0.73	1.00	-0.73
14				
15	U4 in [0.00 - 0.20[	0.87	-0.47	-0.47
16	U4 in [0.20 - 0.40[	0.47	0.13	0.13
17	U4 in [0.40 - 0.60[	-0.13	0.60	-0.13
18	U4 in [0.60 - 0.80[	-0.60	0.67	-0.60
19	U4 in [0.80 - <[	-0.67	1.00	-0.67
20				
21	U1 in [0.00 - 0.20[	0.73	-0.33	-0.33
22	U1 in [0.20 - 0.40[	0.33	0.13	0.13
23	U1 in [0.40 - 0.60[	-0.13	0.53	-0.13
24	U1 in [0.60 - 0.80[	-0.53	0.60	-0.53
25	U1 in [0.80 - <[	-0.60	1.00	-0.60
26				
27	U3 in [0.00 - 0.20[	0.67	0.13	0.13
28	U3 in [0.20 - 0.40[	-0.13	0.27	-0.13
29	U3 in [0.40 - 0.60[	-0.27	0.53	-0.27
30	U3 in [0.60 - 0.80[	-0.53	0.67	-0.53
31	U3 in [0.80 - <[	-0.67	1.00	-0.67

The performance record of University U5 cannot be positively rated into a precise quintile, but both the fourth and the fifth quintile are note positively excluded as rating result. Otherwise, the bipolar-valued sorting characteristics verify the rating-by-ranking result we obtained previously with the KEMENY and COPELAND rating-by-ranking result. The `showActionsSortingResult()` method prints the eventual quintiles rating result:

**Listing 14.7** Showing a quintiles rating-by-sorting result

1	>>> lqr.showActionsSortingResult()			
2	Quantiles sorting result per decision action			
3	[0.20 - 0.40[: U1 with credibility: 0.13 = min(0.33, 0.13)			
4	[0.20 - 0.40[: U2 with credibility: 0.13 = min(0.13, 0.20)			
5	[0.00 - 0.20[: U3 with credibility: 0.13 = min(0.67, 0.13)			

```

6 [0.20 - 0.40[: U4 with credibility: 0.13 = min(0.47,0.13)
7 [0.60 - <[: U5 with credibility: 0.60 = min(0.60,1.00)

```

The quintiles rating-by-sorting result, shown in Listing 14.7 on the previous page, confirms the previous COPELAND rating-by-ranking result. However, University U5 is here sorted conjointly into the fourth and the fifth quintile, and as such is part of the top rated institutions. A result already convincingly illustrated in the ranked heatmap shown in Figure 14.3 on page 200.

Let us conclude this hypothetical rating case study, by saying that we prefer this latter rating-by-sorting approach; perhaps impreciser, due the case given, to missing and contradictory performance data; yet, well grounded in a powerful bipolar-valued logic and epistemic framework. The next Chapter 15 proposes a series of decision problems suitable for exercises and exam questions.

## References

- Bisdorff R (2010) Enumerating chordless circuits in directed graphs. In: ORBEL24-2010, 24th Annual Conference of the Belgian Operational Research Society (ORBEL aka Sogesci-B.V.W.B.), January 28-29, Liège (BE), Université de Liège (BE), pp 1–12, URL <http://hdl.handle.net/10993/23926> 202
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# Chapter 15

## Exercises

**Abstract** In this chapter, we propose a series of decision problems of various difficulties which may serve as exercises and exam questions for an Algorithmic Decision Theory or Multiple-Criteria Decision Analysis course. They cover *selection*, *ranking* and *rating* decision problems.

### Introduction

The difficulty of the exercises is marked as follows: *warming up* (§), *home work* (§§), *research work* (§§§). Solutions should be supported both by computational Python code using the DIGRAPH3 programming resources as well as by methodological and algorithmic arguments from the *Algorithmic Decision Theory* Lectures (Bisdorff, 2020, 2021).

### 15.1 Who will receive the best student award? (§)

#### Data

In Table 15.1 on the next page you find the actual grades obtained by four students : *Ariana* (A), *Bruce* (B), *Clare* (C) and *Daniel* (D) in five courses: C1, C2, C3, C4 and C5 weighted by their respective ECTS points.<sup>1</sup> The student grades are measured on an ordinal performance scale from 0 pts (weakest) to 20 pts (highest). Assume that the grading admits a preference discrimination threshold of 1 point. No considerable performance differences are given. The more ECTS points, the more importance a course takes in the curriculum of the students. An award is to be granted to the student showing the *best* result.

---

<sup>1</sup> ECTS stands for *European Credit Transfer System*. It is a unit used to grade diplomas in all participating countries [https://wwwen.uni.lu/studies/ects\\_credits](https://wwwen.uni.lu/studies/ects_credits)

**Table 15.1** Grades obtained by the students

Course ECTS	CF 2	C2 3	C3 4	C4 2	C5 4
Ariana (A)	11	13	9	15	11
Bruce (B)	12	9	13	10	13
Clare (C)	8	11	14	12	14
Daniel (D)	15	10	12	8	13

## Questions

1. Edit a `PerformanceTableau` instance with the data shown above.
2. Who would you nominate for the grant?
3. Explain and motivate your selection algorithm.
4. Assume that the grading may actually admit an indifference discrimination threshold of 1 point and a preference discrimination threshold of 2 points. How stable is your grant recommendation with respect to this preference discrimination power of the grading scale?

## 15.2 How to fairly rank movies? (§)

### Data

File `graffiti03.py`<sup>2</sup> contains a performance tableau inspired by the star-rating of movies that could be seen in the city of Luxembourg during Spring 2003 (source: [Graffiti magazine, Luxembourg, February 2003](#)). Its content is shown in Fig. 15.1 on the facing page.

```

1  >>> from perfTabs import PerformanceTableau
2  >>> t = PerformanceTableau('graffiti03')
3  >>> t.showHTMLPerformanceHeatmap(WithActionNames=True, \
4  ...                                pageTitle='Graffiti Star wars',
5  ...                                rankingRule=None, colorLevels=5,
6  ...                                ndigits=0)

```

The critic's opinions are expressed on seven ordinal rating levels: -2 (two zeros, *I hate*), -1 (one zero, *I don't like*), 1 (one star, *maybe*), 2 (two stars, *good*), 3 (three stars, *excellent*), 4 (four stars, *not to be missed*), and 5 (five stars, *a master piece*). Notice the many missing data (NA) when a critic had not seen the respective movie. Mind also that the ratings of two movie critics (`jh` and `vt`) are given a higher significance weight.

---

<sup>2</sup> The file is provided in the `examples` directory of the DIGRAPH3 resources.

## Performance table graffiti03

criteria	ap	as	cf	cn	cs	dr	jh	jt	mk	mr	rr	td	vt
weight	1.00	1.00	1.00	1.00	1.00	1.00	2.00	1.00	1.00	1.00	1.00	1.00	2.00
ah	NA	NA	NA	-1	1	NA	1	1	2	NA	1	3	NA
aw	-1	NA	1	NA	2	NA	NA	-1	1	NA	1	NA	NA
bb	2	1	2	2	3	2	2	2	3	2	3	1	1
dl	-1	-1	-1	NA	-1	1	1	1	NA	1	1	1	-1
gny	2	4	2	4	2	3	3	4	2	4	3	2	3
gs	1	NA	-1	NA	1	1	NA	1	NA	-1	-1	-1	NA
hn	3	NA	3	2	2	NA	2	2	3	2	2	NA	1
la	3	2	3	3	2	2	3	3	3	4	3	NA	3
lor	2	3	3	NA	3	4	3	4	1	2	2	2	2
ma	NA	NA	NA	3	2	3	3	3	3	2	2	3	3
md	1	1	-1	NA	NA	-1	NA	1	-1	NA	-1	1	NA
mi	NA	-1	NA	1	-1	NA	1	2	NA	NA	-1	2	1
sa	NA	NA	NA	NA	2	NA	2	1	3	1	NA	NA	NA
sc	1	NA	1	NA	-1	NA	NA	1	1	NA	1	NA	NA
sha	2	-1	1	1	2	2	-1	2	1	1	1	NA	NA
ss	3	3	3	4	2	3	3	3	3	3	1	3	3
vf	NA	NA	NA	1	NA	NA	1	1	NA	NA	1	1	NA

Fig. 15.1 Star-rating of movies from February 2003

### Questions

1. The GRAFFITI magazine suggests a best rated movie by computing the average number of stars it received, ignoring the missing data and any significance weights of the critics. What movie gets the highest average?
2. By taking into account missing data and varying significance weights, how may one find the best rated movie without computing any average of stars?
3. How would one rank these movies so as to at best respect the weighted rating opinions of each movie critic?
4. In what ranking position would appear a movie not seen by any critic? Confirm computationally your answer by adding such a fictive, *not at all evaluated*, movie to the given performance tableau instance.
5. How robust are the preceding results when the significance weights of the movie critics are considered to be only of ordinal type?

### 15.3 What is your best choice recommendation? (§§)

#### Data

A person, who wants to buy a TV set, retains after a first selection, eight potential TV models <sup>3</sup>. To make their best choice, these eight models were evaluated with respect to three decision objectives of equal importance:

1. *Costs* of the set (to be minimised);
2. *Picture and Sound* quality of the TV (to be maximised);
3. *Maintenance contract* quality of the provider (to be maximised).

The *Costs* objective is assessed by the price of the TV set (criterion *Pr* to be minimised). *Picture* quality (criterion *Pq*), *Sound* quality (criterion *Sq*) and *Maintenance contract* quality (criterion *Mq*) are each one assessed on a four levels qualitative performance scale: -1 (*not good*), 0 (*average*), 1 (*good*) and 2 (*very good*).

The actual evaluation data are gathered in Table 15.2.

**Table 15.2** Performance evaluations of the potential TV sets

Criteria Significance	Pr (€) 2	Pq 1	Sq 1	Mq 1
Model T1	-1300	2	2	0
Model T2	-1200	2	2	1
Model T3	-1150	2	1	1
Model T4	-1000	1	1	-1
Model T5	-950	1	1	0
Model T6	-950	0	1	-1
Model T7	-900	1	0	-1
Model T8	-900	0	0	0

The *Price* criterion *Pr* supports furthermore an indifference threshold of 25.00€ and a preference threshold of 75.00€. No considerable performance differences (veto thresholds) are to be considered.

#### Questions

1. Edit a new `PerformanceTableau` instance with the data shown in Table 15.2 and illustrate its content by best showing objectives, criteria, decision alternatives and performance table. If needed, write adequate python code.
2. What is the best TV set to recommend?
3. Illustrate your best choice recommendation with an adequate `graphviz` drawing.

---

<sup>3</sup> A similar didactic decision problem is discussed in [Vincke \(1992, pp.33-35\)](#)

4. Explain and motivate your selection algorithm.
5. Assume that the qualitative criteria: *Picture quality* ( $P_Q$ ), *Sound quality* ( $S_Q$ ), and *Maintenance contract quality* ( $M_Q$ ), are all three considered to be equi-significant and that the significance of the *Price* criterion ( $P_r$ ) equals the significance of these three quality criteria taken together. How stable is your best choice recommendation with thus reviewing the criteria significance weights?

## 15.4 Planning the next holiday activity (§§)

### Data

A family, staying during their holidays in Ronda (Andalusia), is planning the next day's activity (Bisdorff, 2008). The alternatives shown in Table 15.3 are considered as potential actions.

**Table 15.3** The set of potential holiday activities

Identifier	Name	Comment
ant	<i>Antequerra</i>	An afternoon excursion to Antequera and surroundings
ard	<i>Ardales</i>	An afternoon excursion to Ardales and El Chorro
be	<i>Beach</i>	Sun, sea, fun and more
crd	<i>Cordoba</i>	A whole day visit by car to Cordoba
dn	<i>Fa niente</i>	Do nothing
lw	<i>Long walk</i>	A whole day hiking
mal	<i>Malaga</i>	A whole day visit to Malaga
sev	<i>Sevilla</i>	A whole day visit to Sevilla
sw	<i>Short walk</i>	Less than a half day hiking

The family members agree to measure their preferences with respect to a set of seven performance criteria such as the time to attend the place (to be minimised), the required physical investment, the expected quality of the food, touristic interest, relaxation, sun fun & more (see Tab. 15.4 on the following page)

The evaluation, agreed with all family members, of the nine alternatives on all the criteria resulted in the performance tableau shown in Table 15.5 on the next page.

All performances on the qualitative criteria are marked on a same ordinal scale going from 0 (lowest) to 10 (highest). On the quantitative Distance criterion (to be minimised), the required travel time to go to and return from the activity is marked in negative minutes. In order to model only effective preferences, an indifference threshold of 1 point and a preference threshold of 2 points is put on the qualitative performance measures. On the Distance criterion, an indifference threshold of 20 min, and a preference threshold of 45 min. is considered. Furthermore, a difference

**Table 15.4** The set of performance criteria

Identifier	Name	Comment
cult	<i>Cultural Interest</i>	Andalusian heritage
dis	<i>Distance</i>	Minutes by car to go to and return from the activity
food	<i>Food</i>	Quality of the expected food opportunities
sun	<i>Sun, Fun &amp; More</i>	No comment
phy	<i>Physical Investment</i>	Contribution to physical health care
rel	<i>Relaxation</i>	Anti-stress support
tour	<i>Tourist Attraction</i>	How many stars in the guide?

**Table 15.5** The performance tableau

Criteria	Weight	ant	ard	be	crd	dn	lw	mal	sev	sw
cult	1	7	3	0	10	0	0	5	10	0
dis	1	-120	-100	-30	-360	0	-90	-240	-240	0
phy	1	3	7	0	5	0	10	5	5	5
rel	1	1	5	8	3	10	5	3	3	6
food	1	8	10	4	8	10	1	8	10	1
sun	1	0	3	10	3	1	3	8	5	5
tour	1	5	7	3	10	0	8	10	10	5

of more than two hours to go to and return from the activity's place is considered to raise a veto.

The individual criteria each reflect one or the other family member's preferential point of view. Therefore they are judged equi-significant for the best action to be eventually chosen.

## Questions

1. Edit a new `PerformanceTableau` instance with the data shown above and illustrate its content.
2. How do the criteria express their preferential view point on the set of alternatives? For instance the Tourist Attraction criterion appears to be in its preferential judgments somehow positively correlated with both the Cultural Interest and the Food criteria. It is also apparent that the Distance criterion is somehow negatively correlated to these latter criteria. How to explore and illustrate these intuitions?
3. What is the preference modelling effect of the considerable performance difference of 120 minutes assumed for the Distance criterion?
4. How would you rank the potential holiday activities with the fairest balance of the performance criteria?
5. What is the first choice you would recommend to the family members?

## 15.5 What is the best public policy? (§§)

### Data files

- File `perfTab_1.py`<sup>4</sup> contains a 3-Objectives performance tableau with 100 performance records concerning public policies evaluated with respect to an *economic*, a *societal* and an *environmental* decision objective.
- File `historicalData_1.py` contains a performance tableau of the same kind with 2000 historical performance records.

### Questions

1. Illustrate the content of the given `perfTab_1.py` performance tableau by best showing *objectives*, *criteria*, *decision alternatives* (public policies) and *performance evaluations*. If needed, write adequate python code.
2. Construct the corresponding bipolar-valued outranking digraph. How *confident* and/or *robust* are the apparent outranking situations?
3. What are apparently the 5 best-ranked decision alternatives in your decision problem from the different decision objectives point of views and from a global fair compromise view? Justify your ranking approach from a methodological point of view.
4. How would you rate your 100 public policies into relative deciles classes?
5. Using the given historical records in file `historicalData_1.py`, how would you rate your 100 public policies into absolute deciles classes?
6. Explain the differences you may observe between the absolute and the previous relative rating result.
7. Select among your 100 potential public policies a shortlist of up to 15 potential best policies, all reaching an absolute performance quantile of at least 66.67%.
8. Based on the previous best policies shortlist (see Question 7), what is your eventual best-choice recommendation? Is it perhaps a best choice unopposed by all three objectives?

## 15.6 A fair diploma validation decision (§§§)

### Data

Use the `RandomAcademicPerformanceTableau` class from the `randomPerfTab` module for generating realistic random students performance tableaux concerning a curriculum of nine ECTS weighted Courses ([Bisdorff, 2021](#)). Assume

---

<sup>4</sup> Files `perfTab_1.py` and `historicalData_1.py` are provided in the `examples` directory of the DIGRAPH3 resources ([Bisdorff, 2021](#)).

that all the gradings are done on an integer scale from 0 (weakest) to 20 (best). It is known that grading procedures are inevitably somehow imprecise; therefore assume an indifference discrimination threshold of 1 point and a preference discrimination threshold of 2 points. Furthermore, a performance difference of more than 12 points is considerable and will trigger a polarisation situation. To validate eventually their curriculum, the students are required to obtain more or less 10 points in each course.

### Questions

1. Design and implement a fair diploma validation decision rule based on the grades obtained in the nine Courses.
2. Run simulation tests with random students performance tableaux for validating your design and implementation.

### References

- Bisdorff R (2008) On clustering the criteria in an outranking based decision aid approach. In: Thi HAL, Bouvry P, Pham D (eds) Computation and Optimization in Information Systems and Management Sciences, Springer, CCIS, pp 409–418, URL <http://hdl.handle.net/10993/23718> 211
- Bisdorff R (2020) Lectures of the Algorithmic Decision Theory Course, University of Luxembourg. URL <http://hdl.handle.net/10993/37933> 207
- Bisdorff R (2021) Technical documentation of the Digraph3 collection of Python modules. URL <https://digraph3.readthedocs.io/en/latest/techDoc.html> 207, 213
- Graffiti magazine, Luxembourg (February 2003) Star wars 208
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## **Part IV**

### **Advanced topics**

The fourth part gathers five chapters introducing and discussing further going algorithmic developments. In Chapter 16 KENDALL's ordinal correlation index is consistently extended to bipolar-valued digraphs. Chapter 17 explains the important concept of digraph kernel and illustrates the DIGRAPH3 algorithmic approach to their computation. In the following Chapter 18, criteria significance weights are considered to be uncertain and become random variates. This idea opens the way to compute a bipolar-valued likelihood of outranking and outranked situations leading to  $\alpha\%$ -confident outranking digraphs. In Chapter 19, the criteria significance weights are considered to be only of ordinal type. This working hypothesis induces the need to put into place a specific robustness analysis of the corresponding outranking digraphs. A last Chapter 20 makes use of the preceding robustness analysis for introducing and discussing ideas, like two-stage elections with multipartisan primary selection or bipolar voting systems, for tempering plurality tyranny effects in social choice problems.

# Chapter 16

## On measuring the fitness of a multiple-criteria ranking

**Abstract** Starting from a motivating decision problem about how to list, from the best to the worst, a set movies that are star-rated by journalists and movie critics, the chapter shows that KENDALL's ordinal correlation index tau can be extended to a bipolar-valued relational equivalence measure of bipolar-valued digraphs. This finding gives way, on the one hand, to measure the fitness and fairness of multiple-criteria ranking rules. On the other hand, it provides a tool for illustrating preference divergences between decision objectives and criteria.

### 16.1 Listing movies from best star-rated to worst

In a stubborn keeping with a two-valued logic, where every argument can only be true or false, there is no place for efficiently taking into account missing data or logical indeterminateness. These cases are seen as problematic and, at best are simply ignored. Worst, in modern data science, missing data get often replaced with *fictive* values, potentially falsifying hence all subsequent computations.

In social choice problems, voting abstentions are, however, frequently observed and represent a social expression that may be significant for revealing non represented social preferences. And, in marketing studies, interviewees will not always respond to all the submitted questions. Again, such abstentions do sometimes contain nevertheless valid information concerning consumer preferences.

Such a case is given with a list of star-rated movies that could be seen in town in September 2007<sup>1</sup>. The underlying performance tableau data, stored in a file named `graffiti07.py`<sup>2</sup>, is shown below with the `showHTMLPerformanceTableau()` method :

```
1 >>> from outrankingDigraphs import\  
2 ... PerformanceTableau
```

---

<sup>1</sup> *Graffiti Star wars*, Edition Revue Luxembourg, September 2007, p. 30.

<sup>2</sup> to be found in the `examples` directory of the DIGRAPH3 resources

```

3 >>> gt07 = PerformanceTableau('graffiti07')
4 >>> gt07.showHTMLPerformanceTableau(\n
5 ...           title='Graffiti Star wars', \
6 ...           ndigits=0)

```

## Graffiti Star wars

criteria	AP	AS	CF	CS	DR	FG	GS	JH	JPT	MR	RR	SF	SJ	TD	VT
weight	1.00	1.00	1.00	1.00	1.00	1.00	1.00	3.00	1.00	1.00	1.00	1.00	1.00	1.00	3.00
<b>mv_AL</b>	3	-1	2	-1	NA	NA	3	NA	2	NA	NA	NA	2	NA	2
<b>mv BI</b>	1	-1	1	1	-1	NA	NA	NA	2	NA	2	NA	NA	NA	NA
<b>mv_CM</b>	NA	3	3	2	NA	NA	3	2	3	2	1	2	2	NA	2
<b>mv_DF</b>	NA	4	3	1	2	NA	1	3	2	2	2	3	-1	NA	1
<b>mv_DG</b>	3	2	2	3	3	1	4	3	3	3	NA	NA	-1	NA	3
<b>mv_DI</b>	1	2	NA	1	2	2	NA	1	2	2	NA	1	1	NA	NA
<b>mv_DJ</b>	3	1	3	NA	NA	NA	3	NA	2	2	NA	2	4	3	-1
<b>mv_FC</b>	3	2	2	1	3	NA	1	3	3	3	2	NA	2	1	3
<b>mv_FF</b>	2	3	2	1	2	2	NA	1	2	2	1	2	1	3	1
<b>mv_GG</b>	2	3	3	1	NA	NA	1	-1	NA	-1	2	NA	-1	NA	1
<b>mv_GH</b>	1	NA	1	-1	2	2	1	2	1	3	4	NA	NA	NA	NA
<b>mv_HN</b>	3	1	3	1	3	NA	4	3	2	NA	1	NA	3	3	3
<b>mv_HP</b>	1	NA	3	1	NA	NA	NA	1	1	2	3	NA	1	2	NA
<b>mv_HS</b>	2	4	2	3	2	2	2	3	4	2	3	NA	1	3	NA
<b>mv_JB</b>	3	4	3	NA	3	2	3	2	3	2	NA	2	2	NA	3
<b>mv_MB</b>	NA	2	NA	NA	1	NA	1	2	2	1	2	NA	2	2	NA
<b>mv_NP</b>	NA	1	3	1	NA	3	2	3	3	3	2	3	NA	NA	3
<b>mv_PE</b>	3	4	2	NA	3	1	NA	2	4	2	NA	3	3	NA	3
<b>mv_QS</b>	NA	3	2	NA	4	3	NA	4	4	3	3	4	NA	4	4
<b>mv_RG</b>	2	2	2	2	NA	NA	3	1	2	2	1	NA	NA	3	NA
<b>mv_RR</b>	3	2	NA	1	4	NA	4	3	3	4	3	3	NA	4	2
<b>mv_SM</b>	3	3	2	2	2	NA	3	2	2	3	NA	2	2	3	NA
<b>mv_TF</b>	-1	NA	1	1	1	NA	NA	1	2	NA	-1	NA	-1	1	NA
<b>mv_TM</b>	2	1	2	2	NA	NA	2	2	2	3	NA	2	NA	4	NA
<b>mv_TP</b>	2	3	3	1	2	NA	NA	3	2	NA	2	NA	1	NA	2

**Fig. 16.1** Star-ratings of movies from September 2007

Figure 16.1 shows the star-ratings of 25 movies by 15 journalists and movie critics: 5 stars (*masterpiece*), 4 stars (*must be seen*), 3 stars (*excellent*), 2 stars (*good*), 1 star (*could be seen*), -1 star (*I do not like*), -2 stars (*I hate*), NA (*not seen*). Notice in the second row the higher significance (3.00) that is granted to two locally renowned movie critics, namely JH and VT. Their opinion counts for three times the opinion of the other critics. With six times a 4 stars (*must be seen*) mark, mv\_QS is best-rated, followed by mv\_RR with four times a 4 stars mark. Fewest stars obtains movie mv\_TF with three times a -1 star (*don't like*) mark and five times a 1 star

mark. Notice that many movies, like movie `mv_BI`, are only rated by some of the critics.

To aggregate all the critics' star-ratings, the *Graffiti* magazine computes for each movie a global score –the average weighted number of stars it obtained– just ignoring the *not seen* movies. Listing 16.1 illustrates below how to compute these global scores using the data stored in the `gt07` performance tableau. Attribute `gt07.actions`, resp. `gt07.criteria`, contains the description of the 25 movies, resp. the 15 critics. The actual star-ratings are to be found in the `gt07.evaluation` attribute.

**Listing 16.1** Computing the average weighted number of stars per movie

```

1 >>> # gt07 = PerformanceTableau('graffiti07')
2 >>> globalScores = {}
3 >>> for mv in gt07.actions:
4 >>>     globalScores[mv] = Decimal('0')
5 >>>     sumWeights = Decimal('0')
6 >>>     for critic in gt07.criteria:
7 >>>         stars = gt07.evaluation[critic][mv]
8 >>>         if stars != gt07.NA:
9 >>>             weight = gt07.criteria[critic]['weight']
10 >>>             globalScores[mv] += (stars * weight)
11 >>>             sumWeights += weight
12 >>>     globalScores[mv] /= sumWeights
13 >>> graffitiList = [(globalScores[mv],mv) for mv in globalScores]
14 >>> graffitiList.sort(reverse=True)
15 >>> for item in graffitiList:
16 >>>     print('%s: %.2f' % (item[1],item[0]) )
17 mv_QS: 3.60
18 mv_RR: 2.88
19 mv_PE: 2.67
20 mv_JB: 2.62
21 mv_HN: 2.62
22 mv_NP: 2.60
23 mv_HS: 2.60
24 mv_DG: 2.56
25 mv_SM: 2.53
26 mv_FC: 2.41
27 mv_TP: 2.21
28 mv_CM: 2.20
29 mv_TM: 2.17
30 mv_DF: 1.94
31 mv_RG: 1.83
32 mv_MB: 1.73
33 mv_GH: 1.67
34 mv_DJ: 1.67
35 mv_FF: 1.61
36 mv_AL: 1.60
37 mv_HP: 1.55
38 mv_DI: 1.42
39 mv_GG: 0.71
40 mv_BI: 0.71
41 mv_TF: 0.55

```

The global scores ranking confirms in Lines 17-18 both leading movies –`mv_QS` (3.60) and `mv_RR` (2.88) – as well as in Line 41 the weakest rated one –`mv_TF` (0.55). Mind however that these global averages, due to the numerous missing ratings, are not computed with commensurable denominators; some critics do indeed use a more or less extended range of stars. The movies not seen for instance by critic `SJ` are favoured, as this critic is severer than others in her rating. Dropping the movies that were not rated by all the critics is not possible either, as none of the 25

movies was actually rated by all the 15 critics. Providing a fictive value for the many not seen situations, will as well always somehow falsify the global scores. What to do?

A better approach is to rank the movies on the basis of pairwise bipolar-valued “*rated at least as well as*” statements. Under this epistemic argumentation approach, missing evaluations are naturally treated as opinion abstentions and hence do not falsify the logical computations. Such a NETFLOWS ranking from best-rated to weakest-rated is provided by the **heatmap** browser view generated with the `showHTMLPerformanceHeatmap()` method (see List. 16.2).

**Listing 16.2** Showing the movie from best to worst rated in a heatmap view

```

1 >>> gt07.showHTMLPerformanceHeatmap(\n2 ...     pageTitle='Ranking the movies',\n3 ...     rankingRule='NetFlows',\n4 ...     Correlations=True,\n5 ...     ndigits=0)

```

The NETFLOWS ranking shown in Figure 16.2 on the next page confirms again that movie `mv_QS`, with 6 *must be seen* marks, is correctly first-ranked and the movie `mv_TF` is last-ranked with five *don't like* marks.

It is fair, however, to eventually mention here that the *Graffiti* magazine's average stars ranking rule is actually showing a very similar result. Indeed, average scores usually confirm well all evident pairwise comparisons, yet *enforce* comparability for all less evident ones. How to judge now the fitness of a given ranking rule?

This is the purpose of the ordinal correlation *tau* indexes shown in Figure 16.2 on the facing page 3rd row. Computing these ordinal correlation indexes is the subject of the next Section.

## 16.2 KENDALL's ordinal correlation tau index

M. G. Kendall defined his ordinal correlation  $\tau$  (*tau*) index for linear orders of dimension  $n$  as a balancing of the number  $Co$  of correctly oriented pairs against the number  $In$  of incorrectly oriented pairs (Kendall, 1938). The total number of irreflexive pairs being  $n(n-1)$ , in the case of linear orders,  $Co + In = n(n-1)$ . Hence  $\tau = \left( \frac{Co}{n(n-1)} \right) - \left( \frac{In}{n(n-1)} \right)$ . In case  $In$  is zero,  $\tau = +1$  (all pairs are equivalently oriented); inversely, in case  $Co$  is zero,  $\tau = -1$  (all pairs are differently oriented).

Noticing that  $\frac{Co}{n(n-1)} = 1 - \frac{In}{n(n-1)}$ , and recalling that the bipolar-valued negation is operated by changing the sign of the characteristic value:

$$\tau = 1 - 2 \frac{In}{n(n-1)} = -\left( 2 \frac{In}{n(n-1)} - 1 \right) = 2 \frac{Co}{n(n-1)} - 1, \quad (16.1)$$

### Ranking the movies

criteria	JH	JPT	AP	DR	MR	VT	GS	CS	SJ	RR	TD	CF	SF	AS	FG
weights	+3.00	+1.00	+1.00	+1.00	+1.00	+3.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00	+1.00
tau(*)	+0.50	+0.43	+0.32	+0.26	+0.25	+0.23	+0.16	+0.14	+0.14	+0.13	+0.11	+0.11	+0.10	+0.08	+0.03
mv_QS	4	4	NA	4	3	4	NA	NA	NA	3	4	2	4	3	3
mv_RR	3	3	3	4	4	2	4	1	NA	3	4	NA	3	2	NA
mv_DG	3	3	3	3	3	3	4	3	-1	NA	NA	2	NA	2	1
mv_NP	3	3	NA	NA	3	3	2	1	NA	2	NA	3	3	1	3
mv_HN	3	2	3	3	NA	3	4	1	3	1	3	3	NA	1	NA
mv_HS	3	4	2	2	2	NA	2	3	1	3	3	2	NA	4	2
mv_SM	3	2	3	2	2	3	NA	2	2	3	2	2	NA	3	2
mv_JB	2	3	3	3	2	3	3	NA	2	NA	NA	3	2	4	2
mv_PE	2	4	3	3	2	3	NA	NA	3	NA	NA	2	3	4	1
mv_FC	3	3	3	3	3	3	1	1	2	2	1	2	NA	2	NA
mv_TP	3	2	2	2	NA	2	NA	1	1	2	NA	3	NA	3	NA
mv_CM	2	3	NA	NA	2	2	3	2	2	1	NA	3	2	3	NA
mv_DF	3	2	NA	2	2	1	1	1	-1	2	NA	3	3	4	NA
mv_TM	2	2	2	NA	3	NA	2	2	NA	NA	4	2	2	1	NA
mv_DJ	NA	2	3	NA	2	-1	3	NA	4	NA	3	3	2	1	NA
mv_AL	NA	2	3	NA	NA	2	3	-1	2	NA	NA	2	NA	-1	NA
mv_RG	1	2	2	NA	2	NA	3	2	NA	1	3	2	NA	2	NA
mv_MB	2	2	NA	1	1	NA	1	NA	2	2	2	NA	NA	2	NA
mv_GH	2	1	1	2	3	NA	1	-1	NA	4	NA	1	NA	NA	2
mv_HP	1	1	1	NA	2	NA	NA	1	1	3	2	3	NA	NA	NA
mv_BI	NA	2	1	-1	NA	NA	NA	1	NA	2	NA	1	NA	-1	NA
mv_DI	1	2	1	2	2	NA	NA	1	1	NA	NA	NA	1	2	2
mv_FF	1	2	2	2	2	1	NA	1	1	1	3	2	2	3	2
mv_GG	-1	NA	2	NA	-1	1	1	1	-1	2	NA	3	NA	3	NA
mv_TF	1	2	-1	1	NA	NA	NA	1	-1	-1	1	1	NA	NA	NA

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: standard, Ranking rule: NetFlows

Ordinal (Kendall) correlation between global ranking and global outranking relation: +0.780

Mean marginal correlation (a) : +0.234

Standard marginal correlation deviation (b) : +0.147

Ranking fairness (a) - (b) : +0.087

**Fig. 16.2** Star-ratings of movies ranked with the NETFLows rule

KENDALL's original *tau* definition implemented in fact the bipolar-valued negation of the non equivalence of two linear orders, i.e. the normalised majority margin of equivalently oriented irreflexive pairs.

Let  $r_1$  and  $r_2$  be two random crisp relations defined on a same set of 5 alternatives. Computing KENDALL's *tau* index may be done as shown in Listing 16.3.

**Listing 16.3** Computing a relational equivalence digraph

```

1 >>> from randomDigraphs import RandomDigraph
2 >>> r1 = RandomDigraph(order=5, Bipolar=True)
3 >>> r2 = RandomDigraph(order=5, Bipolar=True)
4 >>> from digraphs import EquivalenceDigraph
5 >>> eqd = EquivalenceDigraph(r1,r2)
6 >>> eqd.showRelationTable(ReflexiveTerms=False)
7 * ---- Relation Table ----

```

```

8   r(<=>) | 'a1'  'a2'  'a3'  'a4'  'a5'
9   -----|-----
10  'a1' |   -  -1.00  1.00  -1.00  1.00
11  'a2' |  -1.00   -  -1.00  1.00  -1.00
12  'a3' |  -1.00 -1.00   -  1.00  1.00
13  'a4' |  -1.00  1.00 -1.00   -  1.00
14  'a5' |  -1.00  1.00 -1.00  1.00   -
15  Valuation domain: [-1.00;1.00]
16 >>> eqd.correlation
17   {'correlation': -0.1, 'determination': 1.0}

```

In the table of the equivalence relation ( $r_1 \Leftrightarrow r_2$ ) above (see List. 16.1 on page 219 Lines 10-14), we observe that the normalised majority margin of equivalent versus non equivalent irreflexive pairs amounts to  $(9 - 11)/20 = -0.1$ , i.e. the value of KENDALL's *tau* index in this plainly determined crisp case (see Line 17).

What happens now with more or less determined and even partially indeterminate relations? May we proceed in a similar way?

### 16.3 Bipolar-valued relational equivalence

Two random bipolar-valued digraphs  $d_1$  and  $d_2$  of order five, generated in Listing 16.4, will help exploring this idea.

**Listing 16.4** Two random bipolar-valued digraphs

```

1 >>> from randomDigraphs import RandomValuationDigraph
2 >>> d1 = RandomValuationDigraph(order=5, seed=1)
3 >>> d1.showRelationTable(ReflexiveTerms=False)
4   * ----- Relation Table -----
5   r(d1) | 'a1'  'a2'  'a3'  'a4'  'a5'
6   -----|-----
7   'a1' |   -  -0.66  0.44  0.94  -0.84
8   'a2' |  -0.36   -  -0.70  0.26  0.94
9   'a3' |  0.14  0.20   -  0.66  -0.04
10  'a4' |  -0.48  -0.76  0.24   -  -0.94
11  'a5' |  -0.02  0.10  0.54  0.94   -
12  Valuation domain: [-1.00;1.00]
13 >>> d2 = RandomValuationDigraph(order=5, seed=2)
14 >>> d2.showRelationTable(ReflexiveTerms=False)
15   * ----- Relation Table -----
16   r(d2) | 'a1'  'a2'  'a3'  'a4'  'a5'
17   -----|-----
18   'a1' |   -  -0.86  -0.78  -0.80  -0.08
19   'a2' |  -0.58   -  0.88  0.70  -0.22
20   'a3' |  -0.36  0.54   -  -0.46  0.54
21   'a4' |  -0.92  0.48  0.74   -  -0.60
22   'a5' |  0.10  0.62  0.00  0.84   -
23  Valuation domain: [-1.00;1.00]

```

In the generated random digraphs  $d_1$  and  $d_2$ , 9 pairs, like  $(a_1, a_2)$  or  $(a_3, a_2)$  for instance, appear equivalently oriented (see Lines 7,18 or 9,20). The Equiva-

lenceDigraph class computes this bipolar-valued relational equivalence between digraphs  $d1$  and  $d2$  (see List. 16.5).

**Listing 16.5** Bipolar-valued Equivalence Digraph

```

1 >>> from digraphs import EquivalenceDigraph
2 >>> eqd = EquivalenceDigraph(d1,d2)
3 >>> eqd.showRelationTable(ReflexiveTerms=False)
4   * ----- Relation Table -----
5   r(<=>) | 'a1'  'a2'  'a3'  'a4'  'a5'
6   -----|-----
7   'a1' |   -    0.66  -0.44 -0.80  0.08
8   'a2' |  0.36   -   -0.70  0.26 -0.22
9   'a3' | -0.14  0.20   -   -0.46 -0.04
10  'a4' |  0.48 -0.48  0.24   -    0.60
11  'a5' | -0.02  0.10  0.00  0.84   -
12
12 Valuation domain: [-1.00;1.00]

```

In our bipolar-valued epistemic logic, logical disjunctions and conjunctions are implemented as max, respectively min operations. Notice also that the logical equivalence ( $d1 \Leftrightarrow d2$ ) corresponds to a double implication ( $d1 \Rightarrow d2$ )  $\wedge$  ( $d2 \Rightarrow d1$ ) and that the implication ( $d1 \Rightarrow d2$ ) is logically equivalent to the disjunction ( $\neg d1 \vee d2$ ).

When  $r(xd1y)$  and  $r(xd2y)$  denote the bipolar-valued characteristic values of relation  $d1$ , resp.  $d2$ , we may hence compute as follows a majority margin  $M(d1 \Leftrightarrow d2)$  between equivalently and not equivalently oriented irreflexive pairs  $(x,y)$ :

$$M(d1 \Leftrightarrow d2) = \sum_{(x \neq y)} \left[ \min \left( \max(-r(xd1y), r(xd2y)), \max(-r(xd2y), r(xd1y)) \right) \right]. \quad (16.2)$$

$M(d1 \Leftrightarrow d2)$  is thus given by the sum of the non reflexive terms of the relation table of  $eqd$ , the relational equivalence digraph computed above (see List. 16.5). In the crisp case,  $M(d1 \Leftrightarrow d2)$  is normalised with the maximum number of possible irreflexive pairs, namely  $n(n-1)$ . In the extended  $r$ -valued case, the maximal possible equivalence majority margin  $M$  corresponds to the sum  $D$  of the conjoint determinations of  $(xd1y)$  and  $(xd2y)$  (see Bisdorff, 2012):

$$D = \sum_{x \neq y} \min \left[ \text{abs}(r(xd1y)), \text{abs}(r(xd2y)) \right], \quad (16.3)$$

and we obtain hence in the general  $r$ -valued case:

$$\tau(d1, d2) = \frac{M(d1 \Leftrightarrow d2)}{D}. \quad (16.4)$$

$\tau(d1, d2)$  corresponds so to the classical ordinal correlation index, yet restricted to the conjointly determined parts of the given digraphs  $d1$  and  $d2$ . In the limit case

of two crisp linear orders,  $D$  equals  $n(n - 1)$ , i.e. the number of irreflexive pairs, and we recover KENDALL's original  $\tau$  index definition.

It is worthwhile noticing that the ordinal correlation index  $\tau(d_1, d_2)$  one obtains above corresponds in fact to the ratio of:

- $r(d_1 \Leftrightarrow d_2) = \frac{M(d_1 \Leftrightarrow d_2)}{n(n-1)}$ :

the normalised majority margin of the pairwise *relational* equivalence statements, also called *valued ordinal correlation*, and

- $d = \frac{D}{n(n-1)}$ :

the normalised determination of the corresponding pairwise relational equivalence statements, in fact the *determinateness* of the relational equivalence digraph.

The epistemic determination effect is thus successfully *out-factored* from the ordinal correlation effect. With completely determined relations,  $\tau(d_1, d_2) = r(d_1 \Leftrightarrow d_2)$ . The ordinal correlation with a completely indeterminate digraph, i.e. when  $D = 0$ , is set by convention to the indeterminate correlation value 0.0. With uniformly chosen random  $r$ -valued digraphs, the expected  $\tau$  index is 0.0, denoting in fact an indeterminate relational equivalence. The corresponding expected normalised determination  $d$  is about 0.333 (see [Bisdorff \(2012\)](#)).

We verify Equation 16.4 on the preceding page below with help of the equivalence digraph  $eqd$  computed in Listing 16.5 on the previous page.

**Listing 16.6** Computing the ordinal correlation index from the equivalence digraph

```

1 >>> # eqd = EquivalenceDigraph(d1, d2)
2 >>> M = Decimal('0'); D = Decimal('0')
3 >>> n2 = eqd.order*(eqd.order - 1)
4 >>> for x in eqd.actions:
5     ...     for y in eqd.actions:
6         ...         M += eqd.relation[x][y]
7         ...         D += abs(eqd.relation[x][y])
8 >>> print('r(rd1<=>rd2) = %+.3f, d = %.3f, tau = %+.3f' % \
9           (M/n2,D/n2,M/D))
10    r(rd1<=>rd2) = +0.026, d = 0.356, tau = +0.073

```

The DIGRAPH3 resources directly provide for the preceding computations the `computeOrdinalCorrelation()` method which renders a dictionary with a correlation ( $\tau$ ) and a determination ( $d$ ) attribute.  $r(d_1 \Leftrightarrow d_2)$  is recovered by multiplying  $\tau$  with  $d$  (see List. 16.7 Line 4).

**Listing 16.7** Computing the valued ordinal correlation index

```

1 >>> corrd1d2 = d1.computeOrdinalCorrelation(d2)
2 >>> tau = corrd1d2['correlation']
3 >>> d = corrd1d2['determination']
4 >>> r = tau * d
5 >>> print('tau(d1,d2) = %+.3f, d = %.3f,\ \
6 ...         r(d1<=>d2) = %+.3f' % (tau, d, r))
7   tau(d1,d2) = +0.073, d = 0.356, r(d1<=>d2) = +0.026

```

The DIGRAPH3 resources provide for convenience a direct `showCorrelation()` method:

```

1 >>> d1.showCorrelation(\n
2 ...     d1.computeOrdinalCorrelation(d2) )\n3 Correlation indexes:\n4     Extended Kendall tau      : +0.073\n5     Epistemic determination   :  0.356\n6     Bipolar-valued equivalence : +0.026

```

We are now ready for assessing the quality of the NETFLOWS ranking of the movies shown in the heatmap view of Figure 16.2 on page 221.

## 16.4 Fitness of ranking heuristics

The NETFLOWS ranking of the movies shown in the heatmap view in Figure 16.2 on page 221 is based on the bipolar-valued outranking digraph modelling the pairwise global “*rated at least as well as*” relation among the 25 movies from the performance tableau instance `gt07`.

**Listing 16.8** The bipolar-valued outranking digraph of the star-rated movies

```

1 >>> bod = BipolarOutrankingDigraph(gt07)\n2 *----- Object instance description -----*\n3 Instance class    : BipolarOutrankingDigraph\n4 Instance name     : rel_grafittiPerfTab.xml\n5 Actions           : 25\n6 Criteria          : 15\n7 Size              : 390\n8 Determinateness   : 65%\n9 Valuation domain  : {'min': Decimal('-1.0'),\n10                  'med': Decimal('0.0'),\n11                  'max': Decimal('1.0'),}\n12 >>> g.computeCoSize()\n13 188

```

Listing 16.8 reveals that the outranking digraph `bod` contains 390 positively validated (Line 7), 188 positively invalidated (Line 13), and 22 indeterminate outranking situations from the potential  $25 \times 24 = 600$  irreflexive movie pairs.

Listing 16.9 illustrates with the `NetFlowsOrder` class from the `linearOrders` module how to compute the global NETFLOWS ranking `nf`, as shown in the ordered heat map of Figure 16.2 on page 221 and the bipolar-valued relational equivalence of the `nf` ranking with each one the individual critic’s star-ratings.

**Listing 16.9** Computing marginal criterion correlations with global NETFLOWS ranking

```

1 >>> from linearOrders import NetFlowsOrder\n2 >>> nf = NetFlowsOrder(bod)\n3 >>> nf.netFlowsRanking\n4 ['mv_QS', 'mv_RR', 'mv_DG', 'mv_NP', 'mv_HN', 'mv_HS', 'mv_SM',\n5  'mv_JB', 'mv_PE', 'mv_FC', 'mv_TP', 'mv_CM', 'mv_DF', 'mv_TM',\n6  'mv_DJ', 'mv_AL', 'mv_RG', 'mv_MB', 'mv_GH', 'mv_HP', 'mv_BI',\n7  'mv_DI', 'mv_FF', 'mv_GG', 'mv_TF']\n8 >>> for i, item in enumerate(\n

```

```

9 ...         bod.computeMarginalVersusGlobalRankingCorrelations(\n
10 ...             nf.netFlowsRanking,ValuedCorrelation=True) ):\n
11 ...     print('r(%s<=>nf) = %+.3f' % (item[1],item[0]) )\n
12\n
13 r (JH<=>nf)   = +0.500\n
14 r (JPT<=>nf)  = +0.430\n
15 r (AP<=>nf)   = +0.323\n
16 r (DR<=>nf)   = +0.263\n
17 r (MR<=>nf)   = +0.247\n
18 r (VT<=>nf)   = +0.227\n
19 r (GS<=>nf)   = +0.160\n
20 r (CS<=>nf)   = +0.140\n
21 r (SJ<=>nf)   = +0.137\n
22 r (RR<=>nf)   = +0.133\n
23 r (TD<=>nf)   = +0.110\n
24 r (CF<=>nf)   = +0.110\n
25 r (SF<=>nf)   = +0.103\n
26 r (AS<=>nf)   = +0.080\n
27 r (FG<=>nf)   = +0.027

```

In Listing 16.9 on the preceding page (see Lines 13-27), we obtain the relational equivalence characteristic values shown in the third row of the ranked heatmap (see Fig. 16.2 on page 221). The global NETFLOWS ranking `nf` represents obviously a rather balanced compromise with respect to each movie critic's star-ratings, as there appears no negative correlation with anyone of them. The ranking `nf` apparently takes also correctly in account that the journalist *JH*, a locally renowned movie critic, shows a higher significance weight (see Line 13).

The ordinal correlation between the global NETFLOWS ranking and the outranking digraph `bod` may be furthermore computed as illustrated in Listing 16.10:

**Listing 16.10** Computing the ordinal correlation between NETFLOWS and global outranking digraph

```

1 >>> corrgnf = bod.computeOrdinalCorrelation(nf)\n2 >>> bod.showCorrelation(corrgnf)\n3 Correlation indexes:\n4     Extended Kendall tau      : +0.780\n5     Epistemic determination   :  0.300\n6     Bipolar-valued equivalence : +0.234

```

One may notice in Line 4 that the ordinal correlation  $\tau(\text{bod}, \text{nf})$  index between the NETFLOWS ranking `nf` and the determined part of the outranking digraph `bod` is quite high (+0.78). Due to the rather high number of missing data, the  $r$ -valued relational equivalence between the `nf` and the `bod` digraph, with a characteristics value of only +0.234, may be misleading. Yet, +0.234 still corresponds to a 62% majority support of the movie critics' star-ratings.

It would be interesting to compare similarly the correlations one may obtain with other global ranking heuristics, like the COPELAND ranking rule.

## 16.5 Illustrating preference divergences

The bipolar-valued relational equivalence indexes gives us, via the `showCriteriaCorrelationTable(ValuedCorrelation=True)` method, a further measure for studying how *divergent* may appear the rating opinions expressed by the movie critics.

	AP	AS	CF	CS	DR	FG	GS	JH	JPT	MR	RR	SF	SJ	TD	VT
AP	+0.63	+0.04	+0.19	+0.09	+0.22	-0.01	+0.11	+0.23	+0.25	+0.08	+0.02	+0.04	+0.19	+0.04	+0.12
AS		+0.77	+0.12	+0.12	+0.04	-0.02	-0.06	+0.02	+0.24	-0.08	+0.07	+0.04	-0.07	-0.01	+0.02
CF			+0.77	+0.07	+0.11	+0.03	+0.05	+0.07	+0.10	-0.03	+0.01	+0.00	+0.06	+0.03	-0.04
CS				+0.63	+0.04	-0.02	+0.07	+0.13	+0.25	+0.01	+0.03	+0.00	+0.02	+0.03	+0.07
DR					+0.45	+0.03	+0.07	+0.17	+0.23	+0.16	+0.06	+0.03	+0.10	+0.07	+0.10
FG						+0.15	-0.01	+0.04	+0.01	+0.06	-0.00	+0.02	+0.01	+0.01	+0.02
GS							+0.40	+0.07	+0.07	+0.09	-0.02	+0.00	+0.06	+0.04	+0.04
JH								+0.77	+0.28	+0.26	+0.15	+0.12	+0.10	+0.05	+0.14
JPT									+0.92	+0.15	+0.06	+0.09	+0.08	+0.08	+0.17
MR										+0.63	+0.18	+0.08	+0.03	+0.09	+0.10
RR											+0.51	+0.04	+0.01	+0.05	+0.05
SF												+0.18	+0.01	+0.02	+0.05
SJ													+0.51	+0.03	+0.07
TD														+0.26	+0.00
VT															+0.40

**Fig. 16.3** Pairwise valued correlation of the movie critics

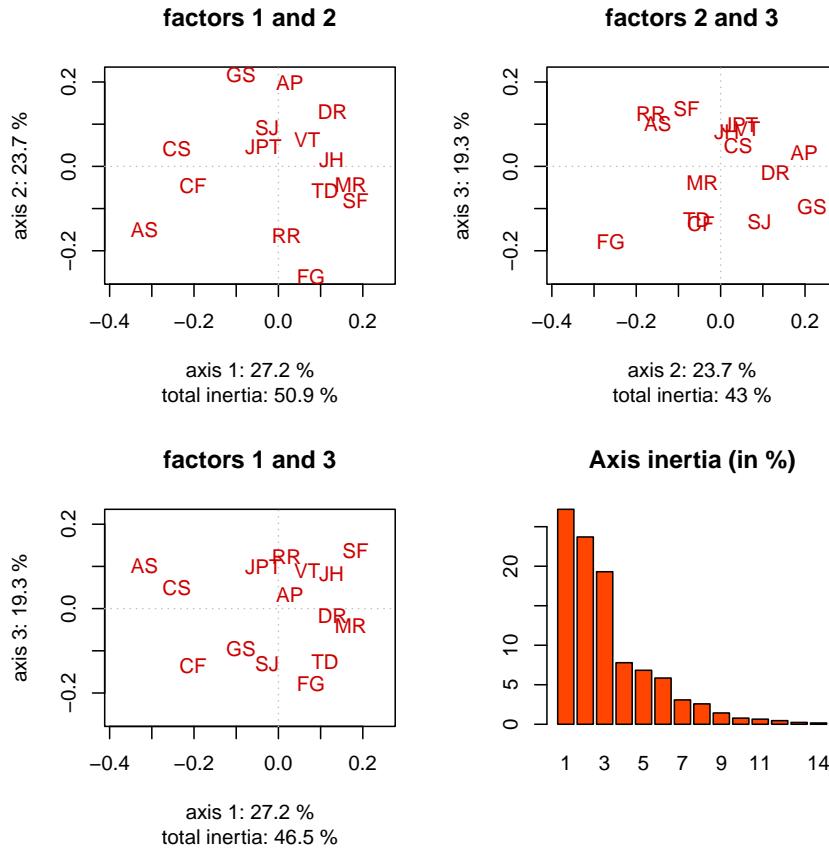
It is remarkable to notice in Figure 16.3 that, due to the quite numerous missing data, all pairwise valued ordinal correlation indexes  $r(x \leftrightarrow y)$  appear to be of low value, except the diagonal ones. These reflexive indexes  $r(x \leftrightarrow x)$  would trivially all amount to +1.0 in a plainly determined case. Here they indicate a reflexive normalised determination score  $d$ , i.e. the proportion of pairs of movies each critic did evaluate. Critic JPT (the editor of the Graffiti magazine), for instance, evaluated all but one ( $d = 24 \times 23 / 600 = 0.92$ ), whereas critic FG evaluated only 10 movies among the 25 in discussion ( $d = 10 \times 9 / 600 = 0.15$ ).

To get a picture of the actual divergence of rating opinions concerning jointly seen pairs of movies, we may develop a Principal Component Analysis of the corresponding  $\tau$  correlation matrix<sup>3</sup>. The 3D plot of the first 3 principal axes is shown in Figure 16.2 on page 221.

```
1 >>> bod.export3DplotOfCriteriaCorrelation(\n2 ...                               ValuedCorrelation=False)
```

The first 3 principal axes support together about 70% of the total inertia. Most eccentric and opposed in their respective rating opinions appear, on the first principal axis with 27.2% inertia, the conservative daily press against labour and public press. On the second principal axis with 23.7.7% inertia, it is the people press versus the cultural critical press. And, on the third axis with still 19.3% inertia, the written media appear most opposed to the radio media.

<sup>3</sup> The 3D PCA plot method requires a running R statistics software (<https://www.r-project.org/>) installation and the Calmat matrix calculator (see the calmat directory in the DIGRAPH3 resources)



**Fig. 16.4** 3D PCA plot of the criteria ordinal correlation matrix

## 16.6 Exploring the “*better rated*” and the “*as well as rated*” opinions

In order to furthermore study the quality of a ranking result, it may be interesting to have a separate view on the asymmetric and symmetric parts of the “*at least as well rated as*” opinions (see Sec. 2.3).

Let us first inspect the pairwise asymmetric part, namely the “*better rated than*” and “*less well rated than*” opinions of the movie critics.

```

1 >>> from digraphs import AsymmetricPartialDigraph
2 >>> ag = AsymmetricPartialDigraph(bod)
3 >>> ag.showHTMLRelationTable(\n4 ...     actionsList=g.computeNetFlowsRanking(), ndigits=0)

```

**Valued Adjacency Matrix**

rks y	mv_QS	mv_RR	mv_NP	mv_HN	mv_HS	mv_SM	mv_JB	mv_PE	mv_FC	mv_TP	mv_CM	mv_DF	mv_TM	mv_DJ	mv_AL	mv_RG	mv_MB	mv_GH	mv_HP	mv_BI	mv_DI	mv_FF	mv_GG	mv_TF
mv_QS	-	11	12	11	10	9	14	9	11	15	9	10	9	9	7	6	9	9	7	6	5	9	15	8
mv_RR	-5	-	0	0	0	0	0	5	0	0	0	9	12	9	10	8	9	6	10	13	10	9	9	9
mv_DG	-8	0	-	0	0	0	0	0	0	7	7	10	10	6	8	10	7	9	5	6	7	9	9	9
mv_NP	-7	0	0	-	0	0	0	0	8	0	0	8	0	8	7	6	4	6	8	6	5	7	12	10
mv_HN	-10	0	0	0	-	0	0	0	0	0	0	0	7	0	6	0	10	7	6	8	5	7	13	9
mv_HS	-3	0	0	0	0	0	-	0	0	0	0	0	5	0	7	0	0	10	9	8	9	7	11	9
mv_SM	-8	0	0	0	0	0	0	-	0	0	0	0	12	9	0	6	9	9	9	0	6	5	11	11
mv_JB	-9	-1	0	0	0	0	0	0	-	0	0	0	6	0	6	8	0	9	9	9	8	8	5	11
mv_PE	-7	0	0	0	0	0	0	0	-	0	0	0	4	10	0	0	6	8	8	0	6	5	9	13
mv_FC	-9	0	0	0	0	0	0	0	-	0	0	0	-	10	6	0	5	3	8	6	9	9	5	7
mv_TP	-7	0	-1	0	0	0	0	-4	-4	0	-	0	0	6	4	1	7	6	7	7	7	7	9	14
mv_CM	-8	-1	-5	-4	-3	-3	-1	0	-2	-2	0	-	0	0	8	8	0	10	0	0	7	3	9	14
mv_DF	-9	-2	-2	-2	0	0	0	0	0	0	0	-	0	4	-1	6	8	6	0	6	8	0	0	
mv_TM	-3	-7	-6	-2	-1	0	0	0	-1	-1	0	-	0	0	0	0	0	9	7	5	7	9	5	
mv_DJ	-7	-8	-4	-3	0	4	0	0	-2	-1	0	0	0	0	0	0	0	4	3	6	4	0	0	
mv_AL	-4	0	-4	-6	-6	0	-1	-3	-4	-2	0	0	3	0	0	-	0	0	0	0	0	0	3	
mv_RG	-7	-7	-6	-4	-1	-2	-1	-5	-4	-2	-3	-2	-2	0	0	0	0	0	0	0	0	0	4	
mv_MB	-9	-8	-5	-4	-4	-9	-4	-1	-2	-1	-2	-1	0	0	0	-2	0	0	0	0	0	0	0	
mv_GH	-5	-8	-7	-6	-6	-8	-4	-3	0	0	-5	-5	0	0	-2	-1	-3	0	0	0	0	0	3	
mv_HP	-4	-5	-5	-2	-4	-3	-3	-4	-3	-3	-1	-3	0	-7	-2	-1	0	-2	-1	0	0	0	0	
mv_BI	-5	-4	-6	-1	-1	-7	-5	-5	-5	-5	-1	-3	0	-3	2	0	-2	0	0	0	0	0	-1	
mv_DI	-9	-6	-5	-5	-3	-3	-3	-7	-7	-6	-1	-7	0	-5	0	0	0	0	0	0	0	0	0	
mv_FF	-11	-11	-7	-10	-5	0	-3	-3	-3	-9	-8	-2	-6	0	-1	0	-1	0	0	0	5	0	-	
mv_GG	-6	-8	-7	-4	-5	-7	-9	-10	-9	-9	-4	-2	-7	0	-3	0	-2	-2	-1	-3	5	-2	0	
mv_TF	-8	-7	-7	-5	-7	-11	-9	-8	-8	-7	-6	-3	-1	-1	-4	-3	0	0	-1	0	-	0	-	

Valuation domain: [-19.00; +19.00]

**Fig. 16.5** Asymmetric part of the “at least as well star-rated as” statements

We notice in Figure 16.5 that the NETFLOWS ranking rule inverts in fact just three “less well rated than” opinions and four “better rated than” ones. A similar look in Figure 16.6 at the symmetric part –the pairwise “as well rated as” opinions—suggests a preordered preference structure in several equivalently rated classes.

```
1 >>> from digraphs import SymmetricPartialDigraph
2 >>> sg = SymmetricPartialDigraph(bod)
3 >>> sg.showHTMLRelationTable()
4 ...
      actionsList=g.computeNetFlowsRanking(), \
5 ...
      ndigits=0)
```

**Valued Adjacency Matrix**

rks y	mv_QS	mv_RR	mv_NP	mv_HN	mv_HS	mv_SM	mv_JB	mv_PE	mv_FC	mv_TP	mv_CM	mv_DF	mv_TM	mv_DJ	mv_AL	mv_RG	mv_MB	mv_GH	mv_HP	mv_BI	mv_DI	mv_FF	mv_GG	mv_TF
mv_QS	-	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
mv_RR	0	-	5	7	8	6	4	0	2	9	10	0	0	0	0	8	0	0	0	0	0	0	0	0
mv_DG	0	9	-	9	10	5	9	7	8	13	0	0	0	0	0	0	0	0	0	0	0	0	0	0
mv_NP	0	5	7	-	10	3	7	9	8	11	9	0	0	12	0	0	0	0	0	0	0	0	0	0
mv_HN	0	4	8	8	-	5	9	9	8	10	10	0	0	10	0	0	8	0	0	0	0	0	0	0
mv_HS	0	2	5	5	3	-	10	2	5	6	9	0	0	10	0	0	1	0	0	0	0	0	14	0
mv_SM	0	6	7	5	7	6	-	1	6	6	10	12	0	0	10	6	4	0	0	0	0	0	0	0
mv_JB	0	0	5	1	3	4	8	-	9	6	0	13	6	8	9	0	0	0	0	0	0	0	0	0
mv_PE	0	2	6	0	4	3	6	11	-	5	0	0	5	7	0	0	0	0	5	0	0	0	0	0
mv_FC	0	5	11	9	8	2	6	8	7	-	10	0	0	11	0	0	0	0	0	0	0	0	0	0
mv_TP	0	4	6	3	2	3	0	0	0	0	-	6	11	0	4	5	0	0	0	0	0	0	0	0
mv_CM	0	0	0	0	0	0	0	5	0	0	4	-	3	8	9	0	7	5	0	0	0	0	0	0
mv_DF	0	0	0	2	2	2	2	0	1	1	5	1	-	0	6	0	0	0	5	6	8	13	0	
mv_TM	0	0	0	0	0	0	0	0	1	0	0	2	0	-	2	2	7	6	0	0	0	0	0	0
mv_DJ	0	0	0	0	0	2	4	2	1	0	0	0	0	0	4	-	3	5	0	0	0	0	4	1
mv_AL	0	0	0	1	0	1	0	0	0	0	3	1	0	0	2	3	-	2	2	0	0	3	1	0
mv_RG	0	0	0	0	0	0	0	0	0	0	0	0	1	3	4	-	1	0	6	0	8	9	0	0
mv_MB	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	2	3	-	2	0	4	4	2	0
mv_GH	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	6	-	0	2	5	3	0
mv_HP	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	2	0	0	3	6	5	0
mv_BI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	2	1	-	1	0
mv_DI	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	4	0	1	8	5	-	6	0
mv_FF	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	5	0	0	9	0	0	0	0
mv_GG	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	1	0	0	0	0	0	0	3	-
mv_TF	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	2	-

Valuation domain: [-19.00; +19.00]

**Fig. 16.6** Symmetric part of the “at least as well star-rated as” statements

Such a preordering of the movies may, for instance, be computed with the `computeRankingByChoosing()` method, where we iteratively extract dominant kernels –best remaining choices– and absorbent kernels –worst remaining choices– (see the next Chapter). We operate therefore on the asymmetric “*better rated than*” opinions, i.e. the codual of the “*at least as well rated as*” opinions<sup>4</sup> (see Line 2 in List. 16.11).

**Listing 16.11** Bipolar ranking-by-choosing the movies

```

1 >>> from transitiveDigraphs import RankingByChoosingDigraph
2 >>> rbc = RankingByChoosingDigraph(bod, CoDual=True)
3 >>> rbc.showRankingByChoosing()
4   Ranking by Choosing and Rejecting
5     1st Best Choice ['mv_QS']
6       2nd Best Choice ['mv_DG', 'mv_FC', 'mv_HN', 'mv_HS', 'mv_NP',
7                     'mv_PE', 'mv_RR', 'mv_SM']
8         3rd Best Choice ['mv_CM', 'mv_JB', 'mv_TM']
9           4th Best Choice ['mv_AL', 'mv_TP']
10          4th Worst Choice ['mv_AL', 'mv_TP']
11        3rd Worst Choice ['mv_GH', 'mv_MB', 'mv_RG']
12      2nd Worst Choice ['mv_DF', 'mv_DJ', 'mv_FF', 'mv_GG']
13    1st Worst Choice ['mv_BI', 'mv_DI', 'mv_HP', 'mv_TF']

```

In the next Chapter 17, we thoroughly discuss the computation of such kernels in bipolar-valued digraphs.

## References

- Bisdorff R (2012) On measuring and testing the ordinal correlation between bipolar outranking relations. In: Mousseau V, Pirlot M (eds) DAP'2012 From Multiple Criteria Decision Aid to Preference Learning, University of Mons (Belgium), pp 91–100, URL <http://hdl.handle.net/10993/23909> 223, 224  
Kendall MG (1938) A new measure of rank correlation. Biometrika 30:81–93 220

---

<sup>4</sup> A kernel in a digraph  $g$  is a clique in the dual digraph  $-g$ .

# Chapter 17

## On computing digraph kernels

**Abstract** We illustrate in this chapter, first, the concept of graph kernel, i.e. maximal independent set of vertices. In non-symmetric digraphs the kernel concept becomes richer and separates into initial and terminal kernels. In, furthermore, lateralised outranking digraphs, initial and terminal kernels become separate and may deliver suitable first resp. last choice recommendations. After commenting the tractability of kernel computations, we close the chapter with the solving of bipolar-valued kernel equation systems.

### 17.1 What is a graph kernel ?

We call *choice* in a graph, respectively a digraph, a subset of its vertices, resp. of its nodes or actions. A choice  $Y$  is called *internally stable* or *independent* when there exist no links –(edges) or relations (arcs)– between its members. Furthermore, a choice  $Y$  is called *externally stable* when for each vertex, node or action  $x$  not in  $Y$ , there exists at least a member  $y$  of  $Y$  such that  $x$  is linked or related to  $y$ . Now, an internally and externally stable choice is called a *kernel*.

A first trivial example is immediately given by the maximal independent vertices sets (MISs) of the  $n$ -cycle graph (see Fig. 21.5 on page 321). Indeed, each MIS in the  $n$ -cycle graph is by definition independent, i.e. *internally stable*, and each non selected vertex in the  $n$ -cycle graph is in relation with either one or even two members of the MIS.

In all graphs or symmetric digraphs, the *maximality condition* imposed on the internal stability is equivalent to the *external stability* condition. Indeed, if there would exist a vertex or node not related to any of the elements of a choice, we may safely add this vertex or node to the given choice without violating its internal stability. All kernels must hence be maximal independent choices. In fact, in a topological sense, they correspond to maximal *holes* in the given graph.

Figure 17.1 on the next page illustrates this coincidence between MISs and kernels in graphs and symmetric digraphs with a random 3-regular graph instance gen-

erated in Listing 17.1. A random MIS in this graph may be computed by using the MISModel class (see Lines 5-8 below).

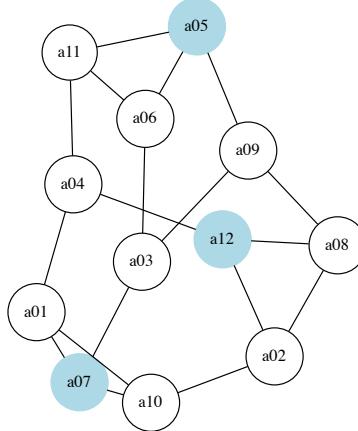
**Listing 17.1** Generating a random 3-regular graph of order 12

```

1 >>> from graphs import RandomRegularGraph
2 >>> r3g12 = RandomRegularGraph(order=12,\ 
3 ...                                degree=3,seed=4)
4 >>> from graphs import MISModel
5 >>> mg = MISModel(r3g12)
6   Iteration: 1
7     Running a Gibbs Sampler for 660 step !
8     {'a05', 'a07', 'a12'} is maximal !
9 >>> mg.exportGraphViz('random3RegularGraph-mis')
10    *---- exporting a dot file for GraphViz tools ---*
11    Exporting to random3RegularGraph-mis.dot
12    fdp -Tpng random3RegularGraph-mis.dot\
13                                -o random3RegularGraph-mis.png

```

**Fig. 17.1** A random MIS coloured in the random 3-regular graph r3g12. All non blue vertices are covered by a blue vertex



Digraph3 (graphviz), R. Bisdorff, 2019

It is easily verified in Figure 17.1, that the computed MIS renders indeed a valid kernel of the given graph. The complete set of kernels of this 3-regular graph instance coincides hence with the set of its MISs.

**Listing 17.2** Printing out all maximal independent sets of the random 3-regular graph

```

1 >>> g.showMIS()
2   *--- Maximal Independent Sets ---*
3   ['a05', 'a07', 'a12']
4   ['a01', 'a06', 'a08']
5   ['a07', 'a08', 'a11']

```

```

6      ['a01', 'a09', 'a11', 'a12']
7      ['a01', 'a02', 'a09', 'a11']
8      ['a01', 'a06', 'a09', 'a12']
9      ['a01', 'a02', 'a06', 'a09']
10     ['a07', 'a09', 'a11', 'a12']
11     ['a02', 'a07', 'a09', 'a11']
12     ['a06', 'a07', 'a09', 'a12']
13     ['a04', 'a06', 'a07', 'a08']
14     ['a02', 'a04', 'a05', 'a07']
15     ['a04', 'a05', 'a07', 'a08']
16     ['a09', 'a10', 'a11', 'a12']
17     ['a06', 'a09', 'a10', 'a12']
18     ['a04', 'a06', 'a08', 'a10']
19     ['a04', 'a06', 'a09', 'a10']
20     ['a01', 'a03', 'a11', 'a12']
21     ['a01', 'a02', 'a03', 'a11']
22     ['a01', 'a03', 'a08', 'a11']
23     ['a01', 'a03', 'a05', 'a12']
24     ['a01', 'a02', 'a03', 'a05']
25     ['a01', 'a03', 'a05', 'a08']
26     ['a02', 'a03', 'a04', 'a05']
27     ['a03', 'a10', 'a11', 'a12']
28     ['a03', 'a08', 'a10', 'a11']
29     ['a03', 'a05', 'a10', 'a12']
30     ['a02', 'a04', 'a06', 'a07', 'a09']
31     ['a03', 'a04', 'a05', 'a08', 'a10']
32 number of solutions: 29
33 cardinality distribution
34 card.: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
35 freq.: [0, 0, 0, 3, 24, 2, 0, 0, 0, 0, 0, 0, 0]
36 stability number : 5
37 execution time: 0.00056 sec.
38 Results in self.misset
39 >>> g.misset
40 [frozenset({'a05', 'a07', 'a12'}),
41  frozenset({'a11', 'a01', 'a12', 'a09'}),
42  frozenset({'a11', 'a01', 'a02', 'a09'}),
43  ...
44  ...
45  frozenset({'a03', 'a10', 'a08', 'a11'}),
46  frozenset({'a03', 'a10', 'a12', 'a05'}),
47  frozenset({'a03', 'a10', 'a04', 'a05', 'a08'})]
```

All graphs and symmetric digraphs admit MISs, hence also kernels. In the context of digraphs, i.e. *oriented* graphs, the kernel concept gets much richer and separates from the symmetric MIS concept.

## 17.2 Initial and terminal kernels

In an oriented graph context, the internal stability condition of the kernel concept remains untouched; however, the external stability condition gets indeed split up by the orientation into two lateral cases:

1. A *dominant* stability condition, where each non selected node is dominated by at least one member of the kernel;
2. An *absorbent* stability condition, where each non selected node is absorbed by at least one member of the kernel.

A both internally stable **and** dominant, resp. absorbent choice is called a dominant or *initial*, resp. an absorbent or *terminal* kernel. From a topological perspective, the initial kernel concept looks from the outside of the digraph into its interior, whereas the terminal kernel looks from the interior of a digraph toward its outside. From an algebraic perspective, the initial kernel is a prefix operand, and the terminal kernel is a postfix operand in the kernel equation systems (see Sec. 17.6 on page 247).

Furthermore, as the kernel concept involves conjointly a positive logical refutation (the internal stability) and a positive logical affirmation (the external stability), it appeared rather quickly necessary in our operational developments to adopt a bipolar characteristic  $[-1.0, 1.0]$  valuation domain, modelling logical negation by a change of numerical sign and including explicitly a third median value (0.0), expressing logical *indeterminateness* –neither positive, nor negative (Bisdorff, 2000, 2002, 2004).

In such a bipolar-valued context, we call *prekernel* a choice which is *externally stable* and for which the internal stability condition is *valid or indeterminate*. We say that the independence condition is in this case only *weakly validated*. Notice that all kernels are hence prekernels, but not vice-versa.

In graphs or symmetric digraphs, where there is essentially no apparent *laterality*; all prekernels are initial and terminal at the same time. A universal example is given by the *complete* digraph .

**Listing 17.3** The prekernels of a complete digraph

```

1 >>> from digraphs import CompleteDigraph
2 >>> u = CompleteDigraph(order=5)
3 >>> u
4     *----- Digraph instance description -----*
5     Instance class    : CompleteDigraph
6     Instance name     : complete
7     Digraph Order     : 5
8     Digraph Size      : 20
9     Valuation domain : [-1.00 ; 1.00]
10    -----
11 >>> u.showPreKernels()
12    *--- Computing preKernels ---*
13    Dominant kernels :
14    ['1'] ind. : 1.0; dom. : 1.0; abs. : 1.0
15    ['2'] ind. : 1.0; dom. : 1.0; abs. : 1.0

```

```

16   ['3'] ind. : 1.0; dom. : 1.0; abs. : 1.0
17   ['4'] ind. : 1.0; dom. : 1.0; abs. : 1.0
18   ['5'] ind. : 1.0; dom. : 1.0; abs. : 1.0
19     Absorbent kernels :
20   ['1'] ind. : 1.0; dom. : 1.0; abs. : 1.0
21   ['2'] ind. : 1.0; dom. : 1.0; abs. : 1.0
22   ['3'] ind. : 1.0; dom. : 1.0; abs. : 1.0
23   ['4'] ind. : 1.0; dom. : 1.0; abs. : 1.0
24   ['5'] ind. : 1.0; dom. : 1.0; abs. : 1.0
25 *----- statistics -----
26   graph name: complete
27   number of solutions
28   dominant kernels : 5
29   absorbent kernels: 5
30   cardinality frequency distributions
31   cardinality : [0, 1, 2, 3, 4, 5]
32   dominant kernel : [0, 5, 0, 0, 0, 0]
33   absorbent kernel: [0, 5, 0, 0, 0, 0]
34   Execution time : 0.00004 sec.
35   Results in sets: dompreKernels
36   and abspreKernels.

```

In a complete digraph, each single node is indeed both an initial and a terminal prekernel candidate and there is no definite begin or end of the digraph to be detected. Laterality is here entirely relative to a specific singleton chosen as reference point of view.

The same absence of laterality is apparent (see List. 17.4) in two other universal digraph models, the *empty* and the *indeterminate* digraph.

**Listing 17.4** The prekernels of the empty or indeterminate digraph

```

1 >>> from digraphs import EmptyDigraph
2 >>> ed = EmptyDigraph(order=5)
3 >>> ed.showPreKernels()
4 *--- Computing preKernels ---*
5   Dominant kernel :
6   ['1', '2', '3', '4', '5']
7     independence : 1.0
8     dominance   : 1.0
9     absorbency  : 1.0
10  Absorbent kernel :
11  ['1', '2', '3', '4', '5']
12    independence : 1.0
13    dominance   : 1.0
14    absorbency  : 1.0
15 >>> from digraphs import IndeterminateDigraph
16 >>> id = IndeterminateDigraph(order=5)
17 >>> id.showPreKernels()
18 *--- Computing preKernels ---*
19   Dominant prekernel :
20   ['1', '2', '3', '4', '5']
21     independence : 0.0
22     dominance   : 1.0
23     absorbency  : 1.0

```

```

24     Absorbent prekernel :
25     ['1', '2', '3', '4', '5']
26     independence : 0.0
27     dominance : 1.0
28     absorbency : 1.0

```

In the empty digraph, the whole set of nodes gives indeed at the same time the unique initial and terminal prekernel (see Lines 6 and 11). Similarly, for the *indeterminate* digraph (see Lines 20 and 25).

Both these results make sense, as in a completely empty or indeterminate digraph, there is no *interior* of the digraph defined, only a *border* which is hence at the same time an initial and terminal prekernel (see Section 2.4 on page 25). Notice however, that in the latter indeterminate case, the complete set of nodes verifies only weakly the internal stability condition (see Lines 21 and 26).

Other common digraph models, although being clearly oriented, may show nevertheless no apparent laterality, like *chordless circuits*, i.e. holes surrounded by an oriented cycle -a circuit- of odd length. They do not admit in fact any initial or terminal prekernel.

**Listing 17.5** The prekernels of the 5-circuit digraph

```

1 >>> from digraphs import CirculantDigraph
2 >>> c5 = CirculantDigraph(order=5,circulants=[1])
3 >>> c5.showPreKernels()
4     ----- statistics -----
5     digraph name: c5
6     number of solutions
7     dominant prekernels : 0
8     absorbent prekernels: 0

```

Chordless circuits of *even* length  $2 \times k$ , with  $k > 1$ , contain however two isomorphic prekernels of cardinality  $k$  which qualify conjointly as initial and terminal candidates.

**Listing 17.6** The prekernels of the 6-circuit digraph

```

1 >>> c6 = CirculantDigraph(order=6,circulants=[1])
2 >>> c6.showPreKernels()
3     ----- Computing preKernels ----*
4     Dominant preKernels :
5     ['1', '3', '5'] ind. : 1.0, dom. : 1.0, abs. : 1.0
6     ['2', '4', '6'] ind. : 1.0, dom. : 1.0, abs. : 1.0
7     Absorbent preKernels :
8     ['1', '3', '5'] ind. : 1.0, dom. : 1.0, abs. : 1.0
9     ['2', '4', '6'] ind. : 1.0, dom. : 1.0, abs. : 1.0

```

Chordless circuits of even length may thus be indifferently oriented along two opposite directions. Notice by the way in Fig. 17.2 on the next page that the duals of all chordless circuits of odd or even length, i.e. *filled* circuits also called *anti-holes* (see Fig. 17.3 on page 238), never contain any potential prekernel candidates.

**Listing 17.7** The prekernels of the dual of the 6-circuit digraph

```

1 >>> dc6 = -c6    # dc6 = DualDigraph(c6)

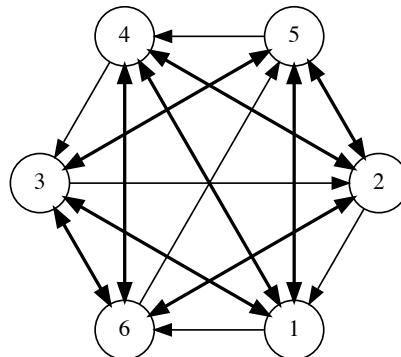
```

```

2 >>> dc6.showPreKernels()
3     *----- statistics -----
4     graph name: dual_c6
5     number of solutions
6         dominant prekernels :  0
7         absorbent prekernels:  0
8 >>> dc6.exportGraphViz(fileName='dualChordlessCircuit')
9     *----- exporting a dot file for GraphViz tools -----
10    Exporting to dualChordlessCircuit.dot
11    circo -Tpng dualChordlessCircuit.dot\
12                      -o dualChordlessCircuit.png

```

**Fig. 17.2** The dual of the chordless 6-circuit. No initial or terminal prekernel –weakly independent and dominant, resp. absorbent subset of nodes– may be found in this kind of digraphs



Digraph3 (graphviz), R. Bisdorff, 2020

We call *weak*, a chordless circuit with indeterminate inner part.

In Listing 17.8 we use the `IndeterminateInnerPart` parameter of the `CirculantDigraph` class for constructing such a weak chordless 6-circuit digraph. It is worth noticing in Fig. 17.3 on the following page that the *dual* version of a weak circuit corresponds to its *converse* version<sup>1</sup>.

**Listing 17.8** The weak 6-circuit digraph

```

1 >>> from digraphs import CirculantDigraph
2 >>> c6 = CirculantDigraph(order=6,circulants=[1], \
3 ...                           IndeterminateInnerPart=True)
4 >>> (-c6).exportGraphViz()
5     *----- exporting a dot file for GraphViz tools -----
6     Exporting to dual_c6.dot

```

<sup>1</sup> Not to be confused with the dual graph of a plane graph  $g$  that has a vertex for each face of  $g$ . Here we mean the *less than* (strict converse) relation corresponding to a *greater or equal* relation, or the *less than or equal* relation corresponding to a (strict converse) *greater than* relation.

```

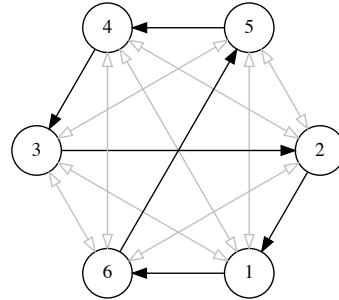
7   circo -Tpng dual_c6.dot -o dual-c6.png
8 >>> (~c6).exportGraphViz()
9   *---- exporting a dot file for GraphViz tools -----*
10  Exporting to converse_c6.dot
11  circo -Tpng converse_c6.dot -o converse-c6.png

```

Dual of weak 6-circuit

*Digraph3 (graphviz), R. Bisdorff, 2020*

Converse of weak 6-circuit

*Digraph3 (graphviz), R. Bisdorff, 2020***Fig. 17.3** Dual and converse transforms of the weak 6-circuit give the same digraph

Weak chordless circuits of length  $n$  are in fact part of the class of digraphs that are invariant under the *codual* transform,  $cn = -(\sim cn) = \sim(-cn)$ <sup>2</sup>. When digraph  $cn$  is a weak chordless  $n$ -circuit,  $cn$ ,  $-cn$ ,  $\sim cn$  and  $\sim(-cn)$  will all admit the same set of prekernels.

### 17.3 Kernels in lateralised digraphs

Humans do live in an apparent physical space of plain transitive lateral orientation, fully empowered in finite geometrical 3D models with linear orders, where first, resp. last ranked, nodes deliver unique initial, resp. terminal, kernels. Similarly, in finite preorders, the first, resp. last, equivalence classes deliver the unique initial, resp. unique terminal, kernels. More generally, in finite partial orders, i.e. asymmetric and transitive digraphs, topological sort algorithms will easily reveal on the first, resp. last, level all unique initial, resp. terminal, kernels.

<sup>2</sup> The class of *self-codual* bipolar-valued digraphs consists of all weakly asymmetric digraphs, i.e. digraphs containing only asymmetric and/or indeterminate links. Limit cases consists of, on the one side, full tournaments with indeterminate reflexive links, and, on the other side, fully indeterminate digraphs. In this class, the converse (inverse  $\sim$ ) operator is indeed identical to the dual (negation  $-$ ) one.

In genuine random digraphs, however, we may need to check for each of its MISs, whether one, both, or none of the lateralised external stability conditions may be satisfied. Consider, for instance in Listing 17.9, the following random digraph instance of order 7 and generated with an arc probability of 30%.

**Listing 17.9** Generating a random digraph `rd` of order 7 and arc probability 0.3

```

1 >>> from randomDigraphs import RandomDigraph
2 >>> rd = RandomDigraph(order=7, arcProbability=0.3, seed=5)
3 >>> rd.exportGraphViz('randomLaterality')
4     *---- exporting a dot file for GraphViz tools ---*
5     Exporting to randomLaterality.dot
6     dot -Grankdir=BT -Tpng randomLaterality.dot \
7                         -o randomLaterality.png

```

**Fig. 17.4** A random digraph instance of order 7 and arc probability 0.3. The digraph instance is neither asymmetric ( $a_3 \leftrightarrow a_6$ ) nor symmetric ( $a_2 \rightarrow a_1$ ,  $a_1 \not\rightarrow a_2$ ) and, the digraph is not transitive ( $a_5 \rightarrow a_2 \rightarrow a_1$ , but  $a_5 \not\rightarrow a_1$ )



Digraph3 (graphviz), R. Bischoff, 2020

The random digraph shown in Figure 17.4 has no apparent special properties, except from being connected (see Line 3 in List. 17.10).

**Listing 17.10** Inspecting the properties of random digraph `rd`

```

1 >>> rd.showComponents()
2     *--- Connected Components ---*
3     1: ['a1', 'a2', 'a3', 'a4', 'a5', 'a6', 'a7']
4 >>> rd.computeSymmetryDegree(Comments=True, InPercents=True)
5     Symmetry degree (%) of digraph <randomDigraph>:
6         arcs x>y: 14, symmetric: 1, asymmetric: 13

```

```

7     symmetric/arcs =  7.1
8 >>> rd.computeChordlessCircuits()
9     [] # no chordless circuits detected
10 >>> rd.computeTransitivityDegree(Comments=True, InPercents=True)
11   Transitivity degree (%) of graph <randomDigraph>:
12     triples x>y>z: 23, closed: 11, open: 12
13     closed/triples =  47.8

```

There are no chordless circuits (see Line 9 above) and more than half of the required transitive closure is missing (see Line 12 above).

Now, we know that its potential prekernels must be among its set of maximal independent choices.

**Listing 17.11** Inspecting the prekernels of random digraph `rd`

```

1 >>> rd.showMIS()
2     *--- Maximal independent choices ---*
3     ['a2', 'a4', 'a6']
4     ['a6', 'a1']
5     ['a5', 'a1']
6     ['a3', 'a1']
7     ['a4', 'a3']
8     ['a7']
9 >>> rd.showPreKernels()
10    *--- Computing preKernels ---*
11    Dominant preKernels :
12      ['a2', 'a4', 'a6']
13        independence : 1.0
14        dominance   : 1.0
15        absorbency   : -1.0
16        covering     : 0.500
17      ['a4', 'a3']
18        independence : 1.0
19        dominance   : 1.0
20        absorbency   : -1.0
21        covering     : 0.600
22    Absorbent preKernels :
23      ['a3', 'a1']
24        independence : 1.0
25        dominance   : -1.0
26        absorbency   : 1.0
27        covering     : 0.500
28      ['a6', 'a1']
29        independence : 1.0
30        dominance   : -1.0
31        absorbency   : 1.0
32        covering     : 0.600

```

Among the six MISs contained in random digraph `rd` (see Lines 3-8 in List. 17.11) we discover two initial and two terminal kernels (Lines 12, 17, 23, 28). Notice by the way the covering values (between 0.0 and 1.0) shown by the `showPreKernels()` method (Lines 16, 21, 27 and 32). The higher this value, the more the corresponding prekernel candidate makes apparent the digraph's laterality. In Fig. 17.5 on the next page, the same digraph `rd` is redrawn by looking into its interior via the best

covering initial prekernel candidate: the dominant choice  $\{a_3, a_4\}$  (coloured in yellow), and looking out of it via the best covered terminal prekernel candidate: the absorbent choice  $\{a_1, a_6\}$  (coloured in blue).

```

1 >>> rd.exportGraphViz(fileName='orientedLaterality', \
2 ...                         firstChoice=set(['a3', 'a4']), \
3 ...                         lastChoice=set(['a1', 'a6'])) \
4     *---- exporting a dot file for GraphViz tools ----*
5     Exporting to orientedLaterality.dot
6     dot -Grankdir=BT -Tpng orientedLaterality.dot \
7             -o orientedLaterality.png

```

**Fig. 17.5** A random digraph oriented by best covering initial and best covered terminal prekernels



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As all reasonable bipolar-valued outranking digraph usually show some laterality –the marginal criteria preferences being not all totally contradictory– initial and terminal prekernels provide convincing fist, resp. last, choice recommendations as illustrated in Chapter 4 on page 49.

## 17.4 Computing first and last choice recommendations

To illustrate this idea, let us finally compute in Listing 17.12 on the next page first and last choice recommendations in a random bipolar-valued outranking digraph.

**Listing 17.12** Generating a random bipolar-valued outranking digraph

```

1 >>> from outrankingDigraphs import\
2 ...           RandomBipolarOutrankingDigraph
3 >>> g = RandomBipolarOutrankingDigraph(seed=5)
4 >>> g
5      *----- Object instance description -----*
6      Instance class    : RandomBipolarOutrankingDigraph
7      Instance name     : randomOutranking
8      Actions          : 7
9      Criteria         : 7
10     Size             : 26
11     Determinateness   : 34.275
12     Valuation domain : {'min': -100.0, 'med': 0.0, 'max': 100.0}
13 >>> g.showHTMLPerformanceTableau()

```

## Performance table randomOutranking

criterion	g1	g2	g3	g4	g5	g6	g7
a1	64.90	1.31	13.88	98.24	94.10	14.57	31.00
a2	NA	NA	61.75	87.24	69.06	6.51	81.85
a3	11.32	27.95	12.67	28.93	96.66	30.14	48.07
a4	46.91	91.63	0.18	96.15	89.37	60.31	31.58
a5	NA	76.57	87.14	53.92	29.88	0.34	48.12
a6	54.38	15.96	20.95	67.78	36.12	67.79	70.47
a7	57.39	79.71	21.55	20.48	16.60	33.79	5.70

**Fig. 17.6** The performance tableau of a random outranking digraph instance

The associated random performance tableau shown in Fig. 17.6 reveals the performance evaluations of 7 potential decision alternatives with respect to 7 decision criteria supporting each an increasing performance scale from 0.0 to 100.0. Notice the missing performance data concerning decision alternatives a2 and a5. The resulting *strict outranking* - i.e. a weighted majority supported – “*better than without considerable counter-performance*” – digraph is shown in Fig. 17.7 on the next page.

```

1 >>> gcd = ~(-g) # Codual: the converse of the negation
2 >>> gcd.exportGraphViz(fileName='tutOutRanking')
3      *---- exporting a dot file for GraphViz tools ----*
4      Exporting to tutOutranking.dot
5      dot -Grankdir=BT -Tpng tutOutranking.dot \
6                      -o tutOutranking.png

```

**Listing 17.13** Computing the prekernels of the strict outranking digraph gcd

```

1 >>> gcd.showPreKernels()
2      *--- Computing preKernels ---*
3      Dominant preKernels :
4      ['a1', 'a2', 'a4']

```

**Fig. 17.7** A random strict outranking digraph instance.

All decision alternatives appear strictly better performing than alternative  $a_7$ . We call it a CONDORCET loser and it is an evident terminal prekernel candidate. On the other side, three alternatives:  $a_1$ ,  $a_2$  and  $a_4$  are not dominated. They give together an initial prekernel candidate



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```

5     independence : 0.00
6     dominance   : 6.98
7     absorbency  : -48.84
8     covering    : 0.667
9     Absorbent preKernels :
10    ['a3', 'a7']
11    independence : 0.00
12    dominance   : -74.42
13    absorbency  : 16.28
14    covered      : 0.800

```

With such unique disjoint initial and terminal prekernels (see Line 4 and 10 in Listing 17.13 on the facing page), the random digraph  $gcd$  is hence clearly lateralised. Indeed, these initial and terminal prekernels of the codual outranking digraph reveal *first*, resp. *last*, choice recommendations one may formulate on the basis of a given outranking digraph instance.

**Listing 17.14** Computing a first and last choice recommendation from digraph  $gcd$

```

1 >>> g.showBestChoiceRecommendation()
2 Rubis best choice recommendation(s) (BCR)
3 (in decreasing order of determinateness)
4 Credibility domain: [-100.00,100.00]
5 === >> potential first choice(s)
6 * choice          : ['a1', 'a2', 'a4']

```

```

7      independence      : 0.00
8      dominance        : 6.98
9      absorbency       : -48.84
10     covering (%)     : 66.67
11     determinateness (%) : 57.97
12     - most credible action(s) =
13           {'a4': 20.93, 'a2': 20.93}
14     ===>> potential last choice(s)
15     * choice          : ['a3', 'a7']
16     independence      : 0.00
17     dominance         : -74.42
18     absorbency        : 16.28
19     covered (%)       : 80.00
20     determinateness (%) : 64.62
21     - most credible action(s) = { 'a7': 48.84, }

```

Notice in Lines 13 and 21 in Listing 17.14 on the previous page that solving the valued kernel equation system provides furthermore a positive characterisation of the most credible decision alternatives in each respective choice recommendation. Alternatives  $a_2$  and  $a_4$  give equivalent candidates for a unique *first* choice, and alternative  $a_7$  is clearly confirmed as the *last* choice.

In Fig. 17.8 on the facing page, we orient the drawing of the strict outranking digraph instance with the help of these first and last choice recommendations.

```

1 >>> gcd.exportGraphViz(fileName='firstLastOrientation', \
2 ...                           firstChoice=['a2', 'a4'], \
3 ...                           lastChoice=['a7'])
4 ----- exporting a dot file for GraphViz tools -----
5   Exporting to firstLastOrientation.dot
6   dot -Grankdir=BT -Tpng firstLastOrientation.dot \
7       -o firstLastOrientation.png

```

It may be interesting to compare this result with a COPELAND ranking of the underlying performance tableau (see Section 8.2 on page 110 on ranking with incommensurable criteria).

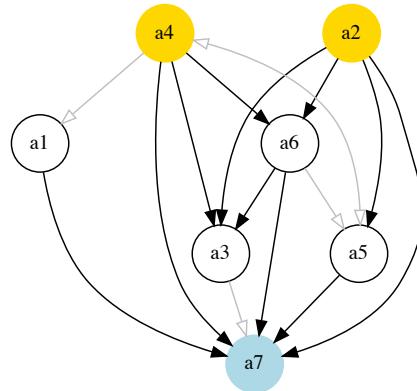
```

1 >>> g.showHTMLPerformanceHeatmap(colorLevels=5, ndigits=0, \
2 ...                           Correlations=True, rankingRule='Copeland')

```

In the resulting linear ranking (see Fig. 17.9 on the facing page), alternative  $a_4$  is set at first rank, followed by alternative  $a_2$ . This makes sense as alternative  $a_4$  shows three performances in the first quintile, whereas  $a_2$  is only partially evaluated and shows only two such excellent performances. But  $a_4$  also shows a very weak performance in the first quintile. Both decision actions, hence, don't show eventually a performance profile that would make apparent a clear preference situation in favour of one or the other. In this sense, the prekernels based best choice recommendations may appear more faithful with respect to the actually definite strict outranking relation than any '*forced*' linear ranking result as shown in Fig. 17.9 on the next page.

**Fig. 17.8** The strict outranking digraph oriented by its first and last choice recommendations. The gray arrows, like the one between alternatives  $a_4$  and  $a_1$ , represent indeterminate preferential situations. Alternative  $a_1$  appears hence to be rather incomparable to all the other, except alternative  $a_7$



Digraph3 (graphviz), R. Bisdorff, 2020

### Heatmap of Performance Tableau 'randomOutranking'

criteria	g4	g7	g5	g6	g1	g2	g3
<b>weights</b>	9	10	6	5	4	8	1
<b>tau(*)</b>	+0.64	+0.40	+0.29	+0.17	+0.02	-0.05	-0.10
<b>a4</b>	96	32	89	60	47	92	0
<b>a2</b>	87	82	69	7	NA	NA	62
<b>a6</b>	68	70	36	68	54	16	21
<b>a1</b>	98	31	94	15	65	1	14
<b>a5</b>	54	48	30	0	NA	77	87
<b>a3</b>	29	48	97	30	11	28	13
<b>a7</b>	20	6	17	34	57	80	22

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Ranking rule: Copeland

Ordinal (Kendall) correlation between global ranking and global outranking relation: +0.848

**Fig. 17.9** Heat map with Copeland ranking of the performance tableau

## 17.5 Tractability of kernel computation

Let us give some hints on the *tractability* of prekernel computations. Detecting all prekernels in a digraph is a computationally difficult problem. Checking external stability conditions for an independent choice is equivalent to checking its maximality and may be done in the linear complexity of the order of the digraph. However, checking all independent choices contained in a digraph may get hard already for tiny sparse digraphs of order  $n > 20$  (see Bisdorff, 2006). Indeed, the worst case is

given by an empty or indeterminate digraph where the set of all potential independent choices to check is in fact the power set of the vertices.

```

1 >>> from digraphs import EmptyDigraph
2 >>> e = EmptyDigraph(order=20)
3 >>> e.showMIS()      # by visiting all  $2^{20}$  independent choices
4     *--- Maximal independent choices ---*
5     [ '1', '2', '3', '4', '5', '6', '7', '8', '9', '10',
6       '11', '12', '13', '14', '15', '16', '17', '18', '19', '20' ]
7     number of solutions: 1
8     execution time: 1.47640 sec.  # <-----
9 >>> 2**20
10    1048576

```

Now, there exist more efficient specialised algorithms for directly enumerating MISs and dominant or absorbent kernels contained in specific digraph models without visiting all independent choices (see [Bisdorff, 2006](#)). *Alain Hertz* provided kindly such a MISs enumeration algorithm for the DIGRAPH3 project (the `showMIS_AH()` method). When the number of independent choices is big compared to the actual number of MISs, like in very sparse or empty digraphs, the performance difference may be dramatic (see Line 7 above and Line 15 below).

**Listing 17.15** Enumerating MISs by visiting only maximal independent choices (A. Hertz)

```

1 >>> e.showMIS_AH()
2     # by visiting only maximal independent choices
3     *-----*
4     * Python implementation of Hertz's *
5     * algorithm for generating all MISs *
6     * R.B. version 7(6)-25-Apr-2006   *
7     *-----*
8     =====>>> Initial solution :
9     [ '1','2','3','4','5','6','7','8','9','10','11',
10      '12','13','14','15','16','17','18','19','20' ]
11     *---- results ----*
12     [ '1','2','3','4','5','6','7','8','9','10','11',
13      '12','13','14','15','16','17','18','19','20' ]
14     *---- statistics ----*
15     MIS solutions : 1
16     execution time : 0.00026 sec. # <-----
17     iteration history: 1

```

For more or less dense strict outranking digraphs of modest order, as facing usually in MCDA applications, enumerating all independent choices remains however in most cases tractable, especially by using a very efficient iterator generator with the `independentChoices()` method shown in Listing 17.16.

**Listing 17.16** Generating all independent choices in a digraph

```

1 def independentChoices(self,U):
2     """
3         Generator for all independent choices with associated
4         dominated, absorbed and independent neighborhoods
5         of digraph instance self.
6         Initiate with U = self.singletons().

```

```

7     Yields [(independent choice, domnb, absnb, indnb)].
8     """
9     if U == []:
10        yield [(frozenset(), set(), set(), set(self.actions)) ]
11    else:
12      x = list(U.pop())
13      for S in self.independentChoices(U):
14        yield S
15        if x[0] <= S[0][3]:
16          Sxgamdom = S[0][1] | x[1]
17          Sxgamabs = S[0][2] | x[2]
18          Sxindep = S[0][3] & x[3]
19          Sxchoice = S[0][0] | x[0]
20          Sx = [(Sxchoice, Sxgamdom, Sxgamabs, Sxindep)]
21          yield Sx

```

And, checking maximality of independent choices via the external stability conditions during their enumeration with the `computePreKernels()` method shown in Listing 17.17 provides the effective advantage of computing all initial and terminal prekernels in a single loop (see Line 10 and [Bisdorff, 2006](#)).

**Listing 17.17** Computing dominant and absorbent preKernels

```

1 def computePreKernels(self):
2     """
3         computing dominant and absorbent preKernels:
4         Result in self.dompreKernels and self.abspreKernels
5     """
6     actions = set(self.actions)
7     n = len(actions)
8     dompreKernels = set()
9     abspreKernels = set()
10    for choice in self.independentChoices(self.singletons()):
11        restactions = actions - choice[0][0]
12        if restactions <= choice[0][1]:
13            dompreKernels.add(choice[0][0])
14        if restactions <= choice[0][2]:
15            abspreKernels.add(choice[0][0])
16    self.dompreKernels = dompreKernels
17    self.abspreKernels = abspreKernels

```

Finally, we use our bipolar-valued epistemic logic framework for computing the credibility that an individual node of the digraph fits with being a member of an initial or terminal prekernel. For this purpose we use kernel equation systems ([Schmidt and Ströhlein, 1985](#)).

## 17.6 Solving kernel equation systems

Let  $G(X, R)$  be a crisp irreflexive digraph defined on a finite set  $X$  of nodes and where  $R$  is the corresponding  $\{-1, +1\}$ -valued adjacency matrix. Let  $Y$  be the  $\{-1, +1\}$ -valued membership characteristic (row) vector of a choice in  $X$ .

When  $Y$  satisfies the following equation system:

$$Y \circ R = -Y, \quad (17.1)$$

where for all  $x$  in  $X$ ,

$$(Y \circ R)(x) = \max_{y \in X, x \neq y} (\min(Y(x), R(x, y))), \quad (17.2)$$

then  $Y$  characterises an *initial kernel* (Bisdorff, Pirlot, and Roubens, 2006; Bisdorff, 2006).

When transposing now the membership characteristic vector  $Y$  into a column vector  $Y^t$ , the following equation system:  $R \circ Y^t = -Y^t$ . makes  $Y^t$  similarly characterise a *terminal kernel*.

Let us verify this result in Listing 17.18 on a tiny random digraph.

**Listing 17.18** Verifying the kernel equation system on a tiny random digraph

```

1 >>> from digraphs import RandomDigraph
2 >>> g = RandomDigraph(order=3, seed=1)
3 >>> g.showRelationTable()
4   * ----- Relation Table -----
5   R | 'a1' 'a2' 'a3'
6   -----|-----
7   'a1' | -1    +1    -1
8   'a2' | -1    -1    +1
9   'a3' | +1    +1    -1
10 >>> g.showPreKernels()
11  *--- Computing preKernels ---*
12  Dominant preKernels :
13  ['a3']
14  Absorbent preKernels :
15  ['a2']

```

It is easy to verify by hand that the characteristic vector  $[-1, -1, +1]$  satisfies the initial kernel equation system; node  $a_3$  gives an *initial kernel*. Similarly, the characteristic vector  $[-1, +1, -1]$  verifies indeed the terminal kernel equation system and node  $a_2$  gives hence a *terminal kernel*.

We succeeded now in generalising crisp kernel equation systems to genuine bipolar-valued digraphs (Bisdorff, Pirlot, and Roubens, 2006; Bisdorff, 2006). The constructive proof, found by *Marc Pirlot*, is based on the following *fixpoint equation* that may be used for computing bipolar-valued kernel membership vectors:

$$T(Y) := -(Y \circ R) = Y. \quad (17.3)$$

*John von Neumann* showed indeed that, when a digraph  $G(X, R)$  is acyclic with a unique initial kernel  $K$  characterised by its membership characteristics vector  $Y_k$ , then the following double fixpoint equation:

$$T^2(Y) := -(-(Y \circ R) \circ R) = Y, \quad (17.4)$$

will admit a stable *high* and a stable *low* fixpoint solution that converge both to  $Y_k$  (Schmidt and Ströhlein, 1985).

Inspired by the crisp double fixpoint equation 17.4 on the preceding page, we observed that for a given bipolar-valued digraph  $G(X, R)$ , each one of its dominant or absorbent prekernels  $K_i$  in  $X$  determines an induced partial digraph  $G(X, R/K_i)$  which is acyclic and admits  $K_i$  as unique prekernel (Bisdorff, 1997).

Following the *von Neumann* fixpoint algorithm, a similar bipolar-valued extended double fixpoint algorithm, applied to  $G(X, R/K_i)$ , allows us to compute the associated bipolar-valued kernel characteristic vectors  $Y_i$  in polynomial complexity.

---

**Algorithm 17.1** Computing bipolar-valued kernel characteristic vectors

---

*in* : bipolar-valued digraph  $G(X, R)$ ,

*out* : set  $\{Y_1, Y_2, \dots\}$  of bipolar-valued kernel membership characteristic vectors.

1. Enumerate all initial and terminal prekernels  $K_1, K_2, \dots$  in the given bipolar-valued digraph (see List. 17.17 on page 247);
  2. For each crisp initial kernel  $K_i$ :
    - a. Construct a partially determined subgraph  $G(X, R/K_i)$  supporting exactly this unique initial kernel  $K_i$ ;
    - b. Use the double fixpoint equation  $T^2$  (Eq. 17.4) with the partially determined adjacency matrix  $R/K_i$  for computing a stable low and a stable high fixpoint;
    - c. Determine the bipolar-valued  $K_i$ -membership characteristic vector  $Y_i$  with an epistemic disjunction of the previous low and high fixpoints;
  3. Repeat step (2) for each terminal kernel  $K_j$  by using the double fixpoint equation  $T^2$  with the transpose of the adjacency matrix  $R/K_j$ .
- 

### Time for a practical illustration

We reconsider the random digraph  $g$  generated in Listing 17.12 on page 242. Digraph  $g$  models the pairwise outranking situations between seven decision alternatives evaluated on seven incommensurable performance criteria. We recompute its corresponding bipolar-valued prekernels on the associated codual digraph  $gcd$ .

```

1 >>> gcd = ~(-g) # strict outranking digraph
2 >>> gcd.showPreKernels()
3     *** Computing prekernels ***
4     Dominant prekernels :
5     ['a1', 'a4', 'a2']
6         independence : +0.000
7         dominance   : +0.070
8         absorbency   : -0.488
9         covering     : +0.667
10    Absorbent prekernels :
11    ['a7', 'a3']
12        independence : +0.000
13        dominance   : -0.744
14        absorbency   : +0.163

```

```

15      covered      : +0.800
16 *----- statistics -----
17   graph name: converse-dual_rel_randomperftab
18   number of solutions
19     dominant kernels : 1
20     absorbent kernels: 1
21   cardinality frequency distributions
22   cardinality      : [0, 1, 2, 3, 4, 5, 6, 7]
23   dominant kernel  : [0, 0, 0, 1, 0, 0, 0, 0]
24   absorbent kernel: [0, 0, 1, 0, 0, 0, 0, 0]
25 Execution time   : 0.00022 sec.

```

The codual outranking digraph  $\text{gcd}$ , modelling a strict outranking relation, admits an initial prekernel  $\{a_1, a_2, a_4\}$  and a terminal one  $\{a_3, a_7\}$  (see Lines 5 and 11 above).

In Listing 17.19 we now compute, with the `domkernelrestrict()` method, the initial prekernel-restricted adjacency table (see Fig. 17.10).

**Listing 17.19** Computing a dominant prekernel restricted adjancency table

```

1 >>> k1Relation = gcd.domkernelrestrict(['a1','a2','a4'])
2 >>> gcd.showHTMLRelationTable(
3 ...     actionsList=['a1','a2','a4','a3','a5','a6','a7'], \
4 ...     relation=k1Relation, \
5 ...     tableTitle='K1 restricted adjacency table')

```

**Fig. 17.10** Initial kernel  $\{a_1, a_2, a_4\}$  restricted adjacency table. The outranking situation between alternatives  $a_4$  and  $a_1$  being *indeterminate*, this initial prekernel is only weakly independent.

## K1 restricted adjacency table

r(x \ y)	a1	a2	a4	a3	a5	a6	a7
a1	-	-0.23	-1.00	0.00	0.00	0.00	0.16
a2	-0.21	-	-0.21	0.21	0.44	0.05	0.49
a4	0.00	-0.21	-	0.21	0.00	0.07	0.58
a3	-0.28	-0.21	-0.74	-	0.00	0.00	0.00
a5	-0.26	-0.67	0.00	0.00	-	0.00	0.00
a6	-0.12	-0.49	-0.49	0.00	0.00	-	0.00
a7	-0.51	-0.49	-0.86	0.00	0.00	0.00	-

Valuation domain: [-1.00; +1.00]

The corresponding initial prekernel membership characteristic vector may be computed with the `computeKernelVector()` method.

**Listing 17.20** Fixpoint iterations for initial prekernel  $\{a_1, a_2, a_4\}$

```

1 >>> gcd.computeKernelVector(['a1','a2','a4'], \
2 ...                           Initial=True,Comments=True)
3 --> Initial prekernel: {'a1', 'a2', 'a4'}
4 initial low vector : [-1.00,-1.00,-1.00,-1.00,-1.00,-1.00,-1.00]
5 initial high vector: [+1.00,+1.00,+1.00,+1.00,+1.00,+1.00,+1.00]
6 1st low vector    : [ 0.00,+0.21,-0.21, 0.00,-0.44,-0.07,-0.58]
7 1st high vector   : [+1.00,+1.00,+1.00,+1.00,+1.00,+1.00,+1.00]
8 2nd low vector    : [ 0.00,+0.21,-0.21, 0.00,-0.44,-0.07,-0.58]
9 2nd high vector   : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.05,-0.21]
10 3rd low vector   : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.07,-0.21]
11 3rd high vector  : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.05,-0.21]

```

```

12 4th low vector      : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.07,-0.21]
13 4th high vector    : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.07,-0.21]
14 Iterations         : 4
15 low & high fusion  : [ 0.00,+0.21,-0.21,+0.21,-0.21,-0.07,-0.21]
16 Choice vector for initial prekernel: {'a1', 'a2', 'a4'}
17   'a2': +0.21
18   'a4': +0.21
19   'a1':  0.00
20   'a6': -0.07
21   'a3': -0.21
22   'a5': -0.21
23   'a7': -0.21

```

In Listing 17.20 on the facing page we start the fixpoint computation with an empty set characterisation as first low vector and a complete set  $X$  characterising high vector. After each iteration, the low vector is set to the negation of the previous high vector and the high vector is set to the negation of the previous low vector.

A unique stable prekernel characteristic vector  $Y_1$  is here attained at the fourth iteration with positive members  $a2: +0.21$  and  $a4: +0.21$  (60.5% criteria significance majority);  $a1: 0.00$  being an ambiguous potential member. Alternatives  $a3$ ,  $a5$ ,  $a6$  and  $a7$  are all negative members, i.e. positive *non members* of this outranking prekernel.

Let us also compute the restricted adjacency table for the outranked, i.e. the *terminal* prekernel  $\{a3, a7\}$ .

```

1 >>> k2Relation = gcd.abskernelrestrict(['a3','a7'])
2 >>> gcd.showHTMLRelationTable(
3 ...   actionsList=['a3','a7','a1','a2','a4','a5','a6'], \
4 ...   relation=k2Relation, \
5 ...   tableTitle='K2 restricted adjacency table')

```

**Fig. 17.11** Terminal prekernel  $\{a3, a7\}$  restricted adjacency table. Again, we notice that this terminal prekernel is only weakly independent.

## K2 restricted adjacency table

r(x S y)	a3	a7	a1	a2	a4	a5	a6
<b>a3</b>	-	0.00	-0.28	-0.21	-0.74	-0.40	-0.53
<b>a7</b>	-1.00	-	-0.51	-0.49	-0.86	-0.67	-0.63
<b>a1</b>	0.00	0.16	-	0.00	0.00	0.00	0.00
<b>a2</b>	0.21	0.49	0.00	-	0.00	0.00	0.00
<b>a4</b>	0.21	0.58	0.00	0.00	-	0.00	0.00
<b>a5</b>	0.00	0.16	0.00	0.00	0.00	-	0.00
<b>a6</b>	0.30	0.26	0.00	0.00	0.00	0.00	-

Valuation domain: [-1.00; +1.00]

The corresponding bipolar-valued characteristic vector  $Y_2$  may be computed as follows:

```

1 >>> gcd.computeKernelVector(['a3','a7'], \
2 ...                           Initial=False,Comments=True)
3 --> Terminal prekernel: {'a3', 'a7'}
4 initial low vector  : [-1.00,-1.00,-1.00,-1.00,-1.00,-1.00,-1.00]
5 initial high vector : [+1.00,+1.00,+1.00,+1.00,+1.00,+1.00,+1.00]
6 1st low vector     : [-0.16,-0.49, 0.00,-0.58,-0.16,-0.30,+0.49]
7 1st high vector    : [+1.00,+1.00,+1.00,+1.00,+1.00,+1.00,+1.00]
8 2nd low vector     : [-0.16,-0.49, 0.00,-0.58,-0.16,-0.30,+0.49]

```

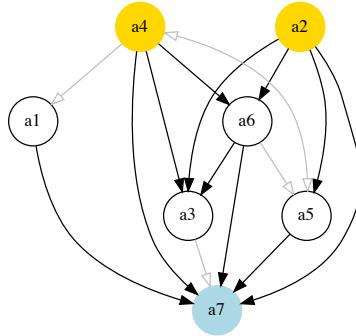
```

9 2nd high vector      : [-0.16,-0.49, 0.00,-0.49,-0.16,-0.26,+0.49]
10 3rd low vector     : [-0.16,-0.49, 0.00,-0.49,-0.16,-0.26,+0.49]
11 3rd high vector    : [-0.16,-0.49, 0.00,-0.49,-0.16,-0.26,+0.49]
12 Iterations          : 3
13 high & low fusion   : [-0.16,-0.49, 0.00,-0.49,-0.16,-0.26,+0.49]
14 Choice vector for terminal prekernel: {'a3','a7'}
15   'a7': +0.49
16   'a3': 0.00
17   'a1': -0.16
18   'a5': -0.16
19   'a6': -0.26
20   'a2': -0.49
21   'a4': -0.49

```

A unique stable bipolar-valued high and low fixpoint is attained at the third iteration with alternative  $a_7$  positively confirmed (about  $(1.0+0.49)/2 = 0.75\%$  criteria significance majority, see Line 13) as member of this terminal prekernel, whereas the membership of alternative  $a_3$  in this prekernel appears indeterminate. All the remaining nodes have negative membership characteristic values and are hence positively excluded from this prekernel.

**Fig. 17.12** The strict outranking digraph oriented by its initial and terminal prekernels. Notice the indeterminate outranking situation between alternatives  $a_4$  and  $a_1$ , and the same indeterminate outranking situation between alternatives  $a_3$  and  $a_7$



Digraph3 (graphviz), R. Bisdorff, 2020

When we reconsider the graphviz drawing of this outranking digraph in Figure 17.12, it becomes obvious here why alternative  $a_1$  is *neither included nor excluded* from the initial prekernel. Same observation is applicable to alternative  $a_3$  that can *neither be included nor excluded* from the terminal prekernel. It may even happen, in case of more indeterminate outranking situations, that no alternative is positively included or excluded from a weakly independent prekernel; the corresponding bipolar-valued membership characteristic vector being completely indeterminate.

To illustrate finally why sometimes it is necessary to operate an epistemic disjunctive fusion of unequal stable low and high membership characteristics vectors (see Alg. 17.1 on page 249, Step 2c), let us consider, for instance, the following crisp 7-cycle graph.

```

1 >>> from digraphs import CirculantDigraph
2 >>> c7 = CirculantDigraph(order=7,circulants=[-1,1])
3 >>> c7
4 *----- Digraph instance description -----*
5 Instance class      : CirculantDigraph
6 Instance name       : c7
7 Digraph Order       : 7
8 Digraph Size        : 14
9 Valuation domain    : [-1.00;1.00]
10 Determinateness (%) : 100.00
11 Attributes          : ['name','order','circulants',
12                           'actions','valuationdomain',
13                           'relation','gamma','notGamma']

```

Digraph  $c_7$  is a symmetric crisp digraph showing, among others, the maximal independent set  $\{2,5,7\}$ , i.e. an initial as well as terminal kernel. The `computeKernelVector()` method computes the corresponding initial kernel characteristic vector.

```

1 >>> c7.computeKernelVector(['2','5','7'],\
2 ...                               Initial=True,Comments=True)
3 --> Initial kernel: {'2': '5', '7'}
4 initial low vector : [-1.0, -1.0, -1.0, -1.0, -1.0, -1.0, -1.0]
5 initial high vector: [+1.0, +1.0, +1.0, +1.0, +1.0, +1.0, +1.0]
6 1 st low vector   : [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
7 1 st high vector : [+1.0, +1.0, +1.0, +1.0, +1.0, +1.0, +1.0]
8 2 nd low vector   : [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
9 2 nd high vector : [ 0.0, +1.0,  0.0,  0.0, +1.0,  0.0, +1.0]
10 stable low vector: [-1.0,  0.0, -1.0, -1.0,  0.0, -1.0,  0.0]
11 stable high vector: [ 0.0, +1.0,  0.0,  0.0, +1.0,  0.0, +1.0]
12 Iterations         : 3
13 low & high fusion : [-1.0, +1.0, -1.0, -1.0, +1.0, -1.0, +1.0]
14 Choice vector for initial prekernel: {'2': '5', '7':
15   '7': +1.00
16   '5': +1.00
17   '2': +1.00
18   '6': -1.00
19   '4': -1.00
20   '3': -1.00
21   '1': -1.00

```

Notice that the stable low vector characterises the *negative membership* part, whereas, the stable high vector characterises the *positive membership* part (see Lines 10-11 above). The bipolar epistemic fusion assembles eventually both stable parts into the correct prekernel characteristic vector (Line 13).

The adjacency matrix of a symmetric digraph staying *unchanged* by the transposition operator, the previous computations, when qualifying the same kernel as a terminal instance, will hence produce exactly the same result.

It is worthwhile noticing the essential computational role, the logical indeterminate value 0.0 is playing in this double fixpoint algorithm. To implement such kind of algorithms without a logical *neutral* term would be like implementing numerical algorithms without a possible usage of the number 0. Infinitely many trivial impossibility theorems and dubious logical results come up.

## Notes

Following the observation that an independent absorbent choice in an acyclic digraph corresponds to the kernel of the associated GRUNDY function, [Riguet \(1948\)](#) introduced the name “noyau” (kernel) for such a choice. Terminal kernels were in the sequel studied by [Berge \(1962\)](#) in the context of Combinatorial Game Theory. Initial kernels –independent and dominating choices– were introduced under the name game solutions by [von Neumann and Morgenstern \(1944\)](#). The absorbent version of the crisp kernel equation system [17.1 on page 248](#) was first introduced by [Schmidt and Ströhlein \(1985\)](#) in the context of their thorough exploration of relational algebra.

The fuzzy version of kernel equation systems was first investigated by [Kitainik \(1993\)](#). Commenting on this work at a meeting in Spring 1995 of the EURO Working Group on Multicriteria Decision Aiding in Lausanne (Switzerland), *Marc Roubens* feared that solving such fuzzy kernel equation systems could be computationally difficult. Triggered by his pessimistic remark and knowing about kernel equation systems and the VON NEUMANN fixpoint theorem ([von Neumann and Morgenstern, 1944; Schmidt and Ströhlein, 1985](#)), I immediately started to implement in Prolog a solver for the valued version of Equation [17.1 on page 248](#); the equation system serving as constraints for a discrete labelling of all possible rational solution vectors. And in Summer 1995, we luckily obtained with a commercial finite domain solver the very first valued outranking kernel from a didactical outranking digraph of order 8, well known in the multiple-criteria decision aiding community. The computation took several seconds on a CRAY 6412 superserver with 12 processors operating in a nowadays ridiculous CPU speed of 90 Mhz. The labelled solution vectors, obtained in the sequel for any outranking digraphs with a single initial kernel, were structured in a way that suggested the converging stages of the VON NEUMANN fixpoint algorithm and so gave the initial idea for eventually computing all prekernels of a bipolar-valued outranking digraph ([Bisdorff, 1996, 1997; Bisdorff and Roubens, 2004](#)).

In our present Python3.9 implementation, such a tiny problem is solved in less than a thousandth of a second on a common laptop. And this remains practically the same for any relevant example of outranking digraph observed in a real decision-aiding problem. Several times we wrote in our personal journal that there is certainly now no more potential for any substantial improvement of this computational efficiency; Only to discover, shortly later, that following a new theoretical idea or choosing a more efficient implementation —using for instance the amazing instrument of iterator generators in Python—, execution times could well be divided by 20.

This nowadays available computational efficiency confers the bipolar-valued kernel concept a methodological premium for solving specific decision problems on the basis of the bipolar-valued outranking digraph (see Chap. 4 and Chap. 12). But it also opens new opportunities for verifying and implementing kernel extraction algorithms for more graph theoretical purposes. New results, like enumerating the non isomorphic maximal independent sets –the kernels– of known difficult graph in-

stances like the  $n$ -cycle, could be obtained (see Sec. 21.7 and Bisdorff and Marichal, 2008).

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# Chapter 18

## On confident outrankings with uncertain criteria significance weights

**Abstract** When modelling preferences following the outranking approach, the signs of the majority margins do sharply distribute validation and invalidation of pairwise outranking situations. How can we be confident in the resulting outranking digraph, when we acknowledge the usual imprecise knowledge of criteria significance weights coupled with small majority margins? In this chapter we propose to link the qualifying significance majority with a required  $\alpha\%$ -confidence level. We model therefore the significance weights as random variables following more or less widespread distributions around an average significance value that corresponds to the given deterministic weight. As the bipolar-valued random credibility of an outranking statement hence results from the simple sum of positive or negative independent random variables, we may apply the Central Limit Theorem (CLT) for computing the bipolar likelihood that the expected majority margin will indeed be positive, respectively negative.

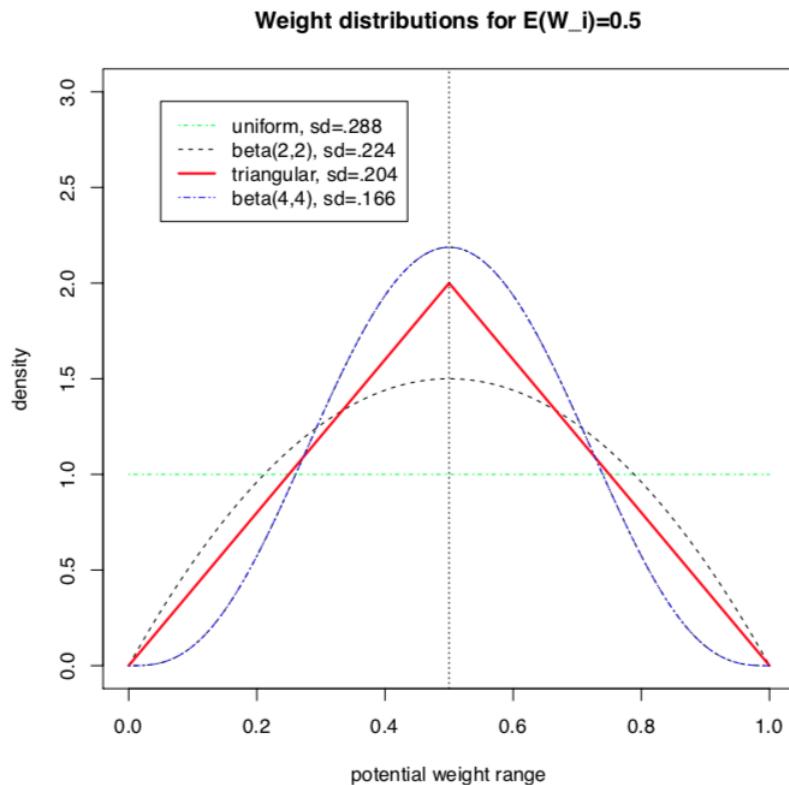
### 18.1 Modelling uncertain criteria significance weights

Let us consider the significance weights of a family  $F$  of  $m$  criteria to be independent random variables  $W_j$ , distributing the potential significance weight of each criterion  $j = 1, \dots, m$  around a mean value  $E(W_j)$  with variance  $V(W_j)$ .

Choosing a specific stochastic model of uncertainty is usually application specific. In the limited scope of this chapter, we will illustrate the consequence of this design decision on the resulting outranking modelling with four slightly different models for taking into account the uncertainty with which we know the numerical significance weights: *uniform*, *triangular*, and two models of *Beta laws*, one more widespread and, the other, more concentrated.

When considering, for instance, that the potential range of a significance weight is distributed between 0 and two times its mean value, we obtain the following random variates (Bisdorff, 2020):

- A continuous *uniform* distribution on the range 0 to  $2E(W_j)$ . Thus  $W_j \sim U(0, 2E(W_j))$  and  $V(W_j) = 1/3(E(W_j))^2$ ;
- A *symmetric beta* distribution with, for instance, parameters  $\alpha = 2$  and  $\beta = 2$ . Thus,  $W_j \sim Beta(2, 2) \times 2E(W_j)$  and  $V(W_j) = 1/5(E(W_j))^2$ ;
- A *symmetric triangular* distribution on the same range with mode  $E(W_j)$ . Thus  $W_j \sim Tr(0, 2E(W_j), E(W_j))$  with  $V(W_j) = 1/6(E(W_j))^2$ ;
- A *narrower beta* distribution with, for instance, parameters  $\alpha = 4$  and  $\beta = 4$ . Thus  $W_j \sim Beta(4, 4) \times 2E(W_j)$ ,  $V(W_j) = 1/9(E(W_j))^2$ .



**Fig. 18.1** Four models of uncertain criteria significance weights

It is worthwhile noticing in Figure 18.1 that these four uncertainty models all admit the same expected value,  $E(W_j)$ , however, with a respective variance which goes decreasing from  $1/3$ , to  $1/9$  of the square of  $E(W_j)$  (Bisdorff, 2014).

## 18.2 Bipolar-valued likelihood of outranking situations

Let  $A = \{x, y, z, \dots\}$  be a finite set of  $n$  potential decision actions, evaluated on  $F = \{1, \dots, m\}$  – a finite and coherent family of  $m$  performance criteria. On each criterion  $j \in F$ , the decision actions are evaluated on a real performance scale  $[0; M_j]$ , supporting an upper-closed indifference threshold  $ind_j$  and a lower-closed preference threshold  $pr_j$  such that  $0 \leq ind_j < pr_j \leq M_j$ . The marginal performance of object  $x$  on criterion  $j$  is denoted  $x_j$ . Each criterion  $j$  is thus characterising a marginal double threshold order  $\geq_j$  on  $A$ :

$$r(x \geq_j y) = \begin{cases} +1 & \text{if } x_j - y_j \geq -ind_j, \\ -1 & \text{if } x_j - y_j \leq -pr_j, \\ 0 & \text{otherwise.} \end{cases} \quad (18.1)$$

Semantics of the marginal bipolar-valued characteristic function:

- $+1$  signifies  $x$  is at least as well evaluated as  $y$  on criterion  $j$ ,
- $-1$  signifies that  $x$  is not at least as well evaluated as  $y$  on criterion  $j$ ,
- $0$  signifies that it is unclear whether, on criterion  $j$ ,  $x$  is at least as well evaluated as  $y$ .



**Fig. 18.2** Bipolar-valued outranking characteristic function

Each criterion  $j$  in  $F$  contributes the random significance  $W_j$  of his “*at least as well evaluated as*” characteristic  $r(x \geqslant_j y)$  to the global characteristic  $\tilde{r}(x \geqslant y)$  in the following way:

$$\tilde{r}(x \geqslant y) = \sum_{j \in F} W_j \times r(x \geqslant_j y) \quad (18.2)$$

Thus,  $\tilde{r}(x \geqslant y)$  becomes a simple sum of positive or negative independent random variables with known means and variances where  $\tilde{r}(x \geqslant y) > 0$  signifies  $x$  is globally at least as well evaluated as  $y$ ,  $\tilde{r}(x \geqslant y) < 0$  signifies that  $x$  is not globally at least as well evaluated as  $y$ , and  $\tilde{r}(x \geqslant y) = 0$  signifies that it is unclear whether  $x$  is globally at least as well evaluated as  $y$ .

From the *Central Limit Theorem* (CLT), we know that such a sum of random variables leads, with  $m$  getting large, to a Gaussian distribution  $Y$  with

$$E(Y) = \sum_{j \in F} (E(W_j) \times r(x \geqslant_j y)), \text{ and} \quad (18.3)$$

$$V(Y) = \sum_{j \in F} (V(W_j) \times |r(x \geqslant_j y)|). \quad (18.4)$$

And the *likelihood of validation*, respectively *invalidation* of an “*at least as well evaluated as*” situation, denoted  $lh(x \geqslant y)$ , may hence be assessed by the probability  $P(Y > 0) = 1.0 - P(Y \leqslant 0)$  that  $Y$  takes a positive, resp.  $P(Y < 0)$  takes a negative value. In the bipolar-valued case here, we can judiciously make usage of the standard Gaussian *error function*, i.e. the bipolar  $2P(Z) - 1.0$  version of the standard Gaussian  $P(Z)$  probability distribution function (Press et al., 2007):

$$lh(x \geqslant y) = -\operatorname{erf}\left(\frac{1}{\sqrt{2}} \frac{-E(Y)}{\sqrt{V(Y)}}\right) \quad (18.5)$$

The range of the bipolar-valued  $lh(x \geqslant y)$  hence becomes  $[-1.0; +1.0]$ , and  $-lh(x \geqslant y) = lh(x \not\geqslant y)$ , i.e. a *negative likelihood* represents the likelihood of the correspondent *negated “at least as well evaluated as”* situation. A likelihood of  $+1.0$  (resp.  $-1.0$ ) means the corresponding preferential situation appears *certainly validated* (resp. invalidated).

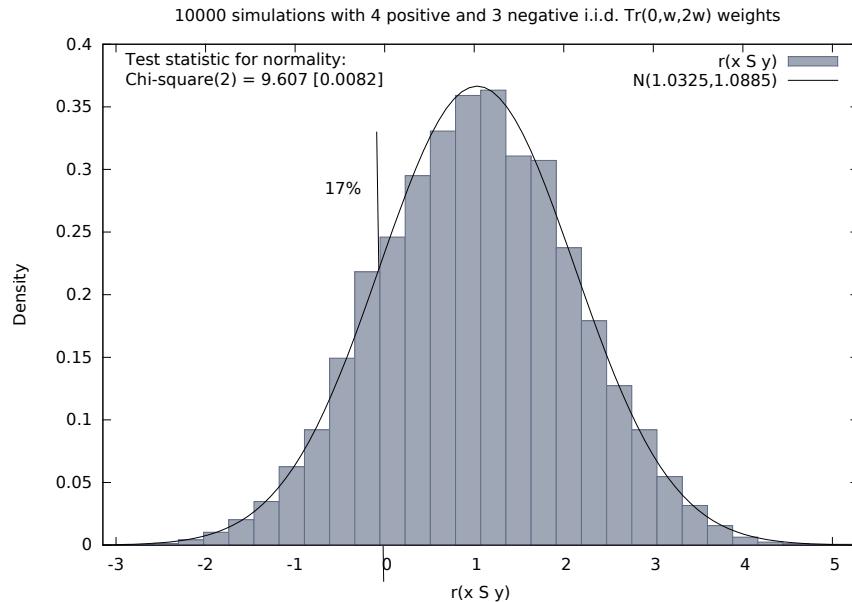
*Example 18.1.* Let  $x$  and  $y$  be evaluated with respect to 7 equisignificant criteria; Four criteria positively support that  $x$  is *as least as well assessed as*  $y$  and three criteria support that  $x$  is *not at least as well evaluated as*  $y$ . Suppose  $E(W_j) = w$  for  $j = 1, \dots, 7$  and  $W_j \sim Tr(0, 2w, w)$  for  $j = 1, \dots, 7$ . The expected value of the global “*at least as well evaluated as*” characteristic value becomes:  $E(\tilde{r}(x \geqslant y)) = 4w - 3w = w$  with a variance  $V(\tilde{r}(x \geqslant y)) = 7\frac{1}{6}w^2$ .

If  $w = 1$ ,  $E(\tilde{r}(x \geqslant y)) = 1$  and  $sd(\tilde{r}(x \geqslant y)) = 1.08$ . By the CLT, the bipolar likelihood of the *at least as well evaluated as* situation becomes:  $lh(x \geqslant y) = 0.66$ , which corresponds to a global support of  $(0.66 + 1.0)/2 = 83\%$  of the criteria significance weights.

A *Monte Carlo* simulation with 10000 runs empirically confirms the effective convergence to a Gaussian (see Fig. 18.3 on the facing page realised with gretl<sup>1</sup>).

---

<sup>1</sup> The Gnu Regression, Econometrics and Time-series Library <http://gretl.sourceforge.net/>.



**Fig. 18.3** Distribution of 10000 random outranking characteristic values

Indeed,  $\tilde{r}(x \geq y) \rightsquigarrow Y = \mathcal{N}(1.03, 1.089)$ , with an empirical probability of observing a negative majority margin of about 17%.

### 18.3 Confidence level of outranking digraphs

#### Definition 18.1 (Confident outranking situation).

Following the classical outranking approach (see (Bisdorff, 2013)), we say, from an epistemic perspective, that decision action  $x$  *outranks* decision action  $y$  at *confidence level*  $\alpha\%$ , when:

- An expected majority of criteria validates, at confidence level  $\alpha\%$  or higher, a global “*at least as well evaluated as*” situation between  $x$  and  $y$ , and
- no considerably less performing is observed on a discordant criterion.

Dually, decision action  $x$  *does not outrank* decision action  $y$  at confidence level  $\alpha\%$ , when:

- An expected minority only of criteria at confidence level  $-\alpha\%$  or lesser, validates a global “*at least as well evaluated as*” situation between  $x$  and  $y$ , and
- no considerably better performing situation is observed on a discordant criterion.

Otherwise, the situation is indeterminate.

#### Time for a coded example

Let us consider the following random performance tableau.

```

1 >>> from randomPerfTabs import RandomPerformanceTableau
2 >>> t = RandomPerformanceTableau(\n
3 ...     numberOfRows=7,\n
4 ...     numberOfCriteria=7, seed=100)\n
5 >>> t.showPerformanceTableau(transposed=True)
6 ----- performance tableau -----*
7 criteria | weights | 'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'\n
8 ----- | ----- |
9    'g1' | 1 | 15.17 44.51 57.87 58.00 24.22 29.10 96.58\n
10   'g2' | 1 | 82.29 43.90 NA 35.84 29.12 34.79 62.22\n
11   'g3' | 1 | 44.23 19.10 27.73 41.46 22.41 21.52 56.90\n
12   'g4' | 1 | 46.37 16.22 21.53 51.16 77.01 39.35 32.06\n
13   'g5' | 1 | 47.67 14.81 79.70 67.48 NA 90.72 80.16\n
14   'g6' | 1 | 69.62 45.49 22.03 33.83 31.83 NA 48.80\n
15   'g7' | 1 | 82.88 41.66 12.82 21.92 75.74 15.45 6.05

```

For the corresponding confident outranking digraph, we assume uncertain criteria significance weights of *triangular* distribution and require a confidence level of  $\alpha = 90\%$ . The `ConfidentBipolarOutrankingDigraph` class provides such a construction.

```

1 >>> from outrankingDigraphs import \
2 ...     ConfidentBipolarOutrankingDigraph
3 >>> g90 = ConfidentBipolarOutrankingDigraph(t,\n
4 ...     distribution ='triangular',confidence=90)\n
5 >>> g90
6 ----- Object instance description -----*
7 Instance class      : ConfidentBipolarOutrankingDigraph
8 Instance name       : rel_randomperftab_CLT
9 Actions            : 7
10 Criteria           : 7
11 Size               : 15
12 Uncertainty model : triangular(a=0,b=2w)
13 Likelihood domain : [-1.0;+1.0]
14 Confidence level  : 0.80 (90.0%)
15 Confident majority : 0.14 (57.1%)
16 Determinateness (%) : 62.07
17 Valuation domain   : [-1.00;1.00]
18 Attributes          : ['name', 'bipolarConfidenceLevel',
19 ...                         'distribution', 'betaParameter', 'actions',
20 ...                         'order', 'valuationdomain', 'criteria',
21 ...                         'evaluation', 'concordanceRelation',
22 ...                         'vetos', 'negativeVetos',
23 ...                         'largePerformanceDifferencesCount',
24 ...                         'likelihoods', 'confidenceCutLevel',
25 ...                         'relation', 'gamma', 'notGamma']

```

The resulting 90%-confident expected outranking relation is shown below.

**Listing 18.1** 90%-confident outranking relation with triangular distributed significance weights

```

1 >>> g90.showRelationTable(LikelihoodDenotation=True)
2 * ---- Outranking Relation Table -----
3 r/(lh) | 'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'\n
4 ----- | ----- |
5   'a1' | +0.00  +0.71  +0.29  +0.29  +0.29  +0.29  +0.00\n
6   | ( - )  (+1.00) (+0.95) (+0.95) (+0.95) (+0.95) (+0.65)\n
7   'a2' | -0.71  +0.00  -0.29  +0.00  +0.00  +0.29  -0.57

```

```

8      | (-1.00)   ( - )   (-0.95)   (-0.65)   (+0.73)   (+0.95)   (-1.00)
9      'a3'     | -0.29   +0.29   +0.00   -0.29   +0.00   +0.00   -0.29
10     | (-0.95)   (+0.95)   ( - )   (-0.95)   (-0.73)   (-0.00)   (-0.95)
11     'a4'     | +0.00   +0.00   +0.57   +0.00   +0.29   +0.57   -0.43
12     | (-0.00)   (+0.65)   (+1.00)   ( - )   (+0.95)   (+1.00)   (-0.99)
13     'a5'     | -0.29   +0.00   +0.00   +0.00   +0.00   +0.29   -0.29
14     | (-0.95)   (-0.00)   (+0.73)   (-0.00)   ( - )   (+0.99)   (-0.95)
15     'a6'     | -0.29   +0.00   +0.00   -0.29   +0.00   +0.00   +0.00
16     | (-0.95)   (-0.00)   (+0.73)   (-0.95)   (+0.73)   ( - )   (-0.00)
17     'a7'     | +0.00   +0.71   +0.57   +0.43   +0.29   +0.00   +0.00
18     | (-0.65)   (+1.00)   (+1.00)   (+0.99)   (+0.95)   (-0.00)   ( - )
19 Valuation domain : [-1.000; +1.000]
20 Uncertainty model : triangular(a=2.0,b=2.0)
21 Likelihood domain : [-1.0;+1.0]
22 Confidence level : 0.80 (90.0%)
23 Confident majority : 0.14 (57.1%)
24 Determinateness : 0.24 (62.1%)

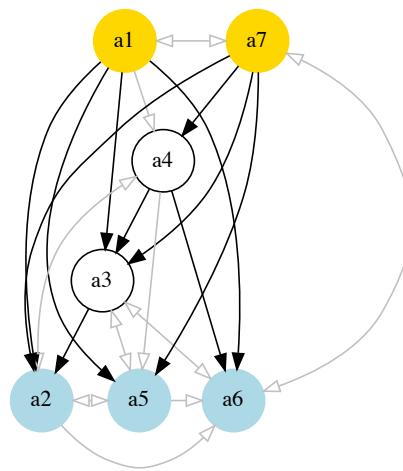
```

The  $(lh)$  figures, indicated in Listing 18.1 above, correspond to bipolar likelihoods and the required bipolar confidence level equals  $(0.90 + 1.0)/2 = 0.80$  (see Line 22 above). Action  $a_1$  thus confidently outranks all other actions, except  $a_7$  where the actual likelihood (+0.65) is lower than the required one (+0.80) and we furthermore observe a considerable counter-performance on criterion  $g_1$ .

Notice also the lack of confidence in the outranking situations we observe between action  $a_2$  and actions  $a_4$  and  $a_5$ . In the deterministic case we would have  $r(a_2 \geq a_4) = -0.143$  and  $r(a_2 \geq a_5) = +0.143$ . All outranking situations with a characteristic value lower or equal to  $\text{abs}(0.143)$ , i.e. a majority support of  $1.143/2 = 57.1\%$  and less, appear indeed to be *not confident* at  $\alpha$  level 90% (see Line 23 above).

Figure 18.4 on the next page shows the corresponding strict 90%-confident outranking digraph, oriented by its initial and terminal strict prekernels.

**Fig. 18.4** 90%-confident strict outranking digraph oriented by its initial and terminal prekernels



Digraph3 (graphviz), R. Bisdorff, 2020

Now, what becomes this 90%-confident outranking digraph when we require a stronger confidence level of, say 99%?

**Listing 18.2** 99%-confident outranking relation

```

1 >>> g99 = ConfidentBipolarOutrankingDigraph(t,\n2 ...     distribution='trinangular',confidence=99)\n3 >>> g99.showRelationTable()\n4 * ----- Outranking Relation Table -----*\n5 r/(lh) | 'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'\n6 -----|\n7 'a1' | +0.00  +0.71  +0.00  +0.00  +0.00  +0.00  +0.00\n8     | (-)  (+1.00)  (+0.95)  (+0.95)  (+0.95)  (+0.95)  (+0.65)\n9 'a2' | -0.71  +0.00  +0.00  +0.00  +0.00  +0.00  -0.57\n10    | (-1.00)  (-)  (-0.95)  (-0.65)  (+0.73)  (+0.95)  (-1.00)\n11 'a3' | +0.00  +0.00  +0.00  +0.00  +0.00  +0.00  +0.00\n12     | (-0.95)  (+0.95)  (-)  (-0.95)  (-0.73)  (-0.00)  (-0.95)\n13 'a4' | +0.00  +0.00  +0.57  +0.00  +0.00  +0.57  -0.43\n14     | (-0.00)  (+0.65)  (+1.00)  (-)  (+0.95)  (+1.00)  (-0.99)\n15 'a5' | +0.00  +0.00  +0.00  +0.00  +0.00  +0.29  +0.00\n16     | (-0.95)  (-0.00)  (+0.73)  (-0.00)  (-)  (+0.99)  (-0.95)\n17 'a6' | +0.00  +0.00  +0.00  +0.00  +0.00  +0.00  +0.00\n18     | (-0.95)  (-0.00)  (+0.73)  (-0.95)  (+0.73)  (-)  (-0.00)\n19 'a7' | +0.00  +0.71  +0.57  +0.43  +0.00  +0.00  +0.00\n20     | (-0.65)  (+1.00)  (+1.00)  (+0.99)  (+0.95)  (-0.00)  (-)\n21 Valuation domain : [-1.000; +1.000]\n22 Uncertainty model : triangular(a=0.0,b=2.0)\n23 Likelihood domain : [-1.0;+1.0]\n24 Confidence level : 0.98 (99.0%)\n25 Confident majority : 0.29 (64.3%)\n26 Determinateness : 0.13 (56.6%)
```

At 99% confidence, the minimal required significance majority support amounts to 64.3% (see List. 18.2 Line 25 above). As a result, most outranking situations don't get anymore validated, like the outranking situations between action  $a_1$  and actions  $a_3, a_4, a_5$  and  $a_6$  (see Line 5 above). The overall epistemic determination of the digraph consequently drops from 62.1% to 56.6% (see Line 26).

Finally, what becomes the previous 90%-confident outranking digraph when the uncertainty concerning the criteria significance weights is modelled with a larger variance, like *uniform* variates (see Line 2 below).

**Listing 18.3** 90%-confident outranking digraph with uniform variates

```

1 >>> gu90 = ConfidentBipolarOutrankingDigraph(t,\n2 ...           confidence=90,distribution='uniform')
3 >>> gu90.showRelationTable()
4   * ----- Outranking Relation Table -----
5   r/(lh) | 'a1'   'a2'   'a3'   'a4'   'a5'   'a6'   'a7'
6   -----|-----
7   'a1' | +0.00   +0.71   +0.29   +0.29   +0.29   +0.29   +0.00
8   | (-) (+1.00) (+0.84) (+0.84) (+0.84) (+0.84) (+0.49)
9   'a2' | -0.71   +0.00   -0.29   +0.00   +0.00   +0.29   -0.57
10  | (-1.00) (-) (-0.84) (-0.49) (+0.56) (+0.84) (-1.00)
11  'a3' | -0.29   +0.29   +0.00   -0.29   +0.00   +0.00   -0.29
12  | (-0.84) (+0.84) (-) (-0.84) (-0.56) (-0.00) (-0.84)
13  'a4' | +0.00   +0.00   +0.57   +0.00   +0.29   +0.57   -0.43
14  | (-0.00) (+0.49) (+1.00) (-) (+0.84) (+1.00) (-0.95)
15  'a5' | -0.29   +0.00   +0.00   +0.00   +0.00   +0.29   -0.29
16  | (-0.84) (-0.00) (+0.56) (-0.00) (-) (+0.92) (-0.84)
17  'a6' | -0.29   +0.00   +0.00   -0.29   +0.00   +0.00   +0.00
18  | (-0.84) (-0.00) (+0.56) (-0.84) (+0.56) (-) (-0.00)
19  'a7' | +0.00   +0.71   +0.57   +0.43   +0.29   +0.00   +0.00
20  | (-0.49) (+1.00) (+1.00) (+0.95) (+0.84) (-0.00) (-)
21 Valuation domain : [-1.000; +1.000]
22 Uncertainty model : uniform(a=0.0,b=2.0)
23 Likelihood domain : [-1.0;+1.0]
24 Confidence level : 0.80 (90.0%)
25 Confident majority : 0.14 (57.1%)
26 Determinateness : 0.24 (62.1%)

```

Despite lower likelihood values (see the *g90* digraph in List. 18.3), we keep here the same confident majority level of 57.1% (Line 25) and, hence, also the same 90%-confident outranking digraph.

For concluding, it is worthwhile noticing again that it is the *neutral* value of the bipolar-valued epistemic logic that allows to easily handle  $\alpha\%$  confidence or not of outranking situations when confronted with uncertain criteria significance weights. Remarkable furthermore is the usage, the standard Gaussian error function (*erf*) provides by delivering *signed* likelihood values immediately concerning either a positive relational statement, or when negative, its negated version (Press et al., 2007).

## References

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# Chapter 19

## Robustness analysis of outranking digraphs

**Abstract** The required cardinal significance weights of the performance criteria represent the '*Achilles'* heel of the outranking approach. Rarely will indeed a decision maker be cognitively competent for suggesting precise decimal-valued criteria significance weights. More often, the decision problem will involve more or less equally important decision objectives with more or less equi-significant criteria per objective. In this chapter, we study the robustness of the outranking digraph when the criteria significance weights faithfully indicate solely an order of importance.

### 19.1 Cardinal or ordinal criteria significance weights?

A random example of such a decision problem with equally important decision objectives and equi-significant criteria may be generated with the Random3ObjectivesPerformanceTableau class .

**Listing 19.1** Generate a Random 3 Objectives Performance Tableau

```
1 >>> from randomPerfTabs import \
2 ...             Random3ObjectivesPerformanceTableau
3 >>> pt = Random3ObjectivesPerformanceTableau(\
4 ...                 numberOfActions=7,\ \
5 ...                 numberOfCriteria=9,seed=102)
6 >>> pt
7      ----- PerformanceTableau instance description -----
8 Instance class    : Random3ObjectivesPerformanceTableau
9 Seed              : 102
10 Instance name   : random3ObjectivesPerfTab
11 Actions          : 7
12 Objectives       : 3
13 Criteria         : 9
14 NA proportion (%): 0.0
15 Attributes       : ['name', 'valueDigits', 'BigData',
16                      'OrdinalScales', 'missingDataProbability',
17                      'negativeWeightProbability', 'randomSeed',
```

```

18         'sumWeights', 'valuationPrecision',
19         'commonScale', 'objectiveSupportingTypes',
20         'actions', 'objectives', 'criteriaWeightMode',
21         'criteria', 'evaluation', 'weightPreorder']
22 >>> pt.showObjectives()
23 ----- show objectives -----
24 Eco: Economical aspect
25     ec1 criterion of objective Eco 8
26     ec4 criterion of objective Eco 8
27     ec8 criterion of objective Eco 8
28     Total weight: 24.00 (3 criteria)
29 Soc: Societal aspect
30     so2 criterion of objective Soc 12
31     so7 criterion of objective Soc 12
32     Total weight: 24.00 (2 criteria)
33 Env: Environmental aspect
34     en3 criterion of objective Env 6
35     en5 criterion of objective Env 6
36     en6 criterion of objective Env 6
37     en9 criterion of objective Env 6
38     Total weight: 24.00 (4 criteria)

```

In Listing 19.1 above one may notice a performance tableau `pt` describing seven decision alternatives that are assessed with respect to three equally important decision objectives concerning:

1. An *economical* aspect with a coalition of three performance criteria of significance weight 8 (Line 24),
2. A *societal* aspect with a coalition of two performance criteria of significance weight 12 (Line 29), and
3. An *environmental* aspect with a coalition of four performance criteria of significance weight 6 (Line 33).

The question we tackle in this chapter is the following: How *dependent* on the actual values of the significance weights appears to be the corresponding bipolar-valued outranking digraph? In the previous Chapter 18 we assumed that the criteria significance weights were random variables. Here, we shall assume that we know for sure only the preordering of the significance weights.

In the random performance tableau `pt` we observe three increasing weight equivalence classes.

#### **Listing 19.2** The significance weights preorder

```

1 >>> pt.showWeightPreorder()
2     ['en3', 'en5', 'en6', 'en9'] (6) <
3     ['ec1', 'ec4', 'ec8'] (8) <
4     ['so2', 'so7'] (12)

```

How robust will appear now the validated outranking and outranked situations when allowing all possible significance weights that are compatible with the weights preorder shown above ([Bisdorff, Meyer, and Veneziano, 2009, 2014](#))?

## 19.2 Qualifying the stability of outranking situations

Let us construct the bipolar-valued outranking digraph corresponding to the random 3-Objectives performance tableau pt.

**Listing 19.3** Example Bipolar Outranking Digraph

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> g = BipolarOutrankingDigraph(pt)
3 >>> g.showRelationTable()
4 * ----- Relation Table -----
5   r(>=) | 'p1'  'p2'  'p3'  'p4'  'p5'  'p6'  'p7'
6   -----|-----
7   'p1' | +1.00 -0.42 +0.00 -0.69 +0.39 +0.11 -0.06
8   'p2' | +0.58 +1.00 +0.83 +0.00 +0.58 +0.58 +0.58
9   'p3' | +0.25 -0.33 +1.00 +0.00 +0.50 +1.00 +0.25
10  'p4' | +0.78 +0.00 +0.61 +1.00 +1.00 +1.00 +0.67
11  'p5' | -0.11 -0.50 -0.25 -0.89 +1.00 +0.11 -0.14
12  'p6' | +0.22 -0.42 +0.00 -1.00 +0.17 +1.00 -0.11
13  'p7' | +0.22 -0.50 +0.17 -0.06 +0.78 +0.42 +1.00

```

In Listing 19.3 Lines 7-13, we notice on the principal diagonal of the relation table the certainly validated reflexive terms +1.00; they are trivially independent of any significance weights. Now, we know for sure that *unanimous* outranking situations are also completely independent of the significance weights. And, all outranking situations that are supported by a majority significance in each coalition of equi-significant criteria are as well independent of the actual importance we attach to each individual criteria coalition. We are furthermore able to effectively test if an outranking situation remains valid with all potential significance weights that are compatible with the given preordering shown in Listing 19.2 on the facing page (Bisdorff, 2014). Mind that usually one obtains more or less numerous outranking situations that are in fact *dependent* on the precise cardinal significance weights given to the criteria.

Such a *stability denotation* of outranking situations can be inspected using the StabilityDenotation=True flag with the showRelationTable() method.

**Listing 19.4** Bipolar-valued outranking relation table with stability denotation

```

1 >>> g.showRelationTable(StabilityDenotation=True)
2 * ----- Relation Table -----
3   r/(stab) | 'p1'  'p2'  'p3'  'p4'  'p5'  'p6'  'p7'
4   -----|-----
5   'p1' | +1.00 -0.42 +0.00 -0.69 +0.39 +0.11 -0.06
6   | (+4)  (-2)  (+0)  (-3)  (+2)  (+2)  (-1)
7   'p2' | +0.58 +1.00 +0.83  0.00 +0.58 +0.58 +0.58
8   | (+2)  (+4)  (+3)  (+2)  (+2)  (+2)  (+2)
9   'p3' | +0.25 -0.33 +1.00  0.00 +0.50 +1.00 +0.25
10  | (+2)  (-2)  (+4)  (0)  (+2)  (+2)  (+1)
11  'p4' | +0.78  0.00 +0.61 +1.00 +1.00 +1.00 +0.67
12  | (+3)  (-1)  (+3)  (+4)  (+4)  (+4)  (+2)
13  'p5' | -0.11 -0.50 -0.25 -0.89 +1.00 +0.11 -0.14

```

14		(-2) (-2) (-2) (-3) (+4) (+2) (-2)
15	'p6'	+0.22 -0.42 0.00 -1.00 +0.17 +1.00 -0.11
16		(+2) (-2) (+1) (-2) (+2) (+4) (-2)
17	'p7'	+0.22 -0.50 +0.17 -0.06 +0.78 +0.42 +1.00
18		(+2) (-2) (+1) (-1) (+3) (+2) (+4)

In Listing 19.4 above we may hence distinguish the following bipolar-valued stability levels:

- ±4: *unanimous* outranking, resp. outranked, situation. The pairwise trivial reflexive outrankings, for instance, all show this stability level;
- ±3: validated outranking, resp. outranked, situation in *each* coalition of equi-significant criteria. This is, for instance, the case for the outranking situation observed between alternatives p1 and p4 (see Lines 6 and 12);
- ±2: validated outranking, resp. outranked situation with *all* potential significance weights that are *compatible* With the given significance preorder (see List. 19.2 on page 268). This is the case when comparing alternatives p1 and p2 (see Lines 6 and 8);
- ±1: validated outranking, resp. outranked situation with the *precisely given decimal* significance weights, a situation we may observe between alternatives p3 and p7 (see Lines 10 and 16);
- 0: indeterminate outranking situation, like the one between alternatives p1 and p3 (see Lines 6 and 10).

It is worthwhile noticing that, in the one limit case where all performance criteria appear equi-significant –there is given a single equivalence class containing all the criteria significance weights– one may only distinguish stability levels ±4 and ±3. In the other limit case, when all performance criteria admit different significance weights, the significance weights may be linearly ordered without ties and no stability level ±3 can be observed.

As mentioned above, all reflexive comparisons trivially confirm an unanimous outranking situation: all decision alternatives are indeed always “*at least as well evaluated as*” themselves. But there appear also two non reflexive unanimous outranking situations: when comparing, for instance, alternative p4 with alternatives p5 and p6 (see List. 19.4 on the preceding page Lines 14 and 16). Let us inspect the details of how alternatives p4 and p5 compare.

1	>>> g.showPairwiseComparison('p4','p5')	
2	*----- pairwise comparison -----*	
3	Comparing actions : (p4, p5)	
4	crit. wght. g(x) g(y) diff   ind pref r()	
5	ec1 8.00 85.19 46.75 +38.44   5.00 10.00 +8.00	
6	ec4 8.00 72.26 8.96 +63.30   5.00 10.00 +8.00	
7	ec8 8.00 44.62 35.91 +8.71   5.00 10.00 +8.00	
8	en3 6.00 80.81 31.05 +49.76   5.00 10.00 +6.00	
9	en5 6.00 49.69 29.52 +20.17   5.00 10.00 +6.00	
10	en6 6.00 66.21 31.22 +34.99   5.00 10.00 +6.00	
11	en9 6.00 50.92 9.83 +41.09   5.00 10.00 +6.00	
12	so2 12.00 49.05 12.36 +36.69   5.00 10.00 +12.00	
13	so7 12.00 55.57 44.92 +10.65   5.00 10.00 +12.00	

```

14     Valuation in range: -72.00 to +72.00;      -----
15             global concordance: +72.00

```

Alternative p4 is indeed unanimously “*at least as well evaluated as*” alternative p5 and  $r(p4 \succsim p5) = 72/72 = +1.00$  (see List. 19.4 on page 269 Line 11). The converse comparison does not, however, reveal such an unanimous outranked situation.

```

1 >>> g.showPairwiseComparison('p5','p4')
2 *----- pairwise comparison -----
3 Comparing actions : (p5, p4)
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ec1 8.00 46.75 85.19 -38.44 | 5.00 10.00 -8.00
6 ec4 8.00 8.96 72.26 -63.30 | 5.00 10.00 -8.00
7 ec8 8.00 35.91 44.62 -8.71 | 5.00 10.00 +0.00
8 en3 6.00 31.05 80.81 -49.76 | 5.00 10.00 -6.00
9 en5 6.00 29.52 49.69 -20.17 | 5.00 10.00 -6.00
10 en6 6.00 31.22 66.21 -34.99 | 5.00 10.00 -6.00
11 en9 6.00 9.83 50.92 -41.09 | 5.00 10.00 -6.00
12 so2 12.00 12.36 49.05 -36.69 | 5.00 10.00 -12.00
13 so7 12.00 44.92 55.57 -10.65 | 5.00 10.00 -12.00
14 Valuation in range: -72.00 to +72.00;      -----
15             global concordance: -64.00

```

The converse comparison only qualifies at stability level  $-3$  (see List. 19.4 on page 269 Line 13);  $r(p5 \succsim p4) = -64/72 = -0.89$ . On criterion ec8 we observe in fact a small negative performance difference of  $-8.71$  (see Line 7 above) which is effectively below the supposed preference discrimination threshold of  $10.00$ . Yet, the outranked situation is supported by a majority of criteria in each decision objective. Hence, the reported preferential situation is completely independent of any chosen significance weights.

Let us now consider a comparison, like the one between alternatives p2 and p1, that is qualified at stability level  $+2$ , resp.  $-2$ .

**Listing 19.5** Comparison of alternatives p2 and p1

```

1 >>> g.showPairwiseOutrankings('p2','p1')
2 *----- pairwise comparison -----
3 Comparing actions : (p2, p1)
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ec1 8.00 89.77 38.11 +51.66 | 5.00 10.00 +8.00
6 ec4 8.00 86.00 22.65 +63.35 | 5.00 10.00 +8.00
7 ec8 8.00 89.43 77.02 +12.41 | 5.00 10.00 +8.00
8 en3 6.00 20.79 58.16 -37.37 | 5.00 10.00 -6.00
9 en5 6.00 23.83 31.40 -7.57 | 5.00 10.00 +0.00
10 en6 6.00 18.66 11.41 +7.25 | 5.00 10.00 +6.00
11 en9 6.00 26.65 44.37 -17.72 | 5.00 10.00 -6.00
12 so2 12.00 89.12 22.43 +66.69 | 5.00 10.00 +12.00
13 so7 12.00 84.73 28.41 +56.32 | 5.00 10.00 +12.00
14 Valuation in range: -72.00 to +72.00;      -----
15             global concordance: +42.00
16 *----- pairwise comparison -----
17 Comparing actions : ('p1', 'p2')
18 crit. wght. g(x) g(y) diff | ind pref r()

```

```

19      ec1    8.00   38.11  89.77  -51.66 | 5.00  10.00  -8.00
20      ec4    8.00   22.65  86.00  -63.35 | 5.00  10.00  -8.00
21      ec8    8.00   77.02  89.43  -12.41 | 5.00  10.00  -8.00
22      en3    6.00   58.16  20.79  +37.37 | 5.00  10.00  +6.00
23      en5    6.00   31.40  23.83  +7.57 | 5.00  10.00  +6.00
24      en6    6.00   11.41  18.66  -7.25 | 5.00  10.00  +0.00
25      en9    6.00   44.37  26.65  +17.72 | 5.00  10.00  +6.00
26      so2   12.00   22.43  89.12  -66.69 | 5.00  10.00 -12.00
27      so7   12.00   28.41  84.73  -56.32 | 5.00  10.00 -12.00
28      Valuation in range: -72.00 to +72.00; -----
29                                         global concordance: -30.00

```

In both comparisons, the evaluations observed with respect to the environmental decision objective are not validating with a significant majority the outranking, resp. outranked, situation. Hence, the stability of the reported preferential situations is in fact dependent on choosing significance weights that are compatible with the given significance weights preorder (see List. 19.2 on page 268).

Let us finally inspect a comparison that is only qualified at stability level +1, like the one between alternatives p7 and p3.

**Listing 19.6** Comparison of alternatives p7 and p3

```

1 >>> g.showPairwiseOutrankings('p7','p3')
2 *----- pairwise comparison -----*
3 Comparing actions : ('p7', 'p3')
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ec1 8.00 15.33 80.19 -64.86 | 5.00 10.00 -8.00
6 ec4 8.00 36.31 68.70 -32.39 | 5.00 10.00 -8.00
7 ec8 8.00 38.31 91.94 -53.63 | 5.00 10.00 -8.00
8 en3 6.00 30.70 46.78 -16.08 | 5.00 10.00 -6.00
9 en5 6.00 35.52 27.25 +8.27 | 5.00 10.00 +6.00
10 en6 6.00 69.71 1.65 +68.06 | 5.00 10.00 +6.00
11 en9 6.00 13.10 14.85 -1.75 | 5.00 10.00 +6.00
12 so2 12.00 68.06 58.85 +9.21 | 5.00 10.00 +12.00
13 so7 12.00 58.45 15.49 +42.96 | 5.00 10.00 +12.00
14 Valuation in range: -72.00 to +72.00; -----
15                                         global concordance: +12.00
16 *----- pairwise comparison -----*
17 Comparing actions : ('p3', 'p7')
18 crit. wght. g(x) g(y) diff | ind pref r()
19 ec1 8.00 80.19 15.33 +64.86 | 5.00 10.00 +8.00
20 ec4 8.00 68.70 36.31 +32.39 | 5.00 10.00 +8.00
21 ec8 8.00 91.94 38.31 +53.63 | 5.00 10.00 +8.00
22 en3 6.00 46.78 30.70 +16.08 | 5.00 10.00 +6.00
23 en5 6.00 27.25 35.52 -8.27 | 5.00 10.00 +0.00
24 en6 6.00 1.65 69.71 -68.06 | 5.00 10.00 -6.00
25 en9 6.00 14.85 13.10 +1.75 | 5.00 10.00 +6.00
26 so2 12.00 58.85 68.06 -9.21 | 5.00 10.00 +0.00
27 so7 12.00 15.49 58.45 -42.96 | 5.00 10.00 -12.00
28 Valuation in range: -72.00 to +72.00; -----
29                                         global concordance: +18.00

```

In both cases, choosing only significances that are compatible with the given weights preorder will not always result in positively validated outranking situations.

### 19.3 Computing the stability denotation of outranking situations

Stability levels  $\pm 4$  and  $\pm 3$  are, the case given, easy to detect. Detecting a stability level  $\pm 2$  is far less obvious. Now, it is precisely again the bipolar-valued epistemic characteristic domain that will give us a way to implement an effective test for stability level  $+2$  and  $-2$  (Bisdorff, 2004a,b).

Let us consider the significance equivalence classes we observe in the given weights preorder. Here we observe three weight classes: 6, 8, and 12, in increasing order (see List. 19.2 on page 268). In the pairwise comparisons, shown above, these equivalence classes may appear positively or negatively, besides the indeterminate significance of value 0.00. We thus get the following ordered bipolar list of significance weights:  $W = [-12, -8, -6, 0, 6, 8, 12]$ .

In all the pairwise marginal comparisons shown in the previous Section, we may observe that each one of the nine criteria assigns one precise item out of this list  $W$ . Let us denote  $q[i]$  the number of criteria assigning item  $W[i]$ , and  $Q[i]$  the cumulative sums of these  $q[i]$  counts, where  $i$  is an index in the range of the length of list  $W$ .

In the comparison of alternatives  $p_2$  and  $p_1$ , for instance (see List. 19.5 on page 271), we observe the following counts:

$W[i]$	-12	-8	-6	0	6	8	12
$q[i]$	0	0	2	1	1	3	2
$Q[i]$	0	0	2	3	4	7	9

Let us denote  $-q$  and  $-Q$  the reversed versions of the  $q$  and the  $Q$  lists. We thus obtain the following result.

$W[i]$	-12	-8	-6	0	6	8	12
$-q[i]$	2	3	1	1	2	0	0
$-Q[i]$	2	5	6	7	9	9	9

Now, a pairwise outranking situation will be qualified at stability level  $+2$ , i.e. positively validated with any significance weights that are compatible with the given weights preorder, when for all  $i$ , we observe  $Q[i] \leq -Q[i]$  and there exists one  $i$  such that  $Q[i] < -Q[i]$ . Similarly, a pairwise outranked situation will be qualified at stability level  $-2$ , when for all  $i$ , we observe  $Q[i] \geq -Q[i]$  and there exists one  $i$  such that  $Q[i] > -Q[i]$  (Bisdorff, 2004b).

Let us verify that the outranking situation observed, for instance, between alternatives  $p_2$  and  $p_1$  does indeed verify this *first order distributional dominance* condition.

$W[i]$	-12	-8	-6	0	6	8	12
$Q[i]$	0	0	2	3	4	7	9
$-Q[i]$	2	5	6	7	9	9	9

Notice that outranking situations qualified at stability levels  $\pm 4$  and  $\pm 3$ , evidently verify the stability level  $\pm 2$  test above. The outranking situation between alternatives  $p_7$  and  $p_3$  does not, however, verify this test (see List. 19.6 on page 272).

$W[i]$	-12	-8	-6	0	6	8	12
$q[i]$	0	3	1	0	3	0	2
$Q[i]$	0	3	4	4	7	7	9
$-Q[i]$	2	2	5	5	6	9	9

This time, not all the  $Q[i]$  are *lower or equal* than the corresponding  $-Q[i]$  terms. Hence the outranking situation between  $p_7$  and  $p_3$  is not positively validated with all potential significance weights that are compatible with the given weights pre-order.

Using this stability denotation, we may, hence, define the following *robust* version of a bipolar-valued outranking digraph.

## 19.4 Robust bipolar-valued outranking digraphs

### Definition 19.1 (Robust outranking situation).

- We say that decision alternative  $x$  *robustly outranks* decision alternative  $y$  when:
  - $x$  positively outranks  $y$  at stability level  $+2$  or higher and
  - we may not observe any considerable counter-performance of  $x$  on a discordant criterion.
- Dually, we say that decision alternative  $x$  *does not robustly outrank* decision alternative  $y$  when:
  - $x$  is positively outranked by  $y$  at stability level  $-2$  or lower and
  - we may not observe any considerable better performance of  $x$  on a discordant criterion.
- Otherwise the outranking situation is indeterminate.

The corresponding *robust* outranking digraph can be computed as follows with the RobustOutrankingDigraph class:

### Listing 19.7 Computing a robust outranking digraph

```

1 >>> from outrankingDigraphs import \
2 ...           RobustOutrankingDigraph
3 >>> rg = RobustOutrankingDigraph(pt) # same pt
4 >>> rg
5      ----- Object instance description -----
6      Instance class       : RobustOutrankingDigraph
7      Instance name        : robust_random3ObjectivesPerfTab
8      Actions              : 7
9      Criteria             : 9
10     Size                 : 22

```

```

11 Determinateness (%) : 68.45
12 Valuation domain   : [-1.00;1.00]
13 Attributes        : ['name', 'methodData', 'actions',
14     'order', 'criteria', 'evaluation',
15     'vetos', 'valuationdomain',
16     'cardinalRelation', 'ordinalRelation',
17     'equisignificantRelation', 'unanimousRelation',
18     'relation', 'gamma', 'notGamma']
19 >>> rg.showRelationTable()
20 * ---- Relation Table -----
21 r/(stab) | 'p1'  'p2'  'p3'  'p4'  'p5'  'p6'  'p7'
22 -----|-----
23 'p1'    | +1.00 -0.42 +0.00 -0.69 +0.39 +0.11 +0.00
24      | (+4)  (-2)  (+0)  (-3)  (+2)  (+2)  (-1)
25 'p2'    | +0.58 +1.00 +0.83 +0.00 +0.58 +0.58 +0.58
26      | (+2)  (+4)  (+3)  (+2)  (+2)  (+2)  (+2)
27 'p3'    | +0.25 -0.33 +1.00 +0.00 +0.50 +1.00 +0.00
28      | (+2)  (-2)  (+4)  (+0)  (+2)  (+2)  (+1)
29 'p4'    | +0.78 +0.00 +0.61 +1.00 +1.00 +1.00 +0.67
30      | (+3)  (-1)  (+3)  (+4)  (+4)  (+4)  (+2)
31 'p5'    | -0.11 -0.50 -0.25 -0.89 +1.00 +0.11 -0.14
32      | (-2)  (-2)  (-2)  (-3)  (+4)  (+2)  (-2)
33 'p6'    | +0.22 -0.42 +0.00 -1.00 +0.17 +1.00 -0.11
34      | (+2)  (-2)  (+1)  (-2)  (+2)  (+4)  (-2)
35 'p7'    | +0.22 -0.50 +0.00 +0.00 +0.78 +0.42 +1.00
36      | (+2)  (-2)  (+1)  (-1)  (+3)  (+2)  (+4)

```

All outranking situations, qualified at stability level  $\pm 1$ , are now put to an *indeterminate* status (see List. 19.7 Lines 23-24 Column p7). In the example here, three positive outrankings get dropped: between p3 and p7, between p7 and p3, and between p6 and p3, where the last situation is actually already doubtful because of a veto situation (see List. 19.7 on the preceding page Lines 22-35). Three negative outrankings get dropped as well: between p1 and p7, between p4 and p2, and between p7 and p4 (see Lines 22-35).

Notice by the way that outranking or outranked situations, although qualified at level  $\pm 2$  or  $\pm 3$  may nevertheless become doubtful because of considerable performance differences. We observe such a doubtful situation when comparing, for instance, alternatives p2 and p4 (see List. 19.7 on the facing page Lines 24-25).

**Listing 19.8** Comparing alternatives p2 and p4

```

1 >>> rg.showPairwiseComparison('p2','p4')
2 *----- pairwise comparison -----
3 Comparing actions : (p2, p4)
4 crit. wght. g(x) g(y) diff    | ind pref r() | v  veto
5 -----
6 ec1  8.00  89.77  85.19 +4.58  | 5.00 10.00 +8.00 |
7 ec4  8.00  86.00  72.26 +13.74 | 5.00 10.00 +8.00 |
8 ec8  8.00  89.43  44.62 +44.81 | 5.00 10.00 +8.00 |
9 en3  6.00  20.79  80.81 -60.02 | 5.00 10.00 -6.00 | 60.00 -1.00
10 en5  6.00  23.83  49.69 -25.86 | 5.00 10.00 -6.00 |
11 en6  6.00  18.66  66.21 -47.55 | 5.00 10.00 -6.00 |
12 en9  6.00  26.65  50.92 -24.27 | 5.00 10.00 -6.00 |
13 so2  12.00 89.12  49.05 +40.07 | 5.00 10.00 +12.00 |
14 so7  12.00 84.73  55.57 +29.16 | 5.00 10.00 +12.00 |
15 Valuation in range: -72.00 to +72.00;

```

```
16 global concordance: +24.00
```

Despite being robust, the apparent positive outranking situation between alternatives  $p_2$  and  $p_4$  becomes doubtful because of a considerable counter-performance ( $-60.02$ ) of  $p_2$  observed on criterion  $en3$ , a negative difference which exceeds slightly the assumed veto discrimination threshold  $v = 60.00$  (see List. 19.8 on the previous page Line 9).

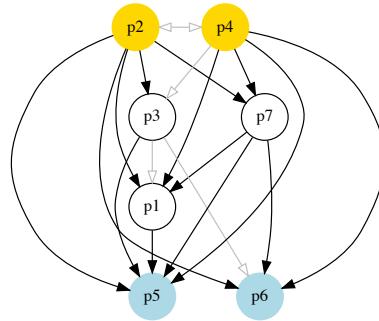
One may finally compare in Figure 19.1 the *standard* and the *robust* version of the corresponding strict outranking digraphs, both oriented by their respective identical initial and terminal prekernels.

```

1 >>> rg.showPreKernels()
2     *--- Computing preKernels ---*
3     Dominant preKernels :
4     ['p2', 'p4']
5     independence : 0.00
6     dominance    : 0.667
7     absorbency   : -0.50
8     covering      : 1.000
9     Absorbent preKernels :
10    ['p5']
11    independence : 1.0
12    dominance    : -0.889
13    absorbency   : 0.167
14    covered       : 1.000
15    ['p6']
16    independence : 1.0
17    dominance    : -1.0
18    absorbency   : 0.111
19    covered       : 1.000

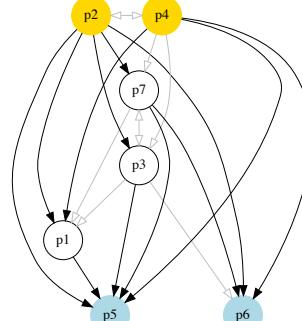
```

Standard strict outranking digraph



*Digraph3 (graphviz), R. Biedorff, 2020*

Robust strict outranking digraph



*Digraph3 (graphviz), R. Biedorff, 2020*

**Fig. 19.1** Standard versus robust strict outranking digraphs oriented by their initial and terminal prekernels

The robust version (right in Fig. 19.1 on the preceding page) drops two strict outranking situations: between p<sub>4</sub> and p<sub>7</sub> and between p<sub>7</sub> and p<sub>1</sub>. The remaining 14 strict outranking (resp. outranked) situations are now all verified at a stability level of +2 and more (resp. -2 and less). They remain valid, hence, with all potential significance weights that are compatible with the given significance weights preordering (see List. 19.2 on page 268).

To appreciate the apparent partial ordering of both the standard and the robust strict outranking digraphs shown in Figure 19.1 on the preceding page, let us have in Figure 19.2 a final heatmap view on the underlying performance tableau ordered by the NETFLOWS ranking rule actually applied to the robust version of the outranking digraph (see the `outrankingModel='this'` flag in Line 4 below).

**Listing 19.9** Computing a robust performance heatmap view

```

1 >>> rg.showHTMLPerformanceHeatmap(\n2 ...           Correlations=True,\n3 ...           colorLevels=5,\n4 ...           outrankingModel='this',\n5 ...           rankingRule='NetFlows')
```

### Heatmap of Performance Tableau 'robust\_random3ObjectivesPerfTab'

criteria	ec4	so2	ec1	ec8	en9	en3	so7	en6	en5
weights	+8.00	+12.00	+8.00	+8.00	+6.00	+6.00	+12.00	+6.00	+6.00
tau <sup>(*)</sup>	+0.55	+0.52	+0.45	+0.38	+0.31	+0.29	+0.24	+0.05	+0.05
p4	72.26	49.05	85.19	44.62	50.92	80.81	55.57	66.21	49.69
p2	86.00	89.12	89.77	89.43	26.65	20.79	84.73	18.66	23.83
p3	68.70	58.85	80.19	91.94	14.85	46.78	15.49	1.65	27.25
p7	36.31	68.06	15.33	38.31	13.10	30.70	58.45	69.71	35.52
p1	22.65	22.43	38.11	77.02	44.37	58.16	28.41	11.41	31.40
p6	75.76	48.59	21.06	29.63	32.96	12.54	26.40	36.04	43.09
p5	8.96	12.36	46.75	35.91	9.83	31.05	44.92	31.22	29.52

Color legend:

quantile	20.00%	40.00%	60.00%	80.00%	100.00%
----------	--------	--------	--------	--------	---------

(\*) tau: Ordinal (Kendall) correlation between marginal criterion and global ranking relation

Outranking model: **this**, Ranking rule: **NetFlows**

Ordinal (Kendall) correlation between global ranking and global outranking relation: **+0.960**

Mean marginal correlation (a) : **+0.338**

Standard marginal correlation deviation (b) : **+0.168**

Ranking fairness (a) - (b) : **+0.170**

**Fig. 19.2** Robust heatmap of the random 3 objectives performance tableau ordered by the NETFLOWS ranking rule

As the initial prekernel {p<sub>2</sub>, p<sub>4</sub>} is in the robust outranking digraph validated at least at stability level  $\pm 2$ , recommending alternatives p<sub>4</sub>, as well as p<sub>2</sub>, as potential best choices, appears robustly justified. Alternative p<sub>4</sub> represents indeed an overall *best compromise choice* between all decision objectives, whereas alternative p<sub>2</sub>

gives an unanimous best choice with respect to two out of three decision objectives. Up to the decision maker to make his final choice.

## 19.5 Characterising unopposed multiobjective outranking situations

When facing a performance tableau involving multiple decision objectives, the robustness level  $\pm 3$  may lead to distinguishing what we call *unopposed* outranking situations, like the one shown in the previous section between alternative  $p_4$  and  $p_1$  ( $r(p_4 \succ p_1) = +0.78$ , see List. 19.4 on page 269 Line 11), namely preferential situations that are more or less validated or invalidated by all the decision objectives.

**Definition 19.2 (Unopposed outranking situation).**

- We say that decision alternative  $x$  is *unopposedly outranked* decision alternative  $y$  when  $x$  positively outranks  $y$  on one or more decision objectives without  $x$  being positively outranked by  $y$  on any decision objective.
- Dually, we say that decision alternative  $x$  is *unopposedly outranked by* decision alternative  $y$  when  $x$  is positively outranked by  $y$  on one or more decision objectives without  $x$  positively outranking  $y$  on any decision objective.

Let us reconsider, for instance, the performance tableau  $pt$  with three decision objectives already seen in Listing 19.1 on page 267:

```

1 >>> pt.showObjectives()
2      ----- show objectives -----
3      Eco: Economical aspect
4          ec1 criterion of objective Eco 8
5          ec4 criterion of objective Eco 8
6          ec8 criterion of objective Eco 8
7          Total weight: 24.00 (3 criteria)
8      Soc: Societal aspect
9          so2 criterion of objective Soc 12
10         so7 criterion of objective Soc 12
11         Total weight: 24.00 (2 criteria)
12      Env: Environmental aspect
13         en3 criterion of objective Env 6
14         en5 criterion of objective Env 6
15         en6 criterion of objective Env 6
16         en9 criterion of objective Env 6
17         Total weight: 24.00 (4 criteria)
```

We notice in this example three decision objectives of equal importance (see Lines 3,13,17). What will be the outranking situations that are positively (resp. negatively) validated for each one of the decision objectives taken individually ?

Such unopposed multiobjective outranking situations may be obtained by operating an epistemic *average fusion* (see the `symmetricAverage()` method) of the marginal outranking digraphs restricted to the coalition of criteria supporting each one of the decision objectives (see List. 19.10 on the next page).

**Listing 19.10** Computing unopposed outranking situations

```

1 >>> from outrankingDigraphs import\                                BipolarOutrankingDigraph
2 ...                                                              
3 >>> geco = BipolarOutrankingDigraph(pt,\                         objectivesSubset=['Eco'])
4 ...                                                              
5 >>> gsoc = BipolarOutrankingDigraph(pt,\                         objectivesSubset=['Soc'])
6 ...                                                              
7 >>> genv = BipolarOutrankingDigraph(pt,\                         objectivesSubset=['Env'])
8 ...                                                              
9 >>> from digraphs import FusionLDigraph
10 >>> objectiveWeights = \
11 ...                           [pt.objectives[obj]['weight']\ \
12 ...                               for obj in t.objectives]
13 >>> uopg = FusionLDigraph([geco,gsoc,genv],\ \
14 ...                               operator='o-average',\ \
15 ...                               weights=objectiveWeights)
16 >>> uopg.showRelationTable(ReflexiveTerms=False)
17 * ---- Relation Table ----
18   r | 'p1'   'p2'   'p3'   'p4'   'p5'   'p6'   'p7'
19 -----
20 'p1' |     - +0.00 +0.00 -0.69 +0.39 +0.11 +0.00
21 'p2' | +0.00     - +0.83 +0.00 +0.00 +0.00 +0.00
22 'p3' | +0.00 -0.33     - +0.00 +0.50 +0.00 +0.00
23 'p4' | +0.78 +0.00 +0.61     - +1.00 +1.00 +0.67
24 'p5' | -0.11 +0.00 +0.00 -0.89     - +0.11 +0.00
25 'p6' | +0.00 +0.00 +0.00 -0.44 +0.17     - +0.00
26 'p7' | +0.00 +0.00 +0.00 +0.00 +0.78 +0.42     -
27 Valuation domain: [-1.0; 1.0]

```

Positive (resp. negative)  $r(x \succsim y)$  characteristic values, like  $r(p1 \succsim p5) = +0.39$  (see List. 19.10 Line 20), show hence only outranking situations being validated (resp. invalidated) by one or more decision objectives without being invalidated (resp. validated) by any other decision objective.

For easily computing this kind of *unopposed multiobjective* outranking digraphs, the `outrankingDigraphs` module conveniently provides a corresponding `UnOpposedBipolarOutrankingDigraph` constructor .

**Listing 19.11** Computing unopposed outranking digraphs

```

16         'criteria', 'methodData',
17         'evaluation', 'order', 'runTimes',
18         'relation', ...
19         'gamma', 'notGamma']
20 >>> uopg.computeOppositeness(InPercents=True)
21 {'standardSize': 23, 'unopposedSize': 13,
22  'oppositeness': 43.47826086956522}

```

The resulting *unopposed* outranking digraph keeps in fact 13 (see [List.19.11 on the previous page](#) Lines 20-22) out of the 23 positively validated *standard* outranking situations, leading to a degree of *oppositeness* –preferential disagreement between decision objectives– of  $(1.0 - 13/23) = 0.4348$ .

We can now, for instance, verify the unopposed status of the outranking situation observed between alternatives p1 and p5.

**Listing 19.12** Example of unopposed multiobjective outranking situation

```

1 >>> uopg.showPairwiseComparison('p1','p5')
2 *----- pairwise comparison -----
3 Comparing actions : ('p1', 'p5')
4 crit. wght. g(x) g(y) diff | ind pref r()
5 ec1 8.00 38.11 46.75 -8.64 | 5.00 10.00 +0.00
6 ec4 8.00 22.65 8.96 +13.69 | 5.00 10.00 +8.00
7 ec8 8.00 77.02 35.91 +41.11 | 5.00 10.00 +8.00
8 en3 6.00 58.16 31.05 +27.11 | 5.00 10.00 +6.00
9 en5 6.00 31.40 29.52 +1.88 | 5.00 10.00 +6.00
10 en6 6.00 11.41 31.22 -19.81 | 5.00 10.00 -6.00
11 en9 6.00 44.37 9.83 +34.54 | 5.00 10.00 +6.00
12 so2 12.00 22.43 12.36 +10.07 | 5.00 10.00 +12.00
13 so7 12.00 28.41 44.92 -16.51 | 5.00 10.00 -12.00
14 Valuation in range: -72.00 to +72.00; -----
15                                     global concordance: +28.00

```

In Listing 19.12 we see that alternative p1 does indeed positively outrank alternative p5 from the economic perspective ( $r(p1 \succsim_{Eco} p5) = +16/24$ ) as well as from the environmental perspective ( $r(p1 \succsim_{Env} p5) = +12/24$ ). Whereas, from the societal perspective, both alternatives appear incomparable ( $r(p1 \succsim_{Soc} p5) = 0/24$ ).

When proportionally equal criteria significance weights per objective are given, these outranking situations appear hence *stable* with respect to all possible importance weights we could allocate to the decision objectives.

This gives way for computing multiobjective *Pareto efficient* choice recommendations.

## 19.6 Computing Pareto efficient multiobjective choices

Indeed, best choice recommendations, computed from an *unopposed multiobjective* outranking digraph, deliver the case given *Pareto efficient* choices.

**Listing 19.13** Pareto efficient multiobjective choice

```

1 >>> uopg.showBestChoiceRecommendation()

```

```

2 Best choice recommendation(s) (BCR)
3 (in decreasing order of determinateness)
4 Credibility domain: [-1.00,1.00]
5 === >> potential first choice(s)
6 choice : ['p2', 'p4', 'p7']
7 independence : 0.00
8 dominance : 0.33
9 absorbency : 0.00
10 covering (%) : 33.33
11 determinateness (%) : 50.00
12 === >> potential last choice(s)
13 choice : ['p3', 'p5', 'p6', 'p7']
14 independence : 0.00
15 dominance : -0.61
16 absorbency : 0.11
17 covered (%) : 33.33
18 determinateness (%) : 50.00

```

Our previous *robust* best choice recommendation (p2 and p4, see Fig. 19.1 on page 276) remains, in this example here, *stable*. We recover indeed the best choice recommendation [p2, p4, p7] (see List. 19.12 on the facing page Line 6). Yet, notice that decision alternative p7 appears to be at the same time a potential *first* as well as a potential *last* choice recommendation (see Line 13), a consequence of p7 being completely *incomparable* to the other decision alternatives when restricting the comparability to only unopposed strict outranking situations.

This kind of Pareto efficient result is shown in Figure 19.3 on the next page.

```

1 >>> (~(-uopg)).exportGraphViz(fileName = 'unopDigraph', \
2 ...                               firstChoice = ['p2','p4'], \
3 ...                               lastChoice = ['p3','p5','p6'])
4 ----- exporting a dot file for GraphViz tools -----
5 Exporting to unopDigraph.dot
6 dot -Grankdir=BT -Tpng unopDigraph.dot -o unopDigraph.png

```

In order to make now an eventual best unique choice, a decision maker will necessarily have to weigh, in a second stage of the decision aiding process, the relative importance of the individual decision objectives.

In a *social choice* context, where decision objectives would match different political parties, Pareto efficient best choice recommendations represent in fact *multipartisan* social choices that may judiciously temper plurality tyranny effects. This idea is developed in the following Chapter 20.

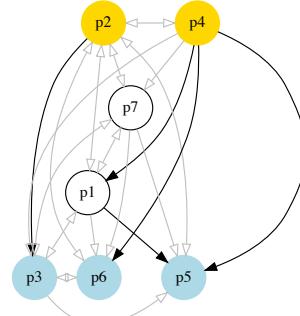
For concluding, let us mention that it is precisely again our bipolar-valued logical characteristic framework that provides us here with a first order distributional dominance test for effectively qualifying the stability level  $\pm 2$  robustness of an outranking digraph when facing performance tableaux with criteria of only ordinal-valued significance. A real-world application of our stability analysis with such a kind of performance tableau may be consulted in (Bisdorff, 2015).

Standard strict outranking digraph



Digraph3 (graphviz), R. Bisдорff, 2020

Unopposed strict outranking digraph



Digraph3 (graphviz), R. Bisдорff, 2020

**Fig. 19.3** Standard versus *unopposed* strict outranking digraphs oriented by first and last choice recommendations

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## Chapter 20

# Tempering plurality tyranny effects in social choice

*The choice of a voting procedure shapes the democracy in which we live.*

– Baujard, Gavrel, Iggersheim, Laslier, and Lebon (2013).

**Abstract** In a *social choice* context, where decision objectives would match different political parties, Pareto efficient choice recommendations represent in fact *multipartisan* social choices that may judiciously deliver the primary selection in a two stage election system. Our bipolar-valued outranking model is based on approvals-disapprovals of “*at least as well evaluated as*” statements. A similar approach is put into practice with approval-disapproval voting systems. When converting such approval-disapproval voting ballots into corresponding performance records, we obtain a  $(-1, 0, 1)$ -valued evaluative voting system. We eventually show that in such a approval-disapproval voting system, the winner tends to be among the more or less multipartisan candidates.

## Introduction

From the seminal work by *Bernard Roy* on, tempering plurality tyranny effects is achieved in the outranking approach by taking into account considerable negative performance differences that render doubtful otherwise positive “*at least as well evaluated as*” situations (Benayoun, Roy, and Sussmann, 1966). In the social choice context of general elections, such a polarisation is not feasible as the genuine unnominal voting ballots do not contain a rich enough preferential information. Yet, when matching decision objectives with political parties, the Pareto efficient outranking situations seen in Section 19.6, correspond to multipartisan “*at least as well approved as*” situations when pairwisely comparing the eligible candidates. A best choice recommendation based of the corresponding multipartisan strict “*better approved as*” situations may, the case given, deliver a convincing primary selection in a two-stage election.

## 20.1 Two-stage elections with multipartisan primary selection

To compute multipartisan social choices we need to, first, convert a given linear voting profile with pre-election polls into a corresponding performance tableau. We shall illustrate this point with a voting profile we discussed already in Chapter 7.

**Listing 20.1** Example of a 3 parties voting profile

```

1 >>> from votingProfiles import RandomLinearVotingProfile
2 >>> lvp = RandomLinearVotingProfile(numberOfCandidates=15,
3 ...                               numberOfVoters=1000,
4 ...                               WithPolls=True,
5 ...                               partyRepartition=0.5,
6 ...                               other=0.1,
7 ...                               seed=0.9189670954954139)
8 >>> lvp
9      *----- VotingProfile instance description -----*
10     Instance class    : RandomLinearVotingProfile
11     Instance name     : randLinearProfile
12     Candidates       : 15
13     Voters           : 1000
14     Attributes        : ['name', 'seed', 'candidates',
15                           'voters', 'WithPolls', 'RandomWeights',
16                           'sumWeights', 'poll1', 'poll2',
17                           'other', 'partyRepartition',
18                           'linearBallot', 'ballot']
19 >>> lvp.showRandomPolls()
20 Random repartition of voters
21 Party_1 supporters : 460 (46.0%)
22 Party_2 supporters : 436 (43.6%)
23 Other voters       : 104 (10.4%)
24 *----- random polls -----
25 Party-1(46.0%) | Party-(43.6%) | expected
26 -----
27   c06 : 19.91% | c11 : 22.94% | c06 : 15.00%
28   c07 : 14.27% | c08 : 15.65% | c11 : 13.08%
29   c03 : 10.02% | c04 : 15.07% | c08 : 09.01%
30   c13 : 08.39% | c06 : 13.40% | c07 : 08.79%
31   c15 : 08.39% | c03 : 06.49% | c03 : 07.44%
32   c11 : 06.70% | c09 : 05.63% | c04 : 07.11%
33   c01 : 06.17% | c07 : 05.10% | c01 : 05.06%
34   c12 : 04.81% | c01 : 05.09% | c13 : 05.04%
35   c08 : 04.75% | c12 : 03.43% | c15 : 04.23%
36   c10 : 04.66% | c13 : 02.71% | c12 : 03.71%
37   c14 : 04.42% | c14 : 02.70% | c14 : 03.21%
38   c05 : 04.01% | c15 : 00.86% | c09 : 03.10%
39   c09 : 01.40% | c10 : 00.44% | c10 : 02.34%
40   c04 : 01.18% | c05 : 00.29% | c05 : 01.97%
41   c02 : 00.90% | c02 : 00.21% | c02 : 00.51%
```

In this example (see List. 20.1 Lines 19-41), we observe 460 Party-1 supporters (46%), 436 Party-2 supporters (43.6%) and 104 other voters (10.4%). Favorite candidates of Party-1 supporters, with more than 10%, are c06 (19.91%), c07

(14.27%) and c03 (10.02%). Whereas for Party-2 supporters, favorite candidates are c11 (22.94%), followed by c08 (15.65%), c04 (15.07%) and c06 (13.4%).

We can convert this linear voting profile into a `PerformanceTableau` object where each political party matches a decision objective.

**Listing 20.2** Converting a voting profile into a performance tableau

```

1 >>> lvp.save2PerfTab('votingPerfTab')
2 >>> from perfTabs import PerformanceTableau
3 >>> vpt = PerformanceTableau('votingPerfTab')
4 >>> vpt
5     *----- PerformanceTableau instance description -----
6     Instance class      : PerformanceTableau
7     Instance name       : votingPerfTab
8     Actions             : 15
9     Objectives          : 3
10    Criteria            : 1000
11    Attributes          : ['name', 'actions', 'objectives',
12                           'criteria', 'weightPreorder', 'evaluation']
13 >>> vpt.objectives
14 OrderedDict([
15     ('party0', {'name': 'other', 'weight': Decimal('104'),
16      'criteria': ['v0003', 'v0008', 'v0011', ...]}),
17     ('party1', {'name': 'party 1', 'weight': Decimal('460'),
18      'criteria': ['v0002', 'v0006', 'v0007', ...]}),
19     ('party2', {'name': 'party 2', 'weight': Decimal('436'),
20      'criteria': ['v0001', 'v0004', 'v0005', ...]})]
21 )

```

In Listing 20.2 the linear voting profile `lvp` is first stored in `PerformanceTableau` format (see Line 1). In Line 3, the `PerformanceTableau` class reloads this stored performance tableau data. The three parties of the linear voting profile represent three decision objectives and the 1000 voters are distributed as 1000 performance criteria according to the party they support.

In order to operate now a *primary multipartisan selection* of potential election winners, we compute the corresponding unopposed multiobjective outranking digraph (see Sec. 19.5).

**Listing 20.3** Computing unopposed multiobjective outranking situations

```

1 >>> from outrankingDigraphs import \
2 ...      UnOpposedBipolarOutrankingDigraph
3 >>> uog = UnOpposedBipolarOutrankingDigraph(vpt)
4 >>> uog
5     *----- Object instance description -----
6     Instance class      : UnOpposedBipolarOutrankingDigraph
7     Instance name       : unopposed_outrankings
8     Actions             : 15
9     Criteria            : 1000
10    Size                : 34
11    Oppositeness (%)   : 67.31
12    Determinateness (%) : 57.61
13    Valuation domain    : [-1.00;1.00]
14    Attributes          : ['name', 'actions', 'valuationdomain',

```

```

15             'objectives', 'criteria',
16             'methodData',
17             'evaluation', 'order', 'runTimes',
18             'relation',
             'marginalRelationsRelations',
             'gamma', 'notGamma']

```

From the potential 105 pairwise outranking situations, we keep 34 positively validated outranking situations, leading to a degree of *oppositeness* between political parties of 67.31% (see List. 20.3 Line 11).

The corresponding bipolar-valued relation table can be shown by orienting the list of candidates with the help of the initial and terminal prekernels.

**Listing 20.4** Computing unopposed multiobjective outranking situations

```

1 >>> uog.showPreKernels()
2     *--- Computing preKernels ---*
3     Dominant preKernels :
4     ['c11', 'c06', 'c13', 'c15']
5         independence : 0.0
6         dominance : 0.18
7         absorbency : -0.66
8         covering : 0.43
9     Absorbent preKernels :
10    ['c02', 'c04', 'c14', 'c03']
11    independence : 0.0
12    dominance : 0.0
13    absorbency : 0.37
14    covered : 0.46
15 >>> orientedCandidatesList = ['c06','c11','c13','c15',\
16 ...           'c01','c05','c07','c08','c09','c10','c12',\
17 ...           'c02','c03','c04','c14']
18 >>> uog.showHTMLRelationTable(\n19 ...     actionsList=orientedCandidatesList,\n20 ...     tableTitle='Unopposed three-partisan outrankings')

```

In Figure 20.1 on the facing page, we may notice that the dominating outranking prekernel  $\{c_06, c_{11}, c_{13}, c_{15}\}$  gathers in fact a multipartisan selection of potential election winners. It is worthwhile noticing that the majority margins obtained from a linear voting profile do verify the zero-sum rule:  $(r(x \succ y) + r(y \succ x)) = 0.0$ . To each positive outranking situation corresponds an equivalent negative converse situation and the resulting outranking and strict outranking digraphs are the same.

When restricting now, in a secondary election stage, the set of eligible candidates to this dominating prekernel, we may compute the actual best social choice.

**Listing 20.5** Recommending the secondary election winner

```

1 >>> from outrankingDigraphs import BipolarOutrankingDigraph
2 >>> g2 = BipolarOutrankingDigraph(vpt,\n3 ...           actionsSubset=['c06','c11','c13','c15'])
4 >>> g2.showRelationTable(ReflexiveTerms=False)
5     * ---- Relation Table ----
6     r      | 'c06'  'c11'  'c13'  'c15'

```

## Multipartisan outranking situations

<b>r(x S y)</b>	<b>a06</b>	<b>a11</b>	<b>a13</b>	<b>a15</b>	<b>a01</b>	<b>a05</b>	<b>a07</b>	<b>a08</b>	<b>a09</b>	<b>a10</b>	<b>a12</b>	<b>a02</b>	<b>a03</b>	<b>a04</b>	<b>a14</b>
<b>a06</b>	-	0.00	0.00	0.00	0.44	0.00	0.25	0.00	0.00	0.00	0.56	0.86	0.29	0.00	0.57
<b>a11</b>	0.00	-	0.00	0.00	0.00	0.55	0.00	0.18	0.59	0.51	0.39	0.80	0.00	0.42	0.47
<b>a13</b>	0.00	0.00	-	0.00	0.00	0.52	-0.27	0.00	0.00	0.00	0.77	0.00	0.00	0.16	
<b>a15</b>	0.00	0.00	0.00	-	0.00	0.39	0.00	0.00	0.00	0.00	0.66	0.00	0.00	0.00	
<b>a01</b>	-0.44	0.00	0.00	0.00	-	0.00	0.00	0.00	0.00	0.00	0.77	0.00	0.00	0.20	
<b>a05</b>	0.00	-0.55	-0.52	-0.39	0.00	-	0.00	-0.47	0.00	-0.12	0.00	0.37	0.00	0.00	0.00
<b>a07</b>	-0.25	0.00	0.27	0.00	0.00	0.00	-	0.00	0.00	0.00	0.30	0.83	0.00	0.00	0.38
<b>a08</b>	0.00	-0.18	0.00	0.00	0.00	0.47	0.00	-	0.00	0.00	0.77	0.00	0.29	0.00	
<b>a09</b>	0.00	-0.59	0.00	0.00	0.00	0.00	0.00	0.00	-	0.00	0.00	0.55	0.00	0.00	0.00
<b>a10</b>	0.00	-0.51	0.00	0.00	0.00	0.12	0.00	0.00	0.00	-	0.00	0.50	0.00	0.00	0.00
<b>a12</b>	-0.56	-0.39	0.00	0.00	0.00	0.00	-0.30	0.00	0.00	0.00	-	0.72	0.00	0.00	0.10
<b>a02</b>	-0.86	-0.80	-0.77	-0.66	-0.77	-0.37	-0.83	-0.77	-0.55	-0.50	-0.72	-	0.00	0.00	0.00
<b>a03</b>	-0.29	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	-	0.00	0.00	
<b>a04</b>	0.00	-0.42	0.00	0.00	0.00	0.00	0.00	-0.29	0.00	0.00	0.00	0.00	0.00	-	0.00
<b>a14</b>	-0.57	-0.47	-0.16	0.00	-0.20	0.00	-0.38	0.00	0.00	0.00	-0.10	0.00	0.00	0.00	-

Valuation domain: [-1.00; +1.00]

**Fig. 20.1** Relation table of multipartisan outranking digraph

```

7   ----- | -----
8   'c06' |   -    +0.10  +0.48  +0.52
9   'c11' | -0.10   -    +0.27  +0.29
10  'c13' | -0.48  -0.27   -    +0.19
11  'c15' | -0.52  -0.29  -0.19   -
12  Valuation domain: [-1.0; 1.0]
13 >>> g2.computeCondorcetWinners()
14  ['c06']
15 >>> g2.computeCopelandRanking()
16  ['c06', 'c11', 'c13', 'c15']

```

Candidate  $c_{06}$  appears clearly to be the winner of this election. Notice by the way that the restricted pairwise outranking relation, shown in Listing 20.5 Lines 8–11, models a linear ordering of the preselected candidates.

We can eventually check the quality of this best choice by noticing that Candidate  $c_{06}$  represents indeed the *simple majority*, the *instant runoff*, the BORDA, as well as the CONDORCET winner of the initially given linear voting profile  $lvp$ .

```

1 >>> lvp.computeSimpleMajorityWinner()
2  ['c06']
3 >>> lvp.computeInstantRunoffWinner()
4  ['c06']
5 >>> lvp.computeBordaWinners()
6  ['c06']
7 >>> from votingProfiles import MajorityMarginsDigraph
8 >>> cd = MajorityMarginsDigraph(lvp)
9 >>> cd.computeCondorcetWinners()
10 ['c06']

```

In our example voting profile here, the multipartisan primary selection stage appears quite effective in reducing the number of eligible candidates to four out of a set of 15 candidates without by the way rejecting the actual winning candidate.

However, in a very *divisive* two major parties system, like in the US, where preferences of the supporters of one major party are opposite to the preferences of the supporters of the other major party, the multipartisan outranking digraph will become nearly indeterminate.

In Listing 20.6 below we generate such a divisive kind of linear voting profile with the help of the `DivisivePolitics` flag (see Lines 4 and 14-20). When now converting the voting profile into a performance tableau (Lines 20-21), we can compute the corresponding unopposed outranking digraph.

**Listing 20.6** A divisive two-party example of a random linear voting profile

```

1 >>> from votingProfiles import RandomLinearVotingProfile
2
3 >>> lvp = RandomLinearVotingProfile(\ 
4 ...     numberOfCandidates=7,numberOfVoters=500,\ 
5 ...     WithPolls=True, partyRepartition=0.4,other=0.2,\ 
6 ...     DivisivePolitics=True, seed=1)
7
8     Random repartition of voters
9     Party-1 supporters : 240 (48.00%)
10    Party-2 supporters : 160 (32.00%)
11    Other voters       : 100 (20.00%)
12
13    *----- random polls -----
14    Party_1(48.0%) | Party_2(32.0%) | expected
15    -----
16    c2 : 30.84%   | c1 : 30.84%   | c2 : 15.56%
17    c3 : 23.67%   | c4 : 23.67%   | c3 : 12.91%
18    c7 : 17.29%   | c6 : 17.29%   | c7 : 11.43%
19    c5 : 11.22%   | c5 : 11.22%   | c1 : 11.00%
20    c6 : 09.79%   | c7 : 09.79%   | c6 : 10.23%
21    c4 : 04.83%   | c3 : 04.83%   | c4 : 09.89%
22    c1 : 02.37%   | c2 : 02.37%   | c5 : 08.98%
23
24 >>> lvp.save2PerfTab('divisiveExample')
25 >>> dvp = PerformanceTableau('divisiveExample')
26 >>> from outrankingDigraphs import \
27 ...     UnOpposedBipolarOutrankingDigraph
28 >>> uodg = UnOpposedBipolarOutrankingDigraph(dvp)
29 >>> uodg
30
31    *----- Object instance description -----*
32    Instance class : UnOpposedBipolarOutrankingDigraph
33    Instance name  : divisiveExample
34    Actions        : 7
35    Criteria       : 500
36    Size           : 0
37    Oppositeness (%) : 100.00
38    Determinateness (%) : 50.00
39    Valuation domain : [-1.00;1.00]
```

With an oppositeness degree of 100.0% (see List. 20.6 Lines 32-33), the preferential disagreement between the political parties is complete, and the unopposed

outranking digraph `uodg` becomes completely indeterminate as shown in the relation table below.

```

1 >>> uodg.showRelationTable(ReflexiveTerms=False)
2     * ---- Relation Table ----
3     r | 'c1'  'c2'  'c3'  'c4'  'c5'  'c6'  'c7'
4     -----|-----
5     'c1' |   - +0.00 +0.00 +0.00 +0.00 +0.00 +0.00
6     'c2' | +0.00   - +0.00 +0.00 +0.00 +0.00 +0.00
7     'c3' | +0.00 +0.00   - +0.00 +0.00 +0.00 +0.00
8     'c4' | +0.00 +0.00 +0.00   - +0.00 +0.00 +0.00
9     'c5' | +0.00 +0.00 +0.00 +0.00   - +0.00 +0.00
10    'c6' | +0.00 +0.00 +0.00 +0.00 +0.00   - +0.00
11    'c7' | +0.00 +0.00 +0.00 +0.00 +0.00 +0.00   -
12
12 Valuation domain: [-1.0; 1.0]

```

As a consequence, a multipartisan primary selection, computed with a `showBestChoiceRecommendation()` method, will keep the complete initial set of eligible candidates and, hence, become *ineffective* (see List. 20.7 Line 6).

#### **Listing 20.7** Example of ineffective primary multipartisan selection

```

1 >>> uodg.showBestChoiceRecommendation()
2 Rubis best choice recommendation(s) (BCR)
3   (in decreasing order of determinateness)
4   Credibility domain: [-1.00,1.00]
5 === >> ambiguous choice(s)
6   choice          : ['c1','c2','c3','c4','c5','c6','c7']
7   independence    : 0.00
8   dominance       : 1.00
9   absorbency      : 1.00
10  covered (%)     : 100.00
11  determinateness (%) : 50.00
12  - most credible action(s) = { }

```

With such kind of divisive voting profile, there may indeed not always exist an obvious winner. In Listing 20.8, we see, for instance, that the simple majority winner is `c2` (Line 2), whereas the instant-run-off winner is `c6` (Line 4).

#### **Listing 20.8** Example of non obvious secondary selection

```

1 >>> lvp.computeSimpleMajorityWinner()
2   ['c2']
3 >>> lvp.computeInstantRunoffWinner()
4   ['c6']
5 >>> from votingProfiles import MajorityMarginsDigraph
6 >>> cg = MajorityMarginsDigraph(lvp)
7 >>> cg.showRelationTable(ReflexiveTerms=False)
8     * ---- Relation Table ----
9     r() | 'c1' 'c2' 'c3' 'c4' 'c5' 'c6' 'c7'
10    -----|-----
11    'c1' |   - -68 -90 -46 -68 -88 -84
12    'c2' | +68   - -32 +80 +46 -6 -24
13    'c3' | +90 +32   - +58 +46 +4 +8
14    'c4' | +4 -80 -58   - -16 -68 -72
15    'c5' | +68 -46 -46 +16   - -26 -64

```

```

16      'c6' | +88   +6   -4   +68   +26   -     -2
17      'c7' | +84   +24   -8   +72   +64   +2   -
18 Valuation domain: [-500;+500]
19 >>> cg.computeCondorcetWinners()
20      ['c3']
21 >>> lvp.computeBordaWinners()
22      ['c3','c7']
23 >>> cg.computeCopelandRanking()
24      ['c3', 'c7', 'c6', 'c2', 'c5', 'c4', 'c1']

```

But in our example here, we are lucky. When constructing with the pairwise majority margins digraph (Line 6), a CONDORCET winner, namely  $a_3$ , becomes apparent (Lines 13 and 20), which is also one of the two BORDA winners (Line 22). More interesting even is to notice that the apparent majority margins digraph models in fact a linear ranking  $[a_3, a_7, a_6, a_2, a_5, a_4, a_1]$  of all the eligible candidates, as shown with a COPELAND ranking rule (Line 24).

In Figure 20.2, this linear ranking is shown with a graphviz drawing where all transitive arcs are dropped and the drawing is oriented with the CONDORCET winner  $a_3$  and loser  $a_1$  (Line 3 below).

```

1 >>> cg.closeTransitive(Reverse=True)
2 >>> cg.exportGraphViz('divGraph', \
3 ...           firstChoice=['c3'],lastChoice=['c1'])
4     *----- exporting a dot file for GraphViz tools -----*
5     Exporting to divGraph.dot
6     dot -Grankdir=BT -Tpng divGraph.dot -o divGraph.png

```

**Fig. 20.2** The linear ranking modelled by the majority margins digraph



Digraph3 (graphviz), R. Bisendorff, 2020

## 20.2 Bipolar approval-disapproval voting systems

Traditionally, most of the voting systems in use in the World, do only collect approval votes and abstentions, dismissing potentially strong disapproval opinions, not to be assimilated to voting abstentions. Collecting both explicit approvals (+1) and explicit disapprovals (-1) essentially enriches the expression of voters' preferences.

In the `votingProfiles` module we provide a `BipolarApprovalVotingProfile` class for handling voting results where, for each eligible Candidate  $x$ , the voters are invited to *approve* (+1), *disapprove* (-1), or *ignore* (0) the statement that Candidate  $c$  should win the election (Baujard et al., 2013).

File `bpApVotingProfile.py` contains an example of such a approval-disapproval voting profile concerning 100 voters and 15 eligible candidates<sup>1</sup>. We can inspect its content with the `BipolarApprovalVotingProfile` class:

**Listing 20.9** Bipolar approval voting profiles

```

1 >>> from votingProfiles import \
2 ...                               BipolarApprovalVotingProfile
3 >>> bavp = BipolarApprovalVotingProfile('bpApVotingProfile')
4 >>> bavp
5      *----- VotingProfile instance description -----*
6      Instance class   : BipolarApprovalVotingProfile
7      Instance name    : bpApVotingProfile
8      Candidates       : 15
9      Voters          : 100
10     Attributes       : ['name', 'candidates', 'voters',
11                           'approvalBallot', 'netApprovalScores',
12                           'ballot']
```

Beside the `candidates` and `voters` attributes, we discover in Listing 20.10 the `approvalBallot` attribute which gathers approval-disapproval votes. Its content is the following.

**Listing 20.10** Inspecting an approval-disapproval ballot

```

1 >>> bavp.approvalBallot
2     {'v001':
3         {'c01': Decimal('0'),
4          ...
5         'c04': Decimal('1'),
6          ...
7         'c15': Decimal('0')
8         },
9     'v002':
10        {'c01': Decimal('-1'),
11         'c02': Decimal('0'),
12         ...
13         'c15': Decimal('1')
14         },
15     ...
```

---

<sup>1</sup> The file `bpApVotingProfile.py` may be found in the `examples` directory of the DI-GRAFH3 resources.

```

16     'v100':
17     {'c01': Decimal('0'),
18      'c02': Decimal('1'),
19      ...
20      'c15': Decimal('1')
21    }
22 }
```

Let us denote  $A_v$  the set of candidates approved by voter  $v$ . In the approval-Ballot attribute we hence record in fact the bipolar-valued truth characteristic values  $r(c \in A_v)$  of the statements that Candidate  $c$  is *approved* by voter  $v$ . In Listing 20.10 on the previous page Line 5, we observe for instance that voter v001 positively approves Candidate c04. And, in Line 10, we see that voter v002 negatively approves, i.e. positively disapproves Candidate c01.

The `showApprovalResults()` method and the `showDisapprovalResults()` method show how many approvals, respectively disapprovals, each candidate receives.

```

1 >>> bavp.showApprovalResults()
2     Approval results
3     Candidate: 'c12' obtains 34 votes
4     Candidate: 'c05' obtains 30 votes
5     Candidate: 'c03' obtains 28 votes
6     Candidate: 'c14' obtains 27 votes
7     Candidate: 'c11' obtains 27 votes
8     Candidate: 'c04' obtains 27 votes
9     Candidate: 'c01' obtains 27 votes
10    Candidate: 'c13' obtains 24 votes
11    Candidate: 'c07' obtains 24 votes
12    Candidate: 'c15' obtains 23 votes
13    Candidate: 'c02' obtains 23 votes
14    Candidate: 'c09' obtains 22 votes
15    Candidate: 'c08' obtains 22 votes
16    Candidate: 'c10' obtains 21 votes
17    Candidate: 'c06' obtains 21 votes
18    Total approval votes: 380
19    Approval proportion: 380/1500 = 0.25
20 >>> bavp.showDisapprovalResults()
21     Disapproval results
22     Candidate: 'c12' obtains 16 votes
23     Candidate: 'c03' obtains 22 votes
24     Candidate: 'c09' obtains 23 votes
25     Candidate: 'c04' obtains 24 votes
26     Candidate: 'c06' obtains 24 votes
27     Candidate: 'c13' obtains 24 votes
28     Candidate: 'c11' obtains 25 votes
29     Candidate: 'c02' obtains 26 votes
30     Candidate: 'c07' obtains 26 votes
31     Candidate: 'c08' obtains 26 votes
32     Candidate: 'c05' obtains 27 votes
33     Candidate: 'c10' obtains 27 votes
34     Candidate: 'c14' obtains 27 votes
35     Candidate: 'c15' obtains 27 votes
```

```

36 Candidate: 'c01' obtains 32 votes
37 Total disapproval votes: 376
38 Disapproval proportion: 376/1500 = 0.25

```

In Lines 3 and 22 above, we notice that, of all eligible candidates, it is Candidate c12 who receives the highest number of approval votes (34) and the lowest number of disapproval votes (16). Total number of approval, respectively disapproval, votes approaches more or less a proportion of 25% of the  $100 \times 15 = 1500$  potential approval votes. About 50% of the latter remain hence ignored.

When operating now, for each Candidate  $x$ , the difference between the number of approval and the number of disapproval votes they receive, we obtain per candidate a corresponding *net approval* score; in fact, the bipolar truth characteristic value of the statement “*Candidate  $x$  should win the election*”.

$$r(\text{Candidate } x \text{ should win the election}) = \sum_v (r(x \in A_v)). \quad (20.1)$$

These bipolar characteristic values are stored in the `netApprovalScores` attribute and may be printed out with the `showNetApprovalScores()` method:

```

1 >>> bavp.showNetApprovalScores()
2 Net Approval Scores
3 Candidate: 'c12' obtains 18 net approvals
4 Candidate: 'c03' obtains 6 net approvals
5 Candidate: 'c05' obtains 3 net approvals
6 Candidate: 'c04' obtains 3 net approvals
7 Candidate: 'c11' obtains 2 net approvals
8 Candidate: 'c14' obtains 0 net approvals
9 Candidate: 'c13' obtains 0 net approvals
10 Candidate: 'c09' obtains -1 net approvals
11 Candidate: 'c07' obtains -2 net approvals
12 Candidate: 'c06' obtains -3 net approvals
13 Candidate: 'c02' obtains -3 net approvals
14 Candidate: 'c15' obtains -4 net approvals
15 Candidate: 'c08' obtains -4 net approvals
16 Candidate: 'c01' obtains -5 net approvals
17 Candidate: 'c10' obtains -6 net approvals

```

We observe in Line 3 above that Candidate c12, with a net approval score of  $34 - 16 = 18$ , represents indeed the *best approved* candidate for winning the election. With a net approval score of  $28 - 22 = 6$ , Candidate c03 appears 2nd-best approved. The net approval scores define hence a potentially weak ranking on the set of eligible election candidates, and the winner(s) of the election is(are) determined by the first-ranked candidate(s).

## 20.3 Pairwise comparison of approval-disapproval votes

The approval votes of each voter define now on the set of eligible candidates three ordered categories: his approved (+1), his ignored (0) and his disapproved (-1)

ones. Within each of these three categories we consider the voter's actual preferences as *not communicated*, i.e. as missing data. This gives for each voter a partially determined strict order which we find in the `ballot` attribute.

```

1 >>> bavp.ballot['v001']['c12']
2     {'c02': Decimal('1'), 'c11': Decimal('1'),
3      'c14': Decimal('1'), 'c04': Decimal('0'),
4      'c06': Decimal('1'), 'c05': Decimal('1'),
5      'c12': Decimal('0'), 'c13': Decimal('0'),
6      'c15': Decimal('1'), 'c01': Decimal('1'),
7      'c08': Decimal('1'), 'c07': Decimal('1'),
8      'c09': Decimal('0'), 'c03': Decimal('1'),
9      'c10': Decimal('0')}

```

For voter v001, for instance, the best approved Candidate c12 is strictly preferred to Candidates: c01, c02, c03, c05, c06, c07, c08, c11, c14 and c15. No candidate is preferred to c12 and the comparison with c04, c09, c10 and c13 is not communicated, hence indeterminate. Mind by the way that the reflexive comparison of c12 with itself is, as usual, ignored, i.e. indeterminate. Each voter  $v$  defines thus a partially determined transitive strict preference relation denoted  $\succ_v$  on the eligible candidates.

For each pair of eligible candidates, we aggregate the previous individual voter's preferences into a truth characteristic of the statement: "Candidate  $x$  is *better approved than* Candidate  $y$ ", denoted  $r(x \succ y)$ :

$$r(x \succ y) = \sum_v (r(x \succ_v y)). \quad (20.2)$$

We say that Candidate  $x$  is *better approved than* Candidate  $y$  when  $r(x \succ y) > 0$ , i.e. there is a majority of voters who approve *more* and disapprove *less*  $x$  than  $y$ . Vice-versa, we say that Candidate  $x$  is *not* better approved than Candidate  $y$  when  $r(x \succ y) < 0$ , i.e. there is a majority of voters who disapprove more and approve less  $x$  than  $y$ . This computation is achieved with the `MajorityMarginsDigraph` constructor.

```

1 >>> from votingProfiles import MajorityMarginsDigraph
2 >>> m = MajorityMarginsDigraph(bavp)
3 >>> m
4     *----- Digraph instance description -----*
5     Instance class      : MajorityMarginsDigraph
6     Instance name       : rel_bpApVotingProfile
7     Digraph Order       : 15
8     Digraph Size        : 97
9     Valuation domain   : [-100.00;100.00]
10    Determinateness (%) : 52.55
11    Attributes          : ['name', 'actions',
12                           'criteria', 'ballot',
13                           'valuationdomain', 'relation',
14                           'order', 'gamma', 'notGamma']

```

The resulting digraph `m` contains 97 positively validated relations (see Line 8 above) and for all pairs  $(x,y)$  of eligible candidates,  $r(x \succ y)$  takes value in an val-

uation range from  $-100.00$  (unanimously rejected) to  $+100.00$  (unanimously approved).

These pairwise  $r(x \succ y)$  values can be inspected in a browser view with the `showHTMLRelationTable()` method:

```
1 >>> m.showHTMLRelationTable(relationName='r(x > y)')
```

$r(x > y)$	c01	c02	c03	c04	c05	c06	c07	c08	c09	c10	c11	c12	c13	c14	c15
c01	-	1	-10	-10	-8	-2	-2	-3	-3	2	-5	-19	-4	-2	2
c02	-1	-	-5	-4	-9	-2	5	1	-7	3	-2	-20	-1	-4	-2
c03	10	5	-	1	4	6	5	10	7	11	0	-5	2	6	9
c04	10	4	-1	-	2	5	7	8	-1	4	0	-7	8	2	8
c05	8	9	-4	-2	-	4	6	7	4	2	1	-17	6	0	3
c06	2	2	-6	-5	-4	-	-1	2	-3	5	-4	-13	-1	-1	2
c07	2	-5	-5	-7	-6	1	-	7	-4	2	-5	-17	-2	-2	4
c08	3	-1	-10	-8	-7	-2	-7	-	-2	2	-2	-16	0	0	-1
c09	3	7	-7	1	-4	3	4	2	-	3	0	-18	-4	0	2
c10	-2	-3	-11	-4	-2	-5	-2	-2	-3	-	-6	-15	-4	-4	2
c11	5	2	0	0	-1	4	5	2	0	6	-	-15	4	0	5
c12	19	20	5	7	17	13	17	16	18	15	15	-	12	13	18
c13	4	1	-2	-8	-6	1	2	0	4	4	-4	-12	-	1	4
c14	2	4	-6	-2	0	1	2	0	0	4	0	-13	-1	-	-1
c15	-2	2	-9	-8	-3	-2	-4	1	-2	-2	-5	-18	-4	1	-

Valuation domain: [-100; +100]

**Fig. 20.3** The bipolar-valued pairwise majority margins

In Figure 20.3 it gets apparent that Candidate c12 is a CONDORCET winner, i.e. the candidate who beats all the other candidates and, with the given voting profile gavp, should without doubt win the election. This strongly confirms the first-ranked result obtained with the previous net approval scoring.

Let us eventually compute, with the help of the NETFLOWS ranking rule<sup>2</sup>, a linear ranking of the 15 eligible candidates and compare the result with the net approval scores ranking.

**Listing 20.11** Comparing the net approval and the NETFLOWS rankings

```
1 >>> from linearOrders import NetFlowsOrder
2 >>> nf = NetFlowsOrder(m, Comments=True)
3 >>> print('NetFlows versus Net Approval Ranking')
4 >>> print('Candidate\tNetFlows score\tNet Approval score')
5 >>> for item in nf.netFlows:
```

---

<sup>2</sup> See Sec. 8.3.

```

6     ...     print( '%9s\t %+.3f\t %+.1f' %
7     ...             (item[1], item[0], \
8     ...              bavp.netApprovalScores[item[1]] ) )
9     NetFlows versus Net Approval Ranking
10    Candidate   NetFlows score  Net Approval score
11    'c12'        +410.000      +18.0
12    'c03'        +142.000      +6.0
13    'c04'        +98.000       +3.0
14    'c05'        +54.000       +3.0
15    'c11'        +34.000       +2.0
16    'c09'        -16.000       -1.0
17    'c14'        -20.000       +0.0
18    'c13'        -22.000       +0.0
19    'c06'        -50.000       -3.0
20    'c07'        -74.000       -2.0
21    'c02'        -96.000       -3.0
22    'c08'        -102.000      -4.0
23    'c15'        -110.000      -4.0
24    'c10'        -122.000      -6.0
25    'c01'        -126.000      -5.0

```

In Listing 20.11 on the previous page we may notice that the NETFLOWS rule delivers a ranking that is very similar to the one previously obtained with the corresponding *Net Approval* scores. Only minor inversions do appear, like in the midfield, where Candidate *c09* advances before Candidates *c13* and *c14* (see Line 16), and *c06* and *c07* swap their positions 9 and 10 (see Line 19). The two last-ranked Candidates *c10* and *c01* also swap their positions.

This confirms again the pertinence of the net approval scoring approach for finding the winner in an approval-disapproval voting system. Yet, voting by approving (+1), disapproving (-1) or ignoring (0) eligible candidates, may also be seen as a performance evaluation of the eligible candidates on a {-1,0,1}-valued ordinal scale.

## 20.4 Three-valued evaluative voting systems

Following such an epistemic perspective, we may effectively convert the given `BipolarApprovalVotingProfile` instance into a `PerformanceTableau` instance, so as to get access to a corresponding outranking decision aiding approach.

Mind that, contrary to the majority margins of the “*better approved than*” relation, all voters consider now the approved candidates to be equivalent (+1). Same is true for the disapproved (-1), respectively the ignored (0) candidates. The voter’s marginal preferences model this time a complete preorder with three equivalence classes.

From the saved file `AVPerfTab.py` (see Line 1 below), we may construct an outranking relation on the eligible candidates with our standard `BipolarOutrankingDigraph` class constructor. The semantics of this outranking relation are the following:

- We say that Candidate  $x$  *outranks* Candidate  $y$  when there is a majority of voters who consider  $x$  *at least as well evaluated as*  $y$ ;
- We say that Candidate  $x$  is *outranked by* Candidate  $y$  when there is a majority of voters who consider  $x$  *not at least as well evaluated as*  $y$ ;
- Otherwise, the outranking situation is indeterminate.

**Listing 20.12** Computing the outranking digraph

```

1 >>> bavp.save2PerfTab(fileName='AVPerfTab',valueDigits=0)
2     *--- Saving as performance tableau in AVPerfTab.py ---*
3 >>> from outrankingDigraphs import \
4             BipolarOutrankingDigraph
5 >>> odg = BipolarOutrankingDigraph('AVPerfTab')
6 >>> odg
7     *----- Object instance description -----*
8     Instance class      : BipolarOutrankingDigraph
9     Instance name       : rel_AVPerfTab
10    Actions            : 15
11    Criteria           : 100
12    Size               : 210
13    Determinateness (%) : 69.29
14    Valuation domain   : [-1.00;1.00]
15    Attributes         : ['name', 'actions', 'order',
16                           'criteria', 'evaluation', 'NA',
17                           'valuationdomain', 'relation',
18                           'gamma', 'notGamma', ...]
```

The size ( $210 = 15 \times 14$ ) of the resulting outranking digraph `odg`, shown in Listing 20.12 Line 12 above, reveals that the corresponding “*at least as good evaluated as*” relation models actually a trivial *complete* digraph. All candidates appear to be *equally* at least as well evaluated and the strict “*better evaluated than*” (codual) outranking digraph becomes in fact empty. The converted performance tableau does apparently not contain sufficiently discriminatory performance evaluations for supporting any strict preference situations.

Yet, we may nevertheless try to apply again the NETFLOWS ranking rule to this complete outranking digraph `odg` and print side by side the corresponding NETFLOWS scores and the previous Net Approval scores.

**Listing 20.13** Comparing the NETFLOWS and the Net Approval rankings

```

1 >>> from linearOrders import NetFlowsOrder
2 >>> nf = NetFlowsOrder(odg)
3 >>> print('NetFlows versus Net Approval Ranking')
4 >>> print('Candidate\tNetFlows Score\tNet Approval Score')
5 >>> for item in nf.netFlows:
6     ...     print('%9s\t%+.3f\t%+.0f' %
7     ...           (item[1],item[0],\
8     ...            bavp.netApprovalScores[item[1]]))
9     NetFlows versus Net Approval Ranking
10    Candidate    NetFlows score Net Approval score
11    c12          +4.100        +18
12    c03          +1.420        +6
13    c04          +0.980        +3
```

14	c05	+0.540	+3
15	c11	+0.340	+2
16	c09	-0.160	-1
17	c14	-0.200	0
18	c13	-0.220	0
19	c06	-0.500	-3
20	c07	-0.740	-2
21	c02	-0.960	-3
22	c08	-1.020	-4
23	c15	-1.100	-4
24	c10	-1.220	-6
25	c01	-1.260	-5

Despite its apparent poor strict preference discriminating power, we obtain in Listing 20.13 on the previous page here NETFLOWS scores that are directly proportional (divided by 100) to the scores obtained with the *better approved than* majority margins digraph  $m$  (see List. 20.11 on page 295).

Encouraged by this positive result, we may furthermore compute a RUBIS best choice recommendation (see Chap. 4).

**Listing 20.14** Computing a best social choice recommendation

```

1 >>> odg.showBestChoiceRecommendation()
2 Rubis best choice recommendation(s) (BCR)
3   (in decreasing order of determinateness)
4   Credibility domain: [-1.00,1.00]
5 === >> ambiguous first choice(s)
6   * choice      : ['c01','c02','c03','c04','c05',
7                     'c06','c07','c08','c09','c10',
8                     'c11','c12','c13','c14','c15']
9   independence   : 0.06
10  dominance     : 1.00
11  absorbency    : 1.00
12  covering (%)  : 100.00
13  determinateness (%) : 61.13
14  - most credible action(s) = {
15    'c12': 0.44, 'c03': 0.34, 'c04': 0.30,
16    'c14': 0.28, 'c13': 0.24, 'c06': 0.24,
17    'c11': 0.20, 'c10': 0.20, 'c07': 0.20,
18    'c01': 0.20, 'c08': 0.18, 'c05': 0.18,
19    'c15': 0.14, 'c09': 0.14, 'c02': 0.06, }
20 === >> ambiguous last choice(s)
21   * choice      : ['c01','c02','c03','c04','c05',
22                     'c06','c07','c08','c09','c10',
23                     'c11','c12','c13','c14','c15']
24   independence   : 0.06
25   dominance     : 1.00
26   absorbency    : 1.00
27   covered (%)   : 100.00
28   determinateness (%) : 63.73
29   - most credible action(s) = {
30    'c13': 0.36, 'c06': 0.36, 'c15': 0.34,
31    'c01': 0.34, 'c08': 0.32, 'c07': 0.30,
32    'c02': 0.30, 'c14': 0.28, 'c11': 0.28,
```

```

33     'c09': 0.28, 'c04': 0.26, 'c10': 0.24,
34     'c05': 0.20, 'c03': 0.20, 'c12': 0.06, }
```

The strict outranking digraph ( $\sim (-odg)$ ) being actually *empty*, we obtain a unique *ambiguous* –first as well as last– choice recommendation which trivially retains all fifteen candidates (see List. 20.14 on the preceding page Lines 6 - 8 above). Yet, the bipolar-valued best choice membership characteristic vector reveals that, among all the fifteen potential winners, it is indeed Candidate  $c_{12}$  the most credible one with a 72% majority of voters' support (see Line 15,  $(0.44 + 1.0)/2 = 0.72$ ); followed by Candidate  $c_{03}$  (67%) and Candidate  $c_{04}$  (65%). Similarly, Candidates  $c_{13}$  and  $c_{06}$  represent the most credible losers with a 68% majority voters' support (Line 30).

We observe here empirically that *evaluative* voting systems, using three-valued ordinal performance scales, match closely corresponding approval-disapproval voting systems. The latter systems model, however, more faithfully the very preferential information that is expressed with *approved*, *disapproved* and *ignored* statements. The corresponding evaluation on a three-level scale, being value (numbers) based, cannot express the fact that in approval-disapproval voting system there is no preferential information given concerning the pairwise comparison of all approved, respectively disapproved or ignored candidates.

Let us finally illustrate how approval-disapproval voting systems may favour multipartisan supported candidates. We shall therefore compare *approval-disapproval* versus *uninominal plurality* election results when considering a highly divisive and partisan political context.

## 20.5 Favouring multipartisan candidates

In modern democracy, politics are largely structured by political parties and activists. Let us so consider an approval-disapproval voting profile  $dvp$  where the random voter behaviour is simulated from two pre-electoral polls concerning a political scene with essentially two major highly competing parties, like the one existing in the US.

**Listing 20.15** A random approval-disapproval voting profile in a divisive political context

```

1 >>> dvp = RandomBipolarApprovalVotingProfile(\ \
2 ...                               numberOfCandidates=15,\ \
3 ...                               numberOfVoters=100,\ \
4 ...                               approvalProbability=0.25,\ \
5 ...                               disapprovalProbability=0.25,\ \
6 ...                               WithPolls=True,\ \
7 ...                               partyRepartition=0.5,\ \
8 ...                               other=0.05,\ \
9 ...                               DivisivePolitics=True,\ \
10 ...                              seed=200)
11 >>> dvp.showRandomPolls()
12 Random repartition of voters
```

```

13   Party-1 supporters : 45 (45.00%)
14   Party-2 supporters : 49 (49.00%)
15   Other voters       : 6 (06.00%)
16   *----- random polls -----
17   Party-1(45.0%) | Party-2(49.0%) | expected
18   -----
19   'c05' : 24.10% | 'c07' : 24.10% | 'c07' : 11.87%
20   'c14' : 23.48% | 'c10' : 23.48% | 'c10' : 11.60%
21   'c03' : 15.13% | 'c01' : 15.13% | 'c05' : 10.91%
22   'c12' : 07.55% | 'c04' : 07.55% | 'c14' : 10.67%
23   'c08' : 07.11% | 'c09' : 07.11% | 'c01' : 07.67%
24   'c15' : 04.37% | 'c13' : 04.37% | 'c03' : 07.09%
25   'c11' : 03.99% | 'c02' : 03.99% | 'c04' : 04.55%
26   'c06' : 03.80% | 'c06' : 03.80% | 'c09' : 04.49%
27   'c02' : 02.79% | 'c11' : 02.79% | 'c12' : 04.32%
28   'c13' : 02.63% | 'c15' : 02.63% | 'c08' : 04.30%
29   'c09' : 02.24% | 'c08' : 02.24% | 'c06' : 03.57%
30   'c04' : 01.89% | 'c12' : 01.89% | 'c13' : 03.32%
31   'c01' : 00.57% | 'c03' : 00.57% | 'c15' : 03.25%
32   'c10' : 00.20% | 'c14' : 00.20% | 'c02' : 03.21%
33   'c07' : 00.14% | 'c05' : 00.14% | 'c11' : 03.16%

```

In Listing 20.15 on the preceding page, the divisive political situation is reflected by the fact that Party-1 and Party-2 supporters show strict opposite preferences. The leading candidates of Party-1 (`c05` and `c14`) are last choices for Party-2 supporters and, Candidates `c07` and `c10`, leading candidates for Party-2 supporters, are similarly the least choices for Party-1 supporters.

No clear winner may be guessed from these pre-election polls. As Party-2 shows however slightly more supporters than Party-1, the expected winner in an uninominal plurality or instant-run-off voting system will be Candidate `c07`, i.e., the leading candidate of majority Party-2 (see below).

```

1 >>> dvp.computeSimpleMajorityWinner()
2   ['c07']
3 >>> dvp.computeInstantRunoffWinner()
4   ['c07']

```

Now, in a corresponding approval-disapproval voting system, Party-1 supporters will usually approve their leading candidates and disapprove the leading candidates of Party-2. Vice versa, Party-2 supporters will usually approve their leading candidates and disapprove the leading candidates of Party-1. Let us consult the resulting approval votes per candidate.

```

1 >>> dvp.showApprovalResults()
2   Candidate: 'c07' obtains 30 votes
3   Candidate: 'c10' obtains 28 votes
4   Candidate: 'c05' obtains 28 votes
5   Candidate: 'c01' obtains 28 votes
6   Candidate: 'c03' obtains 26 votes
7   Candidate: 'c02' obtains 26 votes
8   Candidate: 'c12' obtains 25 votes
9   Candidate: 'c14' obtains 24 votes
10  Candidate: 'c13' obtains 24 votes

```

```

11 Candidate: 'c09' obtains 21 votes
12 Candidate: 'c04' obtains 21 votes
13 Candidate: 'c08' obtains 19 votes
14 Candidate: 'c06' obtains 17 votes
15 Candidate: 'c15' obtains 15 votes
16 Candidate: 'c11' obtains 12 votes
17 Total approval votes: 344
18 Approval proportion: 344/1500 = 0.23

```

When considering only the approval votes, we find confirmed above that the leading candidate of Party-2 obtains in this simulation a plurality of approval votes. In uninominal plurality or instant-runoff voting systems, this candidate wins hence the election, quite to the despair of Party-1 supporters. As a foreseeable consequence, this election result will be more or less aggressively contested which leads to a loss of popular trust in democratic elections and institutions.

If we look however on the corresponding disapprovals, we discover that, not surprisingly, the leading candidates of both parties collect by far the highest number of disapproval votes.

```

1 >>> dvp.showDisapprovalResults()
2     Candidate: 'c02' obtains 14 votes
3     Candidate: 'c04' obtains 14 votes
4     Candidate: 'c13' obtains 14 votes
5     Candidate: 'c06' obtains 15 votes
6     Candidate: 'c09' obtains 15 votes
7     Candidate: 'c08' obtains 16 votes
8     Candidate: 'c11' obtains 16 votes
9     Candidate: 'c15' obtains 18 votes
10    Candidate: 'c12' obtains 20 votes
11    Candidate: 'c01' obtains 29 votes
12    Candidate: 'c03' obtains 30 votes
13    Candidate: 'c10' obtains 37 votes
14    Candidate: 'c07' obtains 44 votes
15    Candidate: 'c14' obtains 45 votes
16    Candidate: 'c05' obtains 49 votes
17 Total disapproval votes: 376
18 Disapproval proportion: 376/1500 = 0.25

```

Balancing now approval against disapproval votes will favour the moderate, bipartisan supported, candidates.

```

1 >>> dvp.showNetApprovalScores()
2 Net Approval Scores
3     Candidate: 'c02' obtains 12 net approvals
4     Candidate: 'c13' obtains 10 net approvals
5     Candidate: 'c04' obtains 7 net approvals
6     Candidate: 'c09' obtains 6 net approvals
7     Candidate: 'c12' obtains 5 net approvals
8     Candidate: 'c08' obtains 3 net approvals
9     Candidate: 'c06' obtains 2 net approvals
10    Candidate: 'c01' obtains -1 net approvals
11    Candidate: 'c15' obtains -3 net approvals
12    Candidate: 'c11' obtains -4 net approvals
13    Candidate: 'c03' obtains -4 net approvals

```

```

14 Candidate: 'c10' obtains -9 net approvals
15 Candidate: 'c07' obtains -14 net approvals
16 Candidate: 'c14' obtains -21 net approvals
17 Candidate: 'c05' obtains -21 net approvals

```

Candidate c02, appearing in the pre-electoral polls in the midfield (in position 7 for Party-2 and in position 9 for Party-1 supporters, see List.20.15 on page 299), shows indeed the highest net approval score. Second highest net approval score obtains Candidate c13, in position 6 for Party-2 and in position 10 for Party-1 supporters.

Figure 20.4, showing the NETFLOWS ranked relation table of the “*better approved than*” majority margins digraph, confirms below this net approval scoring result.

```

1 >>> m = MajorityMarginsDigraph(dvp)
2 >>> m.showHTMLRelationTable(\'
3 ...     actionsList=m.computeNetFlowsRanking(),
4 ...     relationName='r(x > y)')

```

r(x > y)	c02	c13	c04	c09	c12	c08	c06	c01	c11	c15	c03	c10	c07	c14	c05
c02	-	6	5	6	9	5	10	12	12	14	11	15	22	22	23
c13	-6	-	2	5	2	5	8	10	14	10	9	13	18	23	20
c04	-5	-2	-	0	2	2	3	7	7	8	11	13	18	21	18
c09	-6	-5	0	-	0	2	5	5	11	9	5	13	16	21	16
c12	-9	-2	-2	0	-	4	6	2	9	6	10	5	10	23	25
c08	-5	-5	-2	-2	-4	-	4	0	8	9	7	5	11	21	20
c06	-10	-8	-3	-5	-6	-4	-	-2	5	5	5	6	13	17	18
c01	-12	-10	-7	-5	-2	0	2	-	1	-1	3	8	11	9	13
c11	-12	-14	-7	-11	-9	-8	-5	-1	-	1	-2	7	14	13	16
c15	-14	-10	-8	-9	-6	-9	-5	1	-1	-	0	3	10	14	14
c03	-11	-9	-11	-5	-10	-7	-5	-3	2	0	-	-3	7	16	16
c10	-15	-13	-13	-13	-5	-5	-6	-8	-7	-3	3	-	3	6	7
c07	-22	-18	-18	-16	-10	-11	-13	-11	-14	-10	-7	-3	-	3	5
c14	-22	-23	-21	-21	-23	-21	-17	-9	-13	-14	-16	-6	-3	-	0
c05	-23	-20	-18	-16	-25	-20	-18	-13	-16	-14	-16	-7	-5	0	-

Valuation domain: [-100; +100]

Fig. 20.4 The pairwise *better approved than* majority margins

Candidate c02 appears indeed *better approved than* any other candidate (CONDORCET winner); and, the leading Candidates of Party-1, c05 and c14, are *less approved than* any other candidates (weak CONDORCET losers).

```

1 >>> m.computeCondorcetWinners()
2 ['c02']

```

```

3 >>> m.computeWeakCondorcetLosers()
4     ['c05','c14']

```

We see this result furthermore confirmed when computing the corresponding first, respectively last choice recommendation.

```

1 >>> m.showBestChoiceRecommendation()
2     Rubis best choice recommendation(s) (BCR)
3     (in decreasing order of determinateness)
4     Credibility domain: [-100.00,100.00]
5     === >> potential first choice(s)
6         * choice          : ['c02']
7             independence   : 100.00
8             dominance      : 5.00
9             absorbency     : -23.00
10            covering (%) : 100.00
11            determinateness (%) : 52.50
12            - most credible action(s) = { 'c02': 5.00, }
13     === >> potential last choice(s)
14         * choice          : ['c05', 'c14']
15             independence   : 0.00
16             dominance      : -23.00
17             absorbency     : 5.00
18             covered (%)    : 100.00
19             determinateness (%) : 50.00
20             - most credible action(s) = { }

```

Candidate c02, being actually a CONDORCET winner, gives an initial prekernel of digraph m, whereas Party-1 leading Candidates c05 and c14, both being weak CONDORCET losers, give together a terminal prekernel. They hence represent our *first choice*, respectively, *last choice* recommendations for winning this simulated election.

It is worthwhile noticing again the essential structural and computational role, the ignored value is playing in approval-disapproval voting systems. This epistemic and logical *neutral* term is needed indeed for handling in a consistent and efficient manner *not communicated votes* and/or *indeterminate* preferential statements.

Let us conclude by predicting that, for leading political candidates in an aggressively divisive political context, the perspective to easily fail election within approval-disapproval voting system, might or will induce a change in the usual way of running electoral campaigns. Political parties and politicians, who avoid aggressive competitive propaganda and instead propose multipartisan collaborative social choices, will be rewarded with better election results than any kind of extremism. It could mean the end of sterile political obstructions and war like electoral battles. *Let's do it !.*

## References

- Baujard A, Gavrel F, Iggersheim H, Laslier JF, Lebon I (2013) Approval Voting, Evaluation Voting: An Experiment during the 2012 French Presidential Election. *Revue Économique* (Presses de Sciences Po) 64(2):345–356 [283](#), [291](#)
- Benayoun R, Roy B, Sussmann B (1966) ELECTRE: une méthode pour guider le choix en présence de points de vue multiples. Tech. Rep. 49, Société d’Economie et de Mathématique Appliquée, Direction Scientifique [283](#)

**Part V**

**Working with undirected graphs**

The last part introduces Python resources for working with undirected graphs. Its aim is to eventually show the operational benefits one may get when implementing vertex adjacency with bipolar-valued characteristics. Several special graph models and algorithms are illustrated: –Q-coloring, –MIS and clique enumeration, –line graphs and computing maximal matchings, –grid graphs and computing the Ising model, – $n$ -cycle graphs and computing their non-isomorphic MISs, –Spanning tree graphs and graph forests, –Generating and recognising split, interval and permutation graphs.

# Chapter 21

## Bipolar-valued undirected graphs

**Abstract** The chapter introduces bipolar-valued undirected graphs and illustrates several special graph models and algorithms like Q-coloring, MIS and clique enumeration, line graphs and maximal matchings, grid graphs, and  $n$ -cycle graphs with their non-isomorphic maximal independent sets of vertices.

### 21.1 Implementing simple graphs

In the DIGRAPH3 `graphs` module, the root `Graph` class provides a generic simple graph model, without loops and multiple links. A given object of this `Graph` type contains at least the following attributes:

1. `name`: usually the name of the stored instance,
2. `vertices`: a dictionary with `name` and `shortName` attributes,
3. `order`: the number of vertices,
4. `valuationDomain`: a dictionary with three entries: the minimum ( $-1$ , means certainly no link), the median ( $0$ , means missing information) and the maximum characteristic value ( $+1$ , means certainly a link),
5. `edges` : a dictionary with *frozensets*<sup>1</sup> of pairs of vertices as keys carrying a characteristic value in the range of the previous valuation domain,
6. `size`: the number of positive edges,
7. `gamma`: a dictionary containing the direct neighbours of each vertex, automatically added by the object constructor.

#### Example Python terminal session

A random crisp graph instance can be generated with the `RandomGraph` class (see Lines 1-2 in List. 21.1 on the next page below).

---

<sup>1</sup> [Python documentation](#)

**Listing 21.1** Generating a random graph instance

```

1 >>> from graphs import RandomGraph
2 >>> g = RandomGraph(order=7, edgeProbability=0.5)
3 >>> g.save(fileName='tutorialGraph')

```

The saved Graph instance, named `tutorialGraph.py`, is encoded as shown in Listing 21.2.

**Listing 21.2** Stored instance of a random graph

```

1 # Graph instance saved in Python format
2 from decimal import Decimal
3 vertices = {
4     'v1': {'shortName': 'v1', 'name': 'random vertex'},
5     'v2': {'shortName': 'v2', 'name': 'random vertex'},
6     'v3': {'shortName': 'v3', 'name': 'random vertex'},
7     'v4': {'shortName': 'v4', 'name': 'random vertex'},
8     'v5': {'shortName': 'v5', 'name': 'random vertex'},
9     'v6': {'shortName': 'v6', 'name': 'random vertex'},
10    'v7': {'shortName': 'v7', 'name': 'random vertex'},
11 }
12 valuationDomain = {
13     'min':Decimal('-1'), 'med':Decimal('0'),
14     'max':Decimal('1')}
15 edges = {
16     frozenset(['v1','v2']) : Decimal('-1'),
17     frozenset(['v1','v3']) : Decimal('-1'),
18     frozenset(['v1','v4']) : Decimal('-1'),
19     frozenset(['v1','v5']) : Decimal('1'),
20     frozenset(['v1','v6']) : Decimal('-1'),
21     frozenset(['v1','v7']) : Decimal('-1'),
22     frozenset(['v2','v3']) : Decimal('1'),
23     frozenset(['v2','v4']) : Decimal('1'),
24     frozenset(['v2','v5']) : Decimal('-1'),
25     frozenset(['v2','v6']) : Decimal('1'),
26     frozenset(['v2','v7']) : Decimal('-1'),
27     frozenset(['v3','v4']) : Decimal('-1'),
28     frozenset(['v3','v5']) : Decimal('-1'),
29     frozenset(['v3','v6']) : Decimal('-1'),
30     frozenset(['v3','v7']) : Decimal('-1'),
31     frozenset(['v4','v5']) : Decimal('1'),
32     frozenset(['v4','v6']) : Decimal('-1'),
33     frozenset(['v4','v7']) : Decimal('1'),
34     frozenset(['v5','v6']) : Decimal('1'),
35     frozenset(['v5','v7']) : Decimal('-1'),
36     frozenset(['v6','v7']) : Decimal('-1'),
37 }

```

The stored graph instance may be reloaded and plotted with the `exportGraphViz()` method (see Fig. 21.1 on the next page):

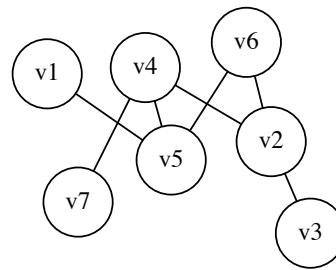
```

1 >>> g = Graph('tutorialGraph')
2 >>> g.exportGraphViz()
3     ----- exporting a dot file for GraphViz tools -----
4     Exporting to tutorialGraph.dot

```

```
5   fdp -Tpng tutorialGraph.dot -o tutorialGraph.png
```

**Fig. 21.1** Example simple graph instance



*Digraph3 (graphviz), R. Bisdorff, 2019*

Properties, like the gamma function –the cover relation–, vertex degrees and neighbourhood depths are shown with a `showShort()` method.

**Listing 21.3** Inspecting a graph instance

```

1 >>> g.showShort()
2     *---- short description of the graph ----*
3     Name : 'tutorialGraph'
4     Vertices : ['v1','v2','v3','v4','v5','v6','v7']
5     Valuation domain : {'min': -1, 'med': 0, 'max': 1}
6     Gamma function :
7         v1 -> ['v5']
8         v2 -> ['v6', 'v4', 'v3']
9         v3 -> ['v2']
10        v4 -> ['v5', 'v2', 'v7']
11        v5 -> ['v1', 'v6', 'v4']
12        v6 -> ['v2', 'v5']
13        v7 -> ['v4']
14     degrees      : [0, 1, 2, 3, 4, 5, 6]
15     distribution  : [0, 3, 1, 3, 0, 0, 0]
16     nbh depths   : [0, 1, 2, 3, 4, 5, 6, 'inf.']
17     distribution  : [0, 0, 1, 4, 2, 0, 0]
```

A Graph instance corresponds bijectively to a symmetric Digraph instance and we can convert from one to the other with the `graph2Digraph()`, and vice versa, with the `digraph2Graph()` method (see List. 21.4). Thus, all computing resources of the Digraph class suitable for symmetric digraphs become readily available, and vice versa.

**Listing 21.4** Conversion between graphs and digraphs

```
1 >>> dg = g.graph2Digraph()
```

```

2 >>> dg.showRelationTable(ndigits=0,ReflexiveTerms=False)
3     * ---- Relation Table ----
4     S | 'v1'  'v2'  'v3'  'v4'  'v5'  'v6'  'v7'
5     ---|-----
6     'v1' |   -    -1    -1    -1     1    -1    -1
7     'v2' |   -1    -    1     1    -1     1    -1
8     'v3' |   -1    1    -    -1    -1    -1    -1
9     'v4' |   -1    1    -1    -    1    -1     1
10    'v5' |    1   -1    -1     1    -    1    -1
11    'v6' |   -1    1    -1    -1     1    -    -1
12    'v7' |   -1   -1    -1     1    -1    -1    -
13 >>> g1 = dg.digraph2Graph()
14 >>> g1
15     *----- Digraph instance description -----*
16     Instance class      : Digraph
17     Instance name       : tutorialGraph
18     Digraph Order       : 7
19     Digraph Size        : 14
20     Valuation domain   : [-1.00;1.00]
21     Determinateness (%) : 100.00
22     Attributes          : ['name', 'order', 'actions',
23                               'valuationdomain', 'relation',
24                               'gamma', 'notGamma']

```

## 21.2 Q-coloring of a graph

With the `Q_Coloring` class, a 3-coloring of the tutorial graph `g` may be computed with a Gibbs sampler ([Geman and Geman, 1984](#)) and plotted as shown in Listing 21.5.

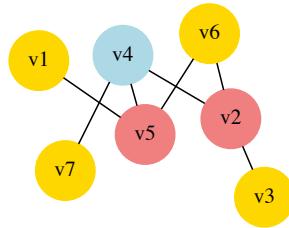
**Listing 21.5** Computing a 3-coloring of the random graph  $g$

Actually, with the given tutorial graph instance, a 2-coloring is already feasible (see Fig. 21.2).

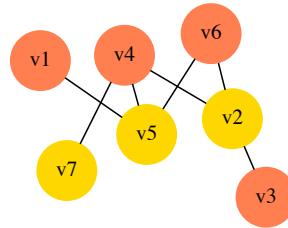
```

1 >>> qc = Q_Coloring(g,colors=['gold','coral'])
2   Running a Gibbs Sampler for 42 step !
3   The q-coloring with 2 colors is feasible !!
4 >>> qc.showConfiguration()
5     v5 gold
6     v3 coral
7     v7 gold
8     v2 gold
9     v4 coral
10    v1 coral
11    v6 coral
12 >>> qc.exportGraphViz('tutorial-2-coloring')
13   Exporting to tutorial-2-coloring.dot
14   fdp -Tpng tutorial-2-coloring.dot \
15           -o tutorial-2-coloring.png

```



*Digraph3 (graphviz), R. Bisdorff, 2019*



*Digraph3 (graphviz), R. Bisdorff, 2019*

**Fig. 21.2** 3-Coloring and 2-Coloring of the tutorial graph

### 21.3 MIS and clique enumeration

2-colorings define sets of independent vertices that are maximal in cardinality –the MISs. The `showMIS()` method computes and prints out such sets. The result is stored in a `misset` attribute (see List. 21.6)

**Listing 21.6** Computing and printing the maximal independent sets of graph `g`

```

1 >>> g = Graph('tutorialGraph')
2 >>> g.showMIS()
3   *--- Maximal Independent Sets ---*
4   ['v2', 'v5', 'v7']

```

```

5      ['v3', 'v5', 'v7']
6      ['v1', 'v2', 'v7']
7      ['v1', 'v3', 'v6', 'v7']
8      ['v1', 'v3', 'v4', 'v6']
9      number of solutions: 5
10     cardinality distribution
11     card.: [0, 1, 2, 3, 4, 5, 6, 7]
12     freq.: [0, 0, 0, 3, 2, 0, 0, 0]
13     execution time: 0.00032 sec.
14     Results in self.misset
15 >>> g.misset
16     [frozenset({'v7', 'v2', 'v5'}),
17      frozenset({'v3', 'v7', 'v5'}),
18      frozenset({'v1', 'v2', 'v7'}),
19      frozenset({'v1', 'v6', 'v7', 'v3'}),
20      frozenset({'v1', 'v6', 'v4', 'v3'})]
```

A MIS in the dual `-g` of a graph instance `g`, corresponds to a maximal *clique*, i.e. a maximal complete subgraph in `g`. Maximal cliques may be computed and printed with the `showCliques()` method. The result is stored in a `cliques` attribute as shown in Listing 21.7.

**Listing 21.7** Computing and printing the maximal independent sets of graph `g`

```

1 >>> g.showCliques()
2     *--- Maximal Cliques ---*
3     ['v2', 'v3']
4     ['v4', 'v7']
5     ['v2', 'v4']
6     ['v4', 'v5']
7     ['v1', 'v5']
8     ['v2', 'v6']
9     ['v5', 'v6']
10    number of solutions: 7
11    cardinality distribution
12    card.: [0, 1, 2, 3, 4, 5, 6, 7]
13    freq.: [0, 0, 7, 0, 0, 0, 0, 0]
14    execution time: 0.00049 sec.
15    Results in self.cliques
16 >>> g.cliques
17     [frozenset({'v2', 'v3'}), frozenset({'v4', 'v7'}),
18      frozenset({'v2', 'v4'}), frozenset({'v4', 'v5'}),
19      frozenset({'v1', 'v5'}), frozenset({'v6', 'v2'}),
20      frozenset({'v6', 'v5'})]
```

## 21.4 Line graphs and maximal matchings

The `graphs` module also provides a `LineGraph` constructor. A *line graph* represents the adjacencies between edges of the given graph instance. In Listing 21.8 on the next page we compute for instance the line graph of the 5-cycle graph.

**Listing 21.8** Computing the line graph of the 5-cycle graph

```

1 >>> from graphs import CycleGraph, LineGraph
2 >>> g = CycleGraph(order=5)
3 >>> g
4     *----- Graph instance description -----*
5     Instance class    : CycleGraph
6     Instance name     : cycleGraph
7     Graph Order       : 5
8     Graph Size        : 5
9     Valuation domain : [-1.00; 1.00]
10    Attributes       : ['name', 'order',
11                      'vertices', 'valuationDomain',
12                      'edges', 'size', 'gamma']
13 >>> lg = LineGraph(g)
14 >>> lg
15     *----- Graph instance description -----*
16     Instance class    : LineGraph
17     Instance name     : line-cycleGraph
18     Graph Order       : 5
19     Graph Size        : 5
20     Valuation domain : [-1.00; 1.00]
21     Attributes       : ['name', 'graph',
22                      'valuationDomain', 'vertices',
23                      'order', 'edges', 'size', 'gamma']
24 >>> lg.showShort()
25     *---- short description of the graph ----*
26     Name              : 'line-cycleGraph'
27     Vertices          : [frozenset({'v1', 'v2'}),
28                           frozenset({'v1', 'v5'}), frozenset({'v2', 'v3'}),
29                           frozenset({'v3', 'v4'}), frozenset({'v4', 'v5'})]
30     Valuation domain : {'min': Decimal('-1'),
31                           'med': Decimal('0'), 'max': Decimal('1')}
32     Gamma function   :
33     frozenset({'v1', 'v2'}) ->
34         [frozenset({'v2', 'v3'}), frozenset({'v1', 'v5'})]
35     frozenset({'v1', 'v5'}) ->
36         [frozenset({'v1', 'v2'}), frozenset({'v4', 'v5'})]
37     frozenset({'v2', 'v3'}) ->
38         [frozenset({'v1', 'v2'}), frozenset({'v3', 'v4'})]
39     frozenset({'v3', 'v4'}) ->
40         [frozenset({'v2', 'v3'}), frozenset({'v4', 'v5'})]
41     frozenset({'v4', 'v5'}) ->
42         [frozenset({'v4', 'v3'}), frozenset({'v1', 'v5'})]
43     degrees          : [0, 1, 2, 3, 4]
44     distribution     : [0, 0, 5, 0, 0]
45     nbh depths      : [0, 1, 2, 3, 4, 'inf.']
46     distribution     : [0, 0, 5, 0, 0, 0]
```

Iterated line graph constructions are usually expanding, except for chordless cycles, where the same cycle is repeated, and for non-closed paths, where iterated line graphs progressively reduce one by one the number of vertices and edges and eventually become an empty graph.

Notice in Listing 21.9 that the MISs in a line graph provide *maximal matchings* –maximal sets of independent edges– of the original graph.

**Listing 21.9** Computing the MISs of the line graph of the 8-cycle graph

```

1 >>> c8 = CycleGraph(order=8)
2 >>> lc8 = LineGraph(c8)
3 >>> lc8.showMIS()
4     Maximal Independent Sets ---
5     [{frozenset({'v3', 'v4'}), frozenset({'v5', 'v6'}), frozenset({'v1', 'v8'})}]
6     [{frozenset({'v2', 'v3'}), frozenset({'v5', 'v6'}), frozenset({'v1', 'v8'})}]
7     [{frozenset({'v8', 'v7'}), frozenset({'v2', 'v3'}), frozenset({'v5', 'v6'})}]
8     [{frozenset({'v8', 'v7'}), frozenset({'v2', 'v3'}), frozenset({'v4', 'v5'})}]
9     [{frozenset({'v7', 'v6'}), frozenset({'v3', 'v4'}), frozenset({'v1', 'v8'})}]
10    [{frozenset({'v2', 'v1'}), frozenset({'v8', 'v7'}), frozenset({'v4', 'v5'})}]
11    [{frozenset({'v2', 'v1'}), frozenset({'v7', 'v6'}), frozenset({'v4', 'v5'})}]
12    [{frozenset({'v2', 'v1'}), frozenset({'v7', 'v6'}), frozenset({'v3', 'v4'})}]
13    [{frozenset({'v7', 'v6'}), frozenset({'v2', 'v3'}), frozenset({'v1', 'v8'})},
14      frozenset({'v4', 'v5'})]
15    [{frozenset({'v2', 'v1'}), frozenset({'v8', 'v7'}), frozenset({'v3', 'v4'})},
16      frozenset({'v5', 'v6'})]
17    number of solutions: 10
18    cardinality distribution
19    card.: [0, 1, 2, 3, 4, 5, 6, 7, 8]
20    freq.: [0, 0, 0, 8, 2, 0, 0, 0, 0]
21    execution time: 0.00029 sec.

```

**Fig. 21.3** A perfect maximum matching of the 8-cycle graph.

Every vertex of the 8-cycle is adjacent to a matching edge.

Odd cycle graphs do not admit any perfect matching



Digraph3 (graphviz), R. Bisdomff, 2019

The two last MISs of cardinality 4 (see Lines 13-16 in List. 21.9) give perfect maximum matchings of the 8-cycle graph as shown in Figure 21.3.

**Listing 21.10** Computing maximum matchings in the 8-cycle graph

```

1 >>> maxMatching = c8.computeMaximumMatching()
2 >>> c8.exportGraphViz(fileName='maxMatchingcycleGraph', \

```

```

3 ...                               matching=maxMatching)
4 *---- exporting a dot file for GraphViz tools ----*
5 Exporting to maxMatchingcycleGraph.dot
6 Matching: {frozenset({'v1', 'v2'}),
7             frozenset({'v5', 'v6'}),
8             frozenset({'v3', 'v4'}),
9             frozenset({'v7', 'v8'}) }
10 circo -Tpng maxMatchingcycleGraph.dot\
11           -o maxMatchingcycleGraph.png

```

## 21.5 Grids and the ISING model

Special classes of graphs, like the `GridGraph` class provide  $n \times m$  rectangular or triangular grids. With a *Gibbs* sampler<sup>2</sup> the `IsingModel` class can simulate in Figure 21.4 on the next page an ISING model on, for instance, a  $15 \times 15$  grid (Ising, 1925).

**Listing 21.11** Simulating an Ising model on a the  $15 \times 15$  rectangular grid

```

1 >>> from graphs import GridGraph, IsingModel
2 >>> g = GridGraph(n=15, m=15)
3 >>> g.showShort()
4 *---- show short -----*
5   Grid graph    : grid-6-6
6   n            : 6
7   m            : 6
8   order        : 36
9 >>> im = IsingModel(g, beta=0.3, nSim=100000)
10  Running a Gibbs Sampler for 100000 step !
11 >>> im.exportGraphViz(colors=['lightblue', 'lightcoral'])
12 *---- exporting a dot file for GraphViz tools -----*
13   Exporting to grid-15-15-ising.dot
14   fdp -Tpng grid-15-15-ising.dot -o grid-15-15-ising.png

```

## 21.6 Simulating METROPOLIS random walks

Finally, we provide the `MetropolisChain` class, a specialisation of the `Graph` class, implementing a generic Monte Carlo Markov Chain (MCMC) sampler for simulating random walks on a graph following given transition probabilities `probs = {'v1': x, 'v2': y, ...}` for visiting each vertex (see List. 21.12) (Metropolis et al., 1953).

**Listing 21.12** Simulating random walks on a graph

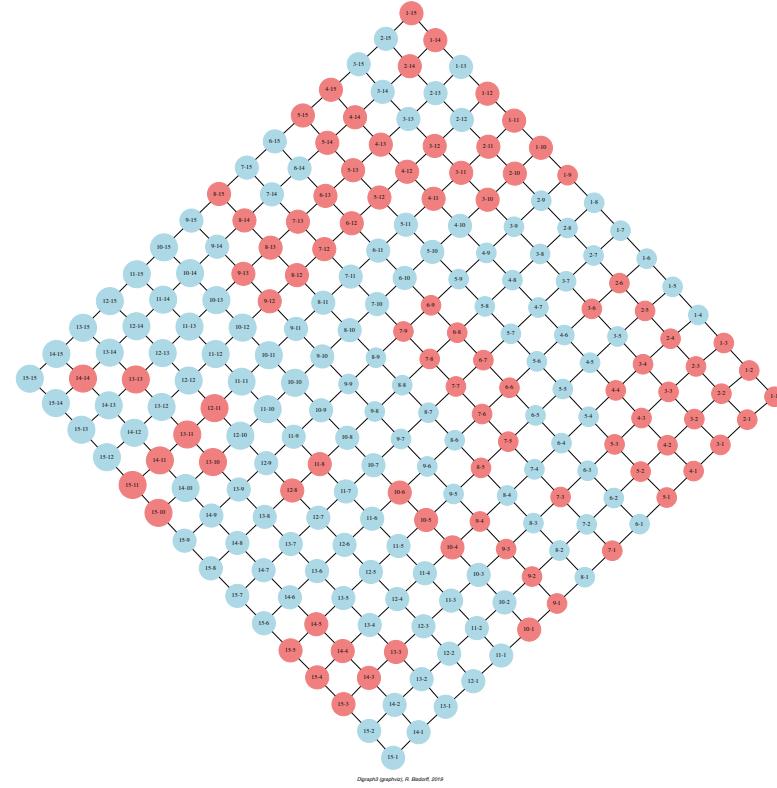
```

1 >>> from graphs import Graph

```

---

<sup>2</sup> Geman and Geman (1984)



**Fig. 21.4** Ising model of the 15x15 grid graph

```

2 >>> g = Graph(numberOfVertices=5,edgeProbability=0.5)
3 >>> g.showShort()
4     *----- short description of the graph -----*
5     Name          : 'randomGraph'
6     Vertices      : ['v1', 'v2', 'v3', 'v4', 'v5']
7     Valuation domain : {'max': 1, 'med': 0, 'min': -1}
8     Gamma function :
9     v1 -> ['v2', 'v3', 'v4']
10    v2 -> ['v1', 'v4']
11    v3 -> ['v5', 'v1']
12    v4 -> ['v2', 'v5', 'v1']
13    v5 -> ['v3', 'v4']
14 >>> # initialize a potential stationary probability vector
15 >>> n = g.order
16 >>> i = 0
17 >>> for v in g.vertices:

```

```

18 ...     probs[v] = (n - i) / (n * (n+1) / 2)
19 ...     i += 1

```

The `checkSampling()` method of the `MetropolisChain` class (see below) generates a random walk of  $nSim = 30000$  steps on the given graph and records by the way the observed relative frequency with which each vertex is passed by. In Listing 21.13 we notice how well the random walk is matching the given stationary probability vector.

**Listing 21.13** Checking the quality of the MCMC sampler

```

1 >>> from graphs import MetropolisChain
2 >>> met = MetropolisChain(g,probs)
3 >>> frequency = met.checkSampling(verticesList[0],nSim=30000)
4 >>> for v in verticesList:
5 ...     print(v,probs[v],frequency[v])
6     v1 0.3333 0.3343
7     v2 0.2666 0.2680
8     v3 0.2      0.2030
9     v4 0.1333 0.1311
10    v5 0.0666 0.0635

```

In this example, the stationary transition probability distribution (see List. 21.13 above), shown by the `showTransitionMatrix()` method in Listing 21.14 is quite adequately simulated.

**Listing 21.14** Printing the transition probability distribution

```

1 >>> met.showTransitionMatrix()
2 * ----- Transition Matrix -----
3   Pij | 'v1'   'v2'   'v3'   'v4'   'v5'
4   ----|-----
5   'v1' | 0.23   0.33   0.30   0.13   0.00
6   'v2' | 0.42   0.42   0.00   0.17   0.00
7   'v3' | 0.50   0.00   0.33   0.00   0.17
8   'v4' | 0.33   0.33   0.00   0.08   0.25
9   'v5' | 0.00   0.00   0.50   0.50   0.00

```

For the reader interested in algorithmic applications of Markov Chains we recommend consulting [Häggström \(2002\)](#).

## 21.7 Computing the non isomorphic MISs of the n-cycle graph

Due to the public success of our common work with Jean-Luc Marichal ([Bisdorff and Marichal, 2008](#)), we present in this chapter an example Python session for computing the non isomorphic maximal independent sets (MISs) of the 12-cycle graph, i.e. a `CirculantDigraph` class instance of order 12 and symmetric circulants 1 and  $-1$ .

```

1 >>> from digraphs import CirculantDigraph
2 >>> c12 = CirculantDigraph(order=12,circulants=[1,-1])

```

```

3 >>> c12 # 12-cycle digraph instance
4 *----- Digraph instance description -----*
5 Instance class : CirculantDigraph
6 Instance name : c12
7 Digraph Order : 12
8 Digraph Size : 24
9 Valuation domain : [-1.0, 1.0]
10 Determinateness : 100.000
11 Attributes : ['name', 'order', 'circulants',
12                 'actions', 'valuationdomain',
13                 'relation', 'gamma', 'notGamma']

```

Such  $n$ -cycle graphs – see the 12-cycle graph in Figure 21.5 on page 321– are also provided as undirected graph instances by the `CycleGraph` class.

```

1 >>> from graphs import CycleGraph
2 >>> cg12 = CycleGraph(order=12)
3 >>> cg12
4 *----- Graph instance description -----*
5 Instance class : CycleGraph
6 Instance name : cycleGraph
7 Graph Order : 12
8 Graph Size : 12
9 Valuation domain : [-1.0, 1.0]
10 Attributes : ['name', 'order', 'vertices',
11                  'valuationDomain', 'edges',
12                  'size', 'gamma']

```

A non-isomorphic MIS of the 12-cycle graph corresponds in fact to a set of isomorphic MISs, i.e. an orbit of MISs under the automorphism group of the 12-cycle graph. In Listing 21.15 we are now first computing all maximal independent sets that are detectable in the 12-cycle digraph with the `showMIS()` method.

**Listing 21.15** Computing the MISs of the 12-cycle graph

```

1 >>> c12.showMIS(withListing=False)
2 *--- Maximal independent choices ---
3 number of solutions: 29
4 cardinality distribution
5 card.: [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]
6 freq.: [0, 0, 0, 0, 3, 24, 2, 0, 0, 0, 0, 0]
7 Results in c12.misset

```

In the 12-cycle graph, we observe 29 labelled MISs: 3 of cardinality 4, 24 of cardinality 5, and 2 of cardinality 6. In case of  $n$ -cycle graphs with  $n > 20$ , as the cardinality of the MISs becomes big, it is preferable to use, as shown in Listing 21.16 on the facing page, the `perrinMIS` shall command<sup>3</sup> compiled from C and installed along with all the DIGRAPH3 python modules for computing the set of MISs observed in the graph.

---

<sup>3</sup> The `perrinMIS` shell command may be installed system wide with the command `textrm{make installPerrin}` from the main DIGRAPH3 directory. It is stored by default into `/usr/local/bin/`. This may be changed with the `INSTALLDIR` flag. The command `make installPerrinUser` installs it instead without sudo into the user's private `.bin` directory.

**Listing 21.16** Computing the MISs with the perrinMIS shell command

```

1 ...$ echo 12 | /usr/local/bin/perrinMIS
2 # -----
3 # Generating MIS set of Cn with the      #
4 # Perrin sequence algorithm.           #
5 # Temporary files used.               #
6 # even versus odd order optimised.   #
7 # RB December 2006                   #
8 # Current revision Dec 2018          #
9 # -----
10 Input cycle order ? <-- 12
11 mis 1 : 100100100100
12 mis 2 : 010010010010
13 mis 3 : 001001001001
14 ...
15 ...
16 ...
17 mis 27 : 001001010101
18 mis 28 : 101010101010
19 mis 29 : 010101010101
20 Cardinalities:
21 0 : 0
22 1 : 0
23 2 : 0
24 3 : 0
25 4 : 3
26 5 : 24
27 6 : 2
28 7 : 0
29 8 : 0
30 9 : 0
31 10 : 0
32 11 : 0
33 12 : 0
34 Total: 29
35 execution time: 0 sec. and 2 millisec.
```

Reading in the result of the perrinMIS shell command, stored in a file called by default curd.dat, may be operated with the readPerrinMisset () method .

```

1 >>> c12.readPerrinMisset(file='curd.dat')
2 >>> c12.misset
3     {frozenset({'5', '7', '10', '1', '3'}),
4      frozenset({'9', '11', '5', '2', '7'}),
5      frozenset({'7', '2', '4', '10', '12'}),
6      ...
7      ...
8      ...
9      frozenset({'8', '4', '10', '1', '6'}),
10     frozenset({'11', '4', '1', '9', '6'}),
11     frozenset({'8', '2', '4', '10', '12', '6'})}
```

For computing the corresponding non-isomorphic MISs, we actually need the automorphism group of the 12-cycle graph. The Digraph class therefore provides

the `automorphismGenerators()` method which adds automorphism group generators to a `Digraph` class instance with the help of the external `dreadnaut` shell command from the *nauty* software package<sup>4</sup>.

**Listing 21.17** Computing the automorphism group generators

```

1 >>> c12.automorphismGenerators()
2     ...
3     Permutations
4     {'1':'1', '2':'12', '3':'11', '4':'10',
5      '5':'9', '6':'8', '7':'7', '8':'6', '9':'5',
6      '10':'4', '11':'3', '12':'2'}
7     {'1':'2', '2':'1', '3':'12', '4':'11', '5':'10',
8      '6':'9', '7':'8', '8':'7', '9':'6', '10':'5',
9      '11':'4', '12':'3'}
10 >>> print('grpsize = ', c12.automorphismGroupSize)
11     grpsize = 24

```

The 12-cycle graph's automorphism group, of size 24, is generated with both the permutations shown in Listing 21.17.

The `showOrbits()` method renders now the labelled representatives of each of the four orbits of isomorphic MISs observed in the 12-cycle graph (see Lines 7-10 below).

**Listing 21.18** Computing the MISs orbits of the 12-cycle graph

```

1 >>> c12.showOrbits(c12.misset, withListing=False)
2     ...
3     *---- Global result ----
4     Number of MIS: 29
5     Number of orbits : 4
6     Labelled representatives and cardinality:
7     1: ['2','4','6','8','10','12'], 2
8     2: ['2','5','8','11'], 3
9     3: ['2','4','6','9','11'], 12
10    4: ['1','4','7','9','11'], 12
11    Symmetry vector
12    stabilizer size: [1, 2, 3, ..., 8, 9, ..., 12, 13, ...]
13    frequency       : [0, 2, 0, ..., 1, 0, ..., 1, 0, ...]

```

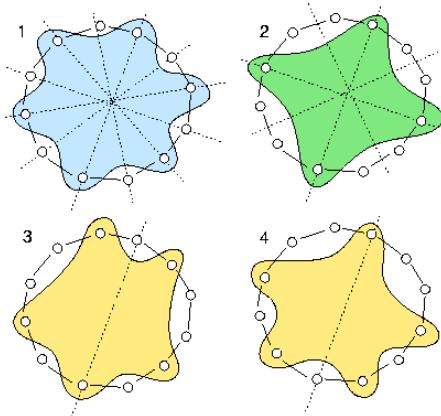
The corresponding group stabilisers' sizes and frequencies: orbit 1 with 6 symmetry axes, orbit 2 with 4 symmetry axes, and orbits 3 and 4 both with one symmetry axis (see Lines 12-13 in List. 21.18), are illustrated in the corresponding unlabelled graphs of Figure 21.5 on the next page.

The non isomorphic MISs in the 12-cycle graph represent in fact all the ways one may write the number 12 as the circular sum of '2's and '3's without distinguishing opposite directions of writing. The first orbit corresponds to writing six times a '2'; the second orbit corresponds to writing four times a '3'. The third and fourth orbit

---

<sup>4</sup> The `automorphismGenerators` method uses the `dreadnaut` shell command from the *nauty* software package. See <https://www3.cs.stonybrook.edu/~algorith/implement/nauty/implementation.shtml>. On Mac OS there exist `dmg` installers and on Ubuntu Linux or Debian, one may easily install it with `...$ sudo apt-get install nauty`.

**Fig. 21.5** Symmetry axes of the four non isomorphic MISs of the 12-cycle graph



correspond to writing two times a '3' and three times a '2'. There are two non isomorphic ways to do this latter circular sum. Either separating the '3's by one and two '2's, or by zero and three '2's (Bisdorff and Marichal, 2008).

The next chapter is devoted more specifically to tree graphs and graph forests.

## References

- Bisdorff R, Marichal J (2008) Counting non-isomorphic maximal independent sets of the n-cycle graph. *Journal of Integer Sequences* 11(Art. 08.5.7):1–16, URL <https://cs.uwaterloo.ca/journals/JIS/VOL11/Marichal/marichal.html> 317, 321
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# Chapter 22

## On tree graphs and graph forests

**Abstract** The chapter specifically addresses working with tree graphs and graph forests. We illustrate how to generate and recognise random tree graphs and how to compute the centres of a tree and draw a rooted and oriented tree. Finally, algorithms for computing spanning trees and forests are presented.

### 22.1 Generating random tree graphs

Using the `RandomTree` class from the `graphs` module, we may, for instance, generate a random tree graph with 9 vertices.

**Listing 22.1** Generating a random tree graph

```
1 >>> from graphs import RandomTree
2 >>> rt = RandomTree(order=9, seed=100)
3 >>> rt
4     ----- Graph instance description -----
5     Instance class    : RandomTree
6     Instance name     : randomTree
7     Graph Order       : 9
8     Graph Size        : 8
9     Valuation domain : [-1.00; 1.00]
10    Attributes       : ['name', 'order',
11                          'vertices', 'valuationDomain',
12                          'edges', 'prueferCode',
13                          'size', 'gamma']
14    ----- RandomTree specific data -----
15    Pruefer code   : ['v3','v8','v8','v3','v7','v6','v7']
16 >>> rt.exportGraphViz('tutRandomTree')
17     ----- exporting a dot file for GraphViz tools ---
18     Exporting to tutRandomTree.dot
19     neato -Tpng tutRandomTree.dot -o tutRandomTree.png
```

In Fig. 22.1 on the following page one may notice that a tree graph of order  $n > 2$  always contains  $n - 1$  edges (see Lines 7 and 8 in List. 22.1) and its structure

**Fig. 22.1** Random tree graph instance of order 9. One may distinguish vertices like  $v_1$ ,  $v_2$ ,  $v_4$ ,  $v_5$  or  $v_9$  of degree 1, called the *leaves* of the tree, and vertices like  $v_3$ ,  $v_6$ ,  $v_7$  or  $v_8$  of degree 2 or more, called the *nodes* of the tree



Digraph3 (graphviz), R. Bischoff, 2019

is entirely characterised by a corresponding PRÜFER code, i.e. a list of vertices keys of length  $n - 2$ . See, for instance in Line 15 the code `['v3', 'v8', 'v8', 'v3', 'v7', 'v6', 'v7']` corresponding to our sample tree graph `rt`.

Each position of the code indicates the parent of the remaining leaf with the smallest vertex label. Vertex  $v_3$  is thus the parent of  $v_1$  and we drop leaf  $v_1$ ,  $v_8$  is now the parent of leaf  $v_2$  and we drop  $v_2$ , vertex  $v_8$  is again the parent of leaf  $v_4$  and we drop  $v_4$ , vertex  $v_3$  is the parent of leaf  $v_5$  and we drop  $v_5$ ,  $v_7$  is now the parent of leaf  $v_3$  and we may drop  $v_3$ ,  $v_6$  becomes the parent of leaf  $v_8$  and we drop  $v_8$ ,  $v_7$  becomes the parent of leaf  $v_6$  and we may drop  $v_6$ . The two eventually remaining vertices,  $v_7$  and  $v_9$ , give the last link in the reconstructed tree (Barthélemy and Guenoche, 1991).

It is, as well possible to, first, generate a random PRÜFER code of length  $n - 2$  from a set of  $n$  vertices (see List. 22.2 Lines 1-9) and then, construct the corresponding tree of order  $n$  by reversing the procedure illustrated above. The resulting tree graph is shown in Fig. 22.2 on the next page.

**Listing 22.2** Generating a tree graph with a random PRÜFER code.

```

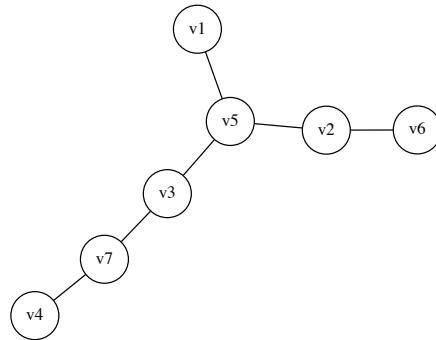
1 >>> verticesList = ['v1', 'v2', 'v3', 'v4', 'v5', 'v6', 'v7']
2 >>> n = len(verticesList)
3 >>> from random import seed, choice
4 >>> seed(101)
5 >>> code = []
6 >>> for k in range(n-2):
7 ...     code.append( choice(verticesList) )
8 >>> print(code)
  
```

```

9      ['v5', 'v7', 'v2', 'v5', 'v3']
10 >>> rt = RandomTree(prueferCode=code)
11 >>> rt
12 *----- Graph instance description -----*
13 Instance class    : RandomTree
14 Instance name     : randomTree
15 Graph Order       : 7
16 Graph Size        : 6
17 Valuation domain  : [-1.00; 1.00]
18 Attributes : ['name', 'order', 'vertices',
19                 'valuationDomain', 'edges',
20                 'prueferCode', 'size', 'gamma']
21 *---- RandomTree specific data ----*
22 Pruefer code : ['v5', 'v7', 'v2', 'v5', 'v3']
23 >>> rt.exportGraphViz('tutPruefTree')
24 *---- exporting a dot file for GraphViz tools -----*
25 Exporting to tutPruefTree.dot
26 neato -Tpng tutPruefTree.dot -o tutPruefTree.png

```

**Fig. 22.2** Tree graph instance generated with a random PRÜFER code ['v5', 'v7', 'v2', 'v5', 'v3']



Digraph3 (graphviz), R. Bisdorff, 2019

Following from the bijection between a labelled tree and its PRÜFER code, we actually know that there exist  $n^{n-2}$  different tree graphs with the same  $n$  vertices.

Given a genuine graph, how can we recognise that it is in fact a tree instance ?

## 22.2 Recognising tree graphs

Given a graph  $g$  of order  $n$  and size  $s$ , the following 5 assertions A1, A2, A3, A4 and A5 are all equivalent (see (Barthélemy and Guenoche, 1991)):

A1  $g$  is a tree;

- A2  $g$  is without (chordless) cycles and  $n = s + 1$ ;
- A3  $g$  is connected and  $n = s + 1$ ;
- A4 Any two vertices of  $g$  are always connected by a unique path;
- A5  $g$  is connected and dropping any single edge will always disconnect  $g$ .

Assertion A3, for instance, gives a simple test for recognising a tree graph. In case of a *lazy evaluation* of the test in Listing 22.3 Line 3 below, it is opportune, from a computational complexity perspective, to first, check the order and size of the graph, before checking its potential connectedness. We provide the `isTree()` method for computing both these tests (see Line 8 in List. 22.3).

**Listing 22.3** Recognizing a tree graph.

```

1 >>> from graphs import RandomGraph
2 >>> g = RandomGraph(order=8,edgeProbability=0.3,seed=62)
3 >>> if g.order == (g.size +1) and g.isConnected():
4 ...     print('The graph is a tree ?', True)
5 ... else:
6 ...     print('The graph is a tree ?',False)
7     The graph is a tree ? True
8 >>> g.isTree()
9     True

```

The random graph of order 8 and edge probability 30%, generated with seed 62, is actually a tree graph instance, as confirmed by its graphviz drawing shown in Fig. 22.3 on the facing page.

```

1 >>> g.exportGraphViz('test62')
2      ----- exporting a dot file for GraphViz tools ---
3      Exporting to test62.dot
4      fdp -Tpng test62.dot -o test62.png

```

Yet, we still have to recover its corresponding PRÜFER code. Therefore, we may use the `tree2Pruefer()` method from the `TreeGraph` class. But, first, the instance class of graph  $g$  is changed to the `TreeGraph` type (see Line 2 in List. 22.4).

**Listing 22.4** Computing the PRÜFER code of a tree graph instance.

```

1 >>> from graphs import TreeGraph
2 >>> g.__class__ = TreeGraph
3 >>> g.tree2Pruefer()
4     ['v6', 'v1', 'v2', 'v1', 'v2', 'v5']

```

In Fig. 22.3 on the facing page, we noticed that vertex  $v_2$  is actually situated in the *centre* of the tree with a neighbourhood depth of 2. Centres of a graph are the vertices with minimal neighbourhood depth. For finding such centre(s), the `Graph` class provides the `computeGraphCentres()` method. (see Lines 1-2 in List. 22.4). Knowing now the centre of graph  $g$ , we may draw a correspondingly rooted and oriented tree with the `exportOrientedTreeGraphViz()` method from the `TreeGraph` class (see Fig. 22.4 on the facing page).

**Listing 22.5** Computing the centres of a tree and drawing a rooted and oriented tree.

```

1 >>> g.computeGraphCentres()

```

**Fig. 22.3** Recognising a tree graph. We may notice that vertex v2 is actually situated in the *centre* of the tree with a neighbourhood depth of 2



Digraph3 (graphviz), R. Bisдорff, 2019

```

2   {'v2': 2}
3 >>> g.exportOrientedTreeGraphViz(\n4 ...     fileName='rootedTree', root='v2')\n5 *---- exporting a dot file for GraphViz tools\n6 Exporting to rootedTree.dot\n7 dot -Grankdir=TB -Tpng rootedTree.dot -o rootedTree.png

```

**Fig. 22.4** Drawing an oriented tree rooted at its centre



Digraph3 (graphviz)  
R. Bisдорff, 2020

Let us now turn our attention toward a major application of tree graphs, namely *spanning trees* and *forests* related to graph traversals.

## 22.3 Spanning trees and forests

With the `RandomSpanningTree` class we may generate, from a given connected graph `g` instance, *uniform* random instances of a spanning tree by using WILSON's algorithm (see List. 22.6 and Fig. 22.5 on the facing page).

**Listing 22.6** Generating uniform random spanning trees.

```

1 >>> from graphs import RandomGraph,\n2 ...             RandomSpanningTree\n3 >>> g = RandomGraph(order=9,\n4 ...                 edgeProbability=0.4, seed=100)\n5 >>> spt = RandomSpanningTree(g)\n6 >>> spt\n7     *----- Graph instance description -----*\n8     Instance class      : RandomSpanningTree\n9     Instance name       : randomGraph_randomSpanningTree\n10    Graph Order         : 9\n11    Graph Size          : 8\n12    Valuation domain   : [-1.00; 1.00]\n13    Attributes          : ['name','vertices','order',\n14                           'valuationDomain',\n15                           'edges','size','gamma',\n16                           'dfs','date', 'dfsx',\n17                           'prueferCode']\n18    *---- RandomTree specific data ----*\n19    Pruefer code        : ['v7','v9','v5','v1','v8','v4','v9']\n20 >>> spt.exportGraphViz(fileName='randomSpanningTree',\n21 ...                         WithSpanningTree=True)\n22    *---- exporting a dot file for GraphViz tools ----*\n23    Exporting to randomSpanningTree.dot\n24    [[v1','v5','v6','v5','v1','v8','v9','v3','v9','v4',\n25    'v7','v2','v7','v4','v9','v8','v1']]\n26    neato -Tpng randomSpanningTree.dot\\
27    -o randomSpanningTree.png

```

WILSON's algorithm only works for connected graphs (Wilson, 1996). More general, and in case of a not connected graph, the `RandomSpanningForest` class generates a not necessarily uniform random instance of a *spanning forest* – one or more random tree graphs– generated from a random *depth first search* of the graph components' traversals (see List. 22.7 and Fig. 22.6 on page 330).

**Listing 22.7** Computing spanning forests over disconnected graphs.

```

1 >>> g = RandomGraph(order=15,\n2 ...                 edgeProbability=0.1, seed=140)\n3 >>> g.computeComponents()\n4    [{v12','v01','v13'}, {'v02','v06'}, ...

```

**Fig. 22.5** Random spanning tree



Diagram3 (graphviz), R. Bisdorff, 2019

```

5      {'v08','v03','v07'}, {'v15','v11','v10','v04','v05'},
6      {'v09','v14'}]
7 >>> spf = RandomSpanningForest(g, seed=100)
8 >>> spf.exportGraphViz(fileName='spanningForest', \
9 ...                           WithSpanningTree=True)
10     *----- exporting a dot file for GraphViz tools -----*
11     Exporting to spanningForest.dot
12     [['v03','v07','v08','v07','v03'],
13     ['v13','v12','v13','v01','v13'],
14     ['v02','v06','v02'],
15     ['v15','v11','v04','v11','v15',
16     'v10','v05','v10','v15'],
17     ['v09','v14','v09']]
18     neato -Tpng spanningForest.dot -o spanningForest.png

```

## 22.4 Maximum determined spanning forests

In case of valued graphs –supporting weighted edges– we may finally construct a *most determined* spanning tree (or forest if not connected) using KRUSKAL’s greedy minimum-spanning-tree algorithm on the dual valuation of the graph (Kruskal, 1956)<sup>1</sup>.

In Listing 22.8 we generate, for instance, a randomly valued graph with five vertices and seven edges bipolar-valued in [-1.0; 1.0].

---

<sup>1</sup> KRUSKAL’s algorithm is a minimum-spanning-tree algorithm which finds an edge of the least possible weight that connects any two trees in the forest.

**Fig. 22.6** Random spanning forest



Digraph3 (graphviz), R. Bisдорff, 2019

**Listing 22.8** Generating randomly bipolar-valued graphs.

```

1 >>> from graphs import RandomValuationGraph
2 >>> g = RandomValuationGraph(order=5, seed=2)
3 >>> g
4 *----- Graph instance description -----*
5 Instance class      : RandomValuationGraph
6 Instance name       : randomGraph
7 Graph Order         : 5
8 Graph Size          : 7
9 Valuation domain   : [-1.00; 1.00]
10 Attributes         : ['name', 'order',
11                  'vertices', 'valuationDomain',
12                  'edges', 'size', 'gamma']

```

To inspect the edges' actual weights, we first transform the graph into a corresponding digraph (see Line 1 in List 22.9) and use the `showRelationTable()` method (see Line 2) for printing its symmetric adjacency matrix.

**Listing 22.9** Symmetric relation table

```

1 >>> dg = g.graph2Digraph()
2 >>> dg.showRelationTable()
3 *----- Relation Table -----*
4 S | 'v1' 'v2' 'v3' 'v4' 'v5'
5 -----|-----
6 'v1' |  0.00  0.91  0.90 -0.89 -0.83
7 'v2' |  0.91  0.00  0.67  0.47  0.34
8 'v3' |  0.90  0.67  0.00 -0.38  0.21
9 'v4' | -0.89  0.47 -0.38  0.00  0.21
10 'v5' | -0.83  0.34  0.21  0.21  0.00
11 Valuation domain: [-1.00;1.00]

```

To compute the most determined spanning tree or forest, we can use the `BestDeterminedSpanningForest` constructor.

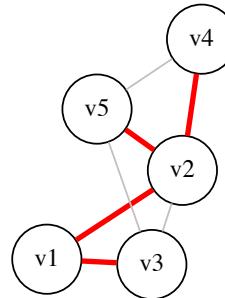
**Listing 22.10** Computing best determined spanning forests.

```

1 >>> from graphs import\
2             BestDeterminedSpanningForest
3 >>> mt = BestDeterminedSpanningForest(g)
4 >>> print(mt)
5     *----- Graph instance description -----*
6     Instance class      : BestDeterminedSpanningForest
7     Instance name       : bdSpanningForest
8     Graph Order         : 5
9     Graph Size          : 4
10    Valuation domain   : [-1.00; 1.00]
11    Attributes          : ['name','vertices','order',
12                           'valuationDomain',
13                           'edges','size','gamma',
14                           'dfs','date',
15                           'averageTreeDetermination']
16    *---- best determined spanning tree ----*
17    Depth first search path(s) :
18    [['v1','v2','v4','v2','v5','v2','v1','v3','v1']]
19    Average determination(s) : [Decimal('0.655')]

```

The random graph  $g$  is connected and, hence, admits a single spanning tree of maximum mean determination  $= (0.47 + 0.91 + 0.90 + 0.34)/4 = 0.655$  (see Lines 9, 6 and 10 in List. 22.8 on page 329 and Fig. 22.7).

**Fig. 22.7** Best determined spanning tree

*Digraph3 (graphviz), R. Bisendorff, 2019*

```

1 >>> mt.exportGraphViz(\n2     ...     fileName='bestDeterminedspanningTree',\n3     ...     WithSpanningTree=True)\n4     *---- exporting a dot file for GraphViz tools ---*\n5     Exporting to spanningTree.dot\n6     [['v4','v2','v1','v3','v1','v2','v5','v2','v4']]

```

```
7   neato -Tpng bestDeterminedSpanningTree.dot \
8       -o bestDeterminedSpanningTree.png
```

One may easily verify that all other potential spanning trees, including instead the edges  $\{v3, v5\}$  and/or  $\{v4, v5\}$ , will show a lower average determination.

Chapter 23, the last on undirected graphs, is devoted to different models of perfect graphs, namely split, interval, comparability and permutation graphs.

## References

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[324](#), [325](#)
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- Wilson DB (1996) Generating random spanning trees more quickly than the cover time. In: Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia PA), ACM New York, pp 296–303 [328](#)

## Chapter 23

# About split, interval and permutation graphs

**Abstract** The last chapter of this book eventually presents some famous classes of perfect graphs, namely comparability, interval, permutation and split graphs. We first present an example of a graph which is at the same time a triangulated, a comparability, a split and a permutation graph. The importance to be an interval is illustrated with BERGE's mystery story. We discuss furthermore the generation of permutation graphs and close with how to recognise that a given graph is in fact a permutation graph.

### 23.1 A 'multiply' perfect graph

A graph  $g$  is called:

- BERGE or *perfect* when  $g$  and its dual  $-g$  both don't contain any chordless odd cycles of length greater than 3 (Berge, 1963; Chudnovsky, Robertson, Seymour, and Robin, 2006);
- *Triangulated* graph when  $g$  does not contain any *chordless cycle* of length 4 and more;

**Definition 23.1 (Some perfect graph classes).** Following Martin Golumbic (see Golumbic, 2004, p. 149), we call a given graph  $g$ :

- *Comparability* graph when  $g$  is *transitively orientable*;
- *Interval* graph when  $g$  is *triangulated* and its dual  $-g$  is a *comparability* graph;
- *Permutation* graph when  $g$  and its dual  $-g$  both are *comparability* graphs;
- *Split* graph when  $g$  and its dual  $-g$  both are *triangulated* graphs.

To illustrate these *perfect* graph classes, we will generate from 8 intervals, randomly chosen in the default integer range  $[0, 10]$ , a `RandomIntervalIntersectionsGraph` instance  $g$  (see Line 2 below).

```
1 >>> from graphs import \
2 ...           RandomIntervalIntersectionsGraph
```

```

3 >>> g = RandomIntervalIntersectionsGraph(\
4 ...                               order=8, seed=100)
5 >>> g
6 *----- Graph instance description -----*
7 Instance class      : RandomIntervalIntersectionsGraph
8 Instance name       : randIntervalIntersections
9 Seed                : 100
10 Graph Order        : 8
11 Graph Size         : 23
12 Valuation domain   : [-1.0; 1.0]
13 Attributes          : ['seed', 'name', 'order',
14   'intervals', 'vertices', 'valuationDomain',
15   'edges', 'size', 'gamma']
16 >>> print(g.intervals)
17 [(2,7), (2,7), (5,6), (6,8), (1,8), (1,1), (4,7), (0,10)]

```

With  $\text{seed} = 100$ , we obtain here an *interval graph*, in fact a *perfect graph*, which is *conjointly* a *triangulated*, a *comparability*, a *split* and a *permutation graph* (see List. 23.1 Lines 6, 10, 14 and Fig. 23.1 on the facing page).

**Listing 23.1** Testing perfect graph categories.

```

1 >>> g.isPerfectGraph(Comments=True)
2 Graph 'randIntervalIntersections' is perfect !
3 >>> g.isIntervalGraph(Comments=True)
4 Graph 'randIntervalIntersections' is triangulated.
5 Graph 'dual_randIntervalIntersections' is transitively orientable.
6 => Graph 'randIntervalIntersections' is an interval graph.
7 >>> g.isSplitGraph(Comments=True)
8 Graph 'randIntervalIntersections' is triangulated.
9 Graph 'dual_randIntervalIntersections' is triangulated.
10 => Graph 'randIntervalIntersections' is a split graph.
11 >>> g.isPermutationGraph(Comments=True)
12 Graph 'randIntervalIntersections' is transitively orientable.
13 Graph 'dual_randIntervalIntersections' is transitively orientable.
14 => Graph 'randIntervalIntersections' is a permutation graph.
15 >>> print(g.computePermutation())
16 ['v5', 'v6', 'v4', 'v2', 'v1', 'v3', 'v7', 'v8']
17 ['v8', 'v6', 'v1', 'v2', 'v3', 'v4', 'v7', 'v5']
18 [8, 2, 6, 5, 7, 4, 3, 1]
19 >>> g.exportGraphViz('randomSplitGraph')
20 *---- exporting a dot file for GraphViz tools -----
21 Exporting to randomSplitGraph.dot
22 fdp -Tpng randomSplitGraph.dot -o randomSplitGraph.png

```

Notice however that the four properties:

1.  $g$  is a *comparability* graph;
  2.  $g$  is a *co-comparability* graph, i.e.  $-g$  is a comparability graph;
  3.  $g$  is a *triangulated* graph;
  4.  $g$  is a *co-triangulated* graph, i.e.  $-g$  is a comparability graph;
- are independent of one another [Golumbic \(2004, p. 275\)](#).

**Fig. 23.1** A conjointly triangulated, comparability, interval, permutation and split graph. In the Figure here one may readily recognise the essential characteristic of split graphs, namely being always splittable into two disjoint sub-graphs: an *independent choice*  $\{v_6\}$  and a *clique*  $\{v_1, v_2, v_3, v_4, v_5, v_7, v_8\}$  – which explains their name



Digraph3 (graphviz), R. Bisdorff, 2019

## 23.2 Who is the liar?

Claude Berge's famous mystery story related by [Golumbic \(2004\)](#), p. 20) may well illustrate the importance of being an *interval graph*.

Suppose the file `berge.py`<sup>1</sup> contains the following Graph instance data:

```

1  vertices = {
2      'A': {'name': 'Abe', 'shortName': 'A'},
3      'B': {'name': 'Burt', 'shortName': 'B'},
4      'C': {'name': 'Charlotte', 'shortName': 'C'},
5      'D': {'name': 'Desmond', 'shortName': 'D'},
6      'E': {'name': 'Eddie', 'shortName': 'E'},
7      'I': {'name': 'Ida', 'shortName': 'I'},
8  }
9  valuationDomain = {'min':-1,'med':0,'max':1}
10 edges = {
11     frozenset(['A','B']) : 1,
12     frozenset(['A','C']) : -1,
13     frozenset(['A','D']) : 1,
14     frozenset(['A','E']) : 1,
15     frozenset(['A','I']) : -1,
16     frozenset(['B','C']) : -1,
17     frozenset(['B','D']) : -1,
18     frozenset(['B','E']) : 1,
19     frozenset(['B','I']) : 1,
20     frozenset(['C','D']) : 1,
21     frozenset(['C','E']) : 1,
22     frozenset(['C','I']) : 1,
```

---

<sup>1</sup> The file `berge.py` is provided in the `examples` directory of the DIGRAPH3 resources.

```

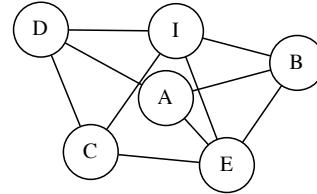
23     frozenset(['D','E']) : -1,
24     frozenset(['D','I']) : 1,
25     frozenset(['E','I']) : 1,
26 }
```

Six professors (labeled A, B, C, D, E and I) had been to the library on the day that a rare tractate was stolen. Each entered once, stayed for some time, and then left. If two professors were in the library at the same time, then at least one of them saw the other. Detectives questioned the professors and gathered the testimonies that *Abe* saw *Burt* and *Eddie*; *Burt* saw *Abe* and *Ida*; *Charlotte* saw *Desmond* and *Ida*; *Desmond* saw *Abe* and *Ida*; *Eddie* saw *Burt* and *Ida*; and *Ida* saw *Charlotte* and *Eddie*. This data is gathered in the previous file, where each positive edge  $\{x,y\}$  models the testimony that, either  $x$  saw  $y$ , or  $y$  saw  $x$ .

```

1 >>> from graphs import Graph
2 >>> g = Graph('berge')
3 >>> g.showShort()
4     ----- short description of the graph -----
5     Name          : 'berge'
6     Vertices      : ['A', 'B', 'C', 'D', 'E', 'I']
7     Valuation domain : {'min': -1, 'med': 0, 'max': 1}
8     Gamma function :
9         A -> ['D', 'B', 'E']
10        B -> ['E', 'I', 'A']
11        C -> ['E', 'D', 'I']
12        D -> ['C', 'I', 'A']
13        E -> ['C', 'B', 'I', 'A']
14        I -> ['C', 'E', 'B', 'D']
15 >>> g.exportGraphViz('bergel')
16     ----- exporting a dot file for GraphViz tools -----
17     Exporting to berge1.dot
18     fdp -Tpng berge1.dot -o berge1.png
```

**Fig. 23.2** Graph representation of the testimonies of the professors



*DiGraph3 (graphviz), R. Bisdom, 2019*

From graph theory we know that time interval intersections graphs must in fact be interval graph instances, i.e. *triangulated* and *co-comparative* graphs. The testimonies graph shown in Figure 23.2 should therefore not contain any chordless cy-

cle of four and more vertices. Now, the presence or not of such chordless cycles in the testimonies graph may be checked with the `computeChordlessCycles()` method (Bisdorff, 2010).

```

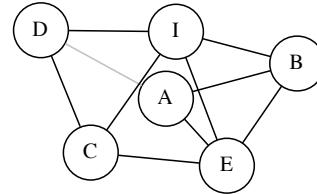
1 >>> g.computeChordlessCycles()
2 Chordless cycle certificate: ['D','C','E','A','D']
3 Chordless cycle certificate: ['D','I','E','A','D']
4 Chordless cycle certificate: ['D','I','B','A','D']
5 [[['D','C','E','A','D'],frozenset({'C','D','E','A'}))],
6 ([['D','I','E','A','D'],frozenset({'D','E','I','A'}))],
7 ([['D','I','B','A','D'],frozenset({'D','B','I','A'})])

```

We see in the Listing above three intersection cycles of length 4, which is impossible to occur on the linear time line. Obviously one professor lied!

And it is *Desmond*. If we doubt his testimony that he saw *Abe* (see Line 1 below), we obtain indeed in Figure 23.3 a triangulated graph instance whose dual is a comparability graph.

**Fig. 23.3** The triangulated testimonies graph



Digraph3 (graphviz), R. Bisdorff, 2019

```

1 >>> g.setEdgeValue( ('D','A'), 0)
2 >>> g.showShort()
3     *---- short description of the graph ----*
4     Name          : 'berge'
5     Vertices      : ['A','B','C','D','E','I']
6     Valuation domain : {'med': 0,'min': -1,'max': 1}
7     Gamma function :
8         A -> ['B', 'E']
9         B -> ['A', 'I', 'E']
10        C -> ['I', 'E', 'D']
11        D -> ['I', 'C']
12        E -> ['A', 'I', 'B', 'C']
13        I -> ['B', 'E', 'D', 'C']
14 >>> g.isIntervalGraph(Comments=True)
15 Graph 'berge' is triangulated.
16 Graph 'dual_berge' is transitively orientable.
17 => Graph 'berge' is an interval graph.
18 >>> g.exportGraphViz('berge2')

```

```

19      *---- exporting a dot file for GraphViz tools -----*
20      Exporting to berge2.dot
21      fdp -Tpng berge2.dot -o berge2.png

```

### 23.3 Generating permutation graphs

A graph is called a *permutation* or *inversion* graph if there exists a permutation of its list of vertices such that the graph is isomorphic to the inversions operated by the permutation in this list (see [Golumbic, 2004](#), Chapter 7, pp 157-170). This kind of graphs (see Fig. 23.4 on the next page) is also part of the class of perfect graphs.

```

1 >>> from graphs import PermutationGraph
2 >>> g = PermutationGraph(permuation = [4, 3, 6, 1, 5, 2])
3 >>> g
4     *----- Graph instance description -----*
5     Instance class    : PermutationGraph
6     Instance name     : permutationGraph
7     Graph Order       : 6
8     Permutation       : [4, 3, 6, 1, 5, 2]
9     Graph Size        : 9
10    Valuation domain : [-1.00; 1.00]
11    Attributes        : ['name', 'vertices', 'order',
12                          'permuation', 'valuationDomain',
13                          'edges', 'size', 'gamma']
14 >>> g.isPerfectGraph()
15 True
16 >>> g.exportGraphViz(fileName='permutationGraph')
17     *---- exporting a dot file for GraphViz tools -----*
18     Exporting to permutationGraph.dot
19     fdp -Tpng permutationGraph.dot \
20             -o permutationGraph.png

```

By using color sorting queues, the minimal vertex colouring shown in Figure 23.5 on the facing page is computable in  $O(n \log(n))$  ([Golumbic, 2004](#)).

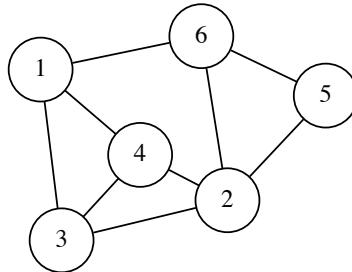
```

1 >>> g.computeMinimalVertexColoring(Comments=True)
2     vertex 1: lightcoral
3     vertex 2: lightcoral
4     vertex 3: lightblue
5     vertex 4: gold
6     vertex 5: lightblue
7     vertex 6: gold
8 >>> g.exportGraphViz(fileName='coloredPermutationGraph', \
9                         WithVertexColoring=True)
10    *---- exporting a dot file for GraphViz tools -----*
11    Exporting to coloredPermutationGraph.dot
12    fdp -Tpng coloredPermutationGraph.dot \
13          -o coloredPermutationGraph.png

```

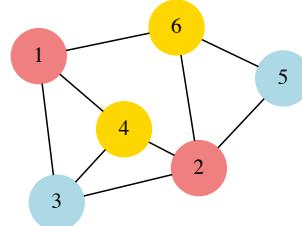
The correspondingly coloured *matching diagram* of the nine inversions –the actual edges of the permutation graph–, which are induced by the given [4,3,6,1,5,2]

**Fig. 23.4** The default permutation graph



Digraph3 (graphviz), R. Bisдорff, 2019

**Fig. 23.5** Minimal vertex coloring of the permutation graph



Digraph3 (graphviz), R. Bisдорff, 2019

permutation, may as well be drawn with the graphviz neato layout and explicitly positioned horizontal lists of vertices as shown in Figure 23.6 on the next page.

```

1 >>> g.exportPermutationGraphViz('matchingDiagram', \
2 ...                               WithEdgeColoring=True)
3 *----- exporting a dot file for GraphViz tools -----*
4   Exporting to matchingDiagram.dot
5   neato -n -Tpng matchingDiagram.dot\
6       -o matchingDiagram.png

```

As mentioned before, a permutation graph and its dual are *transitively orientable*. The `transitiveOrientation()` method constructs from a given permutation graph a digraph where each edge of the permutation graph is converted into an arc oriented in increasing alphabetic order of the adjacent vertices' keys (see Golumbic, 2004). This orientation of the edges of a permutation graph is always transitive and delivers a *transitive ordering* of the vertices as shown in Figure 23.7 on page 341.

```

1 >>> dg = g.transitiveOrientation()

```

**Fig. 23.6** Coloured matching diagram of the permutation  
[4,3,6,1,5,2]



Digraph3 (graphviz), R. Bisдорff, 2019

```

2 >>> dg
3 *----- Digraph instance description -----*
4   Instance class    : TransitiveDigraph
5   Instance name     : oriented_permutationGraph
6   Digraph Order     : 6
7   Digraph Size      : 9
8   Valuation domain : [-1.00; 1.00]
9   Determinateness  : 100.000
10  Attributes       : ['name', 'order', 'actions',
11      'valuationdomain', 'relation',
12      'gamma', 'notGamma', 'size']
13 >>> print('Transitivity degree: %.3f' %\
14 ...          dgd.computeTransitivityDegree() )
15   Transitivity degree: 1.000
16 >>> dg.exportGraphViz(fileName='orientedPermGraph')
17 *---- exporting a dot file for GraphViz tools ----*
18   Exporting to orientedPermGraph.dot
19   dot -Grankdir=TB -Tpng orientedPermGraph.dot\
20           -o orientedPermGraph.png

```

The dual of a permutation graph is again a permutation graph and as such also transitively orientable.

```

1 >>> dgd = (-g).transitiveOrientation()
2 >>> print('Dual transitivity degree: %.3f' %\
3 ...          dgd.computeTransitivityDegree() )
4   Dual transitivity degree: 1.00

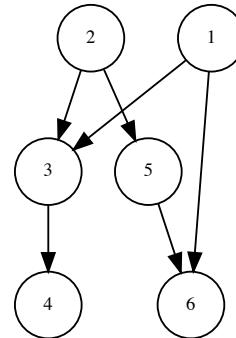
```

## 23.4 Recognising permutation graphs

Now, a given graph  $g$  is a permutation graph if and only if both  $g$  and  $-g$  are transitively orientable. This property gives a polynomial test procedure (in  $O(n^3)$ ) due to the transitivity check) for recognising permutation graphs.

Let us consider, for instance, the following random graph of order 8 generated with an edge probability of 40% and a random seed equal to 4335.

**Fig. 23.7** The transitive orientation of the permutation graph



DiGraph3 (graphviz)  
R. Bisdorff, 2020

```

1 >>> from graphs import RandomGraph
2 >>> g = RandomGraph(order=8,\n3 ...                                edgeProbability=0.4,seed=4335)
4 >>> g
5 *----- Graph instance description -----*
6     Instance class      : RandomGraph
7     Instance name       : randomGraph
8     Seed                : 4335
9     Edge probability   : 0.4
10    Graph Order        : 8
11    Graph Size         : 10
12    Valuation domain  : [-1.00; 1.00]
13    Attributes         : ['name', 'order', 'vertices',
14                           'valuationDomain', 'seed',
15                           'edges', 'size', 'gamma',
16                           'edgeProbability']
17 >>> g.isPerfectGraph()
18     True
19 >>> g.exportGraphViz(fileName='randomGraph4335')
20 *---- exporting a dot file for GraphViz tools ----*
21     Exporting to randomGraph4335.dot
22     fpdf -Tpdf randomGraph4335.dot -o randomGraph4335.pdf

```

If the random perfect graph instance `g`, shown in Figure 23.8 on the next page, is indeed a permutation graph, `g` and its dual `-g` are both *transitively orientable*, i.e. comparability graphs (Golumbic, 2004). With the `isComparabilityGraph()` test, we may easily check this fact. This method proceeds indeed by trying to construct a transitive neighbourhood decomposition of a given graph instance and, if successful, stores the resulting edge orientations into a `edgeOrientations` attribute (Golumbic, 2004, p.129-132).

**Fig. 23.8** Random graph of order 8 generated with edge probability 0.4



Digraph3 (graphviz), R. Bischoff, 2019

```

1 >>> if g.isComparabilityGraph():
2 ...     print(g.edgeOrientations)
3 {('v1','v1'): 0, ('v1','v2'): 1, ('v2','v1'): -1, ('v1','v3'): 1,
4 ('v3','v1'): -1, ('v1','v4'): 1, ('v4','v1'): -1, ('v1','v5'): 0,
5 ('v5','v1'): 0, ('v1','v6'): 1, ('v6','v1'): -1, ('v1','v7'): 0,
6 ('v7','v1'): 0, ('v1','v8'): 1, ('v8','v1'): -1, ('v2','v2'): 0,
7 ('v2','v3'): 0, ('v3','v2'): 0, ('v2','v4'): 0, ('v4','v2'): 0,
8 ('v2','v5'): 0, ('v5','v2'): 0, ('v2','v6'): 0, ('v6','v2'): 0,
9 ('v2','v7'): 0, ('v7','v2'): 0, ('v2','v8'): 0, ('v8','v2'): 0,
10 ('v3','v3'): 0, ('v3','v4'): 0, ('v4','v3'): 0, ('v3','v5'): 0,
11 ('v5','v3'): 0, ('v3','v6'): 0, ('v6','v3'): 0, ('v3','v7'): 0,
12 ('v7','v3'): 0, ('v3','v8'): 0, ('v8','v3'): 0, ('v4','v4'): 0,
13 ('v4','v5'): 0, ('v5','v4'): 0, ('v4','v6'): 0, ('v6','v4'): 0,
14 ('v4','v7'): 0, ('v7','v4'): 0, ('v4','v8'): 0, ('v8','v4'): 0,
15 ('v5','v5'): 0, ('v5','v6'): 1, ('v6','v5'): -1, ('v5','v7'): 1,
16 ('v7','v5'): -1, ('v5','v8'): 1, ('v8','v5'): -1, ('v6','v6'): 0,
17 ('v6','v7'): 0, ('v7','v6'): 0, ('v6','v8'): 1, ('v8','v6'): -1,
18 ('v7','v7'): 0, ('v7','v8'): 1, ('v8','v7'): -1, ('v8','v8'): 0}
19 >>> g.exportEdgeOrientationsGraphViz('transOrientGraph')
20 *----- exporting a dot file for GraphViz tools -----*
21 Exporting to transOrientGraph.dot
22 fdp -Tpng transOrientGraph.dot \
      -o transOrientGraph.png

```

The resulting orientation of the edges of  $g$ , shown in Figure 23.9 on the facing page, is indeed transitive. The same procedure applied to the dual graph  $gd = -g$  gives as well a transitive orientation to the edges of  $-g$  shown in Figure 23.10 on the next page.

```

1 >>> gd = -g
2 >>> if gd.isComparabilityGraph():
3 ...     print(gd.edgeOrientations)
4 {('v1','v1'): 0, ('v1','v2'): 0, ('v2','v1'): 0, ('v1','v3'): 0,
5 ('v3','v1'): 0, ('v1','v4'): 0, ('v4','v1'): 0, ('v1','v5'): 1,
6 ('v5','v1'): -1, ('v1','v6'): 0, ('v6','v1'): 0, ('v1','v7'): 1,
7 ('v7','v1'): -1, ('v1','v8'): 0, ('v8','v1'): 0, ('v2','v2'): 0,
8 ('v2','v3'): -2, ('v3','v2'): 2, ('v2','v4'): -3, ('v4','v2'): 3,
9 ('v2','v5'): 1, ('v5','v2'): -1, ('v2','v6'): 1, ('v6','v2'): -1,
10 ('v2','v7'): 1, ('v7','v2'): -1, ('v2','v8'): 1, ('v8','v2'): -1,
11 ('v3','v3'): 0, ('v3','v4'): -3, ('v4','v3'): 3, ('v3','v5'): 1,

```

**Fig. 23.9** Transitive neighbourhoods of the graph  $g$



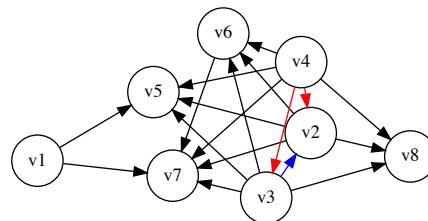
Digraph3 (graphviz), R. Bisдорff, 2019

```

12   ('v5','v3'): -1, ('v3','v6'): 1, ('v6','v3'): -1, ('v3','v7'): 1,
13   ('v7','v3'): -1, ('v3','v8'): 1, ('v8','v3'): -1, ('v4','v4'): 0,
14   ('v4','v5'): 1, ('v5','v4'): -1, ('v4','v6'): 1, ('v6','v4'): -1,
15   ('v4','v7'): 1, ('v7','v4'): -1, ('v4','v8'): 1, ('v8','v4'): -1,
16   ('v5','v5'): 0, ('v5','v6'): 0, ('v6','v5'): 0, ('v5','v7'): 0,
17   ('v7','v5'): 0, ('v5','v8'): 0, ('v8','v5'): 0, ('v6','v6'): 0,
18   ('v6','v7'): 1, ('v7','v6'): -1, ('v6','v8'): 0, ('v8','v6'): 0,
19   ('v7','v7'): 0, ('v7','v8'): 0, ('v8','v7'): 0, ('v8','v8'): 0}
20 >>> gd.exportEdgeOrientationsGraphViz('transOrientDualGraph')
21 *---- exporting a dot file for GraphViz tools ----*
22 Exporting to transOrientDualGraph.dot
23 fdp -Tpng transOrientDualGraph.dot \
24   -o transOrientDualGraph.png
  
```

**Fig. 23.10** Transitive neighbourhoods of the dual graph

$-g$ . It is worthwhile noticing that the orientation of  $g$  is achieved with a single neighbourhood decomposition, covering all the vertices. Whereas, the orientation of the dual graph  $-g$  here needs a decomposition into three subsequent neighbourhoods marked in black, red and blue



Digraph3 (graphviz), R. Bisдорff, 2019

Let us recheck these facts by explicitly constructing transitively oriented digraph instances with the `computeTransitivelyOrientedDigraph()` method.

```

1 >>> og = g.computeTransitivelyOrientedDigraph(\n2 ...           PartiallyDetermined = True)\n3 >>> print('Transitivity degree: %.3f' %\n4 ...           (og.transitivityDegree))\n5   Transitivity degree: 1.000\n6 >>> ogd = (-g).computeTransitivelyOrientedDigraph(\n7 ...           PartiallyDetermined = True)\n8 >>> print('Transitivity degree: %.3f' %\n9 ...           (ogd.transitivityDegree))\n10  Transitivity degree: 1.000

```

The `PartiallyDetermined = True` flag (see Lines 2 and 7 above) is required here in order to orient only the actual edges of the graphs. Relations between vertices not linked by an edge are put to the indeterminate characteristic value 0. This allows us to compute, later on, convenient disjunctive digraph fusions.

As both graphs are indeed transitively orientable (see Lines 5 and 10 above), we may conclude that the given random graph `g` is actually a permutation graph instance. Yet, we still need to find now its corresponding permutation. We therefore implement a recipee given by Martin Golumbic ([Golumbic, 2004](#), p. 159).

We will first *fuse* both `og` and `ogd` orientations above with an *epistemic disjunction* operated with the symmetric o-max operator (see Sec. 2.5).

```

1 >>> from digraphs import FusionDigraph\n2 >>> f1 = FusionDigraph(og, ogd, operator='o-max')\n3 >>> s1 = f1.computeCopelandRanking()\n4 >>> print(s1)\n5   ['v5','v7','v1','v6','v8','v4','v3','v2']

```

With the `computeCopelandRanking()` method, we obtain the linear ordering `[v5, v7, v1, v6, v8, v4, v3, v2]` of the vertices (see Line 5 above).

We reverse now the orientation of the edges in `og` (see `-og` in Line 1 below) in order to generate, again by disjunctive fusion, the *inversions* that are produced by the permutation we are looking for.

Computing again a ranking with the COPELAND rule, reveals the permuted list of vertices we are looking for (see Line 4 below).

```

1 >>> f2 = FusionDigraph((-og), ogd, operator='o-max')\n2 >>> s2 = f2.computeCopelandRanking()\n3 >>> print(s2)\n4   ['v8','v7','v6','v5','v4','v3','v2','v1']

```

Vertex `v8` is put from position 5 to position 1, vertex `v7` is put from position 2 to position 1, vertex `v6` from position 4 to position 3, vertex `v5` from position 1 to position 4, etc ... . We generate these position swaps for all vertices and obtain thus the required permutation (see Line 5 below).

```

1 >>> permutation = [0 for j in range(g.order)]\n2 >>> for j in range(g.order):\n3   ...     permutation[s2.index(s1[j])] = j+1\n4 >>> print(permutation)

```

```
5 [5, 2, 4, 1, 6, 7, 8, 3]
```

It is worthwhile noticing by the way that transitive orientations of a given graph and its dual are usually *not unique* and, so may also be the resulting permutations. However, they all correspond to isomorphic graphs (Golumbic, 2004). In our case here, we observe two different permutations and their reverses:

```
1 s1: ['v1', 'v4', 'v3', 'v2', 'v5', 'v6', 'v7', 'v8']
2 s2: ['v4', 'v3', 'v2', 'v8', 'v6', 'v1', 'v7', 'v5']
3 (s1 -> s2): [2, 3, 4, 8, 6, 1, 7, 5]
4 (s2 -> s1): [6, 1, 2, 3, 8, 5, 7, 4]
```

And

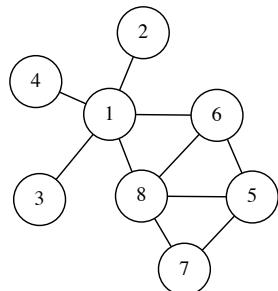
```
1 s3: ['v5', 'v7', 'v1', 'v6', 'v8', 'v4', 'v3', 'v2']
2 s4: ['v8', 'v7', 'v6', 'v5', 'v4', 'v3', 'v2', 'v1']
3 (s3 -> s4): [5, 2, 4, 1, 6, 7, 8, 3]
4 (s4 -> s3) = [4, 2, 8, 3, 1, 5, 6, 7]
```

The `computePermutation()` method does directly operate all these steps: - computing transitive orientations, - ranking their epistemic fusion and, - delivering a corresponding permutation.

```
1 >>> g.computePermutation(Comments=True)
2 ['v1', 'v2', 'v3', 'v4', 'v5', 'v6', 'v7', 'v8']
3 ['v2', 'v3', 'v4', 'v8', 'v6', 'v1', 'v7', 'v5']
4 [2, 3, 4, 8, 6, 1, 7, 5]
```

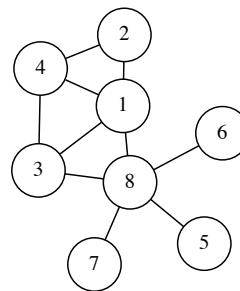
Let us finally check in Figure 23.11 that the two permutations [2, 3, 4, 8, 6, 1, 7, 5] and [4, 2, 8, 3, 1, 5, 6, 7], observed above, will generate corresponding *isomorphic* permutation graphs.

[2, 3, 4, 8, 6, 1, 7, 5]



*Digraph3 (graphviz), R. Bisдорff, 2019*

[4, 2, 8, 3, 1, 5, 6, 7]



*Digraph3 (graphviz), R. Bisдорff, 2019*

**Fig. 23.11** Isomorphic permutation graphs

```

1 >>> gtesta = PermutationGraph(\n2 ...           permutation=[2, 3, 4, 8, 6, 1, 7, 5])\n3 >>> gtestb = PermutationGraph(\n4 ...           permutation=[4, 2, 8, 3, 1, 5, 6, 7])\n5 >>> gtesta.exportGraphViz('gtesta')\n6 >>> gtestb.exportGraphViz('gtestb')

```

And, we indeed recover indeed two isomorphic copies of the original random graph (compare with Fig. 23.8 on page 342).

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## Appendix

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