A Time-Split MacCormack Scheme for Two-Dimensional Nonlinear Reaction-Diffusion Equations

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Abstract. A three-level explicit time-split MacCormack scheme is proposed for solving the two-dimensional nonlinear reaction-diffusion equations. The computational cost is reduced thank to the splitting and the explicit MacCormack scheme. Under the well known condition of Courant-Friedrich-Lewy (CFL) for stability of explicit numerical schemes applied to linear parabolic partial differential equations, we prove the stability and convergence of the method in $L^{\infty}(0,T;L^2)$ -norm. A wide set of numerical evidences which provide the convergence rate of the new algorithm are presented and critically discussed.

Keywords: 2D nonlinear reaction-diffusion equations, locally one-dimensional operators (splitting), explicit MacCormack scheme, a three-level explicit time-split MacCormack method, stability and convergence rate.

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1 Introduction and motivation

A large number of biological problems of significant interest are modeled by parabolic equations [9]. The general framework is a set of biological entities (either ions, molecules, proteins or cells) that interact with each other and diffuse within a given domain. So it becomes possible to build some models via reaction-diffusion equations. For example, the dendritic spines possess a twitching motion which are described by the reaction-diffusion models [12]. In this paper, we consider the following two-dimensional reaction-diffusion equations,

$$u_t - a\Delta u = f(u), \quad (x, y) \in \Omega, \quad t \in (0, T]; \tag{1}$$

with the initial condition

$$u(x, y, 0) = u_0(x, y), \quad (x, y) \in \overline{\Omega}; \tag{2}$$

and the boundary condition

$$u(x, y, t) = \varphi(x, y, t), \quad (x, y) \in \partial\Omega, \quad t \in (0, T];$$
 (3)

where a is the diffusive coefficient, $f \in C^1(\mathbb{R})$ is a Lipschitz function, $\Omega = (0,1)^2$, Δ denotes the Laplacian operator, $\partial\Omega$ is the boundary of Ω and u_t designates $\frac{\partial u}{\partial t}$. The initial condition u_0 and the boundary condition φ are assumed to be regular enough and satisfy the requirement $\varphi(x, y, 0) = u_0(x, y)$, for every $(x, y) \in \partial\Omega$, so that the initial value problem (1)-(3), admits a smooth solution.

In the last decades [23, 19, 32], MacCormack approach which is a predictor-corrector, finite difference scheme has been used to solve certain classes of nonlinear partial differential equations (PDEs). There exist both explicit and implicit versions of the method, but the explicit predates the implicit by more than a

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decade, and it is considered as one of the milestones of computational fluid dynamics. Both versions facilitate the solution of parabolic and hyperbolic equations by marching forward in time [19, 20, 21]. The popularity of MacCormack explicit method is due in part to its simplicity and ease of implementation. The predictor and corrector phases each uses forward differencing for first-order time derivatives, with alternate one-side differencing for first-order space derivatives. This is especially convenient for systems of equations with nonlinear advertive jacobian matrices associated with one-side explicit schemes, such as Lax-Wendroff approach (for instance, see [13, 25, 32]). However, the explicit MacCormack is not a suitable method for solving high Reynolds numbers flows, where the viscous regions become very thin (see [2], P. 630). To overcome this difficulty, MacCormack [18] developed a hybrid version of his scheme, known as the MacCormack rapid solver method. The new algorithm is an explicit-implicit method. For example, in a search of an efficient solution, the authors [26, 27, 31, 29] applied this hybrid method to some complex PDEs (such as: mixed Stokes-Darcy model and 2D incompressible Navier-Stokes equations) and they obtained satisfactory results regarding both stability and convergence rate of the method. It is worth noticing to mention that the rapid solver algorithm has a good stability condition and it is too faster than a large set of numerical methods for solving steady and unsteady flows at high to low Reynolds numbers [18]. So, the hybrid method of MacCormack will be used to solve the 2D reaction-diffusion equations (1)-(3), in our future works.

Armed with the information gleaned from both MacCormack and MacCormack rapid solver methods, we can now analyze a time-split MacCormack technique applied to problem (1)-(3). Firstly, it's worth noting to recall that the problem considered in this paper has been solved in literature by a wide set of explicit, implicit and coupled explicit-implicit numerical schemes. While some explicit methods usually suffer the severely restricted temporal step size [17, 35], the fully implicit methods although unconditionally stable, provide a large system of nonlinear equations at every time level [3, 15]. These systems lead to a considerable computational cost in practical applications. A possible improvement is to use the second time discretization such as the linearized Cank-Nicolson, implicit-explicit and collocation approaches [10, 14, 5, 30, 34, 33, 38, 9, 16]. Unfortunately, at least two starting values are needed to begin these algorithms. These values can be obtained by the initial and boundary conditions and an additional iterative method. To overcome this difficulty, the authors [37] applied a two-level linearized compact ADI scheme to problem (1)-(3). The main results of their work (namely Theorems 1-2) have been proved under the assumptions that the time step $\Delta t = k$, mesh size $\Delta x = \Delta y = h$ and the ratio $\frac{\Delta t}{h^2} \leq C$, must be sufficiently small ([37], page 9, line above (3.26) and page 10, Theorem 2). These requirements in general are less restrictive and can make the method even more impractical. Furthermore, this paper represents an extension of the work in [28].

The time-split MacCormack approach we study for the initial-boundary value problem (1)-(3) is new, a three-level explicit predictor-corrector method, second order accurate in time and fourth order convergent in space, under the time step restriction: $\frac{ak}{h^2} \leq \frac{1}{2}$, and it is motivated by this time step restriction (indeed, lots of explicit schemes for solving equation (1)-(3), are stable under the well-known condition of Courant-Friedrich-Lewy: $\frac{4ak}{h^2} \leq 1$) and its efficiency and effectiveness. From this observation, it is obvious that: (a) a time-split MacCormack approach is more practical, (b) although the new algorithm and a two-level linearized compact ADI method have the same order of convergence, the linearized compact ADI scheme requires substantially more computer times to solve problem (1)-(3), than does a time-split MacCormack. An explicit time-split MacCormack algorithm [21, 28, 22] "splits" the original MacCormack scheme into a sequence of one-dimensional operations, thereby achieving a good stability condition. In other words, the splitting makes it possible to advance the solution in each direction with the maximum allowable time step. This is particularly advantageous if the allowable time steps Δt_x and Δt_y , are much different because of differences in the mesh spacings Δx and Δy . In order to explain this method, we will make use of the 1D difference operators $L_x(\Delta t_x)$ and $L_y(\Delta t_y)$. Setting $u_{ij}^n = u(x_i, y_j, t^n)$, the $L_x(\Delta t_x)$ operator applied to u_{ij}^n ,

$$u_{ij}^* = L_x(\Delta t_x)u_{ij}^n,\tag{4}$$

is by definition equivalent to the two-step predictor-corrector MacCormack formulation. The $L_y(\Delta t_y)$ operator is defined in a similar manner, that is,

$$u_{ij}^* = L_y(\Delta t_y) u_{ij}^n. (5)$$

These expressions make use of a dummy time index, which is denoted by the asterisk. Now, letting $\Delta t_x = \Delta t$ and $\Delta t_y = \frac{\Delta t}{2m}$, where m is a positive integer, a second order accurate scheme can be constructed by applying

the L_x and Ly operators to u_{ij}^n , in the following way:

$$u_{ij}^{n+1} = \left[L_y \left(\frac{\Delta t}{2m} \right) \right]^m L_x(\Delta t) \left[L_y \left(\frac{\Delta t}{2m} \right) \right]^m u_{ij}^n.$$

This sequence is quite useful for the case $\Delta y \ll \Delta x$. In general, a scheme formed by a sequence of these operators is: (1) stable, if the time step of each operator does not exceed the allowable time step for that operator; (2) consistent, if the sums of the time steps for each of the operators are equal: and (3) second-order accurate, if the sequence is symmetric.

In this paper, we are interested in a numerical solution of the initial-boundary value problem (1)-(3), using a time-split MacCormack approach. Specifically, the work is focused on the following four items:

- 1. full description of a three-level explicit time-split MacCormack scheme for solving the nonlinear reaction-diffusion equations (1)-(3);
- 2. stability analysis of the numerical scheme;
- **3.** error estimates of the method;
- **4.** a wide set of numerical examples which provide the convergence rate, confirms the theoretical results and shows the efficiency and effectiveness of the method.

Items 1, 2 and 3, are our original contributions since as far as we know, there is no available work in literature which solves the reaction-diffusion model (1)-(3), using a time-split MacCormack method.

The paper is organized as follows: Section 2 considers a detailed description of a three-level explicit time-split MacCormack method applied to problem (1)-(3). In section 3, we study the stability of the numerical scheme under the condition given above, while section 4 analyzes the error estimates and the convergence of the method. A large set of numerical examples which provides the convergence rate of the new algorithm and confirms the theoretical result (on the stability) are presented and discussed in section 5. We draw the general conclusion and present our future works in section 6.

2 Full description of a time-split MacCormack method

This section deals with the description of a three-level explicit time-split MacCormack method applied to two-dimensional nonlinear reaction-diffusion equations (1)-(3).

Let N and M be two positive integers. Set $k:=\Delta t=\frac{T}{N};\ h:=\Delta x=\Delta y=\frac{1}{M},$ be the time step and mesh size, respectively. Put $t^n=kn,\ t^*=(n+r)k,\ t^{**}=(n+s)k,$ where 0< r< s<1, so $t^*\in (t^n,t^{n+1}),\ t^{**}\in (t^*,t^{n+1});\ n=0,1,2,...,N-1;\ x_i=ih;\ y_j=jh;\ 0\leq i,j\leq M.$ Also, let $\Omega_k=\{t^n,0\leq n\leq N\};\ \overline{\Omega}_h=\{(x_i,y_j),0\leq i,j\leq M\};\ \Omega_h=\overline{\Omega}_h\cap\Omega$ and $\partial\Omega_h=\overline{\Omega}_h\cap\partial\Omega.$

Consider $U_h = \{u_{ij}^n, n = 0, 1, ..., N; i, j = 0, 1, 2, ..., M\}$ be the space of grid functions defined on $\Omega_h \times \Omega_k$. Letting

$$\delta_{t}u_{ij}^{*} = \frac{u_{ij}^{*} - u_{ij}^{n}}{k/2}; \quad \delta_{t}u_{ij}^{**} = \frac{u_{ij}^{**} - u_{ij}^{*}}{k}; \quad \delta_{t}u_{ij}^{n+1} = \frac{u_{ij}^{n+1} - u_{ij}^{**}}{k/2}; \quad \delta_{x}u_{i+\frac{1}{2},j}^{n} = \frac{u_{i+1,j}^{n} - u_{ij}^{n}}{h};$$

$$\delta_{y}u_{i,j+\frac{1}{2}}^{n} = \frac{u_{i,j+1}^{n} - u_{ij}^{n}}{h}; \quad \delta_{x}^{2}u_{ij}^{n} = \frac{\delta_{x}u_{i+\frac{1}{2},j}^{n} - \delta_{x}u_{i-\frac{1}{2},j}^{n}}{h}; \quad \delta_{y}^{2}u_{ij}^{n} = \frac{\delta_{y}u_{i,j+\frac{1}{2}}^{n} - \delta_{y}u_{i,j-\frac{1}{2}}^{n}}{h}.$$

$$(6)$$

Using this, we define the following norms and scalar products.

$$||u^n||_{L^2} = h\left(\sum_{i,j=1}^{M-1} |u_{ij}^n|^2\right)^{\frac{1}{2}}; ||\delta_x u^n||_{L^2} = h\left(\sum_{j=1}^{M-1} \sum_{i=0}^{M-1} |\delta_x u_{i+\frac{1}{2},j}^n|^2\right)^{\frac{1}{2}};$$

$$\|\delta_y u^n\|_{L^2} = h \left(\sum_{j=0}^{M-1} \sum_{i=1}^{M-1} |\delta_y u_{i,j+\frac{1}{2}}^n|^2 \right)^{\frac{1}{2}}; \|\delta_\lambda^2 u^n\|_{L^2} = h \left(\sum_{i,j=1}^{M-1} |\delta_\lambda^2 u_{ij}^n|^2 \right)^{\frac{1}{2}}, \tag{7}$$

where $\lambda = x$ or y. Furthermore, the scalar products are defined as

$$(u^n, v^n) = h^2 \sum_{i,j=1}^{M-1} u_{ij}^n v_{ij}^n; < \delta_x u^n, \delta_x v^n >_x = h^2 \sum_{j=1}^{M-1} \sum_{i=0}^{M-1} \delta_x u_{i+\frac{1}{2},j}^n \delta_x v_{i+\frac{1}{2},j}^n;$$

and

$$<\delta_y u^n, \delta_y v^n>_y = h^2 \sum_{i=0}^{M-1} \sum_{i=1}^{M-1} \delta_y u^n_{i,j+\frac{1}{2}} \delta_y v^n_{i,j+\frac{1}{2}}.$$
 (8)

The space $H^1(\Omega)$ is endowed with the norm $|\cdot|_{H^1}$ (respectively, $|\cdot|_{H^1}$) defined as

$$|u^n|_{H^1} = \left(\|\delta_x u^n\|_{L^2}^2 + \|\delta_y u^n\|_{L^2}^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|u^n\|_{H^1} = \left(\|u^n\|_{L^2}^2 + \|\delta_x u^n\|_{L^2}^2 + \|\delta_y u^n\|_{L^2}^2\right)^{\frac{1}{2}}.$$
 (9)

It is worth noticing to mention that a time-split MacCormack [22, 21] "splits" the original MacCormack scheme into a sequence of 1D operators, thereby achieving a less restrictive stability condition. In order words, the splitting makes it possible to advance the solution in each direction with the maximum allowable time step ([2], page 231).

In other to give a detailed description of this method, we consider the 1D difference operators $L_x(\Delta t_x)$ and $L_y(\Delta t_y)$ defined by equations (4) and (5), respectively. Following the approach presented in ([2], page 231), a second-order accurate scheme can be constructed by applying the L_x and L_y operators to u_{ij}^n in the following manner:

$$u_{ij}^{n+1} = L_y(k/2)L_x(k)L_y(k/2)u_{ij}^n. (10)$$

Using these tools, we are able to provide a three-level explicit time-split MacCormack method for solving the initial-boundary value problem (1)-(3). Putting $\Delta t_x = k$, $\Delta t_y = \frac{k}{2}$ and $\Delta x = \Delta y := h$, it comes from equations (4), (5) and (10) that

$$u_{ij}^* = L_y(k/2)u_{ij}^n; \ u_{ij}^{**} = L_x(k)u_{ij}^* = L_x(k)L_y(k/2)u_{ij}^n \quad \text{and} \quad u_{ij}^{n+1} = L_y(k/2)u_{ij}^{**}.$$
 (11)

In the following, we should find explicit expressions of equations $u_{ij}^* = L_y(k/2)u_{ij}^n$ and $u_{ij}^{**} = L_x(k)u_{ij}^*$. This will help to give an explicit formula of the equation $u_{ij}^{n+1} = L_y(k/2)u_{ij}^{**}$, which represents a "one-step time-split MacCormack algorithm". For the sake of simplicity, we use both notations: $u_{ij}^n = u_{ij}^n$ and $[u+v]_{ij}^n = u_{ij}^n + v_{ij}^n$.

The application of the Taylor series expansion about (x_i, y_j, t^n) at the predictor and corrector steps with time step k/2 yields

$$u_{ij}^{\overline{*}} = u_{ij}^{n} + \frac{k}{2} u_{t} \Big|_{ij}^{n} + \frac{k^{2}}{8} u_{2t} \Big|_{ij}^{n} + O(k^{3}); \quad u_{ij}^{\overline{*}} = u_{ij}^{n} + \frac{k}{2} u_{t} \Big|_{ij}^{\overline{*}} + \frac{k^{2}}{8} u_{2t} \Big|_{ij}^{\overline{*}} + O(k^{3}). \tag{12}$$

From the definition of the operator $L_{\nu}(k/2)$, let consider the equation

$$u_t - au_{yy} = f(u)$$
, which is equivalent to $u_t = au_{yy} + f(u)$. (13)

Using equation (13), it is not difficult to see that

$$u_{2t} = (au_{yy} + f(u))_{t} = a^{2}u_{4y} + a(f(u))_{yy} + (au_{yy} + f(u))f'(u).$$

This fact together with equation (12) provide

$$u_{ij}^{\overline{*}} = u_{ij}^{n} + \frac{k}{2} [au_{yy} + f(u)]_{ij}^{n} + \frac{k^{2}}{8} \left[a^{2}u_{4y} + a(f(u))_{yy} + (au_{yy} + f(u))f'(u) \right]_{ij}^{n} + O(k^{3});$$
(14)

$$u_{ij}^{\overline{\overline{*}}} = u_{ij}^{n} + \frac{k}{2} [au_{yy} + f(u)]_{ij}^{\overline{*}} + \frac{k^{2}}{8} \left[a^{2}u_{4y} + a(f(u))_{yy} + (au_{yy} + f(u))f'(u) \right]_{ij}^{\overline{*}} + O(k^{3}).$$
 (15)

Now, expanding the Taylor series about (x_i, y_j, t^n) with mesh size h using central difference representation to get

$$u_{yy,ij}^{n} = \delta_{y}^{2} u_{ij}^{n} + O(h^{2}); \quad (f(u))_{yy,ij}^{n} = \delta_{y}^{2} \left(f(u_{ij}^{n}) \right) + O(h^{2}); \quad u_{4y,ij}^{n} = \delta_{y}^{2} (\delta_{y}^{2} u_{ij}^{n}) + O(h^{2});$$

$$u_{yy,ij}^{\overline{*}} = \delta_{y}^{2} u_{ij}^{\overline{*}} + O(h^{2}); \quad (f(u))_{yy,ij}^{\overline{*}} = \delta_{y}^{2} \left(f(u_{ij}^{\overline{*}}) \right) + O(h^{2}); \quad u_{4y,ij}^{\overline{*}} = \delta_{y}^{2} (\delta_{y}^{2} u_{ij}^{\overline{*}}) + O(h^{2}), \tag{16}$$

where $\delta_y^2 w_{ij}^l$ is given by relation (6). Substituting equations (16) into equations (14) and (15) to obtain

$$u_{ij}^{\overline{*}} = u_{ij}^{n} + \frac{k}{2} [a\delta_{y}^{2} u_{ij}^{n} + f(u_{ij}^{n})] + k^{2} \rho_{ij}^{n} + O(k^{3} + kh^{2});$$
(17)

and

$$u_{ij}^{\overline{\overline{*}}} = u_{ij}^{n} + \frac{k}{2} [a \delta_{y}^{2} u_{ij}^{\overline{*}} + f(u_{ij}^{\overline{*}})] + k^{2} \rho_{ij}^{\overline{*}} + O(k^{3} + kh^{2}), \tag{18}$$

where

$$\rho_{ij}^{\alpha} = \frac{1}{8} \left[a^2 \delta_y^2 (\delta_y^2 u_{ij}^{\alpha}) + a \delta_y^2 f(u_{ij}^{\alpha}) + a \delta_y^2 u_{ij}^{\alpha} f'(u_{ij}^{\alpha}) + f(u_{ij}^{\alpha}) f'(u_{ij}^{\alpha}) \right], \tag{19}$$

where $\alpha = n, \overline{*}$. The term $f(u_{ij}^{\overline{*}})$ should be expressed as a function of $f(u_{ij}^n)$, $f'(u_{ij}^n)$ and $u_{t,ij}^n$. Applying the Taylor expansion about (x_i, y_j, t^n) with time step k/2 using forward difference representation to get

$$f(u_{ij}^{\overline{*}}) = f(u_{ij}^n) + \frac{k}{2} u_{t,ij}^n f'(u_{ij}^n) + O(k^2).$$
(20)

But, it comes from equation (13) and relations (16) that

$$u_{t,ij}^n = au_{yy,ij}^n + f(u_{ij}^n) = a\delta_y^2 u_{ij}^n + f(u_{ij}^n) + O(h^2).$$
(21)

This fact, together with equation (20) result in

$$f(u_{ij}^{\overline{*}}) = f(u_{ij}^n) + \frac{k}{2} \left[a \delta_y^2 u_{ij}^n + f(u_{ij}^n) \right] f'(u_{ij}^n) + O(k^2 + kh^2).$$
 (22)

Plugging equations (17), (18) and (22), straightforward computations give

$$u_{ij}^{\overline{\overline{*}}} = u_{ij}^n + \frac{k}{2} \left(a \delta_y^2 u_{ij}^n + f(u_{ij}^n) \right) + k^2 (2\rho_{ij}^n + \rho_{ij}^{\overline{*}}) + O(k^3 + kh^2).$$
 (23)

Taking the average of $u_{ij}^{\overline{*}}$ and $u_{ij}^{\overline{\overline{*}}}$ to get

$$\frac{u_{ij}^{\overline{*}} + u_{ij}^{\overline{*}}}{2} = u_{ij}^{n} + \frac{k}{2} \left(a \delta_{y}^{2} u_{ij}^{n} + f(u_{ij}^{n}) \right) + \frac{1}{2} k^{2} (3 \rho_{ij}^{n} + \rho_{ij}^{\overline{*}}) + O(k^{3} + kh^{2}), \tag{24}$$

where ρ_{ij}^{α} is given by relation (19).

On the other hand, to define the operator $L_x(k)$, we should consider the equation

$$u_t = au_{xx}. (25)$$

It comes from equation (25), that

$$u_{2t} = au_{xx,t} = a^2 u_{4x}. (26)$$

Applying the Taylor series expansion about (x_i, y_i, t^*) (where $t^* \in (t^n, t^{n+1})$, is the time used at the beginning of the next step in a time-split MacCormack scheme) with mesh size h using central difference representation, we obtain

$$u_{xx,ij}^* = \delta_x^2 u_{ij}^* + O(h^2); \ u_{4x,ij}^* = \delta_x^2 (\delta_x^2 u_{ij}^*) + O(h^2); \ u_{xx,ij}^{\overline{**}} = \delta_x^2 u_{ij}^{\overline{**}} + O(h^2); \ u_{4x,ij}^{\overline{**}} = \delta_x^2 (\delta_x^2 u_{ij}^{\overline{**}}) + O(h^2), \ (27)$$

where $\delta_x^2 u_{ij}^l$ is defined by equation (6). Also, expanding the Taylor series at the predictor and corrector steps about (x_i, y_j, t^*) with time step k using forward difference, it is not difficult to observe that

$$u_{ij}^{\overline{**}} = u_{ij}^* + ku_t)_{ij}^* + \frac{k^2}{2} u_{2t})_{ij}^* + O(k^3); \ u_{ij}^{\overline{**}} = u_{ij}^* + ku_t)_{ij}^{\overline{**}} + \frac{k^2}{2} u_{2t})_{ij}^{\overline{**}} + O(k^3).$$
 (28)

A combination of equations (28), (27), (25) and (26) provides

$$u_{ij}^{\overline{***}} = u_{ij}^* + ak\delta_x^2 u_{ij}^* + \frac{a^2 k^2}{2} \delta_x^2 (\delta_x^2 u_{ij}^*) + O(k^3 + kh^2); \quad u_{ij}^{\overline{***}} = u_{ij}^* + ak\delta_x^2 u_{ij}^{\overline{***}} + \frac{a^2 k^2}{2} \delta_x^2 (\delta_x^2 u_{ij}^{\overline{***}}) + O(k^3 + kh^2). \quad (29)$$

In order to obtain a simple expression of $\delta_x^2 u_{ij}^{\frac{\pi}{**}}$, we should use the first equation in (29). Tracking the infinitesimal term in this equation, direct computations give

$$\delta_x^2 u_{ij}^{**} = \delta_x^2 u_{ij}^* + ak \delta_x^2 \left(\delta_x^2 u_{ij}^* \right) + \frac{a^2 k^2}{2} \delta_x^2 (\delta_x^4 u_{ij}^*).$$

The truncation of this error term does not compromise the result. This fact, together with relation (29) yield

$$u_{ij}^{\overline{**}} = u_{ij}^* + ak\delta_x^2 u_{ij}^* + \frac{3a^2k^2}{2}\delta_x^2(\delta_x^2 u_{ij}^*) + O(k^3 + kh^2).$$
(30)

Taking the average of $u_{ij}^{\overline{**}}$ and $u_{ij}^{\overline{\overline{**}}}$, it is not hard to see that

$$\frac{u_{ij}^{\overline{**}} + u_{ij}^{\overline{**}}}{2} = u_{ij}^* + ak\delta_x^2 u_{ij}^* + a^2 k^2 \delta_x^2 (\delta_x^2 u_{ij}^*) + O(k^3 + kh^2). \tag{31}$$

In way similar, starting with the one-dimensional equation: $u_t - au_{yy} = f(u)$, expanding the Taylor series about (x_i, y_j, t^{**}) (where t^{**} represents the time used at the last step in a time-split MacCormack approach) at the predictor and corrector steps with time step k/2 and mesh size h, using forward difference representations to get

$$\frac{u_{ij}^{\overline{n+1}} + u_{ij}^{\overline{n+1}}}{2} = u_{ij}^{**} + \frac{k}{2} \left(a \delta_y^2 u_{ij}^{**} + f(u_{ij}^{**}) \right) + \frac{k^2}{2} (3 \gamma_{ij}^{**} + \gamma_{ij}^{\overline{n+1}}) + O(k^3 + kh^2), \tag{32}$$

where we set $\mu = **, \overline{n+1}$, and

$$\gamma_{ij}^{\mu} = \frac{1}{8} \left[a^2 \delta_y^2 (\delta_y^2 u_{ij}^{\mu}) + a \delta_y^2 f(u_{ij}^{\mu}) + a \delta_y^2 u_{ij}^{\mu} f'(u_{ij}^{\mu}) + f(u_{ij}^{\mu}) f'(u_{ij}^{\mu}) \right]. \tag{33}$$

To construct a three-level explicit time-split MacCormack method for solving the nonlinear reaction-diffusion equation (1)-(3), we must follow the ideas presented in the literature to construct the explicit MacCormack scheme[18, 19, 21, 22]. Specifically, we should neglect the terms of second order together with the infinitesimal term $O(k^3 + kh^2)$ in equations (24), (31) and (32). In addition, the terms u_{ij}^* , u_{ij}^{**} and u_{ij}^{n+1} must be defined as the average of predicted and corrected values, that is,

$$u_{ij}^* = \frac{u_{ij}^{\overline{*}} + u_{ij}^{\overline{*}}}{2}; \ u_{ij}^{**} = \frac{u_{ij}^{\overline{**}} + u_{ij}^{\overline{**}}}{2} \quad \text{and} \quad u_{ij}^{n+1} = \frac{u_{ij}^{\overline{n+1}} + u_{ij}^{\overline{n+1}}}{2}.$$
(34)

Thus, equations

$$u_{ij}^* = L_y(k/2)u_{ij}^n; \ u_{ij}^{**} = L_x(k)u_{ij}^* \quad \text{and} \quad u_{ij}^{n+1} = L_y(k/2)u_{ij}^{**},$$
 (35)

are by definition equivalent to

$$u_{ij}^* = u_{ij}^n + \frac{k}{2} \left(a \delta_y^2 u_{ij}^n + f(u_{ij}^n) \right); \quad u_{ij}^{**} = u_{ij}^* + ak \delta_x^2 u_{ij}^* \quad \text{and} \quad u_{ij}^{n+1} = u_{ij}^{**} + \frac{k}{2} \left(a \delta_y^2 u_{ij}^{**} + f(u_{ij}^{**}) \right). \tag{36}$$

Since the operator $L_y(k/2)L_x(k)L_y(k/2)$ is symmetric, this fact together with relations (24), (31) and (32) show that the obtained method is a three-level technique, an explicit predictor-corrector scheme, second order accurate in time and fourth order convergent in space. This theoretical result is confirmed by a wide

set of numerical examples (we refer the readers to section 5). From the definition of the linear operators " δ_x^2 " and " δ_y^2 " given by (6), equation (36) can be rewritten as follows. For n = 0, 1, ..., N - 1;

$$u_{ij}^* = u_{ij}^n + \frac{k}{2} \left(\frac{a}{h^2} (u_{i,j+1}^n - 2u_{ij}^n + u_{i,j-1}^n) + f(u_{ij}^n) \right), \quad i = 0, 1, ..., M; \quad j = 1, 2, ..., M - 1;$$
 (37)

$$u_{ij}^{**} = u_{ij}^{*} + \frac{ak}{h^{2}} (u_{i+1,j}^{*} - 2u_{ij}^{*} + u_{i-1,j}^{*}), \quad i = 1, 2, ..., M - 1; \quad j = 0, 1, ..., M;$$
(38)

$$u_{ij}^{n+1} = u_{ij}^{**} + \frac{k}{2} \left(\frac{a}{h^2} (u_{i,j+1}^{**} - 2u_{ij}^{**} + u_{i,j-1}^{**}) + f(u_{ij}^{**}) \right), \quad i = 0, 1, ..., M; \quad j = 1, 2, ..., M - 1,$$
(39)

with the initial and boundary conditions. For i, j = 0, 1, ..., M,

$$u_{ij}^{0} = u_{0}(x_{i}, y_{j}); \ u_{i0}^{n} = \varphi_{i0}^{n}; \ u_{iM}^{n} = \varphi_{iM}^{n}; \ u_{0j}^{n} = \varphi_{0j}^{n}; \ u_{Mj}^{n} = \varphi_{Mj}^{n}; \ u_{0j}^{*} = \varphi_{0j}^{n+1}; \ u_{Mj}^{*} = \varphi_{Mj}^{n+1}; \ u_{j0}^{*} = \varphi_{j0}^{n+1}; \ u_{Mj}^{*} = \varphi_{jM}^{n+1}; \ u_{i0}^{*} = \varphi_{i0}^{n}; \ u_{Mj}^{N} = \varphi_{iM}^{N};$$

$$u_{0j}^{*} = \varphi_{0j}^{N}; \ u_{Mj}^{*} = \varphi_{Mj}^{N}; \ u_{i0}^{*} = \varphi_{iM}^{N}; \ u_{i0}^{N} = \varphi_{i0}^{N}; \ u_{iM}^{N} = \varphi_{iM}^{N};$$

$$u_{0j}^{N} = \varphi_{0j}^{N}; \ u_{Mj}^{N} = \varphi_{Mj}^{N},$$

$$(40)$$

which represent a detailed description of a three-level explicit time-split MacCormack method applied to problem (1)-(3).

In the rest of this paper, we prove the stability, the error estimates and the convergence rate of a three-level time-split MacCormack approach under the time step restriction

$$\frac{2ak}{h^2} \le 1,\tag{41}$$

where a is the diffusive coefficient given in equation (1). We recall that k is the time step and h is the grid size. Estimate (41) is well known in literature as CFL condition for stability of the explicit schemes when solving linear parabolic equations. We assume that the analytical solution $\overline{u} \in L^{\infty}(0,T;L^{2}(\Omega)) \cap H^{1}(0,T;H^{3}(\Omega)) \cap H^{2}(0,T;H^{1}(\Omega)) \cap H^{2}(0,T;L^{2}(\Omega)) \cap L^{2}(0,T;H^{4}(\Omega))$, that is, there exists a positive constant \widetilde{C} , independent of both time step k and mesh size h, so that

$$\||\overline{u}|\|_{L^{\infty}(0,T;L^{2}(\Omega))} + \||\overline{u}|\|_{H^{1}(0,T;H^{3}(\Omega))} + \||\overline{u}|\|_{H^{2}(0,T;H^{1}(\Omega))} + \||\overline{u}|\|_{H^{2}(0,T;L^{2}(\Omega))} + \||\overline{u}|\|_{L^{2}(0,T;H^{4}(\Omega))} \leq \widetilde{C}. \tag{42}$$

3 Stability analysis of a three-level time-split MacCormack scheme

This section considers a deep analysis of the stability of a three-level time-split MacCormack scheme (37)-(40) for solving equations (1)-(3).

Theorem 3.1. Let u be the solution provided by the numerical scheme (37)-(40). Under the time step restriction (41), the following estimate holds

$$\max_{0 \le n \le N} ||u^n||_{L^2(\Omega)} \le \widetilde{C} + \exp\left(CT \sum_{l=0}^3 (Ck)^l\right),$$

where C is a positive constant independent of the time step k and mesh size h and \widetilde{C} is given by relation (42).

The following result (namely Lemmas 3.1) plays a crucial role in the proof of Theorem 3.1.

Lemma 3.1. Setting $u_{ij}^n=u(x_i,y_j,t^n)$ be the numerical solution provided by the scheme (37)-(40), $\overline{u}_{ij}^n=\overline{u}(x_i,y_j,t^n)$ be the exact one and let $e_{ij}^n=u_{ij}^n-\overline{u}_{ij}^n$ be the error. We recall that $\overline{u}_{ij}^*=\frac{\overline{u}_{ij}^*+\overline{u}_{ij}^*}{2}$, $\overline{u}_{ij}^{**}=\frac{\overline{u}_{ij}^{**}+\overline{u}_{ij}^{**}}{2}$, satisfy relations (24) and (31), respectively. u_{ij}^* and u_{ij}^{**} are given by equations (37) and (38), respectively. The following equalities hold:

$$a < \delta_x^2 e_{ij}^n, e_{ij}^n >_x = h^2 \sum_{j,i=1}^{M-1} \frac{a}{h^2} \left(e_{i+1,j}^n - 2e_{ij}^n + e_{i-1,j}^n \right) e_{ij}^n = -a \| \delta_x e^n \|_{L^2(\Omega)}^2, \tag{43}$$

and

$$a < \delta_y^2 e_{ij}^n, e_{ij}^n >_y = h^2 \sum_{i=1}^{M-1} \frac{a}{h^2} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n = -a \| \delta_y e^n \|_{L^2(\Omega)}^2, \tag{44}$$

where the operators δ_x and δ_y are given by relation (6).

Proof. (of Lemma 3.1). Firstly, it is not hard to observe that

$$\frac{a}{h^2} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n = \frac{a}{h} (\delta_y e_{i,j+\frac{1}{2}}^n - \delta_y e_{i,j-\frac{1}{2}}^n) e_{ij}^n,$$

for i = 0, 1, ..., M and j = 1, 2, ..., M - 1. We should prove only equation (43). The proof of relation (44) is similar.

It follows from the definition of the operator δ_x^2 and the scalar product $\langle \cdot, \cdot \rangle_x$ given by (6) and (8), respectively, that

$$a < \delta_{x}^{2} e_{ij}^{n}, e_{ij}^{n} >_{x} = h^{2} \sum_{i,j=1}^{M-1} \frac{a}{h^{2}} \left(e_{i+1,j}^{n} - 2e_{ij}^{n} + e_{i-1,j}^{n} \right) e_{ij}^{n} = a \sum_{i,j=1}^{M-1} \left[\left(e_{i+1,j}^{n} - e_{ij}^{n} \right) e_{ij}^{n} - \left(e_{i,j}^{n} - e_{i-1,j}^{n} \right) e_{ij}^{n} \right] = a h \sum_{i,j=1}^{M-1} \left[\left(\delta_{x} e_{i+\frac{1}{2},j}^{n} \right) e_{ij}^{n} - \left(\delta_{x} e_{i-\frac{1}{2},j}^{n} \right) e_{ij}^{n} \right] = a h \sum_{i,j=1}^{M-1} \left[\left(\delta_{x} e_{i+\frac{1}{2},j}^{n} \right) e_{ij}^{n} - \left(\delta_{x} e_{i-\frac{1}{2},j}^{n} \right) e_{ij}^{n} \right] = a h \sum_{j=1}^{M-1} \left\{ \left(\left(\delta_{x} e_{\frac{3}{2},j}^{n} \right) e_{1j}^{n} - \left(\delta_{x} e_{1,j}^{n} \right) + \left(\left(\delta_{x} e_{\frac{5}{2},j}^{n} \right) e_{2j}^{n} - \left(\delta_{x} e_{\frac{3}{2},j}^{n} \right) e_{2j}^{n} \right) + \left(\left(\delta_{x} e_{\frac{3}{2},j}^{n} \right) e_{3j}^{n} - \left(\delta_{x} e_{\frac{5}{2},j}^{n} \right) e_{5j}^{n} \right) + \dots + \left(\left(\delta_{x} e_{M-\frac{3}{2},j}^{n} \right) e_{M-2,j}^{n} - \left(\delta_{x} e_{M-\frac{5}{2},j}^{n} \right) e_{M-2,j}^{n} \right) + \left(\left(\delta_{x} e_{M-\frac{1}{2},j}^{n} \right) e_{M-1,j}^{n} - \left(\delta_{x} e_{M-\frac{3}{2},j}^{n} \right) e_{M-1,j}^{n} \right) \right\} = a h \sum_{j=1}^{M-1} \left\{ - \left(e_{2j}^{n} - e_{1j}^{n} \right) \delta_{x} e_{\frac{3}{2},j}^{n} - \left(e_{3j}^{n} - e_{2j}^{n} \right) \delta_{x} e_{\frac{5}{2},j}^{n} - \left(e_{4j}^{n} - e_{3j}^{n} \right) \delta_{x} e_{\frac{7}{2},j}^{n} - \dots - \left(e_{M-1,j}^{n} - e_{M-2,j}^{n} \right) \delta_{x} e_{M-\frac{3}{2},j}^{n} + \left(\delta_{x} e_{M-\frac{1}{2},j}^{n} \right) e_{M-1,j}^{n} - \left(\delta_{x} e_{\frac{1}{2},j}^{n} \right) e_{1,j}^{n} \right\}.$$

$$(45)$$

It comes from the boundary condition (40) that $e_{Mj}^n = e_{0j}^n = 0$. So $(\delta_x e_{M-\frac{1}{2},j}^n) e_{Mj}^n = 0$ and $(\delta_x e_{\frac{1}{2},j}^n) e_{0j}^n = 0$. This fact, together with equation (45) provide

$$h^{2} \sum_{i,j=1}^{M-1} \frac{a}{h^{2}} \left(e_{i+1,j}^{n} - 2e_{ij}^{n} + e_{i-1,j}^{n} \right) e_{ij}^{n} = ah^{2} \sum_{j,i=1}^{M-1} \left\{ - \left(\delta_{x} e_{\frac{3}{2},j}^{n} \right)^{2} - \left(\delta_{x} e_{\frac{5}{2},j}^{n} \right)^{2} - \dots - \left(\delta_{x} e_{M-\frac{3}{2},j}^{n} \right)^{2} - \left(\delta_{x} e_{M-\frac{3}{2},j}^{n} \right)^{2} - \dots - \left(\delta_{x} e_{M-\frac{3}{2},j}^{n} \right)^{2} - \left(\delta_{x} e_{M-\frac{3}{2},j}^{n} \right)^{2} - \dots - \left(\delta_{x} e_{M-\frac{3}{2},j}^{n} \right)^{2} - \left(\delta_{x} e_{M-\frac{3}{2},j}^{n} \right)^{2} - \dots - \left(\delta_{x$$

Proof. (of Theorem 3.1). A combination of equations (24), (34) and (37) gives

$$e_{ij}^* = e_{ij}^n + \frac{k}{2} \left(a \delta_y^2 e_{ij}^n + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) + \frac{k^2}{2} (3\rho_{ij}^n + \rho_{ij}^{\overline{*}}) + O(k^3 + kh^2), \tag{46}$$

where ρ_{ij}^{α} is defined by (19). Utilizing the definition of the operator " δ_y^2 ", equation (46) is equivalent to

$$e_{ij}^* = e_{ij}^n + \frac{k}{2} \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) + \frac{k^2}{2} (3\rho_{ij}^n + \rho_{ij}^{\overline{*}}) + O(k^3 + kh^2). \tag{47}$$

Of course, the aim of this study is to give a general picture of the stability analysis of the numerical scheme (37)-(40). Since the formulas can become quite heavy, for the sake of readability, we should neglect the

higher order terms in time step k and grid spacing h. However, the truncation of the infinitesimal terms does not compromise the result on the stability analysis. Using this, equation (47) becomes

$$e_{ij}^* = e_{ij}^n + \frac{k}{2} \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right).$$

Taking the square, it holds

$$(e_{ij}^*)^2 = (e_{ij}^n)^2 + k \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) e_{ij}^n$$

$$+ \frac{k^2}{4} \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right)^2.$$

$$(48)$$

Now, using equality $2(a-b)b = a^2 - b^2 - (a-b)^2$ and inequality $(a \pm b)^2 \le 2(a^2 + b^2)$, for any $a, b \in \mathbb{R}$, and by simple computations, it is not hard to see that

$$(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n)^2 \le 2\left[(e_{i,j+1}^n - e_{ij}^n)^2 + (e_{i,j-1}^n - e_{ij}^n)^2 \right]; \tag{49}$$

$$\left[\frac{a}{h^{2}}(e_{i,j+1}^{n} - 2e_{ij}^{n} + e_{i,j-1}^{n}) + f(u_{ij}^{n}) - f(\overline{u}_{ij}^{n})\right]^{2} \leq 2\left[\frac{2a^{2}}{h^{4}}\left[\left(e_{i,j+1}^{n} - e_{i,j}^{n}\right)^{2} + \left(e_{i,j}^{n} - e_{i,j-1}^{n}\right)^{2}\right] + \left(f(u_{ij}^{n}) - f(\overline{u}_{ij}^{n})\right)^{2}\right].$$
(50)

 $f \in \mathcal{C}^1(\mathbb{R})$ is a Lipschitz function, so there is a positive constant C independent of the time step k and the mesh size h so that

$$|f(u_{ij}^n) - f(\overline{u}_{ij}^n)| \le C|e_{ij}^n|. \tag{51}$$

From inequality (51), it is easy to see that

$$(f(u_{ij}^n) - f(\overline{u}_{ij}^n)) e_{ij}^n \le C(e_{ij}^n)^2 \text{ and } (f(u_{ij}^n) - f(\overline{u}_{ij}^n))^2 \le C^2(e_{ij}^n)^2.$$
 (52)

A combination of estimates (48)-(52) results in

$$(e_{ij}^*)^2 \le (e_{ij}^n)^2 + k \left\{ \frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) e_{ij}^n + C(e_{ij}^n)^2 \right\} + \frac{k^2}{2} \left\{ \frac{2a^2}{h^2} \left[\left(\delta_y e_{i,j+\frac{1}{2}}^n \right)^2 + \left(\delta_y e_{i,j-\frac{1}{2}}^n \right)^2 \right] + C^2(e_{ij}^n)^2 \right\}.$$

$$(53)$$

Summing this up from i, j = 1, 2, ..., M - 1, and rearranging terms, this provides

$$\sum_{i,j=1}^{M-1} (e_{ij}^*)^2 \le \sum_{i,j=1}^{M-1} (e_{ij}^n)^2 + Ck \left(1 + \frac{Ck}{2} \right) \sum_{i,j=1}^{M-1} (e_{ij}^n)^2 + \frac{ak}{h^2} \sum_{i,j=1}^{M-1} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) e_{ij}^n + \frac{a^2k^2}{h^2} \sum_{i,j=1}^{M-1} \left[\left(\delta_y e_{i,j+\frac{1}{2}}^n \right)^2 + \left(\delta_y e_{i,j-\frac{1}{2}}^n \right)^2 \right],$$

which implies

$$\sum_{i,j=1}^{M-1} (e_{ij}^*)^2 \le \sum_{i,j=1}^{M-1} (e_{ij}^n)^2 + Ck \left(1 + \frac{Ck}{2} \right) \sum_{i,j=1}^{M-1} (e_{ij}^n)^2 + \frac{ak}{h^2} \sum_{i,j=1}^{M-1} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) e_{ij}^n + \frac{2a^2k^2}{h^2} \sum_{i=1}^{M-1} \sum_{j=0}^{M-1} \left(\delta_y e_{i,j+\frac{1}{2}}^n \right)^2.$$
(54)

Multiplying both sides of inequality (54) by h^2 , and using equation (44) to get

$$||e^*||_{L^2(\Omega)}^2 \le ||e^n||_{L^2(\Omega)}^2 + Ck\left(1 + \frac{Ck}{2}\right) ||e^n||_{L^2(\Omega)}^2 - ak||\delta_y e^n||_{L^2(\Omega)}^2 + \frac{2a^2k^2}{h^2} ||\delta_y e^n||_{L^2(\Omega)}^2.$$

From the time step restriction (41), $-1 + \frac{2ak}{h^2} \le 0$, utilizing this, it follows

$$\|e^*\|_{L^2(\Omega)}^2 \le \|e^n\|_{L^2(\Omega)}^2 + Ck\left(1 + \frac{Ck}{2}\right) \|e^n\|_{L^2(\Omega)}^2.$$
(55)

In way similar, combining equations (32), (34) and (39), it is not hard to show that

$$||e^{n+1}||_{L^{2}(\Omega)}^{2} \le ||e^{**}||_{L^{2}(\Omega)}^{2} + Ck\left(1 + \frac{Ck}{2}\right)||e^{**}||_{L^{2}(\Omega)}^{2}.$$

$$(56)$$

We must find a similar estimate associated with $||e^{**}||_{L^2(\Omega)}^2$ and $||e^*||_{L^2(\Omega)}^2$. Using equations (31), (34) and (38), it holds

$$e_{ij}^{**} = e_{ij}^{*} + \frac{ak}{h^{2}}(e_{i+1,j}^{*} - 2e_{ij}^{*} + e_{i-1,j}^{*}),$$

for i = 1, 2, ..., M - 1, and j = 0, 1, ..., M. Taking the square, we obtain

$$(e_{ij}^{**})^2 = (e_{ij}^*)^2 + \frac{2ak}{h^2}(e_{i+1,j}^* - 2e_{ij}^* + e_{i-1,j}^*)e_{ij}^* + \frac{a^2k^2}{h^4}(e_{i+1,j}^* - 2e_{ij}^* + e_{i-1,j}^*)^2,$$

which implies

$$(e_{ij}^{**})^2 \le (e_{ij}^*)^2 + \frac{2ak}{h^2} (e_{i+1,j}^* - 2e_{ij}^* + e_{i-1,j}^*) e_{ij}^* + \frac{2a^2k^2}{h^4} \left[(e_{i+1,j}^* - e_{ij}^*)^2 + (e_{i-1,j}^* - e_{ij}^*)^2 \right], \tag{57}$$

Now, summing relation (57) for i, j = 1, 2, ..., M - 1, and multiplying the obtained equation by h^2 , this results in

$$h^{2} \sum_{i,j=1}^{M-1} (e_{ij}^{**})^{2} \leq h^{2} \sum_{i,j=1}^{M-1} (e_{ij}^{*})^{2} + 2ak \sum_{i,j=1}^{M-1} (e_{i+1,j}^{*} - 2e_{ij}^{*} + e_{i-1,j}^{*}) e_{ij}^{*} + 2a^{2}k^{2} \sum_{i,j=1}^{M-1} \left[(\delta_{x} e_{i+\frac{1}{2},j}^{*})^{2} + (\delta_{x} e_{i-\frac{1}{2},j}^{*})^{2} \right],$$

which implies

$$h^2 \sum_{i,j=1}^{M-1} (e_{ij}^{**})^2 \le h^2 \sum_{i,j=1}^{M-1} (e_{ij}^{*})^2 + 2ak \sum_{i,j=1}^{M-1} (e_{i+1,j}^{*} - 2e_{ij}^{*} + e_{i-1,j}^{*}) e_{ij}^{*} + 4a^2k^2 \sum_{i=0}^{M-1} \sum_{j=1}^{M-1} (\delta_x e_{i+\frac{1}{2},j}^{*})^2.$$

which is equivalent to

$$\|e^{**}\|_{L^2(\Omega)}^2 \leq \|e^*\|_{L^2(\Omega)}^2 + 2ak\sum_{i,j=1}^{M-1} (e_{i+1,j}^* - 2e_{ij}^* + e_{i-1,j}^*)e_{ij}^* + 4a^2k^2h^{-2}\|\delta_x e^*\|_{L^2(\Omega)}^2.$$

Utilizing equality (43), this gives

$$||e^{**}||_{L^2(\Omega)}^2 \le ||e^*||_{L^2(\Omega)}^2 + 2ak\left(-1 + \frac{2ak}{h^2}\right) ||\delta_x e^*||_{L^2(\Omega)}^2.$$
(58)

It comes from the time step restriction (41) that $\frac{2ak}{h^2} \leq 1$. So, estimate (58) provides

$$\|e^{**}\|_{L^2(\Omega)}^2 \le \|e^*\|_{L^2(\Omega)}^2.$$
 (59)

Now, plugging inequalities (55), (56) and (59), straightforward calculations yield

$$\|e^{n+1}\|_{L^2(\Omega)}^2 \le \left[1 + Ck\left(1 + \frac{Ck}{2}\right)\right]^2 \|e^n\|_{L^2(\Omega)}^2 = \|e^n\|_{L^2(\Omega)}^2 + Ck\left[2 + 2Ck + C^2k^2 + \frac{1}{4}C^3k^3\right] \|e^n\|_{L^2(\Omega)}^2.$$

Summing this up from n = 0, 1, 2, ..., p - 1, for any nonnegative integer p satisfying $1 \le p \le N$, to get

$$||e^{p}||_{L^{2}(\Omega)}^{2} \leq ||e^{0}||_{L^{2}(\Omega)}^{2} + Ck\left[2 + 2Ck + C^{2}k^{2} + \frac{1}{4}C^{3}k^{3}\right] \sum_{n=0}^{p-1} ||e^{n}||_{L^{2}(\Omega)}^{2}.$$

$$(60)$$

It comes from the initial condition given in (40), that $e_{ij}^0 = 0$, for $0 \le i, j \le M$. Applying the discrete Gronwall Lemma, estimate (60) gives

$$||e^p||_{L^2(\Omega)}^2 \le \exp\left\{Ckp\left(2 + 2Ck + C^2k^2 + \frac{1}{4}C^3k^3\right)\right\}.$$
 (61)

But $k = \frac{T}{N}$, so $Ckp = CT\frac{p}{N} \leq CT$ (since $p \leq N$). This fact, together with estimate (61) result in

$$||e^p||_{L^2(\Omega)}^2 \le \exp\left\{CT\left(2 + 2Ck + C^2k^2 + \frac{1}{4}C^3k^3\right)\right\}.$$

Taking the square root, it is easy to see that

$$||e^p||_{L^2(\Omega)} \le \exp\left\{CT\left(1 + Ck + \frac{1}{2}C^2k^2 + \frac{1}{8}C^3k^3\right)\right\}.$$
 (62)

We have that $||u^p||_{L^2(\Omega)} - ||\overline{u}^p||_{L^2(\Omega)} \le ||u^p - \overline{u}^p||_{L^2(\Omega)} = ||e^p||_{L^2(\Omega)}$. A combination of this inequality together with relation (62) yields

$$||u^{p}||_{L^{2}(\Omega)} \leq ||\overline{u}^{p}||_{L^{2}(\Omega)} + \exp\left\{CT\left(1 + Ck + \frac{1}{2}C^{2}k^{2} + \frac{1}{8}C^{3}k^{3}\right)\right\} \leq ||\overline{u}^{p}||_{L^{2}(\Omega)} + \exp\left\{CT\sum_{l=0}^{3}(Ck)^{l}\right\}.$$

Since \overline{u} is the exact solution, the proof of Theorem 3.1 is completed thanks to estimate (42).

4 Convergence of the method

This section deals with the error estimates of a three-level time-split MacCormack method (37)-(40) applied to equations (1)-(3), under the time step restriction (41). We assume that the exact solution \overline{u} satisfies estimate (42). We recall that

$$\mathcal{U}_h = \{u_{ij}^n, n = 0, 1, 2, ..., N; i, j = 0, 1, 2, ..., M\},$$
(63)

is the space of grid functions defined on $\Omega_h \times \Omega_k$, where $\Omega_k = \{t^n, 0 \le n \le N\}$ and $\Omega_h = \{(x_i, y_j), 0 \le i, j \le M\} \cap \Omega$.

Let introduce the following discrete norms

$$|||u|||_{L^{\infty}(0,T;L^{2}(\Omega))} = \max_{0 \le n \le N} ||u^{n}||_{L^{2}(\Omega)}; \quad |||u|||_{L^{2}(0,T;L^{2}(\Omega))} = \left(k \sum_{n=0}^{N} ||u^{n}||_{L^{2}(\Omega)}^{2}\right)^{\frac{1}{2}},$$

and

$$|||u|||_{L^1(0,T;L^2(\Omega))} = k \sum_{n=0}^N ||u^n||_{L^2(\Omega)}; \text{ for } u \in \mathcal{U}_h.$$
(64)

Theorem 4.1. Suppose u be the solution provided by a three-level time-split MacCormack approach (37)-(40). Under the time step restriction (41), the error term $e = u - \overline{u}$, satisfies

$$|||e|||_{L^{\infty}(0,T;L^{2}(\Omega))} \le O(k+kh^{2}) = O(k+h^{4}).$$

The proof of Theorem 4.1 requires some intermediate results (namely Lemmas 3.1, 4.1 and 4.2).

Lemma 4.1. Consider $v \in H^4(\Omega)$, be a function satisfying $v|_{[x_i,x_{i+1}]} \in C^6[x_i,x_{i+1}]$, for i = 0,1,2,...,M-1. Then, it holds

$$\frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) - v_{2x,i} = \frac{h^2}{12}v_{4x,i} - \frac{h^4}{720}\left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)})\right], \quad for \quad i = 1, 2, ..., M - 1$$

where $\theta_i^{(4)} \in (x_{i-1}, x_i)$, $\theta_i^{(3)} \in (x_i, x_{i+1})$ and v_{mx} denotes the derivative of order m of v. Furthermore, for i = 2, 3, ..., M-2,

$$\frac{1}{h^4}(v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}) - v_{4x,i} = h^2 \left\{ \frac{1}{720} \left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)}) \right] + \frac{241}{3220} \left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)}) \right] \right\},$$

where
$$\theta_i^{(2)} \in (x_{i-2}, x_{i-1}), \ \theta_i^{(4)} \in (x_{i-1}, x_i), \ \theta_i^{(3)} \in (x_i, x_{i+1}) \ and \ \theta_i^{(1)} \in (x_{i+1}, x_{i+2}).$$

Proof. (of Lemma 4.1) Expanding the Taylor series about x_i with grid spacing h using both forward and backward differences to obtain

$$v_{i+2} = v_{i+1} + hv_{x,i+1} + \frac{h^2}{2}v_{2x,i+1} + \frac{h^3}{6}v_{3x,i+1} + \frac{h^4}{24}v_{4x,i+1} + \frac{h^5}{120}v_{5x,i+1} + \frac{h^6}{720}v_{6x}(\theta_i^{(1)}), \tag{65}$$

where $\theta_i^{(1)} \in (x_{i+1}, x_{i+2});$

$$v_{i-2} = v_{i-1} - hv_{x,i-1} + \frac{h^2}{2}v_{2x,i-1} - \frac{h^3}{6}v_{3x,i-1} + \frac{h^4}{24}v_{4x,i-1} - \frac{h^5}{120}v_{5x,i-1} + \frac{h^6}{720}v_{6x}(\theta_i^{(2)}), \tag{66}$$

where $\theta_i^{(2)} \in (x_{i-2}, x_{i-1});$

$$v_{i} = v_{i+1} - hv_{x,i+1} + \frac{h^{2}}{2}v_{2x,i+1} - \frac{h^{3}}{6}v_{3x,i+1} + \frac{h^{4}}{24}v_{4x,i+1} - \frac{h^{5}}{120}v_{5x,i+1} + \frac{h^{6}}{720}v_{6x}(\theta_{i}^{(3)}), \tag{67}$$

where $\theta_i^{(3)} \in (x_i, x_{i+1});$

$$v_{i} = v_{i-1} + hv_{x,i-1} + \frac{h^{2}}{2}v_{2x,i-1} + \frac{h^{3}}{6}v_{3x,i-1} + \frac{h^{4}}{24}v_{4x,i-1} + \frac{h^{5}}{120}v_{5x,i-1} + \frac{h^{6}}{720}v_{6x}(\theta_{i}^{(4)}), \tag{68}$$

where $\theta_i^{(4)} \in (x_{i-1}, x_i)$.

In way similar, applying the Taylor expansion for both derivative and higher order derivatives of v to obtain

$$v_{x,i+1} = v_{x,i} + hv_{2x,i} + \frac{h^2}{2}v_{3x,i} + \frac{h^3}{6}v_{4x,i} + \frac{h^4}{24}v_{5x,i} + \frac{h^5}{120}v_{6x}(\theta_i^{(5)}), \tag{69}$$

where $\theta_i^{(5)} \in (x_i, x_{i+1});$

$$v_{x,i-1} = v_{x,i} - hv_{2x,i} + \frac{h^2}{2}v_{3x,i} - \frac{h^3}{6}v_{4x,i} + \frac{h^4}{24}v_{5x,i} - \frac{h^5}{120}v_{6x}(\theta_i^{(6)}), \tag{70}$$

where $\theta_i^{(6)} \in (x_{i-1}, x_i);$

$$v_{2x,i+1} = v_{2x,i} + hv_{3x,i} + \frac{h^2}{2}v_{4x,i} + \frac{h^3}{6}v_{5x,i} + \frac{h^4}{24}v_{6x}(\theta_i^{(7)}), \tag{71}$$

where $\theta_i^{(7)} \in (x_i, x_{i+1});$

$$v_{2x,i-1} = v_{2x,i} - hv_{3x,i} + \frac{h^2}{2}v_{4x,i} - \frac{h^3}{6}v_{5x,i} + \frac{h^4}{24}v_{6x}(\theta_i^{(8)}), \tag{72}$$

where $\theta_i^{(8)} \in (x_{i-1}, x_i);$

$$v_{3x,i+1} = v_{3x,i} + hv_{4x,i} + \frac{h^2}{2}v_{5x,i} + \frac{h^3}{6}v_{6x}(\theta_i^{(9)}), \quad v_{3x,i-1} = v_{3x,i} - hv_{4x,i} + \frac{h^2}{2}v_{5x,i} - \frac{h^3}{6}v_{6x}(\theta_i^{(10)}), \quad (73)$$

where $\theta_i^{(9)} \in (x_i, x_{i+1}), \, \theta_i^{(10)} \in (x_{i-1}, x_i);$

$$v_{4x,i+1} = v_{4x,i} + hv_{5x,i} + \frac{h^2}{2}v_{6x}(\theta_i^{(11)}), \quad v_{4x,i-1} = v_{4x,i} - hv_{5x,i} + \frac{h^2}{2}v_{6x}(\theta_i^{(12)}), \tag{74}$$

where $\theta_i^{(11)} \in (x_i, x_{i+1}), \ \theta_i^{(12)} \in (x_{i-1}, x_i);$

$$v_{5x,i+1} = v_{5x,i} + hv_{6x}(\theta_i^{(13)}), \quad v_{5x,i-1} = v_{5x,i} - hv_{6x}(\theta_i^{(12)}), \tag{75}$$

where $\theta_i^{(13)} \in (x_i, x_{i+1}), \, \theta_i^{(14)} \in (x_{i-1}, x_i).$

Now, adding equations (67)-(68) side by side, this gives

$$2v_{i} = v_{i+1} + v_{i-1} - h(v_{x,i+1} - v_{x,i-1}) + \frac{h^{2}}{2}(v_{2x,i+1} + v_{2x,i-1}) - \frac{h^{3}}{6}(v_{3x,i+1} - v_{3x,i-1}) + \frac{h^{4}}{24}(v_{4x,i+1} + v_{4x,i-1}) - \frac{h^{5}}{120}(v_{5x,i+1} - v_{5x,i-1}) + \frac{h^{6}}{720}(v_{6x}(\theta_{i}^{(3)}) + v_{6x}(\theta_{i}^{(4)})).$$
 (76)

Subtracting (70) from (69) and adding side by side (71) and (72), using also equations (73), (74) and (81), simple calculations provide

$$v_{x,i+1} - v_{x,i-1} = 2hv_{2x,i} + \frac{h^3}{3}v_{4x,i} + \frac{h^5}{720}\left(v_{6x}(\theta_i^{(5)}) + v_{6x}(\theta_i^{(6)})\right); \tag{77}$$

$$v_{2x,i+1} + v_{2x,i-1} = 2v_{2x,i} + h^2 v_{4x,i} + \frac{h^4}{24} \left(v_{6x}(\theta_i^{(7)}) + v_{6x}(\theta_i^{(8)}) \right); \tag{78}$$

$$v_{3x,i+1} - v_{3x,i-1} = 2hv_{4x,i} + \frac{h^3}{6} \left(v_{6x}(\theta_i^{(9)}) + v_{6x}(\theta_i^{(10)}) \right); \tag{79}$$

$$v_{4x,i+1} + v_{4x,i-1} = 2v_{4x,i} + \frac{h^2}{2} \left(v_{6x}(\theta_i^{(11)}) + v_{6x}(\theta_i^{(12)}) \right); \quad v_{5x,i+1} - v_{5x,i-1} = h \left(v_{6x}(\theta_i^{(13)}) + v_{6x}(\theta_i^{(14)}) \right). \quad (80)$$

Combining equations (69)-(80), straightforward computations result in

$$2v_{i} = v_{i+1} + v_{i-1} - h^{2}v_{2x,i} - \frac{h^{4}}{12}v_{4x,i} + h^{6} \left\{ \frac{1}{720} \left(v_{6x}(\theta_{i}^{(3)}) + v_{6x}(\theta_{i}^{(4)}) \right) - \frac{1}{120} \left(v_{6x}(\theta_{i}^{(5)}) + v_{6x}(\theta_{i}^{(6)}) \right) + \frac{1}{48} \left(v_{6x}(\theta_{i}^{(7)}) + v_{6x}(\theta_{i}^{(8)}) \right) - \frac{1}{36} \left(v_{6x}(\theta_{i}^{(9)}) + v_{6x}(\theta_{i}^{(10)}) \right) + \frac{1}{48} \left(v_{6x}(\theta_{i}^{(11)}) + v_{6x}(\theta_{i}^{(12)}) \right) - \frac{1}{120} \left(v_{6x}(\theta_{i}^{(13)}) + v_{6x}(\theta_{i}^{(13)}) \right) \right\}. \tag{81}$$

Since $\theta_i^{(3)}, \theta_i^{(5)}, \theta_i^{(7)}, \theta_i^{(9)}, \theta_i^{(11)}, \theta_i^{(13)} \in (x_i, x_{i+1})$ and $\theta_i^{(4)}, \theta_i^{(6)}, \theta_i^{(8)}, \theta_i^{(10)}, \theta_i^{(12)}, \theta_i^{(12)}, \theta_i^{(14)} \in (x_{i-1}, x_i)$, without loss of generality, we can assume that $\theta_i^{(3)} = \theta_i^{(5)} = \theta_i^{(7)} = \theta_i^{(9)} = \theta_i^{(11)} = \theta_i^{(13)}$ and $\theta_i^{(4)} = \theta_i^{(6)} = \theta_i^{(8)} = \theta_i^{(10)} = \theta_i^{(12)} = \theta_i^{(14)}$. Using this, relation (81) becomes

$$\frac{1}{h^2}(v_{i+1} - 2v_i + v_{i-1}) - v_{2x,i} = \frac{h^2}{12}v_{4x,i} - \frac{h^4}{720}\left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)})\right].$$

This completes the proof of the first item of Lemma 4.1. Now, let prove the second item of Lemma 4.1.

Plugging equations (65) and (67), (66) and (68), (67) and (68), respectively, it is not hard to see that

$$v_{i+2} - 2v_{i+1} + v_i = h^2 v_{2x,i+1} + \frac{h^4}{12} v_{4x,i+1} + \frac{h^6}{720} \left(v_{6x}(\theta_i^{(1)}) + (\theta_i^{(3)}) \right); \tag{82}$$

$$v_{i} - 2v_{i-1} + v_{i-2} = h^{2}v_{2x,i-1} + \frac{h^{4}}{12}v_{4x,i-1} + \frac{h^{6}}{720}\left(v_{6x}(\theta_{i}^{(2)}) + v_{6x}(\theta_{i}^{(3)})\right);$$
(83)

and

$$4v_{i} = 2(v_{i+1} + v_{i-1}) - 2h(v_{x,i+1} - v_{x,i-1}) + h^{2}(v_{2x,i+1} + v_{2x,i-1}) - \frac{h^{3}}{3}(v_{3x,i+1} - v_{3x,i-1}) + \frac{h^{4}}{12}(v_{4x,i+1} + v_{4x,i-1}) - \frac{h^{5}}{60}(v_{5x,i+1} - v_{5x,i-1}) + \frac{h^{6}}{720}\left(v_{6x}(\theta_{i}^{(3)}) + v_{6x}(\theta_{i}^{(4)})\right).$$
(84)

A combination of equations (82)-(84) yields

$$v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2} = -2h(v_{x,i+1} - v_{x,i-1}) + 2h^2(v_{2x,i+1} + v_{2x,i-1}) - \frac{h^3}{3}(v_{3x,i+1} - v_{3x,i-1}) + 2h^2(v_{2x,i+1} + v_{2x,i-1}) - \frac{h^3}{3}(v_{3x,i+1} - v_{3x,i-1}) + 2h^2(v_{3x,i+1} + v_{3x,i-1}) - \frac{h^3}{3}(v_{3x,i+1} - v_{3x,i-1}) + 2h^2(v_{3x,i+1} + v_{3x,i-1}) - \frac{h^3}{3}(v_{3x,i+1} - v_{3x,i-1}) + 2h^2(v_{3x,i+1} - - v_{3x,i-1}) + 2h^2(v_{$$

$$\frac{h^4}{6}(v_{4x,i+1} + v_{4x,i-1}) - \frac{h^5}{60}(v_{5x,i+1} - v_{5x,i-1}) + \frac{h^6}{720}\left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)}) + 3\left(v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)})\right)\right]. \tag{85}$$

Substituting (77)-(80) into (85), simple computations result in

$$v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2} = -4h^2v_{2x,i} - \frac{2h^4}{3}v_{4x,i} + 4h^2v_{2x,i} + 2h^4v_{4x,i} - \frac{2h^4}{3}v_{4x,i} + h^6 \left\{ \frac{1}{720} \left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)}) + 3\left(v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)})\right) \right] - \frac{1}{60} \left(v_{6x}(\theta_i^{(5)}) + v_{6x}(\theta_i^{(6)}) \right) + \frac{1}{12} \left(v_{6x}(\theta_i^{(7)}) + v_{6x}(\theta_i^{(8)}) \right) - \frac{1}{18} \left(v_{6x}(\theta_i^{(9)}) + v_{6x}(\theta_i^{(10)}) \right) + \frac{1}{12} \left(v_{6x}(\theta_i^{(11)}) + v_{6x}(\theta_i^{(12)}) \right) - \frac{1}{60} \left(v_{6x}(\theta_i^{(13)}) + v_{6x}(\theta_i^{(14)}) \right) \right\},$$

which is equivalent to

$$v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2} = h^4 v_{4x,i} + h^6 \left\{ \frac{1}{720} \left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)}) + 3 \left(v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)}) \right) \right] - \frac{1}{60} \left(v_{6x}(\theta_i^{(5)}) + v_{6x}(\theta_i^{(6)}) \right) + \frac{1}{12} \left(v_{6x}(\theta_i^{(7)}) + v_{6x}(\theta_i^{(8)}) \right) - \frac{1}{18} \left(v_{6x}(\theta_i^{(9)}) + v_{6x}(\theta_i^{(10)}) \right) + \frac{1}{12} \left(v_{6x}(\theta_i^{(11)}) + v_{6x}(\theta_i^{(12)}) \right) - \frac{1}{60} \left(v_{6x}(\theta_i^{(13)}) + v_{6x}(\theta_i^{(14)}) \right) \right\}.$$

$$(86)$$

Assuming that $\theta_i^{(3)} = \theta_i^{(5)} = \theta_i^{(7)} = \theta_i^{(9)} = \theta_i^{(11)} = \theta_i^{(13)}$ and $\theta_i^{(4)} = \theta_i^{(6)} = \theta_i^{(8)} = \theta_i^{(10)} = \theta_i^{(12)} = \theta_i^{(14)}$, equation (86) becomes

$$v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2} = h^4 v_{4x,i} + h^6 \left\{ \frac{1}{720} \left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)}) + \right] + \frac{241}{3220} \left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)}) \right] \right\},$$

which is equivalent to

$$\frac{1}{h^4}(v_{i+2} - 4v_{i+1} + 6v_i - 4v_{i-1} + v_{i-2}) - v_{4x,i} = h^2 \left\{ \frac{1}{720} \left[v_{6x}(\theta_i^{(1)}) + v_{6x}(\theta_i^{(2)}) + \right] + \frac{241}{3220} \left[v_{6x}(\theta_i^{(3)}) + v_{6x}(\theta_i^{(4)}) \right] \right\}.$$

This ends the proof of Lemma 4.1.

Lemma 4.2. The term ρ_{ij}^n given by equation (19) can be bounded as

$$|\rho_{ij}^n| \le \widehat{C}_1 \left[1 + \widehat{C}_2 h^2 + \widehat{C}_3 h^4 \right],$$
 (87)

where \widehat{C}_l , l = 1, 2, 3, are positive constant independent of the time step k and the mesh size h.

Proof. (of Lemma 4.2). It comes from relation (19) that

$$\rho_{ij}^{n} = \frac{1}{8} \left[a^{2} \delta_{y}^{2} (\delta_{y}^{2} \overline{u}_{ij}^{n}) + a \delta_{y}^{2} f(\overline{u}_{ij}^{n}) + a \delta_{y}^{2} \overline{u}_{ij}^{n} f'(\overline{u}_{ij}^{n}) + f(\overline{u}_{ij}^{n}) f'(\overline{u}_{ij}^{n}) \right].$$

From the definition of the operator " δ_y^2 ", this is equivalent to

$$\begin{split} \rho_{ij}^n &= \frac{1}{8} \left[\frac{a^2}{h^4} \left(\overline{u}_{i,j+2}^n - 4 \overline{u}_{i,j+1}^n + 6 \overline{u}_{ij}^n - 4 \overline{u}_{i,j-1}^n + \overline{u}_{i,j-2} \right) + \frac{a}{h^2} \left[f(\overline{u}_{i,j+1}^n) - 2 f(\overline{u}_{ij}^n) + f(\overline{u}_{i,j-1}^n) + \left(\overline{u}_{i,j+1}^n - 2 \overline{u}_{ij}^n + \overline{u}_{i,j-1}^n \right) f'(\overline{u}_{ij}^n) \right] + f(\overline{u}_{ij}^n) f'(\overline{u}_{ij}^n) \right]. \end{split}$$

Combining this together with Lemma 4.1, it is easy to see that

$$\rho_{ij}^{n} = \frac{1}{8} \left\{ a^{2} \left[\overline{u}_{4y,ij}^{n} + h^{2} \left(\frac{1}{720} \left[\overline{u}_{6y}^{n}(x_{i}, \theta_{j}^{(1)}) + \overline{u}_{6y}^{n}(x_{i}, \theta_{j}^{(2)}) \right] + \frac{241}{3220} \left[\overline{u}_{6y}^{n}(x_{i}, \theta_{j}^{(3)}) + \overline{u}_{6y}^{n}(x_{i}, \theta_{j}^{(4)}) \right] \right) \right] + a \left[f o \overline{u} \right]_{2y,ij}^{n} + \frac{h^{2}}{12} f o \overline{u} \right]_{4y,ij}^{n} - \frac{h^{4}}{720} \left(f o \overline{u} \right]_{6y}^{n}(x_{i}, \theta_{j}^{(3)}) + f o \overline{u} \right]_{6y}^{n}(x_{i}, \theta_{j}^{(4)}) + \left(\overline{u}_{2y,ij}^{n} + \frac{h^{2}}{12} \overline{u}_{4y,ij}^{n} - \frac{h^{4}}{720} \left(\overline{u}_{6y}^{n}(x_{i}, \theta_{j}^{(3)}) + \overline{u}_{6y}^{n}(x_{i}, \theta_{j}^{(4)}) \right) \right) f'(\overline{u}_{ij}^{n}) \right] + f(\overline{u}_{ij}^{n}) f'(\overline{u}_{ij}^{n}) \right\}.$$

On the other hand, $\overline{u}(x,\cdot,t)|_{[y_j,y_{j+1}]}$, $fo\overline{u}(x,\cdot,t)|_{[y_j,y_{j+1}]} \in \mathcal{C}^6([y_j,y_{j+1}])$, for every $x \in (0,1)$, $t \in (0,T)$ and j=0,1,2,...,M-1; $|||\overline{u}||_{L^{\infty}(0,T;L^2(\Omega))} \leq \widetilde{C}$ (according to estimate (42)) and f (the derivative of f) is continuous. Taking the absolute value of ρ_{ij}^n , there exist positive constants \widehat{C}_l , l=1,2,3, independent of the time step k and the mesh grid h so that

$$|\rho_{ij}^n| \le \hat{C}_1 \left[1 + \hat{C}_2 h^2 + \hat{C}_3 h^4 \right].$$

This completes the proof of Lemma 4.2.

Armed with the results provided by Lemmas 3.1, 4.1 and 4.2, we are ready to prove Theorem 4.1.

Proof. (of Theorem 4.1) We recall that the error term provided by the scheme (37)-(40) is denoted by $e_{ij}^n = u_{ij}^n - \overline{u}_{ij}^n$, where \overline{u} satisfies equations (24), (31) and (32) and u is given by relations (37)-(40). So, it comes from equation (47) that

$$e_{ij}^* = e_{ij}^n + \frac{k}{2} \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) + \frac{1}{2} k^2 (3\rho_{ij}^n + \rho_{ij}^{\overline{*}}) + O(k^3 + kh^2),$$

which is equivalent to

$$e_{ij}^* = e_{ij}^n + \frac{k}{2} \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) + \frac{1}{2} k^2 (3\rho_{ij}^n + \rho_{ij}^{\overline{*}}) + C_r(k^3 + kh^2),$$

where C_r is a parameter that does not depend neither on the time step k nor the grid spacing h and ρ_{ij}^{α} is defined by (19). Taking the square, it is not hard to see that

$$(e_{ij}^*)^2 = (e_{ij}^n)^2 + k \left\{ \frac{a}{h^2} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n + \left(f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) e_{ij}^n \right\} + k^2 (3\rho_{ij}^n + \rho_{ij}^{\overline{*}}) e_{ij}^n + 2C_r(k^3 + kh^2) e_{ij}^n$$

$$+ \frac{k^2}{4} \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right)^2 + \frac{k^3}{2} \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) (3\rho_{ij}^n + \rho_{ij}^{\overline{*}}) + C_r \left(\frac{a}{h^2} (e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n) + f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) (k^4 + k^2h^2) + \frac{k^4}{4} (3\rho_{ij}^n + \rho_{ij}^{\overline{*}})^2 + C_r^2(k^3 + kh^2)^2 + C_r(k^5 + k^3h^2) (3\rho_{ij}^n + \rho_{ij}^{\overline{*}}).$$

$$(88)$$

Applying the inequalities: $2ab \le a^2 + b^2$, $(a \pm b)^2 \le 2(a^2 + b^2)$ and $(a \pm b \pm c)^2 \le 3(a^2 + b^2 + c^2)$, for every $a, b, c \in \mathbb{R}$, together with the time step restriction (41) (that is, $2ak \le h^2$), relation (88) becomes

$$(e_{ij}^*)^2 \leq (e_{ij}^n)^2 + k \left\{ \frac{a}{h^2} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n + \left(f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) e_{ij}^n \right\} + k^2 |(3\rho_{ij}^n + \rho_{ij}^{\overline{*}}) e_{ij}^n| + 2C_r(k^3 + kh^2) |e_{ij}^n| + k^2 |(2\rho_{ij}^n + \rho_{ij}^n) e_{ij}^n| + k^2 |(2\rho_{ij}^n + \rho_{ij}^n + \rho_{ij}^n) e_{ij}^n| + k^2 |(2\rho_{ij}^n + \rho_{ij}^n + \rho_{ij}^n) e_{ij}^n|$$

$$\begin{split} & + \frac{k^2}{4} \left\{ \left[\frac{a}{h^2} (e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}) \right]^2 + \left[f(u^n_{ij}) - f(\overline{u}^n_{ij}) \right]^2 + 2 \frac{a}{h^2} (e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}) \left[f(u^n_{ij}) - f(\overline{u}^n_{ij}) \right] \right\} + \\ & \frac{k^2}{4} \left(|e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}| + \frac{h^2}{a} |f(u^n_{ij}) - f(\overline{u}^n_{ij})| \right) |3\rho^n_{ij} + \rho^{\overline{*}}_{ij}| + \frac{C_r}{2} \left(|e^n_{i,j+1} - 2e^n_{ij} + e^n_{i,j-1}| + \frac{h^2}{a} |f(u^n_{ij}) - f(\overline{u}^n_{ij})| \right) \\ & f(\overline{u}^n_{ij})| \right) (k^3 + kh^2) + \frac{k^4}{2} (3\rho^n_{ij} + \rho^{\overline{*}}_{ij})^2 + 2C_r^2 (k^3 + kh^2)^2. \end{split}$$

which implies

$$\begin{split} (e_{ij}^*)^2 &\leq (e_{ij}^n)^2 + k \left\{ \frac{a}{h^2} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n + \left(f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) e_{ij}^n \right\} + k^2 |(3\rho_{ij}^n + \rho_{ij}^{\overline{\imath}}) e_{ij}^n| + 2C_r(k^3 + kh^2) |e_{ij}^n| \\ &\quad + \frac{k^2}{4} \left\{ \frac{2a^2}{h^4} \left[(e_{i,j+1}^n - e_{ij}^n)^2 + (e_{ij}^n - e_{i,j-1}^n)^2 \right] + \left[f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right]^2 + \frac{a}{h^2} \left[(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n)^2 + \left(f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right)^2 \right] \right\} + \frac{k^2}{4} \left(|e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n| + \frac{h^2}{a} |f(u_{ij}^n) - f(\overline{u}_{ij}^n)| \right) |3\rho_{ij}^n + \rho_{ij}^{\overline{\imath}}| + \frac{C_r}{2} \left(|e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n| + \frac{h^2}{a} |f(u_{ij}^n) - f(\overline{u}_{ij}^n)| \right) |3\rho_{ij}^n + \rho_{ij}^{\overline{\imath}}| + \frac{C_r}{2} \left(|e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n| + \frac{h^2}{a} |f(u_{ij}^n) - f(\overline{u}_{ij}^n)| \right) |\delta(k^3 + kh^2) + \frac{k^4}{2} (3\rho_{ij}^n + \rho_{ij}^{\overline{\imath}})^2 + 2C_r^2(k^3 + kh^2)^2. \\ &\leq (e_{ij}^n)^2 + k \left\{ \frac{a}{h^2} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n + \left(f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right) e_{ij}^n \right\} + \frac{1}{2} \left[k^3 (3\rho_{ij}^n + \rho_{ij}^{\overline{\imath}})^2 + 8C_r^2(k^5 + kh^4) \right] \\ &+ k (e_{ij}^n)^2 + \frac{k^2}{4} \left\{ \frac{2a^2}{h^2} \left[\left(\delta_y e_{i,j+\frac{1}{2}}^n \right)^2 + \left(\delta_y e_{i,j-\frac{1}{2}}^n \right)^2 \right] + \left[f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right]^2 + \frac{a}{h^2} \left[3 \left((e_{i,j+1}^n)^2 + 4(e_{ij}^n)^2 + (e_{i,j-1}^n)^2 \right) + \left(f(u_{ij}^n) - f(\overline{u}_{ij}^n) \right)^2 \right] \right\} \\ &+ C_r^2 (k^{\frac{3}{2}} + k^{\frac{1}{2}}h^2)^2 + \frac{3k}{8} \left((e_{i,j+1}^n)^2 + 4(e_{ij}^n)^2 + (e_{i,j-1}^n)^2 \right) + \frac{kh^4}{8a^2} (f(u_{ij}^n) - f(\overline{u}_{ij}^n))^2 + \frac{k^4}{2} (3\rho_{ij}^n + \rho_{ij}^{\overline{\imath}})^2 \\ &+ 2C_r^2 (k^3 + kh^2)^2. \end{split}$$

From estimates (51)-(52), we have that

$$|f(u_{ij}^n) - f(\overline{u}_{ij}^n)| \le C|e_{ij}^n|; \quad (f(u_{ij}^n) - f(\overline{u}_{ij}^n)) e_{ij}^n \le C(e_{ij}^n)^2 \text{ and } (f(u_{ij}^n) - f(\overline{u}_{ij}^n))^2 \le C^2(e_{ij}^n)^2.$$

This fact, together with estimate (89) results in

$$\begin{split} (e_{ij}^*)^2 &\leq (e_{ij}^n)^2 + k \left\{ \frac{a}{h^2} \left(e_{i,j+1}^n - 2 e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n + C(e_{ij}^n)^2 \right\} + \frac{1}{2} \left[k^3 (3 \rho_{ij}^n + \rho_{ij}^{\overline{*}})^2 + 8 C_r^2 (k^5 + k h^4) \right] \\ &+ k (e_{ij}^n)^2 + \frac{k^2}{4} \left\{ \frac{2a^2}{h^2} \left[(\delta_y e_{i,j+\frac{1}{2}}^n)^2 + (\delta_y e_{i,j-\frac{1}{2}}^n)^2 \right] + C^2 (e_{ij}^n)^2 + \frac{a}{h^2} \left[3 \left((e_{i,j+1}^n)^2 + 4 (e_{ij}^n)^2 + (e_{i,j-1}^n)^2 \right) + C^2 (e_{ij}^n)^2 \right] \right\} \\ &+ \frac{k^3}{4} (3 \rho_{ij}^n + \rho_{ij}^{\overline{*}})^2 + \frac{3k}{4} \left[(e_{i,j+1}^n)^2 + 4 (e_{ij}^n)^2 + (e_{i,j-1}^n)^2 \right] + C^2 \frac{kh^4}{4a^2} (e_{ij}^n)^2 + 2 C_r^2 (k^3 + k h^4) \\ &+ \frac{k^4}{2} (3 \rho_{ij}^n + \rho_{ij}^{\overline{*}})^2 + 2 C_r^2 (k^3 + k h^2)^2. \end{split}$$

Utilizing the time step restriction (41), $\frac{2ak}{h^2} \leq 1$, this implies

$$\begin{split} (e_{ij}^*)^2 &\leq (e_{ij}^n)^2 + \frac{ak}{h^2} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n + \frac{ak}{4} \left[(\delta_x e_{i,j+\frac{1}{2}}^n)^2 + (\delta_y e_{i,j-\frac{1}{2}}^n)^2 \right] + k \left[1 + C + \frac{C^2}{8} + \frac{C^2 k^4}{4} + \frac{C^2 h^4}{4a^2} \right] (e_{ij}^n)^2 + \frac{9k}{8} \left[(e_{i,j+1}^n)^2 + 4(e_{ij}^n)^2 + (e_{i,j-1}^n)^2 \right] + 2k C_r^2 (k^2 + h^4) + 4k^2 C_r^2 (k^4 + h^4) + \frac{k^3}{4} (3 + 2k) (3\rho_{ij}^n + \rho_{ij}^{\overline{*}})^2. \end{split}$$

Summing this up from i, j = 1, 2, ...M - 1, provides

$$\sum_{i,j=1}^{M-1} (e_{ij}^*)^2 \le \sum_{i,j=1}^{M-1} (e_{ij}^n)^2 + k \left[\frac{a}{h^2} \sum_{i,j=1}^{M-1} \left(e_{i,j+1}^n - 2e_{ij}^n + e_{i,j-1}^n \right) e_{ij}^n \right] + \frac{ak}{4} \sum_{i,j=1}^{M-1} \left[\left(\delta_y e_{i,j+\frac{1}{2}}^n \right)^2 + \left(\delta_y e_{i,j-\frac{1}{2}}^n \right)^2 \right] + k \left[1 + C + \frac{C^2}{8} + \frac{C^2 k}{4} + \frac{C^2 h^4}{4a^2} \right] \sum_{i,j=1}^{M-1} (e_{ij}^n)^2 + \frac{9k}{8} \sum_{i,j=1}^{M-1} \left[\left(e_{i,j+1}^n \right)^2 + 4(e_{ij}^n)^2 + \left(e_{i,j-1}^n \right)^2 \right] + \frac{k^3}{4} (3 + 2k) \sum_{i,j=1}^{M-1} \left[9(\rho_{ij}^n)^2 + (\rho_{ij}^{\overline{*}})^2 \right] + 2C_r^2 k \sum_{i,j=1}^{M-1} \left[k^2 + 4k^4 + 5h^4 + 2k(k^4 + h^4) \right].$$
(90)

Combining the boundary condition (40), $e_{Mj}^n = e_{0j}^n = 0$, for all j = 0, 1, ..., M, Lemmas 3.1 and 4.2, and multiplying both sides of inequality (90) by h^2 , straightforward computations yield

$$h^{2} \sum_{i,j=1}^{M-1} (e_{ij}^{*})^{2} \leq h^{2} \sum_{i,j=1}^{M-1} (e_{ij}^{n})^{2} - ak \|\delta_{y}e^{n}\|_{L^{2}(\Omega)}^{2} + \frac{ak}{2} \|\delta_{y}e^{n}\|_{L^{2}(\Omega)}^{2} + k \left[\frac{31}{4} + C + \frac{C^{2}}{8} + \frac{C^{2}k}{4} + \frac{C^{2}h^{4}}{4a^{2}}\right]$$

$$h^{2} \sum_{i,j=1}^{M-1} (e_{ij}^{n})^{2} + \frac{k^{3}h^{2}}{4} (3+2k)(M-1)^{2} \left[9\widehat{C}_{1}^{2}(1+\widehat{C}_{2}h^{2}+\widehat{C}_{3}h^{4})^{2} + \widehat{C}_{1}^{2}(1+\widehat{C}_{2}h^{2}+\widehat{C}_{3}h^{4})^{2}\right]$$

$$+2C_{r}^{2}kh^{2}(M-1)^{2} \left[k^{2} + 4k^{4} + 5h^{4} + 2k(k^{4} + h^{4})\right].$$

Since $h = \frac{1}{M}$, $k \le 1 + k^2$ and $h^2 \le 1 + h^4$, this becomes

$$h^{2} \sum_{j,i=1}^{M-1} (e_{ij}^{*})^{2} \leq h^{2} \sum_{j,i=1}^{M-1} (e_{ij}^{n})^{2} - \frac{ak}{2} \|\delta_{y}e^{n}\|_{L^{2}(\Omega)}^{2} + k \left[\frac{31}{2} + C + \frac{C^{2}}{8} + \frac{C^{2}k}{4} + \frac{C^{2}h^{4}}{4a^{2}} \right] h^{2} \sum_{j,i=1}^{M-1} (e_{ij}^{n})^{2} + \frac{5\widehat{C}_{1}^{2}k^{3}}{2} (5 + 2k^{2})(1 + \widehat{C}_{2} + \widehat{C}_{2}h^{4} + \widehat{C}_{3}h^{4})^{2} + 2C_{r}^{2}k \left[k^{2} + 4k^{4} + 5h^{4} + 2k(k^{4} + h^{4}) \right].$$

which implies

$$||e^*||_{L^2(\Omega)}^2 \le ||e^n||_{L^2(\Omega)}^2 + \widehat{C}_4 \left\{ k \left[1 + k + h^4 \right] ||e^n||_{L^2(\Omega)}^2 + k^3 \left[1 + k^2 + h^4 + h^6 + h^8 + k^2 h^2 + k^2 h^4 + k^2 h^6 + k^2 h^8 \right] + k \left(1 + k \right) \left(k^4 + h^4 \right) \right\}, \tag{91}$$

where we absorbed all the constants into a constant \widehat{C}_4 .

Similarly, one shows that

$$||e^{**}||_{L^{2}(\Omega)}^{2} \leq ||e^{*}||_{L^{2}(\Omega)}^{2} + \widehat{C}_{5} \left\{ k \left[1 + k + h^{4} \right] ||e^{*}||_{L^{2}(\Omega)}^{2} + k^{3} \left[1 + k^{2} + h^{4} + h^{6} + h^{8} + k^{2}h^{2} + k^{2}h^{4} + k^{2}h^{6} + k^{2}h^{8} \right] + k \left(1 + k \right) \left(k^{4} + h^{4} \right) \right\},$$

$$(92)$$

where all the constants have been absorbed into a constant \widehat{C}_5 , and

$$\|e^{n+1}\|_{L^{2}(\Omega)}^{2} \leq \|e^{**}\|_{L^{2}(\Omega)}^{2} + \widehat{C}_{6} \left\{ k \left[1 + k + h^{4} \right] \|e^{**}\|_{L^{2}(\Omega)}^{2} + k^{3} \left[1 + k^{2} + h^{4} + h^{6} + h^{8} + k^{2}h^{2} + k^{2}h^{4} + k^{2}h^{6} + k^{2}h^{8} \right] + k \left(1 + k \right) \left(k^{4} + h^{4} \right) \right\},$$

$$(93)$$

where all the constants have been absorbed into a constant \widehat{C}_6 .

Now, setting

$$\varphi_1(k,h) = 1 + k + h^2 + h^4, \tag{94}$$

and

$$\varphi_2(k,h) = k^3 \left[1 + k^2 + h^4 + h^6 + h^8 + k^2 h^2 + k^2 h^4 + k^2 h^6 + k^2 h^8 \right] + k(1+k)(k^4 + h^4), \tag{95}$$

plugging estimates (91)-(93), straightforward calculations yield

$$\|e^{n+1}\|_{L^2(\Omega)}^2 \leq \|e^n\|_{L^2(\Omega)}^2 + k\left\{\widehat{C}_4 + \widehat{C}_5 + \widehat{C}_6 + k\left[\widehat{C}_4\widehat{C}_5 + \widehat{C}_6(\widehat{C}_4 + \widehat{C}_5) + k\widehat{C}_4\widehat{C}_5\widehat{C}_6\varphi_1(k,h)\right]\varphi_1(k,h)\right\}$$

$$\varphi_{1}(k,h)\|e^{n}\|_{L^{2}(\Omega)}^{2} + \left[\widehat{C}_{4} + \widehat{C}_{5} + \widehat{C}_{6} + k\left[\widehat{C}_{4}\widehat{C}_{5} + \widehat{C}_{4}\widehat{C}_{6} + \widehat{C}_{5}\widehat{C}_{6} + k\widehat{C}_{4}\widehat{C}_{5}\widehat{C}_{6}\varphi_{1}(k,h)\right]\varphi_{1}(k,h)\right]\varphi_{2}(k,h).$$

Absorbing all the constants into a constant \widehat{C}_7 , this yields

$$||e^{n+1}||_{L^{2}(\Omega)}^{2} \leq ||e^{n}||_{L^{2}(\Omega)}^{2} + \widehat{C}_{7} \left\{ k \left[1 + k \left(1 + k\varphi_{1}(k,h) \right) \varphi_{1}(k,h) \right] \varphi_{1}(k,h) \right\} \varphi_{1}(k,h) + \left[1 + k \left[1 + k\varphi_{1}(k,h) \right] \varphi_{1}(k,h) \right] \varphi_{2}(k,h) \right\}.$$

Summing this up from n = 0, 1, 2, ..., p - 1, for any nonnegative integer p such that $1 \le p \le N$, we obtain

$$||e^{p}||_{L^{2}(\Omega)}^{2} \leq ||e^{0}||_{L^{2}(\Omega)}^{2} + \widehat{C}_{7} \left\{ pk \left[1 + k \left(1 + k\varphi_{1}(k, h) \right) \varphi_{1}(k, h) \right] \varphi_{1}(k, h) \sum_{n=0}^{p-1} ||e^{n}||_{L^{2}(\Omega)}^{2} + p \left[1 + k \left[1 + k\varphi_{1}(k, h) \right] \varphi_{1}(k, h) \right] \varphi_{2}(k, h) \right\}.$$

$$(96)$$

It comes from the initial condition given in (40), that $e_{ij}^0 = 0$, for $0 \le i, j \le M$. Applying the Gronwall Lemma, estimate (96) provides

$$\|e^{p}\|_{L^{2}(\Omega)}^{2} \leq \widehat{C}_{7} \exp\left\{\widehat{C}_{7} p k \left[1 + k \left(1 + k \varphi_{1}(k, h)\right) \varphi_{1}(k, h)\right] \varphi_{1}(k, h)\right\} p \left[1 + k \left[1 + k \varphi_{1}(k, h)\right] \varphi_{1}(k, h)\right] \varphi_{2}(k, h).$$
(97)

But $k = \frac{T}{N}$, so $\widehat{C}_7 kp = \widehat{C}_7 T \frac{p}{N} \le \widehat{C}_7 T$ (since $p \le N$). This fact, together with inequality (97) result in

$$\|e^p\|_{L^2(\Omega)}^2 \leq \widehat{C}_7 T \exp\left\{\widehat{C}_7 T \left[1 + k\left(1 + k\varphi_1(k,h)\right)\varphi_1(k,h)\right] \varphi_1(k,h)\right\} \left[1 + k\left[1 + k\varphi_1(k,h)\right]\varphi_1(k,h)\right] \varphi_3(k,h)^2,$$

where $\varphi_3(k,h)^2 = k^{-1}\varphi_2(k,h)$, $\varphi_2(k,h)$ is given by equation (95). Taking the square root, it is easy to see that

$$||e^{p}||_{L^{2}(\Omega)} \leq \sqrt{\widehat{C}_{7}T\left[1+k\left[1+k\varphi_{1}(k,h)\right]\varphi_{1}(k,h)\right]} \exp\left\{\frac{\widehat{C}_{7}T}{2}\left[1+k\left(1+k\varphi_{1}(k,h)\right)\varphi_{1}(k,h)\right]\varphi_{1}(k,h)\right\} \varphi_{3}(k,h). \tag{98}$$

It comes from equality $\varphi_3(k,h)^2 = k^{-1}\varphi_2(k,h)$, and equation (95) that

$$\varphi_3(k,h)^2 = k^2 \left[1 + k^2 + h^4 + h^6 + h^8 + k^2 h^2 + k^2 h^4 + k^2 h^6 + k^2 h^8 \right] + (1+k)(k^4 + h^4) \le (k+kh^2)^2 (\widetilde{C}_8 + \varphi_4(k,h)),$$

where \widetilde{C}_8 is a positive constant independent of k and h, and $\varphi_4(k,h)$ tends to zero when $k,h \to 0$. Taking the maximum over p of estimate (98), for $0 \le p \le N$, the proof of Theorem 4.1 is completed thanks to equation (64).

5 Numerical experiments and Convergence rate

In this section we construct an exact solution to the initial-boundary value problem (1)-(3) for a specific source term f. Furthermore, using Matlab we perform some numerical experiments in bidimensional case. In that case we obtain satisfactory results, so our algorithm performances are not worse for multidimensional problems. We consider two cases which are physical examples associated with the diffusive coefficient a = 1, together with the example introduced in [8]. We confirm the predicted convergence rate from the theory (see Section 2, Page 6, last paragraph). This convergence rate is obtained by listing in Tables 1-6 the

errors between the computed solution and the exact one with different values of mesh size h and time step k, satisfying $k = \frac{1}{2}h^2$. Finally, we look at the error estimates of our proposed method for the parameter T = 1.

Assuming that the exact solution to problem (1)-(3) is of the form $\overline{u}(x,y,t) = [1 + \exp(ct + dx + by)]^{-n}$, where n is an integer. By simple calculations, it holds

$$\overline{u}_t(x, y, t) = -nc \exp(ct + dx + by) \left[1 + \exp(ct + dx + by) \right]^{-n-1},$$
(99)

$$\overline{u}_x(x,y,t) = -nd\exp(ct + dx + by)\left[1 + \exp(ct + dx + by)\right]^{-n-1}$$

and

$$\overline{u}_{xx}(x,y,t) = -nd^2 \exp(ct + dx + by) \left[1 - n \exp(ct + dx + by) \right] \left[1 + \exp(ct + dx + by) \right]^{-n-2}.$$
 (100)

In way similar

$$\overline{u}_{yy}(x,y,t) = -nb^2 \exp(ct + dx + by) \left[1 - n \exp(ct + dx + by) \right] \left[1 + \exp(ct + dx + by) \right]^{-n-2}.$$
 (101)

Combining equations (99)-(101), it is not hard to see that

$$\overline{u}_t - (\overline{u}_{xx} + \overline{u}_{yy}) = -n \exp(ct + dx + by) (1 + \exp(ct + dx + by))^{-n-1} \left\{ c - (d^2 + b^2) \left[1 - n \exp(ct + dx + by) \right] (1 + \exp(ct + dx + by))^{-1} \right\}$$

Setting $c = -(d^2 + b^2)$, this becomes

$$\overline{u}_t - (\overline{u}_{xx} + \overline{u}_{yy}) = n(d^2 + b^2) \exp(ct + dx + by) (1 + \exp(ct + dx + by))^{-n-1} \{1 + [1 - n \exp(ct + dx + by)]$$

$$(1 + \exp(ct + dx + by))^{-1} \} = n(d^2 + b^2) \exp(ct + dx + by) [2 + (1 - n) \exp(ct + dx + by))]$$

$$(1 + \exp(ct + dx + by))^{-n-2}.$$

$$(102)$$

•: Case 1: n = 1. In this case, equation (102) becomes

$$\overline{u}_t - (\overline{u}_{xx} + \overline{u}_{yy}) = 2(d^2 + b^2) \exp(ct + dx + by) (1 + \exp(ct + dx + by))^{-3}.$$

Now, taking $2(d^2+b^2)=1$, this gives $b^2=\frac{1}{2}-d^2$. Since b^2 must be strictly greater than zero, this implies $d^2<\frac{1}{2}$. For $d=\pm\frac{\sqrt{3}}{3}$, this implies $b=\pm\frac{\sqrt{6}}{6}$ and $c=-\frac{1}{2}$. Letting $f(\overline{u})=(1-\overline{u})\overline{u}^2$, our exact solution is given by $\overline{u}(x,y,t)=\left[1+\exp\left(-\frac{1}{2}t+\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{6}y\right)\right]^{-1}$, for $t\in[0,1]$ and $(x,y)\in[0,1]^2$. The initial and boundary conditions are determined by this solution.

•: Case 2: n = -1. It comes from equation (102) that

$$\overline{u}_t - (\overline{u}_{xx} + \overline{u}_{yy}) = -2(d^2 + b^2) \exp(ct + dx + by).$$

Since $\overline{u}=1+\exp(ct+dx+by)$, so $-\exp(ct+dx+by)=1-u$. Taking $-2(d^2+b^2)=-1$, it holds $d=\pm\frac{\sqrt{3}}{3},\ b=\pm\frac{\sqrt{6}}{6}$ and $c=-\frac{1}{2}$. Setting $f(\overline{u})=1-\overline{u}$, we consider the exact solution defined as $\overline{u}(x,y,t)=1+\exp\left(-\frac{1}{2}t+\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{6}y\right)$, for $t\in[0,1]$ and $(x,y)\in[0,1]^2$. The initial and boundary conditions are determined by this solution.

To analyze the convergence rate of our numerical scheme, we take the mesh size $h \in \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}\}$ and time step $k \in \{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \frac{1}{2^{10}}, \frac{1}{2^{11}}\}$, by a mid-point refinement. Under the time step restriction (41), we set $k = \frac{1}{2}h^2$ and we compute the error estimates: $||E(u)||_{L^2(0,T;L^2)}$, $||E(u)||_{L^\infty(0,T;L^2)}$ and $||E(u)||_{L^1(0,T;L^2)}$ related to the time-split method to see that the algorithm is stable, second order accuracy in time and fourth order convergent in space. In addition, we plot the approximate solution, the exact one and the errors versus n. From this analysis, a three-level explicit time-split MaCormack method is both efficient and effective than a two-level linearized compact ADI approach. In fact, although the two-level

linearized compact ADI scheme has the same convergent rate (see [37], Theorem 6.6, p. 19) this method requires too much computer times to achieve the solution. Furthermore, when h varies in the given range, we observe from Tables 1-6 that the approximation errors $O(k^{\beta}) + O(h^{\theta})$ are dominated by the h-terms $O(h^{\theta})$ (or k-terms $O(k^{\beta})$). So, the ratio r_u^m , where $m = 1, 2, \infty$, of the approximation errors on two adjacent mesh levels Ω_{2h} and Ω_h is approximately $(2h)^{\theta}/h^{\theta} = 2^{\theta}$, where m refers to the $L^m(0,T;L^2(\Omega))$ -error norm. Hence, we can simply use r_u^m to estimate the corresponding convergence rate with respect to h. Define the norms for the approximate solution u, the exact one \overline{u} , and the errors E(u), as follows

$$||u||_{L^{2}(0,T;L^{2})} = \left[k \sum_{n=0}^{N} ||u^{n}||_{L_{f}^{2}}^{2}\right]^{\frac{1}{2}}; \ |||\overline{u}||_{L^{2}(0,T;L^{2})} = \left[k \sum_{n=0}^{N} ||\overline{u}^{n}||_{L_{f}^{2}}^{2}\right]^{\frac{1}{2}};$$

$$|||E(u)|||_{L^{2}(0,T;L^{2})} = \left[k \sum_{n=0}^{N} ||u^{n} - \overline{u}^{n}||_{L_{f}^{2}}^{2}\right]^{\frac{1}{2}}; |||E(u)||_{L^{1}(0,T;L^{2})} = k \sum_{n=0}^{N} ||u^{n} - \overline{u}^{n}||_{L_{f}^{2}};$$

and

$$\||E(u)|\|_{L^{\infty}(0,T;L^{2})} = \max_{0 \leq n \leq N} \|u^{n} - \overline{u}^{n}\|_{L_{f}^{2}}.$$

• Test 1. Let Ω be the unit square $(0,1) \times (0,1)$ and T be the final time, T=1. We assume that the diffusive coefficient a=1, and we choose the force $f(\overline{u})=(1-\overline{u})\overline{u}^2$, in such a way that the exact solution \overline{u} is given by

$$\overline{u}(x, y, t) = \left[1 + \exp\left(-\frac{1}{2}t + \frac{\sqrt{3}}{3}x + \frac{\sqrt{6}}{6}y\right)\right]^{-1}.$$

The initial and boundary conditions are given by this solution. We take the mesh size and time step: $h \in \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}\}$ and $k \in \{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^{10}}, \frac{1}{2^{11}}\}$.

Tables 1,2. Analyzing of convergence rate $O(h^{\theta} + \Delta t^{\beta})$ for time-split MacCormack by r_u^m , with varying time step $k = \Delta t$ and mesh grid $h = \Delta x$.

Case: $k = \frac{1}{2}h^2$.

h	$ E(u) _{L^2}$	r_u^2	$ E(u) _{L^{\infty}}$	r_u^{∞}	$ E(u) _{L^1}$	r_u^1
2^{-1}	0.0054	—-	0.0058	—-	0.0053	-
2^{-2}	0.0014	3.8571	0.0014	4.1429	0.0014	3.7857
2^{-3}	0.372×10^{-3}	3.7634	0.3849×10^{-3}	3.6373	0.3717×10^{-3}	3.7665
2^{-4}	0.966×10^{-4}	3.8509	0.995×10^{-4}	3.8683	0.963×10^{-4}	3.8598
2^{-5}	0.2459×10^{-4}	3.9284	0.2529×10^{-4}	4.0963	0.2450×10^{-4}	3.9306

Case: $k = h^2$.

h	$ E(u) _{L^2}$	r_u^2	$ E(u) _{L^{\infty}}$	r_u^{∞}	$ E(u) _{L^1}$	r_u^1
2^{-1}	0.0200		0.0227		0.0189	
2^{-2}	0.0050	4.0000	0.0069	3.2899	0.0049	3.8571
2^{-3}	NAN	— -	inf		Nan	— -

• Test 2. Now, let Ω be the unit square $(0,1)^2$ and T=1. The diffusive term a is assumed equals 1. We choose the force f such that the analytic solution \overline{u} is defined as

$$\overline{u}(x, y, t) = 1 + \exp\left(-\frac{1}{2}t + \frac{\sqrt{3}}{3}x + \frac{\sqrt{6}}{6}y\right), \text{ and } f(\overline{u}) = 1 - \overline{u}.$$

The initial and boundary conditions also are given by the exact solution \overline{u} . Similar to **Test** 1, we take the mesh size and time step: $h \in \{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}\}$ and $k \in \{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^6}, \frac{1}{2^7}, \frac{1}{2^8}, \frac{1}{2^9}, \frac{1}{2^{10}}, \frac{1}{2^{11}}\}$.

Tables 3,4. Convergence rates $O(h^{\theta} + \Delta t^{\beta})$ for time-split MacCormack by r_u^m , with varying spacing h and time step k.

Case: $k = \frac{1}{2}h^2$.

h	$ E(u) _{L^2}$	r_u^2	$ E(u) _{L^{\infty}}$	r_u^{∞}	$ E(u) _{L^1}$	r_u^1
2^{-1}	0.0245		0.0310		0.0232	
2^{-2}	0.0060	4.0833	0.0072	4.3056	0.0060	3.8667
2^{-3}	0.16×10^{-2}	3.75	0.19×10^{-2}	3.7895	0.16×10^{-2}	3.75
2^{-4}	0.4×10^{-3}	4.0000	0.5×10^{-3}	3.8000	0.4×10^{-3}	4.0000
2^{-5}	0.1061×10^{-3}	3.7700	0.1248×10^{-3}	4.0064	0.1052×10^{-3}	3.8023

Case: $k = h^2$.

h	$ E(u) _{L^2}$	r_u^2	$ E(u) _{L^{\infty}}$	r_u^{∞}	$ E(u) _{L^1}$	r_u^1
2^{-1}	0.0892		0.1163		0.0800	
2^{-2}	0.0364	2.4505	0.0814	1.4287	0.0320	2.5000
2^{-3}	0.2132×10^{20}		1.5596×10^{20}		0.0409×10^{20}	

• Test 3. Finally, let Ω be the unit square $(0,1) \times (0,1)$ and T=1. We assume that a=1, and the force f is chosen such that the exact solution u is given by

$$\overline{u}(x,y,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{3}{4}t + \frac{1}{4}x + \frac{1}{4}y\right), \quad \text{and} \quad f(\overline{u}) = (1 - \overline{u}^2)\overline{u}.$$

The initial and boundary conditions are given by the exact solution \overline{u} .

Similar to both **Tests** 1, 2 the mesh size and time step are chosen such that: $h \in \{\frac{1}{2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}{2^4}, \frac{1}{2^5}\}$ and $k \in \{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \frac{1}{2^5}, \frac{1}$

Tables 5,6. Convergence rates $O(h^{\theta} + \Delta t^{\beta})$ for time-split MacCormack by r_u^m , with varying spacing h and time step Δt .

Case: $k = \frac{1}{2}h^2$.

h	$ E(u) _{L^2}$	r_u^2	$ E(u) _{L^{\infty}}$	r_u^{∞}	$ E(u) _{L^1}$	r_u^1
2^{-1}	0.0112		0.0151		0.0106	
2^{-2}	0.0029	3.8621	0.0036	4.1944	0.0028	3.7857
2^{-3}	0.8×10^{-3}	3.6250	0.1×10^{-2}	3.6000	0.8×10^{-3}	3.5000
2^{-4}	0.2001×10^{-3}	3.9980	0.2506×10^{-3}	3.9904	0.5496×10^{-3}	4.0796
2^{-5}	0.509×10^{-4}	3.9312	0.638×10^{-4}	3.9279	0.5000×10^{-4}	3.9220

Case: $k = h^2$.

h	$ E(u) _{L^2}$	r_u^2	$ E(u) _{L^{\infty}}$	r_u^{∞}	$ E(u) _{L^1}$	r_u^1
2^{-1}	0.0400		0.0546		0.0363	
2^{-2}	0.0150	2.6667	0.6601	1.6957	0.5821	2.6691
2^{-3}	NaN		Inf		NaN	

The analysis on the convergence of the numerical scheme presented in Section 4, has suggested that our algorithm is first order convergent in time and fourth order accurate in space. If the result provided Analysis of stability and convergence of a three-level explicit time-split MacCormack method with a=1.

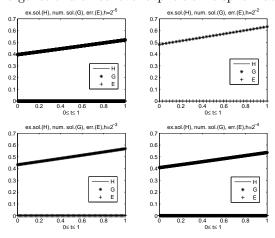


Figure 1:
$$\overline{u}(x,y,t) = \left[1 + \exp\left(-\frac{1}{2}t + \frac{\sqrt{3}}{3}x + \frac{\sqrt{6}}{6}y\right)\right]^{-1}$$
 and $f(\overline{u}) = (1 - \overline{u})\overline{u}^2$

in Section 2, page 6, last paragraph is to believe, this shows that the time-split MacCormack scheme is inconsistent. Surprisingly, it comes from **Tests 1-3**, more precisely Figures 1-3 and **Tables 1-6**, that the three-level explicit time-split MacCormack technique is stable, second order accurate in time and fourth order convergent in space under the time step restriction (41), which confirms the theoretical result provided in Section 2, page 6, last paragraph. Thus, the considered method applied to initial-boundary value problem (1)-(3) is: stable, consistent, second order convergent in time and fourth order accurate in space.

6 General conclusion and future works

We have studied in detail the stability, error estimates and convergence rate of a three-level explicit time-split MacCormack method for solving the 2D nonlinear reaction-diffusion equation (1)-(3). The analysis has suggested that our method is stable, consistent, second order accuracy in time and fourth order convergent in space under the time step restriction (41). This convergence rate is confirmed by a large set of numerical experiments (see both Figures 1-3 and **Tables 1-6**). Numerical evidences also show that the new algorithm is: (1) more efficient and effective than a two-level linearized compact ADI method, (2) fast and robust tools for the integration of general systems of parabolic PDEs. However, the time-split MacCormack method is not is a satisfactory approach for solving high Reynolds number flows where the viscous region becomes very thin. For these flows, the mesh grid must be highly refined in order to accurately resolve the viscous regions. This leads to very small time steps and subsequently long computing times. To overcome this difficulty, MacCormack developed a hybrid version of his scheme, which is known as MacCormack rapid solver method [18]. This hybrid scheme is an explicit-implicit method which has been proved to be from 10 to 100 more faster than a time-split MacCormack algorithm (see [2], P. 632). The rapid solver method will be applied to the two-dimensional nonlinear reaction-diffusion equations in our future works.

Analysis of stability and convergence of a three-level explicit time-split MacCormack method with a = 1.

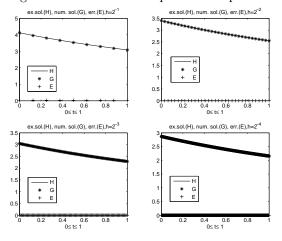


Figure 2: $\overline{u}(x,y,t)=1+\exp\left(-\frac{1}{2}t+\frac{\sqrt{3}}{3}x+\frac{\sqrt{6}}{6}y\right)$ and $f(\overline{u})=1-\overline{u}$

Analysis of stability and convergence of a three-level explicit time-split MacCormack method with a = 1.

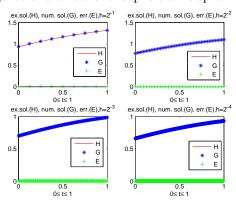


Figure 3: $\overline{u}(x,y,t)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{3}{4}t+\frac{1}{4}x+\frac{1}{4}y\right)$ and $f(\overline{u})=(1-\overline{u}^2)\overline{u}$

References

- [1] B. Alberts et al.. "Molecular biology of the cell", 4th ed. Garland Science, USA (2002).
- [2] F. A. Anderson, R. H. Pletcher, J. C. Tannehill. "Computational fluid mechanics and Heat Transfer". Second Edition, Taylor and Francis, New York, (1997).
- [3] A. Araujo, S. Barbeiro, P. Serrhano. "Convergence of finite difference schemes for nonlinear complex reaction-diffusion processes", SIAM J. Numer. Anal., 53 (2015) 228-250.
- [4] D. G. Aronson, H. F. Weinberger. "Multidimensional nonlinear diffusion arising in population genetics", Advance mathematics, vol. 30, (1978) 33-76.
- [5] U. M. Ascher, S. J. Ruuth, B. Wetton. "Implicit-explicit methods for time-dependent partial differential equations", SIAM J. Num. Anal., 32 (1995) 797-823.
- [6] I. Barras, E. J. Cranspin, P. K. Maini. "Mode transitions in a model reaction-diffusion system driven by domain growth and noise", Bull. Math. Biol., vol. 68 (2006) 981-995.
- [7] R. Casten, R. Holland. "Instability results for reaction-diffusion equation with Neumann boundary condition". J. Diff. Eq., vol. 27 (1978) 266-273.
- [8] N. Chafee, E. F. Infante. "A bifurcation problem for a nonlinear partial differential equation of parabolic type". Appl. Anal., 4(1) (1974) 17-37.
- [9] V. G. Danilov, V. P. Maslov, K. A. Volosov. "Mathematical modeling of Heat Transfer Processes", Kluwer, Dordrecht, (1995).
- [10] R. I. Fernandes, B. Bialecki, G. Fairweather. "An ADI extrapolated Crank-Nicolson orthogonal spline collocation method for nonlinear reaction-diffusion systems on evolving domain", J. comput. Phys., 299 (2015) 561-580.
- [11] J. S. Guo, Y. Morita. "Entire solutions of reaction-diffusion equations and an application to discrete diffusive equations", Discrete Contin. Dyn. Sys., vol. 12 (2005) 193-212.
- [12] D. Holcman, Z. Schuss. "Modeling calcium dynamics in dendritic spines". SIAM J. Appl. Math., vol 65 No 2, (2005) (1006)-(1026).
- [13] P. D. Lax, B. Wendroff. "Systems of conservation laws", Comm. Pure & Appl. Math. 13 (160) 217-237.
- [14] B. Li, H. Gao, W. Sun. "Unconditionally optimal error estimates of a Crank-Nicolson Galerkin method for the nonlinear thermistor equations", SIAM J. Numer. Anal., 53 (2014) 933-954.
- [15] D. Li, C. Zhang. "Split Newton iterative algorithm and it applications", Appl. Math. Comp., 217 (2010) 2260-22265.
- [16] D. Li, C. Zhang, W. Wang, Y. Zhang. "Implicit-explicit predictor-corrector schemes for nonlinear parabolic differential equations", Appl. Math. Model., 35(6) (2011) 2711-2722.
- [17] N. Li, J. Steiner, S. Tang. "Convergence and stability analysis of an explicit finite difference method for 2D reaction-diffusion equations", J. Aust. Math. Soc. Ser. B. 36 (1994) 234-241.
- [18] R. W. MacCormack. "An efficient numerical method for solving the time-dependent compressible Navier-Stokes equations at high Reynolds numbers", NASA TM (1976) 73-129.
- [19] R. W. MacCormack. "A numerical method for solving the equations of compressible viscous-flows", AIAA paper 81-0110, St. Louis, Missouri (1981).
- [20] R. W. MacCormack. "Current status of numerical solutions Navier-Stokes equations", AIAA paper 85-0032, Reno, Nevada (1985).

- [21] R. W. MacCormack, B. S. Baldwin. "A numerical method for solving the Navier-Stokes equations with applications to Shock-Boundary Layer Interactions", AIAA paper 75-1, Pasadena, California (1975).
- [22] R. W. MacCormack, A. J. Paullay. "Computational efficiency achieved by time splitting of finite difference operators", AIAA paper 72-154, San Diego, California (1972).
- [23] R. W. MacCormack, A. J. Paullay. "The effect of viscosity in hyrvelocity impact cratering", AIAA paper 69-354, American Institute of Aeronautics and astrophysics, Cincinnati (1969).
- [24] J. D. Murray. "Mathematical biology: Spatial models and Biomedical Applications", 3th ed., Springer-Verlag, (2003).
- [25] F. T. Namio, E. Ngondiep, R. Ntchantcho, J. C. Ntonga. "Mathematical models of complete shallow water equations with source terms, stability analysis of Lax-Wendroff scheme", J. Theor. Comput. Sci., Vol. 2(132) (2015).
- [26] E. Ngondiep. "Stability analysis of MacCormack rapid solver method for evolutionary Stokes-Darcy problem", J. Comput. Appl. Math. 345(2019), 269-285, 17 pages.
- [27] E. Ngondiep, "Long Time Stability and Convergence Rate of MacCormack Rapid Solver Method for Nonstationary Stokes-Darcy Problem", Comput. Math. Appl., Vol 75, (2018), 3663-3684, 22 pages.
- [28] E. Ngondiep. "An efficient three-level explicit time-split method for solving 2D heat conduction equations", submitted.
- [29] E. Ngondiep, "Long time unconditional stability of a two-level hybrid method for nonstationary incompressible Navier-Stokes equations", J. Comput. Appl. Math. 345(2019), 501-514, 14 pages.
- [30] E. Ngondiep. "Asymptotic growth of the spectral radii of collocation matrices approximating elliptic boundary problems", Int. J. Appl. Math. Comput., 4(2012), 199-219, 20 pages.
- [31] E. Ngondiep, "Error Estimate of MacCormack Rapid Solver Method for 2D Incompressible Navier-Stokes Problems", submitted.
- [32] E. Ngondiep, R. Alqahtani and J. C. Ntonga, "Stability Analysis and Convergence Rate of MacCormack Scheme for Complete Shallow Water Equations with Source Terms", submitted.
- [33] K. M. Owolabi. "Robust IMEX schemes for solving two-dimensional reaction-diffusion models", Int. J. Nonl. SC. Numer. Sim., 15 (2015) 271-284.
- [34] K. M. Owolabi, K. C. Patidar. "High order time stepping methods for solving time-dependent reaction-diffusion equations arising in biology", Appl. Math. Comput. 240 (2014) 30-50.
- [35] A. Quarteroni, A. Valli. "Numerical approximations of partial differential equations", Springer-Verlag, New-York (1997).
- [36] A. M. Turing. "The Chemical Basis of Morphogenesis", Phil. Trans. R. Soc. London, vol. 237 (1995) 37-72.
- [37] F. Wu, X. Cheng, D. Li, J. Duan. "A two-level linearized compact ADI scheme for two-dimensional nonlinear reaction-diffusion equations", Comput. Math. Appl., (2018).
- [38] A. Xiao, G. Zhang, X. Yi. "Two classes of implicit-explicit multistep methods for nonlinear stiff initial-value problems", Appl. Math. Comput., 247 (2014) 47-60.