Criteria significance elicitation from the CONDORCET robustness of bipolar-valued outranking relations

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Abstract

We propose an indirect approach for assessing robust criteria significance weights from global outranking statements that a decision maker may acknowledge in a Multiple Criteria Decision Aid process. Our approach takes advantage of the bipolar characteristic valuation of the global outranking and the eventual robustness of its associated CONDORCET, i.e. median cut, crisp outranking relation. Numerical criteria significance weights are inferred with the help of a mixed integer linear programming model of which we present run tests results under different configurations and tunings.

Keywords: inverse multiple criteria decision analysis, criteria significance elicitation, robust bipolar-valued outranking

1. Introduction

We consider a decision situation in which a finite set of decision alternatives is evaluated on a finite family of performance criteria. A decision maker is willing to pairwisely compare these alternatives according to the outranking paradigm. One considers indeed that an alternative *a outranks* an alternative *b* when a *significant majority* of criteria validates the fact that *a* is performing at least as good as *b* and there is no criterion where *b* seriously outperforms *a* [1]. To assess when such a significant majority of criteria validates an outranking situation requires a more or less precise numerical knowledge of the significance of each criterion in the multiple criteria preference aggregation. Two different approaches exist to specify theses values:

- either via *direct* preference information, where the criteria significance is first assessed and then the aggregated outranking situations are computed,

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- or, via *indirect* preference information, where some a priori partial knowledge of the resulting aggregated outranking is used in order to infer plausible estimators of the criteria significance.

In this article we exclusively concentrate on the indirect preference information approach. Similar approaches, mostly in the domain of Multiple Attribute Value Theory, already appeared in the literature where they are generally called *disaggregation/aggregation* or *ordinal regression* methods [2, 3, 4, 5, 6, 7, 8]. In analogy with corresponding techniques in inferential statistics, we prefer to group all indirect preference information modeling techniques under the generic term *inverse Multiple Criteria Decision Analysis.* The innovative a priori knowledge on which we focus our inverse analysis here is the robustness of the significant majorities that the decision maker acknowledges for his pairwise comparisons with respect to all potential significance weights, a fact we call the CONDORCET robustness of the outranking situation in the sequel of this article. The main result of our article is to show that this kind of a priori knowledge is sufficient on its own for estimating numerical significance weights.

The article is organised as follows: Firstly, we recall the CONDORCET robustness denotation of valued outranking relations and the way of computing it. Afterwards, in Section 3 we present a mathematical model for estimating significance weights, followed in Section 4, by some experiments on the impact of using CONDORCET constraints.

2. Defining and computing the CONDORCET robustness denotation of valued outranking relations

In this section, we briefly recall the definition of the CONDORCET robustness denotation and the way how to compute it. Further details can be found in [9, 10].

Let $A = \{x, y, z, \ldots\}$ be a finite set of $n > 1$ potential decision alternatives and $F = \{g_1, \ldots, g_m\}$ a coherent finite family of $m > 1$ criteria.

The alternatives are evaluated on each criterion on real performance scales to which an indifference q_i and a preference p_i discrimination threshold (for all g_i in F) are associated [1]. The performance of alternative x on criterion g_i is denoted x_i.

In order to characterize a marginal *at least as good as* situation [11, 12] between any two alternatives x and y of A, a double threshold order S_i is associated with each criterion g_i whose numerical representation is given by:

$$S_i(x,y) = \begin{cases} 1 & \text{if} \quad x_i + q_i \geqslant y_i \ , \\ -1 & \text{if} \quad x_i + p_i \leqslant y_i \ , \\ 0 & \text{otherwise.} \end{cases}$$

We associate furthermore with each criterion $g_i \in F$ a rational *significance weight* w_i which represents the contribution of g_i to the *overall warrant or not* of the *at least as good as* preference situation between all pairs of alternatives. Let $W = (w_1, .., w_m)$ be the vector of relative significance weights associated with F such that $0 < w_i < 1$ ($\forall g_i \in F$) and $\sum_{g_i \in F} w_i = 1$ and let $\mathcal{W}$ be the set of such significance weights vectors.

In this paper, as it will be shown later, we may without loss of generality ignore the veto principle normally taken into account when dealing with classical outranking relation [1]. The *overall valued outranking relation*, denoted $\widetilde{S}^W$, aggregating the partial *at least as good as* situations, is then given by:

$$\widetilde{S}^W(x,y) = \sum_{w_i \in W} w_i \cdot S_i(x,y), \ \forall (x,y) \in A \times A.$$

$\widetilde{S}^W(x,y)$ is thus evaluated in the rational interval $[-1,1]$ with the following semantics [11]:

- $\widetilde{S}^W(x,y) = 1$ indicates that all criteria warrant unanimously the "*at least as good as*" preference situation between x and y;
- $\widetilde{S}^W(x,y) > 0$ indicates that a majority of criteria warrant the "*at least as good as*" preference situation between x and y;
- $\widetilde{S}^W(x,y) = 0$ indicates a balanced situation where the criteria warranting the "*at least as good as*" preference situation between x and y are exactly as significant as those who do not warrant this situation;
- $\widetilde{S}^W(x,y) < 0$ indicates that a majority of criteria do not warrant the "*at least as good as*" preference situation between x and y;
- $\widetilde{S}^W(x,y) = -1$ indicates that all criteria do not warrant unanimously the "*at least as good as*" preference situation between x and y.

Let $\succsim_W$ be the preorder[1] on F associated with the natural $\geqslant$ relation on the vector of significance weights W. $\sim_W$ induces r ordered equivalence classes $\Pi_1^W \succ_W \ldots \succ_W \Pi_r^W$ $(1 \leq r \leq m)$. The criteria gathered in each equivalence class have the same significance weight in W and for $i < j$, those of Π_i^W have a higher significance weight than those of Π_j^W. Let $\mathcal{W}_{\succsim_W} \subset \mathcal{W}$ denote the set of all significance weights vectors that are preorder-compatible with $\succsim_W$.

Let $W \in \mathcal{W}$. The Condorcet *robustness denotation*[2] [9] of $\widetilde{S}^W$, denoted $[\![\widetilde{S}^W]\!]$, is defined, for all $(x,y) \in A \times A$, as follows:

$$[\![\widetilde{S}^W]\!](x,y) = \begin{cases} 3 & \text{if } \widetilde{S}^V(x,y) = 1 \ \ \forall V \in \mathcal{W}\,; \\ 2 & \text{if } \big[\widetilde{S}^V(x,y) > 0 \ \ \forall V \in \mathcal{W}_{\succsim_W}\big] \ \wedge \ \big[\exists V' \in \mathcal{W} \ : \ \widetilde{S}^{V'}(x,y) < 1\big]\,; \\ 1 & \text{if } \big[\widetilde{S}^W(x,y) > 0\big] \ \wedge \ \big[\exists V' \in \mathcal{W}_{\succsim_W} \ : \ \widetilde{S}^{V'}(x,y) \leqslant 0\big]\,; \\ 0 & \text{if } \widetilde{S}^W(x,y) = 0\,; \\ -1 & \text{if } \big[\widetilde{S}^W(x,y) < 0\big] \ \wedge \ \big[\exists V' \in \mathcal{W}_{\succsim_W} \ : \ \widetilde{S}^{V'}(x,y) \geqslant 0\big]\,; \\ -2 & \text{if } \big[\widetilde{S}^V(x,y) < 0 \ \ \forall V \in \mathcal{W}_{\succsim_W}\big] \ \wedge \ \big[\exists V' \in \mathcal{W} \ : \ \widetilde{S}^{V'}(x,y) > 0\big]\,; \\ -3 & \text{if } \widetilde{S}^V(x,y) = -1 \ \ \forall V \in \mathcal{W}\,; \end{cases}$$

[1]As classically done, $\succ_W$ denotes the asymmetric part of $\succsim_W$, whereas $\sim$ denotes its symmetric part.

[2]The simple majority validated outranking relation $S^W(x,y)$ such that $\widetilde{S}^W(x,y) > 0$ is generally called the Condorcet *relation* (see Barbut [13]), in honours of the Marquis de Condorcet (1743–1794) who first promoted social choice procedures based on pairwise simple majority votings.

with the following semantics:

- $[\![\widetilde{S}^W]\!](x,y) = \pm 3$ if all criteria *unanimously warrant* (resp. *do not warrant*) the outranking situation between x and y;
- $[\![\widetilde{S}^W]\!](x,y) = \pm 2$ if a *significant majority* of criteria *warrants* (resp. *does not warrant*) the outranking situation between x and y for all $\succsim_W$-compatible weights vectors;
- $[\![\widetilde{S}^W]\!](x,y) = \pm 1$ if a significant majority of criteria *warrants* (respectively *does not warrant*) this outranking situation for W but not for all $\succsim_W$-compatible weights vectors;
- $[\![\widetilde{S}^W]\!](x,y) = 0$ if the total significance of the warranting criteria is *exactly balanced* by the total significance of the not warranting criteria for W.

An outranking situation between two alternatives is said to be Condorcet *robust* when the associated value is equal to ± 3 or ± 2. The careful reader will notice that, in the presence of veto thresholds as defined in [12], if a veto situation occurs in the comparison of a couple of alternatives, the associated Condorcet robustness denotation is fixed by convention to -3, as the overall outranking relation $\widetilde{S}^W$ equals -1, disregarding the criteria significance weights. In our context of inverse analysis, as no preferential information from this situation can help us to capture criteria weights, we may without loss of generality ignore the veto principle.

Let us start by presenting the notation which allows us to detail the construction of the Condorcet robustness denotation associated with a valued outranking relation $\widetilde{S}^W$ and a significance weights vector W.

Let $c_k^W(x,y)$ be the sum of "*at least as good as*" characteristics $S_i(x,y)$ for all criteria $g_i \in \Pi_k^W$. Furthermore, let $C_k^W(x,y) = \sum_{i=1}^{k} c_i^W(x,y)$ be the cumulative sum of "*at least as good as*" characteristics for all criteria having significance at least equal to the one associated with Π_k^W, for all k in $\{1,\ldots,r\}$.

In the absence of ± 3 denotations, the following proposition gives us a test for the presence of a ± 2 denotation:

Proposition 1 (Bisdorff [9]).

$$[\![\widetilde{S}^W]\!](x,y) = 2 \iff \begin{cases} \forall k \in 1,...,r : C_k^W(x,y) \geqslant 0\,; \\ \exists k \in 1,...,r : C_k^W(x,y) > 0. \end{cases}$$

The negative -2 denotation corresponds to similar conditions with reversed inequalities.

The ± 2 denotation test of Proposition 1 corresponds in fact to the verification of *stochastic dominance*-like conditions (see [9]).

A ± 1 Condorcet robustness denotation, corresponding to the observation of a weighted majority (resp. minority) in the absence of the ± 2 case, is simply verified as follows:

$$[\![\widetilde{S}^W]\!](x,y) = \pm 1 \iff \big((\widetilde{S}^W(x,y) \gtrless 0) \,\wedge\, [\![\widetilde{S}^W]\!](x,y) \neq \pm 2 \big).$$

Table 1: Performance table

	g_1	g_2	g_3
a	3	2	1
b	2	3	3
c	1	1	2

Table 2: $W = \{4, 3, 2\}$

	$\widetilde{S}^W$			$[\![\widetilde{S}^W]\!]$		
	a	b	c	a	b	c
a	1.00	-0.11	0.55	3	-1	2
b	0.11	1.00	1.00	1	3	3
c	-0.55	-1.00	1.00	-2	-3	3

Example 1. *Consider a set $A = \{a, b, c\}$ of three alternatives and a coherent family F of three criteria $\{g_1, g_2, g_3\}$ measuring performances on a discrete ordinal scale from 1 to 3. All criteria have to be maximized. In order to simplify the example, we are not considering any discrimination thresholds.*

Table 1 presents the performances of the alternatives on each of the criteria. Table 2 shows the outranking relation and the associated CONDORCET *relation as a result of the use of the weights vector $W = (4, 3, 2)$. W induces the significance ordering $\{g_1\} \succ_W \{g_2\} \succ_W \{g_3\}$, i.e. $\Pi_1^W = \{g_1\}$, $\Pi_2^W = \{g_2\}$ and $\Pi_3^W = \{g_3\}$.*

As b is unanimously dominating c, $[\![\widetilde{S}^W]\!](b, c) = 3$ and $[\![\widetilde{S}^W]\!](c, b) = -3$. Considering the ordered pair (a, b), as $C_1^W(a, b) = 1 > 0$, $C_2^W(a, b) = 0$ and $C_3^W(a, b) = -1 < 0$, the outranking statement is not robustly warranted. As $\widetilde{S}^W(a, b) < 0$, $[\![\widetilde{S}^W]\!](a, b) = -1$.

We define $\succsim_{W'}$ as a *less discriminated preorder* than $\succsim_W$ on F if and only if it respects the two following conditions:

$$g_i \sim_W g_j \Rightarrow g_i \sim_{W'} g_j \quad \forall g_i, g_j \in F,$$
$$g_i \succ_W g_j \Rightarrow g_i \succsim_{W'} g_j \quad \forall g_i, g_j \in F.$$

In other words, $\succsim_{W'}$ is obtained by joining some adjacent classes in $\succsim_W$ together, reducing the discrimination of any associated weights vectors W'. We then deduce the following corollary:

Corollary 1. *If an outranking situation between two alternatives is associated with a robust* CONDORCET *denotation based on a preorder $\succsim_W$, then this denotation won't change when replacing $\succsim_W$ by any less discriminating preorder $\succsim_{W'}$.*

The proof is obvious when noticing that the set of constraints on the cumulated sums to be verified in order to validate a robust statement using any weights vector W' compatible with $\succsim_{W'}$ is included in the set of constraints validating a robust statement using any weights vector W compatible with $\succsim_W$. Consequently, it is natural to use criteria weights vectors the less possibly discriminated (*i.e.* allowing a difference between two criteria weights only if mandatory to validate some given statements) in order to ensure the outranking relation the greatest number of robust statements.

Example 2. *Back to our example, we now compute, in Table 3, the outranking relation and the associated* CONDORCET *relation, using a weights vectore $W' = (2, 2, 1)$, less discriminated than W. Notice that it reconstructs the same median cut outranking*

Table 3: $W' = \{2, 2, 1\}$

	$\widetilde{S}^{W'}$			$[\![\widetilde{S}^{W'}]\!]$		
	a	b	c	a	b	c
a	1.00	-0.20	0.60	3	**-2**	2
b	0.20	1.00	1.00	**2**	3	3
c	-0.60	-1.00	1.00	-2	-3	3

relation than W, but improves the robustness of the statements between a and b. Indeed, as $\Pi_1^{W'} = \{g_1, g_2\}$ and $\Pi_2^{W'} = \{g_3\}$, $C_1^{W'}(a, b) = 0 \leqslant 0$ and $C_2^{W'}(a, b) = -1 < 0$, then $[\![\widetilde{S}^{W'}]\!](a, b) = -2$.

Another important property for our purpose is noted in [14]. Let $W_{\mathbf{1}}$ be the weights vector for which all the criteria weights equal 1. Then:

Proposition 2.

$$\widetilde{S}^{W_{\mathbf{1}}}(x, y) < 0 \Rightarrow \forall W \in \mathcal{W},\ [\![\widetilde{S}^{W}]\!](x, y) \leq 1$$
$$\widetilde{S}^{W_{\mathbf{1}}}(x, y) > 0 \Rightarrow \forall W \in \mathcal{W},\ [\![\widetilde{S}^{W}]\!](x, y) \geq -1$$

In other words, when more than half of the criteria are invalidating an outranking situation, it is impossible to find a Condorcet-robust vector of criteria weights for warranting this situation. Similarly, when more than half of the criteria are validating an outranking situation, it is impossible to find a Condorcet-robust vector of criteria weights for warranting the absence of this situation. Indeed, it is simply impossible to robustly warrant an outranking statement going against the one obtained with an equi-significant weights vector.

3. Inverse analysis for weight elicitation

We present in this section a mathematical model eliciting criteria weights from partial global outrankings confirmed by a decision maker, using the Condorcet denotation in order to improve the overall robustness of the resulting complete outranking relation. Firstly, we recall the modeling of the Condorcet robustness test (see Proposition 1) by linear constraints that may force the robustness denotation of some global outranking statements, as it appears in [10]. Then, we recall the kind of preferential information on alternatives a decision maker can provide and how it can be integrated in the model. Finally, we enrich the model by potentially adding preferential information on the relative significance of the criteria.

3.1. Condorcet *robustness denotation constraints*

We may notice that no constraint can be formulated in order to force a ± 3 Condorcet denotation. Indeed, we can ignore such unanimous situations, positive or negative, as they concern a trivial pairwise comparison situation between Pareto dominant (resp. dominated) alternatives. The outranking situation is anyhow then unanimously warranted (resp. unwarranted), disregarding every possible significance of the criteria. Those denotations don't give us any specific information for the elicitation of the significance weights.

Let $A^2_{\pm 2}$ (resp. $A^2_{\pm 1}$ or A^2_0) bet the set of ordered pairs (x, y) of alternatives such that $[\![\widetilde{S}^w]\!](x, y) = \pm 2$ (resp. ± 1 or 0).

As criteria significance weights are supposed to be rational, we can, without any lost of generality, restrict our assessment problem to integer weights vectors. Hence, an integer weight $w_i \in [1, M]$ will be associated with each criterion g_i, M standing for the maximal admissible value. For the practical resolution of real decision problems, this bound may be set equal to the number m of criteria.

Let $P_{m\times M}$ be a Boolean matrix with generic term $[p_{i,u}]$, characterizing, for each line i, the number of weights units allocated to criterion g_i. Formally, line i^{th} represents the decomposition of the weight associated with g_i on M bits in a unary base, in such a way that $\sum_{u=1}^{M} p_{i,u} = w_i$. For example, if g_i is associated with an integer weights equal to 3, and if $M = 5$, then the i^{th} line of the matrix $P_{m\times 5}$ will be $(1, 1, 1, 0, 0)$.

As each criterion weight must be strictly positive, we easily deduce that at least one weight unit is allocated to each criterion, *i.e.* $p_{i,1} = 1$ for all $g_i \in F$. We obtain the following constraint:

$$\sum_{g_i \in F} p_{i,1} = m.$$

The following constraints warrant the integrity of P, the most significant bits grouped together on the left side of the lines:

$$p_{i,u} \geqslant p_{i,u+1} \quad (\forall i = 1, ..., m,\ \forall u = 1, ..., M - 1).$$

Let us introduce, for every pair $(x, y) \in A^2_{\pm 2}$, the following set of constraints, allowing to represent the $[\![\widetilde{S}^w]\!](x, y) = \pm 2$ condition:

$$\sum_{g_i \in F} \Big(p_{i,u} \cdot \pm S_i(x, y)\Big) \ \geqslant\ b_u(x, y) \quad (\forall u = 1, ..., M),$$

where $b_u(x, y)$ are boolean variables defined for every pair of alternatives and every equi-significance level $u \in \{1, \ldots, M\}$. Those binary variables dictate at least one strict inequality case for every pair of alternatives, as requested in Proposition 1, via the following constraints:

$$\sum_{u=1}^{m} b_u(x, y) \ \geqslant\ 1, \quad \big(\forall (x, y) \in A^2_{\pm 2}\big).$$

In order to model the $[\![\widetilde{S}^w]\!](x, y) = \pm 1$ situation, we formulate the following constraint, for every pair $(x, y) \in A^2_{\pm 1}$:

$$\sum_{g_i \in F} \big(\sum_{u=1}^{M} p_{i,u}\big) \cdot \pm S_i(x, y) \ \geqslant\ 1 \ \ \forall (x, y) \in A^2_{\pm 1},$$

where $\sum_{u=1}^{M} p_{i,u}$ represents the estimated weight of criterion g_i.

Similarly, for every pair $(x, y) \in A_0^2$, we formulate the following constraint:

$$\sum_{g_i \in F} \left(\sum_{u=1}^{M} p_{i,u} \right) \cdot S_i(x, y) = 0.$$

Notice that using such a constraint imposes a balanced situation for a couple of alternatives. In practice, if a decision maker is unable to express a relation between two alternatives, it may be more advisable not to constrain a balanced statement.

3.2. Constraint relaxation using slack variables

In practice, when dealing with preferential information, we will have to face incompatibilities (one can express a set of preferences that are impossible to realize simultaneously) and difficulties to warrant all the desired robust outranking statements, when those are incompatible or, according to Proposition 2, when a robust statement can't ever be obtained.

We then introduce the notion of *relaxed constraints*, obtained from *original constraints* by adding *slack variables*. Those positive real variables, which have to be minimized, avoid not finding any solution by relaxing the constraints incompatible with the underlying problem. Consequently, the problem can be solved and the problematic constraints can be easily identified when analyzing the solution, by looking for non zero slack variables. Depending on the value of the denotation, consequences are different:

- For a ± 1 or 0 denotation constraint, a non zero slack variable indicates that the outranking statement has not been satisfied, *i.e.* it is impossible to find a vector of criteria weights satisfying both desired outranking statements;
- For a ± 2 denotation constraint, a slack variable allows to ensure the resolution of the mathematical problem even if robustness can't be reached for some statements.

In consequence, slack variables are used for ± 1 and 0 denotation constraints when the decision maker isn't positive about an outranking statement or when he is ready to modify his judgement about some statements which are incompatible with others that he consider more credible. For ± 2 denotation constraints, slack variables allow the resolution of the mathematical problem even if it fails to ensure the robustness for some constraints. We should notice that, when a ± 2 denotationconstraint is violated, *i.e.* the slack variable associated with the constraint has reached a non zero value, the outranking statement isn't warranted: to warrant the outranking statement, we have to add a ± 1 denotation original constraint.

Let us now present the constraints in their relaxed versions:

$$\sum_{g_i \in F} \Big(p_{i,u} \cdot \pm S_i(x,y) \Big) \ \pm\ s^{\pm 2}(x,y) \ \geqslant\ b_u(x,y) \quad \forall u = 1,...,M \quad \forall (x,y) \in A^2_{\pm 2}$$
$$\sum_{g_i \in F} \big(\textstyle\sum_{u=1}^{M} p_{i,u} \big) \cdot \pm S_i(x,y) \ \pm\ s^{\pm 1}(x,y) \ \geqslant\ 1 \qquad \forall (x,y) \in A^2_{\pm 1}$$
$$\sum_{g_i \in F} \big(\textstyle\sum_{u=1}^{M} p_{i,u} \big) \cdot S_i(x,y) \ +\ s^0_+(x,y) \ -\ s^0_-(x,y) \ =\ 0 \qquad \forall (x,y) \in A^2_0$$

where $s^{\pm 2}(x,y)$ (resp. $s^{\pm 1}(x,y)$) are slack variables associated with the ordered pair (x,y). A careful reader may notice the fact that, for a 0 denotation constraint, two slack variables are needed, $s^0_+(x,y)$ and $s^0_-(x,y)$ in order to know whether the outranking statement associated with the violated constraint has become positive or negative.

3.3. Taking into account apparent decision maker's preferences

We propose to identify preferential information that a decision maker can provide, in the context of our decision problem, in order to integrate them in the inverse analysis model. These information can take the form of:

- a subset $E \subseteq A \times A$ of couples of alternatives (a,b) for which a decision maker is able to express a strict preference or an indifference;
  Example: a is preferred to b, c and d are indifferent;
- a partial preorder $\succeq_N$ over the weights of a subset of criteria $N \subseteq F$;
  Example: criterion g_1 is more valuable than criterion g_4;
- some numerical values associated with the importance of some criteria;
  Example: criterion g_2 weight value is equal to 3;
- some constraints over numerical values associated with some criteria weights;
  Example: criterion g_1 weight value is between 2 and 4;
- a partial preorder between some sets of criteria, expressing preferences about the sum of some criteria weights;
  Example: the coalition of criteria g_1 and g_3 is more important than g_2;
- some sets of criteria able to validate or invalidate an outranking statement;
  Example: when an alternative x is at least as good as y over criteria g_1, g_2 and g_3, the decision maker considers that x outranks y.

The first type of preferential information concerns strict preferences and indifferences over a subset of ordered pairs of alternatives E. Let P be the strict preference relation over E and I the indifference relation. If the decision maker expresses that aPb then, it necessarily results in $\widetilde{S}^W(a,b) > 0$ and $\widetilde{S}^W(b,a) < 0$, where W is the vector of criteria weights provided by the resolution of the problem. In the same way, if the decision maker expresses that cId then this results in $\widetilde{S}^W(c,d) > 0$ and $\widetilde{S}^W(d,c) > 0$.

In order to provide a solution as robust as possible to the decision maker, we decide to translate the statement aPb as follows into linear constraints:

- an original $+1$ constraint for the ordered pair (a,b) so as to warrant the outranking statement;

- a relaxed $+2$ constraint for the ordered pair (a, b) to intend a robust outranking statement;
- an original -1 constraint for the ordered pair (b, a);
- a relaxed -2 constraint for the ordered pair (b, a).

Similarly, we translate an indifference judgement between alternatives c and d by an original $+1$ constraint and a relaxed $+2$ one for the each ordered pair (c, d) and (d, c).

Let us notice the fact that, according to Proposition 2, when an outranking statement goes against the elementary outranking statement, it will not be possible to warrant CONDORCET robustness. It is then useless to add relaxed ± 2 constraints.

Let us also notice that, if two pairs of alternatives (a, b) and (c, d) are compared similarly on all criteria (*i.e.* $S_i(a, b) = S_i(c, d), \ \forall g_i \in F$), then their concordance value (and also, in absence of veto situation, their overall valued outranking value $\widetilde{S}$) will always be the same, regardless to any weights vector. Consequently, it is unnecessary to take into account the decision maker's preferences on both couples, as it will end in adding twice the same constraints in the model[3].

Information on criteria weights are easy to translate into linear constraints. Thus, if a decision maker express the fact that the weights of criterion g_i is equal to the integer value v, we add the constraint:

$$\sum_{c=1}^{M} p_{ic} = v$$

If the decision maker wants to restrict the value of the weight of criterion g_i between two integers u and v, we add the following constraints:

$$\sum_{c=1}^{M} p_{ic} \geq u \qquad \text{and} \qquad \sum_{c=1}^{M} p_{ic} \leq v$$

A decision maker's statement "criterion g_i is more important that criterion g_j" will be taken into account by adding the constraint:

$$\sum_{c=1}^{M} p_{ic} \geq \sum_{c=1}^{M} p_{jc} + 1$$

This formula can be generalized for subsets of criteria: If a subset of criteria H is more important than a subset K, then we add the following constraint:

$$\sum_{g_i \in H} (\sum_{c=1}^{M} p_{ic}) \geq \sum_{g_j \in K} (\sum_{c=1}^{M} p_{jc}) + 1$$

[3]If a decision maker gives discordant preferences, either a veto has to be raised, or another method has to be considered.

We can also model the fact that a subset H of criteria is, according to the decision maker, sufficient to validate an outranking statement (the sum of its criteria weights is strictly greater than half of the sum of all criteria):

$$\sum_{g_i \in H} (\sum_{c=1}^{M} p_{ic}) \geq \frac{1}{2} \cdot \sum_{g_j \in F} (\sum_{c=1}^{M} p_{jc}) + 1$$

3.4. Inverse analysis mathematical program

We choose on purpose not to explore the whole admissible solution polytope, but to only determine a vector of criteria weights W^*:

- satisfying each constraint $[\![\widetilde{S}^{W^*}]\!](x,y) = \pm 1$,
- which best match the constraints $[\![\widetilde{S}^{W^*}]\!](x,y) = \pm 2$, and
- minimizing the weights $w_i^* \in W^*$ ($\forall g_i \in F$) (which, in practice, means to use the smallest possible number of equi-significance classes).

As a result, we formulate a first objective function, which has to be minimized:

$$Min_1 \;:\; K_1 (\sum_{g_i \in F} \sum_{u=1}^{M} p_{i,u}) \;+\; K_2 (\sum_{(x,y) \in A^2_{\pm 2}} s^{\pm 2}(x,y))$$

K_1 and K_2 are parametric constants used to correctly order both sub-objectives.

The second part of the objective function tends to minimize the sum of the slack variables. As we will show in the following section, this increases the number of validated $[\![\widetilde{S}^{W^*}]\!](x,y) = \pm 2$ constraints, but is not fully optimal. We then slightly modify the $[\![\widetilde{S}^{W^*}]\!](x,y) = \pm 2$ relaxed constraints:

$$\sum_{g_i \in F} \Big(p_{i,u} \cdot \pm S_i(x,y) \Big) \;\pm\; s^{\pm 2}_{bool}(x,y) \cdot m \;\geqslant\; b_u(x,y) \;\forall u = 1, ..., M \;\forall (x,y) \in A^2_{\pm 2}$$

and substitute the real variable $s^{\pm 2}(x,y)$ by a boolean one, $s^{\pm 2}_{bool}(x,y)$, multiplied by the number of criteria[4], m, in order to introduce a second objective function:

$$Min_2 \;:\; K_1 (\sum_{g_i \in F} \sum_{u=1}^{M} p_{i,u}) \;+\; K_2 (\sum_{(x,y) \in A^2_{\pm 2}} s^{\pm 2}_{bool}(x,y))$$

To summarize, let us present the mixed integer linear program obtained using the second objective function. In order to make the lecture easier, we define $\mathfrak{S}$ (resp. $\overline{\mathfrak{S}}$) as the set of ordered pairs of alternatives (x,y) such as x outranks (resp. does not outrank) y:

[4]Every ± 2 constraint slack variable is bounded by the number of criteria. In the worst case for a $+2$ constraint, $C^W_k(x,y) = -m$.

MILP

Variables:

$$p_{i,u} \in \{0,1\} \qquad \forall g_i \in F,\ \forall u = 1..M$$
$$b_u(x,y) \in \{0,1\} \qquad \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}, \forall u = 1..M$$
$$s^{\pm 2}_{bool}(x,y) \geqslant 0 \qquad \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$

Parameters:

$$K_1 > 0,\ \ K_2 > 0$$

Objective function:

$$Min_2 \quad K_1\big(\sum_{g_i \in F}\sum_{u=1}^{M} p_{i,u}\big) \ + \ K_2\big(\sum_{(x,y)\in A^2_{\pm 2}} s^{\pm 2}_{bool}(x,y)\big)$$

Constraints:

s.t.
$$\sum_{g_i \in F} p_{i,1} = m$$
$$p_{i,u} \geqslant p_{i,u+1} \qquad \forall g_i \in F,\ \forall u = 1..M-1$$
$$\sum_{u=1}^{M} b_u(x,y) \ \geqslant \ 1 \qquad \forall (x,y) \in \mathfrak{S} \cup \overline{\mathfrak{S}}$$
$$\sum_{g_i \in F}\Big(\big(\textstyle\sum_{u=1}^{M} p_{i,u}\big)\cdot S_i(x,y)\Big) \ \geqslant \ 1 \qquad \forall (x,y) \in \mathfrak{S}$$
$$\sum_{g_i \in F}\Big(\big(\textstyle\sum_{u=1}^{M} p_{i,u}\big)\cdot S_i(x,y)\Big) \ \leqslant \ -1 \qquad \forall (x,y) \in \overline{\mathfrak{S}}$$
$$\sum_{g_i \in F}\Big(p_{i,u}\cdot S_i(x,y)\Big) + s^{+2}_{bool}(x,y).m \ \geq \ b_u(x,y) \qquad \forall (x,y) \in \mathfrak{S}, \forall u = 1..M$$
$$\sum_{g_i \in F}\Big(p_{i,u}\cdot S_i(x,y)\Big) - s^{-2}_{bool}(x,y).m \ \leq \ -b_u(x,y) \qquad \forall (x,y) \in \overline{\mathfrak{S}}, \forall u = 1..M$$

Constraints (informal) on the weights allowing to model decision maker's preferences:

$$\sum_{c=1}^{M} p_{ic} = v_i \qquad \text{For some criteria } g_i$$
$$\sum_{c=1}^{M} p_{ic} \geq u \quad \text{et} \quad \sum_{c=1}^{M} p_{ic} \leq v \qquad \text{For some criteria } g_i$$
$$\sum_{c=1}^{M} p_{ic} \geq \sum_{c=1}^{M} p_{jc} + 1 \qquad \text{For some couples } (g_i, g_j)$$
$$\sum_{g_i \in H}\big(\sum_{c=1}^{M} p_{ic}\big) \geq \tfrac{1}{2}.\sum_{g_j \in F}\big(\sum_{c=1}^{M} p_{jc}\big) + 1 \qquad \text{For some criteria subsets}$$
$$\sum_{g_i \in H}\big(\sum_{c=1}^{M} p_{ic}\big) \geq \sum_{g_j \in K}\big(\sum_{c=1}^{M} p_{jc}\big) + 1 \qquad \text{For some criteria subsets}$$

4. Experiments on the use of Condorcet robustness constraints

As explained in Corollary 1, decreasing the number of importance classes of a vector of weights advantages the increase of the number of robust arcs. Therefore, in order to measure the behavior of the algorithm and the robustness benefit of using Condorcet constraints, we consider the following experiment: starting from a richly discriminated vector of weights W (ultimately resulting in a preferential linear order on the criteria), we compute another vector W' with the lowest possible level of discrimination, allowing to reconstruct the same median cut outranking digraph than W and compatible with the preferential information on the criteria inherent to W by simply adding to the model the following constraints:

$$w_i > w_j \Rightarrow w'_i \geq w'_j \quad \text{and} \quad w_i = w_j \Rightarrow w'_i = w'_j$$

We design two versions of the experiment:

- In the first one, we consider the complete set of outranking statements between all the alternatives pairs as constraints in the model. The relation is then reconstructed without any interaction and shows an increase in the number of robust statements;
- In the second one, we try to find the quantity of preferential information on the alternatives necessary to reconstruct the complete relation. We iteratively add constraints on outranking statement of alternatives pairs until the relation is totally reconstructed.

We then define three versions of the algorithm:

- A_{con}, a control algorithm obtained by not taking into account the ± 2 constraints (*i.e.* by simply validating the outranking statements for the given vector of integer criteria weights, minimizing the sum of its components);
- Rob_1, using the first defined objective function (*i.e.* minimizing the sum of real slack variables);
- Rob_2, using the second defined objective function (*i.e.* minimizing the number of non-robust constraints).

We also define 25 different sizes of problems, by varying the number of alternatives and criteria according to the following values: 7, 10, 13, 16 and 19. For each size, we randomly generate 200 problems: a performance table and also a vector of integer criteria weights, allowing to compute an original outranking digraph, modeling a decision maker's set of preferences.

4.1. First experiment: Reducing the discrimination of a given weights vector

In the first experiment, starting from a given richly discriminated vector of weights W and the resulting outranking digraph, we search for the less possible discriminated vector W' reconstructing the exact complete median cut relation and respecting the preferential information inherent to the weights vector W. It allows to minimize the mandatory number of importance classes satisfying the relation, which significantly increases the number of robust arcs, as we will show on Table 5.

Notice that, if a balanced situation occurs in the original digraph (*i.e.* $\widetilde{S}_w(x, y) = 0$), we decide not to take it into account when creating the set of constraints, as it express a very anecdotic situation.

As a large value for the parameter M implies an exponential increase of the running time, we decided to fix it to 7 for each problem at the beginning and to increase it when a solution can't be reached with such a low parameter (about 8% when $M = 7$, only 4% when $M = 8, \dots$), until a solution is found. To compare our solutions to the optimal ones, we run again all the problems, taking the number of criteria as the value of M. No real quality improvement can be noticed: most of the time, the solutions were the same or improved by only 1 or 2 percents, for a running time ten to a hundred times longer.

Table 5 summarizes, for some sizes of problems[5], the average percentage of robust outranking statements on the original outranking relations and those obtained with the three defined algorithms.

Table 5: Exp_1 – Robustness increase and running time

		Mean percentage of robust outranking statements				Median running time (s)	
m	n	Orig	A_{con}	Rob_1	Rob_2	Rob_1	Rob_2
7	7	65	79	79	84	0.0	0.0
7	13	65	73	73	76	0.3	0.3
7	19	65	72	72	75	2.3	1.8
13	7	45	72	72	81	0.3	0.3
13	13	46	63	64	71	2.1	2.0
13	19	47	59	60	63	10.7	10.1
16	7	43	73	74	88	1.3	1.2
16	13	40	67	70	78	9.8	8.6
16	19	42	61	61	68	40.2	37.7
19	7	36	68	69	80	0.9	0.8
19	13	37	59	63	72	7.9	6.6
19	19	39	47	49	59	34.0	18.7

One may notice that each method improves the average percentage of robust arcs compared to the original outranking digraph (Orig). A_{con}, by simply minimizing the sum of the weights, tends to minimize the number of equi-significance classes, inducing an increase of the robust statements of 18 percents in average. As it always runs between 0.1 and 0.3 seconds, its running time isn't represented here. Rob_2, by taking into account CONDORCET constraints, allows to obtain a higher percentage of robust statements (an average of 26 percents more than in the original digraph), in an acceptable time, even for large instances. In this particular experiment, Rob_1 gives us results similar to A_{con}, with a slightly longer execution time than Rob_2. This is due to the fact that the solver (Cplex 11.0) uses efficient Branch and Bound algorithms as the admissible solutions domain is strongly reduced by preferential constraints on criteria weights. But, if we are not taking into account these constraints, we notice that Rob_1 clearly improves the number of robust arcs obtained with A_{con} and still runs in acceptable time, unlike Rob_2 whose running time exponentially increases. We may also notice that, on large instances, as the number of constraints increases, the admissible solutions space is reduced, which explains for example the fact that 19×19 problems are in average faster than 16×19 ones.

We showed in this experiment the importance of reducing the weights vector discriminations to the lowest level which reconstruct the decision maker's pref-

[5] As all tests tend to similar conclusion, we decided no to present all of them in order to improve the reading of this article

erences on the couples of alternatives. This preliminary work ensures a maximal enhancement of the outranking digraph and subsequently improves its use in order to solve a problematique. Furthermore, it helps the decision maker to focus on the validation of the preorder of the weights and not on some precise weights values.

4.2. Second experiment: An iterative construction of a preference information set allowing to reconstruct the outranking relation

Getting closer to a practical questioning protocol, we are here considering a context where a fictitious decision maker is asked to give preferential information about selected couples of alternatives which aims at assessing iteratively enough information to infer weights parameters reconstructing his preferences (modeled by the initial complete outranking digraph, or with a few percentage of changes). We are interested in trying to define an adequate number of couples to be selected, using the Rob_2 algorithm for intermediate resolutions, as it previously gave us the best results in an acceptable time.

The algorithm is then the following:

Require:
  A: Alternatives set; F: criteria vector; P: performance table
  G: Outranking digraph, modeling DM's preferences on the alternatives
1: $constraints \leftarrow \emptyset$
2: $weights \leftarrow W_{\mathbf{1}}$ {The equi-significant vector of weights}
3: **while not** is_recapturing_digraph($weights, G$) **do**
4:   $(a_1, a_2) \leftarrow$ select_alternatives_couple(A)
5:   $constraints \leftarrow$ add_preferential_constraints($G(a_1, a_2)$)
6:   $weights \leftarrow$ Rob_2_solve($A, F, P, constraints$)
7: **end while**
8: **return** $weights$

The key steps of the algorithm are the selection of the alternatives couples (step 4) and the loop-ending condition (step 3). For selecting a couple of alternatives, we test three natural heuristics:

- a *random selection* (RS),
- a selection of a couple among the *most represented class* (MRC) of couples having the same behavior on each criterion, in order to fix the greatest number of arcs at each iteration, and
- a selection of the couple with the *worst determined outranking value* (WDV), *i.e.* arcs associated with overall values close to 0 (balanced situation) which are, in the absence of a CONDORCET ± 2 denotation, anecdotic and very sensitive to criteria weights changes.

For the loop-ending condition, we test a complete reconstruction, and also a 95% reconstruction (*i.e.* the algorithm stops when at least a 95% of the arcs, between the outranking digraph obtained by the current vector of weights and the original one, are similar). Notice that, when the preferential information obtained in step 5 are already robustly granted by the current weights vector, there is no need to compute a new vector: step 6 is then ignored and another pair of alternatives is selected.

Table 6 summarizes, for some sizes of problems, the average number of selected couples (*i.e.* an idea on the number of questions we should ask the decision maker before presenting him the final outranking relation) and the average number of effective resolutions. We only present the result using the WDV heuristic, as RS and MRC heuristics results aren't satisfying (the number of selected couples and the running time were two to four times higher compared to the WDV heuristic).

Table 6: Exp_2 – Mean values for Rob_2 iterative reconstruction, using the WDV heuristic

			100% reconstruction			95% reconstruction		
m	n	nb pairs	nb select	nb solve	%rob	nb select	nb solve	%rob
7	7	21	3.7 / 17.6%	1.2	83	2.0	0.8	87
7	13	78	8.7 / 11.2%	2.2	76	4.5	1.2	82
7	19	171	10.2 / 6.0%	2.4	76	5.0	1.3	84
13	7	21	5.2 / 24.8%	2.1	79	2.9	1.3	83
13	13	78	16.8 / 21.5%	4.5	70	7.3	2.2	79
13	19	171	25.8 / 15.1%	6.2	63	8.0	2.3	78
19	7	21	5.9 / 28.1%	2.5	77	3.8	1.7	78
19	13	78	18.2 / 23.3%	5.6	70	7.8	2.5	78
19	19	171	30.7 / 18.0%	8.4	59	10.3	3.1	76

On the left part of Table 6, we detail the results for a complete reconstruction of the original median cut outranking digraph. This experiment corresponds to an iterative version of the first one, without having to consider all couples of alternatives. Consequently, it significantly reduces the running time (even for large instances, it runs under a second for each iteration on a standard computer), showing that such a process can be used in a real-time decision aid process to select couples of alternatives and to reconstruct iteratively a satisfying criteria weights vector. Notice that the robustness is slightly lower than for the first experiment, due to the fact that the algorithm only forces the robustness for some selected couples, but tries to minimize the sum of the weights and so tends to reduce the weights vector discrimination. One can run again the first experiment at the end of the iterative protocol, once the decision maker validates the outranking relation. Notice also that the WDV heuristic considers less than 30% of the whole couples in the worst case (a large number of criteria and a few alternatives) and only 6% in the best ones (few criteria and a large number of alternatives), helping us to find the weights parameters in a relatively fast questioning protocol.

On the right part of Table 6, we detail the results of a 95% reconstruction of the original outranking digraph, running again the previous experiment by modifying the ending condition in order to stop the iterative process when at least 95% of the original outranking digraph has been reconstructed. We can easily see the decrease of the running time, the number of selected pairs and the number of resolution. Notice that the robustness is even better, as the outranking relation is less constrained.

Comparing theses results with the ones from the first experiment (see Table 5), we obtain here a similar percentage of robust arcs, but with a significant decrease of the running time and the use of a very reduced set of given preferential information.

5. Conclusion and future works

In this article, we have shown the potential of the Condorcet robustness denotation, as a powerful tool to focus on sensitive outranking statements and also to reinforce them. We demonstrate the possibility to reconstruct a complete bipolar-valued outranking digraph, starting from a complete set of preferences on the pairs of alternatives, or building iteratively the set of preferences, always improving the robustness of the arcs.

This improvement on building the bipolar-valued outranking relation induces a gain of its credibility: the more robust is the relation, the less critical becomes the actual choice of precise numerical criteria significance weights. As a consequence, the quality of a solution provided by a multiple criteria method exploiting this robust bipolar-valued outranking relation may be much enhanced.

We have furthermore shown that it is possible to create an iterative process for taking interactively into account some of the decision maker's a priori global preferences. In a practical case, this interactive step may naturally lead to the eventual validation of the outranking digraph by the decision maker. Indeed, he may disagree with some of the appearing preference statements and so add fresh preferential information to the model. Noticing here that the number of potential questions is quite high compared to the number of required computational resolutions, especially for large instances, we anticipate being able to significantly reduce the number of interaction cycles needed in practice for: 1.- discussing with the decision maker the eventual refinement of the resulting preferences, and: 2.- eventually fixing the quasi-totality of a bipolar-valued characterisation of a global outranking relation.

Future work will be devoted to these issues, especially the design of adequate practical interaction protocols and questionnaires, as well as the design and implementation of a set of tools helping the decision maker explore and visualize the involved preferential information.

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