Parabolized Navier-Stokes SOlution for Axisymmetric Ogive-Cylinder

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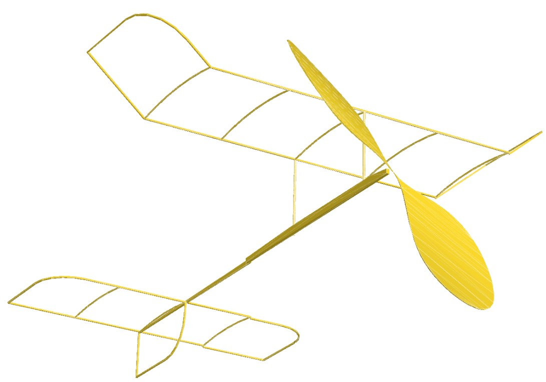

1 Derivation of the Cylinderical Navier Stokes Equations

In this section the system of governing equations given previously will be expanded in a cylinderical coordiniate system.

For cylinderical (x,r,ϕ) coordinates, the velocity vector $\overrightarrow{V}$ is given by:

$$\overrightarrow{V} = u\hat{e_x} + v\hat{e_r} + w\hat{e_\phi} \tag{1}$$

where $\hat{e_x}$, $\hat{e_r}$, and $\hat{e_\phi}$ are the unit vectors.

The del $\overrightarrow{\nabla}$ operator may be given as:

$$\overrightarrow{\nabla} = \hat{e_x}\frac{\partial}{\partial x} + \hat{e_r}\frac{\partial}{\partial r} + \frac{\hat{e_\phi}}{r}\frac{\partial}{\partial \phi} \tag{2}$$

Since the analysis that follows will require taking derivatives of the unit vectors, these expressions will be derived next.

The position vector $\overrightarrow{r}$ may be written in cartesian coordinates as:

$$\begin{aligned} \overrightarrow{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ &= x\hat{i} + r\cos\phi\hat{j} + r\sin\phi\hat{k} \end{aligned} \tag{3}$$

Now the equations for the unit vectors are:

$$\hat{e_x} = \frac{\partial \vec{\mathbf{r}}}{\partial x} = \hat{i} \tag{4}$$

$$\hat{e_r} = \frac{\partial \vec{\mathbf{r}}}{\partial r} = \cos\phi\hat{j} + \sin\phi\hat{k} \tag{5}$$

$$\hat{e_\phi} = \frac{\partial \vec{\mathbf{r}}}{\partial \phi} = -\sin\phi\hat{j} + \cos\phi\hat{k} \tag{6}$$

From these equations, the derivatives of the unit vectors can be found:

$$\frac{\partial \hat{e_x}}{\partial x} = \frac{\partial \hat{e_x}}{\partial r} = \frac{\partial \hat{e_x}}{\partial \phi} = 0 \tag{7}$$

$$\frac{\partial \hat{e_r}}{\partial x} = \frac{\partial \hat{e_r}}{\partial r} = 0 \tag{8}$$

$$\frac{\partial \hat{e_r}}{\partial \phi} = -\sin\phi\hat{j} + \cos\phi\hat{k} = \hat{e_\phi} \tag{9}$$

$$\frac{\partial \hat{e_\phi}}{\partial x} = \frac{\partial \hat{e_\phi}}{\partial r} = 0 \tag{10}$$

$$\frac{\partial \hat{e_\phi}}{\partial \phi} - -\cos\phi\hat{j} - \sin\phi\hat{k} = -\hat{e_r} \tag{11}$$

Using these expressions, the derivatives of a general vector function become:

$$
\begin{aligned}
\frac{\partial \vec{\mathbf{F}}}{\partial x} &= \frac{\partial F_1}{\partial x}\hat{e_x} + \frac{\partial F_2}{\partial x}\hat{e_r} + \frac{\partial F_3}{\partial x}\hat{e_\phi} \\
\frac{\partial \vec{\mathbf{F}}}{\partial r} &= \frac{\partial F_1}{\partial r}\hat{e_x} + \frac{\partial F_2}{\partial r}\hat{e_r} + \frac{\partial F_3}{\partial r}\hat{e_\phi} \\
\frac{\partial \vec{\mathbf{F}}}{\partial \phi} &= \frac{\partial F_1}{\partial \phi}\hat{e_x} + \left(\frac{\partial F_2}{\partial \phi} - F_3\right)\hat{e_r} + \left(\frac{\partial F_3}{\partial \phi} + F2\right)\hat{e_\phi}
\end{aligned}
\tag{12}
$$

where F_1, F_2, and F_3 are the vector components.

1.1 Continuity Equation

The continuity equation will be expanded first:

$$
\frac{D\rho}{Dt} + \rho\overrightarrow{\nabla} \cdot \overrightarrow{V} = 0 \tag{13}
$$

$$
\frac{\partial \rho}{\partial t} + \left(\overrightarrow{V} \cdot \overrightarrow{\nabla}\rho\right) + \rho\left(\overrightarrow{\nabla} \cdot \overrightarrow{V}\right) = 0 \tag{14}
$$

$$
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial r} + \frac{w}{r}\frac{\partial \rho}{\partial \phi} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial w}{\partial \phi} + \frac{v}{r}\right) = 0 \tag{15}
$$

After rearranging and combining terms, we get this form:

$$
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{1}{r}\frac{\partial}{\partial r}(\rho v r) + \frac{1}{r}\frac{\partial}{\partial \phi}(\rho w) = 0 \tag{16}
$$

1.2 Momentum Equation

Before expanding the momentum equation, it will be convenient to use the following for the shear stress matrix:

$$
\bar{\bar{\tau}} = \overrightarrow{\tau_1}\hat{e_x} + \overrightarrow{\tau_2}\hat{e_r} + \overrightarrow{\tau_3}\hat{e_\phi} \tag{17}
$$

Where:

$$
\begin{aligned}
\overrightarrow{\tau_1} &= \tau_{xx}\hat{e_x} + \tau_{xr}\hat{e_r} + \tau_{x\phi}\hat{e_\phi} \\
\overrightarrow{\tau_2} &= \tau_{rx}\hat{e_x} + \tau_{rr}\hat{e_r} + \tau_{r\phi}\hat{e_\phi} \\
\overrightarrow{\tau_3} &= \tau_{\phi x}\hat{e_x} + \tau_{\phi r}\hat{e_r} + \tau_{\phi\phi}\hat{e_\phi}
\end{aligned}
\tag{18}
$$

In this investigation, body forces will be neglected. The momentum equation becomes:

$$
\begin{aligned}
\rho\frac{D\vec{\mathbf{V}}}{Dt} &= \vec{\boldsymbol{\nabla}} \cdot \bar{\bar{\tau}} \\
\rho\{\frac{\partial\vec{\mathbf{V}}}{\partial t} + \vec{\mathbf{V}}\vec{\boldsymbol{\nabla}}\vec{\mathbf{V}}\} &= \vec{\boldsymbol{\nabla}} \cdot \bar{\bar{\tau}}
\end{aligned}
\tag{19}
$$

or

$$
\begin{aligned}
\rho\left\{\frac{\partial u}{\partial t}\hat{e_x} + \frac{\partial v}{\partial t}\hat{e_r} + \frac{\partial w}{\partial t}\hat{e_\phi} + u\frac{\partial\vec{\mathbf{V}}}{\partial t}\hat{e_x} + v\frac{\partial\vec{\mathbf{V}}}{\partial r}\hat{e_r} + \frac{w}{r}\frac{\partial\vec{\mathbf{V}}}{\partial\phi}\hat{e_\phi}\right\} \\
&= \hat{e_x}\cdot\frac{\partial}{\partial x}\{\overrightarrow{\tau_1}\hat{e_x} + \overrightarrow{\tau_2}\hat{e_r} + \overrightarrow{\tau_3}\hat{e_\phi}\} \\
&+ \hat{e_r}\cdot\frac{\partial}{\partial r}\{\overrightarrow{\tau_1}\hat{e_x} + \overrightarrow{\tau_2}\hat{e_r} + \overrightarrow{\tau_3}\hat{e_\phi}\} \\
&+ \frac{\hat{e_\phi}}{r}\cdot\frac{\partial}{\partial\phi}\{\overrightarrow{\tau_1}\hat{e_x} + \overrightarrow{\tau_2}\hat{e_r} + \overrightarrow{\tau_3}\hat{e_\phi}\} \\
&= \frac{\partial\overrightarrow{\tau_1}}{\partial x} + \frac{\partial\partial\overrightarrow{\tau_2}}{\partial r} + \frac{1}{r}\left\{\frac{\partial\overrightarrow{\tau_3}}{\partial\phi} + \overrightarrow{\tau_2}\right\}
\end{aligned}
\tag{20}
$$

By expanding this last expression and rearranging terms, the following form results for the momentum equation

$$\rho\left\{\frac{\partial u}{\partial t}\hat{e_x}+\frac{\partial v}{\partial t}\hat{e_r}+\frac{\partial w}{\partial t}\hat{e_\phi}+u\frac{\overrightarrow{V}}{\partial t}\hat{e_x}+v\frac{\overrightarrow{V}}{\partial r}\hat{e_r}+\frac{w}{r}\frac{\overrightarrow{V}}{\partial \phi}\hat{e_\phi}\right\}a$$

$$\begin{aligned}
&+\rho u\left\{\frac{\partial u}{\partial x}\hat{e_x}+\frac{\partial v}{\partial x}\hat{e_r}+\frac{\partial w}{\partial x}\hat{e_\phi}\right\}\\
&+\rho v\left\{\frac{\partial u}{\partial r}\hat{e_x}+\frac{\partial v}{\partial r}\hat{e_r}+\frac{\partial w}{\partial r}\hat{e_\phi}\right\}\\
&+\frac{\rho w}{r}\left\{\frac{\partial u}{\partial \phi}\hat{e_x}+\left(\frac{\partial v}{\partial \phi}-r\right)\hat{e_r}+\left(\frac{\partial w}{\partial \phi}+v\right)\hat{e_\phi}\right.\\
&=\tau_{xx}\hat{e_x}+\left\{\tau_{xr}\hat{e_r}+\tau_{x\phi}\hat{e_\phi}\right.\\
\overrightarrow{\tau_2}&=\tau_{rx}\hat{e_x}+\tau_{rr}\hat{e_r}+\tau_{r\phi}\hat{e_\phi}\\
\overrightarrow{\tau_3}&=\tau_{\phi x}\hat{e_x}+\tau_{\phi r}\hat{e_r}+\left.\tau_{\phi\phi}\hat{e_\phi}\right\}
\end{aligned} \tag{21}$$

Contents

References

[] W. Durand. *Aerodynamic Theory*, volume IV. Dover Publications, Inc, 1963.