# Foundational Pluralism and Semantic Embedding

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## Abstract

Version 2

The discussion here begins with the philosophy of Rudolf Carnap and the impact of his pluralism, made explicit through the "principle of tolerance" in his "Logical Syntax of Language", and later more carefully elaborated in his "Empiricism Semantics and Ontology". The principle has been seen by some to mark a transition in Carnap's philosophy of mathematics from "Universalism" to pluralism, but this terminology is suggestive in ways which do not faithfully represent the development of Carnap's philosophy, which was, both before and after these developments both foundationalist and pluralistic, both in respect of a priori and empirical sciences and seems contradicted by Carnap's explicit claim that the tolerance he advocated was held by previous logicist philosophers. The principle was not a new doctrine which displaced Carnap's previous views, but an explicit statement of a position he held since his student days.

Carnap's pluralism was primarily an anti-metaphysical doctrine. It was the affirmation that choice of language should be independent of constraints supposedly imposed by metaphysics (in the special sense in which Carnap uses that term). Beyond the narrow and irrelevant confines of metaphysics, choice of language (and hence of foundations) was to be a pragmatic matter. In the context of some domain of application (perhaps the development of a particular science, possibly mathematics) it is advocated that a language be chosen on the basis of its practical suitability for the purpose. Thus Carnap was implicitly advocating that comparisons be made between foundation systems.

Version 3

In this paper philosophic insights concerning the comparative evaluation of logical foundation systems are approached through discussion of two apparently diverse (but nevertheless sympathetic) foundational perspectives.

The first of these appears in philosophy of Rudolf Carnap, with its central programme aiming to facilitate logical rigour in scientific philosophy, the a priori sciences, and empirical science through the use of formal deductive systems. This kind of engagement by philosophers in the development of scientific method became less common after logical positivism fell from grace. Since then Carnap has been more an object of scholarly exegesis than a light to be followed in contemporary research. His central thrust, in seeking to provide intellectual tools for a certain kind of science, is most closely (if perhaps unwittingly) followed by the continuing development and application, in departments of computer science and outside academia, of software supporting the formal derivation of mathematics and its application to engineering problems such as the formal verification of designs and implementations of digital electronic hardware and computer software.

These two distinct but connected perspectives, can be mined for their special views on the role of logical foundation systems and the ways on which logical foundation systems can be evaluated and compared in relation to those roles.

## Introduction

I propose here to discuss ways in which foundations for mathematics can be evaluated and compared.

The discussion comes in four parts. The first part delineates the notion of “foundation system” which the remaining parts will be addressing, partly by reference to the historical development. The second is concerned with some of the kinds of comparison between foundation systems which are best known to mathematical logicians and philosophers. The third introduces considerations which become more significant in the development (principally by computer scientists) and application (by academics and certain kinds of industry) of languages, methods and tools (mostly software) supporting the use of formal foundation systems. The last part ties it all together.

## Scope and Definitions

The discussion is confined to comparisons between logical foundation systems, a term of art for which I will give a definition shortly. To motivate the definition I refer to a conception of mathematical foundations which comes from Frege, and is then subsequently refined through the twentieth century, by philosophers, mathematicians and computer scientists.

The reason for *choosing* particular definitions of the notion of a foundation system, is because certain kinds of comparison are only possible between systems which share certain characteristics, and certain types of language support software which we will discuss later, have their own native language system and support other language systems which are related to that language systems in special ways (and hence share the necessary characteristics for such relations to be possible).

There are two major kinds of foundation system which concern us here, within which kinds we will be examining ways of evaluation and comparison. The first I call syntactically definite, the second semantically definite (though not too much significance should be attached to the use of the word definite here), and I will take the latter term to entail the former for present purposes, i.e. the first kind is a superset of the second.

In describing these classes it is helpful to consider a conception of logical foundation system which comes from Frege:

*A logical foundation system for mathematics is a formal deductive system, in the context of which, the concepts of mathematics can be defined and the theorems of mathematics derived.*

We may go further into what may be meant by “formal deductive system” but, on what may be the most natural contemporary understanding of the phrase, we now know that no such system will prove all the truths of mathematics, and there are good grounds for suspecting that each such system is also *semantically* limited, and hence may fails to define all concepts of interest.

These two failures of completeness (proof theoretic and semantic) suggest ways in which foundations systems can be compared, in terms of what concepts a system can define, and which theorems about these concepts they prove, but of greater interest here is what they tell us about what a foundation system *is*, or what concept of foundation system may be most fruitfully studied.

## Academic Considerations

Consistency strength, semantic expressiveness, other matters.

## Practical Considerations