On weakly ordering by choosing from valued pairwise outranking situations

Raymond Bisdorff

Université du Luxembourg
FSTC/ILAS

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Content

Sample performance tableau

Let $X = \{a_1, ..., a_7\}$ be seven potential decision actions evaluated on three cost criteria (g_1, g_4, g_5) of equi-significance $1/6$ and two benefit criteria (g_2, g_3) of equi-signifiance $1/4$. The given performance tableau is shown below.

Objectives	**Costs**			**Benefits**	
Criteria	$g_1(\downarrow)$	$g_4(\downarrow)$	$g_5(\downarrow)$	$g_2(\uparrow)$	$g_3(\uparrow)$
weights×12	2.0	2.0	2.0	3.0	3.0
indifference	3.41	4.91	-	-	2.32
preference	6.31	8.31	-	-	5.06
veto	60.17	67.75	-	-	48.24
a_1	22.49	36.84	7	8	43.44
a_2	16.18	19.21	2	8	19.35
a_3	29.41	54.43	3	4	33.37
a_4	82.66	86.96	8	6	48.50
a_5	47.77	82.27	7	7	81.61
a_6	32.50	16.56	6	8	34.06
a_7	35.91	27.52	2	1	50.82

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Sample outranking relation

The resulting bipolar outranking relation S is shown below.

Table : r-valued bipolar outranking relation

$r(S) \times 12$	a_1	a_2	a_3	a_4	a_5	a_6	a_7
a_1	$-$	**0**	$+8$	$+12$	$+6$	$+4$	-2
a_2	$+6$	$-$	$+6$	$+12$	**0**	$+6$	$+6$
a_3	-8	-6	$-$	**0**	-12	$+2$	-2
a_4	-12	-12	**0**	$-$	-8	-12	**0**
a_5	-2	**0**	$+12$	$+12$	$-$	-6	**0**
a_6	$+2$	$+4$	$+8$	$+12$	$+6$	$-$	$+2$
a_7	$+2$	-2	$+2$	$+6$	**0**	$+2$	$-$

1. a_6 is a Condorcet winner,
2. a_2 is a weak Condorcet winner,
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a_2	+ 6	–	+ 6	+ 12	**0**	+ 6	+ 6
a_3	−8	−6	–	**0**	−12	+2	−2
a_4	− 12	− 12	**0**	–	− 8	− 12	**0**
a_5	−2	**0**	+12	+12	–	−6	**0**
a_6	+ 2	+ 4	+ 8	+ 12	+ 6	–	+ 2
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a_3	−8	−6	–	**0**	−12	+2	−2
a_4	− 12	− 12	**0**	–	− 8	− 12	**0**
a_5	−2	**0**	+12	+12	–	−6	**0**
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a_3	−8	−6	–	**0**	−12	+2	−2
a_4	– 12	– 12	**0**	–	– 8	– 12	**0**
a_5	−2	**0**	+12	+12	–	−6	**0**
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Ranking by RUBIS best and worst choosing

- Let X_1 be the set X of potential decision actions we wish to rank.
- While the remaining set X_i ($i = 1, 2, ...$) of decision actions to be ranked is not empty, we extract from X_i the best (B_i), respectively worst (W_i) RUBIS choice recommendations and set $X_{i+1} = X_i - B_i$, respectively $X_{i+1} = X_i - W_i$.
- Both iterations determine, hence, two – usually slightly different – opposite weak orderings on X:
    1. a ranking-y-best-choosing order and,
    2. a ranking-by-worst-rejecting order.

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Fusion of best and worst choice rankings

Ranking by recursively choosing: Ranking by recursively rejecting:

```
1st Best Choice ['a02','a05']
 2nd Best Choice ['a06']
   3rd Best Choice ['a07']
     4th Best Choice ['a01']
       5th Best Choice ['a03','a04']
```

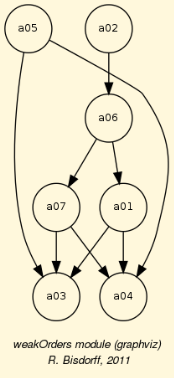
```
Last Choice ['a03','a04']
 2nd Last Choice ['a05','a07']
   3rd Last Choice ['a01']
     4th Last Choice ['a06']
       5th Last Choice ['a02']
```

We may fuse both rankings, the first and the converse of the second, with the help of the epistemic conjunction operator ($\oslash$) to make apparent a valued relation R which represents a weakly complete and transitive closure of the given bipolar valued outranking.
Let ϕ and ψ be two logical formulas:

$$\phi \oslash \psi = \begin{cases} (\phi \wedge \psi) & \text{if} \quad (\phi \wedge \psi) \text{ is true;} \\ (\phi \vee \psi) & \text{if} \quad (\neg\phi \wedge \neg\psi) \text{ is true;} \\ \text{Indeterminate} & \text{otherwise.} \end{cases}$$

Table : Weakly complete transitive closure of S

$r(R)$	a_2	a_5	a_6	a_1	a_7	a_3	a_4
a_2	–	**0**	+6	+6	+6	+6	+12
a_5	**0**	–	**0**	**0**	**0**	+12	+12
a_6	−4	**0**	–	+2	+2	+8	+12
a_1	0	**0**	−4	–	**0**	+8	+12
a_7	−2	**0**	−2	**0**	–	+2	+6
a_3	−6	−12	−2	−8	−2	–	**0**
a_4	−12	−8	−12	−12	0	**0**	–



Notice the contrasted ranks of action a_5 (first best as well as second last), indicating a lack of comparability, which becomes apparent in the conjunctive epistemic fusion R of both weak orderings shown in the Table above and illustrated in the corresponding Hasse diagram.

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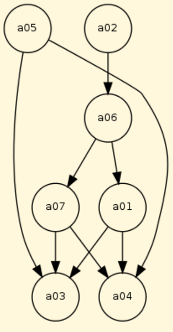
$r(R)$	a_2	a_5	a_6	a_1	a_7	a_3	a_4
a_2	–	**0**	+6	+6	+6	+6	+12
a_5	**0**	–	**0**	**0**	**0**	+12	+12
a_6	−4	**0**	–	+2	+2	+8	+12
a_1	0	**0**	−4	–	**0**	+8	+12
a_7	−2	**0**	−2	**0**	–	+2	+6
a_3	−6	−12	−2	−8	−2	–	**0**
a_4	−12	−8	−12	−12	0	**0**	–



weakOrders module (graphviz)
R. Bisdorff, 2011

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Bipolar characteristic function r

- $X = \{x, y, z, ...\}$ is a finite set of m decision alternatives;
- We define a binary relation R on X with the help of a bipolar characteristic function r taking values in the rational interval $[-1.0; 1.0]$.
- **Bipolar semantics**: For any pair $(x, y) \in X^2$,
  1. $r(x\,R\,y) = +1.0$ means $x\,R\,y$ valid for sure;
  2. $r(x\,R\,y) > 0.0$ means $x\,R\,y$ more or less valid;
  3. $r(x\,R\,y) = 0.0$ means both $x\,R\,y$ and $x\,\not R\,y$ indeterminate;
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- **Boolean operations**: Let ϕ and ψ be two relational propositions.
  1. $r(\neg\phi) = -r(\phi)$;
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- **Boolean operations**: Let ϕ and ψ be two relational propositions.
  1. $r(\neg\phi) = -r(\phi)$.
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Bipolar characteristic function r

- $X = \{x, y, z, ...\}$ is a finite set of m decision alternatives;
- We define a binary relation R on X with the help of a bipolar characteristic function r taking values in the rational interval $[-1.0; 1.0]$.
- **Bipolar semantics**: For any pair $(x, y) \in X^2$,
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Weakly complete binary relations

Let R be an r-valued binary relation defined on X.

Definition

We say that R is weakly complete on X if, for all $(x, y) \in X^2$, **either** $r(x \,\mathrm{R}\, y) \geqslant 0.0$ **or** $r(y \,\mathrm{R}\, x) \geqslant 0.0$.

Examples

1. Marginal semi-orders observed on each criterion,
2. Weighted concordance relations,
3. Polarised outranking relations,
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Universal properties

Let $\mathcal{R}$ denote the set of all possible weakly complete relations definable on X.

Property ($\mathcal{R}$-internal operations)

Examples: Concordance of linear, weak orders or semiorders, bipolar outranking (concordance-discordance) relations.

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1. *The convex combination of any finite set of such weakly complete relations remains a weakly complete relation.*
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We say that a binary relation $R \in \mathcal{R}$ verifies the *coduality principle* ($> \equiv \not\leqslant$), if the converse of its negation equals its asymetric part : $\min\big(r(x\,\mathrm{R}\,y), -r(y\,\mathrm{R}\,x)\big) \;=\; -r(y\,\mathrm{R}\,x)$. Let $\mathcal{R}^{cd}$ denote the set of all possible relations $R \in \mathcal{R}$ that verify the coduality principle.

Property (Coduality principle)

The convex and epistemic-disjunctive (resp. -conjunctive) combinations of a finite set of relations in $\mathcal{R}^{cd}$ verify again the coduality principle.

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Pragmatic principles of the RUBIS choice

$\mathcal{P}_1$: Elimination for well motivated reasons:
Each eliminated alternative has to be outranked by (resp. is outranking) at least one alternative in the RUBIS choice (RC).

$\mathcal{P}_2$: Minimal size:
The RC must be as limited in cardinality as possible.

$\mathcal{P}_3$: Stable and efficient:
The RC must not contain a self-contained sub-RC.

$\mathcal{P}_4$: Effectively better (resp. worse):
The RC must not be ambiguous in the sense that it is not both a best choice as well as a worst choice recommendation.

$\mathcal{P}_5$: Maximally significant:
The RC is, of all potential best (resp. worst) choices, the one that is most significantly supported by the marginal *"at least as good as"* relations.

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Qualifications of a choice in X

Let S be an r-valued outranking relation defined on X and let Y be a non empty subset of X, called a choice in X.

- Y is called outranking (resp. outranked) iff for all non retained alternative x there exists an alternative y retained such that $r(y \,\mathrm{S}\, x) > 0.0$ (resp. $r(x \,\mathrm{S}\, y) > 0.0$).
- Y is called independent iff for all $x \neq y$ in Y, we observe $r(x \,\mathrm{S}\, y) \leqslant 0.0$.
- Y is an outranking kernel (resp. outranked kernel) iff Y is an outranking (resp. outranked) and independent choice.
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Translating the pragmatic RUBIS principles in terms of choice qualifications

$\mathcal{P}_1$: Elimination for well motivated reasons.
The RC is an outranking choice (resp. outranked choice).

$\mathcal{P}_{2+3}$: Minimal and stable choice.
The RC is a hyper-kernel.

$\mathcal{P}_4$: Effectivity.
The RC is a choice which is strictly more outranking than outranked (resp. strictly more outranked than outranking) .

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The RC is the most determined one in the set of potential outranking (resp. outranked) hyper-kernels observed in a given r-valued outranking relation.

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Properties of the RUBIS choice

Property (decisiveness)

Every r-valued (strict) outranking relation admits at least one outranking and one outranked hyper-kernel.

Definition

Let S and S' be two r-valued outranking relations defined on X.

1. We say that S' upgrades action $x \in X$, denoted $S^{x\uparrow}$, if $r(x\,S'\,y) \geqslant r(x\,S\,y)$, and $r(y\,S'\,x) \leqslant r(y\,S\,x)$, and $r(y\,S'\,z) = r(y\,S\,z)$ for all $y, z \in X - \{x\}$.
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Properties of the RUBIS choice

Let A be a subset of X. Let $RBC(S_{|A})$ (resp.$RBC(S'_{|A})$) be the RUBIS best choice wrt to S (resp. S') restricted to A and let $RWC(S_{|A})$ (resp.$RWC(S'_{|A})$) be the RUBIS worst choice wrt to S (resp. S') restricted to A.

Property

1. $S_{|A} = S'_{|A} \Rightarrow RBC(S_{|A}) = RBC(S'_{|A})$ *(RBC local)*,
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3. $x \in RBC(S_{|A}) \Rightarrow x \in RBC(S^{x\uparrow}_{|A})$ *(RBC weakly monotonic)*,
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Let A be a subset of X. Let $RBC(S_{|A})$ (resp.$RBC(S'_{|A})$) be the RUBIS best choice wrt to S (resp. S') restricted to A and let $RWC(S_{|A})$ (resp.$RWC(S'_{|A})$) be the RUBIS worst choice wrt to S (resp. S') restricted to A.

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Ranking-by-Choosing Algorithm

1. Let X_1 be the set X of potential decision actions we wish to rank on the basis of a given outranking relation S.
2. While the remaining set X_i ($i = 1, 2, \ldots$) of decision actions to be ranked is not empty, we extract from X_i the best (B_i), respectively worst (W_i), Rubis choice recommendation and set $X_{i+1} = X_i - B_i$, respectively $X_{i+1} = X_i - W_i$.
3. Both independent iterations determine, hence, two – usually slightly different – opposite weak orderings on X: a ranking-y-best-choosing – and a ranking-by-worst-choosing order.
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Transitive S-closure

Definition

We call a ranking procedure transitive if the ranking procedure renders a (partial) strict ordering $\succ_S$ on X with a given r-valued outranking relation S such that for all $x, y, z \in X$: $r(x \succ_S y) > 0$ and $r(y \succ_S z) > 0$ imply $r(x \succ_S z) > 0$.

Property

Both the RUBIS *ranking-by-best-choosing, as well as the* RUBIS *ranking-by-worst-choosing procedures, are transitive ranking procedures.*

Corollary

i) The fusion of the ranking by RUBIS *best choice and the converse of the ranking by* RUBIS *worst choice of a given r-valued outranking relation S is a transitive ranking procedure.*
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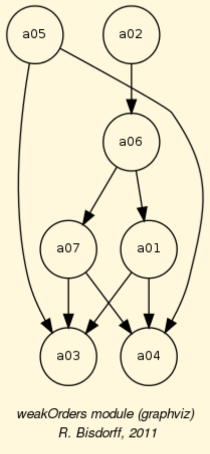
Introductory example

Comparing ranking-by-choosing result with Kohler's and Tideman's:



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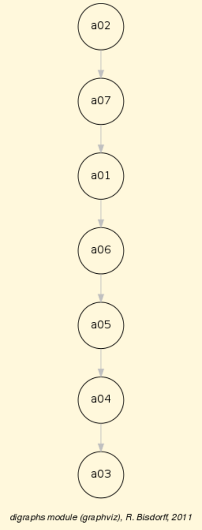

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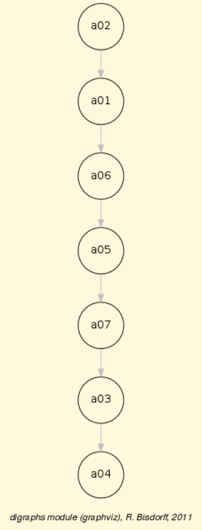

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Sample performance tableau

Let $X = \{a_1, ..., a_7\}$ be seven potential decision actions evaluated on three cost criteria (g_1, g_4, g_5) of equi-significance 1/6 and two benefit criteria (g_2, g_3) of equi-signifiance 1/4. The given performance tableau is shown below.

Objectives	**Costs**			**Benefits**	
Criteria	$g_1(\downarrow)$	$g_4(\downarrow)$	$g_5(\downarrow)$	$g_2(\uparrow)$	$g_3(\uparrow)$
weights×12	2.0	2.0	2.0	3.0	3.0
indifference	3.41	4.91	-	-	2.32
preference	6.31	8.31	-	-	5.06
veto	60.17	67.75	-	-	48.24
a_1	22.49	36.84	7	8	43.44
a_2	16.18	19.21	2	8	19.35
a_3	29.41	54.43	3	4	33.37
a_4	82.66	86.96	8	6	48.50
a_5	47.77	82.27	7	7	81.61
a_6	32.50	16.56	6	8	34.06
a_7	35.91	27.52	2	1	50.82

Quality of ranking result

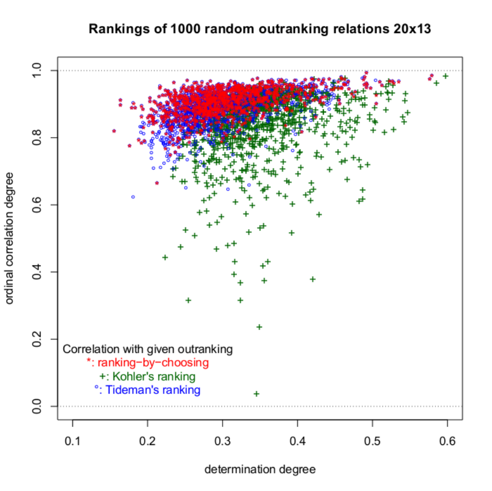
Comparing rankings of a sample of 1000 random r-valued outranking relations defined on 20 actions and evaluated on 13 criteria obtained with RUBIS **ranking-by-choosing**, **Kohler's**, and **Tideman's** (ranked pairs) procedure.

Mean extended Kendall τ correlations with r-valued outranking relation:

Ranking-by-choosing: $+.906$

Tideman's ranking: $+.875$

Kohler's ranking: $+.835$



Quality of ranking-by-choosing result

r-valued determination of ranking result:

- Mean outranking significance:
  0.351 (67.5% of total criteria support),
- Mean Ranking-by-choosing significance:
  0.268 (63.4% of total criteria support),
- Mean covered part of significance:
  $0.268/0.351 =$ **76%**.

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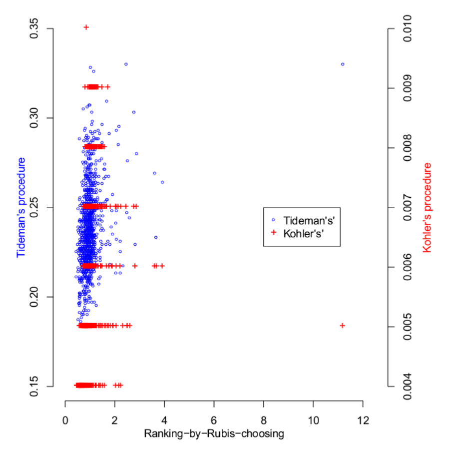
Scalability of ranking procedures

Ranking execution times (in sec.) for 1000 random 20x13 outrankings:

- **Kohler**'s procedure on the right y-axis (less than **1/100** sec.),
- **Tideman**'s procedure on the left y-axis (less than **1/3** sec.),
- the Rubis **ranking-by-choosing** procedure on the x-axis (mostly less than 2 sec.). But, heavy right tail (up to 11 sec. !).



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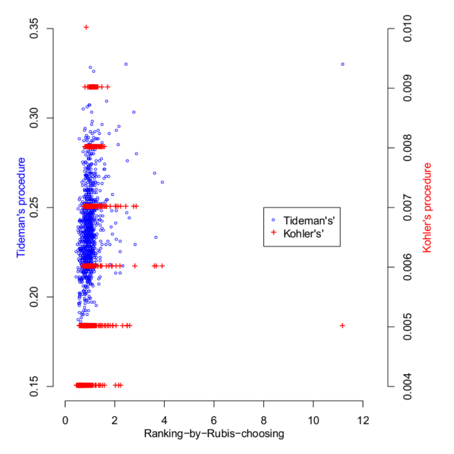

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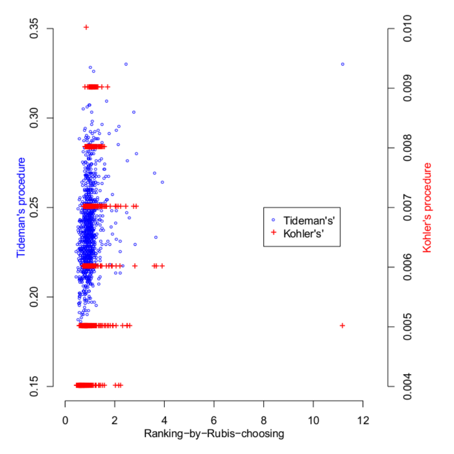

Practical application

- Spiegel (DE) On-line Students' Survey (2004) about the quality of 41 German universities in 15 academic disciplines;
- XMCDA 2.0 encoding of performace tableau;
- Ranking-by-choosing result.

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