IMPLICATIONS OF OPTIMUM PROPELLER DESIGN

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Three conditions must be satisfied simultaneously in designing a propeller: (1) The propeller must supply the amount of thrust required for a given flight condition of a given airplane; (2) The torque required by the propeller must equal the torque available from the powerplant; (3) The propeller efficiency must be as high as possible. Any propeller analysis which fails to consider all three of these conditions jointly is valueless, and is probably highly misleading.

Assume that air density ρ, airspeed V, thrust required T_R, and motor torque Q_M, are given. T_R may or may not be the amount of thrust needed to provide airplane equilibrium at V; it may, for example, be greater than the equilibrium thrust, thus causing the plane to accelerate forward. Q_M is not a function of prop speed n (RPS), but need not be constant; in particular, motor torque may decline as a function of time or as a function of the integral of n. Then the first two conditions may be expressed as:

$$T = T_R \tag{1}$$

$$Q = Q_M \tag{2}$$

Deferring the third condition for a moment, note that dimensional analysis provides the expansions of thrust and torque in terms of coefficients:

$$T = \rho n^2 D^4\, C_T(J,\beta) \tag{3}$$

$$Q = \rho n^2 D^5 C_Q\,(J,\beta) \tag{4}$$

For a propeller of given geometry, plots of C_T, C_Q, as functions of advance ratio J and propeller blade angle at the 0.75-of-maximum-radius station ρ constitute the propeller data, much as the lift and drag coefficients of a wing might be plotted against angle of attack. Conventional variables are the power coefficient C_P, the efficiency η, and the advance ratio J, defined by:

$$C_P(J,\beta) \triangleq 2\pi C_Q(J,\beta) \tag{5}$$

$$\eta(J,\beta)\, C_P(J,\beta) \triangleq J\, C_T(J,\beta) \tag{6}$$

$$V \triangleq JnD \tag{7}$$

The power coefficient C_P is more often found plotted than the torque coefficient C_Q; knowing one is equivalent to knowing the other. Efficiency is merely the ratio of power-out-of the prop in the form of TV to power-into the prop in the form of nQ. The advance ratio is equivalent to angle of attack for an airfoil section with one major difference: an increase in advance ratio corresponds to a decrease in the angle of attack on the blade airfoil sections; roughly speaking, J is like the negative of angle of attack. Note that (7) links four main variables in a definition that must always be satisified.

Overlooking (7) is a rich source of errors. For example, (3) seems to say that thrust is proportional to the square of prop speed. That is true if C_T happens to be constant. But the thrust coefficient is a function of J, and J is a function of n. Therefore, changing n generally changes both the n2 factor and the C_T factor in (3). The same thing happens if, after using (7) to eliminate n2 from (3) and to introduce therein V2, we are tempted to say that thrust varies as the square of speed: in general, changing the speed changes J, and that changes CT. Any propeller analysis which contemplates changes in n or V but assumes the thrust and torque (and power) coefficients are unaffected will at best provide a poor approximation, and at worst, nonsense.

For a propeller of given configuration and for all practical values of J, an optimum blade angle βx can be such that:

$$\eta(J,\beta) \leq (J,\beta_x) \triangleq \eta_x(J) \tag{8}$$

This is the third of the basic conditions which must simultaneously be satisfied in designing a propeller. It says that for a given J, βx provides higher efficiency than any other blade angle. This maximum efficiency is, in principle, a function of J, hence it is so defined at the right. With βx determined as a function of J, for any given propeller, it is convenient to define the thrust and power coefficients at this maximum efficiency condition as:

$$C_{Tx}(J) \triangleq C_T(J,\beta_x(J)) \tag{9}$$

$$C_{Px}(J) \triangleq C_P(J,\beta_x(J)) \tag{10}$$

From typical propeller data as in Figure 1 it is possible to deduce these empirical, and approximate, relations under the condition of maximum efficiency:

$$\eta_x(J) = \eta_x = \text{const} \tag{11}$$

$$C_{Tx}(J) = a_{Tx}J \qquad (a_{Tx} = \text{const}) \tag{12}$$

$$C_{Px}(J) = a_{Px}J^2 \qquad (a_{Px} = \text{const}) \tag{13}$$

These approximations are not independent. In particular, they must necessarily be consistent with the general relation (6) which defines efficiency. If ηx is claimed to be constant, then substituting $C_{Tx}(J)$ and $C_{Px}(J)$ into (6) perforce must yield a constant. In fact it does, providing the constraint on the empirical constants:

$$\eta_x = a_{Tx}/a_{Px} \tag{14}$$

Typical figures are $a_{Tx} = 0.04$, $a_{Px} = 0.05$, and $\eta_x = 0.80$ ... which values are seen to be consistent with (14). These empirical expressions are valuable because they enable closed-form results to be obtained. Ordinarily, the propeller charts are so intrinsic to the overall problem that all insight into the design process is submerged in a plentitude of charts and graphs.

We can now proceed to design an optimum propeller. Combining (1), (3) and (7) yields:

$$D^2 C_T(J,\beta)/J^2 = T_R/\rho V^2 \triangleq K_T \quad (15)$$

Combining (2), (4), (5), and (7) yields:

$$D^3 C_P(J,\beta)/J^2 = 2\pi Q_M/\rho V^2 \triangleq K_Q \quad (16)$$

The propeller diameter D, which is unknown at this point, may be eliminated between (15) and (16), giving:

$$J\, C_P(J,\beta) C_T^{-3/2}(J,\beta) = K_Q K_T^{-3/2} \quad (17)$$

Using (6), either the power or the thrust coefficient may be eliminated from (17). Choosing to eliminate C_P gives:

$$J^2 C_T^{-1/2}(J,\beta)\,\eta^{-1}(J,\beta) = K_Q\, K_T^{-3/2} \quad (17a)$$

Assume now that the propeller is designed for maximum efficiency. Then (9), (11) and (12) may be substituted into (17a) to give, after some simplification, a solution for J in terms of hypotehtically known quantities:

$$J = a_{Tx}^{-1/3} (\eta_x K_Q)^{2/3} / K_T \quad (18)$$

This can be regarded as a solution of (1) subject to the requirement of maximum efficiency given in (8). A similar solution of (2) can be obtained by returning to (15), impressing optimality conditions (9) and (12), and solving for propeller diameter:

$$D^2 = J\, K_T / a_{Tx} \quad (19)$$

Knowing J and D, n can be obtained by substituting (18) and (19) into definition (7):

$$n = T_R V / 2\pi Q_M \eta_x \quad (20)$$

Finally, β must be selected to satisfy (8), and the optimum propeller is completely defined. Had empirical relation (13) been used instead of (12) the results would have been equivalent in the sense of (14). Propeller speed is already expressed in terms of the original data in (20). Eliminating K_T and K_Q according to their definitions of (15) and (16) gives comparable expressions for J and D:

$$J = (a_{Tx}\rho)^{1/3} (2\pi \eta_x Q_M V)^{2/3} / T_R \quad (18a)$$

$$D = (2\pi \eta_x Q_M / a_{Tx} \rho V^2)^{1/3} \quad (19a)$$

The implications of these results will perhaps be clearer if the various constants are suppressed and n, J, and D are shown as proportional only to the major design data. Thus we can say:

$$n \sim T_R V / Q_M \quad (21)$$

$$J \sim (Q_M V)^{2/3} / T_R \quad (22)$$

$$D \sim (Q_M / V^2)^{1/3} \quad (23)$$

It will be immediately obvious that these findings are nothing like the oft-quoted results that assume propeller coefficients constant and independent of the operating conditions, arbitrarily fixed propeller efficiency, of non-optimum propeller efficiency, or that ignore the necessity of matching propeller thrust to that required by the airplane and propeller torque to that available from the motor.

We have assumed that Q_M is not a function of n (if it were, (18) would not be a literal solution for (J), but it may be a function of the number of propeller turns, the integral of n. Consider the case where Q_M declines with turns released. In general, we can expect $\dot{Q}_M$ negative and $\dot{V}$ positive during such a climb, or most of it. How can we arrange for the propeller to be at least approximately optimal under these varying conditions? Or is this possible at all? The most common solution is to vary the blade angle, one school of thought recommending a high pitch at the start, declining as the climb progresses, the idea being to "save" precious motor turns at the beginning of the climb and to expend them more effectively later on; another school of thought recommends the exact opposite, starting with low β and increasing the pitch with time, the idea being to operate nearer to maximum propeller efficiency during the first critical few seconds when the motor is expending the bulk of its limited energy. Let us consider these policies in light of (21), (22) and (23).

For a propeller to satisfy (1) and (2) the advance ratio must be as in (22) and this J implies the specific β_x which must be selected to maximize efficiency. If torque speed and thrust vary jointly in such way that J varies, then blade angle must be made to vary as required to maximize η. But if torque, speed and thrust vary in any way which leaves the right hand side of (22) constant, there is no reason to vary propeller pitch. Observe that if torque decreases while speed increases, the numerator in (22) tends to remain more nearly constant than either variable by itself. How T_R varies is a matter of flight path optimization that has nothing directly to do with propeller efficiency. It seems plausible that both the numerator and denominator of (22) decline as the climb progresses, in which case, the required advance ratio may be nearly constant ... implying that propeller pitch should be constant. As seen in Figure 1, if J is larger, so too is β_x; if J is smaller, a lower blade angle is indicated for maximum efficiency. This helps explain why both the increasing-pitch and decreasing-pitch schools of thought have some merit when compared one to the other, for neither approach makes a direct attack on the complete problem of matching thrust, matching torque, and maximizing propeller efficiency all at the same time.

To do all three of these things, (23) tells us that propeller diameter must vary during the climb. Since Q_M is decreasing and V is increasing, both effects combine to indicate a decreasing D. Here there is no way to argue (as in (22)) that the changes might tend to offset one another; here the changes reinforce one another. Notice, however, that the optimum propeller diameter does not depend on the thrust required, only on motor torque and airplane speed. If more thrust is required at fixed torque and speed, the optimum design calls for a faster-turning prop, (21), operating at a lower advance ratio, (22), and consequently a lower blade angle. If torque varies by 5:1 and speed varies by 1:2, prop diameter should vary by 2.71:1 (equivalently, 1:0.369). Thus propeller diameter near the end of the climb should be only 37% of the propeller diameter at launch. Assuming constant thrust for illustration, the propeller speed at the end of the climb should be ten times faster than at launch. Since J is changing, and (22) says J near end-climb is only 0.543 of J at launch, propeller pitch must decrease.

It is, of course, not obvious that constant thrust is optimum from the flight-path point of view. If we assume that T_R declines in proportion to $(Q_M V)^{2/3}$, then thrust near end climb is 54.3% of thrust at launch, J is constant, the optimum blade angle is constant, and n at end-climb is 5.43 times as large as n at launch. Propeller diameter still decreases by 63.1% from the beginning to the end of the climb.

Ideally we know a pattern of thrust-required which optimizes the airplane trajectory and we know motor torque, $\neq f(n)$; the propeller must be designed to match both these conditions and simultaneously achieve maximum propeller efficiency. In general, this necessitates a propeller diameter that decreases as the climb progresses. If our propeller does not do this, then its efficiency is sub-optimal climb. We can also expect in general that propeller pitch must vary during the climb, but this is probably a much less significant effect.

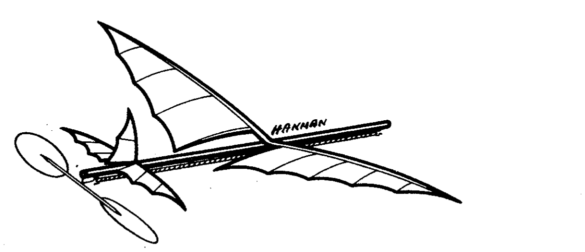
To match propeller torque to motor torque and to attain maximum efficiency when the ideal pattern of thrust-required is not known, we can regard T_R as being merely the thrust that is actually developed, and the plane will climb in accordance with whatever this happens to turn out to be. Perhaps the most significant implication of (21), (22) and (23) is that even in this much simplified situation — where all we are really asking is that torques be matched and propeller efficiency be maximized — the propeller diameter must still vary during the climb. Pitch may be constant, but diameter must vary. This follows because D in (23) does not depend upon T_R, only upon torque and speed. Varying ρ during the climb may or may not be desirable, or even necessary, but it is essential that D decrease as Q_M decreases and V increases.

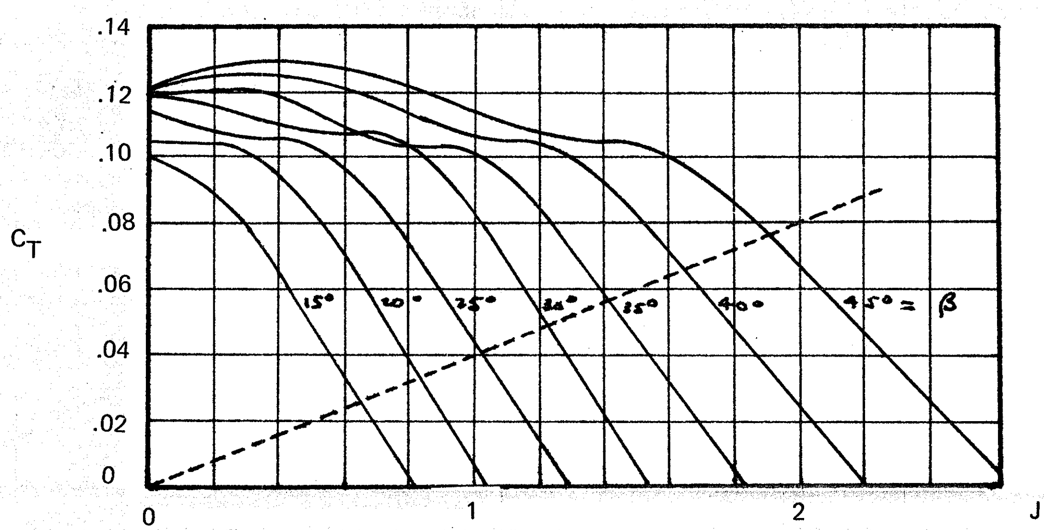
The case of variable-pitch, constant-speed propellers is irrelevant since we assume motor torque, while variable, is not a function of n. Hence, n tails along at whatever value is needed to satisfy conditions on the other parameters and variables.

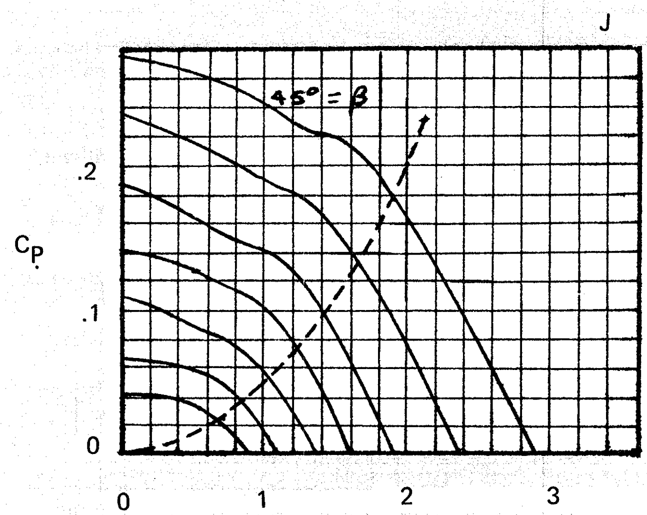
The essential conclusion of this analysis is that to absorb high launch-torque while maintaining propeller efficiency, the propeller diameter must be large at launch and small towards the end of the climb. Simply absorbing the torque while ignoring the efficiency is utterly trivial: one need merely increase β at launch until the blades are fully stalled. This provides an excellent ice-cream-mixer, but a very poor propeller indeed.

Probably the simplest way to approximate the requirements of (21), (22), (23) is to use an asymmetrical two-bladed propeller, retracting the larger blade at some pre-set torque level after it has served to utilize the initial torque-burst efficiently; the climb would then continue on the one smaller blade until it too was finally retracted as motor torque neared zero. If Q_M as a function of turns consists of a sharp exponential drop followed by a relatively flat plateau, this two-stage approximation should provide much better performance than anything achievable with a fixed-diameter propeller.

Recapitualting the two main features of this analysis will aid in evaluating it and in comparing its conclusions to those obtained elsewise. First, we simultaneously impose matching propeller torque to motor torque (which is not a function of n), maximizing propeller efficiency, and matching propeller thrust to thrust required (this last may be simplified away without affecting the main result). It seems obvious that all three must be done and that any study which does not consider all three together incomplete. Second, we utilize empirical propeller relations (11), (12) and (13) to gain two critically needed equations that enable closed-form results to be obtained. The results of (18a), (19a) and (20) are as good approximations as are (11) and (12) which were used in developing them. It seems obvious that saying ηx is independent of J is a far better approximation than saying η is independent of J; similarly, saying C_{Tx} is proportional to J seems a far better approximation than saying C_T is constant. Hence it appears that the important conclusion (19a) is not an artifact due to logical inconsistencies or empirical implausibilities in the analysis, but is qualitatively correct and provides a reasonable quantitative approximation to propeller diameter when motor torque is variable.







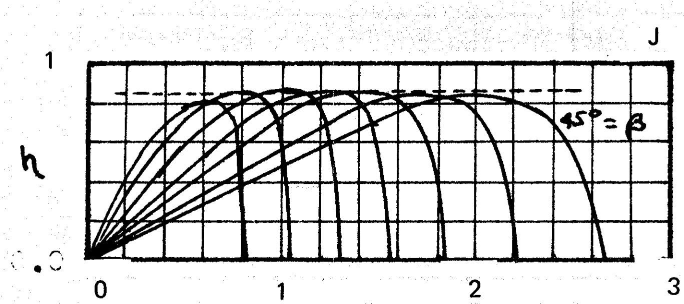


Figure 1

0 Thrust and Power Coefficients and Efficiency vs. Advance Ratio for Two-Blade NACA 5869-9 Propeller with Clark Y Airfoil 9% Thick at 0.75R.

(Adapted from NACA TR 640)

Note: Dashed curves illustrate empirical relations of Equations (12) and (13).

Numerical Example

In the equations n is in units of revolutions/second. All other variables are in consistent units of feet, second and pounds (which implies mass in slugs and density in slugs/ft3). Data must be converted to these units if they given differently and results, of course, may be finally converted to whatever units are convenient. Assume ηx = 0.8, a Tx = 0.04, and ρ = 0.0023 slg/ft3. Taking data for a typical Wakefield:

Launch: V = 20 ft/sec

QM = 46 in-oz.

TR = 9 oz.

Then: D = 38.4 in

J = .672

N = 560 rev/min

βx $\approx$ 17 deg

End of Climb:

V = 40 ft/sec

QM = 9 in-oz

TR = 5 oz

Then: D = 14.04 in

J = .646

N = 3182 rev/min

βx $\approx$ 16 deg

Optimum blade angles, based on Figure 1, are merely indicative. Note negligible change in advance ratio and optimum blade angle. A conventional propeller D is 24 to 30 in; its N starts at about 1000 rev/min and declines with time, averaging 200 to 700 rev/min for the entire climb.